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NULL CONTROLLABILITY AND FINITE-TIME STABILIZATION IN MINIMAL TIME OF ONE-DIMENSIONAL FIRST-ORDER 2×2 LINEAR HYPERBOLIC SYSTEMS

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Abstract. The goal of this article is to present the minimal time needed for the null controllability and finite-time stabilization of one-dimensional first-order 2×2 linear hyperbolic systems. The main technical point is to show that we cannot obtain a better time. The proof combines the backstepping method with the Titchmarsh convolution theorem.

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1. Introduction and main result

1.1. Problem description

In this paper we are interested in the characterization of the minimal time needed for the controllability of the following class of one-dimensional first-order 2×2 linear hyperbolic systems:

$$\begin{cases} \dfrac{\partial y_1}{\partial t}(t,x) + \lambda_1(x)\dfrac{\partial y_1}{\partial x}(t,x) = a(x)y_1(t,x) + b(x)y_2(t,x), \\ \dfrac{\partial y_2}{\partial t}(t,x) + \lambda_2(x)\dfrac{\partial y_2}{\partial x}(t,x) = c(x)y_1(t,x) + d(x)y_2(t,x), & t \in (0,+\infty),\ x \in (0,1), \\ y_1(t,1) = u(t), \quad y_2(t,0) = 0, \\ y_1(0,x) = y_1^0(x), \quad y_2(0,x) = y_2^0(x), \end{cases} \tag{1.1}$$

where $(y_1(t,\cdot), y_2(t,\cdot))$ is the state at time t, (y_1^0, y_2^0) is the initial data and $u(t)$ is the control at time t. We assume that the speeds $\lambda_1, \lambda_2 \in C^{0,1}([0,1])$ are such that

$$\lambda_1(x) < 0 < \lambda_2(x), \quad \forall x \in [0,1].$$

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Finally, $a, b, c, d \in L^\infty(0,1)$ couple the equations of the system inside the domain (the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ will also be referred in the sequel to as the internal coupling matrix).

The boundary condition at $x = 0$ is a particular case of the more general boundary condition

$$y_2(t,0) = q y_1(t,0), \quad (q \in \mathbb{R}), \tag{1.2}$$

and the goal of this paper is to investigate what happens when $q = 0$.

The aforementioned systems appear in linearized versions of various physical models of balance laws, see *e.g.* Chapter 1 of [2]. For instance, the telegrapher equations of Heaviside form a linear system of the form (1.1) for some parameters (see *e.g.* [2], Sect. 1.2 and (1.20) with $-1 + \lambda R_0 C_\ell = 0$).

We recall that the system (1.1) is well-posed: for every $u \in L^2_{\mathrm{loc}}(0,+\infty)$ and $(y_1^0, y_2^0) \in L^2(0,1)^2$, there exists a unique solution $(y_1, y_2) \in C^0([0,+\infty); L^2(0,1)^2)$ to the system (1.1). By solution we mean "solution along the characteristics" or "broad solution" (see *e.g.* Appendix A of [6]). The same statement remains true if, in the boundary condition at $x = 1$, u is replaced by

$$u(t) = \int_0^1 \left(f_1(\xi) y_1(t,\xi) + f_2(\xi) y_2(t,\xi) \right) \mathrm{d}\xi, \tag{1.3}$$

for any $f_1, f_2 \in L^\infty(0,1)$. The relation (1.3) is called the "feedback law".

Let us now introduce the notions of controllability that we are interested in:

Definition 1.1. Let $T > 0$. We say that the system (1.1) is:

- **finite-time stable with settling time** T if, for every $y_1^0, y_2^0 \in L^2(0,1)$, the corresponding solution to the system (1.1) with $u = 0$ satisfies

$$y_1(T,\cdot) = y_2(T,\cdot) = 0. \tag{1.4}$$

- **finite-time stabilizable with settling time** T if there exist $f_1, f_2 \in L^\infty(0,1)$ such that, for every $y_1^0, y_2^0 \in L^2(0,1)$, the corresponding solution to the system (1.1) with u given by (1.3) satisfies (1.4).
- **null controllable in time** T if, for every $y_1^0, y_2^0 \in L^2(0,1)$, there exists $u \in L^2_{\mathrm{loc}}(0,+\infty)$ such that the corresponding solution to the system (1.1) satisfies (1.4).

Obviously, finite-time stability implies finite-time stabilization, which in turn implies null controllability.

Remark 1.2. As we are trying to bring the solution of the system (1.1) to the state zero, let us first mention that, in general, $u = 0$ does not work. Not only this, but in fact any static boundary output feedback laws, that is of the form $u(t) = k y_2(t,1)$ with $k \in \mathbb{R}$, does not work either in general. A simple example is provided by the following 2×2 system with constant coefficients (see also [2], Sect. 5.6 when $y_2(t,0) = y_1(t,0)$):

$$\begin{cases} \dfrac{\partial y_1}{\partial t}(t,x) - \dfrac{\partial y_1}{\partial x}(t,x) = \pi y_2(t,x), \\ \dfrac{\partial y_2}{\partial t}(t,x) + \dfrac{\partial y_2}{\partial x}(t,x) = \pi y_1(t,x), \\ y_1(t,1) = k y_2(t,1), \quad y_2(t,0) = 0, \\ y_1(0,x) = y_1^0(x), \qquad y_2(0,x) = y_2^0(x), \end{cases} \qquad t \in (0,+\infty),\ x \in (0,1). \tag{1.5}$$

Indeed, for this system we can always construct a smooth initial data (y_1^0, y_2^0) which is an eigenfunction of the operator associated with (1.5) and whose corresponding eigenvalue σ is a positive real number, which makes the system (1.5) exponentially unstable. This can be done as follows. We take

$$y_1^0(x) = \frac{1}{\pi}\left(\sigma y_2^0(x) + \frac{\partial y_2^0}{\partial x}(x)\right),$$

(so that the second equation in (1.5) will always be satisfied) and

- If $k < 1 + 1/\pi$, then we take $\sigma = \pi\sqrt{1-\theta^2}$ and $y_2^0(x) = \sin(\theta\pi x)$, where $\theta \in (0,1)$ is any solution to the equation $\sqrt{1-\theta^2} + \theta\cot(\theta\pi) = k$.
- If $k = 1 + 1/\pi$, then we take $\sigma = \pi$ and $y_2^0(x) = \pi x$.
- If $k > 1 + 1/\pi$, then we take $\sigma = \pi\sqrt{1+\theta^2}$ and $y_2^0(x) = 2\sinh(\theta\pi x)$, where $\theta > 0$ is any solution to the equation $\sqrt{1+\theta^2} + \theta\coth(\theta\pi) = k$.

The goal of this work is to establish a necessary and sufficient condition on the time T for the system (1.1) to be null controllable in time T (resp. finite-time stabilizable with settling time T).

Let us now introduce some notations that will be used all along the rest of this article. Let $\phi_1, \phi_2 \in C^{1,1}([0,1])$ be the increasing functions defined for every $x \in [0,1]$ by

$$\phi_1(x) = \int_0^x \frac{1}{-\lambda_1(\xi)}\,\mathrm{d}\xi, \quad \phi_2(x) = \int_0^x \frac{1}{\lambda_2(\xi)}\,\mathrm{d}\xi. \tag{1.6}$$

We then denote by

$$T_1(\Lambda) = \phi_1(1) = \int_0^1 \frac{1}{-\lambda_1(\xi)}\,\mathrm{d}\xi, \quad T_2(\Lambda) = \phi_2(1) = \int_0^1 \frac{1}{\lambda_2(\xi)}\,\mathrm{d}\xi.$$

Finally, we set

$$T_{\min}(\Lambda) = \max\{T_1(\Lambda), T_2(\Lambda)\}, \quad T_{\max}(\Lambda) = T_1(\Lambda) + T_2(\Lambda). \tag{1.7}$$

The naming of the notations in (1.7) will be explained in Remark 1.9 below.

