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MINIMAX SOLUTIONS OF HAMILTON–JACOBI EQUATIONS WITH FRACTIONAL COINVARIANT DERIVATIVES*

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Abstract. We consider a Cauchy problem for a Hamilton–Jacobi equation with coinvariant derivatives of an order $\alpha \in (0, 1)$. Such problems arise naturally in optimal control problems for dynamical systems which evolution is described by differential equations with the Caputo fractional derivatives of the order α. We propose a notion of a generalized in the minimax sense solution of the considered problem. We prove that a minimax solution exists, is unique, and is consistent with a classical solution of this problem. In particular, we give a special attention to the proof of a comparison principle, which requires construction of a suitable Lyapunov–Krasovskii functional.

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1. Introduction

Nowadays, the theory of differential equations with fractional-order derivatives (see, *e.g.*, [8, 22, 41, 43, 46]) is an actively developing branch of mathematics, which attracts the interest of many researchers. In particular, attention is paid to optimal control problems for dynamical systems which evolution is described by differential equations with the Caputo fractional derivatives. Such problems appear in various fields of knowledge including, *e.g.*, chemistry [10], biology [50], electrical engineering [19], and medicine [21]. Main directions of research here are related to necessary optimality conditions (see, *e.g.*, [4, 30] and the references therein) and numerical methods for constructing optimal controls (see, *e.g.*, [29, 45, 51] and the references therein). In addition, note that several problems for linear systems are considered and studied in detail in, *e.g.*, [2, 13, 18, 20, 27, 40]. The reader is also referred to [5] for an overview of works on various control problems for fractional-order systems.

In [12], the dynamic programming principle was extended to a Bolza-type optimal control problem for a dynamical system described by a fractional differential equation with the Caputo derivative of an order $\alpha \in (0, 1)$. In particular, it was shown that the value of this problem should be introduced as a functional in a suitable space of paths. Further, the problem was associated to a Hamilton–Jacobi equation with coinvariant (*ci*-) derivatives of the order α. Note that these derivatives can be considered as an extension of the notion of *ci*-derivatives (of the first order) proposed and developed in, *e.g.*, [23, 37]. It was proved that if the value functional is smooth enough (namely, if it is *ci*-smooth of the order α), then it satisfies the Hamilton–Jacobi equation and the natural

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boundary condition, and, therefore, the value functional can be treated as a solution of this Cauchy problem in the classical sense. However, by analogy with the case of optimal control problems for dynamical systems described by ordinary differential equations (*i.e.*, when $\alpha = 1$), the value functional usually does not possess the required smoothness properties, which leads to the need to introduce and study generalized solutions of the obtained Cauchy problem.

In the paper, we consider a Cauchy problem for a Hamilton–Jacobi equation with *ci*-derivatives of an order $\alpha \in (0,1)$ and propose a notion of a minimax solution of this problem. The technique of minimax solutions originates in the positional differential games theory (see, *e.g.*, [24, 26]) and can be seen as the development of the unification constructions of differential games [25]. Minimax solutions of Hamilton–Jacobi equations with first-order partial derivatives were proposed and comprehensively studied in [48] (see also [49]). Further, this technique was extended to Hamilton–Jacobi equations with first-order *ci*-derivatives, which arise in optimization problems for dynamical systems described by functional differential equations of a retarded type [37] (see also [17, 31–34, 36], and [3] for an infinite dimensional case) and of a neutral type [38, 39, 42]. Note that the minimax approach was also applied to investigate generalized solutions of systems of equations arising in mean field games [1].

Following the general methodology, we define a minimax solution of the considered Cauchy problem in terms of a pair of non-local stability properties of this solution with respect to so-called characteristic differential inclusions, which in this case become fractional differential inclusions with the Caputo derivatives of the order α. We prove that a minimax solution exists, is unique, and is consistent with a classical solution of the problem. In particular, we establish a comparison principle. In general, the proofs of these results are carried out by the schemes of the proofs of the corresponding statements for Hamilton–Jacobi equations with partial derivatives [48] and with first-order *ci*-derivatives [37] (see also [31] and [3]). They are based on properties [15] of the sets of solutions of the characteristic differential inclusions. However, in order to prove the comparison principle, it is required to construct a suitable Lyapunov–Krasovskii functional with a number of prescribed properties (in this connection, see, *e.g.*, Sect. 15 of [37] and also Sect. 5 of [33], Sect. 4.1 of [17]). Observe that the functionals proposed earlier in [17, 31, 36] for the case of first-order *ci*-derivatives cannot be applied in the fractional setting due to formal reasons, and, furthermore, it turns out that dealing with direct fractional counterparts of these functionals also does not lead to a satisfactory result (see Sect. 7.3 for discussion). To overcome this difficulty, we develop a technique that explicitly takes into account features of fractional-order integrals and derivatives and build the required functional *via* a finite sum (the number of terms depends on α) of some integral functionals with weakly singular kernels. In addition, note that each of these functionals can be treated as a modification of the quadratic functional used in, *e.g.*, [11] (see also the references therein). Thus, the construction of the Lyapunov–Krasovskii functional substantially differs from the previous studies and can be considered as the main contribution of the paper.

The paper is organized as follows. In Section 2, we recall definitions of Riemann–Liouville integrals and Caputo derivatives of a fractional order, describe some of their properties, and introduce special functional spaces. Auxiliary facts from the theory of differential inclusions with the Caputo fractional derivatives are presented in Section 3. In Section 4, we discuss a notion of *ci*-derivatives of a fractional order. In Section 5, a Cauchy problem for a Hamilton–Jacobi equation with *ci*-derivatives of a fractional order is considered, and a definition of a minimax solution of this problem is given. Consistency of minimax and classical solutions of the problem is studied in Section 6. A comparison principle is established in Section 7. Existence and uniqueness of a minimax solution are proved in Section 8. Concluding remarks are given in Section 9.

2. Preliminaries

Fix $n \in \mathbb{N}$ and $T > 0$. By $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, we denote the Euclidean norm and the inner product in $\mathbb{R}^n$.

For a given $t \in [0,T]$, let $\mathrm{L}^\infty([0,t],\mathbb{R}^n)$ be the set of (Lebesgue) measurable and essentially bounded functions from $[0,t]$ to $\mathbb{R}^n$. For a function $\psi(\cdot) \in \mathrm{L}^\infty([0,t],\mathbb{R}^n)$, the (left-sided) Riemann–Liouville fractional integral of an order $\alpha > 0$ is defined by

$$(I^\alpha \psi)(\tau) \triangleq \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{\psi(\xi)}{(\tau - \xi)^{1-\alpha}} \, \mathrm{d}\xi, \quad \tau \in [0, t], \tag{2.1}$$

where Γ is the gamma function. In the case $\alpha = 0$, we formally define $(I^0 \psi)(\tau) \triangleq \psi(\tau)$, $\tau \in [0, t]$.

Note that, for every $\alpha \geq 0$, $\beta \geq 0$, and $\psi(\cdot) \in \mathrm{L}^\infty([0, t], \mathbb{R}^n)$, the following semigroup property holds (see, *e.g.*, (2.21) of [46] and also Thm. 2.2 of [8]):

$$\big(I^\alpha (I^\beta \psi)\big)(\tau) = (I^{\alpha+\beta} \psi)(\tau) \quad \forall \tau \in [0, t]. \tag{2.2}$$

Further, according to, *e.g.*, Theorem 3.6 and Remark 3.3 of [46] (see also [8], Thm. 2.6), for any $\alpha \in (0, 1]$ and $\psi(\cdot) \in \mathrm{L}^\infty([0, t], \mathbb{R}^n)$, the inequalities below are valid:

$$\|(I^\alpha \psi)(\tau)\| \leq \frac{\tau^\alpha}{\Gamma(\alpha+1)} \operatorname*{ess\,sup}_{\xi \in [0, \tau]} \|\psi(\xi)\|, \quad \|(I^\alpha \psi)(\tau) - (I^\alpha \psi)(\tau')\| \leq \frac{2|\tau - \tau'|^\alpha}{\Gamma(\alpha+1)} \operatorname*{ess\,sup}_{\xi \in [0, t]} \|\psi(\xi)\|, \tag{2.3}$$

where τ, $\tau' \in [0, t]$. In particular, we have the inclusion $(I^\alpha \psi)(\cdot) \in \mathrm{C}([0, t], \mathbb{R}^n)$. Here and below, by $\mathrm{C}([0, t], \mathbb{R}^n)$, we denote the space of continuous functions from $[0, t]$ to $\mathbb{R}^n$ endowed with the norm

$$\|x(\cdot)\|_{[0,t]} \triangleq \max_{\tau \in [0, t]} \|x(\tau)\|, \quad x(\cdot) \in \mathrm{C}([0, t], \mathbb{R}^n).$$

For $\alpha \in (0, 1]$, let $\mathrm{AC}^\alpha([0, t], \mathbb{R}^n)$ be the set of functions $x : [0, t] \to \mathbb{R}^n$ that can be represented in the form

$$x(\tau) = x(0) + (I^\alpha \psi)(\tau) \quad \forall \tau \in [0, t] \tag{2.4}$$

for some function $\psi(\cdot) \in \mathrm{L}^\infty([0, t], \mathbb{R}^n)$. The set $\mathrm{AC}^\alpha([0, t], \mathbb{R}^n)$ is a subset of $\mathrm{C}([0, t], \mathbb{R}^n)$. Note that, in the case $\alpha = 1$, the set $\mathrm{AC}^1([0, t], \mathbb{R}^n)$ coincides with the set $\mathrm{Lip}([0, t], \mathbb{R}^n)$ of Lipschitz continuous functions from $[0, t]$ to $\mathbb{R}^n$.

Let $\alpha \in (0, 1]$ and $x(\cdot) \in \mathrm{AC}^\alpha([0, t], \mathbb{R}^n)$. It follows from (2.2) that, for every $\beta \in [0, 1 - \alpha]$, the inclusion $(I^\beta (x(\cdot) - x(0)))(\cdot) \in \mathrm{AC}^{\alpha+\beta}([0, t], \mathbb{R}^n)$ holds. In particular, we obtain $(I^{1-\alpha}(x(\cdot) - x(0)))(\cdot) \in \mathrm{Lip}([0, t], \mathbb{R}^n)$. Hence, the (left-sided) Caputo fractional derivative of $x(\cdot)$ of the order α, which is defined by

$$({}^C D^\alpha x)(\tau) \triangleq \frac{\mathrm{d}}{\mathrm{d}\tau} \big(I^{1-\alpha}(x(\cdot) - x(0))\big)(\tau), \tag{2.5}$$

exists for almost every (a.e.) $\tau \in [0, t]$, and, moreover, the equality $({}^C D^\alpha x)(\tau) = \psi(\tau)$ is valid for a.e. $\tau \in [0, t]$, where $\psi(\cdot) \in \mathrm{L}^\infty([0, t], \mathbb{R}^n)$ is the function from (2.4). If $\alpha = 1$, then the Caputo derivative $({}^C D^1 x)(\tau)$ is the usual first-order derivative $\dot{x}(\tau) \triangleq \frac{\mathrm{d}}{\mathrm{d}\tau} x(\tau)$.

Now, consider the set G_n of pairs $(t, w(\cdot))$ such that $t \in [0, T]$ and $w(\cdot) \in \mathrm{C}([0, t], \mathbb{R}^n)$. For $x(\cdot) \in \mathrm{C}([0, T], \mathbb{R}^n)$ and $t \in [0, T]$, let $x_t(\cdot) \in \mathrm{C}([0, t], \mathbb{R}^n)$ denote the restriction of the function $x(\cdot)$ to the interval $[0, t]$:

$$x_t(\tau) \triangleq x(\tau), \quad \tau \in [0, t]. \tag{2.6}$$

Then, we have $(t, x_t(\cdot)) \in G_n$. In accordance with Section 1 of [37] (see also, *e.g.*, [31]), the set G_n is endowed with the metric

$$\operatorname{dist}\big((t, w(\cdot)), (t', w'(\cdot))\big) \triangleq \max\big\{\operatorname{dist}^*\big((t, w(\cdot)), (t', w'(\cdot))\big), \operatorname{dist}^*\big((t', w'(\cdot)), (t, w(\cdot))\big)\big\}, \tag{2.7}$$

where $(t, w(\cdot))$, $(t', w'(\cdot)) \in G_n$ and

$$\operatorname{dist}^*\big((t, w(\cdot)), (t', w'(\cdot))\big) \triangleq \max_{\tau \in [0,t]} \min_{\tau' \in [0,t']} \sqrt{|\tau - \tau'|^2 + \|w(\tau) - w'(\tau')\|^2}.$$

Note that the value $\operatorname{dist}((t, w(\cdot)), (t', w'(\cdot)))$ is the Hausdorff distance between the graphics of the functions $w(\cdot)$ and $w'(\cdot)$ as compact subsets of $\mathbb{R}^{n+1}$.

Let us describe some properties of this metric. By Proposition 8.2 of [12], for any $(t, w(\cdot))$, $(t', w'(\cdot)) \in G_n$ such that $t' \leq t$, the inequalities

$$\operatorname{dist} \leq t - t' + \varkappa(t - t') + \max_{\tau \in [0,t']} \|w(\tau) - w'(\tau)\|, \quad t - t' \leq \operatorname{dist}, \quad \max_{\tau \in [0,t']} \|w(\tau) - w'(\tau)\| \leq \operatorname{dist} + \varkappa(\operatorname{dist}) \tag{2.8}$$

are valid, where $\operatorname{dist} \triangleq \operatorname{dist}((t, w(\cdot)), (t', w'(\cdot)))$ and $\varkappa$ is the modulus of continuity of $w(\cdot)$ on $[0, t]$ given by

$$\varkappa(\delta) \triangleq \max\big\{\|w(\tau) - w(\tau')\| : \tau, \tau' \in [0, t], |\tau - \tau'| \leq \delta\big\}, \quad \delta \geq 0.$$

In particular, if sequences $\{x^{[k]}(\cdot)\}_{k\in\mathbb{N}} \subset \mathrm{C}([0, T], \mathbb{R}^n)$ and $\{t_k\}_{k\in\mathbb{N}} \subset [0, T]$ converge to $x^{[0]}(\cdot) \in \mathrm{C}([0, T], \mathbb{R}^n)$ and $t_0 \in [0, T]$, respectively, then $(t_k, x^{[k]}_{t_k}(\cdot)) \to (t_0, x^{[0]}_{t_0}(\cdot))$ as $k \to \infty$ with respect to the metric dist.

Moreover, note also that if a sequence $\{(t_k, w_k(\cdot))\}_{k\in\mathbb{N}} \subset G_n$ is convergent, then the functions $w_k(\cdot)$, $k \in \mathbb{N}$, are uniformly bounded and equicontinuous (see, *e.g.*, [15], Assert. 6). Namely, there exists $R > 0$ such that $\|w_k(\cdot)\|_{[0,t_k]} \leq R$ for any $k \in \mathbb{N}$ and the function $\varkappa_*(\delta) \triangleq \sup\{\varkappa_k(\delta) : k \in \mathbb{N}\}$, $\delta \geq 0$, satisfies the relation $\varkappa_*(\delta) \to 0$ as $\delta \to 0^+$, where $\varkappa_k$ is the modulus of continuity of $w_k(\cdot)$ on $[0, t_k]$ and the notation $\delta \to 0^+$ means that δ approaches 0 from the right.

Finally, for every $\alpha \in (0, 1]$, we introduce the following two subsets of G_n:

$$G_n^\alpha \triangleq \big\{(t, w(\cdot)) \in G_n : w(\cdot) \in \mathrm{AC}^\alpha([0, t], \mathbb{R}^n)\big\}, \quad G_n^{\alpha\circ} \triangleq \big\{(t, w(\cdot)) \in G_n^\alpha : t < T\big\}. \tag{2.9}$$

Remark 2.1. Instead of the metric dist from (2.7), the set G_n can be endowed with the metric (see, *e.g.*, [3])

$$\operatorname{dist}_0\big((t, w(\cdot)), (t', w'(\cdot))\big) \triangleq |t - t'| + \max_{\tau \in [0,T]} \|w(\tau \wedge t) - w'(\tau \wedge t')\|, \quad (t, w(\cdot)), (t', w'(\cdot)) \in G_n,$$

where $a \wedge b \triangleq \min\{a, b\}$ for all a, $b \in \mathbb{R}$. It can be shown that the metrics dist and dist_0 are not strongly equivalent, but they induce the same topology on G_n (see, *e.g.*, [47], Chap. 2). In particular, we obtain that any functional $\varphi : G_n \to \mathbb{R}$ is continuous with respect to the metric dist if and only if it is continuous with respect to the metric dist_0. The reader is referred to Section 5.1 of [17] for details.

3. Differential inclusions with fractional derivatives

This section deals with ordinary and functional differential inclusions with the Caputo fractional derivatives of an order $\alpha \in (0, 1)$ and provides auxiliary results concerning properties of the sets of solutions of such differential inclusions. The presented notions and statements are useful in order to give a definition of a minimax solution in Section 5.2 below and constitute a basis for the proofs of the main results of the paper.

3.1. Ordinary differential inclusions with fractional derivatives

Suppose that a set-valued function $[0, T] \times \mathbb{R}^n \ni (t, x) \mapsto F(t, x) \subset \mathbb{R}^n \times \mathbb{R}$ satisfies the following conditions:

$(F.1)$ For every $t \in [0, T]$ and $x \in \mathbb{R}^n$, the set $F(t, x)$ is nonempty, convex, and compact in $\mathbb{R}^n \times \mathbb{R}$.

$(F.2)$ The set-valued function F is upper semicontinuous (in the Hausdorff sense). It means that, for every $(t, x) \in [0, T] \times \mathbb{R}^n$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $(t', x') \in [0, T] \times \mathbb{R}^n$, the inequality

$|t-t'|^2 + \|x-x'\|^2 \leq \delta^2$ implies the inclusion $F(t',x') \subset [F(t,x)]^\varepsilon$. Here and below, for $\varepsilon > 0$ and $F \subset \mathbb{R}^n \times \mathbb{R}$, the symbol $[F]^\varepsilon$ stands for the ε-neighbourhood of F, given by

$$[F]^\varepsilon \triangleq \Big\{(f,h) \in \mathbb{R}^n \times \mathbb{R} : \inf_{(f',h') \in F} \big(\|f-f'\|^2 + |h-h'|^2\big)^{1/2} \leq \varepsilon\Big\}.$$

$(F.3)$ There exists $c_F > 0$ such that, for any $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$\sup\big\{\|f\| : (f,h) \in F(t,x)\big\} \leq c_F(1 + \|x\|).$$

Given $(t_0, w_0(\cdot)) \in G_n^\alpha$ and $z_0 \in \mathbb{R}$, consider the Cauchy problem for the differential inclusion

$$\big(({}^C D^\alpha x)(t), \dot{z}(t)\big) \in F(t, x(t)), \tag{3.1}$$

where $(x(t), z(t)) \in \mathbb{R}^n \times \mathbb{R}$ and $t \in [t_0, T]$, under the initial condition

$$x(t) = w_0(t), \quad z(t) = z_0 \quad \forall t \in [0, t_0]. \tag{3.2}$$

Let $XZ^\alpha(t_0, w_0(\cdot), z_0)$ be the set of pairs of functions $(x(\cdot), z(\cdot)) \in \mathrm{AC}^\alpha([0,T], \mathbb{R}^n) \times \mathrm{Lip}([0,T], \mathbb{R})$ satisfying (3.2). Note that it is convenient to identify any such pair $(x(\cdot), z(\cdot))$ with the corresponding function from $[0,T]$ to $\mathbb{R}^n \times \mathbb{R}$. In this sense, taking into account that $\mathrm{AC}^\alpha([0,T],\mathbb{R}^n) \subset \mathrm{C}([0,T],\mathbb{R}^n)$ and $\mathrm{Lip}([0,T],\mathbb{R}) \subset \mathrm{C}([0,T],\mathbb{R})$, the set $XZ^\alpha(t_0, w_0(\cdot), z_0)$ can be considered as a subset of $\mathrm{C}([0,T], \mathbb{R}^n \times \mathbb{R})$.

By a solution of problem (3.1), (3.2), we mean a pair of functions $(x(\cdot), z(\cdot)) \in XZ^\alpha(t_0, w_0(\cdot), z_0)$ such that differential inclusion (3.1) is fulfilled for a.e. $t \in [t_0, T]$. Let $XZ_0^\alpha(t_0, w_0(\cdot), z_0)$ denote the set of such solutions.

Proposition 3.1. *For any $(t_0, w_0(\cdot)) \in G_n^\alpha$ and $z_0 \in \mathbb{R}$, the set $XZ_0^\alpha(t_0, w_0(\cdot), z_0)$ is nonempty and compact in* $\mathrm{C}([0,T], \mathbb{R}^n \times \mathbb{R})$.

