


ON THE GRAPHON MEAN FIELD GAME EQUATIONS: INDIVIDUAL AGENT AFFINE DYNAMICS AND MEAN FIELD DEPENDENT PERFORMANCE FUNCTIONS*

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Abstract. This paper establishes unique solvability of a class of Graphon Mean Field Game equations. The special case of a constant graphon yields the result for the Mean Field Game equations.

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1. INTRODUCTION

Mean Field Game (MFG) theory establishes Nash equilibrium conditions for large populations of asymptotically negligible non-cooperating agents *via* an analysis of the infinite limit population (Huang *et al.* [9–11]; Lasry and Lions [15]). The resulting PDEs (Partial Differential Equations) consist of a backward Hamilton-Jacobi-Bellman (HJB) equation and a forward Fokker-Planck-Kolmogorov (FPK) equation for each generic agent. These equations are linked by the state distribution of a generic agent which is called the mean field of the system.

The basic structure of standard MFG theory assumes a symmetry in the connections of the agents but not necessarily of their dynamics. However, in the recent studies [1–3] asymmetric graph connections in large population games are considered. Large subpopulations (or clusters) of agents are placed at their particular nodes and communicate with the neighbouring subpopulations *via* the graph edges. The graphs are heterogeneous with the edges having not necessarily identical weights. In the network limit, a graphon gives the communication weights $g(\alpha, \beta)$, see for instance the introductions to each of [1–3, 7] for the Graphon MFG (GMFG) framework and [16] for Graphon theory. Along with [1–3], this paper proposes a new type of MFG PDE system associated to the Graphon Mean Field Game system. Our goal here is to establish the unique solvability of the GMFG equation in an appropriate function space.

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The GMFG equations consist of a collection of parameterized Hamilton-Jacobi-Bellman equations, $HJB(\alpha)$, $\alpha \in [0, 1]$, and a collection of parameterized Fokker-Planck-Kolmogorov equations, $FPK(\alpha)$ with $\alpha \in [0, 1]$. The solution of a set of GMFG equations is a parameterized pair (v, μ) , where $v[\alpha] = v(t, \alpha, x)$ solves the $HJB(\alpha)$ equation and $\mu[\alpha] = \mu(t, \alpha, x)$ solves the $FPK(\alpha)$ equation. The coupling of the system PDEs in this paper has the following features (see [3] for a more general framework subject to different hypotheses):

- $FPK(\alpha)$ depends upon $HJB(\alpha)$ through its first order coefficient ∇v .
- $HJB(\alpha)$ depends upon $FPK(\alpha')$ for all $\alpha' \in [0, 1]$ through the graphon g acting on $\mu[\alpha']$; this is the major difference from MFG.

The GMFG equations with a constant graphon reduce to the classical MFG system as a special case, and the original methods to establish solvability of the classical MFG equations are helpful in the present case. In [10, 17], a Banach fixed point analysis is used depending on a contraction argument; this is based on assumptions on the Lipschitz continuity of the functions appearing in the MFG equations and their derivatives, and yields uniqueness as well as existence. This approach is used in the parallel study [3] of the solvability of the GMFG equations. On the other hand, [4, 18] carry out the existence analysis using the Schauder fixed point theorem based upon regularity assumptions and then obtain uniqueness *via* a monotonicity assumption on the running cost.

In this work, similar to the aforementioned analyses, we will establish the existence of solutions *via* the application of a fixed point theorem. Our existence proof adopts Schauder's argument on the fixed point theorem and is more closely relevant to [4, 8, 18] in this sense. Unlike [8] on the solvability in Sobolev space, our solvability is to answer the existence in Hölder space along the lines of [4, 18]. Nevertheless, different from all aforementioned papers, our proof on the continuity of the gradient of the value function with respect to the coefficient functions relies on probabilistic estimates rather than the theory of viscosity solutions. The main advantage of our approach is that we can conclude the local Lipschitz continuity of the solution map, which is stronger than continuity and beneficial to the subsequent analysis of the GMFG.

Having said that, the major difficulty generalizing existence from the MFG case to the GMFG case is to obtain the regularity of the solution with respect to the variable α , which is essential for the existence result by Schauder's fixed point theorem. To be more illustrative, for instance, to obtain a uniform first order estimate of $|\nabla v(t, \alpha, x) - \nabla v(t, \alpha', x)|$ for the solution v of the HJB equation, one has to compare the solutions from two different HJBs parameterized by α and α' . This leads to a study of the sensitivity with respect to coefficient functions of corresponding PDEs. Therefore, the local Lipschitz continuity of the HJB solution map becomes essential for this procedure.

The paper is organized as follows. Section 2 gives the problem set up. Section 3 presents the regularity of parabolic PDE and applies this to the FPK. Section 4 presents the existence result and Section 5 treats uniqueness. Section 6 presents a summary and extensions of the main result. For better clarity, all notations used in this paper have been collected and explained in the Appendix A.

2. PROBLEM SETUP

Let \mathbb{T}^d be a d -torus. $\mathcal{P}_1(\mathbb{T}^d)$ is the Wasserstein space of probability measures on \mathbb{T}^d satisfying

$$\int_{\mathbb{T}^d} |x| d\mu(x) < \infty$$

endowed with 1-Wasserstein metric $d_1(\cdot, \cdot)$ defined by

$$d_1(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{T}^d \times \mathbb{T}^d} |x - y| d\pi(x, y),$$

where $\Pi(\mu, \nu)$ is the collection of all probability measures on $\mathbb{T}^d \times \mathbb{T}^d$ with its marginals agreeing with μ and ν .

We consider the following large system of multi-agent problems. A generic agent can be identified by its state pair $(\alpha, x) \in [0, 1] \times \mathbb{T}^d$, where α is the cluster index and x is a \mathbb{T}^d valued state. The weights of connections between clusters are given by a symmetric measurable function $g : [0, 1]^2 \mapsto \mathbb{R}$, which is commonly referred to a graphon [16]. The population density at the cluster α at time t will be given by $\mu(t, \alpha) \in \mathcal{P}_1(\mathbb{T}^d)$.

Example. Two examples of graphons are given in the following discussion, while the reader is referred to [16] for the fundamental theory of this subject. A uniform graphon which corresponds to the limit of a sequence of Erdős-Rényi graphs with parameter p , $0 \leq p \leq 1$, is given by

$$g(\alpha, \alpha') = p, \quad \forall \alpha, \alpha' \in [0, 1] \quad (2.1)$$

and the uniform attachment graph limit has the graphon

$$g(\alpha, \alpha') = 1 - \max\{\alpha, \alpha'\}, \quad \forall \alpha, \alpha' \in [0, 1]. \quad (2.2)$$

□

A running cost incurred to the generic agent of (α, x) with a feedback control exertion $\mathbf{a} : [0, T] \times [0, 1] \times \mathbb{T}^d \mapsto \mathbb{R}^d$ at time t is given by

$$\ell(\mu, g, \mathbf{a}, t, \alpha, x) = \frac{1}{2} |\mathbf{a}(t, \alpha, x)|^2 + \ell_1(\mu, g, t, \alpha, x) \quad (2.3)$$

for some given function $\ell_1(\cdot, \cdot, \cdot, \cdot, \cdot)$. The following cost can be considered as an example for ℓ_1

$$\ell_1(\mu, g, t, \alpha, x) = \int_0^1 \int_{\mathbb{T}^d} \ell_2(x, y) \mu(t, \alpha', dy) g(\alpha, \alpha') \, d\alpha' \quad (2.4)$$

for some $\ell_2 : \mathbb{T}^d \times \mathbb{T}^d \mapsto \mathbb{R}$.

Let $b : [0, T] \times [0, 1] \times \mathbb{T}^d \mapsto \mathbb{R}$ and $m_0 : [0, 1] \times \mathbb{T}^d \mapsto \mathbb{R}^+$ be two given smooth enough functions. By ∇b , we denote the gradient of b on the domain \mathbb{T}^d , which is mapping $[0, T] \times [0, 1] \times \mathbb{T}^d \mapsto \mathbb{R}^d$. Finding a solution of the GMFG equations consists of solving for the unknown triples (v, \mathbf{a}^*, μ) :

- the value function $v : [0, T] \times [0, 1] \times \mathbb{T}^d \mapsto \mathbb{R}$,
- optimal control $\mathbf{a}^* : [0, T] \times [0, 1] \times \mathbb{T}^d \mapsto \mathbb{R}^d$,
- and the density $\mu : [0, T] \times [0, 1] \times \mathbb{T}^d \mapsto \mathbb{R}^+$,

satisfying the α parameterized family

$$\begin{cases} \partial_t v + (\nabla b + \mathbf{a}^*) \cdot \nabla v + \frac{1}{2} \Delta v + \ell(\mu, g, \mathbf{a}^*) = 0 \\ \mathbf{a}^*(t, \alpha, x) = \arg \min_{a \in \mathbb{R}^d} \{a \cdot \nabla v(t, \alpha, x) + \frac{1}{2} |a|^2\} \\ \partial_t \mu = -\operatorname{div}_x((\nabla b + \mathbf{a}^*)\mu) + \frac{1}{2} \Delta \mu \\ v(T, \alpha, x) = 0, \quad \mu(0, \alpha, x) = m_0(\alpha, x). \end{cases} \quad (2.5)$$

In the first and third equation of (2.5), each term is a function of (t, α, x) without further specification. In particular, the $\ell(\mu, g, \mathbf{a}^*)$ shall be understood as a mapping

$$(t, \alpha, x) \mapsto \ell(\mu, g, \mathbf{a}^*)(t, \alpha, x) := \ell(\mu, g, \mathbf{a}^*, t, \alpha, x).$$

