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LONG TIME DYNAMICS AND UPPER SEMI-CONTINUITY OF ATTRACTORS FOR PIEZOELECTRIC BEAMS WITH NONLINEAR BOUNDARY FEEDBACK*,**,***

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Abstract. A system of boundary-controlled piezoelectric beam equations, accounting for the interactions between mechanical vibrations and the fully-dynamic electromagnetic fields, is considered. Even though electrostatic and quasi-static electromagnetic field approximations of Maxwell's equations are sufficient for most models of piezoelectric systems, where the magnetic permeability is completely discarded, the PDE model considered here retains the pronounced wave behavior of electromagnetic fields to accurately describe the dynamics for the most piezoelectric acoustic devices. It is also crucial to investigate whether the closed-loop dynamics of the fully-dynamic piezoelectric beam equations, with nonlinear state feedback and nonlinear external sources, is close to the one described by the electrostatic/quasi-static equations, when the magnetic permeability μ is small. Therefore, the asymptotic behavior is analyzed for the fully-dynamic model at first. The existence of global attractors with finite fractal dimension and the existence of exponential attractors are proved. Finally, the upper-semicontinuity of attractors with respect to magnetic permeability to the ones of the electrostatic/quasi-static beam equations is shown.

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1. Introduction

Piezoelectric material is a multi-functional smart material to develop electric displacement that is directly proportional to an applied mechanical stress [38], see Figure 1. These materials can be used as actuators requiring an electrical input (voltage, current, or charge) [32, 34]. One of the main components of the electrical input is the drive frequency which determines how fast a piezoelectric beam vibrates or changes its state. Periodic (regularly repeating) and arbitrary signals can be used to drive a piezoelectric beam, which corresponds to continuous control of vibrational modes. Piezoelectric materials are also used as sensors [38] or energy harvesters [13] in

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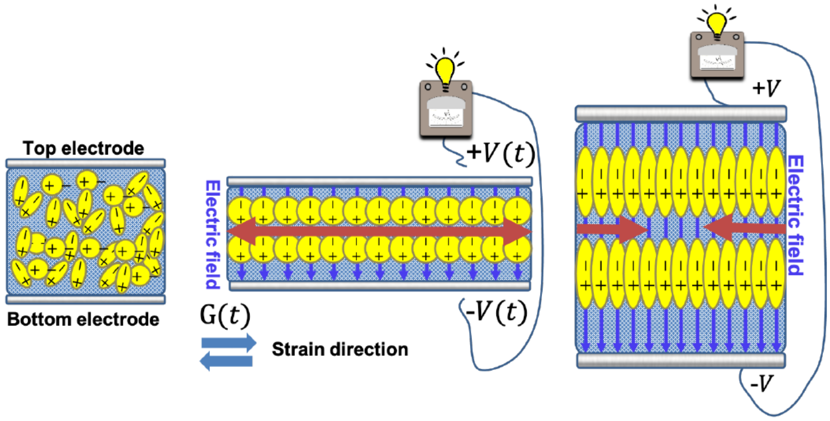

FIGURE 1. (a) A piezoelectric beam is an elastic beam with electrodes at their top and bottom surfaces, and they are connected to an external electric circuit. As voltage $V(t)$ is applied to its electrodes or a mechanical strain controller $G(t)$ are used, the beam is actively (b) stretched or (c) compressed in the longitudinal direction. Therefore, charges are separated and lined up in the vertical direction. Even though the magnetic energy (stored/produced) is smaller in comparison to the electro-mechanical energy, it has direct contribution to the electric field across the electrodes and longitudinal vibrations.

key applications, *i.e.* wearable human-machine interface based on PVDF sensors [2, 11], nano-positioners and micro-sensors [10, 12, 14], and bio-compatible piezoelectric sensors [20].

For many applications of piezoelectricity, electrostatic and quasi-static approximations due to Maxwell's equations are sufficient to describe low-frequency vibrations. However, for piezoelectric acoustic wave devices, there are situations in which the full electromagnetic coupling needs to be considered [9, 39]. Note that as electromagnetic waves are involved, the complete set of Maxwell equations needs to be coupled to the mechanical equations of motion. Such a fully-dynamic theory has been called piezo-electro-magnetism by some researchers [41].