1.2. Literature

Boundary null controllability and stabilization of hyperbolic systems of balance laws have attracted numerous attention of both mathematicians and engineers during the last decades. In the pioneering work [29], the author established the null controllability of general $n \times n$ coupled linear hyperbolic systems of the form (1.1) in a control time that is given by the sum of the two largest times from the states convecting in opposite directions ([29], Thm. 3.2). It was also observed that this time can be shorten in some cases ([29], Prop. 3.4), and the problem to find the minimal control time for hyperbolic partial differential equations (PDEs) was then raised ([29], Rem. p. 656).

For systems of linear conservation laws (*i.e.* when no internal coupling matrix is present in the system), this problem was completely solved few years later in [31], where the minimal control time has been characterized in terms of the boundary coupling matrix, that is the matrix coupling the equations at the boundary on the uncontrolled side. For systems of balance laws, the story is far from over. A first improvement of the control time of [29] was recently obtained in [7] thanks to the introduction of some rank condition on the boundary coupling matrix. However, this was first done for some generic internal coupling matrices or under rather stringent conditions ([7], Thms. 1.1 and 1.5). The same authors were then able to remove some of these restrictions in [10]. For the present paper it is especially important to emphasize that the new time introduced in [7, 10] is only shown to be sufficient for the null controllability in these works. On the other hand, the minimal control time needed to achieve the exact controllability property (that is when we want to reach any final data and not only zero), was completely characterized in Theorem 1.12 of [19] by a simple and calculable formula. It is also pointed out that null and exact controllability are equivalent properties if the boundary coupling matrix has a full row rank. For quasilinear systems, it has been shown in Theorem 3.2 of [22] that the time of [29] yields the (local) exact controllability of such systems if the linearization of the boundary coupling matrix has a full row rank in a neighborhood of the state zero (see also [23] concerning local null controllability). For homogeneous quasilinear systems, a smaller control time was then obtained in Theorem 1.1 of [16].

Concerning now the stabilization property, the first works seem [13, 26] for the exponential stabilization of homogeneous quasilinear hyperbolic systems in a C^1 framework by using the method of characteristics. To the best of our knowledge, the weakest sufficient condition using this technique can be found in Theorem 1.3, p. 173 of [24]. This condition was then improved in Theorem 2.3 of [4] in a H^2 framework thanks to the construction of an explicit strict Lyapunov function. In all the previous references, the feedback laws were static boundary output feedback laws (that is, depending only on the state values at the boundaries). However, due to the locality of such kind of feedback laws, these two strategies may not be effective to deal with general systems of balance laws ([2], Sect. 5.6 and Rem. 1.2). Another method was then used to address this problem, the backstepping method. For PDEs, this method now consists in transforming our initial system into another system – called target system – for which the stabilization properties are simpler to study. The transformation used is usually a Volterra transformation of the second kind. One can refer to the tutorial book [21] to design boundary feedback laws stabilizing systems modeled by various PDEs and to the introduction of [6] for a complementary state of the art on this method. This technique turned out to be a powerful tool to stabilize general coupled hyperbolic systems, moreover in finite time. In [11] the authors adapted this technique to obtain the first finite-time stabilization result for 2×2 linear hyperbolic system. This method was then developed, notably with a more careful choice of the target system, to treat 3×3 systems in [17] and then to treat general $n \times n$ systems in [18, 20]. However, the control time obtained in these works was larger than the one in [29] and it was only shown in [1, 5] that we can stabilize with the same time as the one of [29]. These works have recently been generalized to time-dependent systems in [6]. Finally, let us also mention the two recent works [8, 9] concerning the finite-time stabilization of homogeneous quasilinear systems, with the same control time as in [7, 10].

In spite of quite a number of contributions dealing with these two problems (controllability and stabilization), we see that there are no references concerning the optimality of the control time for systems of linear balance laws with spatial-varying internal coupling matrix, especially when null and exact controllability are not equivalent, so that the results in [11, 19] cannot be considered. This is of course a nontrivial task and it requires the addition of new techniques as we shall see below. The goal of this article is to fill this gap, at least for 2×2 systems. We will provide an explicit formula of the minimal control time for any 2×2 system of linear balance laws with spacial-varying internal coupling matrix. We will see that one of the main differences between null and exact controllability is that such a critical time is sensitive to the behavior of the internal coupling matrix for the null controllability, whereas it is known to never be the case for the exact controllability [11, 19].

1.3. Main result and comments

The important quantity in the present work is the following:

Definition 1.3. For $\varepsilon > 0$ and a function $f : (0, \varepsilon) \longrightarrow \mathbb{R}$, we denote by

$$\ell_\varepsilon(f) = \begin{cases} \sup I_\varepsilon(f) & \text{if } I_\varepsilon(f) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where $I_\varepsilon(f) = \{\ell \in (0, \varepsilon) \quad | \quad f = 0 \text{ a.e. in } (0, \ell)\}$.

The quantity $\ell_\varepsilon(f)$ is the length of the largest interval of the form $(0, \ell)$ where the function f vanishes.

Example 1.4.

(E1) The simplest example of function f with $\ell_\varepsilon(f) = \ell$ ($\ell \in [0,\varepsilon]$) is obviously the step function

$$f(x) = \begin{cases} 0 & \text{if } x \leq \ell, \\ 1 & \text{if } x > \ell. \end{cases}$$

(E2) If $f \in C^k([0,\varepsilon))$ ($k \in \mathbb{N}$) and satisfies $f^{(k)}(0) \neq 0$, then $\ell_\varepsilon(f) = 0$. In particular, if f has an analytic extension in a neighborhood of $x = 0$, then $\ell_\varepsilon(f) = 0$.

(E3) An example of smooth function f with $\ell_\varepsilon(f) = 0$ but that does not satisfy the previous conditions is

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \exp\left(-\dfrac{1}{x}\right) & \text{if } x > 0. \end{cases} \tag{1.8}$$

The main result of this article is the following complete characterization of the controllability properties of the system (1.1):

Theorem 1.5. *Let $T > 0$.*

(i) If the system (1.1) is null controllable in time T, then necessarily

$$T \geq \max\left\{ T_{\min}(\Lambda), \quad \int_{\ell_{x_\Lambda}(c)}^1 \left(\frac{1}{-\lambda_1(\xi)} + \frac{1}{\lambda_2(\xi)}\right) \mathrm{d}\xi \right\}, \tag{1.9}$$

where $x_\Lambda \in (0,1)$ is the unique solution to $\phi_1(x_\Lambda) + \phi_2(x_\Lambda) = T_2(\Lambda)$ $(= \phi_2(1))$.

(ii) If the time T satisfies (1.9), then the system (1.1) is finite-time stabilizable with settling time T.

Note in particular that the system (1.1) is then null controllable in time T if, and only if, it is finite-time stabilizable with settling time T.

