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CONVERGENCE OF THE NATURAL p-MEANS FOR THE p-LAPLACIAN*

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Abstract. We prove uniform convergence in Lipschitz domains of approximations to p-harmonic functions obtained using the natural p-means introduced by Ishiwata, Magnanini, and Wadade [*Calc. Var. Partial Differ. Equ.* **56** (2017) 97]. We also consider convergence of natural means in the Heisenberg group in the case of smooth domains.

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1. Introduction

The characterization of p-harmonic functions by asymptotic mean value expansions from [21], originally motivated by the tug-of-war games approach from [23] for the ∞-Laplacian and [24] for the p-Laplacian, has been extended to more general mean value expansions in [2, 12] including the case of variable exponent $p(x)$, the parabolic case in [15], and the Heisenberg group [10]. We refer to the monograph [6] for historical remarks and more references.

For $1 < p < \infty$ there are several notions of solutions for the p-harmonic equation:

- weak supersolutions (based on integration by parts);
- potential theoretical supersolutions (based on the comparison principle);
- viscosity supersolutions (based on a pointwise comparison principle);
- supersolutions in the sense of means (based on generalized means).

It is easy to see that weak supersolutions are potential theoretic and viscosity supersolutions. That bounded potential theoretic supersolutions are weak supersolutions is not trivial even for $p = 2$, where it follows from the Riesz representation for subhamornic functions [25]. Fore general $p \neq 2$ this key result is due to Lindqvist [18]. The equivalence between viscosity supersolutions and potential theoretic supersolutions is in [14].

The definition of supersolutions in the sense of means and its equivalence to viscosity solutions is in [21] and it is further developed in [15] to include the parabolic p-Laplacian. Let us recall the definition:

*Dedicated with admiration to our friend Enrique Zuazua on his 60^{th}-birthday.

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Definition 1.1. Let $u\colon \Omega \mapsto \mathbb{R}$ be a lower-semicontinous function defined on a domain $\Omega \subset \mathbb{R}^n$. For $1 < p \leq \infty$ set $\alpha = \frac{p-2}{n+p}$ and $\beta = \frac{n+2}{n+p}$. Note that $\alpha + \beta = 1$. The function u is a *supersolution in the sense of means* if whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ touches u from below at x_0; that is, $\phi(x) \leq u(x)$ for $x \in \Omega$ and $\phi(x_0) = u(x_0)$, we have

$$\phi(x_0) \leq \frac{\alpha}{2}\left(\sup_{B_\epsilon}\phi + \inf_{B_\epsilon}\phi\right) + \beta \fint_{B_\epsilon}\phi + o(\epsilon^2), \tag{1.1}$$

as $\varepsilon \to 0^+$.

By $0 \leq h(\epsilon) + o(\epsilon^2)$ as $\epsilon \to 0^+$ we mean that:

$$\lim_{\epsilon \to 0^+} \frac{[h(\epsilon)]^-}{\epsilon^2} = 0.$$

We define subsolution in the sense of means similarly using upper-semicontinuous functions, test functions touching from above, and reversing the sign of (1.1). Hence, solution in the sense of means is defined accordingly.

Definition 1.2. A p-harmonic function $u\colon \Omega \mapsto \mathbb{R}$ *in the sense of means* is a continuous function that is both, a supersolution and a subsolution in the sense of means. We use the symbol $\doteq$ to denote this type of expansions, so that we write

$$u(x_0) \doteq \frac{\alpha}{2}\left(\sup_{B_\epsilon}u + \inf_{B_\epsilon}u\right) + \beta \fint_{B_\epsilon}u + o(\epsilon^2), \tag{1.2}$$

We emphasize that (1.2) means that (1.1) and its analogue for subsolutions must hold.

Clearly if (1.2) holds in the classical sense, it holds in the sense of means. The reciprocal statement holds for p-harmonic functions on the plane [3] for $1 < p < \infty$, it is false for $p = \infty$ ([14]) and it is unknown in $\mathbb{R}^n$ for $n \geq 3$, except for the linear case $p = 2$.

Recall that the homogeneous ∞-Laplacian is the operator

$$\Delta_\infty^H u = \frac{1}{|\nabla u|^2}\langle D^2u\,\nabla u, \nabla u\rangle, \tag{1.3}$$

and the homogeneous p-Laplacian is given by

$$\Delta_p^H u = \Delta u + (p-2)\Delta_\infty^H u = \frac{n+2}{\beta}\left[\beta\frac{1}{n+2}\Delta u + \alpha\,\Delta_\infty^H u\right], \tag{1.4}$$

where $\alpha = \frac{p-2}{n+p}$ and $\beta = \frac{n+2}{n+p}$. We have expressed the homogeneous p-Laplacian as a linear combination of a multiple of the Laplacian and the infinity Laplacian. With this choice of notation we have for $u \in C^2(\Omega)$ with $\nabla u(x) \neq 0$ the expansion

$$\begin{aligned}\frac{\alpha}{2}\left(\sup_{B_\epsilon(x)}u + \inf_{B_\epsilon(x)}u\right) + \beta\fint_{B_\epsilon(x)}u &= u(x) \\ &+ \alpha\Delta_\infty^H u(x) + \beta\tfrac{1}{(n+2)}\Delta u(x) \\ &+ o(\epsilon^2) \\ &= u(x) + \tfrac{\beta}{n+2}\Delta_p^H u + o(\epsilon^2),\end{aligned} \tag{1.5}$$

as $\varepsilon \to 0^+$. We conclude that in the smooth non vanishing gradient case u is p-harmonic if and only if we have the expansion

$$u(x) = \frac{\alpha}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) + \beta \fint_{B_\epsilon(x)} u + o(\epsilon^2)$$

as $\varepsilon \to 0^+$. The same expansion holds for general viscosity solutions replacing $=$ by $\doteq$. This is the main result in [21].

Outline of the paper: In Section 2 we recall the dynamic programming principle (DPP) proved in [9] and its relation with asymptotic mean value properties. In Section 3 we recall the properties of natural p-means following [13]. These are needed in order to prove the DPP for the natural p-means (Thm. 4.3) and the comparison principle for natural p-means in Lipschitz domains (Thm. 4.4) in Section 4. Finally, we develop the theory of p-means in the Heisenberg Group in Section 5. We show that strong uniqueness principle holds for the homogeneous Riemannian p-Laplacian in C^2-domains and for the sub-Riemannian p-Laplacian in the unit ball of $\mathbb{R}^3$.

2. Dynamic programming principle (DPP)

In a general setting, Del Teso, Manfredi and Parviainen [9] have proved that for a large class of average operators, for which it is possible to define the notion of asymptotic mean value property for the p-Laplacian, we have convergence of the approximations given by a suitable dynamic programming principle at scale ϵ as $\epsilon \to 0$. As long as the average operators do satisfy some properties, the limit of these approximations will satisfy the asymptotic mean value for a Dirichlet problem associated to the p-Laplacian, and therefore this limit will be a classical solution.

Consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, a boundary Lipschitz function $g\colon \partial\Omega \mapsto \mathbb{R}$, and a fixed positive ϵ. We define a strip of width ϵ around the boundary of Ω as follows

$$\Gamma_\varepsilon := \{x \in \mathbb{R}^n \setminus \Omega \colon d(x, \partial\Omega) \le \varepsilon\},$$

and set $\Omega_E = \Omega \cup \Gamma_1$ and $B(\Omega)$, $B(\Omega_E)$ the class of bounded real functions defined on Ω and Ω_E respectively.

Recall the definition of average operator and asymptotic mean value property from [9]:

Definition 2.1. We say that an operator $A : B(\Omega_E) \to B(\bar{\Omega})$ is an average if it satisfies the following properties:

- (Stable) $\inf_{\{y\in\Omega_E\}} \phi(y) \le A[\phi](x) \le \sup_{\{y\in\Omega_E\}} \phi(y)$, for all $x \in \bar{\Omega}$;
- (Monotone) If $\phi \le \psi$ in Ω_E then $A[\phi] \le A[\psi]$ in $\bar{\Omega}$;
- (Affine invariance) $A[\lambda\phi + \xi] = \lambda A[\phi] + \xi, \forall \lambda > 0$, for all $\xi \in \mathbb{R}$.