For a beam of length L and thickness h, denoting $v(x,t)$ and $p(x,t)$ by the longitudinal displacements of the centerline of the single piezoelectric beam and the total charge accumulated at the electrodes of the beam, respectively, the linear PDE model (by the small displacement assumptions of Euler-Bernoulli) proposed by [27, 28] is given by

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0 \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \text{ in } \quad (0,L)\times\mathbb{R}^+, \end{cases} \tag{1.1}$$

$$\begin{cases} v(0,t) = 0, \qquad \alpha v_x(L,t) - \gamma\beta p_x(L,t) = \frac{G(t)}{h} \\ p(0,t) = 0, \qquad \beta p_x(L,t) - \gamma\beta v_x(L,t) = -\frac{V(t)}{h}, \ t\in\mathbb{R}^+, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x) \\ p(x,0) = p_0(x), \ p_t(x,0) = p_1(x), \qquad\qquad x\in(0,L) \end{cases} \tag{1.2}$$

where $G(t)$ and $V(t)$ are the strain and voltage controllers, and $\rho,\alpha,\gamma,\mu,\beta > 0$ denote the mass density per unit volume, elastic stiffness, piezoelectric coefficient, magnetic permeability, impermittivity coefficient of the beam, respectively. Moreover, define $\alpha_1 := \alpha - \gamma^2\beta > 0$. This model is obtained by considering the full set of Maxwell's equations, and therefore, accounts for the interaction of the electromagnetic effects with the mechanical vibrations. In practical applications, magnetic effects are completely discarded, *i.e.* the magnetic permeability $\mu \to 0$ and $G(t) \equiv 0$, the second equation is solved for p_{xx} and substituted back to the first equation to obtain

$$\left\{ \rho v_{tt} - \alpha_1 v_{xx} = 0 \text{ in } \quad (0,L) \times \mathbb{R}^+, \right. \tag{1.3}$$

$$\begin{cases} v(0,t) = 0, & \alpha_1 v_x(L,t) = -\frac{\gamma V(t)}{h}, \ \ t \in \mathbb{R}^+, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), & x \in (0,L). \end{cases} \tag{1.4}$$

As a result, although the mechanical equation is dynamic above, the electromagnetic interactions are static. Therefore, the theory of piezoelectricity does not describe the wave behavior of electromagnetic fields as in (1.1). This theory is called the quasi-static or electrostatic theory.

The model (1.3) is a simple wave equation and it is known that as the voltage controller $V(t)$ is chosen in the form of a state feedback $V(t) = kv_t(L,t), k > 0$ in (1.4), which is the tip velocity, an exponential stability result is obtained. Even though the electrostatic and quasi-static assumptions due to the Maxwell's equations are good enough for low-frequency vibrational dynamics, this stability result is not only for the low-frequency vibrations but also for high-frequency vibrations. On the other hand, as the wave behavior of electromagnetic fields is retained as in (1.1), one state feedback controller, voltage $V(t)$ in (1.2), is chosen in the form of total current $V(t) = kp_t(L,t), k > 0$, a lack of exponential stability/exact controllability is proved [27, 28]. Moreover, for certain combinations of the material constants, even approximate controllability/strong stability is at stake. These combinations are mostly corresponding to the high-frequency electromagnetic vibrations. Later, this investigation went deeper in [31] that for certain sub-class of material parameters, an exponential stability is obtained in a more regular state space, which is is the trade-off. An additional mechanical controller is a necessity to make the system exponentially stable [35] without any condition on the material constants or controllers. A class of such linear stabilizing control laws are widely used in controlling piezoelectric beams. Indeed, the actuator design need to be nonlinear due to the addition of magnetic effects (high-frequency solutions). Therefore the choice of the feedback controller must be linear for low-frequency solutions but nonlinear (saturating-type) for high-frequency solutions. For (1.1), a nonlinear boundary feedback is a necessity for the voltage boundary condition in (1.2).

For certain applications, as pointed out in [41], the magnetic effects may become major, and the refined model (1.1) better describes the electromagnetic and mechanical interactions. It is therefore a natural question to ask whether the dynamics of problems described by the fully-dynamic piezoelectric beam model (1.1) is close to the dynamics described by the electrostatic beam model eqreflinear-eletrostatic as the magnetic permeability parameter μ gets smaller. This work answers this question in terms of the convergence of global attractors. Furthermore, it is also important to underline that the analysis of the asymptotic behavior of nonlinear hyperbolic equations with boundary dissipation has been the object of study by many researchers in recent years. Due to the intrinsic difficulties that arise in such models, new analytic techniques have been used for the study of asymptotic dynamics in the sense of compact global attractors (see *i.e.* [3–6, 22, 23, 40], and references therein).