Our goal in this paper is to establish existence, uniqueness for the solution of (2.5) in an appropriate solution space. We close this section with a brief illustration of the probabilistic formulation on the GMFG for the motivational purpose. A generic player in GMFG is identified by a pair $(\alpha, x) \in [0, 1] \times \mathbb{T}^d$, where α is geographical information and x is a state. The population density at index α at time t is denoted by $\mu(t, \alpha) \in \mathcal{P}_1(\mathbb{T}^d)$ and the relation between two generic players in α and α' is given by a graphon $g(\alpha, \alpha')$. Given a population density $(t, \alpha) \mapsto \mu(t, \alpha)$ and a graphon $(\alpha, \alpha') \mapsto g(\alpha, \alpha')$, a generic player exerts its optimal strategy of the following stochastic control problem described below. State evolution of the generic player at α follows a controlled stochastic differential equation:

$$X_t^\alpha = X_0^\alpha + \int_0^t (\nabla b(s, \alpha, X_s^\alpha) + \mathbf{a}(s, \alpha, X_s^\alpha)) ds + W_t^\alpha, \quad (2.6)$$

where the drift is formed by a control process \mathbf{a} and a conservative vector field ∇b , W^α is a Brownian motion in a filtered probability space independent to W^β for any $\beta \neq \alpha$, and X_0^α is an initial random variable with a given distribution $m_0(\alpha)$. In the above, the left hand side is understood as the coset of \mathbb{Z}^d that contains the right hand side by a mapping $\pi(x) = x + \mathbb{Z}^d$. We use $X^\alpha[\mathbf{a}]$ to denote the process with the dependence on \mathbf{a} . The objective of the generic player at α with a given population density flow μ is to minimize the total cost incurred during $[0, T]$ of the form

$$J^\alpha(\mathbf{a}, \mu) = \mathbb{E} \left[\int_0^T \ell(\mu, g, \mathbf{a}, t, \alpha, X_t^\alpha[\mathbf{a}]) dt \right]$$

over a reasonably rich enough control space of \mathbf{a} . Note that the optimal strategy \mathbf{a}^* depends on μ . Given an initial distribution m_0 , the goal of the GMFG is to find the Nash equilibrium μ^* and the corresponding \mathbf{a}^* , *i.e.* the pair (μ^*, \mathbf{a}^*) satisfies

$$J^\alpha(\mathbf{a}^*, \mu^*) \leq J^\alpha(\mathbf{a}, \mu^*), \forall \mathbf{a} \text{ and } \mu^*(t, \alpha) \sim X_t^\alpha[\mathbf{a}^*], \forall (t, \alpha).$$

Indeed, the above formulation poses a class of mean field game problems indexed by $\alpha \in [0, 1]$ and couplings between mean field games are imposed by the running cost ℓ *via* graphon g . For more detailed discussion and various applications are referred to [1–3].

3. SOME REGULARITY RESULTS

We are going to present sensitivity results of the parabolic PDE and FPK equations with respect to their coefficients separately, which eventually serve for the proof of fixed point theorem as key elements. Throughout the paper, we will use $\Psi(\cdot)$ in various places as a generic positive function increasing with respect to its variables. Moreover, all function spaces and relevant norms are sorted out in Appendix A.

3.1. Parabolic equations

Consider the equation

$$\begin{cases} \partial_t u = \frac{1}{2} \Delta u - cu + f, & \text{on } (0, T) \times \mathbb{T}^d \\ u(0, x) = 0, & \text{on } x \in \mathbb{T}^d. \end{cases} \quad (3.1)$$

We will denote the solution map by $u = u[c, f]$ whenever it is necessary to emphasize its dependence on the coefficient functions.

3.1.1. Preliminaries on solvability

If the coefficients c and f are Hölder in both variables (t, x) , then there exists a unique classical solution. Recall that $\Psi(\cdot)$ is a generic function mentioned in the first paragraph of Section 3.

Lemma 3.1. *If $c, f \in C^{\delta/2, \delta}([0, T] \times \mathbb{T}^d)$ holds for some $\delta \in (0, 1)$, then there exists unique solution $u \in C^{1+\delta/2, 2+\delta}([0, T] \times \mathbb{T}^d)$ of (3.1) satisfying*

$$|u|_{1+\delta/2, 2+\delta} \leq \Psi(|c|_{\delta/2, \delta}, |f|_{\delta/2, \delta}).$$

Moreover, $v(t, x) := u(T - t, x)$ has a probabilistic representation $v[c, f]$ of the form

$$v(t, x) = v[c, f](t, x) := \mathbb{E} \left[\int_t^T \exp \left\{ - \int_t^s c(r, X^{t,x}(r)) dr \right\} f(s, X^{t,x}(s)) ds \right], \quad (3.2)$$

where

$$X^{t,x}(s) = x + W(s) - W(t) \quad (3.3)$$

for some Brownian motion W .

Proof. The solvability and its Hölder estimate is from Theorem 8.7.2 and Theorem 8.7.3 of [13], Theorem IV.5.1 of [14]. The probabilistic representation $v[c, f]$ is from Feynman-Kac formula, see [6]. \square

In the above, we remark that, (3.3) reads by $X^{t,x}(s) = \pi(x + W(s) - W(t))$, where π is the generic mapping $\mathbb{R}^d \mapsto \mathbb{R}^d / \mathbb{Z}^d$. Later we also need to use the following definition of weak solution, see [5].

Definition 3.2. A function $u \in L^2([0, T], H^1(\mathbb{T}^d))$ is a weak solution of (3.1) if u satisfies

$$\begin{cases} \int_{\mathbb{T}^d} \phi(-\partial_t u - cu + f) dx = \frac{1}{2} \int_{\mathbb{T}^d} \nabla \phi \cdot \nabla u dx, \quad \forall \phi \in H^1(\mathbb{T}^d) \\ u(0, x) = 0, \quad \text{on } x \in \mathbb{T}^d. \end{cases} \quad (3.4)$$

We have the following uniqueness with the same assumptions as in Lemma 3.1.

Lemma 3.3. *If $c, f \in C^{\delta/2, \delta}([0, T] \times \mathbb{T}^d)$ holds for some $\delta \in (0, 1)$, then there exists unique weak solution of (3.1) in $L^2([0, T], H^1(\mathbb{T}^d))$.*

Proof. By Lemma 3.1, there exists a classical solution u . Together with the compactness of the domain, it yields $u \in L^2([0, T], H^1(\mathbb{T}^d))$. By Theorem 7.4 of [5], uniqueness in $L^2([0, T], H^1(\mathbb{T}^d))$ holds if $c \in L^\infty$ and $f \in L^2$, and this is valid, since all coefficients are continuous on the compact domain. \square

3.1.2. First order regularity and sensitivity of the solution map

Although Lemma 3.1 has an estimation on $|u|_{1,2}$, it is controlled by an upper bound relevant to the Hölder norm of coefficients in the t variable, which is not desirable, see Section 4.5 for further remarks. Next, we will develop an upper bound independent of t -Hölder norm of the coefficients. To proceed, we define a linear operator

$$Lu = \partial_t u - \frac{1}{2} \Delta u. \quad (3.5)$$

The first result is on an estimate of $|u|_0 = \sup_{[0, T] \times \mathbb{T}^d} |u(t, x)|$.

Lemma 3.4. *If $c, f \in C^{\delta/2, \delta}([0, T] \times \mathbb{T}^d)$, then u of (3.1) satisfies $|u|_0 \leq e^{|c|_0 T} |f|_0 T$.*

Proof. If $c = 0$, then with $u_1 = |f|_0 t$,

$$Lu_1 - f = |f|_0 - f \geq 0.$$

If $c \neq 0$, then with $u_2 = \frac{|f|_0(e^{|c|_0 t} - 1)}{|c|_0}$,

$$\begin{aligned} (L + c)u_2 &= |f|_0 e^{|c|_0 t} \left(1 + \frac{c}{|c|_0}\right) - \frac{c}{|c|_0} |f|_0 \\ &= |f|_0 (e^{|c|_0 t} - 1) \left(1 + \frac{c}{|c|_0}\right) + |f|_0 \\ &\geq f. \end{aligned}$$

Note that both u_1 and u_2 are no greater than $e^{|c|_0 T} |f|_0 T$, and finally the comparison principle yields the result. \square

Next we will have the first order estimate independent to the Hölder norm in t of the coefficients. It also gives sensitivity of the solution map with respect to the coefficients.

Lemma 3.5. *Let c, f be in $C^{\delta,1}([0, T] \times \mathbb{T}^d)$ for some $\delta \in (0, 1)$. Then the solution u of (3.1) belongs to $C^{1,2}([0, T] \times \mathbb{T}^d)$ with*

$$|u|_{0,1} \leq \Psi(|c|_{0,1} + |f|_{0,1}).$$

Furthermore, the solution map $u = u[c, f]$ satisfies

$$|u[c_1, f_1] - u[c_2, f_2]|_0 \leq \Psi(K)(|c_1 - c_2|_0 + |f_1 - f_2|_0)$$

for $K := |c_1|_0 + |c_2|_0 + |f_1|_0 + |f_2|_0$.