Definition 2.2. We say that a family of averages $\{A_\varepsilon\}_{\varepsilon>0}$ satisfies the *(asymptotic) mean value property* for the p-Laplacian if for every $\phi \in C^\infty(\Omega_E) \cap B(\Omega_E)$ such that $\nabla\phi \ne 0$ we have

$$\phi(x) = A_\varepsilon[\phi](x) + c\,\varepsilon^2 \left(-\Delta_p^H \phi(x)\right) + o(\varepsilon^2). \tag{2.1}$$

for some constant $c > 0$ independent of ε and ϕ and where the constant in $o(\varepsilon^2)$ can be taken uniformly for all $x \in \overline{\Omega}$. Equivalently, we can write

$$\left\| \frac{\phi - A_\varepsilon[\phi]}{c_{p,n}\varepsilon^2} - \left(-\Delta_p^H \phi\right) \right\|_{L^\infty(\overline{\Omega})} = o(1) \text{ as } \varepsilon \to 0. \tag{2.2}$$

Given $g \in C(\partial\Omega)$, let G be a continuous extension of g to Ω_E. Associated to a mean value property A_ε we have a dynamic programming principle at scale ϵ given by

$$\begin{cases} u_\varepsilon(x) &= A_\varepsilon[u_\varepsilon](x) \quad \text{in } \Omega, \\ u_\varepsilon(x) &= G(x) \quad \text{on } \Gamma_1. \end{cases} \tag{DPP}$$

Next, we describe some conditions on $\{A_\varepsilon\}_{\varepsilon>0}$ so that the solutions to (DPP) converge to the unique viscosity solution to the Dirichlet problem

$$\begin{cases} -\Delta_p u &= 0 \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega, \end{cases} \tag{Dp}$$

for some $g \in C(\partial\Omega)$. Consider the following conditions on the family of averages $\{A_\varepsilon\}_{\varepsilon>0}$:

Uniform Stability:

For all $\varepsilon > 0$ there exists $u_\varepsilon \in B(\Omega_E)$, a solution of (DPP) with a bound on $\|u_\varepsilon\|_{L^\infty(\Omega_E)}$ uniform in ε. $\qquad (\mathrm{A}_s)$

Uniform Boundedness

For all $\varepsilon > 0$ there exists $u_\varepsilon \in B(\Omega_E)$, a solution of (DPP), and
$\inf_{\Omega_E} G \le u_\varepsilon(x) \le \sup_{\Omega_E} G \quad \text{for all} \quad x \in \Omega.$ $\qquad (\mathrm{A}_s')$

Comparison Principle

Let u_ε^1 and u_ε^2 be a subsolution and a supersolution of (DPP) with boundary data G_1 and G_2 respectively.
If $G^1 \le G^2$ on Ω_E then $u_\varepsilon^1 \le u_\varepsilon^2$ in Ω_E. $\qquad (\mathrm{A}_{\mathrm{cp}})$

The conditions (A_s), (A_s') and (A_{cp}) are the natural assumptions satisfied by the dynamic programming principles in the literature. For more information on this subject we refer to [19, 20]. The main result of [9] is:

Theorem 2.3. *Assume $p \in (1,\infty]$, $\Omega \subset \mathbb{R}^n$ be a bounded domain and $g \in C(\partial\Omega)$. Let the family of averages $\{A_\varepsilon\}_{\varepsilon>0}$ satisfies the asymptotic mean value property for the p-Laplacian (2.2). Let also $\{u_\varepsilon\}_{\varepsilon>0}$ be a sequence of solutions of the corresponding (DPP). Then we have that*

- *If the domain Ω is of class C^2 and the family of averages $\{A_\varepsilon\}_{\varepsilon>0}$ satisfies the stability property (A_s), or*
- *If the domain Ω is Lipschitz and the family of averages $\{A_\varepsilon\}_{\varepsilon>0}$ satisfies the uniform boundedness (A_s') and the comparison principle (A_{cp}) properties,*

we obtain the convergence

$$u_\varepsilon \to v \text{ uniformly in } \overline{\Omega} \text{ as } \varepsilon \to 0,$$

where v is the unique viscosity solution of (Dp).

This theorem is an extension of the method in [5] to the p-Laplacian. The first step in the proof is to establish that the p-Laplacian satisfies the *strong uniqueness principle* defined in [5] ([9], Prop. 3.2). The key point here is that the convergence of approximations that satisfy the asymptotic mean value property (2.1), the uniform boundedness property ($\mathrm{A}_s^{'}$), and the comparison principle (A_{cp}) depends only on the strong uniqueness principle for the limit operator, which is the p-Laplacian in our case.

3. Natural p-means

Ishiwata, Magnanini, and Wadade have introduced the notion of natural p-means in their paper [13]. Let X be a compact topological measure space endowed with a positive finite Radon measure ν. For a continuous function $u \in C(X)$ and $1 \leq p \leq \infty$ there exists a unique real value $\mu_p^X(u)$ such that

$$\|u - \mu_p^X(u)\|_p = \min_{\lambda \in \mathbb{R}} \|u - \lambda\|_p \,. \tag{3.1}$$

Existence, uniqueness, and several properties described below are established in [13]. While for general p there is no explicit formula for $\mu_p^X(u)$, this formula exists for the cases $p = 1, 2$, and $p = \infty$:

$$\begin{aligned} \mu_1(u) &= \operatorname{med}(u), \\ \mu_2(u) &= ⨍_X u(y) d\nu, \text{ and} \\ \mu_\infty(u) &= \tfrac{1}{2}(\min_{y \in X} u(y) + \max_{y \in X} u(y)) \end{aligned} \tag{3.2}$$

We now list several properties of the natural mean:

– Continuity in the L^p-norm ([13], Thm. 2.4:

$$\big| \|u - \mu_p^X(u)\|_p - \|v - \mu_p^X(v)\|_p \big| \leq \|u - v\|_p$$

– Monotonicity ([13], Thm. 2.4): If $u \leq v$ a.e. on $L^p(X)$, then we have $\mu_p^X(u) \leq \mu_p^X(v)$.
– Affine invariance ([13], Prop. 2.7): $\mu_p^X(u + c) = c + \mu_p^X(u)$ and $\mu_p^X(\alpha u) = \alpha \mu_p^X(u)$ for $c, \alpha \in \mathbb{R}$.

Note that the natural p-mean is both monotone and continuous (in the L^p-norm), which is not the case for means considered in previous works [2, 15, 21].

Next, we consider the family of natural means $\{\mu_p(u, \epsilon)\}_{0<\varepsilon<1}$ defined on functions $u \in B(\Omega_E)$ as follows. For $x \in \Omega$ and $B_\varepsilon(x)$ the ball of radius ε centered at x we set

$$\mu_p(u, \epsilon)(x) = \mu_p^{B_\varepsilon(x)}(u). \tag{3.3}$$

For any $u \in L^p(\Omega_E)$ the function

$$x \mapsto \mu_p(u, \epsilon)(x)$$

is continuous in Ω. This property is not shared by the tug-of-war means of type (1.2).

We can now rephrase Theorem 3.2 in [13]:

Theorem 3.1. *The family of natural means $\{\mu_p(\phi, \epsilon)\}_{0<\varepsilon<1}$ satisfies the asymptotic mean value property for the p-Laplacian*

$$\mu_p(\phi, \epsilon)(x) = \phi(x) + \frac{\varepsilon^2}{2(n+2)} \Delta_p^H \phi(x) + o(\varepsilon^2) \tag{3.4}$$

as $\varepsilon \to 0$ for all $\phi \in C^2$ such that $\nabla\phi(x) \neq 0$.