In this paper, the main goal is to analyze the limiting behavior of the fully magnetic model (1.1)–(1.2) with nonlinear boundary feedback controllers $G(t) = g_1(v_t(L,t))$ and $V(t) = g_2(p_t(L,t))$, and a nonlinear mechanical source term $f(v)$:

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta v_t + f(v) = 0 \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \end{cases} \tag{1.5}$$

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) + g_1\big(v_t(L,t)\big) = 0 \\ p(0,t) = \beta p_x(L,t) - \gamma\beta v_x(L,t) + g_2\big(p_t(L,t)\big) = 0, \ \ t \in \mathbb{R}^+, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x) \\ p(x,0) = p_0(x), \ p_t(x,0) = p_1(x), & x \in (0,L). \end{cases} \tag{1.6}$$

As $\mu \to 0$, the limit electrostatic (or quasi-static) equation is obtained

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta v_t + f(v) = 0, \\ -\beta p_{xx} + \gamma\beta v_{xx} = 0, \qquad (x,t) \in (0,L) \times \mathbb{R}^+, \end{cases} \tag{1.7}$$

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) + g_1\big(v_t(L,t)\big) = 0 \\ p(0,t) = \beta p_x(L,t) - \gamma\beta v_x(L,t) + g_2\big(p_t(L,t)\big) = 0, \ t \in \mathbb{R}^+, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x) \\ p(x,0) = p_0(x), \qquad x \in (0,L). \end{cases} \tag{1.8}$$

It is crucial to point out that a similar type of problem for the nonlinear elastic beam equations [21], a one-dimensional version of the von-Karman system, is considered in [26] with a small parameter ε for the longitudinal inertia term. As the weak limit is considered $\varepsilon \to 0$, the velocity of longitudinal wave propagation tends to infinity, or in particular, the stretching equation becomes elliptic, the system converges to so-called nonlocal Timoshenko system.

The main contribution of this paper is twofold:

1. It is proved that solutions of fully-dynamic piezoelectric beam equations converge to ones of the electrostatic (quasi-static) equations as the magnetic permeability coefficient $\mu \to 0$.
2. By considering nonlinear dissipations at the boundary, the existence of a global attractor is established. The global attractor is regular and has finite fractal dimension. It is shown that the upper semi-continuity of attractors of (1.5)–(1.6) to the ones of (1.7)–(1.8) as $\mu \to 0$ is obtained. In fact, the hyperbolic dynamics for the fully-dynamic piezoelectric beam equations (1.5)–(1.6) with nonlinear boundary dissipation leads to a significantly more difficult class of problem, and this was never considered rigorously for piezoelectric beams in the literature. The comparison of the attractors of the fully-dynamic and electrostatic closed-loop equations is also established here.

This paper is organized as the following. In Section 1.1, the existence and uniqueness of weak and strong solutions are established by the theory of monotone operators. In Section 2, the existence of a bounded absorbing set, uniformly in μ, is proved. The classical Lyapunov's approach is discarded, and a compactness/uniqueness argument is utilized. In Section 3, the long-time dynamics for the dynamical system associated to (1.5)–(1.6) is studied. The recent quasi-stability theory developed in [7, 8] is adopted on the absorbing set for the first time for piezoelectric beams. This new and powerful theory implies at once the existence of global attractor, regularity and finite fractal dimension of the attractor as well as the existence of exponential attractors. Finally, in Section 4, the upper-semicontinuity of global attractors is proved as $\mu \to 0$.

1.1. Notations and assumptions

The following notations will be used for the rest of the paper:

$$\|u\|_q = \|u\|_{L^q(0,L)}, \quad q \geqslant 1, \qquad \langle u, v\rangle = \langle u, v\rangle_{L^2(0,L)}.$$

In view of the left-end boundary conditions in (1.6), the following Hilbert space is define

$$H^1_*(0,L) = \big\{u \in H^1(0,L) : u(0) = 0\big\},$$

and since $u(0) = 0$ the Poincaré's inequality holds $\lambda_1\|u\|_2^2 \leqslant \|u_x\|_2^2, \quad \forall u \in H^1_*(0,L)$ where $\lambda_1 > 0$ is the Poincaré's constant (the smallest eigenvalue of $-\partial_x^2$). Therefore, $\|u\|_{H^1_*(0,L)} := \|u_x\|_2$ defines a norm in $H^1_*(0,L)$.