Proof. u of (3.1) can be written by $u(t, x) = v[c, f](T - t, x)$ with its probabilistic representation of (3.2). By setting $X^i := X^{t, x_i}$ of (3.3), we have

$$X_s^1 - X_s^2 = x_1 - x_2, \quad \forall s \geq t.$$

If we define

$$\Lambda_s^i = e^{-\int_t^s c(r, X^i(r)) dr},$$

then

$$v[c, f](t, x_i) = \mathbb{E} \left[\int_t^T \Lambda_s^i f(s, X^i(s)) ds \right].$$

We first note that, by mean value theorem

$$\left| \int_t^s c(r, X^1(r)) dr - \int_t^s c(r, X^2(r)) dr \right| \leq T |c|_{0,1} |x_1 - x_2|.$$

Once again by mean value theorem and the fact of $|\int_t^s c(r, X^i(r))dr| \leq T|c|_0$, we obtain

$$\begin{aligned} |\Lambda_s^1 - \Lambda_s^2| &\leq e^{T|c|_0} \left| \int_t^s c(r, X^1(r))dr - \int_t^s c(r, X^2(r))dr \right| \\ &\leq \Psi(|c|_{0,1})|x_1 - x_2| \end{aligned}$$

with probability one for $\Psi = T|c|_{0,1}e^{T|c|_0}$. Therefore, we have

$$\begin{aligned} |v[c, f](t, x_1) - v[c, f](t, x_2)| &\leq \mathbb{E} \int_t^T |\Lambda_s^1 f(s, X^1(s)) - \Lambda_s^2 f(s, X^2(s))| ds \\ &\leq \mathbb{E} \int_t^T (|\Lambda^1|_0 |f(s, X^1(s)) - f(s, X^2(s))| + |f|_0 |\Lambda_s^1 - \Lambda_s^2|) ds \\ &\leq \mathbb{E} \int_t^T \Psi(|c|_0) |\nabla f|_0 |X^1(s) - X^2(s)| ds + \mathbb{E} \int_t^T |f|_0 |\Lambda_s^1 - \Lambda_s^2| ds \\ &\leq T\Psi(|c|_0) |\nabla f|_0 |x_1 - x_2| + T|f|_0 \Psi(|c|_{0,1}) |x_1 - x_2| \\ &\leq \Psi(|c|_{0,1} + |f|_{0,1}) |x_1 - x_2| \end{aligned}$$

This implies that $|\nabla v|_0 \leq \Psi(|c|_{0,1} + |f|_{0,1})$. Together with Lemma 3.4, we conclude that

$$|u|_{0,1} \leq \Psi(|c|_{0,1} + |f|_{0,1}). \quad (3.6)$$

Next, we estimate $|u[c, f_1] - u[c, f_2]|_0$. For any (t, x) , we set $\Lambda_s = e^{-\int_t^s c(r, X(r))dr}$, and note that

$$\begin{aligned} |v[c, f_1](t, x) - v[c, f_2](t, x)| &\leq \mathbb{E} \int_t^T |\Lambda_s f_1(s, X_s) - \Lambda_s f_2(s, X_s)| ds \\ &\leq |f_1 - f_2|_0 \mathbb{E} \int_t^T |\Lambda_s| ds \\ &\leq T e^{T|c|_0} |f_1 - f_2|_0. \end{aligned}$$

This concludes that

$$|u[c, f_1] - u[c, f_2]|_0 \leq \Psi(|c|_0) |f_1 - f_2|_0. \quad (3.7)$$

In the following, we estimate $|u[c_1, f] - u[c_2, f]|_0$. By setting $\Lambda_s^i = e^{-\int_t^s c_i(r, X(r))dr}$, we have

$$|\Lambda_s^1 - \Lambda_s^2| \leq e^{T(|c_1|_0 + |c_2|_0)} \int_t^s |c_1(r, X_r) - c_2(r, X_r)| dr \leq e^{T(|c_1|_0 + |c_2|_0)} |c_1 - c_2|_0 T$$

with probability one. Therefore,

$$\begin{aligned} |v[c_1, f](t, x) - v[c_2, f](t, x)| &\leq \mathbb{E} \int_t^T |\Lambda_s^1 f(s, X_s) - \Lambda_s^2 f(s, X_s)| ds \\ &\leq |f|_0 \mathbb{E} \int_t^T |\Lambda_s^1 - \Lambda_s^2| ds \\ &\leq T^2 e^{T(|c_1|_0 + |c_2|_0)} |f|_0 |c_1 - c_2|_0. \end{aligned}$$

This implies that

$$|u[c_1, f] - u[c_2, f]|_0 \leq \Psi(|c_1|_0 + |c_2|_0 + |f|_0) |c_1 - c_2|_0. \quad (3.8)$$

The conclusion yields from (3.6), (3.7), (3.8). \square

3.1.3. Second order regularity and first order sensitivity

Next, we will see that under better regularity of c and f in x , we can improve regularity and sensitivity. Formally, if u of (3.1) is smooth enough, one can take derivatives of the equation to conclude that $\bar{u}_j = \partial_j u$ is the solution of the following equation depending on c, f and u of (3.1).

$$\begin{cases} \partial_t \bar{u}_j = \frac{1}{2} \Delta \bar{u}_j - c \bar{u}_j - u \partial_j c + \partial_j f, & \text{on } (0, T) \times \mathbb{T}^d \\ \bar{u}_j(0, x) = 0, & \text{on } x \in \mathbb{T}^d \end{cases} \quad (3.9)$$

However, (3.9) is valid only if $u \in C^{1,3}$ is given a priori.

Lemma 3.6. *If $c, f \in C^{\delta,2}([0, T] \times \mathbb{T}^d)$ for some $\delta \in (0, 1)$, then the solution u of (3.1) is in $C^{1,3}([0, T] \times \mathbb{T}^d)$ and $\bar{u}_j = \partial_j u$ is the unique solution of (3.9).*

Proof. By Lemma 3.3, u satisfies, for any $\phi \in H^2(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} \phi(-\partial_t u - cu + f) dx = \frac{1}{2} \int_{\mathbb{T}^d} \nabla \phi \cdot \nabla u dx.$$

Now, if we replace the test function ϕ by $\partial_i \phi$ in the above variational form, then we have

$$\int_{\mathbb{T}^d} \partial_i \phi(-\partial_t u - cu + f) dx = \frac{1}{2} \int_{\mathbb{T}^d} \nabla \partial_i \phi \cdot \nabla u dx.$$

Using integration by parts, we can show that \bar{u}_j solves the variational form of (3.9) for any $\phi \in H^2(\mathbb{T}^d)$. Since $H^2(\mathbb{T}^d)$ is a dense subset in $H^1(\mathbb{T}^d)$, \bar{u}_j is indeed a unique weak solution of (3.9).

Lastly, since the $\nabla c, \nabla f \in C^{\delta,1}([0, T] \times \mathbb{T}^d)$, we conclude that \bar{u}_j is indeed a classical solution from Lemma 3.1. This also implies that $u \in C^{1,3}([0, T] \times \mathbb{T}^d)$. \square

Lemma 3.7. *Let $c, f \in C^{\delta,2}([0, T] \times \mathbb{T}^d)$. Then the solution u of (3.1) belongs to $C^{1,3}([0, T] \times \mathbb{T}^d)$ with*

$$|u[c, f]|_{0,2} \leq \Psi(|c|_{0,2} + |f|_{0,2}).$$

Furthermore, the solution map $u = u[c, f]$ of (3.1) satisfies

$$|u[c_1, f_1] - u[c_2, f_2]|_{0,1} \leq \Psi(K)(|c_1 - c_2|_{0,1} + |f_1 - f_2|_{0,1})$$

for

$$K := |c_1|_{0,1} + |c_2|_{0,1} + |f_1|_{0,1} + |f_2|_{0,1}.$$

Proof. By Lemma 3.6, $\bar{u}_j = \partial_j u$ is the classical solution of (3.9), which satisfies

$$\bar{u}_j = u[c, \bar{f}],$$

where

$$\bar{f} = -u \partial_j c + \partial_j f.$$

Applying Lemma 3.5, we have $|\bar{u}_j|_{0,1} < \Psi(|c|_{0,1} + |\bar{f}|_{0,1})$. Note that, $|\bar{f}|_{0,1}$ is controlled by $|u|_{0,1} + |\partial_j c|_{0,1} + |\partial_j f|_{0,1}$, which implies that $|\bar{f}|_{0,1} \leq \Psi(|c|_{0,2} + |f|_{0,2})$ due to Lemma 3.5. Hence, we conclude that $|u[c, f]|_{0,2} \leq \Psi(|c|_{0,2} + |f|_{0,2})$.

At last, applying Lemma 3.5 on $u[c, \bar{f}]$ again, we have

$$|u[c_1, \bar{f}_1] - u[c_2, \bar{f}_2]|_0 \leq \Psi(K)(|c_1 - c_2|_0 + |\bar{f}_1 - \bar{f}_2|_0)$$

for $K = |c_1|_0 + |\bar{f}_1|_0 + |c_2|_0 + |\bar{f}_2|_0$, which similarly concludes the desired result. \square

3.1.4. Summary on regularity and sensitivity

Now we may summarize and generalize the results above to a PDE with non-zero initial conditions. Consider equation

$$\begin{cases} \partial_t u = \frac{1}{2} \Delta u - cu + f, & \text{on } (0, T) \times \mathbb{T}^d \\ u(0, x) = \psi(x), & \text{on } x \in \mathbb{T}^d. \end{cases} \quad (3.10)$$

To proceed, we recall the following notations:

- $C_{0,n'}^{\delta,n}$ be the space of all functions $f \in C^{\delta,n}([0, T] \times \mathbb{T}^d)$ with the topology induced by the norm $|\cdot|_{0,n'}$.
- $C_{0,1}^{1,3}([0, T] \times \mathbb{T}^d)$ is the space of all $u \in C^{1,3}([0, T] \times \mathbb{T}^d)$ topologized by $|\cdot|_{0,1}$.

