ESAIM: COCV 27 (2021) 80
https://doi.org/10.1051/cocv/2021076

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

BOUNDARY CONTROLLABILITY OF A SYSTEM MODELLING A PARTIALLY IMMERSED OBSTACLE

G. Vergara-Hermosilla[1], G. Leugering[2] and Y. Wang[2,*]

Abstract. In this paper, we address the problem of boundary controllability for the one-dimensional nonlinear shallow water system, describing the free surface flow of water as well as the flow under a fixed gate structure. The system of differential equations considered can be interpreted as a simplified model of a particular type of wave energy device converter called oscillating water column. The physical requirements naturally lead to the problem of exact controllability in a prescribed region. In particular, we use the concept of nodal profile controllability in which at a given point (the node) time-dependent profiles for the states are required to be reachable by boundary controls. By rewriting the system into a hyperbolic system with nonlocal boundary conditions, we at first establish the semi-global classical solutions of the system, then get the local controllability and nodal profile using a constructive method. In addition, based on this constructive process, we provide an algorithmic concept to calculate the required boundary control function for generating a solution for solving these control problem.

Mathematics Subject Classification. 35L50, 35L65, 93B05, 93C20.

Received February 25, 2021. Accepted June 30, 2021.

1. Introduction

1.1. Context

Free surface interactions with fixed or floating structures have been intensively studied by the mathematical community in the last years with respect to modelling, well-posedness, numerical simulations, etc. Recently, Lannes proposed in [7] a new formulation of water-waves problem in order to take into account the presence of a floating body. More precisely, in his work Lannes implemented a method for the full water wave equations and for reduced asymptotic models, such as the Boussinesq and the nonlinear shallow-water equations, where the pressure exerted by the fluid on the partially immersed structure appears as a Lagrange multiplier associated with the constraint that under the floating structure, the surface of the fluid coincides with the bottom of the structure. In the case of the nonlinear shallow water equations, the resulting fluid-structure model with vertical lateral walls has been studied in [7] and in the more general case of non vertical walls in [6]. An extension to

Keywords and phrases: Nonlinear shallow water equations, exact controllability, nodal profile controllability, boundary control, semi-global piecewise C^1 solution.

[1] Institut de Mathématiques de Bordeaux, Université de Bordeaux, 351, cours de la Liberation 33405 Talence, France.
[2] Department of Mathematics, Friedrich-Alexander-Universität Erlangen-Nürnberg Cauerstrasse 11, 91058 Erlangen, Germany.

[*] Corresponding author: yue.wang@fau.de

a system modelling a floating structure on a viscous shallow water regime has been recently studied by Maity *et al.* in [14], and by Matignon, Tucsnak and Vergara-Hermosilla in [15, 16].

Bresch, Lannes and Métivier in [3] treat the derivation and mathematical analysis of a fluid-structure interaction problem with a configuration where the motion of the fluid is governed by the Boussinesq system, which is a dispersive perturbation of the hyperbolic nonlinear shallow water equations, and in the presence of a fixed partially immersed obstacle. They showed that the fluid-structure interaction problem can be reduced to a transmission problem.

In a similar way and motivated for mathematical modeling and simulations of a specific type of wave energy converting device, the so-called oscillating water column (OWC) device, Bocchi, He and Vergara-Hermosilla discuss in [1] about the fluid-structure interaction of the partially submerged fixed wall structure of the OWC device, considering the nonlinear shallow water equations to describe the fluid movement, and obtain explicit transmission conditions for the system and respective reduced transmission problems. Recently, Bocchi, He and Vergara-Hermosilla propose in [2] a new and general approach on the mathematical modelling of the OWC, where include the presence of the time-dependent air pressure in the device and prove a local well-posedness result in a Sobolev setting. Hence, by considering their results, in this work we deep on a particular kind of boundary controllability on the transmission problems studied in [1] on an equivalent physical configuration. More precisely, we deal with the exact boundary controllability of nodal profile on a system modelling a structure partially immersed in a fluid governed by the nonlinear shallow water equations [7], considering a discontinuity in the height of the fluid bottom and the transmission conditions developed in [1]. This physical situation is presented in a graphical sketch in Figure 1.

This kind of boundary controllability was motivated by practical applications on gas networks and introduced recently in the literature by Gugat, Herty and Schleper, in [5]. Their new approach was almost immediately generalized by Li to general 1-D first order quasilinear hyperbolic systems with general nonlinear boundary conditions in [9].

As is well known, the usual exact boundary controllability that asks the solution to the system under certain boundary controls to satisfy a given final state at a suitably large time $t = T$, however, the exact boundary controllability of nodal profile, requires that the value of solution satisfies the given profiles on one or more nodes for $t \geq T$ by using boundary controls. This approach can be established by means of a constructive method with modular structure, by using the following three ingredients: existence and uniqueness of semi-global classical solution to the mixed initial-boundary value problem, exchanging the role of the the space variable x and time variable t, and the uniqueness of classical solution to the one-sided mixed initial-boundary value problem. This approach was synthesized in a systematic way in the book of Li *et al.* [11], where their also deal with the local exact boundary controllability of nodal profile on a tree-like network with general topology. For more details, see [4, 17] and [18].

1.2. General settings

Mathematically speaking, in this work we consider an incompressible, irrotational, inviscid and homogeneous fluid in a shallow water regime. The motion of the fluid is governed by the 1D nonlinear shallow water equations [7], which are given by

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} \right) + gh \partial_x \zeta = 0, \end{cases} \tag{1.1}$$

where $\zeta(t,x)$ is free surface elevation, $h(t,x)$ is the fluid height given by $h_b + \zeta$ with h_b denoting the height of the bottom of the fluid, $q(t,x)$ is the horizontal discharge defined by

$$q(t,x) := \int_{-h_b}^{\zeta(t,x)} u(t,x,z) \mathrm{d}z, \tag{1.2}$$

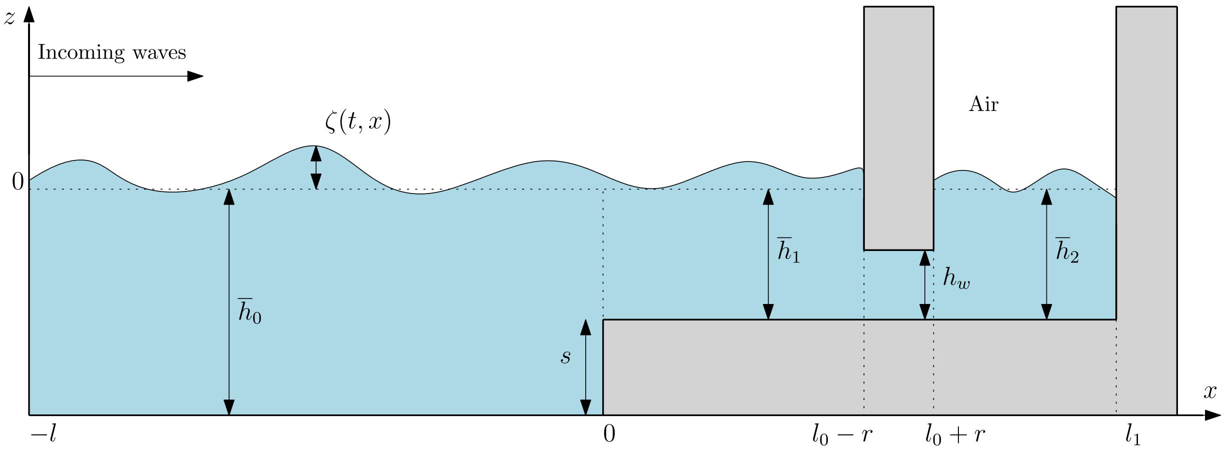

FIGURE 1. Configuration.