Define the following Hilbert space

$$\mathcal{H} = H^1_*(0,L) \times H^1_*(0,L) \times L^2(0,L) \times L^2(0,L)$$

with the inner product

$$(z,\tilde{z})_{\mathcal{H}} = \rho\langle\phi,\tilde{\phi}\rangle + \mu\langle\varphi,\tilde{\varphi}\rangle + \alpha_1\langle v_x,\tilde{v}_x\rangle + \beta\langle\gamma v_x - p_x, \gamma\tilde{v}_x - \tilde{p}_x\rangle, \tag{1.9}$$

where $z = (v,p,\phi,\varphi)$, $\tilde{z} = (\tilde{v},\tilde{p},\tilde{\phi},\tilde{\varphi}) \in \mathcal{H}$. The norm induced by the inner product on $\mathcal{H}$ can now be defined by

$$\|z\|_{\mathcal{H}}^2 = \rho\|\phi\|_2^2 + \mu\|\varphi\|_2^2 + \alpha_1\|v_x\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2. \tag{1.10}$$

The following assumptions are needed for the nonlinear source term f and damping terms g_1 and g_2 in (1.5) and (1.6).

Assumption (i) Let $f:\mathbb{R}\to\mathbb{R}$ be a C^1-function satisfying

$$|f'(v)| \leqslant k_0\big(1+|v|^{\theta-1}\big), \quad \forall v\in\mathbb{R} \tag{1.11}$$

with $f(0)=0$, $k_0>0$ and $\theta\geqslant 1$. Moreover, letting $\hat{f}(v) = \int_0^v f(s)\mathrm{d}s$, assume that there exist constants $k_1 \geqslant 0$, $m_f > 0$ such that

$$\hat{f}(v) \geqslant -k_1|v|^2 - m_f, \qquad \forall v\in\mathbb{R}, \tag{1.12}$$

and

$$f(v)v - \hat{f}(v) \geqslant -k_1|v|^2 - m_f, \qquad \forall v\in\mathbb{R}, \tag{1.13}$$

where $0\leqslant k_1 < \frac{\alpha_1\lambda_1}{2}$.

Assumption (ii) Let $g_i\in C^1(\mathbb{R})$, $i=1,2$, be increasing functions with $g_i(0)=0$. Moreover, there exist constants $m_i, M_i > 0$ such that

$$m_i \leqslant g_i'(s) \leqslant M_i, \quad \forall s\in\mathbb{R}. \tag{1.14}$$

Observe that **Assumption (ii)** above automatically implies the monotonicity property, *i.e.*

$$(g_i(u)-g_i(v))(u-v) \geqslant m_i|u-v|^2, \quad \forall u,v\in\mathbb{R}. \tag{1.15}$$

Remark 1.1. The function f in **Assumptions (i)** can be chosen as $f(s)=4s^3-2s$. In this case, $\hat{f}(s) = s^4 - s^2$. Thus,

$$\hat{f}(v) \geqslant \min_{\xi\in\mathbb{R}}\{\xi^4-\xi^2\} = -\frac{1}{4}$$

so that (1.11) and (1.12) hold with $m_f = \frac{1}{4}$ and $\theta = 3$. Moreover, since

$$f(v)v - \hat{f}(v) \geqslant 3v^4 - v^2 \geqslant -\frac{1}{12} \geqslant -m_f,$$

it is concluded that (1.13) also holds. For other choices of f, refer to [19].