For more details, we refer it to Appendix A.

Theorem 3.8. *The solution map $u : [c, f, \psi] \mapsto u[c, f, \psi]$ given by (3.10) is a locally Lipschitz continuous map*

$$C_{0,1}^{\delta,2} \times C_{0,1}^{\delta,2} \times C_3^4 \mapsto C_{0,1}^{1,3}.$$

Proof. It is enough to show that

$$\begin{aligned} & |u[c_1, f_1, \psi_1] - u[c_2, f_2, \psi_2]|_{0,1} \\ & \leq \Psi(K)(|c_1 - c_2|_{0,1} + |f_1 - f_2|_{0,1} + |\psi_1 - \psi_2|_3) \end{aligned}$$

for $K = |c_1|_{0,1} + |c_2|_{0,1} + |f_1|_{0,1} + |f_2|_{0,1} + |\psi_1|_3 + |\psi_2|_3$. Indeed, setting $\tilde{u}(t, x) = u(t, x) - \psi(x)$, we have

$$\tilde{u} = u[c, f + \frac{1}{2} \Delta \psi - c\psi, 0]$$

for the solution map $u[\cdot, \cdot, \cdot]$ defined via (3.10), and observe that the desired result is a consequence of Lemma 3.7. \square

Note that the local Lipschitz continuity of Theorem 3.8 automatically yields its local boundedness, *i.e.*

$$|u[c, f, \psi]|_{0,1} \leq \Psi(|c|_{0,1} + |f|_{0,1} + |\psi|_3) \quad (3.11)$$

for some positive increasing function Ψ . The following Harnack type inequality will be useful.

Corollary 3.9. *If $f \equiv 0$, $\psi = e^b$ for some $c, b \in C^{\delta,2}([0, T] \times \mathbb{T}^d)$, then the solution u of (3.10) satisfies the inequality*

$$e^{-(|b|_0 + |c|_0 T)} < u(t, x) < e^{|b|_0 + |c|_0 T}, \quad \forall (t, x) \in [0, T] \times \mathbb{T}^d.$$

Proof. The inequalities follow from the representation for $v(t, x) = u(T - t, x)$ in the form of

$$v(t, x) = \mathbb{E} \left[\exp \left\{ - \int_t^T c(r, X^{t,x}(r)) dr \right\} \psi(X^{t,x}(T)) \right],$$

where X is given by (3.3). □

3.2. The FPK equation

We study the weak solution of FPK equation on $[0, T] \times \mathbb{T}^d$:

$$\begin{cases} \partial_t \nu(t, x) = -\operatorname{div}_x(b(t, x)\nu(t, x)) + \frac{1}{2} \Delta \nu(t, x) \\ \nu(0, x) = m_0(x). \end{cases} \quad (3.12)$$

We adopt the conventional notation of

$$\langle m, \psi \rangle := \int_{\mathbb{T}^d} \psi(x) m(dx)$$

for any $m \in \mathcal{P}_1(\mathbb{T}^d)$ and $\psi : \mathbb{T}^d \mapsto \mathbb{R}$ whenever it is well defined.

Definition 3.10. ν is said to be a weak solution of FPK (3.12), if it satisfies, for any $\phi \in C_c^\infty([0, T] \times \mathbb{T}^d)$

$$\langle m_0, \phi(0, x) \rangle + \int_0^T \langle \nu_t, (\partial_t + \mathcal{L})\phi \rangle dt = 0,$$

where

$$\mathcal{L} = b \cdot \nabla + \frac{1}{2} \Delta.$$

We denote the solution map of (3.12) by $\nu = \nu[b, m_0]$. We recall that $C([0, T], \mathcal{P}_1(\mathbb{T}^d))$ is the space of all continuous mappings $\nu : [0, T] \mapsto \mathcal{P}_1(\mathbb{T}^d)$ with a metric given by

$$\operatorname{dist}(\nu_1, \nu_2) = \sup_t d_1(\nu_1(t), \nu_2(t)),$$

where d_1 is 1-Wasserstein metric for \mathcal{P}_1 .

Theorem 3.11. *Let $m_0 \in \mathcal{P}_1(\mathbb{T}^d)$. Then the solution map $b \mapsto \nu[b, m_0]$ of (3.12) is a locally Lipschitz continuous mapping from $C([0, T] \times \mathbb{T}^d)$ to $C([0, T], \mathcal{P}_1(\mathbb{T}^d))$. In particular, if $|b_1|_0 + |b_2|_0 < K$ then*

$$\sup_t d_1(\nu_1(t), \nu_2(t)) \leq \Psi(K) |b_1 - b_2|_0.$$

Moreover, $\nu = \nu[b, m_0]$ satisfies,

$$d_1(\nu(t), \nu(s)) \leq (1 + \sqrt{T} |b|_0) |t - s|^{1/2}, \quad (3.13)$$

$$\sup_t \int_{\mathbb{T}^d} |x| \nu(t, dx) \leq \int_{\mathbb{T}^d} |x| m_0(dx) + |b|_0 T + \sqrt{T}. \quad (3.14)$$

Proof. If $|b|_0 < \infty$ and $m_0 \in \mathcal{P}_1$, then

$$X(t) = X(0) + \int_0^t b(s, X_s) ds + W(t), \quad X(0) \sim m_0$$

has a unique solution. An application of Itô's formula and the definition of the weak solution verifies that $\nu(t) = Law(X(t))$ is the weak solution of (3.12), see [4]. (3.13) also follows from [4].

Next, (3.14) follows from

$$\sup_t \mathbb{E}|X(t)| \leq \mathbb{E}|X(0)| + |b|_0 T + \sqrt{T}.$$

Let's assume $|b_1|_0 + |b_2|_0 < K$ and ν_1 and ν_2 are corresponding solutions of (3.12). We denote by X_1 and X_2 the solutions of the SDE above. Note that

$$\begin{aligned} \mathbb{E}|X_1(t) - X_2(t)| &\leq \mathbb{E} \int_0^t |b_1(s, X_1(s)) - b_2(s, X_2(s))| ds \\ &\leq |b_1 - b_2|_0 T + K \int_0^t \mathbb{E}[|X_1(s) - X_2(s)|] ds. \end{aligned}$$

So, we can use the Gronwall's inequality to have

$$\mathbb{E}|X_1(t) - X_2(t)| \leq |b_1 - b_2|_0 T e^{KT}.$$

Therefore, we can have local Lipschitz of $b \mapsto \nu[b, m_0]$ from

$$d_1(\nu_1(t), \nu_2(t)) \leq \mathbb{E}|X_1(t) - X_2(t)| \leq |b_1 - b_2|_0 T e^{KT}.$$

□

4. EXISTENCE

We now return to the GMFG scheme. First observe that, by using the cost of the form (2.3), the triple (v, \mathbf{a}^*, μ) is the solution of (2.5) if and only if the pair $(\tilde{v} := v - b, \mu)$ is the solution of HJB equation

$$\begin{cases} \partial_t \tilde{v} - \frac{1}{2} |\nabla \tilde{v}|^2 + \frac{1}{2} \Delta \tilde{v} + \tilde{\ell}_1(\mu, g) = 0 \\ \tilde{v}(T, \alpha, x) = -b(T, \alpha, x) \end{cases} \quad (4.1)$$

coupled with FPK equation

$$\begin{cases} \partial_t \mu = \operatorname{div}_x(\mu \nabla \tilde{v}) + \frac{1}{2} \Delta \mu \\ \mu(0, \alpha, x) = m_0(\alpha, x), \end{cases} \quad (4.2)$$

where $\tilde{\ell}_1$ is

$$\tilde{\ell}_1(t, \alpha, x) = \ell_1(t, \alpha, x) + (\partial_t b + \frac{1}{2} |\nabla b|^2 + \frac{1}{2} \Delta b)(t, \alpha, x). \quad (4.3)$$

Next, we outline our approach to the existence as follows. We define an operator

$$\nu = \Phi(\mu) = \Phi_2 \circ \Phi_1(\mu),$$

where

1. $\nabla \tilde{v} = \Phi_1(\mu)$, where \tilde{v} is the solution of (4.4) with a given μ :

$$\begin{cases} \partial_t \tilde{v} - \frac{1}{2} |\nabla \tilde{v}|^2 + \frac{1}{2} \Delta \tilde{v} + \tilde{\ell}_1(\mu, g) = 0 \\ \tilde{v}(T, \alpha, x) = -b(T, \alpha, x) \end{cases} \quad (4.4)$$

2. $\nu = \Phi_2(\bar{v})$ be the function solving (4.5) with a given \bar{v} :

$$\begin{cases} \partial_t \nu = \operatorname{div}_x(\bar{v}\nu) + \frac{1}{2} \Delta \nu \\ \nu(0, \alpha, x) = m_0(\alpha, x). \end{cases} \quad (4.5)$$

The existence of the solution for the GMFG can be accomplished by Schauder's fixed point theorem in an appropriate space to be mentioned below.