The characterization of viscosity solutions via asymptotic expansions then follows:

Theorem 3.2. *[13] For $p \in (1,\infty]$ we have that a continuous function $u \in C(\Omega)$ is p-harmonic if and only if the following expansion holds (in the viscosity sense)*

$$u(x) \doteq \mu_p(u,\epsilon)(x) + o(\varepsilon^2) \tag{3.5}$$

as $\varepsilon \to 0$.

This theorem has recently been extended to the case of Carnot groups [1].

Next, we consider DPP associated to the natural p-means. Suppose that Ω is a bounded Lipschitz domain. Recall that the boundary strip is given by $\Gamma_\epsilon = \{x \in \mathbb{R}^n \setminus \Omega\colon\ \mathrm{dist}(x,\Omega) \leq \epsilon\}$. We also set $\Omega_\epsilon = \Omega \bigcup \Gamma_\epsilon$, which is a compact set. Let $G : \Gamma_\epsilon \to \mathbb{R}$ be continuous and bounded. Consider solutions to the following DPP:

$$\left\{ \begin{array}{lcll} u_\varepsilon(x) & = & \mu_p(u_\varepsilon,\varepsilon)(x) & \text{for } x \in \Omega, \\ u_\varepsilon(x) & = & G(x) & \text{on } \Gamma_\epsilon. \end{array} \right. \tag{3.6}$$

Lemma 3.3. *If $u \in B(\Omega_\epsilon)$ satisfies (3.6), then we have that $u \in C(\Omega)$.*

Note that the proof below shows that the mapping $x \mapsto \mu_\epsilon(u_\epsilon,\epsilon)(x)$ is continuous on $\overline{\Omega}$. However, for $x \in \partial\Omega$ we could have $\mu_p(u_\epsilon,\epsilon)(x) \neq G(x)$. This is already the case when $n = 1$ and $p = 2$ as shown by the explicit solution of K. Brustad [7].

Proof. We use the rescaling property in order to compare values on the same ball. We have

$$\mu_p(u,\epsilon)(x) = \mu_p(u^x_\epsilon, 1)(0),$$

where $u^x_\epsilon(z) = u(x+\epsilon z)$. We then write for points $x_0, x_1 \in \overline{\Omega}$

$$\mu_p(u,\epsilon)(x_0) - \mu_p(u,\epsilon)(x_1) = \mu_p(u^{x_0}_\epsilon, 1)(0) - \mu_p(u^{x_1}_\epsilon, 1)(0).$$

Since $u^{x_1}_\epsilon \mapsto u^{x_0}_\epsilon$ in L^p norm as $x_1 \to x_0$ the conclusion follows form the L^p continuity of the natural p-mean ([13], Thm. 2.4). Thus, the function $u \in C(\Omega)$. □

Existence and uniqueness for (3.6) in the the case $p = \infty$ was proved by Le Gruyer and Archer [**17**].

Theorem 3.4. *For $p \in (1,\infty]$ there exists a unique solution $u_\epsilon \in C(\Omega) \cap B(\Omega_\epsilon)$ to the DPP (3.6).*

Proof. We only need to consider the case $1 < p < \infty$. *Existence:* For $v \in B(\Omega_\epsilon)$ define the operator

$$\left\{ \begin{array}{lcll} T_\epsilon v(x) & = & \mu_p(v,\epsilon)(x), & \text{for } x \in \Omega. \\ T_\epsilon v(x) & = & G(x) & \text{on } \Gamma_\epsilon. \end{array} \right. \tag{3.7}$$

The starting point is the function $u_0 = \chi_{\Gamma_\epsilon} G(x) + \chi_\Omega \inf G$. Note that a minor variation of the proof of the monotonicity Theorem 2.5 in [13] gives $u_0(x) \leq T_\epsilon u_0(x)$ for $x \in \overline{\Omega}$. Define the sequence $u_n = T_\epsilon u_{n-1}$ for $n \geq 1$. Note that the sequence $u_n(x)$ is non-decreasing $u_{n-1} \leq u_n$ and bounded above so that the point-wise limit $u_\epsilon(x) = \lim_{n\to\infty} u_n(x)$ exists for all $x \in \Omega_\epsilon$. From the dominated convergence theorem we also get convergence in the L^p-norm, so that by the continuity with respect to the L^p-norm of the natural p-means we get $\mu_p(u_\epsilon,\epsilon)(x) = \lim_{n\to\infty} \mu_p(u_n,\epsilon)(x)$. This implies that $\mu_p(u_\epsilon,\epsilon)(x) = u_\epsilon(x)$ for all $x \in \Omega$. Therefore u_ϵ is continuous in Ω by Lemma 3.3 and solves (3.6).

Uniqueness: Set

$$M = \sup\{u_\epsilon(x) - v_\epsilon(x)\colon x \in \Omega_\epsilon\}.$$

Choose a sequence $x_n \in \Omega_\epsilon$ such that $u_\epsilon(x_n) - v_\epsilon(x_n) \to M$ as $n \to \infty$. Since $u_\epsilon - v_\epsilon$ is continuous we can select a converging subsequence x_{n_k} such that $x_{n_k} \to x^* \in \Omega_\epsilon$ as $k \to \infty$ and

$$\lim_{k\to\infty} (u_\epsilon(x_{n_k}) - v_\epsilon(x_{n_k})) = u_\epsilon(x^*) - v_\epsilon(x^*) = M.$$

Thus, the set where the maximum of the difference is achieved

$$D = \{x \in \Omega_\epsilon\colon u_\epsilon(x) - v_\epsilon(x) = M\}$$

is non-empty. When $x \in D \cap \Gamma_\epsilon$ we have

$$M = u_\epsilon(x) - v_\epsilon(x) = 0.$$

Let us assume that $M > 0$ and reach a contradiction. In this case we have $\emptyset \neq D \subset \Omega$. By semicontinuity D is closed. Let us next show that D is open. Let $x \in D$. Since we have $u_\epsilon(x) = \mu_p(u_\epsilon, \epsilon)(x)$ we get

$$\int_{B_\epsilon(x)} |u_\epsilon(y) - u_\epsilon(x)|^{p-2} \left(u_\epsilon(y) - u_\epsilon(x)\right) \,\mathrm{d}y = 0, \tag{3.8}$$

and since we have $v_\epsilon(x) = \mu_p(v_\epsilon, \epsilon)(x)$ we get

$$\int_{B_\epsilon(x)} |v_\epsilon(y) - v_\epsilon(x)|^{p-2} \left(v_\epsilon(y) - v_\epsilon(x)\right) \,\mathrm{d}y = 0. \tag{3.9}$$

Next, since we always have $u_\epsilon(y) - v_\epsilon(y) \le u_\epsilon(x) - v_\epsilon(x)$, it follows that

$$u_\epsilon(y) - u_\epsilon(x) \le v_\epsilon(y) - v_\epsilon(x). \tag{3.10}$$

Since the function $t \mapsto |t|^{p-2}t$ is monotone increasing, we obtain

$$|u_\epsilon(y) - u_\epsilon(x)|^{p-2}(u_\epsilon(y) - u_\epsilon(x)) \le |v_\epsilon(y) - v_\epsilon(x)|^{p-2}(v_\epsilon(y) - v_\epsilon(x)).$$