Proposition 3.2. *Let $(t_k, w_k(\cdot)) \in G_n^\alpha$, $z_k \in \mathbb{R}$, and $(x^{[k]}(\cdot), z^{[k]}(\cdot)) \in XZ_0^\alpha(t_k, w_k(\cdot), z_k)$ for every $k \in \mathbb{N}$, and let $(t_k, w_k(\cdot)) \to (t_0, w_0(\cdot)) \in G_n^\alpha$ and $z_k \to z_0 \in \mathbb{R}$ as $k \to \infty$. Then, the sequence $\{(x^{[k]}(\cdot), z^{[k]}(\cdot))\}_{k \in \mathbb{N}}$ contains a subsequence that converges to a solution $(x^{[0]}(\cdot), z^{[0]}(\cdot)) \in XZ_0^\alpha(t_0, w_0(\cdot), z_0)$.*

Proposition 3.3. *Let $(t_0, w_0(\cdot)) \in G_n^\alpha$, $z_0 \in \mathbb{R}$, and $(x(\cdot), z(\cdot)) \in XZ_0^\alpha(t_0, w_0(\cdot), z_0)$. Then, for every $t' \in [t_0, T]$ and $(x'(\cdot), z'(\cdot)) \in XZ_0^\alpha(t', x_{t'}(\cdot), z(t'))$, the inclusion $(x'(\cdot), z''(\cdot)) \in XZ_0^\alpha(t_0, w_0(\cdot), z_0)$ holds, where the function $z''(\cdot)$ is defined by $z''(t) \triangleq z(t)$ for $t \in [0, t']$ and $z''(t) \triangleq z'(t)$ for $t \in (t', T]$.*

In the case when there is no additional variable $z(t)$, similar statements are proved in [15] by adapting the proofs of the corresponding results for ordinary and functional differential inclusions with first-order derivatives (see, *e.g.*, [9] and also [28, 37]). The proofs of Propositions 3.1, 3.2, and 3.3 can be carried out by the same scheme with only minor technical changes, and, therefore, they are omitted.

Finally, note that, since the right-hand side of differential inclusion (3.1) does not depend on $z(t)$, then, for any $(t_0, w_0(\cdot)) \in G_n^\alpha$, $z_0 \in \mathbb{R}$, $(x(\cdot), z(\cdot)) \in XZ_0^\alpha(t_0, w_0(\cdot), z_0)$, and $z_0' \in \mathbb{R}$, we have $(x(\cdot), z'(\cdot)) \in XZ_0^\alpha(t_0, w_0(\cdot), z_0')$ for the function $z'(t) \triangleq z_0' + z(t) - z_0$, $t \in [0,T]$.

3.2. Functional differential inclusions with fractional derivatives

Let us also give an analogue of Proposition 3.1 for the case when the right-hand side of differential inclusion (3.1) depends not only on a single value $x(t)$ of an unknown solution, but on values $x(\tau)$ for all $\tau \in [0,t]$ or, in other words, on the function $x_t(\cdot)$ given by (2.6).

Let a set-valued functional $G_n^\alpha \ni (t, w(\cdot)) \mapsto \mathcal{F}(t, w(\cdot)) \subset \mathbb{R}^n \times \mathbb{R}$ be such that:

$(\mathcal{F}.1)$ For every $(t, w(\cdot)) \in G_n^\alpha$, the set $\mathcal{F}(t, w(\cdot))$ is nonempty, convex, and compact in $\mathbb{R}^n \times \mathbb{R}$.

($\mathcal{F}.2$) The set-valued functional $\mathcal{F}$ is upper semicontinuous. Namely, for every $(t, w(\cdot)) \in G_n^\alpha$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, for any $(t', w'(\cdot)) \in G_n^\alpha$, the inequality $\operatorname{dist}((t, w(\cdot)), (t', w'(\cdot))) \leq \delta$ implies the inclusion $\mathcal{F}(t', w'(\cdot)) \subset [\mathcal{F}(t, w(\cdot))]^\varepsilon$.

($\mathcal{F}.3$) There exists $c_{\mathcal{F}} > 0$ such that, for any $(t, w(\cdot)) \in G_n^\alpha$,

$$\sup\big\{\|f\| : (f, h) \in \mathcal{F}(t, w(\cdot))\big\} \leq c_F(1 + \|w(\cdot)\|_{[0,t]}).$$

Given $(t_0, w_0(\cdot)) \in G_n^\alpha$ and $z_0 \in \mathbb{R}$, consider a Cauchy problem for the functional differential inclusion

$$\big(({}^C D^\alpha x)(t), \dot{z}(t)\big) \in \mathcal{F}(t, x_t(\cdot)), \tag{3.3}$$

where $(x(t), z(t)) \in \mathbb{R}^n \times \mathbb{R}$ and $t \in [t_0, T]$, under initial condition (3.2). A pair $(x(\cdot), z(\cdot)) \in XZ^\alpha(t_0, w_0(\cdot), z_0)$ is a solution of this problem if functional differential inclusion (3.3) holds for a.e. $t \in [t_0, T]$. Let $\mathcal{XZ}_0^\alpha(t_0, w_0(\cdot), z_0)$ be the set of such solutions.

Proposition 3.4. *For any $(t_0, w_0(\cdot)) \in G_n^\alpha$ and $z_0 \in \mathbb{R}$, the set $\mathcal{XZ}_0^\alpha(t_0, w_0(\cdot), z_0)$ is nonempty and compact in* $\mathrm{C}([0, T], \mathbb{R}^n \times \mathbb{R})$.

This proposition can be proved by the scheme from Theorem 1 of [15].

4. Fractional coinvariant derivatives

Let us recall the notion of coinvariant (*ci*-) differentiability of an order $\alpha \in (0, 1]$ of a functional $\varphi : G_n^\alpha \to \mathbb{R}$ introduced in [12].

For $(t_0, w_0(\cdot)) \in G_n^\alpha$, consider the set of admissible extensions $x(\cdot)$ of $w_0(\cdot)$ defined by

$$X^\alpha(t_0, w_0(\cdot)) \triangleq \big\{x(\cdot) \in \mathrm{AC}^\alpha([0, T], \mathbb{R}^n) : x(t) = w_0(t) \ \forall t \in [0, t_0]\big\}. \tag{4.1}$$

A functional $\varphi : G_n^\alpha \to \mathbb{R}$ is called *ci*-differentiable of the order α at a point $(t_0, w_0(\cdot)) \in G_n^{\alpha\circ}$ if there exist $\partial_t^\alpha \varphi(t_0, w_0(\cdot)) \in \mathbb{R}$ and $\nabla^\alpha \varphi(t_0, w_0(\cdot)) \in \mathbb{R}^n$ such that, for every extension $x(\cdot) \in X^\alpha(t_0, w_0(\cdot))$, the relation

$$\varphi(t, x_t(\cdot)) - \varphi(t_0, w_0(\cdot)) = \partial_t^\alpha \varphi(t_0, w_0(\cdot))(t - t_0) + \int_{t_0}^t \langle \nabla^\alpha \varphi(t_0, w_0(\cdot)), ({}^C D^\alpha x)(\tau)\rangle \, \mathrm{d}\tau + o(t - t_0) \tag{4.2}$$

holds for all $t \in (t_0, T)$. Here, $x_t(\cdot)$ is determined by $x(\cdot)$ and t according to (2.6), the function o may depend on t and $x(\cdot)$, and $o(\delta)/\delta \to 0$ as $\delta \to 0^+$. In this case, the quantities $\partial_t^\alpha \varphi(t_0, w_0(\cdot))$ and $\nabla^\alpha \varphi(t_0, w_0(\cdot))$ are called the *ci*-derivatives of the order α of φ at $(t_0, w_0(\cdot))$.

A functional $\varphi : G_n^\alpha \to \mathbb{R}$ is said to be *ci*-smooth of the order α if it is continuous, *ci*-differentiable of the order α at every point $(t, w(\cdot)) \in G_n^{\alpha\circ}$, and the functionals $\partial_t^\alpha \varphi : G_n^{\alpha\circ} \to \mathbb{R}$ and $\nabla^\alpha \varphi : G_n^{\alpha\circ} \to \mathbb{R}^n$ are continuous. Recall that the set $G_n^{\alpha\circ} \subset G_n$ is endowed with the metric dist from (2.7) (see also Rem. 2.1).

Note that, if $\alpha = 1$, then the notion of *ci*-differentiability of the order α (*i.e.*, of the first order) agrees with the notion of *ci*-differentiability from, *e.g.*, [23, 37].

The following proposition expresses one of the key properties of *ci*-smooth of the order α functionals (see, *e.g.*, Lem. 2.1 of [37], Lem. 9.2 of [12], and also Prop. 1 of [17]).

Proposition 4.1. *Let a functional $\varphi : G_n^\alpha \to \mathbb{R}$ be ci-smooth of the order α. Then, for every function $x(\cdot) \in \mathrm{AC}^\alpha([0, T], \mathbb{R}^n)$, the function $\omega(t) \triangleq \varphi(t, x_t(\cdot))$, $t \in [0, T]$, is continuous on $[0, T]$ and is Lipschitz continuous on $[0, \vartheta]$ for any fixed $\vartheta \in [0, T)$. Moreover, the equality below holds:*

$$\dot{\omega}(t) = \partial_t^\alpha \varphi(t, x_t(\cdot)) + \langle \nabla^\alpha \varphi(t, x_t(\cdot)), ({}^C D^\alpha x)(t)\rangle \ \textit{for a.e. } t \in [0, T]. \tag{4.3}$$

Remark 4.2. At first glance, the notion of *ci*-differentiability of the order α (see (4.2)) as well as formula (4.3) seem rather unusual. Nevertheless (see [12]), we need formula (4.3) with precisely the Caputo fractional derivative $({}^C D^\alpha x)(t)$ and not with the first-order derivative $\dot{x}(t)$ in order to derive the Hamilton–Jacobi equation associated with an optimal control problem for a dynamical system described by a differential equation with the Caputo fractional derivative of the order α. Moreover, to the best of our knowledge, this approach to the development of the theory of Hamilton–Jacobi equations corresponding to such fractional-order dynamical systems is the only one that has been proposed so far.

In order to illustrate the notion of *ci*-differentiability of the order α in the fractional setting $\alpha \in (0,1)$, let us present three examples.

Example 4.3. Take a functional $\psi : G_n^1 \to \mathbb{R}$ and, for every $(t, w(\cdot)) \in G_n^\alpha$, denote

$$h(\tau \mid t, w(\cdot)) \triangleq \big(I^{1-\alpha}(w(\cdot) - w(0))\big)(\tau), \quad \tau \in [0,t]. \tag{4.4}$$

Consider the functional $\varphi : G_n^\alpha \to \mathbb{R}$ given by

$$\varphi(t, w(\cdot)) \triangleq \psi\big(t, h(\cdot \mid t, w(\cdot))\big), \quad (t, w(\cdot)) \in G_n^\alpha. \tag{4.5}$$

Then, it can be verified directly that, in view of (2.5), the functional φ is *ci*-differentiable of the order α at a point $(t, w(\cdot)) \in G_n^{\alpha\circ}$ provided that the functional ψ is *ci*-differentiable of the first order at the point $(t, h(\cdot \mid t, w(\cdot)))$. Moreover, in this case, we have $\partial_t^\alpha \varphi(t, w(\cdot)) = \partial_t^1 \psi(t, h(\cdot \mid t, w(\cdot)))$ and $\nabla^\alpha \varphi(t, w(\cdot)) = \nabla^1 \psi(t, h(\cdot \mid t, w(\cdot)))$. Hence, since the mapping $G_n^\alpha \ni (t, w(\cdot)) \mapsto (t, h(\cdot \mid t, w(\cdot))) \in G_n$ is continuous, we obtain that *ci*-smoothness of the first order of the functional ψ implies *ci*-smoothness of the order α of the functional φ. In other words, on the basis of every *ci*-smooth of the first order functional ψ, we can define the *ci*-smooth of the order α functional φ according to (4.5). Observe that the class of *ci*-smooth of the first order functionals is rather wide and well-studied. Some examples of such functionals and formulas for calculating the corresponding *ci*-derivatives can be found in, *e.g.*, Section 2 of [23] and Section 2 of [37].

Remark 4.4. In general, every functional $\varphi : G_n^\alpha \to \mathbb{R}$ can be represented in the form (see, *e.g.*, [14], Sect. 4)

$$\varphi(t, w(\cdot)) = \psi\big(t, w(0), h(\cdot \mid t, w(\cdot))\big) \quad \forall (t, w(\cdot)) \in G_n^\alpha.$$

Here, the function $h(\cdot \mid t, w(\cdot))$ is defined by (4.4), and

$$\psi(t, h_0, h(\cdot)) \triangleq \varphi\big(t, l(\cdot \mid t, h_0, h(\cdot))\big), \quad (t, h(\cdot)) \in G_n^1, \quad h_0 \in \mathbb{R}^n,$$

where we denote

$$l(\tau \mid t, h_0, h(\cdot)) \triangleq h_0 + \frac{1}{\Gamma(\alpha)} \int_0^\tau \frac{\dot{h}(\xi)}{(\tau - \xi)^{1-\alpha}} \,\mathrm{d}\xi, \quad \tau \in [0,t].$$

As in Example 4.3, there is a connection between *ci*-differentiability of the order α of the functional φ and *ci*-differentiability of the first order of the functional ψ (with respect to $(t, h(\cdot))$). However, let us observe that the value $l(\tau \mid t, h_0, h(\cdot))$ depends explicitly on the values of the derivatives $\dot{h}(\xi)$ for a.e. $\xi \in [0, \tau]$, which leads to difficulties with continuity properties of the mapping $G_n^1 \ni (t, h(\cdot)) \mapsto \psi(t, h_0, h(\cdot)) \in \mathbb{R}$, where $h_0 \in \mathbb{R}^n$. In particular, this prevents us from dealing directly with the functional ψ instead of the original functional φ.

Example 4.5. Let $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function for which there exists $c_f > 0$ such that, for any $t \in [0,T]$ and $x \in \mathbb{R}^n$,

$$\|f(t,x)\| \leq c_f(1 + \|x\|).$$

For every point $(t_0, w_0(\cdot)) \in G_n^\alpha$, consider the Cauchy problem for the differential equation

$$({}^C D^\alpha x)(t) = f(t, x(t)), \tag{4.6}$$

where $x(t) \in \mathbb{R}^n$ and $t \in [t_0, T]$, under the initial condition

$$x(t) = w_0(t) \quad \forall t \in [0, t_0]. \tag{4.7}$$

A solution of problem (4.6), (4.7) is defined as a function $x(\cdot) \in X^\alpha(t_0, w_0(\cdot))$ satisfying differential equation (4.6) for a.e. $t \in [t_0, T]$. According to, *e.g.*, Proposition 2 of [14], such a solution exists and is unique, and we denote it by $x(\cdot \mid t_0, w_0(\cdot))$. Now, given a continuously differentiable function $\sigma : \mathbb{R}^n \to \mathbb{R}$, consider the functional

$$\varphi(t, w(\cdot)) \triangleq \sigma\big(x(T \mid t, w(\cdot))\big), \quad (t, w(\cdot)) \in G_n^\alpha.$$

Then, similarly to Theorem 3.1 of [16], it can be proved that the functional φ is *ci*-differentiable of the order α at every point $(t, w(\cdot)) \in G_n^{\alpha\circ}$, and, moreover, formulas for calculating the *ci*-derivatives of the order α of φ can be obtained. In particular, this example illustrates the fact that the notion of *ci*-differentiability of the order α can be a useful tool when dealing with functionals defined in terms of solutions of differential equations with the Caputo fractional derivatives of the order α.

The next example shows that even simplest functionals can be not *ci*-differentiable of the order α.

Example 4.6. Take $\ell \in \mathbb{R}^n \setminus \{0\}$ and consider the functional

$$\varphi(t, w(\cdot)) \triangleq \langle \ell, w(t) \rangle, \quad (t, w(\cdot)) \in G_n^1.$$

This functional φ is *ci*-smooth of the first order, and its *ci*-derivatives of the first order are given by

$$\partial_t^1 \varphi(t, w(\cdot)) = 0, \quad \nabla^1 \varphi(t, w(\cdot)) = \ell \quad \forall (t, w(\cdot)) \in G_n^{1\circ}.$$

Nevertheless, let us prove that the functional φ (more precisely, its restriction to G_n^α) is not *ci*-differentiable of the order α at every point $(t, w(\cdot)) \in G_n^{\alpha\circ}$. Arguing by contradiction, assume that φ is *ci*-differentiable of the order α at some point $(t_0, w_0(\cdot)) \in G_n^{\alpha\circ}$. For a given $f \in \mathbb{R}^n$, let us consider the function $x^{[f]}(\cdot) \in X^\alpha(t_0, w_0(\cdot))$ such that $({}^C D^\alpha x^{[f]})(t) = \Gamma(\alpha + 1) f$ for a.e. $t \in [t_0, T]$. Then, we have

$$x^{[f]}(t) = w(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_0} \frac{({}^C D^\alpha w)(\tau)}{(t - \tau)^{1-\alpha}} \, \mathrm{d}\tau + (t - t_0)^\alpha f \quad \forall t \in [t_0, T].$$

Hence, taking f, $g \in \mathbb{R}^n$ that satisfy the condition $\langle \ell, f - g \rangle \neq 0$, we get

$$\varphi(t, x_t^{[f]}(\cdot)) - \varphi(t, x_t^{[g]}(\cdot)) = (t - t_0)^\alpha \langle \ell, f - g \rangle \quad \forall t \in [t_0, T].$$

At the same time, due to the assumption made, there exists $\nabla^\alpha \varphi(t_0, w_0(\cdot)) \in \mathbb{R}^n$ such that

$$\varphi(t, x_t^{[f]}(\cdot)) - \varphi(t, x_t^{[g]}(\cdot)) = (t - t_0) \langle \nabla^\alpha \varphi(t_0, w_0(\cdot)), f - g \rangle + o(t - t_0) \quad \forall t \in (t_0, T).$$

Thus, we obtain

$$(t - t_0)^\alpha \langle \ell, f - g \rangle = (t - t_0) \langle \nabla^\alpha \varphi(t_0, w_0(\cdot)), f - g \rangle + o(t - t_0) \quad \forall t \in (t_0, T).$$

Dividing this equality by $(t-t_0)^\alpha$ and, after that, passing to the limit as $t \to t_0^+$, we derive $\langle \ell, f-g\rangle = 0$, which contradicts the choice of f and g.

5. Hamilton–Jacobi equation with fractional coinvariant derivatives

In this section, we consider a Cauchy problem for a Hamilton–Jacobi equation with fractional *ci*-derivatives of an order $\alpha \in (0,1)$ and propose a definition of a minimax solution of this problem.

5.1. Hamilton–Jacobi equation

Consider the Cauchy problem for the Hamilton–Jacobi equation with *ci*-derivatives of the order α

$$\partial_t^\alpha \varphi(t, w(\cdot)) + H\big(t, w(t), \nabla^\alpha \varphi(t, w(\cdot))\big) = 0 \quad \forall (t, w(\cdot)) \in G_n^{\alpha\circ} \tag{5.1}$$

and the boundary condition

$$\varphi(T, w(\cdot)) = \sigma(w(\cdot)) \quad \forall w(\cdot) \in \mathrm{AC}^\alpha([0,T], \mathbb{R}^n). \tag{5.2}$$

In this problem, $\varphi : G_n^\alpha \to \mathbb{R}$ is an unknown functional, and the given mappings $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\sigma : \mathrm{AC}^\alpha([0,T], \mathbb{R}^n) \to \mathbb{R}$ are assumed to satisfy the following conditions:

$(H.1)$ The function H is continuous.

$(H.2)$ There exists $c_H > 0$ such that, for any $t \in [0,T]$ and x, s, $s' \in \mathbb{R}^n$,

$$|H(t,x,s) - H(t,x,s')| \le c_H(1 + \|x\|)\|s - s'\|.$$

$(H.3)$ For every $R \ge 0$, there exists $\lambda_H > 0$ such that, for any $t \in [0,T]$ and x, x', $s \in \mathbb{R}^n$, if $\|x\| \le R$ and $\|x'\| \le R$, then

$$|H(t,x,s) - H(t,x',s)| \le \lambda_H(1 + \|s\|)\|x - x'\|.$$

(σ) The functional σ is continuous.

Cauchy problem (5.1), (5.2) arises [12] when studying infinitesimal properties of the value functional in Bolza-type optimal control problems for dynamical systems described by fractional differential equations with the Caputo derivatives of the order α. In this connection, assumptions $(H.1)$–$(H.3)$ and (σ) seem quite natural since they are fulfilled in a sufficiently wide range of such problems. If the value functional is *ci*-smooth of the order α, then, according to Theorem 10.1 of [12], it satisfies Hamilton–Jacobi equation (5.1) and boundary condition (5.2), and, therefore, it can be considered as a solution of problem (5.1), (5.2) in the classical sense. In particular, this allows us to efficiently construct optimal control strategies ([12], Cor. 11.4). However, by analogy with the case of optimal control problems for dynamical systems described by ordinary differential equations (*i.e.*, when $\alpha = 1$), the value functional usually does not possess such smoothness properties, which leads to the need to introduce and study generalized solutions of problem (5.1), (5.2). Let us present an example.