To proceed, we recall that d_1 is the Wasserstein metric on $\mathcal{P}_1(\mathbb{T}^d)$. We define the space $S^{1/2}$ as the collection of $\mu : [0, T] \times [0, 1] \mapsto \mathcal{P}_1(\mathbb{T}^d)$ such that

$$|\mu|_{1/2} = |\mu|_0 + [\mu]_{1/2} < \infty,$$

where

$$|\mu|_0 = \sup_{t, \alpha} \int_{\mathbb{T}^d} |x| \mu(t, \alpha, dx)$$

and

$$[\mu]_{1/2} = \sup_{t_1 \neq t_2, \alpha} \frac{d_1(\mu(t_1, \alpha), \mu(t_2, \alpha))}{|t_1 - t_2|^{1/2}}.$$

Note that, $S^{1/2}$ is metrizable by

$$\rho(\mu_1, \mu_2) = \sup_{t, \alpha} d_1(\mu_1(t, \alpha), \mu_2(t, \alpha)), \quad (4.6)$$

and we denote the space $S^{1/2}$ by $(S^{1/2}, \rho)$ whenever we need to emphasize its underlying metric. Note that $B_r := \{\mu \in S^{1/2} : |\mu|_{1/2} \leq r\}$ is a closed convex compact subset of $(S^{1/2}, \rho)$ by generalized version of Arzelà–Ascoli theorem, see P232 of [12].

It is often useful by the duality representation of Wasserstein metric to write

$$\rho(\mu_1, \mu_2) = \sup_{t, \alpha, Lip(f) \leq 1} \int_{\mathbb{T}^d} f(x) d(\mu_1(t, \alpha) - \mu_2(t, \alpha))(x) \quad (4.7)$$

where $Lip(f)$ is the Lipschitz constant of the function f . Similarly, if $\mu \in B_r$ and $f \in C^1$, then

$$\begin{aligned} \int_{\mathbb{T}^d} f(y) d(\mu(t_1, \alpha) - \mu(t_2, \alpha))(y) &\leq |\nabla f|_0 d_1(\mu(t_1, \alpha), \mu(t_2, \alpha)) \\ &\leq r |\nabla f|_0 |t_1 - t_2|^{1/2}. \end{aligned} \quad (4.8)$$

4.1. Assumptions

To proceed, we define a space $C_{0,0,m'}^{\delta,0,m}$ as the collection of all functions in $C^{\delta,0,m}([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R})$ equipped with a $C^{0,0,m'}([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R})$ norm. For instance, if $f \in C_{0,0,2}^{0.5,0,2}$, then we write its norm as

$$|f|_{0,0,2}^{0.5,0,2} = |f|_{0,0,2} = |f|_0 + \sum_i |\partial_{x_i} f|_0 + \sum_{ij} |\partial_{x_i x_j} f|_0.$$

For more details, we refer to Appendix A.

Assumption 4.1. $b : [0, T] \times [0, 1] \times \mathbb{T}^d \mapsto \mathbb{R}^d$, $g : [0, 1]^2 \mapsto \mathbb{R}$, and $m_0 : [0, 1] \times \mathbb{T}^d \mapsto \mathbb{R}^d$ are infinitely smooth functions in all variables.

We pose the following assumptions for the cost function ℓ_1 . Throughout the paper, since g will be a priori given function, we will suppress g by writing

$$\ell_1(\mu, g, t, \alpha, x) = \ell_1(\mu, t, \alpha, x)$$

if this does not cause any confusion. For convenience, we will write

$$\ell_1[\mu](t, \alpha, x) = \ell_1(\mu, t, \alpha, x) = \ell_1(\mu, g, t, \alpha, x).$$

Assumption 4.2. The mapping $\mu \mapsto \ell_1[\mu]$ is a bounded and Lipschitz continuous mapping from $S^{1/2}$ to $C_{0,0,1}^{0.5,0,2}$, that is, for any $\mu \in S^{1/2}$, $\ell_1[\mu]$ belongs to $C^{0.5,0,2}$ and

$$|\ell_1[\mu]|_{0,0,1} < M, \quad |\ell_1[\mu_1] - \ell_1[\mu_2]|_{0,0,1} \leq M\rho(\mu_1, \mu_2),$$

for some $M > 0$ independent to the choice of μ .

We check that the assumptions are valid for a class of examples given in Lemma 4.3.

Lemma 4.3. *Suppose $\ell_2 \in C^\infty(\mathbb{T}^d \times \mathbb{T}^d, \mathbb{R})$ and g are given smooth enough. Then, the cost ℓ_1 of (2.4) satisfies Assumption 4.2.*

Proof. Let $d = 1$ for the simplicity. For $\mu \in S^{1/2}$, we have

$$|\ell_1[\mu]|_0 \leq |\ell_2|_0 |g|_0,$$

$$|\partial_x \ell_1[\mu]|_0 \leq |\partial_x \ell_2|_0 |g|_0,$$

$$|\partial_{xx} \ell_1[\mu]|_0 \leq |\partial_{xx} \ell_2|_0 |g|_0,$$

$$\begin{aligned} |\ell_1(\mu, t_1, \alpha, x) - \ell_1(\mu, t_2, \alpha, x)| &\leq \int_0^1 \int_{\mathbb{T}^d} |\ell_2(x, y)(\mu(t_1, \alpha', dy) - \mu(t_2, \alpha', dy))g(\alpha, \alpha')| d\alpha' \\ &\leq \int_0^1 |\partial_y \ell_2|_0 d_1(\mu(t_1, \alpha'), \mu(t_2, \alpha')) g(\alpha, \alpha') d\alpha' \\ &\leq |\partial_y \ell_2|_0 |g|_0 |\mu|_{1/2} |t_1 - t_2|^{1/2}. \end{aligned}$$

This implies that $\ell_1[\mu] \in C^{1/2,0,2}$ with estimation

$$|\ell_1[\mu]|_{1/2,0,2} \leq |\ell_2|_{2,0}|g|_0(1 + |\mu|_{1/2}). \quad (4.9)$$

Note that (4.9) does not give a uniform upper bound due to the μ -dependence on the right hand side of the inequality. Nevertheless, we have a uniform upper bound for the weaker norm $|\cdot|_{0,0,1}$:

$$|\ell_1[\mu]|_{0,0,1} \leq |\ell_2|_{1,0}|g|_0, \quad \forall \mu \in S^{1/2}.$$

For $\mu_1, \mu_2 \in S^{1/2}$, we have

$$\begin{aligned} \ell_1(\mu_1, t, \alpha, x) - \ell_1(\mu_2, t, \alpha, x) &= \int_0^1 \int_{\mathbb{T}^d} \ell_2(x, y)(\mu_1(t, \alpha', dy) - \mu_2(t, \alpha', dy))g(\alpha, \alpha')d\alpha' \\ &\leq |\partial_y \ell_2|_0 d_1(\mu_1(t, \alpha), \mu_2(t, \alpha))|g|_0. \end{aligned}$$

This implies that

$$|\ell_1[\mu_1] - \ell_1[\mu_2]|_0 \leq |\partial_y \ell_2|_0 |g|_0 \rho(\mu_1, \mu_2).$$

Similarly, we obtain

$$|\partial_x \ell_1[\mu_1] - \partial_x \ell_1[\mu_2]|_0 \leq |\partial_y \partial_x \ell_2|_0 |g|_0 \rho(\mu_1, \mu_2).$$

Therefore, we have Lipschitz continuity

$$|\ell_1[\mu_1] - \ell_1[\mu_2]|_{0,0,1} \leq |\ell_2|_{1,1}|g|_0 \rho(\mu_1, \mu_2),$$

and this implies Assumption 4.2 with $M = |\ell_2|_{1,1}|g|_0$. \square

4.2. Operator Φ_1

Recall that $\nabla \tilde{v} = \Phi_1(\mu)$, where \tilde{v} is the solution of (4.4) with given μ . By Hopf-Cole transform \tilde{v} is the solution of (4.4) if and only if

$$w = \exp\{-\tilde{v}\} \quad (4.10)$$

is the solution of

$$\begin{cases} \partial_t w + \frac{1}{2} \Delta w - w \tilde{\ell}_1[\mu] = 0 & \text{on } (0, T) \times [0, 1] \times \mathbb{T}^d \\ w(T, \alpha, x) = e^{b(T, \alpha, x)} & \text{on } [0, 1] \times \mathbb{T}^d. \end{cases} \quad (4.11)$$

In addition, we have the following relation by chain rule:

$$\nabla \tilde{v} = -\frac{\nabla w}{w}, \quad \Delta \tilde{v} = \frac{-w \Delta w + |\nabla w|^2}{w^2}.$$

Since w -term appears in the denominator, Harnack type inequality in Corollary 3.9 ensures that $\nabla \tilde{v}$ and $\Delta \tilde{v}$ are well defined.

4.2.1. Estimates of parameterized PDEs

We define

$$w = G(f) \tag{4.12}$$

by the solution of

$$\begin{cases} \partial_t w + \frac{1}{2} \Delta w - wf = 0 & \text{on } (0, T) \times [0, 1] \times \mathbb{T}^d \\ w(T, \alpha, x) = e^{b(T, \alpha, x)} & \text{on } [0, 1] \times \mathbb{T}^d. \end{cases} \tag{4.13}$$

Note that $w = G(\tilde{\ell}_1[\mu])$ is the solution of (4.11).

Lemma 4.4. *The mapping G is a locally Lipschitz continuous mapping from $C_{0,0,1}^{0.5,0,2}$ to $C_{0,0,1}^{1,0,2}$.*

Proof. Let $f \in C^{0.5,0,2}$ and $w = G(f)$. By Theorem 3.8, we have $w(\alpha) \in C^{1,3}$. If $\alpha \rightarrow \alpha_0$, then $f(\alpha) \rightarrow f(\alpha_0)$ holds pointwisely. Together with Dominated Convergence Theorem on the probabilistic representation of w , one can conclude $w(t, \alpha, x) \rightarrow w(t, \alpha_0, x)$ whenever $\alpha \rightarrow \alpha_0$. Therefore, w belongs to $C^{1,0,3}$.