Remark 1.6. As we shall see in the proof below, the most difficult part of this result is the necessary condition, that is the item (i).

Example 1.7. For c satisfying the properties in (E2) or given by the function in (E3), this result shows that the time $T_{\max}(\Lambda)$ cannot be improved. This is not trivial, especially when c is given by the function in (E3).

Remark 1.8. When λ_1, λ_2 do not depend on space, the condition (1.9), in the situation $T_{\min}(\Lambda) \leq T < T_{\max}(\Lambda)$, simply becomes

$$c = 0 \quad \text{in } \left(0, 1 - \frac{T}{T_{\max}(\Lambda)}\right).$$

In particular, we see that we can possibly obtain any intermediate time between $T_{\min}(\Lambda)$ and $T_{\max}(\Lambda)$. Moreover, note that the value $T_{\min}(\Lambda)$ is reachable even when c is not identically equal to zero.

Remark 1.9. Theorem 1.5 says that the time on the right-hand side of (1.9) is the so-called minimal control time, that is it is equal to $T_{\inf}(\lambda_1, \lambda_2, a, b, c, d)$, where $T_{\inf}(\lambda_1, \lambda_2, a, b, c, d) = \inf\{T > 0 \quad \text{system (1.1) is null controllable in time T}\}$, (and the infimum being a minimum here). Let us mention that it is sometimes found in the literature that the time $T_{\max}(\Lambda)$ is called "the theoretical lower bound for control time" or "the optimal time". However, we see from our result that we may have

$$T_{\inf}(\lambda_1, \lambda_2, a, b, c, d) < T_{\max}(\Lambda).$$

Therefore, the minimal control time can be strictly less than what is sometimes called the optimal time. This brings some confusion to our point of view and this is why we prefer to avoid using the naming "optimal" for $T_{\max}(\Lambda)$. Instead, we carefully introduced a different naming and use the notations $T_{\min}(\Lambda)$ and $T_{\max}(\Lambda)$ since we can easily check that

$$T_{\min}(\Lambda) = \min\left\{T_{\inf}(\lambda_1, \lambda_2, a, b, c, d) \quad a, b, c, d \in L^\infty(0,1)\right\},$$
$$T_{\max}(\Lambda) = \max\left\{T_{\inf}(\lambda_1, \lambda_2, a, b, c, d) \quad a, b, c, d \in L^\infty(0,1)\right\}.$$

Remark 1.10. Let us comment other possibilities for the boundary conditions at $x = 0$:

(i) When the boundary condition $y_2(t, 0) = 0$ is replaced by (1.2) with boundary coupling "matrix" $q \neq 0$, the result ([11], Thm. 3.2) shows that the time $T_{\max}(\Lambda)$ is the minimal control time (more precisely, it is shown that the system (1.1) with such a boundary condition is equivalent to the same system with no internal coupling matrix, for which $T_{\max}(\Lambda)$ is clearly minimal). However, when $q = 0$, we see that our time is smaller than the one obtained in this reference.

(ii) When a second control is applied at the boundary $x = 0$, *i.e.* the boundary condition $y_2(t, 0) = 0$ is replaced by $y_2(t, 0) = v(t)$ with $v \in L^2_{\text{loc}}(0, +\infty)$ a second control at our disposal, then the time $T_{\min}(\Lambda)$ is the minimal control time. The null controllability for $T \geq T_{\min}(\Lambda)$ can be shown using for instance the well-known constructive method developed in Theorem 3.1 of [22]. On the other hand, the failure of the null controllability for $T < T_{\min}(\Lambda)$ follows from the backstepping method (by means of Volterra transformation of the second kind) and a simple adaptation of Lemma 3.3 below.

Therefore, combining the previous results of the literature with the new results of the present paper, we see that all the following possibilities for the boundary conditions have been handled:

$$y_1(t, 1) = py_2(t, 1) + ru(t), \quad y_2(t, 0) = qy_1(t, 0) + sv(t),$$
$$p, q, r, s \in \mathbb{R} \text{ with } (r, s) \neq (0, 0).$$

Remark 1.11. As we shall see below during the proof, Theorem 1.5 remains true for more regular initial data $y_1^0, y_2^0 \in L^\infty(0, 1)$ (see in particular Rem. 3.4).

The rest of this article is organized as follows. In Section 2, we use the backstepping method to show that our initial system (1.1) is equivalent to a canonical system from a controllability point of view. In Section 3 we use the Titchmarsh convolution theorem to completely characterize the minimal control time for this canonical system. In Section 4 we characterize this time in terms of the parameters of the initial system. Finally, in Section 5 we discuss possible extensions to systems with more than two equations.

2. Reduction to a canonical form

In this section, we perform some changes of unknown to transform our initial system (1.1) into a new system whose controllability properties will be simpler to study, this is the so-called backstepping method for PDEs. The content of section is quite standard by now, we refer for instance to Section 3.2 of [11] for more details on the computations below.

First of all, we remove the diagonal terms in the system (1.1). Using the invertible spatial transformation (seen as an operator from $L^2(0, 1)^2$ onto itself)

$$\begin{cases} \tilde{y}_1(t, x) = e_1(x) y_1(t, x), \\ \tilde{y}_2(t, x) = e_2(x) y_2(t, x), \end{cases} \tag{2.1}$$

with

$$e_1(x) = \exp\left(-\int_0^x \frac{a(\xi)}{\lambda_1(\xi)}\,\mathrm{d}\xi\right), \quad e_2(x) = \exp\left(-\int_0^x \frac{d(\xi)}{\lambda_2(\xi)}\,\mathrm{d}\xi\right), \tag{2.2}$$

we easily see that the system (1.1) is null controllable in time T (resp. finite-time stabilizable with settling time T) if, and only if, so is the system

$$\begin{cases} \dfrac{\partial \tilde{y}_1}{\partial t}(t,x) + \lambda_1(x)\dfrac{\partial \tilde{y}_1}{\partial x}(t,x) = \tilde{b}(x)\tilde{y}_2(t,x), \\ \dfrac{\partial \tilde{y}_2}{\partial t}(t,x) + \lambda_2(x)\dfrac{\partial \tilde{y}_2}{\partial x}(t,x) = \tilde{c}(x)\tilde{y}_1(t,x), & t \in (0,+\infty),\ x \in (0,1), \\ \tilde{y}_1(t,1) = \tilde{u}(t), \qquad \tilde{y}_2(t,0) = 0, \\ \tilde{y}_1(0,x) = \tilde{y}_1^0(x), \quad \tilde{y}_2(0,x) = \tilde{y}_2^0(x), \end{cases} \tag{2.3}$$

where

$$\tilde{b}(x) = b(x)\frac{e_1(x)}{e_2(x)}, \quad \tilde{c}(x) = c(x)\frac{e_2(x)}{e_1(x)}. \tag{2.4}$$

Let us now remove the coupling term on the first equation of (2.3) thanks to a second transformation. Set

$$\mathcal{T} = \left\{(x,\xi) \in (0,1)\times(0,1) \quad | \quad x > \xi\right\}.$$