From (3.8) and (3.9) we obtain that the nonnegative function

$$|v_\epsilon(y) - v_\epsilon(x)|^{p-2}(v_\epsilon(y) - v_\epsilon(x)) - u_\epsilon(y) - u_\epsilon(x)|^{p-2}(u_\epsilon(y) - u_\epsilon(x)) \ge 0$$

has a vanishing integral on $B_\epsilon(x)$. Thus, we obtain

$$|v_\epsilon(y) - v_\epsilon(x)|^{p-2}(v_\epsilon(y) - v_\epsilon(x)) - |u_\epsilon(y) - u_\epsilon(x)|^{p-2}(u_\epsilon(y) - u_\epsilon(x)) = 0$$

for a. e. $x \in B_\epsilon(x)$. Therefore, we have $v_\epsilon(y) - v_\epsilon(x) = u_\epsilon(y) - u_\epsilon(x)$ for $x \in B_\epsilon(x) \setminus S$, where S is a set of measure zero. Indeed, the identity holds everywhere in $B_\epsilon(x) \cap \Omega$ by continuity. Thus, the set D is open and $D = \Omega$, since Ω is connected. Moreover, if $x \in D$ and $y \in B_\epsilon(x)$ we have $v_\epsilon(y) - v_\epsilon(x) = u_\epsilon(y) - u_\epsilon(x)$, or equivalently

$$u_\epsilon(y) - v_\epsilon(y) = u_\epsilon(x) - v_\epsilon(x) = M, \text{ for a.e. } y \in B_\epsilon(x).$$

Taking a point $x_0 \in \Omega$ such that $|B_\epsilon(x_0) \cap \Gamma_\epsilon| > 0$ and $y \in B_\epsilon(x_0) \cap \Gamma_\epsilon$, we get

$$M = u_\epsilon(x) - v_\epsilon(x) = u_\epsilon(y) - v_\epsilon(y) = 0$$

□

4. The comparison principle

In order to apply the results of [9], we need to consider sub- and super-solutions of the (3.6). These are defined as follows:

Definition 4.1. Let $u_\varepsilon \in B(\Omega_E)$ and upper-semicontinuous in Ω. We say that u_ϵ is a subsolution (3.6) if we have

$$\begin{cases} u_\varepsilon(x) & \leq & \mu_p(u_\varepsilon, \varepsilon)(x) & \text{for } x \in \Omega, \\ u_\varepsilon(x) & \leq & G_1(x) & \text{on } \Gamma_\epsilon. \end{cases} \tag{4.1}$$

Definition 4.2. Let $v_\varepsilon \in B(\Omega_E)$ and lower-semicontinuous in Ω. We say that v_ϵ is a supersolution (3.6) if we have

$$\begin{cases} v_\varepsilon(x) & \geq & \mu_p(v_\varepsilon, \varepsilon)(x) & \text{for } x \in \Omega, \\ v_\varepsilon(x) & \geq & G_2(x) & \text{on } \Gamma_\epsilon. \end{cases} \tag{4.2}$$

Theorem 4.3. *For $1 < p \leq \infty$ the family a family of natural means $\{\mu_p(u,\epsilon)\}_{0<\varepsilon<1}$ satisfies the uniform boundedness ($\mathrm{A}_s^{'}$) and comparison principle (A_{cp}) properties.*

Once again the case $p = \infty$ is due to Le Gruyer; see Theorem 4.1 in [16].

Proof. We consider the case $1 < p < \infty$. The uniform boundedness property ($\mathrm{A}_s^{'}$) follows from Theorem 3.4 and the monotonicity of the natural p-means. To prove the comparison principle, we need again the observation from [13]: for $1 < p < \infty$ the function $(u, \lambda) \mapsto |u - \lambda|^{p-2}(u - \lambda)$ is strictly increasing in u for fixed λ and strictly decreasing in λ for fixed u. Therefore if $u(x) \leq v(x)$ a.e. in a set $O \subset \Omega_\epsilon$ we have

$$\int_O |u(x) - \lambda|^{p-2} \left(u(x) - \lambda\right)\right)\, \mathrm{d}x \leq \int_O |v(x) - \lambda|^{p-2} \left(v(x) - \lambda\right)\right)\, \mathrm{d}x \tag{4.3}$$

Let u_ϵ satisfy (4.1), let v_ϵ satisfy (4.2), and suppose that $G_1 \leq G_2$. We need to prove that $u_\epsilon \leq v_\epsilon$. Set

$$M = \sup\{u_\epsilon(x) - v_\epsilon(x) \colon x \in \Omega_\epsilon\}.$$

Choose a sequence $x_n \in \Omega_\epsilon$ such that $u_\epsilon(x_n) - v_\epsilon(x_n) \to M$ as $n \to \infty$. Since $u_\epsilon - v_\epsilon$ is upper-semi-continuos we can select a converging subsequence x_{n_k} such that $x_{n_k} \to x^* \in \Omega_\epsilon$ as $k \to \infty$ and

$$\lim_{k\to\infty} \left(u_\epsilon(x_{n_k}) - v_\epsilon(x_{n_k})\right) = u_\epsilon(x^*) - v_\epsilon(x^*) = M.$$

Thus, the set where the maximum of the difference is achieved

$$D = \{x \in \Omega_\epsilon \colon u_\epsilon(x) - v_\epsilon(x) = M\}$$

is non-empty. When $x \in D \cap \Gamma_\epsilon$ we have

$$M = u_\epsilon(x) - v_\epsilon(x) \leq G_1(x) - G_2(x) \leq 0.$$

Let us assume that $M > 0$ and reach a contradiction. In this case we have $\emptyset \neq D \subset \Omega$. By semicontinuity D is closed. Let us next show that D is open. Let $x \in D$. Since we have $u_\epsilon(x) \leq \mu_p(u_\epsilon, \epsilon)(x)$ we deduce from (4.3) the inequality

$$\int_{B_\epsilon(x)} |u_\epsilon(y) - u_\epsilon(x)|^{p-2} \left(u_\epsilon(y) - u_\epsilon(x)\right) \,\mathrm{d}y \geq 0, \tag{4.4}$$

and since we have $v_\epsilon(x) \geq \mu_p(v_\epsilon, \epsilon)(x)$ we deduce from (4.3) the inequality

$$\int_{B_\epsilon(x)} |v_\epsilon(y) - v_\epsilon(x)|^{p-2} \left(v_\epsilon(y) - v_\epsilon(x)\right) \,\mathrm{d}y \leq 0. \tag{4.5}$$

Next, since we always have $u_\epsilon(y) - v_\epsilon(y) \leq u_\epsilon(x) - v_\epsilon(x)$, it follows that

$$u_\epsilon(y) - u_\epsilon(x) \leq v_\epsilon(y) - v_\epsilon(x). \tag{4.6}$$

Since the function $t \mapsto |t|^{p-2}t$ is monotone increasing, we obtain

$$|u_\epsilon(y) - u_\epsilon(x)|^{p-2}(u_\epsilon(y) - u_\epsilon(x)) \leq |v_\epsilon(y) - v_\epsilon(x)|^{p-2}(v_\epsilon(y) - v_\epsilon(x)).$$

From (4.4) and (4.5) we obtain that the nonnegative function

$$|v_\epsilon(y) - v_\epsilon(x)|^{p-2}(v_\epsilon(y) - v_\epsilon(x)) - u_\epsilon(y) - u_\epsilon(x)|^{p-2}(u_\epsilon(y) - u_\epsilon(x)) \geq 0$$

has a non-positive integral on $B_\epsilon(x)$. Thus, we obtain

$$|v_\epsilon(y) - v_\epsilon(x)|^{p-2}(v_\epsilon(y) - v_\epsilon(x)) - |u_\epsilon(y) - u_\epsilon(x)|^{p-2}(u_\epsilon(y) - u_\epsilon(x)) = 0$$

for a. e. $x \in B_\epsilon(x)$. Therefore, we have $v_\epsilon(y) - v_\epsilon(x) = u_\epsilon(y) - u_\epsilon(x)$ for $x \in B_\epsilon(x) \setminus S$, where S is a set of measure zero. Indeed, the identity holds everywhere in $B_\epsilon(x) \cap \Omega$ since, in addition to (4.6), we have

$$\begin{array}{rcl} v_\epsilon(y) - v_\epsilon(x) & \leq & \liminf\limits_{z \to y} \left(v_\epsilon(z) - v_\epsilon(x)\right) \\ & \leq & \liminf\limits_{z \to y, z \notin S} \left(v_\epsilon(z) - v_\epsilon(x)\right) \\ & \leq & \liminf\limits_{z \to y, z \notin S} \left(u_\epsilon(z) - u_\epsilon(x)\right) \\ & \leq & \limsup\limits_{z \to y, z \notin S} \left(u_\epsilon(z) - u_\epsilon(x)\right) \\ & \leq & \limsup\limits_{z \to y} \left(u_\epsilon(z) - u_\epsilon(x)\right) \\ & \leq & u_\epsilon(y) - u_\epsilon(x). \end{array} \tag{4.7}$$