Given $f_1, f_2 \in C^{0.5,0,2}$ and $w_i = G(f_i)$ with

$$K(\alpha) = |f_1(\alpha)|_{0,1} + |f_2(\alpha)|_{0,1} + |e^{b(T, \alpha)}|_3,$$

we can use local Lipschitz continuity of Theorem 3.8 to obtain local Lipschitz of G ,

$$\begin{aligned} |w_1 - w_2|_{0,0,1} &= \sup_{\alpha} |w_1(\alpha) - w_2(\alpha)|_{0,1} \\ &\leq \sup_{\alpha} \Psi(K(\alpha)) |f_1(\alpha) - f_2(\alpha)|_{0,1} \\ &\leq \Psi(\sup_{\alpha} K(\alpha)) |f_1 - f_2|_{0,0,1}. \end{aligned}$$

In the above, we used the monotonicity of $\Psi(\cdot)$ to switch Ψ and sup. Since $\sup_{\alpha} K(\alpha) \leq \Psi(|f_1|_{0,0,1} + |f_2|_{0,0,1} + |b|_3)$, we can rewrite the above estimations as

$$|w_1 - w_2|_{0,0,1} \leq \Psi(|f_1|_{0,0,1} + |f_2|_{0,0,1} + |b|_3) |f_1 - f_2|_{0,0,1}.$$

□

4.2.2. Φ_1 estimate

Lemma 4.5. *Φ_1 is a uniformly bounded and Lipschitz continuous mapping from $(S^{1/2}, \rho)$ to $C^0([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R}^d)$.*

Proof. If $\mu \in S^{1/2}$, then $\ell_1[\mu] \in C^{0.5,0,2}$ with $|\ell_1[\mu]|_{0,0,1} < M$ by Assumption 4.2. We recall that

$$\tilde{\ell}_1 = \ell_1 + (\partial_t b + \frac{1}{2} |\nabla b|^2 + \frac{1}{2} \Delta b).$$

Due to the smoothness of b and compactness of its domain, we still have $\tilde{\ell}_1[\mu] \in C^{0.5,0,2}$ with $|\tilde{\ell}_1[\mu]|_{0,0,1} < \Psi(M)$. Together with local Lipschitz continuity of $G(\cdot)$ in Lemma 4.4, it implies uniform boundedness of $w = G(\tilde{\ell}_1[\mu])$, i.e.

$$|w|_{0,0,1} < \Psi(M).$$

Moreover, Corollary 3.9 says that the reciprocal of $w = G(\tilde{\ell}_1[\mu])$ is bounded in the sense $|w^{-1}|_0 < \Psi(|\tilde{\ell}_1[\mu]|_0)$. Therefore, we have

$$|w|_{0,0,1} + |w^{-1}|_0 < \Psi(M).$$

Next, we can prove that Φ_1 is uniformly bounded in C^0 :

$$|\Phi_1(\mu)|_0 = |\nabla \tilde{v}|_0 = |w^{-1} \nabla w|_0 \leq |w^{-1}|_0 |\nabla w|_0 \leq |w^{-1}|_0 |w|_{0,0,1} \leq \Psi(M).$$

Finally, we can show the global Lipschitz for Φ_1 by the following estimates:

$$\begin{aligned} |\Phi_1(\mu_1) - \Phi_1(\mu_2)|_0 &= |w_1^{-1} \nabla w_1 - w_2^{-1} \nabla w_2|_0 \\ &= \left| \frac{w_2 \nabla w_1 - w_1 \nabla w_2}{w_1 w_2} \right|_0 \\ &\leq \Psi(M) (|w_2|_0 |\nabla w_1 - \nabla w_2|_0 + |\nabla w_2|_0 |w_1 - w_2|_0) \\ &\leq \Psi(M) |w_1 - w_2|_{0,0,1} \\ &\leq \Psi(M) |\tilde{\ell}_1[\mu_1] - \tilde{\ell}_1[\mu_2]|_{0,0,1} \\ &\leq \Psi(M) \rho(\mu_1, \mu_2). \end{aligned}$$

In the last two steps, we used Lipschitz continuity obtained by Lemma 4.4 and Assumption 4.2. \square

4.3. Operator Φ_2

Next, we will show the properties associated to Φ_2 mapping from $C^0([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R}^d)$ to $S^{1/2}$.

Lemma 4.6. *Φ_2 is a locally Lipschitz continuous mapping from $C^0([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R}^d)$ to $(S^{1/2}, \rho)$. Moreover, $|\Phi_2(\bar{v})|_{1/2} \leq \Psi(|\bar{v}|_0)$ for all $\bar{v} \in C^0([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R}^d)$ for some monotonically increasing positive function Ψ .*

Proof. Given $\bar{v} \in C^0([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R}^d)$ and $\nu = \Phi_2(\bar{v})$, applying (3.14) of Theorem 3.11, it yields that

$$\begin{aligned} |\nu|_0 = \sup_{t, \alpha} \int_{\mathbb{T}^d} |x| \nu(t, \alpha, dx) &= \sup_{\alpha} \sup_t \int_{\mathbb{T}^d} |x| \nu(t, \alpha, dx) \\ &\leq \sup_{\alpha} \left(\int_{\mathbb{T}^d} |x| m_0(\alpha, dx) + |\bar{v}(\alpha)|_0 T + \sqrt{T} \right) \\ &\leq \Psi(|\bar{v}|_0). \end{aligned}$$

Next, we show the following equicontinuity property again by (3.13) of Theorem 3.11:

$$\begin{aligned} \sup_{t_1 \neq t_2, \alpha} d_1(\nu(t_1, \alpha), \nu(t_2, \alpha)) &\leq \sup_{\alpha} (1 + \sqrt{T} |\bar{v}(\alpha)|_0) |t_1 - t_2|^{1/2} \\ &\leq \Psi(|\bar{v}|_0) |t_1 - t_2|^{1/2}. \end{aligned}$$

This proves $\nu \in S^{1/2}$ with

$$|\nu|_{1/2} \leq \Psi(|\bar{v}|_0).$$

For the continuity of Φ_2 , given $\bar{v}_1, \bar{v}_2 \in C^0([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R}^d)$, we set $\nu_i = \Phi_2(\bar{v}_i)$ for $i = 1, 2$. Then, we use the local Lipschitz continuity in Theorem 3.11 to obtain local Lipschitz continuity of Φ_2 as follows:

$$\begin{aligned} \rho(\nu_1, \nu_2) &= \sup_{t, \alpha} d_1(\nu_1(t, \alpha), \nu_2(t, \alpha)) \\ &= \sup_{\alpha} \sup_t d_1(\nu_1(t, \alpha), \nu_2(t, \alpha)) \\ &= \sup_{\alpha} \Psi(|\bar{v}_1(\alpha)|_0 + |\bar{v}_2(\alpha)|_0) |\bar{v}_1(\alpha) - \bar{v}_2(\alpha)|_0 \\ &\leq \Psi(|\bar{v}_1|_0 + |\bar{v}_2|_0) |\bar{v}_1 - \bar{v}_2|_0. \end{aligned}$$

□

4.4. Existence by Schauder's fixed point theorem

Theorem 4.7. *Suppose Assumptions 4.1–4.2 are valid. Then there exists a solution of (2.5) in the space $C^{1,0,2}([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R}) \times C([0, T] \times [0, 1], \mathcal{P}_1(\mathbb{T}^d))$.*

Proof. It is enough to show that $\Phi_2 \circ \Phi_1$ has a fixed point in $S^{1/2}$. Recall that B_r is a convex closed and compact subset of $S^{1/2}$. For simplicity, we denote by \hat{B}_r the closed ball of radius r in $C^0([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R}^d)$.

1. By Lemma 4.5, there exists some positive increasing function Ψ_1 independent to r , such that the mapping

$$\Phi_1 : B_r \mapsto \hat{B}_{\Psi_1(M)}$$

is continuous.

2. By Lemma 4.6, there exists some positive increasing function Ψ_2 such that the mapping

$$\Phi_2 : \hat{B}_{\Psi_1(M)} \mapsto B_{\Psi_2 \circ \Psi_1(M)}$$

is continuous.

Now we take

$$r = \Psi_2(\Psi_1(M))$$

and we have

$$\Phi_2 \circ \Phi_1 : B_r \mapsto B_r$$

is a continuous mapping and this yields the existence of a fixed point for Φ by Schauder's theorem. □

In the above, we have indeed proved the existence in the space $C^{1,0,2}([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R}) \times S^{1/2}$.

4.5. Further remarks on the fixed point theorem

In connection with GMFG, we explain why Theorem 3.8 establishes locally Lipschitz continuity of the solution map $u : [c, f, \psi] \mapsto u[c, f, \psi]$ of (3.10) in the sense of

$$C_{0,1}^{\delta,2} \times C_{0,1}^{\delta,2} \times C_3^4 \mapsto C_{0,1}^{1,3} \quad (4.14)$$

instead of

$$C^{\delta,2} \times C^{\delta,2} \times C^4 \mapsto C^{1,3}. \quad (4.15)$$

For the illustration purpose, if we freeze c, ψ of the solution map u , then local Lipschitz continuity in the sense of (4.14) implies local boundedness

$$|u|_{0,1} \leq \Psi(|f|_{0,1}),$$

while local Lipschitz continuity in the sense of (4.15) implies local boundedness

$$|u|_{0,1} \leq |u|_{1,3} \leq \Psi(|f|_{\delta,2}).$$

The main difference of these two local boundedness properties is that, the first one controls u by f with 0-norm in t -variable while the second one does by f with δ -norm in t -variable, which is not desirable. The main reason is that the running cost $|\ell_1[\mu]|_{1/2,0,1} \leq \Psi(|\mu|_{1/2})$ of (4.9) does not have uniform bound in μ , while $|\ell_1[\mu]|_{0,0,1}$ does. For this reason, we included the regularity results for parabolic PDE solutions by dropping t -regularity while increasing x -regularity as a tradeoff.