1.2. Semigroup formulation

Letting $\phi = v_t, \ \ \varphi = p_t$, and

$$z(t) = (v(t), p(t), \phi(t), \varphi(t)) \in \mathcal{H}, \quad z_0 = (v_0, p_0, v_1, p_1) \in \mathcal{H},$$

the system (1.5) can be reformulated into an equivalent Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}z(t)}{\mathrm{d}t} + (\mathbb{A} + \mathbb{B})\, z(t) = \mathbb{F}(z(t)), \\ z(0) = z_0 \in \mathcal{H}, \end{cases} \tag{1.16}$$

where the operator $\mathbb{A} : D(\mathbb{A}) \subset \mathcal{H} \to \mathcal{H}$ is a linear operator and defined by

$$\mathbb{A}z = \begin{pmatrix} -\phi \\ -\varphi \\ -\frac{\alpha}{\rho} v_{xx} + \frac{\gamma\beta}{\rho} p_{xx} \\ -\frac{\beta}{\mu} p_{xx} + \frac{\gamma\beta}{\mu} v_{xx} \end{pmatrix}$$

with the domain given by

$$D(\mathbb{A}) = \Big\{ z = (v, p, \phi, \varphi) \in \mathcal{H} : v, p \in H^2(0, L), v_x(L) = p_x(L) = 0 \Big\}.$$

The nonlinear operator $\mathbb{B} : \mathcal{H} \to \mathcal{H}$ and the forcing function $\mathbb{F} : \mathcal{H} \to \mathcal{H}$ are defined by

$$\mathbb{B}z = \begin{pmatrix} 0_{2\times 1} \\ \frac{\delta}{\rho}\phi + \frac{1}{\rho}\delta(x - L) g_1(\phi(x)) \\ \frac{1}{\mu}\delta(x - L) g_2(\varphi(x)) \end{pmatrix}, \qquad \mathbb{F}(z) = \begin{pmatrix} 0_{2\times 1} \\ -\frac{1}{\rho} f(v) \\ 0 \end{pmatrix}.$$

Lemma 1.2. *The operator $\mathbb{A} + \mathbb{B}$ is a maximal monotone operator in $\mathcal{H}$.*

Proof. Since $(\mathrm{intD}(\mathbb{A})) \cap \mathrm{D}(\mathbb{B}) \neq \emptyset$, it is enough to show that $\mathbb{A}$ and $\mathbb{B}$ are maximal monotone operators by Theorem 2.6 of [1].

Step 1. $\mathbb{A}$ is maximal monotone. Let $z = (v, p, \phi, \varphi), \tilde{z} = (\tilde{v}, \tilde{p}, \tilde{\phi}, \tilde{\varphi}) \in D(\mathbb{A})$. Then, it is straightforward see that $(\mathbb{A}z - \mathbb{A}\tilde{z}, z - \tilde{z})_{\mathcal{H}} = 0$. Therefore $\mathbb{A}$ is monotone.

Now, we prove that $\mathrm{Range}(I + \mathbb{A}) = \mathcal{H}$. For this, it is sufficient to show that for $h = (h_1, h_2, h_3, h_4) \in \mathcal{H}$, there exists $z = (v, p, \phi, \varphi) \in D(\mathbb{A})$ such that $(I + \mathbb{A})z = h$, *i.e.*

$$\begin{cases} v - \phi = h_1 \in H^1_*(0, L), \\ p - \varphi = h_2 \in H^1_*(0, L), \\ \rho\phi - (\alpha v - \gamma\beta p)_{xx} = \rho h_3 \in L^2(0, L), \\ \mu\varphi - (\beta p - \gamma\beta v)_{xx} = \mu h_4 \in L^2(0, L). \end{cases} \tag{1.17}$$

Observe that the system (1.17) is equivalent to

$$\begin{cases} \rho v - (\alpha v - \gamma\beta p)_{xx} = \rho(h_1 + h_3) \in L^2(0, L), \\ \mu p - (\beta p - \gamma\beta v)_{xx} = \mu(h_2 + h_4) \in L^2(0, L), \end{cases} \tag{1.18}$$

and the system (1.18) is equivalent to the variational problem

$$a\big((v,p),(\tilde{v},\tilde{p})\big) = b(\tilde{v},\tilde{p})$$

where the bilinear form $a : \big(H^1_*(0,L)\times H^1_*(0,L)\big)^2 \to \mathbb{R}$ and the linear form $b : H^1_*(0,L)\times H^1_*(0,L)\to\mathbb{R}$ are given by

$$a\big((v,p),(\tilde{v},\tilde{p})\big) = \rho\langle p,\tilde{p}\rangle + \mu\langle p,\tilde{p}\rangle + \alpha_1\langle v_x,\tilde{v}_x\rangle + \beta\langle\gamma v_x - p_x, \gamma\tilde{v}_x - \tilde{p}_x\rangle, \tag{1.19}$$

$$b(\tilde{v},\tilde{p}) = \rho\langle h_1+h_3,\tilde{v}\rangle + \mu\langle h_2+h_4,\tilde{p}\rangle. \tag{1.20}$$