Let $k_{11}, k_{12}, k_{21}, k_{22} \in L^\infty(\mathcal{T})$. Using the spatial transformation

$$\begin{cases} \hat{y}_1(t,x) = \tilde{y}_1(t,x) - \displaystyle\int_0^x \left(k_{11}(x,\xi)\tilde{y}_1(t,\xi) + k_{12}(x,\xi)\tilde{y}_2(t,\xi)\right)\mathrm{d}\xi, \\ \hat{y}_2(t,x) = \tilde{y}_2(t,x) - \displaystyle\int_0^x \left(k_{21}(x,\xi)\tilde{y}_1(t,\xi) + k_{22}(x,\xi)\tilde{y}_2(t,\xi)\right)\mathrm{d}\xi, \end{cases} \tag{2.5}$$

which is invertible since it is a Volterra transformation of the second kind (see *e.g.* [15], Chap. 2, Thm. 5), we see that the system (2.3) is null controllable in time T (resp. finite-time stabilizable with settling time T) if, and only if, so is the system

$$\begin{cases} \dfrac{\partial \hat{y}_1}{\partial t}(t,x) + \lambda_1(x)\dfrac{\partial \hat{y}_1}{\partial x}(t,x) = 0, \\ \dfrac{\partial \hat{y}_2}{\partial t}(t,x) + \lambda_2(x)\dfrac{\partial \hat{y}_2}{\partial x}(t,x) = g(x)\hat{y}_1(t,0), & t \in (0,+\infty),\ x \in (0,1), \\ \hat{y}_1(t,1) = \hat{u}(t), \qquad \hat{y}_2(t,0) = 0, \\ \hat{y}_1(0,x) = \hat{y}_1^0(x), \quad \hat{y}_2(0,x) = \hat{y}_2^0(x), \end{cases} \tag{2.6}$$

with g given by

$$g(x) = -k_{21}(x,0)\lambda_1(0), \tag{2.7}$$

provided that the kernels $k_{11}, k_{12}, k_{21}, k_{22}$ satisfy the so-called kernel equations:

$$\begin{cases} \lambda_1(x)\dfrac{\partial k_{11}}{\partial x}(x,\xi) + \dfrac{\partial k_{11}}{\partial \xi}(x,\xi)\lambda_1(\xi) + k_{11}(x,\xi)\dfrac{\partial \lambda_1}{\partial \xi}(\xi) + k_{12}(x,\xi)\tilde{c}(\xi) = 0, \\ \lambda_1(x)\dfrac{\partial k_{12}}{\partial x}(x,\xi) + \dfrac{\partial k_{12}}{\partial \xi}(x,\xi)\lambda_2(\xi) + k_{11}(x,\xi)\tilde{b}(\xi) + k_{12}(x,\xi)\dfrac{\partial \lambda_2}{\partial \xi}(\xi) = 0, \\ k_{11}(x,0) = 0, \\ k_{12}(x,x) = \dfrac{\tilde{b}(x)}{\lambda_1(x) - \lambda_2(x)}, \end{cases} \quad (x,\xi) \in \mathcal{T}, \tag{2.8}$$

and

$$\begin{cases} \lambda_2(x)\dfrac{\partial k_{21}}{\partial x}(x,\xi) + \dfrac{\partial k_{21}}{\partial \xi}(x,\xi)\lambda_1(\xi) + k_{21}(x,\xi)\dfrac{\partial \lambda_1}{\partial \xi}(\xi) + k_{22}(x,\xi)\tilde{c}(\xi) = 0, \\ \lambda_2(x)\dfrac{\partial k_{22}}{\partial x}(x,\xi) + \dfrac{\partial k_{22}}{\partial \xi}(x,\xi)\lambda_2(\xi) + k_{21}(x,\xi)\tilde{b}(\xi) + k_{22}(x,\xi)\dfrac{\partial \lambda_2}{\partial \xi}(\xi) = 0, \\ k_{21}(x,x) = \dfrac{\tilde{c}(x)}{\lambda_2(x) - \lambda_1(x)}, \end{cases} \quad (x,\xi) \in \mathcal{T}. \tag{2.9}$$

Note that (2.8) and (2.9) are not coupled.

From Theorem A.1 of [11], we know that the kernel equations (2.8)–(2.9) have a solution. More precisely, we have the following result:

Theorem 2.1. *For every $k^0 \in L^\infty(0,1)$, there exists a unique solution $(k_{11}, k_{12}, k_{21}, k_{22}) \in L^\infty(\mathcal{T})^4$ to the kernel equations (2.8)–(2.9) with*

$$k_{22}(x,0) = k^0(x), \quad x \in (0,1).$$

In the aforementioned reference this result is stated in a C^0 framework (assuming that $a, b, c, d \in C^0([0,1])$) but its proof readily shows that it is valid in L^∞ as well. As before, the notion of solution is to be understood in the sense of solution along the characteristics. The boundary terms such as $k_{21}(x,0)$, which defines g (see (2.7)), or $k_{11}(1,\xi), k_{12}(1,\xi)$, that will appear shortly below in our feedback law (see (3.2)), etc. are also understood in this sense. We refer for instance to the formula (4.6) below for the precise meaning of $k_{21}(x,0)$.

3. Study of the canonical system

We call the system (2.6) the "control canonical form of the system (1.1)" or "canonical system" in short, by analogy with [3, 28] and since we will see in this section that we are able to directly read its controllability properties (a task that seems impossible on the initial system (1.1)).

The goal of this section is to establish the following result:

Theorem 3.1. *Let $T > 0$ and $g \in L^\infty(0,1)$.*

(i) If the system (2.6) is null controllable in time T, then necessarily

$$T \geq \max\left\{ T_1(\Lambda) + \int_{\ell_1(g)}^1 \frac{1}{\lambda_2(\xi)}\,\mathrm{d}\xi, \quad T_2(\Lambda) \right\}. \tag{3.1}$$

(ii) If the time T satisfies (3.1), then the system (2.6) is finite-time stable with settling time T.

Let us emphasize once again that the difficult point is the first item.

Remark 3.2. Since $\hat{u}=0$ stabilizes the canonical system (2.6) by (ii) of Theorem 3.1, we see from the formula (2.1) and (2.5) that our feedback for the system (1.1) is then

$$u(t)=\int_0^1 \frac{k_{11}(1,\xi)e_1(\xi)}{e_1(1)}y_1(t,\xi)\,\mathrm{d}\xi+\int_0^1 \frac{k_{12}(1,\xi)e_2(\xi)}{e_1(1)}y_2(t,\xi)\,\mathrm{d}\xi. \tag{3.2}$$

Note that $u\in C^0([0,+\infty))$.

3.1. The characteristics

Before proving Theorem 3.1 we need to introduce the characteristic curves associated with the system (2.6) and recall some useful properties.

First of all, it is convenient to extend λ_1,λ_2 to functions of $\mathbb{R}$ (still denoted by the same) such that $\lambda_1,\lambda_2\in C^{0,1}(\mathbb{R})$ and

$$\lambda_1(x)\leq-\varepsilon<0<\varepsilon<\lambda_2(x),\quad \forall x\in\mathbb{R}, \tag{3.3}$$

for some $\varepsilon>0$ small enough. Since all the results of the present paper depend only on the values of λ_1,λ_2 in $[0,1]$, they do not depend on such an extension.