Thus, the set D is open and $D = \Omega$, since Ω is connected. Moreover, if $x \in D$ and $y \in B_\epsilon(x)$ we have $v_\epsilon(y) - v_\epsilon(x) = u_\epsilon(y) - u_\epsilon(x)$, or equivalently

$$u_\epsilon(y) - v_\epsilon(y) = u_\epsilon(x) - v_\epsilon(x) = M, \text{ for a.e. } y \in B_\epsilon(x).$$

Taking a point $x_0 \in \Omega$ such that $|B_\epsilon(x_0) \cap \Gamma_\epsilon| > 0$ and $y \in B_\epsilon(x_0) \cap \Gamma_\epsilon$, we get

$$M = u_\epsilon(x) - v_\epsilon(x) = u_\epsilon(y) - v_\epsilon(y) \leq G_1(y) - G_2(y) \leq 0.$$

□

We can now combine Theorems 2.3, 3.1, and 4.3 to conclude

Theorem 4.4. *Assume $p \in (1, \infty]$, $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $g \in C(\partial\Omega)$. For $\varepsilon > 0$ let $\mu_p(u_\varepsilon, \varepsilon)$ be the natural p-means solutions to the DPP (3.6). Then, we have*

$$u_\varepsilon \to u \text{ uniformly in } \overline{\Omega} \text{ as } \varepsilon \to 0,$$

where u is the unique solution of the Dirichlet problem (Dp).

5. Natural p-means in the Heisenberg group

Since the definition of natural means makes sense in general metric-measure spaces, we can consider the case of the Heisenberg group. We begin our set up describing a class of equations and consider whether they satisfy the strong uniqueness principle.

Let $\Omega \subset \mathbb{R}^n$ be a domain. Consider the Dirichlet problem

$$\begin{cases} F(x, \nabla u(x), D^2u(x)) &= 0 \text{ in } \Omega \\ u(x) &= g(x) \text{ on } \partial\Omega, \end{cases} \tag{5.1}$$

where $g \in C(\partial\Omega)$. Assume temporarily that F is degenerate elliptic and smooth and that Ω is also smooth. Let us consider the possibly degenerate linear non-divergence form case, where

$$F(x, \nabla u(x), D^2u(x)) = -\operatorname{trace}\left(\mathbb{A}(x)D^2u(x)\right), \tag{5.2}$$

where $\mathbb{A}$ is an $n \times n$ symmetric positive semi-definite matrix

$$\langle \mathbb{A}(x)\xi, \xi\rangle \geq 0, \text{ for all } \xi \in \mathbb{R}^n.$$

When the matrix of coefficients has a zero eigenvalue we cannot expect the boundary values to be achieved at all points of $\partial\Omega$ due to the presence of characteristic points

$$C = \{y \in \partial\Omega \colon \langle \mathbb{A}(y) \cdot \vec{n}(y), \vec{n}(y)\rangle = 0\},$$

where $\vec{n}(y)$ is the unit normal to $\partial\Omega$ at y. See [22] for an analytic presentation of the theory of equations with nonnegative characteristic form and [11] for a stochastic presentation.

We need to consider the definition of generalized viscosity solution. This corresponds to assuming that at any boundary point the sub- and super-solutions satisfies either the boundary condition or the equation.

Definition 5.1. A bounded upper-semicontinuous function u is a generalized viscosity sub-solution of (5.1) if and only for any $\phi \in C^2(\overline{\Omega})$ and for any point x_0 local maximum of $u - \phi$ in $\overline{\Omega}$ we have

$$\begin{aligned} F(x_0, \nabla\phi(x_0), D^2\phi(x_0)) &\leq 0, \text{ if } x_0 \in \Omega \\ \min\{F(x_0, \nabla\phi(x_0), D^2\phi(x_0)), u(x_0) - g(x_0)\} &\leq 0, \text{ if } x_0 \in \partial\Omega \end{aligned} \tag{5.3}$$

Definition 5.2. A bounded lower-semicontinuous function v is a generalized viscosity super-solution of (5.1) if and only for any $\phi \in C^2(\overline{\Omega})$ and for any point x_0 local minimum of $v - \phi$ in $\overline{\Omega}$ we have

$$\begin{aligned} F(x_0, \nabla\phi(x_0), D^2\phi(x_0)) &\geq 0, \text{ if } x_0 \in \Omega \\ \min\{F(x_0, \nabla\phi(x_0), D^2\phi(x_0)), v(x_0) - g(x_0)\} &\geq 0, \text{ if } x_0 \in \partial\Omega \end{aligned} \tag{5.4}$$

Uniqueness of generalized viscosity solution to (5.1) follows from a comparison principle for generalized viscosity sub and super-solutions. This is called a *strong uniqueness principle* by Barles and Souganidis [5]. As noted in [5] the *strong uniqueness principle* gives convergence of monotone, stable, and consistent approximations in C^2-domains. To get convergence in Lipschitz domains, we need to use additional information, for example the fact that the fundamental solution is radial, to set up an iteration at the boundary and get convergence; see [9].

Next, we will present examples of equations and domains for which the strong uniqueness principle hold.

Consider a frame of n linearly independent vector fields $\mathfrak{X} = \{X_1, \ldots, X_n\}$ in $\mathbb{R}^n$. In local coordinates we write

$$X_i = \sum_{i=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_j}.$$

We can define a Riemannian metric in $\mathbb{R}^n$ by declaring that at each point $\mathfrak{X}$ is an orthonormal basis. The corresponding gradient is

$$D_{\mathfrak{X}} u(x) = \mathbb{A}(x) \cdot \nabla u,$$

where $\mathbb{A}(x) = (a_{ij}(x)$. The second derivative matrix $D^2_{\mathfrak{X}} u$ has entries $X_i(X_j(u))$ and the symmetrized second derivative matrix $(D^2_{\mathfrak{X}} u)^\star = \frac{1}{2}\big(D^2_{\mathfrak{X}} u + (D^2_{\mathfrak{X}} u)^t\big)$ is given by

$$\begin{aligned}\langle (D^2_{\mathfrak{X}} u)^\star \cdot h, h\rangle = {} & \langle \mathbb{A} \cdot D^2 u \cdot \mathbb{A}^t \cdot h, h\rangle \\ & + \sum_{k=1}^{n} \langle \mathbb{A}^t \cdot h, \big(\mathbb{A}^t \cdot h\big)_k\rangle \frac{\partial u}{\partial x_k},\end{aligned}$$

for $h \in \mathbb{R}^n$.

Let us state a special case of a Proposition (1.1) in Barles and Bourdeau [4].