Recall that, we have established the existence of a fixed point of a mapping $\Phi = \Phi_2 \circ \Phi_1$ for $\Phi_1 : \mu \mapsto \nabla \tilde{v}$ and $\Phi_2 : \nabla \tilde{v} \mapsto \nu$. Our approach is along the the Schuader's fixed point theorem with estimates

$$\Phi_1 : B_r \mapsto \hat{B}_{\Psi_1(M)}, \quad \Phi_2 : \hat{B}_{\Psi_1(M)} \mapsto B_{\Psi_2 \circ \Psi_1(M)}.$$

In the above, it is crucial that the Φ_1 is upper bounded by $\Psi_1(M)$ independent to r , and this can be inferred from local boundedness of (4.14) together with uniform boundedness of $|\ell_1[\mu]|_{0,0,1}$.

In contrast, if we use local boundedness in the sense of (4.15), then we have estimations in the form of

$$\Phi_1 : B_r \mapsto \hat{B}_{\Psi_1(r)}, \quad \Phi_2 : \hat{B}_{\Psi_1(r)} \mapsto B_{\Psi_2 \circ \Psi_1(r)}.$$

Since the norm of the running cost $|\ell_1[\mu]|_{1,0,3}$ depends on μ , Φ_1 can not be uniformly bounded. As a result, the choice of $r = \Psi_1(r)$ is infeasible.

5. UNIQUENESS OF GMFG

Assumption 5.1. There exists some $\alpha \in [0, 1]$ satisfying

$$\int_{\mathbb{T}^d} (\ell_1(\mu_1, g, t, \alpha, x) - \ell_1(\mu_2, g, t, \alpha, x))(\mu_1 - \mu_2)(t, \alpha, dx) > 0,$$

for all $\mu_1 \neq \mu_2 \in C([0, T] \times [0, 1], \mathcal{P}_1(\mathbb{T}^d))$ and $t \in [0, T]$.

Theorem 5.2. ([4, 18]) *Suppose Assumptions 4.1–4.2 and 5.1 are valid. Then, there exists a unique solution of (2.5) in the space $C^{1,0,2}([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R}) \times C([0, T] \times [0, 1], \mathcal{P}_1(\mathbb{T}^d))$.*

Proof. For $i = 1, 2$, let (v_i, μ_i) be two different solution pairs, and set

$$\bar{v} = v_1 - v_2, \quad \bar{\mu} = \mu_1 - \mu_2.$$

Note that $\bar{v}(T, \alpha, x) = \bar{\mu}(0, \alpha, x) = 0$ for all (α, x) by their given initial-terminal data. We also write $\ell_1[\mu_i] = \ell_1[\mu_i, g]$ for short. Then \bar{v} satisfies

$$\partial_t \bar{v} + \nabla b \cdot \nabla \bar{v} + \frac{1}{2} \Delta \bar{v} - \frac{1}{2} |\nabla v_1|^2 + \frac{1}{2} |\nabla v_2|^2 + \ell_1[\mu_1] - \ell_1[\mu_2] = 0$$

and $\bar{\mu}$ satisfies

$$-\partial_t \bar{\mu} - \operatorname{div}(\nabla b \bar{\mu}) + \frac{1}{2} \Delta \bar{\mu} + \operatorname{div}(\nabla v_1 \mu_1) - \operatorname{div}(\nabla v_2 \mu_2) = 0.$$

The above two equations can be rewritten as

$$\langle \partial_t \bar{v} + \nabla b \cdot \nabla \bar{v} + \frac{1}{2} \Delta \bar{v}, \bar{\mu} \rangle + \langle -\frac{1}{2} |\nabla v_1|^2 + \frac{1}{2} |\nabla v_2|^2 + \ell_1[\mu_1] - \ell_1[\mu_2], \bar{\mu} \rangle = 0$$

and

$$\langle \partial_t \bar{v} + \nabla b \cdot \nabla \bar{v} + \frac{1}{2} \Delta \bar{v}, \bar{\mu} \rangle + \langle \bar{v}, \operatorname{div}(\nabla v_1 \mu_1) - \operatorname{div}(\nabla v_2 \mu_2) \rangle = 0.$$

By subtracting above two equations, and utilizing the identities

$$\langle \operatorname{div}(\nabla v_1 \mu_1), \bar{v} \rangle = -\langle |\nabla v_1|^2, \mu_1 \rangle + \langle \nabla v_1 \cdot \nabla v_2, \mu_1 \rangle$$

and

$$\langle \operatorname{div}(\nabla v_2 \mu_2), \bar{v} \rangle = \langle |\nabla v_2|^2, \mu_2 \rangle - \langle \nabla v_1 \cdot \nabla v_2, \mu_2 \rangle,$$

we obtain

$$\langle \frac{1}{2} (\mu_1 + \mu_2), |\nabla \bar{v}|^2 \rangle + \langle \ell_1[\mu_1] - \ell_1[\mu_2], \bar{\mu} \rangle = 0.$$

The first term is non-negative and the second term is strictly positive for some $\alpha \in [0, 1]$ by (A3), which implies a contradiction. \square

6. CONCLUDING REMARKS

Our main result of Theorem 5.2 provides existence and uniqueness of the GMFG equation under some assumptions. One limitation of the current setting is that the running cost in the current setup allows to use Hopf-Cole transformation, which is essential to the subsequent analysis on regularities. To deal with the full generalization of the running cost, one must adopt different approaches and it will be in our future research direction. It is also desirable to generalize the result to the whole domain \mathbb{R}^d instead of compact domain \mathbb{T}^d . Another limitation is that, the current setting requires the continuity of the graphon. Note that some graphons are not necessarily continuous. Nevertheless, the continuity condition of the graphon can be relaxed in the following sense by similar arguments with additional complexity of notations, which is sketched below briefly.

To proceed, we define \hat{C}^0 as the collection of bounded measurable functions $f : [0, T] \times [0, 1] \times \mathbb{T}^d \mapsto \mathbb{R}$, and we denote its norm as

$$|f|_0 = \sup_{[0, T] \times [0, 1] \times \mathbb{T}^d} |f(t, \alpha, x)|.$$

With $\hat{C}^{\delta, 0, 2}$, we denote the set of functions $f \in \hat{C}^0$ with finite norm

$$|f|_{\delta, 0, 2} = |f|_0 + \sup_{t_1 < t_2, \alpha, x} \frac{|f(t_1, \alpha, x) - f(t_2, \alpha, x)|}{|t_1 - t_2|^\delta} + \sum_i |\partial_i f|_0 + \sum_{ij} |\partial_{ij} f|_0.$$

By the above definition $\hat{C}^{\delta,0,2}$ allows the discontinuity in α .

Assumption 6.1. 1. b and m_0 are infinitely smooth in their domains.
2. The graphon g is bounded measurable on $[0, 1]^2$ with

$$|g|_0 = \sup_{[0,1]^2} |g(\alpha, \alpha')| < \infty.$$

We recall that B_r is defined in $S^{1/2}$. We use $\hat{C}_{0,0,2}^{\delta,0,2}$ to denote the same set $\hat{C}^{\delta,0,2}$ with the norm $|\cdot|_{0,0,2}$, i.e.

$$|f|_{0,0,2} = |f|_0 + \sum_i |\partial_i f|_0 + \sum_{ij} |\partial_{ij} f|_0.$$

Assumption 6.2. The mapping $\mu \mapsto \ell_1[\mu]$ is a bounded and Lipschitz continuous mapping from $S^{1/2}$ to $\hat{C}_{0,0,1}^{0.5,0,2}$, that is, for any $\mu \in S^{1/2}$, $\ell_1[\mu]$ belongs to $\hat{C}^{0.5,0,2}$ and

$$|\ell_1[\mu]|_{0,0,1} < M, \quad |\ell_1[\mu_1] - \ell_1[\mu_2]|_{0,0,1} \leq M\rho(\mu_1, \mu_2),$$

for some $M > 0$ independent to the choice of μ .

We also define $\hat{C}^{m,0,n}$ as the collection of $f \in \hat{C}^0$ with continuous bounded m -th derivatives in t and n -th derivatives x . For instance, for $f \in \hat{C}^{1,0,2}$, we have finite norm

$$|f|_{1,0,2} = |f|_0 + |\partial_t f|_0 + \sum_i |\partial_i f|_0 + \sum_{ij} |\partial_{ij} f|_0.$$

Now we present a result in parallel to Theorem 5.2. The proof is similar and so omitted.

Corollary 6.3. *Suppose Assumptions 6.1 – 6.2 and 5.1 are valid. Then there exists a unique solution of (2.5) in the space $\hat{C}^{1,0,2}([0, T] \times [0, 1] \times \mathbb{T}^d, \mathbb{R}) \times \hat{C}([0, T] \times [0, 1], \mathcal{P}_1(\mathbb{T}^d))$.*

APPENDIX A.