In what follows, $i\in\{1,2\}$. Let χ_i be the flow associated with λ_i, *i.e.* for every $(t,x)\in\mathbb{R}\times\mathbb{R}$, the function $s\longmapsto\chi_i(s;t,x)$ is the solution to the ODE

$$\begin{cases}\dfrac{\partial\chi_i}{\partial s}(s;t,x)=\lambda_i(\chi_i(s;t,x)),\quad \forall s\in\mathbb{R},\\ \chi_i(t;t,x)=x.\end{cases} \tag{3.4}$$

The existence and uniqueness of a (global) solution to the ODE (3.4) follows from the (global) Cauchy-Lipschitz theorem (see *e.g.* [14], Thm. II.1.1). The uniqueness also yields the important group property

$$\chi_i\left(\sigma;s,\chi_i(s;t,x)\right)=\chi_i(\sigma;t,x),\quad \forall\sigma,s\in\mathbb{R}. \tag{3.5}$$

By classical regularity results on ODEs (see *e.g.* [14], Thm. V.3.1), we have $\chi_i\in C^1(\mathbb{R}^3)$ and

$$\frac{\partial\chi_i}{\partial t}(s;t,x)=-\lambda_i(\chi_i(s;t,x)),\quad \frac{\partial\chi_i}{\partial x}(s;t,x)=\frac{\lambda_i(\chi_i(s;t,x))}{\lambda_i(x)}. \tag{3.6}$$

Let us now introduce the entry and exit times $s_i^{\mathrm{in}}(t,x),s_i^{\mathrm{out}}(t,x)\in\mathbb{R}$ of the flow $\chi_i(\cdot;t,x)$ inside the domain $[0,1]$, *i.e.* the respective unique solutions to

$$\begin{cases}\chi_1(s_1^{\mathrm{in}}(t,x);t,x)=1,\quad \chi_1(s_1^{\mathrm{out}}(t,x);t,x)=0,\\ \chi_2(s_2^{\mathrm{in}}(t,x);t,x)=0,\quad \chi_2(s_2^{\mathrm{out}}(t,x);t,x)=1.\end{cases}$$

Their existence and uniqueness are guaranteed by the condition (3.3). It readily follows from (3.5) and the uniqueness of s_i^{in} that

$$s_i^{\mathrm{in}}\left(s,\chi_i(s;t,x)\right)=s_i^{\mathrm{in}}(t,x),\quad \forall s\in\mathbb{R}. \tag{3.7}$$

By the implicit function theorem we have $s_i^{\text{in}} \in C^1(\mathbb{R}^2)$ with (using (3.6))

$$\begin{cases} \dfrac{\partial s_1^{\text{in}}}{\partial t}(t,x) > 0, & \dfrac{\partial s_1^{\text{in}}}{\partial x}(t,x) > 0, \\ \dfrac{\partial s_2^{\text{in}}}{\partial t}(t,x) > 0, & \dfrac{\partial s_2^{\text{in}}}{\partial x}(t,x) < 0. \end{cases} \tag{3.8}$$

Combined with the group property (3.7), this yields the following inverse formula for every $s, t \in \mathbb{R}$:

$$\begin{cases} s < s_1^{\text{out}}(t,1) \iff s_1^{\text{in}}(s,0) < t, \\ s < s_2^{\text{out}}(t,0) \iff s_2^{\text{in}}(s,1) < t. \end{cases} \tag{3.9}$$

Finally, since λ_i does not depend on time, we have an explicit formula for the inverse function $\theta \longmapsto \chi_i^{-1}(\theta; t, x)$. Indeed, it solves

$$\begin{cases} \dfrac{\partial (\chi_i^{-1})}{\partial \theta}(\theta; t, x) = \dfrac{1}{\frac{\partial \chi_i}{\partial s}\left(\chi_i^{-1}(\theta; t, x); t, x\right)} = \dfrac{1}{\lambda_i(\theta)}, \quad \forall \theta \in \mathbb{R}, \\ \chi_i^{-1}(x; t, x) = t, \end{cases}$$

which gives

$$\chi_i^{-1}(\theta; t, x) = t + \int_x^\theta \frac{1}{\lambda_i(\xi)} \,\mathrm{d}\xi. \tag{3.10}$$

This also yields an explicit formula for $s_1^{\text{in}}, s_2^{\text{in}}$ and $s_1^{\text{out}}, s_2^{\text{out}}$ and, in particular,

$$T_1(\Lambda) = s_1^{\text{out}}(0,1), \quad T_2(\Lambda) = s_2^{\text{out}}(0,0).$$

3.2. Proof of Theorem 3.1

First of all, the solution of the canonical system (2.6) is explicitly given by:

$$\hat{y}_1(t,x) = \begin{cases} \hat{y}_1^0\left(\chi_1(0; t, x)\right) & \text{if } s_1^{\text{in}}(t,x) < 0, \\ \hat{u}\left(s_1^{\text{in}}(t,x)\right) & \text{if } s_1^{\text{in}}(t,x) > 0, \end{cases} \tag{3.11}$$

and

$$\hat{y}_2(t,x) = \begin{cases} \hat{y}_2^0\left(\chi_2(0; t, x)\right) + \displaystyle\int_0^t g\left(\chi_2(s; t, x)\right) \hat{y}_1(s,0) \,\mathrm{d}s & \text{if } s_2^{\text{in}}(t,x) < 0, \\ \displaystyle\int_{s_2^{\text{in}}(t,x)}^t g\left(\chi_2(s; t, x)\right) \hat{y}_1(s,0) \,\mathrm{d}s & \text{if } s_2^{\text{in}}(t,x) > 0. \end{cases} \tag{3.12}$$

Next, we show a uniform lower bound for the control time:

Lemma 3.3. *Let $T > 0$. If the system (2.6) is null controllable in time T, then necessarily*

$$T \geq T_{\min}(\Lambda).$$

This result states that the control time cannot be better than the one of the case $g = 0$.

Proof. For $i \in \{1, 2\}$, let ω_i be the open subset defined by

$$\omega_i = \left\{ x \in (0,1) \quad \middle| \quad s_i^{\text{in}}(T,x) < 0 \right\}.$$

From (3.9) and (3.8), we see that

$$T \geq T_i(\Lambda) \quad \Longleftrightarrow \quad \omega_i = \emptyset. \tag{3.13}$$

Therefore, if $T < T_1(\Lambda)$, then we see from (3.11) that $\hat{y}_1^0$ can be chosen so that $\hat{y}_1(T,x) \neq 0$ for $x \in \omega_1$, whatever $\hat{u}$ is. On the other hand, if $T < T_2(\Lambda)$ and if the system (2.6) is null controllable in time T, then for every $\hat{y}_2^0 \in L^2(0,1)$, there exists $\hat{u} \in L^2(0,T)$ such that, for a.e. $x \in \omega_2$, we have

$$0 = \hat{y}_2^0\left(\chi_2(0;T,x)\right) + \int_0^T g\left(\chi_2(s;T,x)\right) \hat{y}_1(s,0)\,\mathrm{d}s. \tag{3.14}$$

Since $x \in \omega_2 \longmapsto \chi_2(0;T,x)$ is a C^1 diffeomorphism (its inverse is given by $\xi \longmapsto \chi_2(T;0,\xi)$ thanks to (3.5)), this implies that the bounded linear operator $K : L^2(0,T) \longrightarrow L^2(\omega_2)$ defined by

$$(Kh)(x) = -\int_0^T g\left(\chi_2(s;T,x)\right) h(s)\,\mathrm{d}s,$$

is surjective. This is impossible since its range is clearly a subset of $L^\infty(\omega_2)$, which is a proper subset of $L^2(\omega_2)$ (alternatively, one could note that K is compact and therefore it cannot be surjective over an infinite dimensional space, see *e.g.* [27], Thm. 4.18 (b)).