Clearly, a and b are continuous, and moreover, a is coercive since

$$a\big((v,p),(v,p)\big) = \rho\|v\|_2^2 + \mu\|p\|_2^2 + \alpha_1\|v_x\|_2^2 + \|\gamma v_x - p_x\|_2^2.$$

Therefore, the system (1.18) has a unique weak solution $(v,p)\in H^1_*(0,L)\times H^1_*(0,L)$ by the Lax-Milgram Theorem. It follows from (1.18) that

$$\begin{cases}\alpha v_{xx} = \rho v + \gamma\beta p_{xx} - \rho(h_1+h_3),\\ \beta p_{xx} = \mu p + \gamma\beta v_{xx} - \mu(h_2+h_4),\end{cases} \tag{1.21}$$

and since $\alpha_1 = \alpha - \gamma^2\beta$, defined in (1.1)–(1.2),

$$v_{xx} = \frac{\rho}{\alpha_1}v + \frac{\gamma\mu}{\alpha_1}p + \frac{\gamma\mu}{\alpha_1}(h_2+h_4) - \frac{\rho}{\alpha_1}(h_1+h_3)\in L^2(0,L).$$

This leads to $v\in H^2(0,L)\cap H^1_*(0,L)$. Therefore, by the second equation in (1.21), it is concluded that

$$p_{xx} = \frac{\mu}{\beta}p + \gamma v_{xx} - \frac{\mu}{\beta}(h_2+h_4)\in L^2(0,L),$$

and $p\in H^2(0,L)\cap H^1_*(0,L)$. By incorporating the two fists equations in (1.17), $\phi,\varphi\in H^1_*(0,L)$. Hence, it is now shown that there exists $z=(v,p,\phi,\varphi)\in D(\mathbb{A})$ such that $(I+\mathbb{A})z = h$. This proves the maximal monotonicity of $\mathbb{A}$.

Step 2. $\mathbb{B}$ is maximal monotone. It must be shown that the operator is monotone and hemi-continuous by Theorem 2.4 of [1]. Let $z=(v,p,\phi,\varphi), \tilde{z}=(\tilde{v},\tilde{p},\tilde{\phi},\tilde{\varphi})\in D(\mathbb{B})$. From (1.15),

$$\begin{aligned}(\mathbb{B}z-\mathbb{B}\tilde{z}, z-\tilde{z})_{\mathcal{H}} &= \delta\|\phi-\tilde{\phi}\|_2^2 + \big(g_1(\phi(L)) - g_1(\tilde{\phi}(L))\big)(\phi(L)-\tilde{\phi}(L))\\ &\quad + \big(g_2(\varphi(L)) - g_2(\tilde{\varphi}(L))\big)(\varphi(L)-\tilde{\varphi}(L)) \geqslant 0,\end{aligned}$$

and this implies that $\mathbb{B}$ is monotone. Now, letting $z^i = (v^i,p^i,\phi^i,\varphi^i)\in\mathcal{H}$, $i=1,2$ leads to

$$(\mathbb{B}(z^1+tz^2),z)_{\mathcal{H}} = \delta\int_0^L(\phi^1+t\phi^2)\phi\,\mathrm{d}x + g_1(\phi^1(L)+t\phi^2(L))\phi(L) + g_2(\varphi^1(L)+t\varphi^2(L))\varphi(L). \tag{1.22}$$

Now, since

$$(\phi^1+t\phi^2)\phi\to\phi^1\phi \quad\text{pointwise as}\quad t\to 0,$$

by the Lebesgue's Dominated Convergence Theorem,

$$\lim_{t\to 0}\int_0^L(\phi^1+t\phi^2)\phi\,\mathrm{d}x=\int_0^L\phi^1\phi\,\mathrm{d}x. \tag{1.23}$$

By the continuity of g_1 and g_2,

$$\begin{cases}\lim\limits_{t\to 0}g_1(\phi^1(L)+t\phi^2(L))\phi(L)=g_1(\phi^1(L))\phi(L),\\ \lim\limits_{t\to 0}g_2(\varphi^1(L)+t\varphi^2(L))\varphi(L)=g_2(\varphi^1(L))\varphi(L).\end{cases} \tag{1.24}$$

Finally, it follows from (1.22)–(1.24) that

$$\lim_{t\to 0}(\mathbb{B}(z^1+tz^2),z)_{\mathcal{H}}=(\mathbb{B}(z^1),z)_{\mathcal{H}},$$

and thus $\mathbb{B}$ is hemi-continuous; the maximal monotonicity follows. The proof is complete. □

The following definition of weak solution is now introduced, see *e.g.* [3].