Example 5.1. Suppose that $n = 1$ and, following Section 12 of [12], consider the optimal control problem for the dynamical system described by the differential equation with the Caputo fractional derivative of the order α

$$({}^C D^\alpha x)(t) = \Gamma(\alpha + 1) u(t),$$

where $x(t) \in \mathbb{R}$, $u(t) \in [-1, 1]$, and $t \in [0, T]$, and for the cost functional $J = |x(T)|$ to be minimized. It can be verified directly (or, *e.g.*, based on [13], Thm. 1), that the value functional $\rho : G_1^\alpha \to \mathbb{R}$ in this problem is given by

$$\rho(t, w(\cdot)) = \max\{0, |\rho_*(t, w(\cdot))| - (T - t)^\alpha\} \quad \forall (t, w(\cdot)) \in G_1^\alpha,$$

where the auxiliary functional $\rho_* : G_1^\alpha \to \mathbb{R}$ is defined by

$$\rho_*(t, w(\cdot)) \triangleq w(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{({}^C D^\alpha w)(\tau)}{(T - \tau)^{1-\alpha}} \, \mathrm{d}\tau, \quad (t, w(\cdot)) \in G_1^\alpha.$$

Note that (see, *e.g.*, [12], Sect. 12) the functional ρ_* is *ci*-smooth of the order α, and

$$\partial_t^\alpha \rho_*(t, w(\cdot)) = 0, \quad \nabla^\alpha \rho_*(t, w(\cdot)) = \frac{1}{\Gamma(\alpha)(T - t)^{1-\alpha}} \quad \forall (t, w(\cdot)) \in G_1^{\alpha\circ}.$$

However, let us choose a point $(t_0, w_0(\cdot)) \in G_1^{\alpha\circ}$ such that $|\rho_*(t_0, w_0(\cdot))| = (T - t_0)^\alpha$ and prove that the value functional ρ is not *ci*-differentiable of the order α at this point. Suppose that $\rho_*(t_0, w_0(\cdot)) = (T - t_0)^\alpha$ for definiteness. For every $f \in \mathbb{R}$, considering the function $x^{[f]}(\cdot)$ as in Example 4.6, we derive

$$\rho_*(t, x_t^{[f]}(\cdot)) = \rho_*(t_0, w_0(\cdot)) + \big((T - t_0)^\alpha - (T - t)^\alpha\big) f = (T - t_0)^\alpha (1 + f) - (T - t)^\alpha f \quad \forall t \in [t_0, T].$$

Then, for a fixed $f > -1$ and all $t \in [t_0, T]$, we obtain $\rho_*(t, x_t^{[f]}(\cdot)) \geq (T - t)^\alpha$, and, hence,

$$\rho(t, x_t^{[f]}(\cdot)) = \big((T - t_0)^\alpha - (T - t)^\alpha\big)(1 + f).$$

On the other hand, take $g \leq -1$. Put $t_* \triangleq T - (T - t_0)(1/g + 1)^{1/\alpha}$ and observe that $t_* \in (t_0, T]$. Then, for any $t \in [t_0, t_*]$, we have $0 \leq \rho_*(t, x_t^{[g]}(\cdot)) \leq (T - t)^\alpha$, and, consequently, $\rho(t, x_t^{[g]}(\cdot)) = 0$. Thus, if we assume that ρ is *ci*-differentiable of the order α at the point $(t_0, w_0(\cdot))$, we get

$$\big((T - t_0)^\alpha - (T - t)^\alpha\big)(1 + f) = (t - t_0) \nabla^\alpha \rho(t_0, w_0(\cdot))(f - g) + o(t - t_0) \quad \forall t \in (t_0, t_*). \tag{5.3}$$

Dividing (5.3) by $t - t_0$ and passing to the limit as $t \to t_0^+$, we derive

$$\frac{\alpha(1 + f)}{(T - t_0)^{1-\alpha}} = \nabla^\alpha \rho(t_0, w_0(\cdot))(f - g).$$

Since this equality must hold for all $f > -1$ and $g \leq -1$, we come to a contradiction and complete the proof.

Below, we give a definition of a minimax solution of problem (5.1), (5.2), which is a modification of the corresponding definitions in the case of Hamilton–Jacobi equations with partial derivatives (see, *e.g.*, [48], Sect. 6.2) and with first-order *ci*-derivatives (see, *e.g.*, Sect. 6 of [37], [31], and also [3]). We prove that the minimax solution exists, is unique, and is consistent with a classical solution of problem (5.1), (5.2). In particular, we establish a comparison principle.

5.2. Minimax solution

Consider the set-valued function $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n \ni (t, x, s) \mapsto E(t, x, s) \subset \mathbb{R}^n \times \mathbb{R}$, where

$$E(t, x, s) \triangleq \big\{(f, h) \in \mathbb{R}^n \times \mathbb{R} : \|f\| \leq c_H(1 + \|x\|), \, h = \langle s, f \rangle - H(t, x, s)\big\}, \quad t \in [0, T], \quad x, s \in \mathbb{R}^n. \tag{5.4}$$

Note that (see, *e.g.*, Sect. 6.2 of [48]) the set $E(t,x,s)$ is nonempty, convex, and compact in $\mathbb{R}^n \times \mathbb{R}$ for every $t \in [0,T]$ and $x,\ s \in \mathbb{R}^n$, the set-valued function E is continuous (in the Hausdorff sense) due to assumption $(H.1)$, and the inequality below holds:

$$\sup\big\{\|f\| : \ (f,h) \in E(t,x,s)\big\} \le c_H(1+\|x\|) \quad \forall t \in [0,T] \quad \forall x,s \in \mathbb{R}^n.$$

In addition, it follows from $(H.2)$ that $E(t,x,s) \cap E(t,x,s') \neq \emptyset$ for any $t \in [0,T]$ and $x,\ s,\ s' \in \mathbb{R}^n$.

Given $(t_0, w_0(\cdot)) \in G_n^\alpha$, $z_0 \in \mathbb{R}$, and $s \in \mathbb{R}^n$, consider the Cauchy problem for the differential inclusion

$$\big(({}^C D^\alpha x)(t), \dot z(t)\big) \in E(t, x(t), s), \tag{5.5}$$

where $(x(t), z(t)) \in \mathbb{R}^n \times \mathbb{R}$ and $t \in [t_0, T]$, under the initial condition

$$x(t) = w_0(t), \quad z(t) = z_0 \quad \forall t \in [0, t_0]. \tag{5.6}$$

Here, s is treated as a constant parameter. Let $CH(t_0, w_0(\cdot), z_0, s)$ be the set of solutions $(x(\cdot), z(\cdot))$ of problem (5.5), (5.6). According to Proposition 3.1 and the described above properties of the function E, the set $CH(t_0, w_0(\cdot), z_0, s)$ is nonempty and compact in $\mathrm{C}([0,T], \mathbb{R}^n \times \mathbb{R})$. Following the conventional terminology, differential inclusion (5.5) is called a characteristic differential inclusion, and any element of the set $CH(t_0, w_0(\cdot), z_0, s)$ is called a (generalized) characteristic of equation (5.1).

We say that a functional $\varphi : G_n^\alpha \to \mathbb{R}$ is an upper solution of problem (5.1), (5.2) if it is lower semicontinuous, satisfies the boundary condition

$$\varphi(T, w(\cdot)) \ge \sigma(w(\cdot)) \quad \forall w(\cdot) \in \mathrm{AC}^\alpha([0,T], \mathbb{R}^n), \tag{5.7}$$

and possesses the following property:

(φ_+) For every $(t_0, w_0(\cdot)) \in G_n^{\alpha\circ}$, $t \in (t_0, T]$, $s \in \mathbb{R}^n$, and $\varepsilon > 0$, there exists $(x(\cdot), z(\cdot)) \in CH(t_0, w_0(\cdot), 0, s)$ such that $\varphi(t, x_t(\cdot)) - z(t) \le \varphi(t_0, w_0(\cdot)) + \varepsilon$.

Respectively, a lower solution of this problem is an upper semicontinuous functional $\varphi : G_n^\alpha \to \mathbb{R}$ such that

$$\varphi(T, w(\cdot)) \le \sigma(w(\cdot)) \quad \forall w(\cdot) \in \mathrm{AC}^\alpha([0,T], \mathbb{R}^n) \tag{5.8}$$

and the statement below holds:

(φ_-) For every $(t_0, w_0(\cdot)) \in G_n^{\alpha\circ}$, $t \in (t_0, T]$, $s \in \mathbb{R}^n$, and $\varepsilon > 0$, there exists $(x(\cdot), z(\cdot)) \in CH(t_0, w_0(\cdot), 0, s)$ such that $\varphi(t, x_t(\cdot)) - z(t) \ge \varphi(t_0, w_0(\cdot)) - \varepsilon$.

In (φ_+) and (φ_-), as usual, the function $x_t(\cdot)$ is the restriction of the function $x(\cdot)$ to the interval $[0,t]$ (see (2.6)).

A functional $\varphi : G_n^\alpha \to \mathbb{R}$ is called a minimax solution of problem (5.1), (5.2) if it is an upper solution as well as a lower solution of this problem.

Remark 5.2. Conditions (φ_+) and (φ_-) can be reformulated in terms of weak invariance of respectively the epigraph and hypograph of the functional φ with respect to characteristic differential inclusion (5.5) for every $s \in \mathbb{R}^n$ (see, *e.g.*, definitions $(U2)$ and $(L2)$ in Sect. 6.3 of [48]). Note also that, in the terminology of positional differential games theory, statements (φ_+) and (φ_-) express so-called u-stability and v-stability properties of the value function (see, *e.g.*, Sect. 4.2 of [26] and Sect. 8 of [24]).

6. Consistency

This section deals with issues of consistency of a minimax solution of problem (5.1), (5.2) with a solution of this problem in the classical sense.

By a classical solution of problem (5.1), (5.2), we mean a *ci*-smooth of the order α functional $\varphi : G_n^\alpha \to \mathbb{R}$ that satisfies Hamilton–Jacobi equation (5.1) and boundary condition (5.2).

The schemes of the proof of the statements below go back to the proofs of the corresponding results for Hamilton–Jacobi equations with partial derivatives (see, *e.g.*, Sect. 2.4 of [48]) and with first-order *ci*-derivatives (see, *e.g.*, Sects. 4 and 5 of [37], Prop. 5.1 of [31], and also Sect. B.1 of [3]).

Theorem 6.1. *A classical solution of problem* (5.1), (5.2) *is a minimax solution of this problem.*

Proof. Since a classical solution $\varphi : G_n^\alpha \to \mathbb{R}$ is continuous and satisfies (5.2), in order to prove that φ is a minimax solution, it suffices to verify that φ possesses properties (φ_+) and (φ_-). To this end, let us show that, for given $(t_0, w_0(\cdot)) \in G_n^{\alpha\circ}$ and $s \in \mathbb{R}^n$, there exists $(x^*(\cdot), z^*(\cdot)) \in CH(t_0, w_0(\cdot), 0, s)$ such that

$$\varphi(t, x_t^*(\cdot)) - z^*(t) = \varphi(t_0, w_0(\cdot)) \quad \forall t \in [t_0, T]. \tag{6.1}$$

Consider the set-valued functional $G_n^\alpha \ni (t, w(\cdot)) \mapsto \mathcal{E}^*(t, w(\cdot)) \subset \mathbb{R}^n \times \mathbb{R}$, where, for every $(t, w(\cdot)) \in G_n^\alpha$,

$$\mathcal{E}^*(t, w(\cdot)) \triangleq \begin{cases} E(t, w(t), s) \cap E\big(t, w(t), \nabla^\alpha \varphi(t, w(\cdot))\big), & \text{if } t < T, \\ E(t, w(t), s), & \text{if } t = T. \end{cases}$$

Due to the given in Section 5.2 properties of the set-valued function E and continuity of the functionals $\nabla^\alpha \varphi$ and $G_n \ni (t, w(\cdot)) \mapsto w(t) \in \mathbb{R}^n$, the functional $\mathcal{E}^*$ satisfies conditions $(\mathcal{F}.1)$–$(\mathcal{F}.3)$. Then, owing to Proposition 3.4, the Cauchy problem for the functional differential inclusion

$$\big(({}^C D^\alpha x)(t), \dot{z}(t)\big) \in \mathcal{E}^*(t, x_t(\cdot)),$$

where $(x(t), z(t)) \in \mathbb{R}^n \times \mathbb{R}$ and $t \in [t_0, T]$, under the initial condition $x(t) = w_0(t)$ and $z(t) = 0$ for all $t \in [0, t_0]$ admits a solution $(x^*(\cdot), z^*(\cdot))$. By construction, we have $(x^*(\cdot), z^*(\cdot)) \in CH(t_0, w_0(\cdot), 0, s)$ and, in accordance with (5.4),

$$\dot{z}^*(t) = \langle \nabla^\alpha \varphi(t, x_t^*(\cdot)), ({}^C D^\alpha x^*)(t) \rangle - H\big(t, x^*(t), \nabla^\alpha \varphi(t, x_t^*(\cdot))\big) \text{ for a.e. } t \in [t_0, T].$$

Hence, taking into account that φ satisfies equation (5.1), we get

$$\dot{z}^*(t) = \partial_t^\alpha \varphi(t, x_t^*(\cdot)) + \langle \nabla^\alpha \varphi(t, x_t^*(\cdot)), ({}^C D^\alpha x^*)(t) \rangle \text{ for a.e. } t \in [t_0, T].$$

On the other hand, since φ is *ci*-smooth of the order α, by Proposition 4.1, for the function $\omega(t) \triangleq \varphi(t, x_t^*(\cdot))$, $t \in [0, T]$, and a fixed $\vartheta \in [t_0, T)$, we obtain

$$\omega(\vartheta) = \omega(t_0) + \int_{t_0}^{\vartheta} \big(\partial_t^\alpha \varphi(t, x_t^*(\cdot)) + \langle \nabla^\alpha \varphi(t, x_t^*(\cdot)), ({}^C D^\alpha x^*)(t) \rangle\big) \, \mathrm{d}t.$$

Thus, recalling that $x_{t_0}^*(\cdot) = w_0(\cdot)$, $z^*(t_0) = 0$, and $z^*(\cdot) \in \mathrm{Lip}([0, T], \mathbb{R})$, we derive

$$\omega(\vartheta) = \omega(t_0) + \int_{t_0}^{\vartheta} \dot{z}^*(t) \, \mathrm{d}t = \varphi(t_0, w_0(\cdot)) + z^*(\vartheta).$$

This equality is valid for every $\vartheta \in [t_0, T)$, and, therefore, in view of continuity of ω, we conclude (6.1), which completes the proof of the theorem. $\square$

We also establish the following result.

Proposition 6.2. *If a minimax solution of problem (5.1), (5.2) is ci-differentiable of the order α at some point $(t_0, w_0(\cdot)) \in G_n^{\alpha\circ}$, then it satisfies equation (5.1) at this point.*

The proof of this proposition is based on the lemma below.

Lemma 6.3. *If a functional $\varphi : G_n^\alpha \to \mathbb{R}$ is lower semicontinuous, then φ satisfies condition (φ_+) if and only if the following statement holds:*

(φ_+^*) *For any $(t_0, w_0(\cdot)) \in G_n^{\alpha\circ}$ and $s \in \mathbb{R}^n$, there is a characteristic $(x(\cdot), z(\cdot)) \in CH(t_0, w_0(\cdot), 0, s)$ such that $\varphi(t, x_t(\cdot)) - z(t) \le \varphi(t_0, w_0(\cdot))$ for every $t \in [t_0, T]$.*

Respectively, for an upper semicontinuous functional $\varphi : G_n^\alpha \to \mathbb{R}$, condition (φ_-) is equivalent to the following:

(φ_-^*) *For any $(t_0, w_0(\cdot)) \in G_n^{\alpha\circ}$ and $s \in \mathbb{R}^n$, there is a characteristic $(x(\cdot), z(\cdot)) \in CH(t_0, w_0(\cdot), 0, s)$ such that $\varphi(t, x_t(\cdot)) - z(t) \ge \varphi(t_0, w_0(\cdot))$ for every $t \in [t_0, T]$.*

Proof. We prove only the first part of the lemma since the proof of the second one is essentially the same. It is clear that (φ_+) follows from (φ_+^*), so it remains to verify the reverse implication. Let $\varphi : G_n^\alpha \to \mathbb{R}$ be a lower semicontinuous functional satisfying (φ_+), and let $(t_0, w_0(\cdot)) \in G_n^{\alpha\circ}$ and $s \in \mathbb{R}^n$.

Fix $k \in \mathbb{N}$. Denote $t_{k,i} \triangleq t_0 + (T - t_0)i/k$, $i \in \overline{0,k}$. Take arbitrarily $(x^{[k,0]}(\cdot), z^{[k,0]}(\cdot)) \in CH(t_0, w_0(\cdot), 0, s)$ and, applying (φ_+), choose functions $x^{[k,i]} : [0,T] \to \mathbb{R}^n$, $z^{[k,i]} : [0,T] \to \mathbb{R}$, $i \in \overline{1,k}$, such that the following relations hold for every $i \in \overline{1,k}$:

$$(x^{[k,i]}(\cdot), z^{[k,i]}(\cdot)) \in CH\big(t_{k,i-1}, x^{[k,i-1]}_{t_{k,i-1}}(\cdot), 0, s\big), \quad \varphi\big(t_{k,i}, x^{[k,i]}_{t_{k,i}}(\cdot)\big) - z^{[k,i]}(t_{k,i}) \le \varphi\big(t_{k,i-1}, x^{[k,i-1]}_{t_{k,i-1}}(\cdot)\big) + 1/k^2.$$

Further, consider functions $\bar{z}^{[k,i]} : [0,T] \to \mathbb{R}$, $i \in \overline{0,k}$, such that $\bar{z}^{[k,0]}(\cdot) \triangleq z^{[k,0]}(\cdot)$ and, for any $i \in \overline{1,k}$,

$$\bar{z}^{[k,i]}(t) \triangleq \begin{cases} \bar{z}^{[k,i-1]}(t), & \text{if } t \in [0, t_{k,i-1}], \\ z^{[k,i]}(t) + \bar{z}^{[k,i-1]}(t_{k,i-1}), & \text{if } t \in (t_{k,i-1}, T]. \end{cases}$$

Then, by induction, based on Proposition 3.3 (see also the remark in the end of Sect. 3.1), we can prove that, for every $i \in \overline{0,k}$,

$$(x^{[k,i]}(\cdot), \bar{z}^{[k,i]}(\cdot)) \in CH(t_0, w_0(\cdot), 0, s), \quad \varphi\big(t_{k,j}, x^{[k,i]}_{t_{k,j}}(\cdot)\big) - \bar{z}^{[k,i]}(t_{k,j}) \le \varphi(t_0, w_0(\cdot)) + j/k^2 \quad \forall j \in \overline{0,i}.$$

Thus, for the functions $x^{[k]}(\cdot) \triangleq x^{[k,k]}(\cdot)$ and $z^{[k]}(\cdot) \triangleq \bar{z}^{[k,k]}(\cdot)$, we obtain

$$(x^{[k]}(\cdot), z^{[k]}(\cdot)) \in CH(t_0, w_0(\cdot), 0, s), \quad \varphi\big(t_{k,j}, x^{[k]}_{t_{k,j}}(\cdot)\big) - z^{[k]}(t_{k,j}) \le \varphi(t_0, w_0(\cdot)) + 1/k \quad \forall j \in \overline{0,k}.$$

Due to compactness of $CH(t_0, w_0(\cdot), 0, s)$, we can assume that the sequence $\{(x^{[k]}(\cdot), z^{[k]}(\cdot))\}_{k \in \mathbb{N}}$ converges to a characteristic $(x^{[0]}(\cdot), z^{[0]}(\cdot)) \in CH(t_0, w_0(\cdot), 0, s)$. Now, let $t \in [t_0, T]$ be fixed. For every $k \in \mathbb{N}$, denoting $t_k \triangleq \max\{t_{k,i} : t_{k,i} \le t, \, i \in \overline{0,k}\}$, we get

$$\varphi\big(t_k, x^{[k]}_{t_k}(\cdot)\big) - z^{[k]}(t_k) \le \varphi(t_0, w_0(\cdot)) + 1/k. \tag{6.2}$$

As $k \to \infty$, we have $t_k \to t$, $(t_k, x^{[k]}_{t_k}(\cdot)) \to (t, x^{[0]}_t(\cdot))$, and $z^{[k]}(t_k) \to z^{[0]}(t)$. Hence, passing to the limit as $k \to \infty$ in inequality (6.2), in view of lower semicontinuity of φ, we derive

$$\varphi(t, x^{[0]}_t(\cdot)) - z^{[0]}(t) \leq \liminf_{k\to\infty} \big(\varphi\big(t_k, x^{[k]}_{t_k}(\cdot)\big) - z^{[k]}(t_k)\big) \leq \varphi(t_0, w_0(\cdot)).$$

So, the functional φ possesses property (φ^*_+), and the lemma is proved. $\square$

Proof of Proposition 6.2. Assume that $(t_0, w_0(\cdot)) \in G^{\alpha\circ}_n$ and a minimax solution $\varphi : G^\alpha_n \to \mathbb{R}$ of problem (5.1), (5.2) is *ci*-differentiable of the order α at $(t_0, w_0(\cdot))$. Denote $s_0 \triangleq \nabla^\alpha\varphi(t_0, w_0(\cdot))$. Since φ is an upper solution of problem (5.1), (5.2), due to (φ^*_+), there exists a characteristic $(x(\cdot), z(\cdot)) \in CH(t_0, w_0(\cdot), 0, s_0)$ such that $\varphi(t, x_t(\cdot)) - z(t) \leq \varphi(t_0, w_0(\cdot))$ for every $t \in [t_0, T]$. In particular, according to (5.4), we have

$$z(t) = \int_{t_0}^{t} \big(\langle s_0, ({}^C D^\alpha x)(\tau)\rangle - H(\tau, x(\tau), s_0)\big)\,\mathrm{d}\tau \quad \forall t \in [t_0, T].$$

Hence, taking into account that $x(\cdot) \in X^\alpha(t_0, w_0(\cdot))$, in view of (4.2), we derive

$$0 \geq \varphi(t, x_t(\cdot)) - z(t) - \varphi(t_0, w_0(\cdot)) = \partial^\alpha_t\varphi(t_0, w_0(\cdot))(t - t_0) + \int_{t_0}^{t} H(\tau, x(\tau), s_0)\,\mathrm{d}\tau + o(t - t_0) \quad \forall t \in (t_0, T). \tag{6.3}$$

Note that $H(t, x(t), s_0) \to H(t_0, w_0(t_0), s_0)$ as $t \to t_0^+$ by virtue of assumption $(H.1)$. Therefore, dividing (6.3) by $t - t_0$ and, after that, passing to the limit as $t \to t_0^+$, we get

$$0 \geq \partial^\alpha_t\varphi(t_0, w_0(\cdot)) + H(t_0, w_0(t_0), s_0). \tag{6.4}$$

On the other hand, based on the fact that φ is a lower solution of problem (5.1), (5.2), and, consequently, it possesses property (φ^*_-), we can similarly obtain the inequality

$$0 \leq \partial^\alpha_t\varphi(t_0, w_0(\cdot)) + H(t_0, w_0(t_0), s_0). \tag{6.5}$$

It follows from (6.4) and (6.5) that φ satisfies equation (5.1) at $(t_0, w_0(\cdot))$. The proposition is proved. $\square$

In particular, from Proposition 6.2, we derive

Theorem 6.4. *If a minimax solution of problem* (5.1), (5.2) *is ci-smooth of the order α, then it is a classical solution of this problem.*

Theorems 6.1 and 6.4 allow us to conclude that the introduced notion of a minimax solution of problem (5.1), (5.2) is consistent with the notion of a solution of this problem in the classical sense.