□

Remark 3.4. The previous proof can be adapted to show that the condition $T \geq T_{\min}(\Lambda)$ is also necessary for the null controllability in time T with more regular initial data $y_1^0, y_2^0 \in L^\infty(0,1)$. Indeed, doing the change of variable $x = \chi_2(T;t,0)$ in (3.14) we obtain the surjectivity of the operator $\tilde{K} : L^\infty(0,T) \longrightarrow L^\infty(T - T_2(\Lambda), 0)$ defined by $(\tilde{K}h)(t) = (Kh)(\chi_2(T;t,0))$, but this is impossible since its range is in fact included in $C^0(T - T_2(\Lambda), 0)$. To see this, we use that its kernel $(t,s) \longmapsto g(\chi_2(s;t,0))$ is a convolution kernel (see Step 2 in the proof of Thm. 3.1 below) and the continuity of translations in L^1.

The proof of the item (i) of Theorem 3.1 crucially relies on the Titchmarsh convolution theorem ([30], Thm. VII) (see also [25], Chap. XV):

Theorem 3.5. *Let $\alpha, \beta \in L^1(0,\bar{\tau})$ ($\bar{\tau} > 0$). We have*

$$\int_0^\tau \alpha(\tau - \sigma)\beta(\sigma)\,\mathrm{d}\sigma = 0, \qquad \textit{a.e. } 0 < \tau < \bar{\tau}, \tag{3.15}$$

if, and only if,

$$\ell_{\bar{\tau}}(\alpha) + \ell_{\bar{\tau}}(\beta) \geq \bar{\tau}.$$

Remark 3.6. The difficulty in the proof of this result is the necessary condition, *i.e.* the implication "$\Longrightarrow$", just like it is the case for our main result. Let us however mention that its proof is easy in case α satisfies the condition in (E2) of Example 1.4 (by taking derivatives of (3.15) and using the injectivity of Volterra transformations of the second kind). It does not seem trivial for functions of the form (1.8) though.

We are now ready to prove the main result of Section 3:

Proof of Theorem 3.1.

1) Thanks to Lemma 3.3, we can assume that $T \geq T_1(\Lambda)$ and $T \geq T_2(\Lambda)$. This means that $s_1^{\text{in}}(T,x) > 0$ and $s_2^{\text{in}}(T,x) > 0$ for every $x \in (0,1)$ (see (3.13) and (3.8)). It then follows from the explicit formula (3.11) and (3.12) that $\hat{y}_1(T,\cdot) = 0$ if, and only if,

$$\hat{u}\left(s_1^{\text{in}}(T,x)\right) = 0, \quad 0 < x < 1, \tag{3.16}$$

and $\hat{y}_2(T,\cdot) = 0$ if, and only if,

$$\int_{s_2^{\text{in}}(T,x)}^{T} g\left(\chi_2(s;T,x)\right)\hat{y}_1(s,0)\,\mathrm{d}s = 0, \quad 0 < x < 1. \tag{3.17}$$

2) Let us focus on the second condition (3.17). Writing $x = \chi_2(T;t,0)$, which belongs to $(0,1)$ for $t \in (s_2^{\text{in}}(T,1),T)$ (recall in particular (3.9)), and using the group properties (3.5) and (3.7) with the identity $s_2^{\text{in}}(t,0) = t$, we obtain that $\hat{y}_2(T,\cdot) = 0$ if, and only if,

$$\int_{t}^{T} g\left(\chi_2(s;t,0)\right)\hat{y}_1(s,0)\,\mathrm{d}s = 0, \quad s_2^{\text{in}}(T,1) < t < T. \tag{3.18}$$

Now we use the fact that $g\left(\chi_2(s;t,0)\right)$ is actually a function of $s-t$. Indeed, by uniqueness to the solution to the ODE (3.4), we see that the characteristics take the form

$$\chi_i(s;t,x) = \tilde{\chi}_i(s-t;x),$$

where $s \longmapsto \tilde{\chi}_i(s;x)$ is the unique solution to

$$\begin{cases} \dfrac{\partial \tilde{\chi}_i}{\partial s}(s;x) = \lambda_i(\tilde{\chi}_i(s;x)), \quad \forall s \in \mathbb{R}, \\ \tilde{\chi}_i(0;x) = x. \end{cases}$$

Using the change of variables $\sigma = s - t$ and introducing

$$\alpha(\theta) = \hat{y}_1(-\theta + T, 0), \qquad \beta(\theta) = g\left(\tilde{\chi}_2(\theta;0)\right), \quad 0 < \theta < T - s_2^{\text{in}}(T,1),$$

we see that (3.18) is equivalent to (setting $\tau = T - t$)

$$\int_0^{\tau} \alpha(\tau - \sigma)\beta(\sigma)\,\mathrm{d}\sigma = 0, \quad 0 < \tau < \bar{\tau}, \tag{3.19}$$

where

$$\bar{\tau} = T - s_2^{\text{in}}(T,1).$$

3) Applying the Titchmarsh convolution theorem (Thm. 3.5) we deduce that (3.19) is equivalent to

$$\ell_{\bar{\tau}}\left(\alpha\right) + \ell_{\bar{\tau}}\left(\beta\right) \geq \bar{\tau}.$$

From the explicit expression (3.11) and the inverse formula (3.9), we see that

$$\alpha(\theta)=\begin{cases}\hat{y}_1^0\left(\chi_1(0;-\theta+T,0)\right) & \text{if } \theta>T-s_1^{\mathrm{out}}(0,1),\\ \hat{u}\left(s_1^{\mathrm{in}}(-\theta+T,0)\right) & \text{if } \theta<T-s_1^{\mathrm{out}}(0,1).\end{cases}$$

Therefore, we can choose $\hat{y}_1^0$ so that

$$\alpha(\theta)\neq 0,\quad \forall\theta\in\left(T-s_1^{\mathrm{out}}(0,1),T-s_1^{\mathrm{out}}(0,1)+\varepsilon\right),$$

for some $0<\varepsilon<s_1^{\mathrm{out}}(0,1)$. This yields the bound

$$\ell_{\bar{\tau}}\left(\alpha\right)\leq T-s_1^{\mathrm{out}}(0,1).$$

Consequently, we necessarily have

$$\ell_{\bar{\tau}}\left(\beta\right)\geq s_1^{\mathrm{out}}(0,1)-s_2^{\mathrm{in}}(T,1). \tag{3.20}$$

Since $s\longmapsto\tilde{\chi}_2(s;0)$ is increasing with $\tilde{\chi}_2(0;0)=0$, this is equivalent to

$$\begin{aligned}\ell_1\left(g\right)\geq\tilde{\chi}_2\left(s_1^{\mathrm{out}}(0,1)-s_2^{\mathrm{in}}(T,1);0\right)&=\chi_2\left(s_1^{\mathrm{out}}(0,1);s_2^{\mathrm{in}}(T,1),0\right)\\&=\chi_2\left(s_1^{\mathrm{out}}(0,1);T,1\right)\quad\text{(by (3.5) with } s=s_2^{\mathrm{in}}(T,1)).\end{aligned}$$

Since $s\longmapsto\chi_2(s;T,1)$ is increasing, this is also equivalent to

$$\chi_2^{-1}(\ell_1\left(g\right);T,1)\geq s_1^{\mathrm{out}}(0,1)=T_1(\Lambda).$$

Using the explicit expression (3.10), we then obtain the desired condition $T\geq T_1(\Lambda)+\int_{\ell_1(g)}^1\frac{1}{\lambda_2(\xi)}\,\mathrm{d}\xi$.