Proposition 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a C^2-domain. Let u be an upper-semicontinuous sub-solution of (5.1) in the possibly degenerate linear case where $F(x, \nabla u(x), D^2 u(x))$ is given by (5.2). Suppose that $u(x_0) > g(x_0)$ at $x_0 \in \partial\Omega$, then we have*

$$\langle \mathbb{A}(x_0)\, \vec{n}(x_0), \vec{n}(x_0)\rangle = 0 \text{ and} \tag{5.5}$$

$$-\frac{1}{2} \operatorname{trace}\big[\mathbb{A}(x_0) D^2 d(x_0)\big] \le 0. \tag{5.6}$$

Here $d(x) = \operatorname{dist}(x, \partial\Omega)$ and we select an open set U such that $x_0 \in U$ and d has second derivatives bounded in $\overline{U}$. Moreover, we have that $d(x) > 0$ for $x \in \overline{U} \cap \Omega$, $d(x) = 0$ for $x \in \overline{U} \cap \partial\Omega$ and,

$$|\nabla d(x)| = |-\vec{n}(x)| = 1, \qquad \|D^2 d(x)\|_{L^\infty(\overline{U}} < +\infty.$$

Let us consider the case of the Heisenberg group $\mathbb{H}^1$. The horizontal layer $\mathcal{H}$ is generated by $Y_1 = \frac{\partial}{\partial y_1} - \frac{y_2}{2}\frac{\partial}{\partial y_3}$, $Y_2 = \frac{\partial}{\partial y_2} + \frac{y_1}{2}\frac{\partial}{\partial y_3}$. We set $Y_3 = [Y_1, Y_2] = \frac{\partial}{\partial y_3}$. With this notation the Riemannian frame is $\mathfrak{X} = \{Y_1, Y_2, Y_3\}$ and the sub-Riemannian frame is $\mathcal{H} = \{Y_1, Y_2\}$. Consider the matrix $\mathbb{A}(y)$

$$\mathbb{A}(y) = \begin{pmatrix} 1 & 0 & -\frac{y_2}{2} \\ 0 & 1 & \frac{y_1}{2} \\ 0 & 0 & 1 \end{pmatrix} \tag{5.7}$$

In this case we have $\langle \mathbb{A}^t \cdot h, \left(\mathbb{A}^t \cdot h\right)_k \rangle \frac{\partial u}{\partial y_k} = 0$ for $k = 1, 2, 3$. Thus, the Riemannian 3×3 matrix $(D^2_{\mathfrak{X}} u)^\star$ satisfies

$$\langle (D^2_{\mathfrak{X}} u)^\star \cdot h, h \rangle = \langle \mathbb{A} \cdot D^2 u \cdot \mathbb{A}^t \cdot h, h \rangle \tag{5.8}$$

for $h \in \mathbb{R}^3$. The Laplace operator $\triangle_{\mathfrak{X}} u = \operatorname{trace}(D^2_{\mathfrak{X}} u)^\star = \operatorname{trace}\left(\mathbb{A} \cdot D^2 u \cdot \mathbb{A}^t\right)$ can also be written as

$$\triangle_{\mathfrak{X}} u = \operatorname{trace}\left(\mathbb{B} \cdot D^2 u\right),$$

where $\mathbb{B}(y) = \mathbb{A}(y)^t \cdot \mathbb{A}(y)$. The matrix $\mathbb{B}(y)$ is given by

$$\mathbb{B}(y) = \begin{pmatrix} 1 & 0 & -\frac{y_2}{2} \\ 0 & 1 & \frac{y_1}{2} \\ -\frac{y_2}{2} & \frac{y_1}{2} & 1 + \frac{y_1^2 + y_2^2}{4} \end{pmatrix} \tag{5.9}$$

Note that $\mathbb{B}$ is symmetric and $\det \mathbb{B} = 1$. Since the operator $\triangle_{\mathfrak{X}} u$ is uniformly elliptic, Proposition 5.3 show that there are no characteristic points and thus, *the strong uniqueness principle for $\triangle_{\mathfrak{X}}$ holds in C^2 domains.*

Consider next the corresponding ∞-Laplacian

$$\begin{aligned} \triangle^\infty_{\mathfrak{X}} u &= \langle (D^2_{\mathfrak{X}} u)^\star D_{\mathfrak{X}} u, D_{\mathfrak{X}} u \rangle \\ &= \langle \mathbb{A} \cdot D^2 u \cdot \mathbb{A}^t \cdot \mathbb{A} \cdot \nabla u, \mathbb{A} \cdot \nabla u \rangle \\ &= \langle \mathbb{B} \cdot D^2 u \cdot \mathbb{B} \cdot \nabla u, \nabla u \rangle. \end{aligned}$$

The condition for the strong uniqueness for the operator $\triangle^\infty_{\mathfrak{X}}$ follows from a generalization of Proposition 5.3 to non-linear homogeneous operators obtained in [9]. Let Ω be a C^2 domain, let $y \in \partial\Omega$, and $\vec{n}(y)$ the outer unit normal vector at y. The condition for a point $y \in \partial\Omega$ not to be a regular boundary point reads

$$\langle (\vec{n}(y) \otimes \vec{n}(y)) \mathbb{B}(y) \vec{n}(y), \mathbb{B}(y) \vec{n}(y) \rangle = 0. \tag{5.10}$$

We use the following elementary fact, whose proof we add for the sake of completeness.

Lemma 5.4. *Let M be a non singular $n \times n$ matrix and $v \in \mathbb{R}^n$. We have*

$$\langle (v \otimes v) M^t M v, M^t M v \rangle = |Mv|^4. \tag{5.11}$$

Proof.

$$\begin{aligned} \langle (v \otimes v) M^t M v, M^t M v \rangle &= \langle M (v \otimes v) M^t M v, M v \rangle \\ &= \langle (Mv \otimes Mv) M v, M v \rangle \\ &= |Mv|^4. \end{aligned}$$

□

Therefore, using this lemma, we obtain from condition (5.10) the relation

$$\langle (\vec{n}(y) \otimes \vec{n}(y)) \mathbb{B}(y) \vec{n}(y), \mathbb{B}(y) \vec{n}(y) \rangle = |\mathbb{A} \vec{n}(y)|^4.$$

Since $\mathbb{A}$ is non degenerate, we conclude $\vec{n}(y) = 0$, which is contradiction since $\vec{n}(y)$ is a unit vector. Thus, there are no irregular boundary points and *the strong uniqueness principle holds for $\triangle^\infty_{\mathfrak{X}}$ in C^2-domains.*

Next, consider the homogeneous p-Laplacian for $1 < p < \infty$ given by

$$\triangle^p_{\mathfrak{X}} u = \triangle_{\mathfrak{X}} u + (p-2)\frac{\triangle^\infty_{\mathfrak{X}} u}{|\nabla_{\mathfrak{X}} u|^2}.$$

This operator is formally defined whenever $\nabla_{\mathfrak{X}} u \neq 0$, but it can be defined in the viscosity sense as in the Euclidean case (see for example [9]). In local coordinates we write

$$\triangle^p_{\mathfrak{X}} u = \operatorname{trace}(\mathbb{B}\, D^2 u) + (p-2)\frac{\langle \mathbb{B}\, D^2 u\, \mathbb{B}^t \cdot \nabla u, \nabla u\rangle}{\langle \mathbb{B} \cdot \nabla u, \nabla u\rangle}. \tag{5.12}$$

Let Ω be a C^2 domain, let $y \in \partial\Omega$, and $\vec{n}(y)$ the outer unit normal vector at y. The condition for a point $y \in \partial\Omega$ not to be a regular boundary point reads

$$\langle \mathbb{B}(y)\vec{n}(y), \vec{n}(y)\rangle + (p-2)\frac{\langle (\vec{n}(y)\otimes\vec{n}(y))\mathbb{B}(y)\cdot\vec{n}(y), \mathbb{B}(y)\cdot\vec{n}(y)\rangle}{\langle \mathbb{B}(y)\cdot\vec{n}(y), \vec{n}(y)\rangle} = 0. \tag{5.13}$$

From this condition we deduce

$$|\mathbb{A}(y)\cdot\vec{n}(y)|^2 + (p-2)\frac{|A(y)\cdot\vec{n}(y)|^4}{|\mathbb{A}(y)\cdot\vec{n}(y)|^2} = (p-1)|\mathbb{A}(y)\cdot\vec{n}(y)|^2 = 0,$$

so that $\vec{n}(y) = 0$ getting a contradiction. *Therefore, the strong uniqueness principle holds for the homogeneous Riemannian p-Laplacian $\triangle^p_{\mathfrak{X}}$ in C^2-domains in $\mathbb{R}^3$.* This argument works exactly the same for the analogue Riemannian Heisenberg vector fields in $\mathbb{R}^{2n+1}$.