with $u(t,x,z)$ denoting the horizontal component of the fluid velocity vector field. In the following, we consider the interval $(-l,l_1)$ as spacial domain, and we assume that the height of the bottom h_b is given by

$$h_b = \begin{cases} h_0, & x \in (-l,0), \\ h_1, & x \in [0,l_1), \end{cases}$$

where h_1 and h_0 are real constants such that $h_1 < h_0$. Furthermore, we assume that

$$\left|\frac{q}{h}\right| < \sqrt{gh}. \tag{1.3}$$

For our analysis, we divide the domain of the problem $(-l,l_1)$ into two parts: the interior domain $\mathcal{I} = (l_0 - r, l_0 + r)$, and its complement $\mathcal{E} = (-l,l_1)\backslash\overline{\mathcal{I}}$, called exterior domain, and which is the union of three intervals $\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$ with

$$\mathcal{E}_0 = (-l,0), \quad \mathcal{E}_1 = (0, l_0 - r), \quad \mathcal{E}_2 = (l_0 + r, l_1). \tag{1.4}$$

Let

$$\zeta(t,x) = \begin{cases} \zeta_i(t,x), & x \in \mathcal{E}_i,\ i = 0,1,2, \\ \zeta_w \in \mathbb{R}, & x \in \mathcal{I}. \end{cases} \tag{1.5}$$

where ζ_w is the parameterization of the bottom of the solid. Therefore, the fluid heights are defined by

$$h(t,x) = \begin{cases} h_i(t,x) := \bar{h}_i + \zeta_i(t,x), & x \in \mathcal{E}_i,\ i = 0,1,2, \\ h_w \in \mathbb{R}, & x \in \mathcal{I}, \end{cases} \tag{1.6}$$

where $\bar{h}_i (i = 0, 1, 2)$ are the fluid height at rest with $\bar{h}_1 = \bar{h}_2 = \bar{h}_0 - s$, and s is the height of the step. $h_w = \bar{h}_1 + \zeta_w$.

$$q(t,x) = \begin{cases} q_i(t,x), & x \in \mathcal{E}_i,\ i = 0,1,2, \\ q_w(t), & x \in \mathcal{I}. \end{cases} \tag{1.7}$$

where $\partial_t q_w(t) = -\frac{h_w}{\rho}\partial_x P_w$ with the fluid density ρ and an unknown surface pressure P_w.

Thus, the coupled system of nonlinear shallow water equations for $x \in \mathcal{E}_i$ can be written as

$$\begin{cases} \partial_t \zeta_i + \partial_x q_i = 0, \\ \partial_t q_i + \partial_x \left(\dfrac{q_i^2}{h_i} \right) + g h_i \partial_x \zeta_i = 0, \end{cases} \qquad i = 0,1,2, \tag{1.8}$$

with the boundary conditions given as

$$x = -l : \quad \zeta_0 = f(t), \tag{1.9}$$

$$x = l_1 : \quad q_2 = 0, \tag{1.10}$$

where $f(t)$ denotes a prescribed boundary function or a boundary control to be determined.

In addition, we consider the transmission conditions developed in [1], which read as follows:

$$x = 0 : \quad \begin{cases} \zeta_0(t,0) = \zeta_1(t,0), \\ q_0(t,0) = q_1(t,0), \end{cases} \tag{1.11}$$

$$x = l_0 \pm r : \quad \begin{cases} q_2(t, l_0 + r) = q_1(t, l_0 - r) = q_w(t), \\ \left[\dfrac{q_2^2}{2h_2^2} + g\zeta_2 \right] \Bigg|_{x = l_0 + r} - \left[\dfrac{q_1^2}{2h_1^2} + g\zeta_1 \right] \Bigg|_{x = l_0 - r} = -\alpha \dfrac{\mathrm{d}}{\mathrm{d}t} q_w(t), \end{cases} \tag{1.12}$$

where $\alpha = \frac{2r}{h_w}$.

The initial data of the system is given at rest,

$$\zeta(0,x) = \begin{cases} 0, & x \in \mathcal{E}, \\ \zeta_w, & x \in \mathcal{I}, \end{cases} \quad \text{and} \quad q(0,x) = 0. \tag{1.13}$$

1.3. Outline of the paper

In Section 2 we present the main results about exact controllability of final data given on the whole or a part of fluid domain at a given finite time $T > 0$ (see in Sect. 2.1), and exact controllability of nodal profile for a given demand at one end $x = l_1$ (See in Sect. 2.2). In Section 3.1 we prove some results relative to the well-posedness result of semi-global C^1 solutions, and in Section 3.2 we prove the main theorems of the paper by considering a constructive method with modular structure. Finally, in Section 2.3, we proposed an algorithm that synthesise the ideas of the proof of Theorem 2.4, which could be utilised to implement numerically our results on exact controllability of nodal profile.

2. Main results

Now we come to our results about control problems for the model given in (1.8). In order to ensure the well-posedness of quasilinear system (1.8) with (1.9)–(1.12), we consider the exact controllability problem in the neighbourhood of the equilibrium state. In the following, Theorem 2.1 and Theorem 2.4 show the local exact controllability and of nodal profile near the rest state given as (1.13), respectively. The corresponding results near other equilibrium state can be obtained in a similar way, we explain with some words in Remark 2.2.

2.1. Exact controllability with one boundary control

The aim of exact controllability for system (1.8) is looking for a boundary control $f(t)$ acting at $x = -l$ though the boundary condition (1.9), such that the prescribed final data $(\zeta_{iT}(x), q_{iT}(x))$ is attained in the entire water regime at a given time T:

$$U_i(T,x) := (\zeta_i, q_i)(T,x) = (\zeta_{iT}(x), q_{iT}(x)), \qquad x \in \mathcal{E}_i,\ i = 0,1,2, \tag{2.1}$$

where the U_i is generated by the control f as a solution of system (1.8)–(1.12) with the initial condition:

$$(\zeta_i, q_i)(0,x) = (\zeta_{i0}(x), q_{i0}(x)), \qquad x \in \mathcal{E}_i,\ i = 0,1,2. \tag{2.2}$$

In order to ensure the existence and uniqueness of piece-wise C^1 semi-global solution for this problem, here we give some assumptions on the initial data and final data in advance: For $i = 0,1,2$,

(S1) $\zeta_{i0}(x), q_{i0}(x)$ are C^1 functions with small norm $\|(\zeta_{i0}, q_{i0})\|_{C^1[\overline{\mathcal{E}}_i]\times C^1[\overline{\mathcal{E}}_i]}$;

(S2) $\zeta_{i0}(x), q_{i0}(x)$ satisfy the C^1 compatibility at the points $(t,x) = (0,-l), (0,l_1)$ and the piecewise C^1 compatibility at the adjoint points $(t,x) = (0,0), (0,l_0-r), (0,l_0+r)$;

Accordingly,

(S1') $\zeta_{iT}(x), q_{iT}(x)$ are C^1 functions with small norm $\|(\zeta_{iT}, q_{iT})\|_{C^1[\overline{\mathcal{E}}_i]\times C^1[\overline{\mathcal{E}}_i]}$;

(S2') $\zeta_{iT}(x), q_{iT}(x)$ satisfy the C^1 compatibility at the points $(t,x) = (T,-l), (T,l_1)$ and the piecewise C^1 compatibility at the adjoint points $(t,x) = (T,0), (T,l_0-r), (T,l_0+r)$.