Remark 6.5. The question of under what conditions problem (5.1), (5.2) admits a classical solution seems interesting and important, but is beyond the scope of the present paper. Nevertheless, let us note that, based on the results of [12] and [16], it can be verified that the Cauchy problem for the Hamilton–Jacobi equation

$$\partial^\alpha_t\varphi(t, w(\cdot)) + \langle\nabla^\alpha\varphi(t, w(\cdot)), f(t, w(t))\rangle = 0 \quad \forall (t, w(\cdot)) \in G^{\alpha\circ}_n$$

and the boundary condition

$$\varphi(T, w(\cdot)) = \sigma(w(T)) \quad \forall w(\cdot) \in \mathrm{AC}^\alpha([0, T], \mathbb{R}^n) \tag{6.6}$$

has a classical solution provided that the functions f and σ are as in Example 4.5. In addition, applying the results of [13] on the reduction of optimal control problems for linear fractional-order dynamical systems to optimal control problems for ordinary dynamical systems, some particular examples can be constructed when the Cauchy problem for the Hamilton–Jacobi equation

$$\partial_t^\alpha \varphi(t, w(\cdot)) + \langle \nabla^\alpha \varphi(t, w(\cdot)), A(t) w(t) \rangle + \min_{u \in U} \big(\langle \nabla^\alpha \varphi(t, w(\cdot)), f(t, u) \rangle + \chi(t, u) \big) = 0 \quad \forall (t, w(\cdot)) \in G_n^{\alpha \circ} \tag{6.7}$$

and boundary condition (6.6) has a classical solution. In equation (6.7), $A : [0, T] \to \mathbb{R}^{n \times n}$, $f : [0, T] \times U \to \mathbb{R}^n$, and $\chi : [0, T] \times U \to \mathbb{R}$ are continuous functions, $\mathbb{R}^{n \times n}$ is the space of $(n \times n)$-matrices endowed with the norm induced by the Euclidean norm $\| \cdot \|$ in $\mathbb{R}^n$, $U \subset \mathbb{R}^{n_U}$ is a compact set, $n_U \in \mathbb{N}$.

7. Comparison principle

The goal of this section is to prove the result below, which is often called a comparison principle. In the next section, it is used in the proof of existence and uniqueness of a minimax solution of problem (5.1), (5.2).

Theorem 7.1. *Let φ_+ and φ_- be respectively an upper and a lower solutions of problem (5.1), (5.2). Then, the inequality below holds:*

$$\varphi_-(t, w(\cdot)) \leq \varphi_+(t, w(\cdot)) \quad \forall (t, w(\cdot)) \in G_n^\alpha. \tag{7.1}$$

In general, this theorem is proved by the same scheme as the corresponding statements for Hamilton–Jacobi equations with partial derivatives (see, *e.g.*, [48], Thm. 7.3) and with first-order *ci*-derivatives (see, *e.g.*, [31], Lem. 7.7). However, the key point of the proof, which concerns construction of a Lyapunov–Krasovskii functional with a number of prescribed properties (in this connection, see, *e.g.*, Sect. 5 of [33] and Sect. 4.1 of [17]), substantially differs from the previous studies owing to features of fractional-order integrals and derivatives (see Sect. 7.3 for discussion).

7.1. Lyapunov–Krasovskii functionals

The construction of the required Lyapunov–Krasovskii functional is carried out in four steps.

7.1.1. Functional $V_{\gamma,\mu}$

Given $\gamma \in (0, 1)$ and $\mu > 0$, consider the functional

$$G_1 \ni (t, r(\cdot)) \mapsto V_{\gamma,\mu}(t, r(\cdot)) \triangleq \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{e^{-\mu (t-\tau)^\gamma} r(\tau)}{(t-\tau)^\gamma} \, \mathrm{d}\tau \in \mathbb{R}. \tag{7.2}$$

Recall that the set G_1 consists in pairs $(t, r(\cdot))$ such that $t \in [0, T]$ and $r(\cdot) \in \mathrm{C}([0, T], \mathbb{R})$, and it is endowed with the metric dist from (2.7) (see also Rem. 2.1).

Lemma 7.2. *For every $\gamma \in (0, 1)$ and $\mu > 0$, the following statements hold:*

(*V*.1) *The functional $V_{\gamma,\mu}$ is continuous.*

(*V*.2) *If $r(\cdot) \in \mathrm{AC}^\gamma([0, T], \mathbb{R})$ and $r(0) = 0$, then the function $v(t) \triangleq V_{\gamma,\mu}(t, r_t(\cdot))$, $t \in [0, T]$, satisfies the inclusion $v(\cdot) \in \mathrm{Lip}([0, T], \mathbb{R})$, and, for a.e. $t \in [0, T]$,*

$$\dot{v}(t) = ({}^C D^\gamma r)(t) - \frac{\mu}{\Gamma(1-\gamma)} r(t) + \frac{\mu^2 \gamma^2}{\Gamma(1-\gamma)} \int_0^t \frac{r(\tau)}{(t-\tau)^{\gamma+1}} \int_0^{t-\tau} \xi^{2\gamma-1} e^{-\mu \xi^\gamma} \, \mathrm{d}\xi \, \mathrm{d}\tau.$$

If, in addition, the function $r(\cdot)$ is nonnegative, then

$$\dot{v}(t) \leq ({}^C D^\gamma r)(t) - \frac{\mu}{\Gamma(1-\gamma)} r(t) + \frac{\mu^2 \Gamma(\gamma+1)}{2\Gamma(1-\gamma)} (I^\gamma r)(t) \text{ for a.e. } t \in [0,T].$$

Proof. For brevity, denote $V \triangleq V_{\gamma,\mu}$.

1. Let us show that, for any $(t, r(\cdot)) \in G_1$, the function $v(\tau) \triangleq V(\tau, r_\tau(\cdot))$, $\tau \in [0,t]$, satisfies the estimate

$$|v(\tau') - v(\tau)| \leq \frac{\|r(\cdot)\|_{[0,t]}}{\Gamma(2-\gamma)} |\tau' - \tau|^{1-\gamma} + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)} \varkappa(|\tau' - \tau|) \quad \forall \tau, \tau' \in [0,t], \tag{7.3}$$

where $\varkappa$ is the modulus of continuity of $r(\cdot)$ on $[0,t]$.

If $t = 0$, inequality (7.3) holds automatically. So, let $t > 0$. Note that

$$v(\tau) = \frac{1}{\Gamma(1-\gamma)} \int_0^\tau \frac{e^{-\mu\xi^\gamma} r(\tau - \xi)}{\xi^\gamma} \, \mathrm{d}\xi \quad \forall \tau \in [0,t]. \tag{7.4}$$

Fix τ, $\tau' \in [0,t]$ such that $\tau' > \tau$. If $\tau = 0$, then, taking into account that $v(0) = 0$, we obtain

$$|v(\tau') - v(\tau)| = |v(\tau')| \leq \frac{1}{\Gamma(1-\gamma)} \int_0^{\tau'} \frac{e^{-\mu\xi^\gamma} |r(\tau' - \xi)|}{\xi^\gamma} \, \mathrm{d}\xi \leq \frac{\|r(\cdot)\|_{[0,t]}}{\Gamma(1-\gamma)} \int_0^{\tau'} \frac{\mathrm{d}\xi}{\xi^\gamma} = \frac{\|r(\cdot)\|_{[0,t]}}{\Gamma(2-\gamma)} (\tau' - \tau)^{1-\gamma},$$

and, if $\tau > 0$, we derive

$$\begin{aligned} |v(\tau') - v(\tau)| &\leq \frac{1}{\Gamma(1-\gamma)} \int_\tau^{\tau'} \frac{e^{-\mu\xi^\gamma} |r(\tau' - \xi)|}{\xi^\gamma} \, \mathrm{d}\xi + \frac{1}{\Gamma(1-\gamma)} \int_0^\tau \frac{e^{-\mu\xi^\gamma} |r(\tau' - \xi) - r(\tau - \xi)|}{\xi^\gamma} \, \mathrm{d}\xi \\ &\leq \frac{\|r(\cdot)\|_{[0,t]}}{\Gamma(1-\gamma)} \int_\tau^{\tau'} \frac{\mathrm{d}\xi}{\xi^\gamma} + \frac{\varkappa(\tau' - \tau)}{\Gamma(1-\gamma)} \int_0^\tau \frac{\mathrm{d}\xi}{\xi^\gamma} = \frac{\|r(\cdot)\|_{[0,t]}}{\Gamma(2-\gamma)} \big((\tau')^{1-\gamma} - \tau^{1-\gamma}\big) + \frac{\varkappa(\tau' - \tau)}{\Gamma(2-\gamma)} \tau^{1-\gamma} \\ &\leq \frac{\|r(\cdot)\|_{[0,t]}}{\Gamma(2-\gamma)} (\tau' - \tau)^{1-\gamma} + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)} \varkappa(\tau' - \tau). \end{aligned}$$

Thus, inequality (7.3) is valid.

2. Now, let us prove statement $(V.1)$. Let $(t_0, r^{[0]}(\cdot)) \in G_1$ and $\{(t_k, r^{[k]}(\cdot))\}_{k \in \mathbb{N}} \subset G_1$ be such that $\mathrm{dist}_k \triangleq \mathrm{dist}((t_0, r^{[0]}(\cdot)), (t_k, r^{[k]}(\cdot))) \to 0$ as $k \to \infty$. For every $k \in \mathbb{N} \cup \{0\}$, let $\varkappa_k$ be the modulus of continuity of $r^{[k]}(\cdot)$ on $[0, t_k]$. Since the functions $r^{[k]}(\cdot)$, $k \in \mathbb{N} \cup \{0\}$, are uniformly bounded and equicontinuous (see Sect. 2), there exists $R > 0$ such that $\|r^{[k]}(\cdot)\|_{[0,t_k]} \leq R$ for any $k \in \mathbb{N} \cup \{0\}$, and $\varkappa_*(\delta) \triangleq \sup\{\varkappa_k(\delta) : k \in \mathbb{N} \cup \{0\}\} \to 0$ as $\delta \to 0^+$. Hence, in order to establish the required convergence $V(t_k, r^{[k]}(\cdot)) \to V(t_0, r^{[0]}(\cdot))$ as $k \to \infty$, it suffices to prove for every $k \in \mathbb{N}$ the inequality

$$|V(t_0, r^{[0]}(\cdot)) - V(t_k, r^{[k]}(\cdot))| \leq \frac{R}{\Gamma(2-\gamma)} \mathrm{dist}_k^{1-\gamma} + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)} (\mathrm{dist}_k + 2\varkappa_*(\mathrm{dist}_k)). \tag{7.5}$$

Fix $k \in \mathbb{N}$. Assume that $t_0 \leq t_k$. Then, we have

$$|V(t_0, r^{[0]}(\cdot)) - V(t_k, r^{[k]}(\cdot))| \leq |V(t_0, r^{[0]}(\cdot)) - V(t_0, r^{[k]}_{t_0}(\cdot))| + |V(t_0, r^{[k]}_{t_0}(\cdot)) - V(t_k, r^{[k]}(\cdot))|.$$

For the first term, by virtue of (2.8) and (7.4), we derive

$$\begin{aligned}|V(t_0, r^{[0]}(\cdot)) - V(t_0, r^{[k]}_{t_0}(\cdot))| &\leq \frac{1}{\Gamma(1-\gamma)} \int_0^{t_0} \frac{e^{-\mu\xi^\gamma} |r^{[0]}(t_0 - \xi) - r^{[k]}(t_0 - \xi)|}{\xi^\gamma} \, \mathrm{d}\xi \\ &\leq \frac{T^{1-\gamma}}{\Gamma(2-\gamma)} \max_{\xi \in [0, t_0]} |r^{[0]}(\xi) - r^{[k]}(\xi)| \leq \frac{T^{1-\gamma}}{\Gamma(2-\gamma)} (\mathrm{dist}_k + \varkappa_*(\mathrm{dist}_k)),\end{aligned}$$

and for the second term, in view of (2.8) and (7.3), we get

$$\begin{aligned}|V(t_0, r^{[k]}_{t_0}(\cdot)) - V(t_k, r^{[k]}(\cdot))| &\leq \frac{R}{\Gamma(2-\gamma)} (t_k - t_0)^{1-\gamma} + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)} \varkappa_*(t_k - t_0) \\ &\leq \frac{R}{\Gamma(2-\gamma)} \mathrm{dist}_k^{1-\gamma} + \frac{T^{1-\gamma}}{\Gamma(2-\gamma)} \varkappa_*(\mathrm{dist}_k).\end{aligned}$$

Thus, we obtain inequality (7.5). In the case $t_k < t_0$, this inequality can be proved in a similar way.

3. Further, let us prove $(V.2)$. Fix $r(\cdot) \in \mathrm{AC}^\gamma([0,T], \mathbb{R})$ such that $r(0) = 0$ and consider the function $v(t) \triangleq V(t, r_t(\cdot))$, $t \in [0, T]$. For every $\theta \geq 0$, based on the equality

$$e^{-\mu\theta^\gamma} = 1 - \mu\gamma \int_0^\theta \frac{e^{-\mu\xi^\gamma}}{\xi^{1-\gamma}} \, \mathrm{d}\xi, \tag{7.6}$$

which can be verified by direct calculation, we derive

$$e^{-\mu\theta^\gamma} = 1 - \mu\gamma \int_0^\theta \frac{\mathrm{d}\xi}{\xi^{1-\gamma}} - \mu\gamma \int_0^\theta \frac{e^{-\mu\xi^\gamma} - 1}{\xi^{1-\gamma}} \, \mathrm{d}\xi = 1 - \mu\theta^\gamma + \mu\gamma \int_0^\theta \frac{1 - e^{-\mu\xi^\gamma}}{\xi^{1-\gamma}} \, \mathrm{d}\xi,$$

and, consequently, according to (2.1) and (7.2), the function $v(\cdot)$ can be represented as follows:

$$\begin{aligned}v(t) &= (I^{1-\gamma} r)(t) - \frac{\mu}{\Gamma(1-\gamma)} (I^1 r)(t) + \frac{\mu\gamma}{\Gamma(1-\gamma)} \int_0^t \frac{r(\tau)}{(t-\tau)^\gamma} \int_0^{t-\tau} \frac{1 - e^{-\mu\xi^\gamma}}{\xi^{1-\gamma}} \, \mathrm{d}\xi \, \mathrm{d}\tau \\ &\triangleq v_1(t) - \frac{\mu}{\Gamma(1-\gamma)} v_2(t) + \frac{\mu\gamma}{\Gamma(1-\gamma)} v_3(t) \quad \forall t \in [0, T].\end{aligned} \tag{7.7}$$

Since $r(\cdot) \in \mathrm{AC}^\gamma([0,T], \mathbb{R})$ and $r(0) = 0$, for $v_1(t) \triangleq (I^{1-\gamma} r)(t)$, $t \in [0, T]$, we conclude $v_1(\cdot) \in \mathrm{Lip}([0,T], \mathbb{R})$ and $\dot{v}_1(t) = ({}^C D^\gamma r)(t)$ for a.e. $t \in [0, T]$ (see Sect. 2). For $v_2(t) \triangleq (I^1 r)(t)$, $t \in [0, T]$, we have $v_2(\cdot) \in \mathrm{Lip}([0,T], \mathbb{R})$, and $\dot{v}_2(t) = r(t)$ for any $t \in (0, T)$. Thus, it remains to investigate the properties of the function

$$v_3(t) \triangleq \int_0^t \frac{r(\tau)}{(t-\tau)^\gamma} \int_0^{t-\tau} \frac{1 - e^{-\mu\xi^\gamma}}{\xi^{1-\gamma}} \, \mathrm{d}\xi \, \mathrm{d}\tau, \quad t \in [0, T].$$

4. To this end, let us introduce the auxiliary function

$$M(\theta) \triangleq \frac{1}{\theta^\gamma} \int_0^\theta \frac{1 - e^{-\mu\xi^\gamma}}{\xi^{1-\gamma}} \, \mathrm{d}\xi, \quad \theta > 0,$$

and describe some of its properties. It follows from (7.6) that

$$0 \leq 1 - e^{-\mu\xi^\gamma} \leq \mu\gamma \int_0^\xi \frac{\mathrm{d}\eta}{\eta^{1-\gamma}} = \mu\xi^\gamma \quad \forall \xi \geq 0$$

and, hence,

$$0 \leq M(\theta) \leq \frac{\mu}{\theta^{\gamma}} \int_0^{\theta} \xi^{2\gamma-1}\,\mathrm{d}\xi = \frac{\mu}{2\gamma}\theta^{\gamma} \quad \forall \theta > 0. \tag{7.8}$$

In particular, we obtain $M(\theta) \to 0$ as $\theta \to 0^+$. Further, by virtue of the integration by parts formula, we derive

$$\dot{M}(\theta) = -\frac{\gamma}{\theta^{\gamma+1}} \int_0^{\theta} \frac{1 - e^{-\mu\xi^{\gamma}}}{\xi^{1-\gamma}}\,\mathrm{d}\xi + \frac{1 - e^{-\mu\theta^{\gamma}}}{\theta} = \frac{\mu\gamma}{\theta^{\gamma+1}} \int_0^{\theta} \xi^{2\gamma-1} e^{-\mu\xi^{\gamma}}\,\mathrm{d}\xi \quad \forall \theta > 0. \tag{7.9}$$

Consequently, we have

$$0 \leq \dot{M}(\theta) \leq \frac{\mu\gamma}{\theta^{\gamma+1}} \int_0^{\theta} \xi^{2\gamma-1}\,\mathrm{d}\xi = \frac{\mu}{2\theta^{1-\gamma}} \quad \forall \theta > 0 \tag{7.10}$$

and, therefore, for any $\theta > 0$ and $\theta' > \theta$,

$$0 \leq M(\theta') - M(\theta) = \int_{\theta}^{\theta'} \dot{M}(\xi)\,\mathrm{d}\xi \leq \int_{\theta}^{\theta'} \frac{\mu}{2\xi^{1-\gamma}}\,\mathrm{d}\xi \leq \frac{\mu}{2\theta^{1-\gamma}}(\theta' - \theta). \tag{7.11}$$

5. Now, based on the representation

$$v_3(t) = \int_0^t M(t-\tau) r(\tau)\,\mathrm{d}\tau \quad \forall t \in [0,T],$$

let us prove first that the function $v_3(\cdot)$ satisfies the Lipschitz condition $|v_3(t') - v_3(t)| \leq L|t'-t|$ for every t, $t' \in [0,T]$ with the constant

$$L \triangleq \frac{(\gamma+2)\mu\|r(\cdot)\|_{[0,T]} T^{\gamma}}{2(\gamma+1)\gamma}.$$

Fix t, $t' \in [0,T]$ such that $t' > t$. If $t = 0$, then, taking into account that $v_3(0) = 0$ and using (7.8), we derive

$$\begin{aligned} |v_3(t') - v_3(t)| = |v_3(t')| &\leq \int_0^{t'} M(t'-\tau)|r(\tau)|\,\mathrm{d}\tau \leq \|r(\cdot)\|_{[0,T]} \int_0^{t'} \frac{\mu}{2\gamma}(t'-\tau)^{\gamma}\,\mathrm{d}\tau \\ &= \frac{\mu\|r(\cdot)\|_{[0,T]}(t')^{\gamma+1}}{2(\gamma+1)\gamma} \leq \frac{\mu\|r(\cdot)\|_{[0,T]}T^{\gamma}}{2(\gamma+1)\gamma} t' \leq L(t'-t). \end{aligned}$$

Suppose that $t > 0$. Then, we have

$$v_3(t') - v_3(t) = \int_t^{t'} M(t'-\tau) r(\tau)\,\mathrm{d}\tau + \int_0^t (M(t'-\tau) - M(t-\tau)) r(\tau)\,\mathrm{d}\tau. \tag{7.12}$$

For the first term, according to (7.8), we get

$$\begin{aligned} \Big| \int_t^{t'} M(t'-\tau) r(\tau)\,\mathrm{d}\tau \Big| &\leq \int_t^{t'} M(t'-\tau)|r(\tau)|\,\mathrm{d}\tau \leq \|r(\cdot)\|_{[0,T]} \int_t^{t'} \frac{\mu}{2\gamma}(t'-\tau)^{\gamma}\,\mathrm{d}\tau \\ &= \frac{\mu\|r(\cdot)\|_{[0,T]}(t'-t)^{\gamma+1}}{2(\gamma+1)\gamma} \leq \frac{\mu\|r(\cdot)\|_{[0,T]}T^{\gamma}}{2(\gamma+1)\gamma}(t'-t), \end{aligned} \tag{7.13}$$

and, for the second term, by virtue of (7.11), we conclude

$$\begin{aligned}\Big|\int_0^t (M(t'-\tau)-M(t-\tau))r(\tau)\,\mathrm{d}\tau\Big| &\le \int_0^t (M(t'-\tau)-M(t-\tau))|r(\tau)|\,\mathrm{d}\tau\\ &\le \|r(\cdot)\|_{[0,T]}\int_0^t \frac{\mu}{2(t-\tau)^{1-\gamma}}(t'-t)\,\mathrm{d}\tau = \frac{\mu\|r(\cdot)\|_{[0,T]}t^{\gamma}}{2\gamma}(t'-t)\\ &\le \frac{\mu\|r(\cdot)\|_{[0,T]}T^{\gamma}}{2\gamma}(t'-t).\end{aligned}$$

Thus, we obtain the desired estimate.