Consider next the sub-Riemannian case. The horizontal gradient of a function $u\colon \Omega \subset \mathbb{R}^3 \mapsto \mathbb{R}$ is given by

$$D_{\mathfrak{X},0}u(p) = (Y_1 u, Y_2 u)^t = \begin{pmatrix} u_{y_1} - \frac{y_2}{2}u_{y_3} \\ u_{y_2} + \frac{y_1}{2}u_{y_3}\end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{y_2}{2} \\ 0 & 1 & \frac{y_1}{2}\end{pmatrix}\begin{pmatrix} u_x \\ u_y \\ u_z\end{pmatrix}$$

Setting the matrix

$$\sigma(y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{y_2}{2} & \frac{y_1}{2}\end{pmatrix},$$

we write $D_{\mathfrak{X},0}u(y) = \sigma^t(y)\cdot\nabla u(y)$.

The second order horizontal derivative matrix $D^2_{\mathfrak{X},0}u$ is a 2×2 matrix with entries $X_i(X_j u))$ for $i,j = 1,2$. Its symmetric version $(D^2_{\mathfrak{X},0}u)^*$ is the 2×2 principal minor of the matrix $(D^2_{\mathfrak{X}}u)^*$. We compute it in local coordinates

$$\begin{aligned}(D^2_{\mathfrak{X},0}u)^*(y) &= \big((D^2_{\mathfrak{X}}u)^*\big)_{2\times 2}(y) = \big(\mathbb{A}(y)\cdot D^2u(y)\cdot\mathbb{A}^t(y)\big)_{2\times 2} \\ &= \tau^t\cdot\mathbb{A}(y)\cdot D^2u(y)\cdot\mathbb{A}^t(y)\cdot\tau,\end{aligned}$$

where τ is the 2×3 matrix

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0\end{pmatrix}.$$

The horizontal Laplacian is given by

$$\Delta_{\mathfrak{X},0}u(y) = \mathrm{trace}((D^2_{\mathfrak{X},0}u)^*)(y) = \mathrm{trace}(\tau^t \cdot \mathbb{A}(y) \cdot D^2u(y) \cdot \mathbb{A}^t(y) \cdot \tau).$$

Set $\mathbb{A}_0(y) = \tau^t\mathbb{A}(y)$ so that we write

$$\Delta_{\mathfrak{X},0}u(y) = \mathrm{trace}(\mathbb{A}_0(y) \cdot D^2u(y) \cdot \mathbb{A}_0^t(y)) = \mathrm{trace}(\mathbb{B}_0(y) \cdot D^2u(y)),$$

where $\mathbb{B}_0 = \mathbb{A}_0^t \cdot \mathbb{A}_0$ is the 3×3 matrix given by

$$\mathbb{B}_0(y) = \begin{pmatrix} 1 & 0 & -\frac{y_2}{2} \\ 0 & 1 & \frac{y_1}{2} \\ -\frac{y_2}{2} & \frac{y_1}{2} & \frac{y_1^2+y_2^2}{4} \end{pmatrix}$$

The matrix $\mathbb{B}_0$ is symmetric but now it is singular.

The condition for boundary regularity for a C^2-domain Ω in $\mathbb{R}^3$ for the operator $\Delta_{\mathfrak{X},0}$ is given by the equation

$$\langle \mathbb{B}_0\, \vec{n}, \vec{n} \rangle = 0. \tag{5.14}$$

Since $\mathbb{B}_0$ is a singular matrix the equation (5.14) has non-trivial solutions. Since $\mathbb{B}_0 = \mathbb{A}_0^t \cdot \mathbb{A}_0$, these are the solutions of the equation

$$|\mathbb{A}_0 \vec{n}(y)| = 0,$$

where

$$\mathbb{A}_0 = \begin{pmatrix} 1 & 0 & -\frac{y_2}{2} \\ 0 & 1 & \frac{y_1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

Let $\Omega = B_1(0) = \mathbf{B}$ be the Euclidean unit ball in $\mathbb{R}^3$. Write $y = (y_1, y_2, y_3) \in \partial\mathbf{B}$ and $\vec{n}(y) = (a, b, c)$. We have three conditions

$$\begin{aligned} a - \frac{y_2}{2}c &= 0 \\ b + \frac{y_1}{2}c &= 0 \\ a^2 + b^2 + c^2 &= 1 \end{aligned}$$

If c were 0, there would be no solutions, so that we can assume $c \neq 0$. We write $\vec{n}(y) = (c\,\frac{y_2}{2}, -c\,\frac{y_1}{2}, c) = c(\frac{y_2}{2}, -\frac{y_1}{2}, 1)$ and choose c so that $\vec{n}(y)$ is a unit vector. This gives at most two possible irregular boundary points $(y_1, y_2, \pm\sqrt{y_1^2 + y_2^2})$. When $c = 1$ we get $a = b = 0$ so that the North pole $N = (0, 0, 1)$ and the South pole $S = (0, 0, -1)$ are the only possible irregular points. We must now consider condition (5.6) at these points. We have that $d(y) = \mathrm{dist}(y, \partial\mathbf{B}) = 1 - |y|$, so that

$$\nabla u(y) = -\frac{y}{|y|}, \qquad D^2d(y) = -\frac{1}{|y|}I + \frac{y \otimes y}{|y|^3}$$

and,

$$\mathbb{B}(y)D^2d(y) = \begin{pmatrix} 1 & 0 & -\frac{y_2}{2} \\ 0 & 1 & \frac{y_1}{2} \\ -\frac{y_2}{2} & \frac{y_1}{2} & \frac{y_1^2+y_2^2}{4} \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{|y|} + \frac{y_1^2}{|y|^3} & \frac{y_1y_2}{|y|^3} & \frac{y_1y_3}{|y|^3} \\ \frac{y_1y_2}{|y|^3} & -\frac{1}{|y|} + \frac{y_2^2}{|y|^3} & \frac{y_2y_3}{|y|^3} \\ \frac{y_1y_3}{|y|^3} & \frac{y_2y_3}{|y|^3} & -\frac{1}{|y|} + \frac{y_3^2}{|y|^3} \end{pmatrix}$$

A calculation shows that

$$\operatorname{trace}\mathbb{B}(y)D^2d(y) = -\frac{(y_1^4 + 2y_1^2(2+y_2^2) + y_2^2(4+y_2^2) + 8y_3^2)}{4|y|^3}.$$

We conclude that $-\operatorname{trace}\mathbb{B}(y)D^2d(y) > 0$. Thus, there are no irregular boundary points. *Therefore, the strong uniqueness principle holds for the operator $\Delta_{\mathfrak{X},0}$ in the Euclidean unit ball* $\mathbf{B}$.

We next compute the ∞-Laplacian:

$$\begin{aligned}\triangle^{\infty}_{\mathfrak{X},0}u &= \langle \left(D^2_{\mathfrak{X},0}u\right)^{\star} \cdot D_{\mathfrak{X},0}u, D_{\mathfrak{X},0}u\rangle \\ &= \langle \tau^t \cdot \mathbb{A} \cdot D^2u \cdot \mathbb{A}^t \cdot \tau \cdot \sigma^t \cdot \nabla u, \sigma^t \cdot \nabla u\rangle \\ &= \langle \sigma \cdot \tau^t \cdot \mathbb{A} \cdot D^2u \cdot \mathbb{A}^t \cdot \tau \cdot \sigma^t \cdot \nabla u, \nabla u\rangle \\ &= \langle \mathbb{B}_0 \cdot D^2u \cdot \mathbb{B}_0^t \cdot \nabla u, \nabla u\rangle\end{aligned}$$

Let us check the first order condition for the strong uniqueness to hold:

$$\langle (n(y)\otimes n(y))\mathbb{B}_0(y)n(y), \mathbb{B}_0(y)n(y)\rangle = 0 \tag{5.15}$$

Using the identity (5.11), the condition (5.15) reduces to

$$|\mathbb{A}_0 n(y)| = 0.$$

Let us denote $n(y) = (a, b, c)$. As in the previous case, we have possible irregular points.