The following result states the existence of such a control, which can be constructed explicitly in Section 3.

Theorem 2.1. *(Exact Controllability). Let*

$$T > 2\left(\frac{l}{\sqrt{g\bar{h}_0}} + \frac{l_1 - 2r}{\sqrt{g\bar{h}_1}}\right). \tag{2.3}$$

For any given initial data (ζ_{i0}, q_{i0}) and final data (ζ_{iT}, q_{iT}) satisfying the assumptions (S1)-(S2) and (S1')-(S2'), respectively, there exists a boundary control $f(t)$ with small norm $\|f(t)\|_{C^1[0,T]}$, such that mixed initial-boundary value problem for equation (1.8) with the initial condition (2.2), the boundary conditions (1.9)–(1.10) and the transmission conditions (1.11)–(1.12) admits a unique piecewise C^1 solution $(\zeta_i, q_i) = (\zeta_i, q_i)(t,x)\,(i = 0,1,2)$ with small piecewise $C^1 \times C^1$ norm on the domain $\mathcal{R}_i(T) = \{(t,x)\,|\,0 \le t \le T, x \in \mathcal{E}_i\}$, which exactly satisfies the desired final condition (2.1).

In the above result, we consider only the steady-state at rest, but we can easily establish the corresponding local exact controllability in the neighborhood of a stationary subsonic continuously differentiable state.

Remark 2.2. The exact controllability result given in Theorem 2.1 still holds for initial states in a C^1-neighborhood of a stationary subsonic continuously differentiable state $(q_{is}, \zeta_{is}), i = 0,1,2$. For $x \in \mathcal{E}_i, i = 0,1,2$, the stationary solutions $(q_{is}(x), \zeta_{is}(x))$ are given by

$$q_{is}(x) = const, \tag{2.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{q_{is}^2}{\bar{h}_i+\zeta_{is}}\right)+g(\bar{h}_i+\zeta_{is})\frac{\mathrm{d}}{\mathrm{d}x}(\zeta_{is})=0. \tag{2.5}$$

Remark 2.3. The result given in Theorem 2.1 illustrates that after a finite time, the control given at one end $x=-l$ can effect the state functions ζ and q in the entire space horizon $x\in\mathcal{E}$. In fact, this result can be generalized to a special case, where the final condition (2.1) is replaced by a demand given on a part of the horizon:

$$U_2(T,x):=(\zeta_2,q_2)(T,x)=(\zeta_{2T}(x),q_{2T}(x)),\qquad x\in\mathcal{E}_2, \tag{2.6}$$

where $\zeta_{2T}(x),q_{2T}(x)$ are regarded as given final state functions in the 'indoor water regime' $\mathcal{E}_2$. Now, the infimum controllability time (2.9) should be modified by a smaller lower bound:

$$T>\frac{l}{\sqrt{g\bar{h}_0}}+\frac{2l_1-l_0-3r}{\sqrt{g\bar{h}_1}}. \tag{2.7}$$

A limit case, with a given demand only at the end node $x=l_1$, suggests a controllability problem of nodal profile, which will be shown in details in the following subsection.

2.2. Exact controllability of nodal profile: a given demand at the end $x=l_1$

Stimulated by some practical applications, Gugat *et al.* [5] and Li [9] proposed in 2010 another kind of exact boundary controllability, called the nodal profile control. This kind of controllability does not ask the solution to exactly attain any given final state at a suitable time $t=T$ by means of boundary controls, instead it asks the state to exactly fit any given profile function w.r.t time on a node after a suitable time T.

For equation (1.8), we consider a given demand in fluid height and horizontal discharge at the end $x=l_1$ by

$$U_2(t,l_1):=(\zeta_2,q_2)(t,l_1)=(\bar{h}_2+\zeta_B(t),q_B(t)),\qquad t\in[T,\overline{T}]. \tag{2.8}$$

It's worth to mention that the given nodal profile function $U_B:=(\bar{h}_2+\zeta_B(t),q_B(t))$ should be compatible with the boundary condition (1.10) in the time interval $[0,\overline{T}]$ at the node $x=l_1$, which implies that the demand q_B must be set as 0. While, the other one, $\zeta_B(t)$, can be chosen as any given C^1 function of time after a finite time T.

In the following, we give a positive answer to the exact controllability of nodal profile:

Theorem 2.4. *(Exact Controllability of Nodal Profile). Let*

$$T>\left(\frac{l}{\sqrt{g\bar{h}_0}}+\frac{l_1-2r}{\sqrt{g\bar{h}_1}}\right) \tag{2.9}$$

and let $\overline{T}$ be an arbitrarily given number satisfying $\overline{T}>T$. Then for any given initial data (ζ_{i0},q_{i0}) satisfying assumptions (S1)-(S2), and for any given demand of the surface elevation $\zeta_B(t)$ with small norm $\|\zeta_B\|_{C^1[T,\overline{T}]}$, there exists a boundary control $f(t)\in C^1[0,\overline{T}]$, such that the mixed initial-boundary value problem for equation (1.8) with initial condition (2.2), boundary conditions (1.9)–(1.10) and transmission conditions (1.11)–(1.12) admits a unique piecewise C^1 solution $(\zeta_i,q_i)=(\zeta_i,q_i)(t,x)\,(i=0,1,2)$ with small $C^1\times C^1$ norm on the domain

$\mathcal{R}_i(T) = \{(t,x)\,|\,0 \le t \le T, x \in \mathcal{E}_i\}$, *which exactly satisfies the given nodal profile condition at the end* $x = l_1$*:*

$$\zeta_2(t,l_1) = \zeta_B(t), \quad \forall t \in [T, \overline{T}]. \tag{2.10}$$

Remark 2.5. In the above results, we consider only the steady state at rest, but we can easily replace it with any other steady state that is compatible with the boundary and transmission conditions (1.10)–(1.12) and establish local exact controllability of nodal profile near a given subsonic continuously differentiable stationary state $(q_{is}(x), \zeta_{is}(x)), x \in \mathcal{E}_i$, $i = 0,1,2$: There exists a C^1-neighborhood of the stationary state such that for all initial data in this neighborhood that satisfies the C^1-compatibility conditions and for any given smooth subsonic desired fluid height $\bar{h}_2 + \zeta_B(t,l_1)$ and discharge profile $q_B(t,l_1)$ at the end $x = l_1$ that is in a sufficiently small C^1-neighborhood of the boundary data corresponding to the stationary state we can construct a continuously differentiable control $f = f(t)$ such that the demand is fulfilled exactly for all $t \in [T, \overline{T}]$. Moreover, this control generates a continuously differentiable system state in the entire domain.