6. Since $v_3(\cdot)\in \mathrm{Lip}([0,T],\mathbb{R})$, then the derivative $\dot v_3(t)$ exists for a.e. $t\in[0,T]$. In order to obtain an explicit formula for this derivative, let us calculate the right-hand side derivative $\dot v_3^+(t)$ of $v_3(\cdot)$ at every $t\in(0,T)$. For the first term in (7.12), owing to (7.13), we have

$$\Big|\frac{1}{t'-t}\int_t^{t'} M(t'-\tau)r(\tau)\,\mathrm{d}\tau\Big| \le \frac{\mu\|r(\cdot)\|_{[0,T]}(t'-t)^{\gamma}}{2(\gamma+1)\gamma}\quad \forall t'\in(t,T]$$

and, therefore,

$$\lim_{t'\to t^+}\frac{1}{t'-t}\int_t^{t'} M(t'-\tau)r(\tau)\,\mathrm{d}\tau = 0.$$

Let us consider the second term in (7.12). For any $\tau\in[0,t)$, we get

$$\lim_{t'\to t^+}\frac{(M(t'-\tau)-M(t-\tau))r(\tau)}{t'-t} = \dot M(t-\tau)r(\tau)$$

and, moreover, due to (7.11),

$$\frac{|(M(t'-\tau)-M(t-\tau))r(\tau)|}{t'-t} \le \frac{\mu\|r(\cdot)\|_{[0,T]}}{2(t-\tau)^{1-\gamma}}\quad \forall t'\in(t,T].$$

Then, applying Lebesgue's dominated convergence theorem, we conclude

$$\lim_{t'\to t^+}\frac{1}{t'-t}\int_0^t \big(M(t'-\tau)-M(t-\tau)\big)r(\tau)\,\mathrm{d}\tau = \int_0^t \dot M(t-\tau)r(\tau)\,\mathrm{d}\tau.$$

Hence, we derive

$$\dot v_3^+(t) \triangleq \lim_{t'\to t^+}\frac{v_3(t')-v_3(t)}{t'-t} = \int_0^t \dot M(t-\tau)r(\tau)\,\mathrm{d}\tau.$$

As a result, in view of (7.9), we get

$$\dot v_3(t) = \int_0^t \dot M(t-\tau)r(\tau)\,\mathrm{d}\tau = \mu\gamma\int_0^t \frac{r(\tau)}{(t-\tau)^{\gamma+1}}\int_0^{t-\tau}\xi^{2\gamma-1}e^{-\mu\xi^{\gamma}}\,\mathrm{d}\xi\,\mathrm{d}\tau \text{ for a.e. } t\in[0,T]. \tag{7.14}$$

7. Summarizing the above, we obtain that $v(\cdot) \in \operatorname{Lip}([0,T],\mathbb{R})$ and

$$\begin{aligned}\dot{v}(t) &= \dot{v}_1(t) - \frac{\mu}{\Gamma(1-\gamma)}\dot{v}_2(t) + \frac{\mu\gamma}{\Gamma(1-\gamma)}\dot{v}_3(t)\\ &= ({}^C D^\gamma r)(t) - \frac{\mu}{\Gamma(1-\gamma)} r(t) + \frac{\mu^2\gamma^2}{\Gamma(1-\gamma)}\int_0^t \frac{r(\tau)}{(t-\tau)^{\gamma+1}}\int_0^{t-\tau} \xi^{2\gamma-1} e^{-\mu\xi^\gamma}\,\mathrm{d}\xi\,\mathrm{d}\tau \text{ for a.e. } t\in[0,T].\end{aligned}$$

8. If the function $r(\cdot)$ is nonnegative, then, for a.e. $t \in [0,T]$, according to (2.1), (7.10), and (7.14), we derive

$$\dot{v}_3(t) \le \int_0^t \frac{\mu r(\tau)}{2(t-\tau)^{1-\gamma}}\,\mathrm{d}\tau = \frac{\mu\Gamma(\gamma)}{2}(I^\gamma r)(t)$$

and, therefore,

$$\dot{v}(t) \le ({}^C D^\gamma r)(t) - \frac{\mu}{\Gamma(1-\gamma)} r(t) + \frac{\mu^2\Gamma(\gamma+1)}{2\Gamma(1-\gamma)}(I^\gamma r)(t).$$

This completes the proof of the lemma. □

7.1.2. Functional $V^*_{\beta,\mu}$

Let $\beta \in [0, 1-\alpha)$ and $\mu > 0$ be fixed. Note that $\gamma \triangleq \alpha+\beta \in (0,1)$ and take the corresponding functional $V_{\alpha+\beta,\mu}$ from (7.2). For every $(t, w(\cdot)) \in G_n$, denote

$$q(\tau \mid t, w(\cdot)) \triangleq \|w(\tau) - w(0)\|^2, \quad r(\tau \mid t, w(\cdot), \beta) \triangleq \big(I^\beta q(\cdot \mid t, w(\cdot))\big)(\tau), \quad \tau\in[0,t].$$

Consider the functional

$$G_n \ni (t, w(\cdot)) \mapsto V^*_{\beta,\mu}(t, w(\cdot)) \triangleq V_{\alpha+\beta,\mu}\big(t, r(\cdot \mid t, w(\cdot), \beta)\big) \in \mathbb{R}. \tag{7.15}$$

In accordance with the introduced notations, this functional can be defined explicitly by

$$V^*_{\beta,\mu}(t, w(\cdot)) \triangleq \frac{1}{\Gamma(1-\alpha-\beta)\Gamma(\beta)}\int_0^t \frac{e^{-\mu(t-\tau)^{\alpha+\beta}}}{(t-\tau)^{\alpha+\beta}}\int_0^\tau \frac{\|w(\xi)-w(0)\|^2}{(\tau-\xi)^{1-\beta}}\,\mathrm{d}\xi\,\mathrm{d}\tau, \quad (t, w(\cdot)) \in G_n, \tag{7.16}$$

if $\beta > 0$, and, if $\beta = 0$, by

$$V^*_{0,\mu}(t, w(\cdot)) \triangleq \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{e^{-\mu(t-\tau)^\alpha}\|w(\tau)-w(0)\|^2}{(t-\tau)^\alpha}\,\mathrm{d}\tau, \quad (t, w(\cdot)) \in G_n. \tag{7.17}$$

Lemma 7.3. *For every $\beta \in [0, 1-\alpha)$ and $\mu > 0$, the following statements hold:*

(V^*.1) *The functional $V^*_{\beta,\mu}$ is continuous.*
(V^*.2) *For any $(t, w(\cdot)) \in G_n$, the inequality below is valid:*

$$V^*_{\beta,\mu}(t, w(\cdot)) \ge e^{-\mu T^{\alpha+\beta}}\big(I^{1-\alpha} q(\cdot \mid t, w(\cdot))\big)(t).$$

In particular, the functional $V^*_{\beta,\mu}$ is nonnegative.

$(V^*.3)$ *If* $x(\cdot) \in \mathrm{AC}^\alpha([0,T],\mathbb{R}^n)$*, then the function* $v^*(t) \triangleq V^*_{\beta,\mu}(t, x_t(\cdot))$*,* $t \in [0,T]$*, satisfies the inclusion* $v^*(\cdot) \in \mathrm{Lip}([0,T],\mathbb{R})$*, and*

$$\dot{v}^*(t) \leq ({}^C D^\alpha q)(t) - \frac{\mu}{\Gamma(1-\alpha-\beta)}(I^\beta q)(t) + \frac{\mu^2 \Gamma(\alpha+\beta+1)}{2\Gamma(1-\alpha-\beta)}(I^{\alpha+2\beta} q)(t) \text{ for a.e. } t \in [0,T], \tag{7.18}$$

where $q(\cdot) \triangleq q(\cdot \mid T, x(\cdot))$*.*

Proof. For brevity, denote $V \triangleq V_{\alpha+\beta,\mu}$ and $V^* \triangleq V^*_{\beta,\mu}$.

1. Since the functional V is continuous by $(V.1)$, in order to establish $(V^*.1)$, it is sufficient to prove continuity of the mapping

$$G_n \ni (t, w(\cdot)) \mapsto \big(t, r(\cdot \mid t, w(\cdot), \beta)\big) \in G_1. \tag{7.19}$$

Take $(t_0, w_0(\cdot)) \in G_n$ and $\{(t_k, w_k(\cdot))\}_{k\in\mathbb{N}} \subset G_n$ such that $\mathrm{dist}_k \triangleq \mathrm{dist}((t_0, w_0(\cdot)), (t_k, w_k(\cdot))) \to 0$ as $k \to \infty$. Then, in accordance with Section 2, there exists $R > 0$ such that $\|w_k(\cdot)\|_{[0,t_k]} \leq R$ for every $k \in \mathbb{N} \cup \{0\}$, and $\varkappa_*(\delta) \triangleq \sup\{\varkappa_k(\delta) : k \in \mathbb{N} \cup \{0\}\} \to 0$ as $\delta \to 0^+$, where $\varkappa_k$ is the modulus of continuity of $w_k(\cdot)$ on $[0, t_k]$. Denote $q^{[k]}(\cdot) \triangleq q(\cdot \mid t_k, w_k(\cdot))$, $k \in \mathbb{N} \cup \{0\}$. For any $k \in \mathbb{N} \cup \{0\}$, we have

$$|q^{[k]}(\tau) - q^{[k]}(\tau')| \leq \|w_k(\tau) - w_k(\tau')\|(\|w_k(\tau)\| + \|w_k(\tau')\| + 2\|w_k(0)\|) \leq 4R\varkappa_*(|\tau - \tau'|) \quad \forall \tau, \tau' \in [0, t_k].$$

Moreover, for every $k \in \mathbb{N}$, by virtue of (2.8), we derive

$$|q^{[0]}(\tau) - q^{[k]}(\tau)| \leq 4R(\|w_0(\tau) - w_k(\tau)\| + \|w_0(0) - w_k(0)\|) \leq 8R(\mathrm{dist}_k + \varkappa_*(\mathrm{dist}_k)) \quad \forall \tau \in [0, \min\{t_0, t_k\}] \tag{7.20}$$

and, hence,

$$\mathrm{dist}\big((t_0, q^{[0]}(\cdot)), (t_k, q^{[k]}(\cdot))\big) \leq \mathrm{dist}_k + 4R\varkappa_*(\mathrm{dist}_k) + 8R(\mathrm{dist}_k + \varkappa_*(\mathrm{dist}_k)).$$

Thus, the sequence $\{(t_k, q^{[k]}(\cdot))\}_{k\in\mathbb{N}} \subset G_1$ converges to $(t_0, q^{[0]}(\cdot)) \in G_1$. Further, consider the functions $r^{[k]}(\cdot) \triangleq r(\cdot \mid t_k, w_k(\cdot), \beta)$, $k \in \mathbb{N} \cup \{0\}$. If $\beta = 0$, then $r^{[k]}(\cdot) = q^{[k]}(\cdot)$ for all $k \in \mathbb{N} \cup \{0\}$, and, consequently, we get $(t_k, r^{[k]}(\cdot)) \to (t_0, r^{[0]}(\cdot))$ as $k \to \infty$, which proves continuity of mapping (7.19). Let $\beta > 0$. Then, for any $k \in \mathbb{N} \cup \{0\}$, taking into account the estimate

$$|q^{[k]}(\tau)| \leq (\|w_k(\tau)\| + \|w_k(0)\|)^2 \leq 4R^2 \quad \forall \tau \in [0, t_k]$$

and due to (2.3), we obtain

$$|r^{[k]}(\tau) - r^{[k]}(\tau')| \leq \frac{2|\tau - \tau'|^\beta}{\Gamma(\beta+1)} \max_{\xi \in [0,t_k]} |q^{[k]}(\xi)| \leq \frac{8R^2}{\Gamma(\beta+1)}|\tau - \tau'|^\beta \quad \forall \tau, \tau' \in [0, t_k].$$

In addition, for any $k \in \mathbb{N}$, based on (7.20), we derive

$$|r^{[0]}(\tau) - r^{[k]}(\tau)| \leq \frac{\tau^\beta}{\Gamma(\beta+1)} \max_{\xi \in [0,\tau]} |q^{[0]}(\xi) - q^{[k]}(\xi)| \leq \frac{8RT^\beta}{\Gamma(\beta+1)}(\mathrm{dist}_k + \varkappa_*(\mathrm{dist}_k)) \quad \forall \tau \in [0, \min\{t_0, t_k\}],$$

and, therefore, in view of (2.8), we have

$$\mathrm{dist}\big((t_0, r^{[0]}(\cdot)), (t_k, r^{[k]}(\cdot))\big) \leq \mathrm{dist}_k + \frac{8R^2}{\Gamma(\beta+1)}\mathrm{dist}_k^\beta + \frac{8RT^\beta}{\Gamma(\beta+1)}(\mathrm{dist}_k + \varkappa_*(\mathrm{dist}_k)).$$

So, the sequence $\{(t_k, r^{[k]}(\cdot))\}_{k\in\mathbb{N}} \subset G_1$ converges to $(t_0, r^{[0]}(\cdot)) \in G_1$, and, hence, mapping (7.19) is continuous.

2. Further, for every $(t, w(\cdot)) \in G_n$, since the functions $q(\cdot) \triangleq q(\cdot \mid t, w(\cdot))$ and $r(\cdot \mid t, w(\cdot), \beta) = (I^\beta q)(\cdot)$ are nonnegative, in accordance with (2.1) and (2.2), we obtain

$$V^*(t, w(\cdot)) = \frac{1}{\Gamma(1-\alpha-\beta)} \int_0^t \frac{e^{-\mu(t-\tau)^{\alpha+\beta}} (I^\beta q)(\tau)}{(t-\tau)^{\alpha+\beta}} \,\mathrm{d}\tau \geq \frac{e^{-\mu T^{\alpha+\beta}}}{\Gamma(1-\alpha-\beta)} \int_0^t \frac{(I^\beta q)(\tau)}{(t-\tau)^{\alpha+\beta}} \,\mathrm{d}\tau$$
$$= e^{-\mu T^{\alpha+\beta}} \big(I^{1-\alpha-\beta}(I^\beta q)\big)(t) = e^{-\mu T^{\alpha+\beta}} (I^{1-\alpha} q)(t) \geq 0,$$

which proves $(V^*.2)$.

3. Let us prove statement $(V^*.3)$. Fix $x(\cdot) \in \mathrm{AC}^\alpha([0,T], \mathbb{R}^n)$. By virtue of Corollary 4.2 in [11], we have $q(\cdot) \triangleq q(\cdot \mid T, x(\cdot)) \in \mathrm{AC}^\alpha([0,T], \mathbb{R})$, and, therefore, $r(\cdot) \triangleq r(\cdot \mid T, x(\cdot), \beta) = (I^\beta q)(\cdot) \in \mathrm{AC}^{\alpha+\beta}([0,T], \mathbb{R})$ in view of the equality $q(0) = 0$. Since $r(0) = 0$ and the function $r(\cdot)$ is nonnegative, it follows from $(V.2)$ that the function $v^*(t) \triangleq V^*(t, x_t(\cdot)) = V(t, r_t(\cdot))$, $t \in [0,T]$, satisfies the inclusion $v^*(\cdot) \in \mathrm{Lip}([0,T], \mathbb{R})$, and

$$\dot{v}^*(t) \leq ({}^C D^{\alpha+\beta} r)(t) - \frac{\mu}{\Gamma(1-\alpha-\beta)} r(t) + \frac{\mu^2 \Gamma(\alpha+\beta+1)}{2\Gamma(1-\alpha-\beta)} (I^{\alpha+\beta} r)(t) \text{ for a.e. } t \in [0,T]. \tag{7.21}$$

Note that, due to (2.2), we derive $(I^{\alpha+\beta} r)(t) = (I^{\alpha+2\beta} q)(t)$ for any $t \in [0,T]$, and, moreover, according to (2.5),

$$({}^C D^{\alpha+\beta} r)(t) = \frac{\mathrm{d}}{\mathrm{d}t} (I^{1-\alpha-\beta} r)(t) = \frac{\mathrm{d}}{\mathrm{d}t} \big(I^{1-\alpha-\beta}(I^\beta q)\big)(t) = \frac{\mathrm{d}}{\mathrm{d}t} (I^{1-\alpha} q)(t) = ({}^C D^\alpha q)(t) \text{ for a.e. } t \in [0,T].$$

Thus, inequality (7.21) implies estimate (7.18). The lemma is proved. □

7.1.3. Functional V_*

By suitably combining the functionals $V^*_{\beta,\mu}$ from (7.15) for various values of $\beta \in [0, 1-\alpha)$ and $\mu > 0$, we obtain the following result.

Lemma 7.4. *For any $\lambda > 0$, there exist a number $\lambda_* > 0$ and a functional $V_* : G_n \to \mathbb{R}$ such that:*

$(V_*.1)$ *The functional V_* is nonnegative and continuous. In addition, if $(t, w(\cdot)) \in G_n$ and $w(\tau) = w(0)$ for all $\tau \in [0,t]$, then $V_*(t, w(\cdot)) = 0$.*

$(V_*.2)$ *For every function $x(\cdot) \in \mathrm{AC}^\alpha([0,T], \mathbb{R}^n)$, the function $v_*(t) \triangleq V_*(t, x_t(\cdot))$, $t \in [0,T]$, satisfies the inclusion $v_*(\cdot) \in \mathrm{Lip}([0,T], \mathbb{R})$, and*

$$\dot{v}_*(t) \leq e^{-\lambda_* t} \big(2\langle x(t) - x(0), ({}^C D^\alpha x)(t)\rangle - \lambda \|x(t) - x(0)\|^2\big) \textit{ for a.e. } t \in [0,T]. \tag{7.22}$$

$(V_*.3)$ *For any compact set $X \subset \mathrm{C}([0,T], \mathbb{R}^n)$ and any number $\rho > 0$, there exists $\delta > 0$ such that, for every $x(\cdot) \in X$, the inequality $V_*(T, x(\cdot)) \leq \delta$ implies the estimate $\|x(\cdot) - x(0)\|_{[0,T]} \leq \rho$.*

Remark 7.5. In the case $\alpha = 1$, statements $(V_*.1)$ and $(V_*.2)$ are satisfied for the functional

$$V_*(t, w(\cdot)) \triangleq e^{-\lambda t} \|w(t) - w(0)\|^2, \quad (t, w(\cdot)) \in G_n.$$

Thus, Lemma 7.4 provides an analogue of this functional for the case when $\alpha \in (0,1)$.

Before proving Lemma 7.4, we present an auxiliary proposition.

Proposition 7.6. *If $\beta \geq 1-\alpha$ and a function $\psi(\cdot) \in \mathrm{L}^\infty([0,T],\mathbb{R})$ is nonnegative, then*

$$(I^\beta \psi)(t) \leq \frac{\Gamma(1-\alpha)T^{\alpha+\beta-1}}{\Gamma(\beta)}(I^{1-\alpha}\psi)(t) \quad \forall t \in [0,T].$$

Proof. According to (2.1), for every $t \in [0,T]$, we have

$$(I^\beta \psi)(t) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{(t-\tau)^{\alpha+\beta-1}\psi(\tau)}{(t-\tau)^\alpha}\,\mathrm{d}\tau \leq \frac{T^{\alpha+\beta-1}}{\Gamma(\beta)}\int_0^t \frac{\psi(\tau)}{(t-\tau)^\alpha}\,\mathrm{d}\tau \leq \frac{\Gamma(1-\alpha)T^{\alpha+\beta-1}}{\Gamma(\beta)}(I^{1-\alpha}\psi)(t),$$

which proves the proposition. □

Proof of Lemma 7.4. Fix $\lambda > 0$. Choice of the number λ_* and construction of the functional $V_* : G_n \to \mathbb{R}$ depend on the value of α. For the reader's convenience, we first consider in detain the cases when $\alpha \in [1/2, 1)$ and $\alpha \in [1/4, 1/2)$, and, after that, we handle the general case when $\alpha \in [2^{-m}, 2^{-(m-1)})$ for some $m \in \mathbb{N}$.

1. Assume that $\alpha \in [1/2, 1)$. Define

$$\beta_1 \triangleq 0, \quad \mu_1 \triangleq \Gamma(1-\alpha)\lambda, \quad \lambda_* \triangleq \frac{\mu_1^2\Gamma(\alpha+1)T^{2\alpha-1}}{2\Gamma(\alpha)}e^{\mu_1 T^\alpha},$$

take the corresponding functional $V^*_{\beta_1,\mu_1}$ from (7.15), and put

$$V_*(t,w(\cdot)) \triangleq e^{-\lambda_* t}V^*_{\beta_1,\mu_1}(t,w(\cdot)), \quad (t,w(\cdot)) \in G_n. \tag{7.23}$$

Let us show that the specified λ_* and V_* possess properties (V_*.1)–(V_*.3).

Since the functional $V^*_{\beta_1,\mu_1}$ is nonnegative and continuous by (V^*.1) and (V^*.2), we obtain that the functional V_* is nonnegative and continuous, too. Now, let $(t,w(\cdot)) \in G_n$ be such that $w(\tau) = w(0)$ for all $\tau \in [0,t]$. Then, it follows from (7.17) that $V^*_{\beta_1,\mu_1}(t,w(\cdot)) = 0$, and, consequently, $V_*(t,w(\cdot)) = 0$. Thus, statement (V_*.1) is proved.