Let us check the second order condition for $(0, 0, \pm 1)$, which in this case becomes

$$\langle D^2d(y)\mathbb{B}_0(y)n(y), \mathbb{B}_0(y)n(y)\rangle \le 0$$

Since $\mathbb{A}_0(y)n(y) = 0$, then also $\mathbb{B}_0(y)n(y) = \mathbb{A}_0^t(y)\mathbb{A}_0(y)n(y) = 0$ and we do not get a contradiction. Thus, for the ∞-Laplacian Proposition 5.3 does not help to eliminate all irregular boundary points.

On the the other hand, a similar argument to the case $p \neq 2$ gives *the strong uniqueness principle for the sub-Riemannian p-Laplacian*

$$\begin{aligned}\triangle^{p}_{\mathfrak{X},0}u &= \triangle_{\mathfrak{X},0}u + (p-2)\frac{\triangle^{\infty}_{\mathfrak{X},0}u}{|\nabla_{\mathfrak{X},0}u|^2} \\ &= \operatorname{trace}(\mathbb{A}_0 D^2u) + (p-2)\frac{\langle \mathbb{B}_0 \cdot D^2u \cdot \mathbb{B}_0^t \cdot \nabla u, \nabla u\rangle}{|\mathbb{A}_0\nabla u|^2}\end{aligned}$$

in the Euclidean unit ball $\mathbf{B}$.

A left-invariant homogeneous gauge in the Heisenberg group $\mathbb{H}^1$ corresponding to our choice of vector fields $\{Y_1, Y_2, Y_3\}$ is given by

$$|y|_{\mathbb{H}^1} = |(y_1, y_2, y_3)|_{\mathbb{H}^1} = \left((y_1^2 + y_2^2)^2 + 16\, y_3^2\right)^{\frac{1}{4}} \tag{5.16}$$

The distance between two points y and z is then $d_{\mathbb{H}^1}(y,z) = |z^{-1}\cdot y|_{\mathbb{H}^1}$. The corresponding balls are then $B_\varepsilon^{\mathbb{H}^1}(y) = \{z\colon d_{\mathbb{H}^1}(y,z) < \varepsilon\}$ and the natural p-means are given by

$$\mu_p^{\mathbb{H}^1}(\phi,\epsilon)(y) = \mu_p^{B_\varepsilon^{\mathbb{H}^1}(y)}(\phi)$$

as in (3.1), where the Radon measure ν is just the Lebesgue measure in $B_\varepsilon^{\mathbb{H}^1}(y)$. Next, we recall a special case of Lemma 3.1 in [1], which we had proven independently for the case of the Heisenberg group.

Theorem 5.5. *[1] For $p\in(1,\infty]$ we have that a continuous function $u\in C(\Omega)$ is p-harmonic if and only if the following expansion holds (in the viscosity sense)*

$$\mu_p^{\mathbb{H}^1}(u,\epsilon)(y) \doteq u(y) + o(\varepsilon^2) \tag{5.17}$$

as $\varepsilon\to 0$.

This theorem is based on the asymptotic formula

$$\mu_p^{\mathbb{H}^1}(\phi,\epsilon)(y) = \phi(y) + \beta(p)\triangle_{\mathfrak{X},0}^p\phi(y)\epsilon^2 + o(\varepsilon^2) \tag{5.18}$$

that holds for any smooth function such that $\nabla_{\mathfrak{X},0}\phi \neq 0$. For our choice of horizontal vector fields $\{Y_1, Y_2\}$ we have

$$\beta(p) = \frac{2}{(p+2)(p+4)}\Big(\frac{\Gamma(\frac{p}{4}+\frac{3}{2})}{\Gamma(\frac{p}{4}+1)}\Big)^2,$$

where Γ is the Euler Gamma function for $p\in(1,\infty)$ and $\beta(\infty)=\frac{1}{2}$. This expression is different from the value obtained in [1] because we have chosen a different representation for the horizontal vector fields.

Theorem 5.5 is precisely the consistency condition of [5]. Stability and boundedness hold since Theorems 3.4 and 4.3, hold once we have Lemma 3.3 in $\mathbb{H}^1$. The DPP that we consider is

$$\left\{\begin{array}{rcll} u_\varepsilon(x) & = & \mu_p^{\mathbb{H}^1}(u_\varepsilon,\varepsilon)(x) & \text{for } x\in\Omega, \\ u_\varepsilon(x) & = & G(x) & \text{on } \Gamma_\epsilon. \end{array}\right. \tag{5.19}$$

Lemma 5.6. *If $u\in B(\Omega_\epsilon)$ satisfies (5.19), when we have $u\in C(\Omega)$.*

Proof. We use the rescaling property in order to compare values on the same ball. We have

$$\mu_p^{\mathbb{H}^1}(u,\epsilon)(y) = \mu_p^{\mathbb{H}^1}(u_\epsilon^y,1)(0),$$

where $u_\epsilon^y(z) = u((\epsilon z)^{-1}\cdot y)$. We then write for points $y,y'\in\overline{\Omega}$

$$\mu_p^{\mathbb{H}^1}(u,\epsilon)(y) - \mu_p^{\mathbb{H}^1}(u,\epsilon)(y') = \mu_p^{\mathbb{H}^1}(u_\epsilon^y,1)(0) - \mu_p^{\mathbb{H}^1}(u_\epsilon^{y'},1)(0).$$

Since $u_\epsilon^y \mapsto u_\epsilon^{y'}$ in L^p norm as $y\to y'$ the conclusion follows form the L^p continuity of the natural p-mean ([13], Thm. 2.4). Thus, the function $u\in C(\Omega)$. □

In conclusion, we checked above that when $\Omega=\mathbf{B}$ is the Euclidean unit ball in $\mathbb{R}^3$ we have the strong uniqueness for the sub-Riemannian p-Laplacian for $1<p<\infty$. Since consistency follows from [1], we have the following result:

Theorem 5.7. *Let $\mathbf{B}$ be the Euclidean unit ball in $\mathbb{R}^3$ and let $g \in C(\partial\mathbf{B})$. For $\varepsilon > 0$ let $\mu_p^{\mathbb{H}^1}(u_\varepsilon, \varepsilon)$ be the natural p-means solutions to the DPP*

$$\begin{cases} u_\varepsilon(x) &= \mu_p^{\mathbb{H}^1}(u_\varepsilon, \varepsilon)(x) & \text{for } x \in \mathbf{B}, \\ u_\varepsilon(x) &= G(x) & \text{on } \Gamma_\epsilon, \end{cases} \tag{5.20}$$

where G is a continuous extension of g to Γ_ε. Then, we have

$$u_\varepsilon^{\mathbb{H}^1} \to u \text{ uniformly in } \overline{\mathbf{B}} \text{ as } \varepsilon \to 0,$$

where u is the unique solution to the Dirichlet problem for the sub-Riemannian p-Laplacian

$$\begin{cases} -\triangle_{\mathfrak{X},0}^p u &= 0 & \text{in } \mathbf{B} \\ u &= g & \text{on } \partial\mathbf{B}, \end{cases} \tag{5.21}$$

for $1 < p < \infty$.

Added in Proof: We have learned that Chandra, Ishiwata, Magnanini, and Wadade [8] have also proved the convergence of the natural p-means in the Euclidean case. Their approach and our approach differ in the treatment of the boundary, but the main results are the same.

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