In this appendix, we will summarize the notations of Hölder space used in this paper. For this purpose, we will define the following functionals for a function u from a product normed space $S = X \times Y$ to \mathbb{R}^d whenever it is well defined.

- $|u|_0 = \sup_S |u(x, y)|$.
- For nonnegative integers l, m , define

$$|u|_{l,m} = \sum_{i=0}^l \sum_{|\alpha|=i} |D_x^\alpha u|_0 + \sum_{i=0}^m \sum_{|\alpha|=i} |D_y^\alpha u|_0.$$

In the above, α is a multi-index for differential operators. For instance, $|\alpha| = \sum_{i=1}^{d_1} |\alpha_i|$ for a multi-index $\alpha = (\alpha_i : i = 1, \dots, d_1)$.

- For positive numbers $l', m' \in (0, 1)$, define

$$[u]_{l',m'} = [u]_{l',0} + [u]_{0,m'},$$

where

$$[u]_{l',0} = \sup_{x_1 \neq x_2, y} \frac{|u(x_1, y) - u(x_2, y)|}{|x_1 - x_2|^{l'}},$$

and

$$[u]_{0,m'} = \sup_{x, y_1 \neq y_2} \frac{|u(x, y_1) - u(x, y_2)|}{|y_1 - y_2|^{m'}}.$$

- For nonnegative integers l, m and positive number $l' \in (0, 1)$, define

$$|u|_{l+l',m} = |u|_{l,m} + \sum_{|\alpha|=l} [D_x^\alpha u]_{l',0}.$$

- For nonnegative integers l, m and positive numbers $l', m' \in (0, 1)$, define

$$|u|_{l+l',m+m'} = |u|_{l,m} + \sum_{|\alpha|=l} [D_x^\alpha u]_{l',m'} + \sum_{|\alpha|=m} [D_y^\alpha u]_{l',m'}.$$

One can check that the following spaces are Banach spaces:

- $C^{l,m}(X \times Y; \mathbb{R}^d) := \{u : |u|_{l,m} < \infty\}$,
- $C^{l+l',m}(X \times Y; \mathbb{R}^d) := \{u : |u|_{l+l',m} < \infty\}$,
- $C^{l+l',m+m'}(X \times Y; \mathbb{R}^d) := \{u : |u|_{l+l',m+m'} < \infty\}$.

In this paper, we also involve the space $C^{l',0,m}$ of functions with a domain $S = X \times Y \times Z$, whose norm is defined as

$$|u|_{l',0,m} = |u|_{0,0,m} + [D_z^m u]_{l',0,0},$$

where

$$|u|_{0,0,m} = \sum_{i=0}^m \sum_{|\alpha|=i} |D_z^\alpha u|_0, \text{ and } [u]_{l',0,0} = \sup_{x_1 \neq x_2, y, z} \frac{|u(x_1, y, z) - u(x_2, y, z)|}{|x_1 - x_2|^{l'}}.$$

In this paper, our functions involve state domain taking values in d -torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$. For $x \in \mathbb{R}^d$, let $\pi(x)$ be the coset of \mathbb{Z}^d that contains x , *i.e.*

$$\pi(x) = x + \mathbb{Z}^d.$$

A canonical metric on \mathbb{T}^d can be induced from the Euclidean metric by

$$|\pi(x) - \pi(y)|_{\mathbb{T}^d} = \inf\{|x - y - z| : z \in \mathbb{Z}^d\}.$$

For the illustration purpose, we provide a list of representative Hölder spaces used throughout the paper:

- $C^{\delta/2, \delta}([0, T] \times \mathbb{T}^d)$ is a space of functions $u(t, x)$ with a norm defined by

$$|u|_{\delta/2, \delta} = |u|_0 + [u]_{\delta/2, \delta},$$

where $[u]_{\delta/2,\delta}$ is a seminorm defined by

$$[u]_{\delta/2,\delta} = \sup_{t_1 \neq t_2, x} \frac{|u(t_1, x) - u(t_2, x)|}{|t_1 - t_2|^{\delta/2}} + \sup_{t, x_1 \neq x_2} \frac{|u(t, x_1) - u(t, x_2)|}{|x_1 - x_2|^\delta}.$$

This definition may be slightly different from different resources. For instance, the definition given by [13] for the seminorm is

$$[u]'_{\delta/2,\delta} = \sup_{(t_1, x_1) \neq (t_2, x_2) \in [0, T] \times \mathbb{T}^d} \frac{|u(t_1, x_1) - u(t_2, x_2)|}{(|t_1 - t_2|^{1/2} + |x_1 - x_2|)^\delta}.$$

Indeed, two norms induced by $[u]_{\delta/2,\delta}$ and $[u]'_{\delta/2,\delta}$ are equivalent, which can be seen from below:

$$[u]_{\delta/2,\delta} = [u]_{\delta/2,0} + [u]_{0,\delta} \leq 2[u]'_{\delta/2,\delta},$$

and

$$\begin{aligned} [u]'_{\delta/2,\delta} &\leq \sup_{(t_1, x_1) \neq (t_2, x_2)} \frac{|u(t_1, x_1) - u(t_2, x_1)| + |u(t_2, x_1) - u(t_2, x_2)|}{(|t_1 - t_2|^{1/2} + |x_1 - x_2|)^\delta} \\ &\leq \sup_{t_1 \neq t_2} \frac{|u(t_1, x_1) - u(t_2, x_1)|}{|t_1 - t_2|^{\delta/2}} + \sup_{x_1 \neq x_2} \frac{|u(t_2, x_1) - u(t_2, x_2)|}{|x_1 - x_2|^\delta} \\ &\leq [u]_{\delta/2,0} + [u]_{0,\delta} = [u]_{\delta/2,\delta}. \end{aligned}$$

– $C^{0,1}([0, T] \times \mathbb{T}^d)$ is a space of functions $u(t, x)$ with a norm

$$|u|_{0,1} = |u|_0 + \sum_{i=1,\dots,d} |\partial_{x_i} u|_0,$$

and $C^{\delta,1}([0, T] \times \mathbb{T}^d)$ is a space of functions $u(t, x)$ with a norm

$$|u|_{\delta,1} = |u|_{0,1} + \sum_{i=1,\dots,d} [\partial_{x_i} u]_{\delta,0}.$$

– $C^{1,2}([0, T] \times \mathbb{T}^d)$ is a space of functions $u(t, x)$ with a norm

$$|u|_{1,2} = |u|_0 + |\partial_t u|_0 + \sum_{i=1,\dots,d} |\partial_{x_i} u|_0 + \sum_{i,j=1,\dots,d} |\partial_{x_i x_j} u|_0.$$

– $C^{1+\delta/2, 2+2\delta}([0, T] \times \mathbb{T}^d)$ is a space with a norm

$$|u|_{1+\delta/2, 2+2\delta} = |u|_{1,2} + [\partial_t u]_{\delta/2,\delta} + \sum_{i,j=1,\dots,d} [\partial_{x_i x_j} u]_{\delta/2,\delta}.$$

– $C^{0,2}([0, T] \times \mathbb{T}^d)$ is a space with a norm

$$|u|_{0,2} = |u|_0 + \sum_{i=1,\dots,d} |\partial_{x_i} u|_0 + \sum_{i,j=1,\dots,d} |\partial_{x_i x_j} u|_0.$$

$C^{\delta,2}([0, T] \times \mathbb{T}^d)$ is a space with a norm

$$|u|_{\delta,2} = |u|_{0,2} + \sum_{i,j=1,\dots,d} [\partial_{x_i x_j} u]_{\delta,0}.$$

We use $C_{0,2}^{\delta,2}([0, T] \times \mathbb{T}^d)$ to denote the space of all functions in $C^{\delta,2}([0, T] \times \mathbb{T}^d)$ topologized by the norm $|\cdot|_{0,2}$. Such a space $C_{0,2}^{\delta,2}([0, T] \times \mathbb{T}^d)$ is not complete. However, every $|\cdot|_{\delta,2}$ -norm bounded ball in $C_{0,2}^{\delta,2}([0, T] \times \mathbb{T}^d)$ is precompact since $C^{\delta,2}([0, T] \times \mathbb{T}^d)$ is compactly embedded into $C^{0,2}([0, T] \times \mathbb{T}^d)$.

- $C_{0,1}^{1,3}([0, T] \times \mathbb{T}^d)$ is the space of all $u \in C^{1,3}([0, T] \times \mathbb{T}^d)$ topologized by $|\cdot|_{0,1}$, *i.e.*

$$|u|_{0,1} = |u|_0 + \sum_{i=1,\dots,d} |\partial_{x_i} u|_0.$$

- $C^{0,0,2}([0, T] \times [0, 1] \times \mathbb{T}^d)$ is the space of all $u(t, \alpha, x)$ having finite norm of

$$|u|_{0,0,2} = |u|_0 + \sum_{i=1,\dots,d} |\partial_{x_i} u|_0 + \sum_{i,j=1,\dots,d} |\partial_{x_i x_j} u|_0.$$

- $C^{\delta,0,2}([0, T] \times [0, 1] \times \mathbb{T}^d)$ is the space of all $u(t, \alpha, x)$ having finite norm of

$$|u|_{\delta,0,2} = |u|_{0,0,2} + \sum_{i,j=1,\dots,d} [\partial_{x_i x_j} u]_{\delta,0,0}.$$

- $C_{0,0,2}^{\delta,0,2}([0, T] \times [0, 1] \times \mathbb{T}^d)$ is the space of all $u(t, \alpha, x)$ having finite norm of $|u|_{\delta,0,2}$, but topologized by $|\cdot|_{0,0,2}$.

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