Further, fix $x(\cdot) \in \mathrm{AC}^\alpha([0,T],\mathbb{R}^n)$. Introduce the auxiliary function $v_1^*(t) \triangleq V^*_{\beta_1,\mu_1}(t,x_t(\cdot))$, $t \in [0,T]$. Due to (V^*.3), we derive $v_1^*(\cdot) \in \mathrm{Lip}([0,T],\mathbb{R})$, and, since $\beta_1 = 0$,

$$\dot{v}_1^*(t) \leq ({}^C D^\alpha q)(t) - \frac{\mu_1}{\Gamma(1-\alpha)}q(t) + \frac{\mu_1^2\Gamma(\alpha+1)}{2\Gamma(1-\alpha)}(I^\alpha q)(t) \text{ for a.e. } t \in [0,T],$$

where $q(\cdot) \triangleq q(\cdot \mid T, x(\cdot))$. Note that, in the considered case, we have $\alpha \geq 1-\alpha$. Hence, by Proposition 7.6, taking into account (V^*.2), we get

$$(I^\alpha q)(t) \leq \frac{\Gamma(1-\alpha)T^{2\alpha-1}}{\Gamma(\alpha)}(I^{1-\alpha}q)(t) \leq \frac{\Gamma(1-\alpha)T^{2\alpha-1}}{\Gamma(\alpha)}e^{\mu_1 T^\alpha}v_1^*(t) \quad \forall t \in [0,T]. \tag{7.24}$$

Therefore, by virtue of the choice of μ_1 and λ_*, we obtain

$$\dot{v}_1^*(t) \leq ({}^C D^\alpha q)(t) - \lambda q(t) + \lambda_* v_1^*(t) \text{ for a.e. } t \in [0,T].$$

Thus, for the function $v_*(t) \triangleq V_*(t,x_t(\cdot)) = e^{-\lambda_* t}v_1^*(t)$, $t \in [0,T]$, we conclude $v_*(\cdot) \in \mathrm{Lip}([0,T],\mathbb{R})$ and

$$\dot{v}_*(t) = e^{-\lambda_* t}\dot{v}_1^*(t) - \lambda_* e^{-\lambda_* t}v_1^*(t) \leq e^{-\lambda_* t}(({}^C D^\alpha q)(t) - \lambda q(t)) \text{ for a.e. } t \in [0,T].$$

From this estimate and the inequality

$$({}^C D^\alpha q)(t) \leq 2\langle x(t) - x(0), ({}^C D^\alpha x)(t)\rangle \text{ for a.e. } t \in [0, T], \tag{7.25}$$

which is valid by Corollary 4.2 in [11], we derive (7.22). Property $(V_*.2)$ is established.

Let us prove $(V_*.3)$. Arguing by contradiction, suppose that there exist a compact set $X \subset \mathrm{C}([0,T],\mathbb{R}^n)$ and a number $\rho > 0$ such that, for every $k \in \mathbb{N}$, one can choose $x^{[k]}(\cdot) \in X$ such that $V_*(T, x^{[k]}(\cdot)) \leq 1/k$ and

$$\|x^{[k]}(\cdot) - x^{[k]}(0)\|_{[0,T]} \geq \rho. \tag{7.26}$$

Owing to compactness of X, we can assume that the sequence $\{x^{[k]}(\cdot)\}_{k\in\mathbb{N}}$ converges to a function $x^{[0]}(\cdot) \in X$. Denote $q^{[k]}(\cdot) \triangleq q(\cdot \mid T, x^{[k]}(\cdot))$, $k \in \mathbb{N} \cup \{0\}$. Then, the sequence $\{q^{[k]}(\cdot)\}_{k\in\mathbb{N}} \subset \mathrm{C}([0,T],\mathbb{R})$ converges to $q^{[0]}(\cdot)$ (see, *e.g.*, the proof of $(V^*.1)$), and, therefore, in view of (2.3), we have $(I^{1-\alpha} q^{[k]})(T) \to (I^{1-\alpha} q^{[0]})(T)$ as $k \to \infty$. On the other hand, for every $k \in \mathbb{N}$, according to $(V^*.2)$, we derive

$$1/k \geq V_*(T, x^{[k]}(\cdot)) = e^{-\lambda_* T} V^*_{\beta_1,\mu_1}(T, x^{[k]}(\cdot)) \geq e^{-\lambda_* T - \mu_1 T^\alpha} (I^{1-\alpha} q^{[k]})(T) \geq 0,$$

wherefrom it follows that $(I^{1-\alpha} q^{[k]})(T) \to 0$ as $k \to \infty$, and, hence, $(I^{1-\alpha} q^{[0]})(T) = 0$. Since the function $q^{[0]}(\cdot)$ is continuous and nonnegative, this equality yields $q^{[0]}(t) = 0$ for all $t \in [0,T]$. But, passing to the limit as $k \to \infty$ in inequality (7.26), we get $\|q^{[0]}(\cdot)\|_{[0,T]} \geq \rho^2 > 0$ and obtain the contradiction, which proves $(V_*.3)$.

2. Consider the next case when $\alpha \in [1/4, 1/2)$. Define

$$\beta_1 \triangleq 0, \quad \beta_2 \triangleq \alpha, \quad \mu_1 \triangleq 2\Gamma(1-\alpha)\lambda, \quad \mu_2 \triangleq \frac{\mu_1^2 \Gamma(\alpha+1)\Gamma(1-2\alpha)}{2\Gamma(1-\alpha)}$$

and

$$\lambda_* \triangleq \frac{\mu_2^2 \Gamma(2\alpha+1)\Gamma(1-\alpha) T^{4\alpha-1}}{2\Gamma(1-2\alpha)\Gamma(3\alpha)} e^{\mu_1 T^\alpha}.$$

Note that $\beta_2 < 1 - \alpha$ since $\alpha < 1/2$. Take the functionals $V^*_{\beta_1,\mu_1}$ and $V^*_{\beta_2,\mu_2}$ from (7.15) and put

$$V_*(t, w(\cdot)) \triangleq e^{-\lambda_* t} \big(V^*_{\beta_1,\mu_1}(t, w(\cdot)) + V^*_{\beta_2,\mu_2}(t, w(\cdot))\big)/2, \quad (t, w(\cdot)) \in G_n.$$

For the specified λ_* and V_*, statement $(V_*.1)$ is verified in the same way as in the first case. The proof of $(V_*.3)$ also does not differ essentially from the arguments given above, because, due to $(V^*.2)$ and the equality $\beta_1 = 0$, for any $x(\cdot) \in \mathrm{C}([0,T],\mathbb{R}^n)$, the following estimate holds:

$$V_*(T, x(\cdot)) \geq e^{-\lambda_* T} V^*_{\beta_1,\mu_1}(T, x(\cdot))/2 \geq e^{-\lambda_* T - \mu_1 T^\alpha} (I^{1-\alpha} q)(T)/2,$$

where $q(\cdot) \triangleq q(\cdot \mid T, x(\cdot))$. Thus, it remains to prove $(V_*.2)$.

For a given $x(\cdot) \in \mathrm{AC}^\alpha([0,T],\mathbb{R}^n)$, introduce the functions $v_1^*(t) \triangleq V^*_{\beta_1,\mu_1}(t, x_t(\cdot))$, $v_2^*(t) \triangleq V^*_{\beta_2,\mu_2}(t, x_t(\cdot))$, $t \in [0,T]$. According to $(V^*.3)$, we obtain $v_1^*(\cdot)$, $v_2^*(\cdot) \in \mathrm{Lip}([0,T],\mathbb{R})$, and, taking into account the choice of β_1, β_2, μ_1, and μ_2, we derive

$$\begin{aligned} \dot v_1^*(t) + \dot v_2^*(t) &\leq 2({}^C D^\alpha q)(t) - \frac{\mu_1}{\Gamma(1-\alpha)} q(t) + \frac{\mu_1^2 \Gamma(\alpha+1)}{2\Gamma(1-\alpha)} (I^\alpha q)(t) \\ &+ ({}^C D^\alpha q)(t) - \frac{\mu_2}{\Gamma(1-2\alpha)} (I^\alpha q)(t) + \frac{\mu_2^2 \Gamma(2\alpha+1)}{2\Gamma(1-2\alpha)} (I^{3\alpha} q)(t) \end{aligned}$$

$$= 2({}^C D^\alpha q)(t) - 2\lambda q(t) + \frac{\mu_2^2 \Gamma(2\alpha+1)}{2\Gamma(1-2\alpha)} (I^{3\alpha} q)(t) \text{ for a.e. } t \in [0,T],$$

where $q(\cdot) \triangleq q(\cdot \mid T, x(\cdot))$. Since, in the second case, we have $3\alpha \geq 1-\alpha$, then Proposition 7.6 and $(V^*.2)$ yield

$$(I^{3\alpha} q)(t) \leq \frac{\Gamma(1-\alpha) T^{4\alpha-1}}{\Gamma(3\alpha)} (I^{1-\alpha} q)(t) \leq \frac{\Gamma(1-\alpha) T^{4\alpha-1}}{\Gamma(3\alpha)} e^{\mu_1 T^\alpha} v_1^*(t) \quad \forall t \in [0,T].$$

Therefore, owing to the choice of λ_*, we conclude

$$\dot{v}_1^*(t) + \dot{v}_2^*(t) \leq 2({}^C D^\alpha q)(t) - 2\lambda q(t) + \lambda_* v_1^*(t) \text{ for a.e. } t \in [0,T].$$

Thus, for the function $v_*(t) \triangleq V_*(t, x_t(\cdot)) = e^{-\lambda_* t}(v_1^*(t) + v_2^*(t))/2$, $t \in [0,T]$, we get $v_*(\cdot) \in \mathrm{Lip}([0,T],\mathbb{R})$ and

$$\dot{v}_*(t) = e^{-\lambda_* t}(\dot{v}_1^*(t) + \dot{v}_2^*(t))/2 - \lambda_* e^{-\lambda_* t}(v_1^*(t) + v_2^*(t))/2 \leq e^{-\lambda_* t}(({}^C D^\alpha q)(t) - \lambda q(t)) \text{ for a.e. } t \in [0,T].$$

This estimate and (7.25) imply (7.22). The proof of $(V_*.3)$ is completed.

3. In the general case, choose $m \in \mathbb{N}$ such that $\alpha \in [2^{-m}, 2^{-(m-1)})$. The cases $m=1$ and $m=2$ were considered above, so we can assume that $m > 2$. Put

$$\beta_i \triangleq (2^{i-1} - 1)\alpha, \quad i \in \overline{1,m}.$$

Note that $\beta_1 = 0$, and, due to the choice of m,

$$0 \leq \beta_i \leq (2^{m-1} - 1)\alpha < (2^{m-1} - 1)2^{-(m-1)} = 1 - 2^{-(m-1)} < 1 - \alpha \quad \forall i \in \overline{1,m}.$$

Further, define numbers $\mu_i > 0$, $i \in \overline{1,m}$, by the following recurrent relations:

$$\mu_1 \triangleq m\Gamma(1-\alpha)\lambda, \quad \mu_{i+1} \triangleq \frac{\mu_i^2 \Gamma(\alpha+\beta_i+1)\Gamma(1-\alpha-\beta_{i+1})}{2\Gamma(1-\alpha-\beta_i)}, \quad i \in \overline{1,m-1},$$

and set

$$\lambda_* \triangleq \frac{\mu_m^2 \Gamma(\alpha+\beta_m+1)\Gamma(1-\alpha) T^{2\alpha+2\beta_m-1}}{2\Gamma(1-\alpha-\beta_m)\Gamma(\alpha+2\beta_m)} e^{\mu_1 T^\alpha}.$$

Finally, take the corresponding functionals $V^*_{\beta_i,\mu_i}$, $i \in \overline{1,m}$, from (7.15) and put

$$V_*(t, w(\cdot)) \triangleq \frac{e^{-\lambda_* t}}{m} \sum_{i=1}^m V^*_{\beta_i,\mu_i}(t, w(\cdot)), \quad (t, w(\cdot)) \in G_n. \tag{7.27}$$

The proofs of properties $(V_*.1)$ and $(V_*.3)$ for the specified λ_* and V_* are carried out by the same scheme as in the two particular cases considered above, and, therefore, they are omitted. Let us prove $(V_*.2)$.

Let $x(\cdot) \in \mathrm{AC}^\alpha([0,T],\mathbb{R}^n)$ and $q(\cdot) \triangleq q(\cdot \mid T, x(\cdot))$. By virtue of $(V^*.3)$, for every $i \in \overline{1,m}$, the function $v_i^*(t) \triangleq V^*_{\beta_i,\mu_i}(t, x_t(\cdot))$, $t \in [0,T]$, satisfies the inclusion $v_i^*(\cdot) \in \mathrm{Lip}([0,T],\mathbb{R})$, and, taking into account the choice of β_1 and μ_1, we derive

$$\sum_{i=1}^m \dot{v}_i^*(t) \leq m({}^C D^\alpha q)(t) - m\lambda q(t)$$

$$-\sum_{i=2}^{m}\frac{\mu_i}{\Gamma(1-\alpha-\beta_i)}(I^{\beta_i}q)(t)+\sum_{i=1}^{m}\frac{\mu_i^2\Gamma(\alpha+\beta_i+1)}{2\Gamma(1-\alpha-\beta_i)}(I^{\alpha+2\beta_i}q)(t) \text{ for a.e. } t\in[0,T]. \tag{7.28}$$

Let $t\in[0,T]$ be fixed. For any $i\in\overline{1,m-1}$, due to the choice of β_i, β_{i+1}, and μ_{i+1}, we have

$$\frac{\mu_i^2\Gamma(\alpha+\beta_i+1)}{2\Gamma(1-\alpha-\beta_i)}(I^{\alpha+2\beta_i}q)(t)=\frac{\mu_i^2\Gamma(\alpha+\beta_i+1)}{2\Gamma(1-\alpha-\beta_i)}(I^{\beta_{i+1}}q)(t)=\frac{\mu_{i+1}}{\Gamma(1-\alpha-\beta_{i+1})}(I^{\beta_{i+1}}q)(t),$$

and, consequently, for the last two terms in (7.28), we get

$$\sum_{i=1}^{m}\frac{\mu_i^2\Gamma(\alpha+\beta_i+1)}{2\Gamma(1-\alpha-\beta_i)}(I^{\alpha+2\beta_i}q)(t)-\sum_{i=2}^{m}\frac{\mu_i}{\Gamma(1-\alpha-\beta_i)}(I^{\beta_i}q)(t)=\frac{\mu_m^2\Gamma(\alpha+\beta_m+1)}{2\Gamma(1-\alpha-\beta_m)}(I^{\alpha+2\beta_m}q)(t). \tag{7.29}$$

Further, owing to the choice of β_m and the inequality $\alpha\geq 2^{-m}$, we obtain

$$\alpha+2\beta_m=(2^m-1)\alpha\geq(2^m-1)2^{-m}=1-2^{-m}\geq 1-\alpha.$$

Hence, according to Proposition 7.6 and $(V^*.2)$, since $\beta_1=0$, we conclude

$$(I^{\alpha+2\beta_m}q)(t)\leq\frac{\Gamma(1-\alpha)T^{2\alpha+2\beta_m-1}}{\Gamma(\alpha+2\beta_m)}(I^{1-\alpha}q)(t)\leq\frac{\Gamma(1-\alpha)T^{2\alpha+2\beta_m-1}}{\Gamma(\alpha+2\beta_m)}e^{\mu_1T^\alpha}v_1^*(t),$$

and, then, in view of the choice of λ_*, we derive

$$\frac{\mu_m^2\Gamma(\alpha+\beta_m+1)}{2\Gamma(1-\alpha-\beta_m)}(I^{\alpha+2\beta_m}q)(t)\leq\lambda_*v_1^*(t). \tag{7.30}$$

Thus, it follows from (7.28), (7.29), and (7.30) that

$$\sum_{i=1}^{m}\dot v_i^*(t)\leq m({}^CD^\alpha q)(t)-m\lambda q(t)+\lambda_*v_1^*(t) \text{ for a.e. } t\in[0,T].$$

Therefore, we get that the function

$$v_*(t)\triangleq V_*(t,x_t(\cdot))=\frac{e^{-\lambda_*t}}{m}\sum_{i=0}^{m}v_i^*(t),\quad t\in[0,T],$$

satisfies the inclusion $v_*(\cdot)\in\mathrm{Lip}([0,T],\mathbb{R})$, and

$$\dot v_*(t)=\frac{e^{-\lambda_*t}}{m}\sum_{i=0}^{m}\dot v_i^*(t)-\frac{\lambda_*e^{-\lambda_*t}}{m}\sum_{i=0}^{m}v_i^*(t)\leq e^{-\lambda_*t}(({}^CD^\alpha q)(t)-\lambda q(t)) \text{ for a.e. } t\in[0,T].$$

This estimate and (7.25) yield (7.22), which completes the proof of $(V_*.3)$. The lemma is proved. □

7.1.4. Functional $\mathcal{V}_\varepsilon$

Below, following the scheme from Section 7.5 of [48] and based on Lemma 7.4, for every sufficiently small $\varepsilon>0$, we define a functional $\mathcal{V}_\varepsilon: G_n\to\mathbb{R}$ with a number of prescribed properties, which are close to those listed in Section 5 of [33] (see also [17], Sect. 4.1). In this connection, see assumptions $(H.4)'$ in [6], $(A.4)$ in Section 9.2 of [48], and $(F.3)$ in [34].

Let $R > 0$ be fixed, and let λ_H be chosen by R according to assumption $(H.3)$. Set $\lambda \triangleq 4\lambda_H$ and take the corresponding number λ_* and functional V_* from Lemma 7.4. Choose $\varepsilon_0 > 0$ such that $\varepsilon_0 \leq 2e^{-(\lambda_H+\lambda_*/2)T}$. For any $\varepsilon \in (0, \varepsilon_0]$, define

$$G_n \ni (t, w(\cdot)) \mapsto \mathcal{V}_\varepsilon(t, w(\cdot)) \triangleq \frac{e^{-\lambda_H t}}{\varepsilon} \sqrt{\varepsilon^4 + V_*(t, w(\cdot))} \in \mathbb{R}$$

and consider also the auxiliary functionals $p_\varepsilon : G_n \to \mathbb{R}$ and $s_\varepsilon : G_n \to \mathbb{R}^n$ given by

$$p_\varepsilon(t, w(\cdot)) \triangleq -\frac{\lambda_H e^{-\lambda_H t}}{\varepsilon} \sqrt{\varepsilon^4 + V_*(t, w(\cdot))} - \frac{2\lambda_H e^{-(\lambda_H+\lambda_*)t}}{\varepsilon} \frac{\|w(t) - w(0)\|^2}{\sqrt{\varepsilon^4 + V_*(t, w(\cdot))}},$$
$$s_\varepsilon(t, w(\cdot)) \triangleq \frac{e^{-(\lambda_H+\lambda_*)t}}{\varepsilon} \frac{w(t) - w(0)}{\sqrt{\varepsilon^4 + V_*(t, w(\cdot))}}, \quad (t, w(\cdot)) \in G_n.$$

Lemma 7.7. *For every $R > 0$ and $\varepsilon \in (0, \varepsilon_0]$, the following statements hold:*

($\mathcal{V}$.1) *The functional $\mathcal{V}_\varepsilon$ is nonnegative and continuous. In addition, if $(t, w(\cdot)) \in G_n$ and $w(\tau) = w(0)$ for all $\tau \in [0, t]$, then $\mathcal{V}_\varepsilon(t, w(\cdot)) \leq \varepsilon$.*

($\mathcal{V}$.2) *The functionals p_ε and s_ε are continuous. Furthermore, for any $(t, w(\cdot))$, $(t, w'(\cdot)) \in G_n$ such that $\|w(t)\| \leq R$, $\|w'(t)\| \leq R$, and $w(0) = w'(0)$, the inequality below is valid:*

$$p_\varepsilon(t, \Delta w(\cdot)) + H\big(t, w'(t), s_\varepsilon(t, \Delta w(\cdot))\big) - H\big(t, w(t), s_\varepsilon(t, \Delta w(\cdot))\big) \leq 0, \tag{7.31}$$

where $\Delta w(\cdot) \triangleq w'(\cdot) - w(\cdot)$.