Definition 1.3. A function $z=(v,p,v_t,p_t)\in C([0,\infty);\mathcal{H})$ with $z(0)=(v_0,p_0,v_1,p_1)$ is called a weak solution to (1.5)–(1.6) if the following identity is satisfied for all $\xi,\zeta\in H^1_*(0,L)$ in the sense of distributions

$$\begin{aligned}\rho\frac{\mathrm{d}}{\mathrm{d}t}\langle v_t,\xi\rangle+\mu\frac{\mathrm{d}}{\mathrm{d}t}\langle p_t,\zeta\rangle+\alpha_1\langle v_x,\xi_x\rangle+\beta\big\langle\gamma v_x-p_x,\gamma\xi_x-\zeta_x\big\rangle+\langle f(v),\xi\rangle\\ +\delta\langle v_t,\xi\rangle+g_1(v_t(L,t))\xi+g_2(p_t(L,t))\zeta(L)=0.\end{aligned} \tag{1.25}$$

If a weak solution further satisfies

$$z\in C\big([0,\infty);D(\mathbb{A}+\mathbb{B})\big)\cap C^1\big([0,\infty);\mathcal{H}\big),$$

then it is called a strong solution.

Define the total energy of solutions $z=(v,p,v_t,p_t)$ of (1.5) by

$$\mathcal{E}_\mu(t)=E_\mu(t)+\int_0^L\hat{f}(v(t))\,\mathrm{d}x \tag{1.26}$$

where $E_\mu(t)$ is the natural energy corresponding to the system (1.5) as in [28]:

$$E_\mu(t)=\frac{1}{2}\big(\rho\|v_t\|_2^2+\mu\|p_t\|_2^2+\alpha_1\|v_x\|_2^2+\beta\|\gamma v_x-p_x\|_2^2\big). \tag{1.27}$$

Lemma 1.4. *Let $z=\big(v,p,v_t,p_t\big)$ be a strong solution to (1.5)–(1.6). Then, the total energy is dissipative, i.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_\mu(t)=-\delta\|v_t(t)\|_2^2-g_1(v_t(L,t))v_t(L,t)-g_2(p_t(L,t))p_t(L,t)\leqslant 0.\quad\forall t\geqslant 0. \tag{1.28}$$

Moreover, there exist constants $C_0,C_1>0$, independent of μ, such that

$$C_0E_\mu(t)-m_f\leqslant\mathcal{E}_\mu(t)\leqslant C_1(1+(E(t))^{\theta+1}),\quad\forall t\geqslant 0. \tag{1.29}$$

Proof. A straightforward computation yields (1.28) by multiplying the first and second equations in (1.5) by u_t and p_t, respectively. It follows from (1.12) that

$$\int_0^L \hat{f}(v)\,\mathrm{d}x \geqslant -k_1\|v\|_2^2 - Lm_f \geqslant -\frac{k_1}{\lambda_1}\|v_x\|_2^2 - Lm_f \geqslant -\frac{2k_1}{\lambda_1\alpha_1}E_\mu(t) - Lm_f,$$

and therefore,

$$\mathcal{E}_\mu(t) \geqslant C_0 E_\mu(t) - Lm_f$$

by the definition of $\mathcal{E}_\mu(t)$, where

$$C_0 = 1 - \frac{2k_1}{\lambda_1\alpha_1} > 0. \tag{1.30}$$

Hence, the first inequality in (1.29) holds. The second inequality in (1.29) follows easily from (1.11). The proof is now complete. □

Remark 1.5. Observe that, without loss of generality, the energy $\mathcal{E}(t)$ can be assumed positive. Indeed, redefining it as $\tilde{\mathcal{E}}(t) = \mathcal{E}(t) + m_f$, the estimate (1.29) implies that $\tilde{\mathcal{E}}(t) \geqslant 0$ for all $t \geqslant 0$.