3. Proofs

3.1. Existence and uniqueness of semi-global C^1 solution

We reduce the system in the exterior domain $\mathcal{E}$ to a compact form by introducing the couple $U_i = (\zeta_i, q_i)^T, i = 0,1,2$:

$$\partial_t U_i + A_i(U_i)\partial_x U_i = 0, \qquad x \in \mathcal{E}_i \tag{3.1}$$

with

$$A_i(U_i) = \begin{pmatrix} 0 & 1 \\ gh_i - \frac{q_i^2}{h_i^2} & \frac{2q_i}{h_i} \end{pmatrix}, \tag{3.2}$$

which has two distinct eigenvalues:

$$\lambda_i^- = \frac{q_i}{h_i} - \sqrt{gh_i} < 0 < \quad \lambda_i^+ = \frac{q_i}{h_i} + \sqrt{gh_i} \tag{3.3}$$

and the corresponding left eigenvectors can be taken as

$$l_i^- = (-\sqrt{gh_i} - \frac{q_i}{h_i}, 1)^T, \quad l_i^+ = (\sqrt{gh_i} - \frac{q_i}{h_i}, 1)^T, \tag{3.4}$$

Introduce the Riemann invariants to the nonlinear shallow water equations

$$\begin{cases} L_i = 2\Big(\sqrt{gh_i(t,x)} - \sqrt{g\bar{h}_i}\,\Big) - \dfrac{q_i}{h_i}, \\ R_i = 2\Big(\sqrt{gh_i(t,x)} - \sqrt{g\bar{h}_i}\,\Big) + \dfrac{q_i}{h_i}. \end{cases} \qquad i = 0,1,2. \tag{3.5}$$

Then, (3.5) can be equivalently rewritten in

$$\begin{cases} \sqrt{gh_i} = \dfrac{R_i + L_i}{4} + \sqrt{g\bar{h}_i}, \\ \dfrac{q_i}{h_i} = \dfrac{R_i - L_i}{2}. \end{cases} \qquad i = 0,1,2. \tag{3.6}$$

Thus,

$$\begin{cases} \zeta_i = \dfrac{(\frac{R_i+L_i}{4}+\sqrt{g\bar{h}_i})^2}{g} - \bar{h}_i, \\ q_i = \dfrac{R_i-L_i}{2}\dfrac{(\frac{R_i+L_i}{4}+\sqrt{g\bar{h}_i})^2}{g}. \end{cases} \qquad i=0,1,2. \tag{3.7}$$

The 1D nonlinear shallow water equations in $\mathcal{E}$ can be rewritten as the following diagonal form:

$$\begin{cases} \partial_t L_i + \lambda_i^- \partial_x L_i = 0, \\ \partial_t R_i + \lambda_i^+ \partial_x R_i = 0, \end{cases} \qquad i=0,1,2. \tag{3.8}$$

Obviously, for each $i=0,1,2,$ $L_i(t,x)$ is the Riemann invariant corresponding to the negative eigenvalue λ_i^-, while $R_i(t,x)$ is the Riemann invariant corresponding to the positive eigenvalue λ_i^+. λ_i^- (resp. λ_i^+) is the entering (resp. departing) characteristic on the right-side boundary, while λ_i^- (resp. λ_i^+) is the departing (resp. entering) characteristic on the left-side boundary. In order to guarantee the well-posedness of the mixed initial-boundary value problem on this coupled network, the boundary conditions on each boundary must satisfy (see [13]):

(1) the number of the boundary conditions must be equal to that of the entering characteristics;

(2) the boundary conditions can be written in the form that the Riemann invariants corresponding to the entering characteristics can be explicitly expressed by all other Riemann invariants (corresponding to the departing characteristics).

The boundary conditions (1.9) and (1.10) can be rewritten in Riemann invariants as

$$x=-l: \quad R_0 = 2\left(\sqrt{g(f+\bar{h}_0)} - \sqrt{g\bar{h}_0}\right) - \frac{R_0-L_0}{2}, \tag{3.9}$$

and

$$x=l_1: \quad R_2 - L_2 = 0, \tag{3.10}$$

which imply that

$$x=-l: \quad L_0 = 3R_0 - 4\left(\sqrt{g(f+\bar{h}_0)} - \sqrt{g\bar{h}_0}\right) \tag{3.11}$$

and

$$x=l_1: \quad R_2 = L_2, \tag{3.12}$$

where R_0 and L_2 are the Riemann invariants corresponding to entering characteristic on $x=-l$ and $x=l_1$, respectively, and

$$\|f\|_{C^1[0,T]} \text{ small} \Longleftrightarrow \left\|\sqrt{g(f+\bar{h}_0)} - \sqrt{g\bar{h}_0}\right\|_{C^1[0,T]} \text{ small.} \tag{3.13}$$

At $x=0$, the Riemann invariants corresponding to entering characteristic are R_0 and L_1. Let

$$\begin{cases} F_1 = \zeta_0 - \zeta_1, \\ F_2 = q_0 - q_1. \end{cases} \tag{3.14}$$

By a direct calculation and noting (3.6)–(3.7), we have

$$\frac{\partial F_1}{\partial R_0}=\frac{\partial \zeta_0}{\partial R_0}=\frac{1}{g}\cdot 2(\frac{L_0+R_0}{4}+\sqrt{g\bar{h}_0})\cdot\frac{1}{4}=\frac{\sqrt{gh_0}}{2g}. \tag{3.15}$$

Similarly,

$$\begin{aligned}&\frac{\partial F_1}{\partial L_1}=-\frac{\partial \zeta_1}{\partial L_1}=-\frac{\sqrt{gh_1}}{2g},\\&\frac{\partial F_2}{\partial R_0}=\frac{\partial q_0}{\partial R_0}=\frac{h_0}{2}+\frac{q_0}{2h_0g}\sqrt{gh_0}=\frac{\sqrt{h_0}}{2\sqrt{g}}(\sqrt{gh_0}+\frac{q_0}{h_0})>0,\\&\frac{\partial F_2}{\partial L_1}=-\frac{\partial q_1}{\partial L_1}=\frac{h_1}{2}+\frac{q_1}{2h_1g}\sqrt{gh_1}=\frac{\sqrt{h_1}}{2\sqrt{g}}(\sqrt{gh_1}+\frac{q_1}{h_1})>0.\end{aligned} \tag{3.16}$$