($\mathcal{V}$.3) *For every function $x(\cdot) \in \mathrm{AC}^\alpha([0, T], \mathbb{R}^n)$, the function $v(t) \triangleq \mathcal{V}_\varepsilon(t, x_t(\cdot))$, $t \in [0, T]$, satisfies the inclusion $v(\cdot) \in \mathrm{Lip}([0, T], \mathbb{R})$, and*

$$\dot{v}(t) \leq p_\varepsilon(t, x_t(\cdot)) + \langle s_\varepsilon(t, x_t(\cdot)), ({}^C D^\alpha x)(t)\rangle \text{ for a.e. } t \in [0, T]. \tag{7.32}$$

($\mathcal{V}$.4) *For any compact set $X \subset \mathrm{AC}^\alpha([0, T], \mathbb{R}^n)$ and any numbers $K > 0$ and $\kappa > 0$, there exists $\varepsilon_* \in (0, \varepsilon_0]$ such that, if $\varepsilon \in (0, \varepsilon_*]$ and $x(\cdot)$, $x'(\cdot) \in X$ satisfy the relations $x(0) = x'(0)$ and $\mathcal{V}_\varepsilon(T, x'(\cdot) - x(\cdot)) \leq K$, then $|\sigma(x'(\cdot)) - \sigma(x(\cdot))| \leq \kappa$.*

Remark 7.8. If it were shown that, for some $\varepsilon \in (0, \varepsilon_0]$, the functional $\mathcal{V}_\varepsilon$ is *ci*-smooth of the order α (see Sect. 4), then it would follow from Proposition 4.1 that, for every function $x(\cdot) \in \mathrm{AC}^\alpha([0, T], \mathbb{R}^n)$, the function $v(t) \triangleq \mathcal{V}_\varepsilon(t, x_t(\cdot))$, $t \in [0, T]$, would satisfy the equality

$$\dot{v}(t) = \partial_t^\alpha \mathcal{V}_\varepsilon(t, x_t(\cdot)) + \langle \nabla^\alpha \mathcal{V}_\varepsilon(t, x_t(\cdot)), ({}^C D^\alpha x)(t)\rangle \text{ for a.e. } t \in [0, T].$$

Comparing this equality with estimate (7.32), we see that, in some sense, the functionals p_ε and s_ε correspond to the derivatives $\partial_t^\alpha \mathcal{V}_\varepsilon$ and $\nabla^\alpha \mathcal{V}_\varepsilon$, respectively. Hence, in this case, statement ($\mathcal{V}$.2) could be considered as the requirement for the functionals $\partial_t^\alpha \mathcal{V}_\varepsilon$ and $\nabla^\alpha \mathcal{V}_\varepsilon$, which is consistent with the item (d) in Section 5 of [33] (see also [17], Sect. 4.1). However, for the results of the paper to be valid, *ci*-smoothness of the order α of the functional $\mathcal{V}_\varepsilon$ is not necessary, and we only need to establish the properties given in Lemma 7.7.

Proof of Lemma 7.7. 1. Taking into account that the functional V_* is nonnegative and continuous by $(V_*.1)$, we obtain that the functional $\mathcal{V}_\varepsilon$ is nonnegative and continuous, too. Now, let $(t, w(\cdot)) \in G_n$ be such that $w(\tau) = w(0)$ for all $\tau \in [0, t]$. Then, in view of $(V_*.1)$, we derive $\mathcal{V}_\varepsilon(t, w(\cdot)) = e^{-\lambda_H t}\varepsilon \leq \varepsilon$. Thus, property ($\mathcal{V}$.1) is established.

2. Note that continuity of the functionals V_* and $G_n \ni (t, w(\cdot)) \mapsto w(t) - w(0) \in \mathbb{R}^n$ imply also continuity of the functionals p_ε and s_ε. Further, fix $(t, w(\cdot))$, $(t, w'(\cdot)) \in G_n$ such that $\|w(t)\| \leq R$, $\|w'(t)\| \leq R$, and $w(0) = w'(0)$ and denote $\Delta w(\cdot) \triangleq w'(\cdot) - w(\cdot)$. By the choice of λ_H, we have

$$\begin{aligned} &p_\varepsilon(t, \Delta w(\cdot)) + H\big(t, w'(t), s_\varepsilon(t, \Delta w(\cdot))\big) - H\big(t, w(t), s_\varepsilon(t, \Delta w(\cdot))\big) \\ &\quad \leq p_\varepsilon(t, \Delta w(\cdot)) + \lambda_H\big(1 + \|s_\varepsilon(t, \Delta w(\cdot))\|\big)\|\Delta w(t)\| \\ &\quad = -\frac{\lambda_H e^{-\lambda_H t}}{\varepsilon}\sqrt{\varepsilon^4 + V_*(t, \Delta w(\cdot))}\Big(1 - \varepsilon e^{\lambda_H t}\frac{\|\Delta w(t)\|}{\sqrt{\varepsilon^4 + V_*(t, \Delta w(\cdot))}} + e^{-\lambda_* t}\frac{\|\Delta w(t)\|^2}{\varepsilon^4 + V_*(t, \Delta w(\cdot))}\Big). \end{aligned} \tag{7.33}$$

Due to the choice of ε_0, we get $\varepsilon e^{\lambda_H t} \leq \varepsilon_0 e^{\lambda_H T} \leq 2e^{-\lambda_* t/2}$, and, therefore,

$$1 - \varepsilon e^{\lambda_H t}\frac{\|\Delta w(t)\|}{\sqrt{\varepsilon^4 + V_*(t, \Delta w(\cdot))}} + e^{-\lambda_* t}\frac{\|\Delta w(t)\|^2}{\varepsilon^4 + V_*(t, \Delta w(\cdot))} \geq \Big(1 - e^{-\lambda_* t/2}\frac{\|\Delta w(t)\|}{\sqrt{\varepsilon^4 + V_*(t, \Delta w(\cdot))}}\Big)^2 \geq 0. \tag{7.34}$$

From (7.33) and (7.34), we derive (7.31).

3. Now, let $x(\cdot) \in \mathrm{AC}^\alpha([0, T], \mathbb{R}^n)$ and $v(t) \triangleq \mathcal{V}_\varepsilon(t, x_t(\cdot))$, $t \in [0, T]$. Then, it follows from $(V_*.2)$ that $v(\cdot) \in \mathrm{Lip}([0, T], \mathbb{R})$ and, by virtue of the choice of λ,

$$\begin{aligned} \dot{v}(t) &= -\frac{\lambda_H e^{-\lambda_H t}}{\varepsilon}\sqrt{\varepsilon^4 + V_*(t, x_t(\cdot))} + \frac{e^{-\lambda_H t}}{2\varepsilon\sqrt{\varepsilon^4 + V_*(t, x_t(\cdot))}}\frac{\mathrm{d}}{\mathrm{d}t}V_*(t, x_t(\cdot)) \\ &\leq -\frac{\lambda_H e^{-\lambda_H t}}{\varepsilon}\sqrt{\varepsilon^4 + V_*(t, x_t(\cdot))} + \frac{e^{-(\lambda_H + \lambda_*)t}}{\varepsilon\sqrt{\varepsilon^4 + V_*(t, x_t(\cdot))}}\langle x(t) - x(0), ({}^C D^\alpha x)(t)\rangle \\ &\quad - \frac{2\lambda_H e^{-(\lambda_H + \lambda_*)t}}{\varepsilon\sqrt{\varepsilon^4 + V_*(t, x_t(\cdot))}}\|x(t) - x(0)\|^2 \text{ for a.e. } t \in [0, T]. \end{aligned}$$

This estimate, in accordance with the definitions of the functionals p_ε and s_ε, yields (7.32).

4. Finally, let us prove $(\mathcal{V}.4)$. Let $X \subset \mathrm{AC}^\alpha([0, T], \mathbb{R}^n)$ be a compact set, and let $K > 0$ and $\kappa > 0$. Taking into account that the functional σ is continuous by assumption (σ), choose $\rho > 0$ such that, for any $x(\cdot)$, $x'(\cdot) \in X$, the inequality $\|x'(\cdot) - x(\cdot)\|_{[0,T]} \leq \rho$ implies the estimate $|\sigma(x'(\cdot)) - \sigma(x(\cdot))| \leq \kappa$. Consider the set

$$\Delta X \triangleq \big\{\Delta x(\cdot) \triangleq x'(\cdot) - x(\cdot) : \, x(\cdot), x'(\cdot) \in X\big\} \subset \mathrm{AC}^\alpha([0, T], \mathbb{R}^n).$$

Since ΔX is compact, based on $(V_*.3)$, take $\delta > 0$ such that, for every $\Delta x(\cdot) \in \Delta X$, it follows from the inequality $V_*(T, \Delta x(\cdot)) \leq \delta$ that $\|\Delta x(\cdot) - \Delta x(0)\|_{[0,T]} \leq \rho$. Now, choose $\varepsilon_* \in (0, \varepsilon_0]$ from the condition $K^2 e^{2\lambda_H T}\varepsilon_*^2 \leq \delta$. Let us show that statement $(\mathcal{V}.4)$ is valid for the specified ε_*.

Let $\varepsilon \in (0, \varepsilon_*]$ and $x(\cdot)$, $x'(\cdot) \in X$ be fixed such that the function $\Delta x(\cdot) \triangleq x'(\cdot) - x(\cdot) \in \Delta X$ satisfies the relations $\Delta x(0) = 0$ and $\mathcal{V}_\varepsilon(T, \Delta x(\cdot)) \leq K$. Then, we derive

$$V_*(T, \Delta x(\cdot)) \leq \varepsilon^4 + V_*(T, \Delta x(\cdot)) = \varepsilon^2 e^{2\lambda_H T}\mathcal{V}_\varepsilon^2(T, \Delta x(\cdot)) \leq \delta,$$

and, therefore, we have $\|\Delta x(\cdot)\|_{[0,T]} \leq \rho$, wherefrom we obtain $|\sigma(x'(\cdot)) - \sigma(x(\cdot))| \leq \kappa$. The lemma is proved. □

7.2. Proof of Theorem 7.1

1. Fix $(t_0, w_0(\cdot)) \in G_n^\alpha$. If $t_0 = T$, then inequality (7.1) for $(t_0, w_0(\cdot))$ holds due to boundary conditions (5.7) for φ_+ and (5.8) for φ_-. So, we further assume that $t_0 < T$. Put

$$X_*^\alpha(t_0, w_0(\cdot)) \triangleq \big\{x(\cdot) \in X^\alpha(t_0, w_0(\cdot)) : \, \|({}^C D^\alpha x)(t)\| \leq c_H(1 + \|x(t)\|) \text{ for a.e. } t \in [t_0, T]\big\},$$

where the set $X^\alpha(t_0, w_0(\cdot))$ is given by (4.1), and c_H is the constant from assumption $(H.2)$. Owing to Proposition 3.1, the set $X^\alpha_*(t_0, w_0(\cdot))$ is compact. In particular, there exists $R > 0$ such that $\|x(\cdot)\|_{[0,T]} \leq R$ for any $x(\cdot) \in X^\alpha(t_0, w_0(\cdot))$. By this number R, define the number $\varepsilon_0 > 0$ and the functionals $\mathcal{V}_\varepsilon$, p_ε, and s_ε for every $\varepsilon \in (0, \varepsilon_0]$ according to Section 7.1.4.

2. Let $\varepsilon \in (0, \varepsilon_0]$. For any $t \in [0, T]$ and $\bar{w}(\cdot) \triangleq (w(\cdot), w'(\cdot)) \in \mathrm{AC}^\alpha([0, t], \mathbb{R}^n \times \mathbb{R}^n)$, consider the set

$$\begin{aligned}\mathcal{F}_\varepsilon&(t, w(\cdot), w'(\cdot))\\ &\triangleq \Big\{(f, f', h) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} : \ \|f\| \leq c_H(1 + \|w(t)\|), \ \|f'\| \leq c_H(1 + \|w'(t)\|),\\ &\qquad \big|h - \langle s_\varepsilon(t, \Delta w(\cdot)), f - f'\rangle - H\big(t, w'(t), s_\varepsilon(t, \Delta w(\cdot))\big) + H\big(t, w(t), s_\varepsilon(t, \Delta w(\cdot))\big)\big| \leq \varepsilon\Big\}, \end{aligned} \tag{7.35}$$

where we denote $\Delta w(\cdot) \triangleq w'(\cdot) - w(\cdot)$. Thus, in accordance with notation (2.9), we obtain the set-valued functional $G^\alpha_{2n} \ni (t, \bar{w}(\cdot) \triangleq (w(\cdot), w'(\cdot))) \mapsto \mathcal{F}_\varepsilon(t, w(\cdot), w'(\cdot)) \subset \mathbb{R}^{2n} \times \mathbb{R}$. Note that $\mathcal{F}_\varepsilon$ possesses properties $(\mathcal{F}.1)$–$(\mathcal{F}.3)$. Indeed, $(\mathcal{F}.1)$ and $(\mathcal{F}.3)$ can be verified directly, and $(\mathcal{F}.2)$ follows from continuity of the function H (see $(H.1)$) and the functional s_ε (see $(\mathcal{V}.2)$). Further, take $z_0 \triangleq \varphi_+(t_0, w_0(\cdot)) - \varphi_-(t_0, w_0(\cdot))$ and consider the Cauchy problem for the functional differential inclusion

$$\big((^C D^\alpha x)(t), (^C D^\alpha x')(t), \dot{z}(t)\big) \in \mathcal{F}_\varepsilon(t, x_t(\cdot), x'_t(\cdot)), \tag{7.36}$$

where $(x(t), x'(t), z(t)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $t \in [t_0, T]$, under the initial condition

$$x(t) = x'(t) = w_0(t), \quad z(t) = z_0 \quad \forall t \in [0, t_0]. \tag{7.37}$$

By Proposition 3.4, the set $\mathcal{W}_\varepsilon$ of solutions $(x(\cdot), x'(\cdot), z(\cdot))$ of problem (7.36), (7.37) is nonempty and compact in $\mathrm{C}([0, T], \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R})$.

3. Let us show that there are functions $(x^{[\varepsilon]}(\cdot), x'^{[\varepsilon]}(\cdot), z^{[\varepsilon]}(\cdot)) \in \mathcal{W}_\varepsilon$ satisfying the inequality

$$z^{[\varepsilon]}(T) \geq \varphi_+(T, x^{[\varepsilon]}(\cdot)) - \varphi_-(T, x'^{[\varepsilon]}(\cdot)). \tag{7.38}$$

For every $t \in [t_0, T]$, consider the set

$$\mathcal{M}_\varepsilon(t) \triangleq \big\{(x(\cdot), x'(\cdot), z(\cdot)) \in \mathcal{W}_\varepsilon : \ z(t) \geq \varphi_+(t, x_t(\cdot)) - \varphi_-(t, x'_t(\cdot))\big\}. \tag{7.39}$$

Note that $\mathcal{M}_\varepsilon(t_0) \neq \emptyset$ by virtue of initial condition (7.37) and the choice of z_0. Put

$$t_\varepsilon \triangleq \max\big\{t \in [t_0, T] : \ \mathcal{M}_\varepsilon(t) \neq \emptyset\big\}. \tag{7.40}$$

The maximum is achieved here owing to compactness of $\mathcal{W}_\varepsilon$, lower semicontinuity of φ_+, and upper semicontinuity of φ_-. So, in order to complete the proof, it is sufficient to verify that $t_\varepsilon = T$.

Arguing by contradiction, assume that $t_\varepsilon < T$. Take

$$(\hat{x}(\cdot), \hat{x}'(\cdot), \hat{z}(\cdot)) \in \mathcal{M}_\varepsilon(t_\varepsilon) \tag{7.41}$$

and denote $\hat{s} \triangleq s_\varepsilon(t_\varepsilon, \hat{x}'_{t_\varepsilon}(\cdot) - \hat{x}_{t_\varepsilon}(\cdot))$. Since φ_+ and φ_- possess respectively properties (φ^*_+) and (φ^*_-), choose characteristics $(x^+(\cdot), z^+(\cdot)) \in CH(t_\varepsilon, \hat{x}_{t_\varepsilon}(\cdot), 0, \hat{s})$ and $(x^-(\cdot), z^-(\cdot)) \in CH(t_\varepsilon, \hat{x}'_{t_\varepsilon}(\cdot), 0, \hat{s})$ such that

$$\varphi_+(t, x^+_t(\cdot)) - z^+(t) \leq \varphi_+(t_\varepsilon, \hat{x}_{t_\varepsilon}(\cdot)), \quad \varphi_-(t, x^-_t(\cdot)) - z^-(t) \geq \varphi_-(t_\varepsilon, \hat{x}'_{t_\varepsilon}(\cdot)) \quad \forall t \in [t_\varepsilon, T]. \tag{7.42}$$

In accordance with (5.4) and (7.35), continuity of H and s_ε implies that there exists $\delta \in (0, T - t_\varepsilon)$ such that $(({}^C D^\alpha x^+)(t), ({}^C D^\alpha x^-)(t), \dot{z}^+(t) - \dot{z}^-(t)) \in \mathcal{F}_\varepsilon(t, x_t^+(\cdot), x_t^-(\cdot))$ for a.e. $t \in [t_\varepsilon, t_\varepsilon + \delta]$. Then, in view of (7.39), (7.41), and (7.42), for the function $\bar{z}(t) \triangleq \hat{z}(t_\varepsilon) + z^+(t) - z^-(t)$, $t \in [t_\varepsilon, t_\varepsilon + \delta]$, we get

$$\begin{aligned}\bar{z}(t_\varepsilon + \delta) &\geq \varphi_+(t_\varepsilon, \hat{x}_{t_\varepsilon}(\cdot)) - \varphi_-(t_\varepsilon, \hat{x}'_{t_\varepsilon}(\cdot)) + z^+(t_\varepsilon + \delta) - z^-(t_\varepsilon + \delta)\\ &\geq \varphi_+(t_\varepsilon + \delta, x^+_{t_\varepsilon + \delta}(\cdot)) - \varphi_-(t_\varepsilon + \delta, x^-_{t_\varepsilon + \delta}(\cdot)).\end{aligned}$$

Further, let functions $\tilde{x}^\pm(\cdot) \in X^\alpha(t_\varepsilon + \delta, x^\pm_{t_\varepsilon + \delta}(\cdot))$ be such that $({}^C D^\alpha \tilde{x}^\pm)(t) = 0$ for a.e. $t \in [t_\varepsilon + \delta, T]$ (in this connection, see, *e.g.*, [15], Lem. 3). Denote $\tilde{s}(t) \triangleq s_\varepsilon(t, \tilde{x}_t^-(\cdot) - \tilde{x}_t^+(\cdot))$, $t \in [t_\varepsilon + \delta, T]$, and consider the function $\tilde{z} : [0, T] \to \mathbb{R}$ defined by $\tilde{z}(t) \triangleq \hat{z}(t)$ for $t \in [0, t_\varepsilon]$, $\tilde{z}(t) \triangleq \bar{z}(t)$ for $t \in (t_\varepsilon, t_\varepsilon + \delta]$, and

$$\tilde{z}(t) \triangleq \bar{z}(t_\varepsilon + \delta) + \int_{t_\varepsilon + \delta}^t \big(H(\tau, \tilde{x}^-(\tau), \tilde{s}(\tau)) - H(\tau, \tilde{x}^+(\tau), \tilde{s}(\tau))\big)\, \mathrm{d}\tau, \quad t \in (t_\varepsilon + \delta, T].$$

Hence, by construction, we obtain $(\tilde{x}^+(\cdot), \tilde{x}^-(\cdot), \tilde{z}(\cdot)) \in \mathcal{M}_\varepsilon(t_\varepsilon + \delta)$, which contradicts definition (7.40) of t_ε.

4. Note that, for every $\varepsilon \in (0, \varepsilon_0]$, it follows from the inclusion $(x^{[\varepsilon]}(\cdot), x'^{[\varepsilon]}(\cdot), z^{[\varepsilon]}(\cdot)) \in \mathcal{W}_\varepsilon$ that $x^{[\varepsilon]}(\cdot)$, $x'^{[\varepsilon]}(\cdot) \in X_*^\alpha(t_0, w_0(\cdot))$, $z^{[\varepsilon]}(\cdot) \in \mathrm{Lip}([0, T], \mathbb{R})$, $z(t_0) = \varphi_+(t_0, w_0(\cdot)) - \varphi_-(t_0, w_0(\cdot))$, and

$$\begin{aligned}\dot{z}^{[\varepsilon]}(t) &\leq \langle s_\varepsilon(t, \Delta x_t^{[\varepsilon]}(\cdot)), ({}^C D^\alpha x^{[\varepsilon]})(t) - ({}^C D^\alpha x'^{[\varepsilon]})(t)\rangle\\ &\quad + H\big(t, x'^{[\varepsilon]}(t), s_\varepsilon(t, \Delta x_t^{[\varepsilon]}(\cdot)\big) - H\big(t, x^{[\varepsilon]}(t), s_\varepsilon(t, \Delta x_t^{[\varepsilon]}(\cdot))\big) + \varepsilon \text{ for a.e. } t \in [t_0, T],\end{aligned}$$

where $\Delta x^{[\varepsilon]}(\cdot) \triangleq x'^{[\varepsilon]}(\cdot) - x^{[\varepsilon]}(\cdot)$. Hence, for the function $v(t) \triangleq \mathcal{V}_\varepsilon(t, \Delta x_t^{[\varepsilon]}(\cdot)) + z^{[\varepsilon]}(t) - \varepsilon(t - t_0)$, $t \in [t_0, T]$, due to ($\mathcal{V}$.1), we have

$$v(t_0) = \mathcal{V}_\varepsilon(t_0, w_0(\cdot) - w_0(\cdot)) + z^{[\varepsilon]}(t_0) \leq \varepsilon + \varphi_+(t_0, w_0(\cdot)) - \varphi_-(t_0, w_0(\cdot)),$$

and, in accordance with ($\mathcal{V}$.3), we obtain $v(\cdot) \in \mathrm{Lip}([0, T], \mathbb{R})$ and

$$\begin{aligned}\dot{v}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{V}_\varepsilon(t, \Delta x_t^{[\varepsilon]}(\cdot)) + \dot{z}^{[\varepsilon]}(t) - \varepsilon\\ &\leq p_\varepsilon(t, \Delta x_t^{[\varepsilon]}(\cdot)) + H\big(t, x'^{[\varepsilon]}(t), s_\varepsilon(t, \Delta x_t^{[\varepsilon]}(\cdot))\big) - H\big(t, x^{[\varepsilon]}(t), s_\varepsilon(t, \Delta x_t^{[\varepsilon]}(\cdot))\big) \text{ for a.e. } t \in [t_0, T],\end{aligned}$$

wherefrom, by virtue of ($\mathcal{V}$.2) and the choice of R, we derive $\dot{v}(t) \leq 0$ for a.e. $t \in [t_0, T]$. Thus, we conclude

$$\mathcal{V}_\varepsilon(T, \Delta x^{[\varepsilon]}(\cdot)) + z^{[\varepsilon]}(T) - \varepsilon(T - t_0) = v(T) \leq v(t_0) \leq \varepsilon + \varphi_+(t_0, w_0(\cdot)) - \varphi_-(t_0, w_0(\cdot)).$$

Since (7.38) and boundary conditions (5.7) for φ_+ and (5.8) for φ_- imply that $z^{[\varepsilon]}(T) \geq \sigma(x^{[\varepsilon]}(\cdot)) - \sigma(x'^{[\varepsilon]}(\cdot))$, we finally get the estimate

$$\mathcal{V}_\varepsilon(T, \Delta x^{[\varepsilon]}(\cdot)) + \sigma(x^{[\varepsilon]}(\cdot)) - \sigma(x'^{[\varepsilon]}(\cdot)) \leq \varepsilon(1 + T - t_0) + \varphi_+(t_0, w_0(\cdot)) - \varphi_-(t_0, w_0(\cdot)), \tag{7.43}$$

which is valid for every $\varepsilon \in (0, \varepsilon_0]$.