4) Conversely, assume that T satisfies this condition and $T\geq T_2(\Lambda)$. Then, (3.20) holds by the previous equivalences. Taking $\hat{u}=0$, we see that $\alpha=0$ in $(0,T-T_1(\Lambda))$, which yields

$$\ell_{\bar{\tau}}\left(\alpha\right)+\ell_{\bar{\tau}}\left(\beta\right)\geq T-T_1(\Lambda)+s_1^{\mathrm{out}}(0,1)-s_2^{\mathrm{in}}(T,1)=T-s_2^{\mathrm{in}}(T,1)=\bar{\tau}.$$

This implies (3.19) (here we only use the "easy part" of the Titchmarsh convolution theorem) and thus $\hat{y}_2(T,\cdot)=0$. Finally, note that $\hat{u}=0$ also obviously satisfies (3.16) and thus $\hat{y}_1(T,\cdot)=0$ as well.

□

Remark 3.7. Let us point out that the space dependence of the speeds brings up more technical difficulties than the case of constant speeds (especially the step 2)).

4. Proof of the main result

In this section we show how to deduce our main result from Theorem 3.1.

Proof of Theorem 1.5.

1) First of all, let us recall that the initial system (1.1) is null controllable in time T (resp. finite-time stabilizable with settling time T) if, and only if, so is the canonical system (2.6) (with g given by (2.7)).

Therefore, thanks to Theorem 3.1 it suffices to show that

$$T \geq T_1(\Lambda) + \int_{\ell_1(g)}^1 \frac{1}{\lambda_2(\xi)}\,\mathrm{d}\xi \quad \Longleftrightarrow \quad T \geq \int_{\ell_{x_\Lambda}(c)}^1 \left(\frac{1}{-\lambda_1(\xi)} + \frac{1}{\lambda_2(\xi)}\right)\mathrm{d}\xi,$$

which amounts to characterize $\ell_1(g)$ in terms of $\ell_{x_\Lambda}(c)$ (we recall that x_Λ is defined in the statement of Thm. 1.5). To this end, we are going to prove the identity

$$\phi_2(\ell_1(g)) = \phi_1(\ell_{x_\Lambda}(c)) + \phi_2(\ell_{x_\Lambda}(c)), \tag{4.1}$$

where we recall that $\phi_1, \phi_2 \in C^{1,1}([0,1])$ are defined in (1.6).

2) We recall that $g(x) = -k_{21}(x,0)\lambda_1(0)$, where k_{21} is the solution in $\mathcal{T}$ to

$$\begin{cases} \lambda_2(x)\dfrac{\partial k_{21}}{\partial x}(x,\xi) + \dfrac{\partial k_{21}}{\partial \xi}(x,\xi)\lambda_1(\xi) + k_{21}(x,\xi)\dfrac{\partial \lambda_1}{\partial \xi}(\xi) + k_{22}(x,\xi)\tilde{c}(\xi) = 0, \\ k_{21}(x,x) = \dfrac{\tilde{c}(x)}{\lambda_2(x) - \lambda_1(x)}, \end{cases} \tag{4.2}$$

and where $\tilde{c}$ is defined in (2.4) and (2.2) (note that $\ell_\varepsilon(\tilde{c}) = \ell_\varepsilon(c)$ for any $\varepsilon \in (0,1]$). Let $s \longmapsto \chi(s;x)$ be the associated characteristic passing through $(x,\xi) = (x,0)$, *i.e.* the solution to the ODE

$$\begin{cases} \dfrac{\partial \chi}{\partial s}(s;x) = \dfrac{\lambda_1(\chi(s;x))}{\lambda_2(s)}, \quad \forall s \in \mathbb{R}, \\ \chi(x;x) = 0, \end{cases} \tag{4.3}$$

(we recall that λ_1, λ_2 have been extended to $\mathbb{R}$ in Sect. 3.1). We have $\chi \in C^1(\mathbb{R}^2)$ by classical regularity results on ODEs with

$$\frac{\partial \chi}{\partial x}(s;x) = \frac{-\lambda_1(\chi(s;x))}{\lambda_2(x)} > 0.$$

Since $f : s \longmapsto s - \chi(s;x)$ is continuous and increasing with $\lim_{s\to\mp\infty} f(s) = \mp\infty$, there exists a unique solution $s^{\mathrm{in}}(x) \in \mathbb{R}$ to

$$\chi\left(s^{\mathrm{in}}(x);x\right) = s^{\mathrm{in}}(x).$$

Besides, for every $x \in (0,1)$, we have $0 < s^{\mathrm{in}}(x) < x$ and

$$(s, \chi(s;x)) \in \mathcal{T}, \quad \forall s \in (s^{\mathrm{in}}(x), x).$$

By the implicit function theorem we have $s^{\mathrm{in}} \in C^1(\mathbb{R})$ with, for every $x \in \mathbb{R}$,

$$(s^{\mathrm{in}})'(x) = \frac{\frac{\partial \chi}{\partial x}(s^{\mathrm{in}}(x);x)}{1 - \frac{\partial \chi}{\partial s}(s^{\mathrm{in}}(x);x)} > 0. \tag{4.4}$$

In particular, the inverse function $(s^{\mathrm{in}})^{-1} : [0, s^{\mathrm{in}}(1)] \longrightarrow [0,1]$ exists. We are going to show that

$$\ell_{s^{\mathrm{in}}(1)}(\tilde{c}) = s^{\mathrm{in}}(\ell_1(g)). \tag{4.5}$$

Along the characteristics, the solution to (4.2) satisfies, for $s \in \big(s^{\mathrm{in}}(x), x\big)$,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} k_{21}(s, \chi(s;x)) = \dfrac{-\frac{\partial \lambda_1}{\partial \xi}(\chi(s;x))}{\lambda_2(s)} k_{21}(s,\chi(s;x)) + \dfrac{-k_{22}(s,\chi(s;x))}{\lambda_2(s)} \tilde{c}(\chi(s;x)), \\ k_{21}\left(s^{\mathrm{in}}(x), s^{\mathrm{in}}(x)\right) = \dfrac{\tilde{c}(s^{\mathrm{in}}(x))}{\lambda_2(s^{\mathrm{in}}(x)) - \lambda_1(s^{\mathrm{in}}(x))}. \end{cases}$$

Consequently,

$$k_{21}(x,0) = r(x)\tilde{c}(s^{\mathrm{in}}(x)) + \int_{s^{\mathrm{in}}(x)}^{x} h(x,\sigma)\tilde{c}\left(\chi(\sigma;x)\right) \mathrm{d}\sigma, \tag{4.6}$$

with

$$r(x) = \exp\left(\int_{s^{\mathrm{in}}(x)}^{x} \frac{-\frac{\partial \lambda_1}{\partial \xi}(\chi(s;x))}{\lambda_2(s)} \,\mathrm{d}s\right) \frac{1}{\lambda_2(s^{\mathrm{in}}(x)) - \lambda_1(s^{\mathrm{in}}(x))},$$

and

$$h(x,\sigma) = \exp\left(\int_{\sigma}^{x} \frac{-\frac{\partial \lambda_1}{\partial \xi}(\chi(s;x))}{\lambda_2(s)} \,\mathrm{d}s\right) \frac{-k_{22}(\sigma,\chi(\sigma;x))}{\lambda_2(\sigma)}.$$