Thus,

$$\begin{aligned}\left|\frac{\partial(F_1,F_2)}{\partial(R_0,L_1)}\right|_{(R_0,L_1)=(0,0)}&=\left[\frac{\sqrt{h_0h_1}}{4g}(\sqrt{gh_1}+\frac{q_1}{h_1})+\frac{\sqrt{h_0h_1}}{4g}(\sqrt{gh_0}+\frac{q_0}{h_0})\right]_{(\zeta_0,\zeta_1,q_0,q_1)=(0,0,0,0)}\\&=\frac{\sqrt{\bar{h}_0\bar{h}_1}}{4g}(\sqrt{g\bar{h}_1}+\sqrt{g\bar{h}_0})\neq 0,\end{aligned} \tag{3.17}$$

then by implicit function theorem, in a neighborhood of $(R_0,L_1)=(0,0)$, the transmission conditions (1.11) can be equivalently rewritten as

$$\begin{cases}R_0=\hat{F}_1(R_1,L_0),\\L_1=\hat{F}_2(R_1,L_0)\end{cases} \tag{3.18}$$

with $\hat{F}_1(0,0)\equiv\hat{F}_2(0,0)\equiv 0$.

Similarly, at $x=l_0\pm r$, the Riemann invariants corresponding to entering characteristic are R_1 and L_2. Using the initial data and integral from 0 to t, the second transmission condition in (1.12) can be replaced by a nonlocal condition as follows

$$\begin{aligned}-\alpha q_w(t)&=-\alpha q_1(0,l_0-r)+\int_0^t\left[\frac{q_2^2}{2h_2^2}+g\zeta_2\right]_{x=l_0+r}-\left[\frac{q_1^2}{2h_1^2}+g\zeta_1\right]_{x=l_0-r}\mathrm{d}\tau\\&=\int_0^t\left[\frac{q_2^2}{2h_2^2}+g\zeta_2\right]_{x=l_0+r}-\left[\frac{q_1^2}{2h_1^2}+g\zeta_1\right]_{x=l_0-r}\mathrm{d}\tau.\end{aligned} \tag{3.19}$$

Then, by the first transmission condition in (1.12), we rewrite (1.12) into

$$\begin{cases}q_1(t,l_0-r)=q_2(t,l_0+r),\\-\alpha q_1(t,l_0-r)=\displaystyle\int_0^t\left[\frac{q_2^2}{2h_2^2}+g\zeta_2\right]_{x=l_0+r}-\left[\frac{q_1^2}{2h_1^2}+g\zeta_1\right]_{x=l_0-r}\mathrm{d}\tau.\end{cases} \tag{3.20}$$

Now, we introduce two boundary functions G_1 and G_2:

$$\begin{cases} G_1 := q_1(t, l_0 - r) - q_2(t, l_0 + r), \\ G_2 := \alpha q_1(t, l_0 - r) + \displaystyle\int_0^t \left[\frac{q_2^2}{2h_2^2} + g\zeta_2\right]_{x=l_0+r} - \left[\frac{q_1^2}{2h_1^2} + g\zeta_1\right]_{x=l_0-r} \mathrm{d}\tau. \end{cases} \tag{3.21}$$

In fact, by substituting (3.7) into (3.21), G_1 and G_2 can be also regarded as functions with respect to Riemann invariants R_1, L_1, R_2, L_2. For simplicity, we denote $R_1(t, l_0 - r), L_1(t, l_0 - r)$ and $R_2(t, l_0 + r), L_2(t, l_0 + r)$ by R_1, L_1 and R_2, L_2, respectively, in the following expression:

$$\begin{aligned} G_1 &= G_1(R_1, L_1, R_2, L_2) \\ &= \frac{R_1 - L_1}{2} \frac{\left(\frac{R_1+L_1}{4} + \sqrt{g\bar{h}_1}\right)^2}{g} - \frac{R_2 - L_2}{2} \frac{\left(\frac{R_2+L_2}{4} + \sqrt{g\bar{h}_2}\right)^2}{g}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} G_2 &= G_2\left(R_1, L_1, \int_0^t G(R_1, L_1, R_2, L_2)\, \mathrm{d}\tau\right) \\ &= \alpha \left(\frac{R_1 - L_1}{2} \frac{\left(\frac{R_1+L_1}{4} + \sqrt{g\bar{h}_1}\ \right)^2}{g}\right) + \int_0^t G(R_1, L_1, R_2, L_2)\, \mathrm{d}\tau, \end{aligned} \tag{3.23}$$

in which G is the function determined by $\left[\frac{q_2^2}{2h_2^2} + g\zeta_2\right]_{x=l_0+r} - \left[\frac{q_1^2}{2h_1^2} + g\zeta_1\right]_{x=l_0-r}$, and $\int_0^t G(R_1, L_1, R_2, L_2)\, \mathrm{d}\tau$ is regarded as a new variable of G_2.

Thus, if

$$\left|\frac{\partial(G_1, G_2)}{\partial(R_1, L_2)}\right|_{(R_1, L_2)=(0,0)} \neq 0, \tag{3.24}$$

then by implicit function theorem, (3.20) can be solved as

$$\begin{cases} R_1(t) = \hat{G}_1(L_1, R_2, \displaystyle\int_0^t G(L_1, R_1, L_2, R_2)\mathrm{d}\tau), \\ L_2(t) = \hat{G}_2(L_1, R_2, \displaystyle\int_0^t G(L_1, R_1, L_2, R_2)\mathrm{d}\tau). \end{cases} \tag{3.25}$$

Indeed, by calculation, we obtain

$$\begin{aligned} \frac{\partial G_1}{\partial R_1} &= \frac{\partial q_1}{\partial R_1} = \frac{\sqrt{h_1}}{2\sqrt{g}}(\sqrt{gh_1} + \frac{q_1}{h_1}) > 0, \\ \frac{\partial G_1}{\partial L_2} &= -\frac{\partial q_2}{\partial L_2} = \frac{\sqrt{h_2}}{2\sqrt{g}}(\sqrt{gh_2} + \frac{q_2}{h_2}) > 0, \\ \frac{\partial G_2}{\partial R_1} &= \alpha\frac{\partial q_1}{\partial R_1} = \alpha\frac{\sqrt{h_1}}{2\sqrt{g}}(\sqrt{gh_1} + \frac{q_1}{h_1}) > 0, \\ \frac{\partial G_2}{\partial R_2} &= 0. \end{aligned} \tag{3.26}$$

Therefore,

$$\left|\frac{\partial(G_1,G_2)}{\partial(R_1,L_2)}\right| = -\alpha\frac{\sqrt{h_1h_2}}{4g}(\sqrt{gh_1}+\frac{q_1}{h_1})(\sqrt{gh_2}+\frac{q_2}{h_2}) < 0, \tag{3.27}$$

which implies (3.24).

Using a method similar to [13], we get the existence and uniqueness of piecewise local C^1 solution to the mixed initial-boundary value problem (1.8) and (1.9)–(1.12). And then, by a method similar to [8], [10], and the generalized result for a kind of non-local boundary condition as Lemma 4.1 in [12], we can obtain the corresponding results on existence and uniqueness of piecewise semi-global C^1 solution.