In view of compactness of $X_*^\alpha(t_0, w_0(\cdot))$, take $K > 0$ such that $|\sigma(x^{[\varepsilon]}(\cdot)) - \sigma(x'^{[\varepsilon]}(\cdot))| \leq K$ for every $\varepsilon \in (0, \varepsilon_0]$. Then, due to (7.43), we have

$$\mathcal{V}_\varepsilon(T, \Delta x^{[\varepsilon]}(\cdot)) \leq K + \varepsilon_0(1 + T - t_0) + \varphi_+(t_0, w_0(\cdot)) - \varphi_-(t_0, w_0(\cdot)) \quad \forall \varepsilon \in (0, \varepsilon_0],$$

and, therefore, applying ($\mathcal{V}$.4), we obtain $\sigma(x^{[\varepsilon]}(\cdot)) - \sigma(x'^{[\varepsilon]}(\cdot)) \to 0$ as $\varepsilon \to 0^+$. Further, since the functionals $\mathcal{V}_\varepsilon$, $\varepsilon \in (0, \varepsilon_0]$, are nonnegative (see ($\mathcal{V}$.1)), it also follows from (7.43) that

$$\sigma(x^{[\varepsilon]}(\cdot)) - \sigma(x'^{[\varepsilon]}(\cdot)) \leq \varepsilon(1 + T - t_0) + \varphi_+(t_0, w_0(\cdot)) - \varphi_-(t_0, w_0(\cdot)) \quad \forall \varepsilon \in (0, \varepsilon_0].$$

Passing to the limit as $\varepsilon \to 0^+$ in this estimate, we derive inequality (7.1) for $(t_0, w_0(\cdot))$. The theorem is proved.

7.3. Discussion

As can be seen, properties ($\mathcal{V}$.1)–($\mathcal{V}$.4) of the functionals $\mathcal{V}_\varepsilon$, p_ε, and s_ε, established in Lemma 7.7, allow us to prove Theorem 7.1 by following along the same lines as in the case of Hamilton–Jacobi equations with partial derivatives (see, *e.g.*, [48], Thm. 7.3) and with first-order *ci*-derivatives (see, *e.g.*, [31], Lem. 7.7). The fact that Cauchy problem (5.1), (5.2) involves fractional-order *ci*-derivatives is reflected in property ($\mathcal{V}$.3), since relation (7.32) contains the term $\langle s_\varepsilon(t, x_t(\cdot)), ({}^C D^\alpha x)(t)\rangle$. Further, note that properties ($\mathcal{V}$.1)–($\mathcal{V}$.4) are in turn a consequence of the definition of the functional $\mathcal{V}_\varepsilon$, which is close in form to the definitions of the functionals proposed earlier in [17, 31, 36] for the case of first-order *ci*-derivatives, and properties (V_*.1)–(V_*.3) of the auxiliary functional V_* from Lemma 7.4. Thus, the construction of the functional V_* is one of the main features of the presented proof of Theorem 7.1. The purpose of this section is to make some additional remarks concerning this construction.

First of all, observe that an important constituent part of the Lyapunov–Krasovskii functionals used in [17, 31, 36] is the functional

$$V(t, w(\cdot)) \triangleq \|w(t)\|^2, \quad (t, w(\cdot)) \in G_n.$$

This functional clearly has the following property: for every function $x(\cdot) \in \mathrm{AC}^1([0,T], \mathbb{R}^n) = \mathrm{Lip}([0,T], \mathbb{R}^n)$, the function $v(t) \triangleq V(t, x_t(\cdot)) = \|x(t)\|^2$, $t \in [0,T]$, satisfies the inclusion $v(\cdot) \in \mathrm{Lip}([0,T], \mathbb{R})$, and

$$\dot{v}(t) = 2\langle x(t), \dot{x}(t)\rangle \text{ for a.e. } t \in [0,T].$$

However, in the case when $\alpha \in (0,1)$, a function $x(\cdot) \in \mathrm{AC}^\alpha([0,T], \mathbb{R}^n)$ is not necessarily Lipschitz continuous (it is only Hölder continuous with exponent α, see (2.3)), and, furthermore, it can happen that, at all points $t \in [0,T]$, the function $x(\cdot)$ is not first-order differentiable (see, *e.g.*, [44]). Consequently, we cannot expect the corresponding function $v(t) \triangleq V(t, x_t(\cdot)) = \|x(t)\|^2$, $t \in [0,T]$, to have any first-order differentiability properties. This fact does not allow us to directly use the functionals from [17, 31, 36] in the present paper.

A possible way to overcome this difficulty is to modify the functional V as follows (see Ex. 4.3):

$$V_1(t, w(\cdot)) \triangleq \big\|\big(I^{1-\alpha}(w(\cdot) - w(0))\big)(t)\big\|^2 = \Big\|\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{w(\tau) - w(0)}{(t-\tau)^\alpha}\, \mathrm{d}\tau\Big\|^2, \quad (t, w(\cdot)) \in G_n.$$

Indeed (see Sect. 2), if $x(\cdot) \in \mathrm{AC}^\alpha([0,T], \mathbb{R}^n)$ and $v_1(t) \triangleq V_1(t, x_t(\cdot))$, $t \in [0,T]$, then $v_1(\cdot) \in \mathrm{Lip}([0,T], \mathbb{R})$ and

$$\dot{v}_1(t) = 2\Big\langle \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x(\tau) - x(0)}{(t-\tau)^\alpha}\, \mathrm{d}\tau, ({}^C D^\alpha x)(t)\Big\rangle \text{ for a.e. } t \in [0,T].$$

Note also that the functional V_1 (more precisely, its restriction to G_n^α) is *ci*-smooth of the order α. Nevertheless, since, for a given $(t, w(\cdot)) \in G_n$, the value $V_1(t, w(\cdot))$ depends on the function $w(\cdot)$ only *via* the value of the integral $(I^{1-\alpha}(w(\cdot) - w(0)))(t)$, it can happen that $V_1(t, w(\cdot)) = 0$ while $\|w(\cdot) - w(0)\|_{[0,t]} \neq 0$. This circumstance implies that the proposed modification V_1 of the functional V is not entirely suitable for taking it as a basis for constructing the desired functional V_*.

For this reason, in accordance with [11] (see also the references therein), we deal with the another adaptation of the functional V to the fractional setting:

$$V_2(t, w(\cdot)) \triangleq \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\|w(\tau) - w(0)\|^2}{(t-\tau)^\alpha} \, \mathrm{d}\tau, \quad (t, w(\cdot)) \in G_n.$$

Namely, by virtue of Corollary 4.2 in [11], we have that, for any function $x(\cdot) \in \mathrm{AC}^\alpha([0,T], \mathbb{R}^n)$, the function $v_2(t) \triangleq V_2(t, x_t(\cdot))$, $t \in [0,T]$, satisfies the inclusion $v_2(\cdot) \in \mathrm{Lip}([0,T], \mathbb{R})$, and

$$\dot{v}_2(t) \leq 2\langle x(t) - x(0), ({}^C D^\alpha x)(t) \rangle \text{ for a.e. } t \in [0,T], \tag{7.44}$$

which in some sense corresponds to the property of the functional V in the case $\alpha = 1$.

Further, analyzing property (V_*.2) in more detail, let us pay attention to the term $-\lambda\|x(t) - x(0)\|^2$ in inequality (7.22). Let us emphasize that the presence of this term is reflected directly in the definition of the functional p_ε and allows us to obtain property ($\mathcal{V}$.2), establishing the relationship between the functionals s_ε and p_ε and the Hamiltonian H. In view of (7.44), when using only the functional V_2 itself, the required term does not appear, and, therefore, it is necessary to perform some transformations of the functional V_2. In this regard, we can introduce, for example, the functional

$$V_3(t, w(\cdot)) \triangleq e^{-\lambda t} V_2(t, w(\cdot)) = \frac{e^{-\lambda t}}{\Gamma(1-\alpha)} \int_0^t \frac{\|w(\tau) - w(0)\|^2}{(t-\tau)^\alpha} \, \mathrm{d}\tau, \quad (t, w(\cdot)) \in G_n.$$

Then, for the function $v_3(t) \triangleq V_3(t, x_t(\cdot))$, $t \in [0,T]$, where $x(\cdot) \in \mathrm{AC}^\alpha([0,T], \mathbb{R}^n)$, we derive the estimate

$$\dot{v}_3(t) \leq e^{-\lambda t} \Big(2\langle x(t) - x(0), ({}^C D^\alpha x)(t) \rangle - \frac{\lambda}{\Gamma(1-\alpha)} \int_0^t \frac{\|x(\tau) - x(0)\|^2}{(t-\tau)^\alpha} \, \mathrm{d}\tau \Big) \text{ for a.e. } t \in [0,T]. \tag{7.45}$$

However, it can be shown that there is no constant $a > 0$ such that

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\|w(\tau) - w(0)\|^2}{(t-\tau)^\alpha} \, \mathrm{d}\tau \geq a\|w(t) - w(0)\|^2 \quad \forall (t, w(\cdot)) \in G_n,$$

and, therefore, the transformation V_3 of the functional V_2 does not lead to the desired result. As another attempt, we can try to take the functional

$$\begin{aligned} V_4(t, w(\cdot)) &\triangleq V_2(t, w(\cdot)) - \lambda \int_0^t \|w(\tau) - w(0)\|^2 \, \mathrm{d}\tau \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\|w(\tau) - w(0)\|^2}{(t-\tau)^\alpha} \, \mathrm{d}\tau - \lambda \int_0^t \|w(\tau) - w(0)\|^2 \, \mathrm{d}\tau, \quad (t, w(\cdot)) \in G_n. \end{aligned}$$

In this case, for the function $v_4(t) \triangleq V_4(t, x_t(\cdot))$, $t \in [0,T]$, where $x(\cdot) \in \mathrm{AC}^\alpha([0,T], \mathbb{R}^n)$, we get

$$\dot{v}_4(t) \leq 2\langle x(t) - x(0), ({}^C D^\alpha x)(t) \rangle - \lambda\|x(t) - x(0)\|^2 \text{ for a.e. } t \in [0,T], \tag{7.46}$$

and, thus, the term $-\lambda\|x(t) - x(0)\|^2$ does appear. Nevertheless, it can happen that the value $V_4(t, w(\cdot))$ is negative for some $(t, w(\cdot)) \in G_n$ (in this connection, see also Proposition 7.6), which entails difficulties with the other two properties (V_*.1) and (V_*.3).

At this stage, we come to the functionals proposed in Sections 7.1.2 and 7.1.3. Let us have a closer look to them in the context of the present section. In the particular case $\alpha \in [1/2, 1)$, according to (7.23), we have

$V_*(t, w(\cdot)) \triangleq e^{-\lambda_* t} V^*_{0,\mu_1}(t, w(\cdot))$, $(t, w(\cdot)) \in G_n$, where the functional V^*_{0,μ_1} is defined by (7.17). We see that the functional V^*_{0,μ_1} is obtained from the functional V_2 by multiplying the integrand by the function $e^{-\mu_1 (t-\tau)^\alpha}$, $\tau \in [0, t]$. Moreover, observe that, due to (7.7), the following representation holds (recall that $\mu_1 = \Gamma(1-\alpha)\lambda$):

$$\begin{aligned} V^*_{0,\mu_1}(t, w(\cdot)) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\|w(\tau) - w(0)\|^2}{(t-\tau)^\alpha} \, \mathrm{d}\tau - \lambda \int_0^t \|w(\tau) - w(0)\|^2 \, \mathrm{d}\tau \\ &\quad + \lambda\alpha \int_0^t \frac{\|w(\tau) - w(0)\|^2}{(t-\tau)^\alpha} \int_0^{t-\tau} \frac{1 - e^{-\mu_1 \xi^\alpha}}{\xi^{1-\alpha}} \, \mathrm{d}\xi \, \mathrm{d}\tau \\ &= V_4(t, w(\cdot)) + \lambda\alpha \int_0^t \frac{\|w(\tau) - w(0)\|^2}{(t-\tau)^\alpha} \int_0^{t-\tau} \frac{1 - e^{-\mu_1 \xi^\alpha}}{\xi^{1-\alpha}} \, \mathrm{d}\xi \, \mathrm{d}\tau \quad \forall (t, w(\cdot)) \in G_n, \end{aligned} \tag{7.47}$$

and, hence, the functional V^*_{0,μ_1} can be treated as a regularization of the functional V_4, which, in particular, overcomes the difficulties with properties $(V_*.1)$ and $(V_*.3)$. On the other hand, in order to get now property $(V_*.2)$, we need to take into account the new additional term in (7.47). To this end, owing to Proposition 7.6, it suffices to multiply the functional V^*_{0,μ_1} by the function $e^{-\lambda_* t}$, $t \in [0, T]$, for the suitably chosen λ_*.

However, in the remaining case $\alpha \in (0, 1/2)$, it can be shown that there is no constant $b > 0$ such that

$$\frac{1}{\Gamma(\alpha)} \int_0^t \frac{\|w(\tau) - w(0)\|^2}{(t-\tau)^{1-\alpha}} \, \mathrm{d}\tau \le \frac{b}{\Gamma(1-\alpha)} \int_0^t \frac{\|w(\tau) - w(0)\|^2}{(t-\tau)^\alpha} \, \mathrm{d}\tau \quad \forall (t, w(\cdot)) \in G_n, \tag{7.48}$$

which does not allow us to obtain an estimate of type (7.24) and, consequently, to deal with the additional term in (7.47) similarly to above. For this reason, the functionals $V^*_{\beta,\mu}$ are introduced (see (7.16)), which are more complex modifications of the functional V_2, and, finally, according to (7.27), the functional V_* is constructed on the basis of the finite sum of the functionals $V^*_{\beta_i,\mu_i}$. All details can be found in the proof of Lemma 7.4.

8. Existence and uniqueness

The main result of the paper is the following theorem, which is valid under assumptions $(H.1)$–$(H.3)$ and (σ) from Section 5.1.

Theorem 8.1. *There exists a unique minimax solution of problem* (5.1)*,* (5.2)*.*

Proof. Taking into account Theorem 7.1, in order to prove the statement, it is sufficient to show that there exist an upper solution $\varphi_+^\circ : G_n^\alpha \to \mathbb{R}$ and a lower solution $\varphi_-^\circ : G_n^\alpha \to \mathbb{R}$ of problem (5.1), (5.2) such that $\varphi_+^\circ(t, w(\cdot)) \le \varphi_-^\circ(t, w(\cdot))$ for every $(t, w(\cdot)) \in G_n^\alpha$. Construction of such functionals φ_+° and φ_-° repeats essentially the arguments given in Section 7 of [31] and follows the scheme from Theorem 8.2 of [48] (see also Sect. 5 of [3] and [42], Thm. 1). The basis of this construction are the properties of the set-valued function E from (5.4) and the sets of characteristics, which are provided by Propositions 3.1, 3.2, and 3.3. For the reader's convenience, we briefly outline the main steps of the proof below.

Let Φ_+ be the set of functionals $\varphi : G_n^\alpha \to \mathbb{R}$ that satisfy boundary condition (5.7) and possess property (φ_+). Respectively, by Φ_-, we denote the set of functionals $\varphi : G_n^\alpha \to \mathbb{R}$ such that (5.8) and (φ_-) are valid.

1. For a given $s \in \mathbb{R}^n$, consider the functionals $\psi_+^{[s]} : G_n^\alpha \to \mathbb{R}$ and $\psi_-^{[s]} : G_n^\alpha \to \mathbb{R}$ defined by

$$\psi_+^{[s]}(t, w(\cdot)) \triangleq \max_{(x(\cdot), z(\cdot)) \in CH(t, w(\cdot), 0, s)} \big(\sigma(x(\cdot)) - z(T)\big), \quad \psi_-^{[s]}(t, w(\cdot)) \triangleq \min_{(x(\cdot), z(\cdot)) \in CH(t, w(\cdot), 0, s)} \big(\sigma(x(\cdot)) - z(T)\big),$$

where $(t, w(\cdot)) \in G_n^\alpha$. The functionals $\psi_+^{[s]}$ and $\psi_-^{[s]}$ are respectively upper and lower semicontinuous, and

$$\psi_+^{[s]}(T, w(\cdot)) = \psi_-^{[s]}(T, w(\cdot)) = \sigma(w(\cdot)) \quad \forall w(\cdot) \in \mathrm{AC}^\alpha([0, T], \mathbb{R}^n).$$

Further, the inclusion $\psi_+^{[s]} \in \Phi_+$ holds, and, in particular, the set Φ_+ is not empty. In addition, for every functional $\varphi \in \Phi_+$, the inequality below is valid:

$$\varphi(t, w(\cdot)) \geq \psi_-^{[s]}(t, w(\cdot)) \quad \forall (t, w(\cdot)) \in G_n^\alpha.$$

2. Put

$$\varphi^\circ(t, w(\cdot)) \triangleq \inf \big\{ \varphi(t, w(\cdot)) : \varphi \in \Phi_+ \big\}, \quad (t, w(\cdot)) \in G_n^\alpha.$$

For any $s \in \mathbb{R}^n$, we have

$$\psi_-^{[s]}(t, w(\cdot)) \leq \varphi^\circ(t, w(\cdot)) \leq \psi_+^{[s]}(t, w(\cdot)) \quad \forall (t, w(\cdot)) \in G_n^\alpha,$$

and, hence, $\varphi^\circ(T, w(\cdot)) = \sigma(w(\cdot))$ for all $w(\cdot) \in \mathrm{AC}^\alpha([0, T], \mathbb{R}^n)$. Moreover, the functional $\varphi^\circ : G_n^\alpha \to \mathbb{R}$ possesses property (φ_+), and, consequently, we obtain $\varphi^\circ \in \Phi_+$.

3. For every $\vartheta \in [0, T]$ and $s \in \mathbb{R}^n$, the functional $\varphi^{[\vartheta, s]} : G_n^\alpha \to \mathbb{R}$ given by

$$\varphi^{[\vartheta, s]}(t, w(\cdot)) \triangleq \begin{cases} \sup\limits_{(x(\cdot), z(\cdot)) \in CH(t, w(\cdot), 0, s)} \big(\varphi^\circ(\vartheta, x_\vartheta(\cdot)) - z(\vartheta) \big), & \text{if } t \in [0, \vartheta), \\ \varphi^\circ(t, w(\cdot)), & \text{if } t \in [\vartheta, T], \end{cases} \qquad (t, w(\cdot)) \in G_n^\alpha,$$

satisfies the inclusion $\varphi^{[\vartheta, s]} \in \Phi_+$. Based on this fact, we derive that $\varphi^\circ \in \Phi_-$.

4. Finally, we define the required functionals φ_+° and φ_-° as respectively the lower and upper closures of the functional φ°:

$$\varphi_+^\circ(t, w(\cdot)) \triangleq \lim_{\delta \to 0^+} \inf_{(t', w'(\cdot)) \in O_\delta(t, w(\cdot))} \varphi^\circ(t', w'(\cdot)), \quad \varphi_-^\circ(t, w(\cdot)) \triangleq \lim_{\delta \to 0^+} \sup_{(t', w'(\cdot)) \in O_\delta(t, w(\cdot))} \varphi^\circ(t', w'(\cdot)),$$

where $(t, w(\cdot)) \in G_n^\alpha$ and

$$O_\delta(t, w(\cdot)) \triangleq \big\{ (t', w'(\cdot)) \in G_n^\alpha : \operatorname{dist}\big((t, w(\cdot)), (t', w'(\cdot)) \big) \leq \delta \big\}.$$

Then, φ_+° is an upper solution of problem (5.1), (5.2), and φ_-° is a lower solution of this problem. Moreover, by construction, we obtain $\varphi_+^\circ(t, w(\cdot)) \leq \varphi_-^\circ(t, w(\cdot))$ for all $(t, w(\cdot)) \in G_n^\alpha$. The theorem is proved. $\square$

9. Conclusion

In the paper, a Cauchy problem for a Hamilton–Jacobi equation with *ci*-derivatives of an order $\alpha \in (0, 1)$ has been considered. A notion of a generalized in the minimax sense solution of this problem has been proposed. It has been proved that a minimax solution exists, is unique, and is consistent with the classical solution of the problem. A special attention has been given to construction of a suitable Lyapunov–Krasovskii functional needed for the proof of a comparison principle.

Possible directions for further research in this area include but are not limited to the following:

(i) establish a relation between the value functional in an optimal control problem for a dynamical system described by differential equations with the Caputo fractional derivatives and the minimax solution of the associated Hamilton–Jacobi equation; obtain such results for differential games;

(ii) find an infinitesimal criteria for the minimax solution in terms of suitable directional derivatives (see, *e.g.*, Sect. 6.3 of [48] and also [17, 31, 34, 39]);

(iii) develop the theory of generalized in the viscosity sense (see, *e.g.*, [7] and also [35, 36]) solutions of the Cauchy problem considered in the paper.

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