Using the change of variable $\theta = (s^{\mathrm{in}})^{-1}(\chi(\sigma;x))$, we obtain

$$\frac{1}{r(x)} k_{21}(x,0) = \tilde{c}(s^{\mathrm{in}}(x)) + \int_0^x \tilde{h}(x,\theta)\tilde{c}\left(s^{\mathrm{in}}(\theta)\right) \mathrm{d}\theta,$$

with kernel

$$\tilde{h}(x,\theta) = \frac{1}{r(x)} h\left(x, \chi^{-1}(s^{\mathrm{in}}(\theta);x)\right) \frac{(s^{\mathrm{in}})'(\theta)}{\frac{\partial \chi}{\partial s}(\chi^{-1}(s^{\mathrm{in}}(\theta);x);x)}.$$

We can check that $\tilde{h} \in L^\infty(\mathcal{T})$ (recall (4.4)). It follows from the injectivity of Volterra transformations of the second kind that

$$\ell_1(g) = \ell_1\left(\tilde{c} \circ s^{\mathrm{in}}\right),$$

which is equivalent to (4.5) since s^{in} is increasing with $s^{\mathrm{in}}(0) = 0$.

3) To conclude the proof, it remains to observe that the solution to the ODE (4.3) satisfies

$$\phi_1(\chi(s;x)) = \phi_2(x) - \phi_2(s),$$

for every $x \in [0,1]$ and $s \in [s^{\mathrm{in}}(x), x]$. Taking $x = 1$ and $s = s^{\mathrm{in}}(1)$, we see that $s^{\mathrm{in}}(1) = x_\Lambda$ (by uniqueness of the solution to the equation $\phi_1(x_\Lambda) + \phi_2(x_\Lambda) = \phi_2(1)$). Taking then $x = \ell_1(g)$ and $s = s^{\mathrm{in}}(\ell_1(g)) = \ell_{x_\Lambda}(\tilde{c})$ (recall (4.5)), we obtain the desired identity (4.1).

□

Remark 4.1. In the proof of Theorem 1.5, we have not used the apparent freedom for the boundary data of k_{22} provided by Theorem 2.1.

5. Extensions and open problems

The results of this paper can be partially extended to systems of more than 2 equations. More precisely, we can consider the following $n \times n$ systems ($n \geq 2$):

$$\begin{cases} \dfrac{\partial y_1}{\partial t}(t,x) + \lambda_1(x)\dfrac{\partial y_1}{\partial x}(t,x) = a(x)y_1(t,x) + B(x)y_+(t,x), \\ \dfrac{\partial y_+}{\partial t}(t,x) + \Lambda_+(x)\dfrac{\partial y_+}{\partial x}(t,x) = C(x)y_1(t,x) + D(x)y_+(t,x), \\ y_1(t,1) = u(t), \qquad y_+(t,0) = Qy_1(t,0), \\ y_1(0,x) = y_1^0(x), \quad y_+(0,x) = y_+^0(x), \end{cases} \quad t \in (0,+\infty),\ x \in (0,1). \tag{5.1}$$

In (5.1), $(y_1(t,\cdot), y_+(t,\cdot)) \in \mathbb{R} \times \mathbb{R}^{n-1}$ is the state at time t, (y_1^0, y_+^0) is the initial data and $u(t) \in \mathbb{R}$ is the control at time t. We assume that we have one negative speed $\lambda_1 \in C^{0,1}([0,1])$ and $n-1$ positive speeds $\lambda_2, \ldots, \lambda_n \in C^{0,1}([0,1])$ such that:

$$\lambda_1(x) < 0 < \lambda_2(x) < \cdots < \lambda_n(x), \quad \forall x \in [0,1], \tag{5.2}$$

and we use the notation $\Lambda_+ = \operatorname{diag}(\lambda_2, \ldots, \lambda_n)$. Finally, $a \in L^\infty(0,1)$, $B \in L^\infty(0,1)^{1\times(n-1)}$, $C \in L^\infty(0,1)^{n-1}$, $D \in L^\infty(0,1)^{(n-1)\times(n-1)}$ couple the equations of the system inside the domain and the constant matrix $Q \in \mathbb{R}^{n-1}$ couples the equations of the system on the boundary $x = 0$.

Let us now introduce the times defined by

$$T_1(\Lambda) = \int_0^1 \frac{1}{-\lambda_1(\xi)}\,\mathrm{d}\xi, \quad T_i(\Lambda) = \int_0^1 \frac{1}{\lambda_i(\xi)}\,\mathrm{d}\xi, \quad \forall i \in \{2, \ldots, n\}.$$

Note that $T_n(\Lambda) < \ldots < T_2(\Lambda)$ by (5.2).

It was established in [12] and Lemma 3.1 of [18] that the system (5.1) is finite-time stabilizable with setting time T if $T \geq T_{\max}(\Lambda)$, where $T_{\max}(\Lambda)$ is still given by (1.7).

Using the backstepping method (see *e.g.* [20], Sect. 2.2), it can be shown as before that the system (5.1) is null controllable in time T (resp. finite-time stabilizable with settling time T) if, and only if, so is the system

$$\begin{cases} \dfrac{\partial \hat{y}_1}{\partial t}(t,x) + \lambda_1(x)\dfrac{\partial \hat{y}_1}{\partial x}(t,x) = 0, \\ \dfrac{\partial \hat{y}_+}{\partial t}(t,x) + \Lambda_+(x)\dfrac{\partial \hat{y}_+}{\partial x}(t,x) = G(x)\hat{y}_1(t,0), \\ \hat{y}_1(t,1) = \hat{u}(t), \qquad \hat{y}_+(t,0) = Q\hat{y}_1(t,0), \\ \hat{y}_1(0,x) = \hat{y}_1^0(x), \quad \hat{y}_+(0,x) = \hat{y}_+^0(x), \end{cases} \quad t \in (0,+\infty),\ x \in (0,1), \tag{5.3}$$

for some $G \in L^\infty(0,1)^{n-1}$ depending on all the parameters $\lambda_1, \Lambda_+, a, B, C, D$ and Q.

By mimicking the proof of Theorem 3.1, we can obtain the following result:

Theorem 5.1. *Let $T > 0$.*

(i) If the system (5.3) is null controllable in time T, then necessarily

$$T \geq \max\left\{T_1(\Lambda) + \max_{i\in\{2,\ldots,n\}} T(\lambda_i, g_{i-1}, q_{i-1}), \quad T_2(\Lambda)\right\}, \tag{5.4}$$

where

$$T(\lambda_i, g_{i-1}, q_{i-1}) = \begin{cases} \displaystyle\int_{\ell_1(g_{i-1})}^{1} \frac{1}{\lambda_i(\xi)}\,\mathrm{d}\xi & \text{if } q_{i-1} = 0, \\ T_i(\Lambda) & \text{if } q_{i-1} \neq 0. \end{cases}$$

(ii) If the time T satisfies (5.4), then the system (5.3) is finite-time stable with settling time T.

However, we are unable so far to deduce from this result some explicit condition for the initial system (5.1). The main technical problem is that G is heavily coupled on the parameters λ_1, Λ_+, a, B, C, D and Q (see *e.g.* [20], Sect. 2.2). We leave it as an open problem that could be investigated in future works.

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