Theorem 3.1. *Let $T > 0$ be given. For any given initial data (ζ_{i0}, q_{i0}) with small norm $\|\zeta_{i0}(\cdot), q_{i0}(\cdot)\|_{C^1[\mathcal{E}_i]\times C^1[\mathcal{E}_i]}$ $(i = 0,1,2)$ and boundary function $f(t)$ with small norm $\|f(\cdot)\|_{C^1[0,T]}$, satisfying the conditions (S1)-(S2), the forward mixed initial-boundary value problem of the shallow water system (1.8) on the connected water regime with the initial condition (2.2), the boundary conditions (1.9)–(1.10) and the interface conditions (1.11)–(1.12) admits a unique semi-global piecewise C^1 solution $U_i = (\zeta_i(t,x), q_i(t,x))(i = 0,1,2)$ with small norm $\sum_{i=0,1,2}\|(\zeta_i(\cdot,\cdot), q_i(\cdot,\cdot))\|_{C^1[\mathcal{R}_i(T)]\times C^1[\mathcal{R}_i(T)]}$ on the domain*

$$\mathcal{R}(T) = \bigcup_{i=0,1,2}\mathcal{R}_i(T) = \bigcup_{i=0,1,2}\{(t,x)\,|\,0\le t\le T, x\in\mathcal{E}_i\}.$$

Similarly, for the backward mixed initial-boundary value problem, we have

Theorem 3.2. *Let $T > 0$ be given. For any given initial data (ζ_{iT}, q_{iT}) and boundary function $f(t)$ with small norm $\|f(\cdot)\|_{C^1[0,T]}$, satisfying the conditions (S1')-(S2'), the backward mixed initial-boundary value problem of shallow water system (1.8) with the final condition (2.1), the boundary conditions (1.9)–(1.10) and the interface conditions (1.11)–(1.12) admits a unique semi-global piecewise C^1 solution $U_i = (\zeta_i(t,x), q_i(t,x))$, $(i = 0,1,2)$ with small norm $\sum_{i=0,1,2}\|(\zeta_i(\cdot,\cdot), q_i(\cdot,\cdot))\|_{C^1[\mathcal{R}_i(T)]\times C^1[\mathcal{R}_i(T)]}$ on the domain*

$$\mathcal{R}(T) = \bigcup_{i=0,1,2}\mathcal{R}_i(T) = \bigcup_{i=0,1,2}\{(t,x)|0\le t\le T, x\in\mathcal{E}_i\}.$$

3.2. Proof of exact controllability. Constructive method

In order to prove Theorem 2.1, by means of the constructive method with modular structure, it suffices to prove the following.

Lemma 3.3. *Let $T > 0$ be defined by (2.9). For any given initial data (ζ_{i0}, q_{i0}) and final data (ζ_{iT}, q_{iT}) with small norm $\|(\zeta_{i0}, q_{i0})\|_{C^1[\overline{\mathcal{E}}_i]\times C^1[\overline{\mathcal{E}}_i]}$ and $\|(\zeta_{iT}, q_{iT})\|_{C^1[\overline{\mathcal{E}}_i]\times C^1[\overline{\mathcal{E}}_i]}$, the nonlinear system (1.8) with (1.9)–(1.10) and (1.11)–(1.12) admits a piecewise C^1 solution $(\zeta_i, q_i) = (\zeta_i, q_i)(t,x)(i = 0,1,2)$ with small $C^1\times C^1$ norm on the domain $\mathcal{R}_i(T)$, which satisfies simultaneously the initial condition (2.2) and the final condition (2.1).*

Proof. By (2.9) and $\bar{h}_1 = \bar{h}_2$, there exists an $\epsilon_0 > 0$ so small that

$$\frac{1}{2}T > \sup_{|\zeta_0|+|q_0|\le\epsilon_0}\frac{l}{\sqrt{g\bar{h}_0}} + \sup_{|\zeta_1|+|q_1|\le\epsilon_0}\frac{l_0-r}{\sqrt{g\bar{h}_1}} + \sup_{|\zeta_2|+|q_2|\le\epsilon_0}\frac{l_1-l_0-r}{\sqrt{g\bar{h}_2}}. \tag{3.28}$$

Let

$$T_1 = \sup_{|\zeta_0|+|q_0|\le\epsilon_0}\frac{l}{\sqrt{g\bar{h}_0}},\ T_2 = \sup_{|\zeta_1|+|q_1|\le\epsilon_0}\frac{l_0-r}{\sqrt{g\bar{h}_1}},\ T_3 = \sup_{|\zeta_2|+|q_2|\le\epsilon_0}\frac{l_1-l_0-r}{\sqrt{g\bar{h}_2}}. \tag{3.29}$$

Step 1:
On the domain $\mathcal{R}_f = \{(t,x)|0 \le t \le T_1 + T_2 + T_3, x \in \mathcal{E}\}$ and $\mathcal{R}_b = \{(t,x)|0 \le t \le T - (T_1 + T_2 + T_3), x \in \mathcal{E}\}$, by solving the corresponding forward and backward mixed problem (1.8) with the initial data (2.2) (or the final data (2.1)), boundary condition (1.10) at $x = l_1$, transmission conditions (1.11)–(1.12) and artificial boundary condition (1.9) at $x = 0$ (in which f can be taken as any C^1 function with small C^1 norm). By Theorem 3.1 and Theorem 3.2, there exist unique piecewise C^1 solutions $U_{if} = (\zeta_{if}, q_{if})(t,x)$ and $U_{ib} = (\zeta_{ib}, q_{ib})(t,x)$ $(i = 0,1,2)$, respectively.

Furthermore, we get the trace $\zeta_{2f}(t,l_1), q_{2f}(t,l_1)$ and $\zeta_{2b}(t,l_1)$, $q_{2b}(t,l_1)$ at $x = l_1$.

Step 2:
Solve the following leftward problem from $x = l_1$ to $x = l_0 + r$ to get the solution $U_2 = U_2(t,x)$ in the domain $\{0 \le t \le T, x \in \mathcal{E}_2\}$:

$$\begin{cases} \partial_t U_2 + A_2(U_2)\partial_x U_2 = 0, \\ x = l_1: \ U_2 = (\bar{\zeta}_2(t), 0), \qquad 0 \le t \le T, \\ t = 0: \ q_2(0,x) = 0, \qquad x \in \mathcal{E}_2, \\ t = T: \ \zeta_2(T,x) = \zeta_{2T}(t), \qquad x \in \mathcal{E}_2, \end{cases} \tag{3.30}$$

where $\bar{\zeta}(t)$ is a C^1 function satisfying

$$\bar{\zeta}_2(t) = \begin{cases} \zeta_{2f}(t,l_1), & 0 \le t \le T_1 + T_2 + T_3, \\ \zeta_{2b}(t,l_1), & T - T_1 - T_2 - T_3 \le t \le T. \end{cases} \tag{3.31}$$

and $\zeta_{2T}(t)$ is given by the final data (2.1).

The problem (3.30) is well-posed (by exchanging the role of t and x in hyperbolic system) and admits a solution $U_2 = (\zeta_2, q_2)(t,x)$ on the domain $\mathcal{R}_2 := \{(t,x)|0 \le t \le T, x \in \mathcal{E}_2\}$. Then, we get the value of trace $\zeta_2(t, l_0 + r)$ and $q_2(t, l_0 + r)$ at $x = l_0 + r$.

Step 3:
With the known functions $\zeta_2(t, l_0 + r)$ and $q_2(t, l_0 + r)$, we try to determine the $\zeta_1(t, l_0 - r)$ and $q_1(t, l_0 - r)$ by the transmission conditions (1.12) at $x = l_0 \pm r$.

At first, by the continuity condition, we have

$$q_1(t, l_0 - r) = q_2(t, l_0 + r) \tag{3.32}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} q_w(t) = \frac{\mathrm{d}}{\mathrm{d}t} q_2(t, l_0 + r). \tag{3.33}$$

Substituting the value of $q_1(t, l_0 - r)$, $q_2(t, l_0 + r)$, $\frac{\mathrm{d}}{\mathrm{d}t} q_w(t)$ and $\zeta_2(t, l_0 + r)$ into the second formula of (1.12), it becomes an equation for $\zeta_1(t, l_0 - r)$ that

$$F(\zeta_1(t, l_0 - r)) = 0 \tag{3.34}$$

with

$$\frac{\partial F}{\partial \zeta_1} = -g - \frac{q_1^2}{h_1^3} = -\frac{1}{h_1}\left(g h_1 + \frac{q_1^2}{h_1^2}\right) \neq 0, \quad t \in [0,T], \tag{3.35}$$

provided with the hyperbolicity near the rest state. Thus, we can solve (3.34) near the rest state and uniquely determine $\zeta_1(t, l_0 - r)$ by Implicit Theorem.

We denote the values of $\zeta_1(t, l_0 - r)$ and $q_1(t, l_0 - r)$ obtained in above as $\bar{\zeta}_1(t)$ and $\bar{q}_1(t)$. Hence, we can verify that
(1) $\bar{\zeta}_1(t)$ and $\bar{q}_1(t)$ are both C^1 functions;
(2) $U_1 = (\bar{\zeta}_1(t), \bar{q}_1(t))$ is compatible with the value of $U_{1f}(t, l_0 - r)$ and $U_{1b}(t, l_0 - r)$ on the corresponding intervals.

Step 4:
Solve the following leftward problem from $x = l_0 - r$ to $x = 0$ to get the solution $U_1 = U_1(t, x)$ in the domain $\{0 \le t \le T, x \in \mathcal{E}_1\}$:

$$\begin{cases} \partial_t U_1 + A_1(U_1)\partial_x U_1 = 0, \\ x = l_0 - r: \ U_1 = (\bar{\zeta}_1(t), \bar{q}_1(t)), \qquad 0 \le t \le T, \\ t = 0: \ q_1(0, x) = 0, \qquad x \in \mathcal{E}_1, \\ t = T: \ \zeta_1(T, x) = \zeta_{1T}(t), \qquad x \in \mathcal{E}_1, \end{cases} \tag{3.36}$$

where $(\bar{\zeta}_1(t), \bar{q}_1(t))$ is obtained from Step 3 and $\zeta_{2T}(t)$ is given by the final data (2.1).
The problem (3.36) is well-posed and admits a solution $U_1 = (\zeta_1, q_1)(t, x)$ on the domain $\mathcal{R}_1 := \{(t, x)|0 \le t \le T, x \in \mathcal{E}_1\}$. Then, we get the values of trace $\zeta_1(t, 0)$ and $q_1(t, 0)$.

Step 5:
By the transmission conditions (1.11) at $x = 0$, it is easy to get

$$(\bar{\zeta}_0, \bar{q}_0) := (\zeta_0(t, 0), q_0(t, 0)) = (\zeta_1(t, 0), q_1(t, 0)). \tag{3.37}$$

Similarly, we solve the leftward problem from $x = 0$ to $x = -l$ to get the solution $U_0 = U_0(t, x)$ in the domain $\{0 \le t \le T, x \in \mathcal{E}_0\}$:

$$\begin{cases} \partial_t U_0 + A_1(U_0)\partial_x U_0 = 0, \\ x = 0: \ U_0 = (\bar{\zeta}_0(t), \bar{q}_0(t)), \qquad 0 \le t \le T, \\ t = 0: \ q_0(0, x) = 0, \qquad x \in \mathcal{E}_0, \\ t = T: \ \zeta_0(T, x) = \zeta_{1T}(t), \qquad x \in \mathcal{E}_0. \end{cases} \tag{3.38}$$

Then we take the boundary control $f(t)$ as the trace $\zeta_0(t, -l)$. We can verify that the solutions $U_i = U_i(t, x)$, $i = 0, 1, 2$ constructed in the Step 2, 4, 5 satisfy all requirements of the Lemma. □

3.3. Proof of Theorem 2.4

Proof. The prescribed boundary data $\zeta_B(t)$ and a fixed control function $f(t)$ in (1.8) would generate an overdetermined initial-boundary value problem. To prove Theorem (2.4), we will find a piecewise C^1 solution $(\zeta_i(t, x), q_i(t, x))$ for system (1.8) on the domain $\mathcal{R}(\overline{T})$, which satisfies simultaneously the initial condition (2.2), the interface conditions (1.11)–(1.12), the null flux boundary condition (1.10), and the given nodal profile (2.10). Substituting this solution into the boundary condition (1.9), we obtain the desired boundary control $f(t)$.

We construct a solution to the control problem using the following steps explained in more detail below, where T_1, T_2 and T_3 are defined as in (3.29).
Step 1: We construct the solution $U_{if} = (\zeta_{if}, q_{if})(t, x)$ on the domain $\mathcal{R}_f = \{(t, x) \,|\, 0 \le t \le T_1 + T_2 + T_3, x \in \mathcal{E}\}$ by solving system (1.8) with the initial data (2.2), null flux boundary condition (1.10) at $x = l_1$, transmission conditions (1.11)–(1.12) and the boundary condition $\zeta_0(t, -l) = F(t)$, where F is an artificial given function satisfying the C^1-compatibility condition at $(t, x) = (0, -l)$.

Step 2: From the forward solution U_{if}, we denote the trace at $x = l_1$ as $(\zeta_2(t,l_1), q_2(t,l_1)) = (\zeta_{Bf}(t), 0)$, thus ζ_{Bf} is a C^1 function in the time interval $[0, T_1 + T_2 + T_3]$ and $q_2(t,l_1) \equiv 0$ is determined by the boundary condition. Thus, we can find a C^1 function $\bar{\zeta}_2(t) \in C^1[0, \overline{T}]$ such that

$$\bar{\zeta}_2(t) = \begin{cases} \zeta_{Bf}(t,l_1), & 0 \le t \le T_1 + T_2 + T_3, \\ \zeta_B(t), & T \le t \le \overline{T}, \end{cases} \tag{3.39}$$

where ζ_B is the given demand.
Step 3: We first solve the leftward problem (3.30) in the domain $\mathcal{R}_2$ to get the solution $U_2 = (\zeta_2, q_2)(t,x)$ and then do the same construction procedure shown as the Step 3, Step 4 and Step 5 in the proof of Lemma 3.3 to get solution $U_i = (\zeta_i, q_i)(t,x), i = 1, 0$ in the domain $\mathcal{R}_1$ and $\mathcal{R}_0$, respectively.
Step 4: Using the trace $\zeta_0(t,-l)$ from the solution $U_0 = (\zeta_0, q_0)(t,x)$, we finally compute the control function $f(t) = \zeta_0(t,-l)$.
Step 5: Using the uniqueness theorem, we verify that: at time $t = 0$, $U_i(0,x) \equiv U_{if}(0,x), x \in \mathcal{E}_i$. Thus, the solution U_i $(i = 0, 1, 2)$ is the piecewise C^1 solution to solve the overdetermined initial-boundary value problem. □

3.4. Numerical algorithm

In this Subsection we present an algorithm (see in Algorithm 1) that could be utilised to implement numerically our results on the exact controllability of nodal profile of the system modelling the partially immersed structure considered. More precisely, we synthesize the ideas of the proof of Theorem 2.4 in a consecutive scheme, which gives rise to the following algorithm in order to obtain numerical approximations of the control $f(t)$ associated with our problem. The implementation, however, is beyond the scope of this article and is deferred to a forthcoming publication.

Algorithm 1: Nodal profile controllability

Input: Initial data $\zeta(0,x)$, $q(0,x)$, and a boundary condition $\zeta(t,-l)$ satisfying the C^1-compatibility for the C^1 function at $(0,-l)$.
Output: Boundary control $f(t)$.
Steps:
1: By considering the system (1.8), to obtain the forward solution $U_{if}(t,x)$ on the domain $\mathcal{R}_f(T)$ and compute the trace at $x = l_1$: $(\zeta_2(t,l_1), q_2(t,l_1)) = (\zeta_{Bf}(t), 0)$.
2: Considering the given demand ζ_B to compute the values of the function $\bar{\zeta}_2(t) \in C^1[0, \overline{T}]$ such that

$$\bar{\zeta}_2(t) = \zeta_{Bf}(t,l_1), \text{ if } 0 \le t \le T_1 + T_2 + T_3 \text{ and } \bar{\zeta}_2(t) = \zeta_B(t), \text{ if } T \le t \le \overline{T}.$$

3: By considering the system (3.30), to compute the leftward solution $U_2(t,x)$ on the domain $\mathcal{R}_2$ and obtain the value of the traces $\zeta_2(t, l_0 + r)$ and $q_2(t, l_0 + r)$ at $x = l_0 + r$.
4: By considering the values of the traces obtained in Step 3 and the transmission conditions (1.12) at $x = l_0 \pm r$, obtain the trace $q_1(t, l_0 - r)$ and solve the nonlinear equation (3.34) in order to get the trace $\zeta_1(t, l_0 - r)$.
5: By considering the system (3.36) and the values of the traces obtained in Step 4 to compute the leftward solution $U_1(t,x)$ on the domain $\mathcal{R}_1$.
6: By considering the solution obtained in Step 5 and the transmission conditions (1.11) obtain the traces $\zeta_0(t,0)$ and $q_0(t,0)$.
7: By considering the traces obtained in Step 6 and the system (3.38) obtain the leftward solution $U_0(t,x)$ on the domain $\mathcal{R}_0$ and obtain the trace $\zeta_0(t,-l)$.
8: To compute the control $f(t) = \zeta_0$ by using the trace computed in the Step 7.

Acknowledgements. The first author has been partially supported by the European Unions Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No.765579-ConFlex. The second and third authors are supported by DFG L595/31-1.

References

[1] E. Bocchi, J. He and G. Vergara-Hermosilla, Modelling and simulation of a wave energy converter. *ESAIM: PS* **70** (2021) 68–83.

[2] E. Bocchi, J. He and G. Vergara-Hermosilla, Well-posedness of a nonlinear shallow water model for an oscillating water column with time-dependent air pressure. arXiv:2104.11570 (2021).

[3] D. Bresch, D. Lannes and G. Metivier, Waves interacting with a partially immersed obstacle in the boussinesq regime. arXiv: 1902.04837 (2019).

[4] Q.-L. Gu and T.-T. Li, Exact boundary controllability of nodal profile for quasilinear hyperbolic systems in a tree-like network. *Math. Methods Appl. Sci.* **34** (2011) 911–928.

[5] M. Gugat, M. Herty and V. Schleper, Flow control in gas networks: exact controllability to a given demand. *Math. Methods Appl. Sci.* **34** (2011) 745–757.

[6] T. Iguchi and D. Lannes, Hyperbolic free boundary problems and applications to wave-structure interactions. *Indiana Univ. Math. J.* **70** (2021) 353–464.

[7] D. Lannes, On the dynamics of floating structures. *Ann. PDE* **3** (2017) 11.

[8] T.-T. Li, Controllability and observability for quasilinear hyperbolic systems. American Institute of Mathematical Sciences Springfield, Ill, USA (2010).

[9] T.-T. Li, Exact boundary controllability of nodal profile for quasilinear hyperbolic systems. *Math. Methods Appl. Sci.* **33** (2010) 2101–2106.

[10] T.-T. Li and Y. Jin, Semi-global c1 solution to the mixed initial-boundary value problem for quasilinear hyperbolic systems. *Chin. Ann. Math.* **22** (2001) 325–336.

[11] T.-T. Li, K. Wang and Q.-L. Gu, Exact boundary controllability of nodal profile for quasilinear hyperbolic systems. Springer (2016).

[12] T.-T. Li and Y.-L. Xu, Local exact boundary controllability for nonlinear vibrating string equations. *Int. J. Mod. Phys. B* **17** (2003) 4062–4071.

[13] T.-T. Li and W.-C. Yu, Boundary value problems for quasilinear hyperbolic systems. Duke University (1985).

[14] D. Maity, J. San Martín, T. Takahashi and M. Tucsnak, Analysis of a simplified model of rigid structure floating in a viscous fluid. *J. Nonlinear Sci.* **29** (2019) 1975–2020.

[15] G. Vergara-Hermosilla, D. Matignon and M. Tucsnak, Well-posedness and input-output stability for a system modelling rigid structures floating in a viscous fluid. *IFAC-PapersOnLine* **53** (2020) 7491–7496.

[16] G. Vergara-Hermosilla, D. Matignon and M. Tucsnak, Asymptotic behaviour of a system modelling rigid structures floating in a viscous fluid. To appears in *IFAC-PapersOnLine* (2021).

[17] K. Wang, Exact boundary controllability of nodal profile for 1-d quasilinear wave equations. *Front. Math. China* **6** (2011) 545.

[18] K. Wang and Q.-L. Gu, Exact boundary controllability of nodal profile for quasilinear wave equations in a planar tree-like network of strings. *Math. Methods Appl. Sci.* **37** (2014) 1206–1218.