Now the following global existence result for the system (1.5)–(1.6) is stated as the following.

Theorem 1.6 (Hadamard Well-posedness)**.** *Suppose that the assumptions (1.11)–(1.14) hold. Then, for any initial data $z_0 \in \mathcal{H}$, the system (1.5)–(1.6) has a unique weak solution $z = (v, p, v_t, p_t)$ satisfying $z \in C([0,\infty);\mathcal{H})$, $\quad z(0) = z_0$, and it is given by*

$$z(t) = e^{t(\mathbb{A}+\mathbb{B})}z_0 + \int_0^t e^{(t-s)(\mathbb{A}+\mathbb{B})}\mathbb{F}(z(s))\,\mathrm{d}s. \tag{1.31}$$

If $z_0 \in D(\mathbb{A}+\mathbb{B})$, the solution is strong. Moreover, the weak solutions depend continuously on the initial data z_0 in $\mathcal{H}$.

Proof. We split the proof into three steps:
Step 1: Local solutions. Since $\mathbb{A}+\mathbb{B}$ is maximal monotone by Lemma 1.2, and $\mathbb{F}$ is locally Lipschitz on $\mathcal{H}$ by (1.11), there exist $t_{\max} \leqslant \infty$ and a unique strong solution $z(t)$ for (1.16) defined on the interval $[0, t_{\max})$ by Theorem 7.2 of [3] for all $z_0 \in D(\mathbb{A}+\mathbb{B})$. Moreover, if $z_0 \in \mathcal{H}$, (1.16) has a unique weak solution $z \in C([0, t_{\max});\mathcal{H})$, and such solution satisfies $\limsup_{t\to t_{\max}^-}\|z(t)\|_{\mathcal{H}} = \infty$ provided that $t_{\max} < \infty$.
Step 2: Global solutions. Let z be a strong solution defined in $[0, t_{\max})$. Then,

$$\mathcal{E}_\mu(t) \leqslant \mathcal{E}_\mu(0), \quad \forall t \in [0, t_{\max}) \tag{1.32}$$

by (1.28). Therefore,

$$\|z(t)\|_{\mathcal{H}}^2 \leqslant \frac{(\mathcal{E}_\mu(0) + Lm_f)}{C_0}, \quad \forall t \in [0, t_{\max}) \tag{1.33}$$

following from (1.29). By a density argument (weak solutions are limits of strong solutions), the inequality (1.33) also holds for weak solutions. Therefore, $t_{\max} = \infty$.

Step 3: Continuous dependence. Let $z^1=(v^1,p^1,v_t^1,p_t^1)$ and $z^2=(v^2,p^2,v_t^2,p_t^2)$ be two weak solutions of (1.11)–(1.14). Then, (1.31) implies

$$\|z^1(t)-z^2(t)\|_{\mathcal{H}} \leqslant \|e^{t(\mathbb{A}+\mathbb{B})}(z^1(0)-z^2(0))\|_{\mathcal{H}} + \int_0^t \|e^{(t-s)(\mathbb{A}+\mathbb{B})}\big(\mathbb{F}(z^1(s))-\mathbb{F}(z^2(s))\big)\|_{\mathcal{H}}\,\mathrm{d}s.$$

Utilizing that $\mathbb{F}$ is locally Lipschitz and by(1.29), there exists a positive constant $C_0=C(\|z^1(0)\|_{\mathcal{H}},\|z^2(0)\|_{\mathcal{H}})$ such that

$$\|z^1(t)-z^2(t)\|_{\mathcal{H}} \leqslant \|z^1(0)-z^2(0)\|_{\mathcal{H}} + C_0\int_0^t \|z^1(s)-z^2(s)\|_{\mathcal{H}}\,\mathrm{d}s.$$

Applying the Gronwall's inequality, it is concluded that

$$\|z^1(t)-z^2(t)\|_{\mathcal{H}} \leqslant e^{C_0T}\|z^1(0)-z^2(0)\|_{\mathcal{H}}, \quad \forall t\in[0,T], \tag{1.34}$$

which proves the continuous dependence of weak solutions. The proof is now complete. □

Remark 1.7. By Theorem 1.6, a one-parameter family of operators $S_\mu(t):\mathcal{H}\to\mathcal{H}$ cna be defined by

$$S_\mu(t)z_0 = z(t) \quad t\geqslant 0 \tag{1.35}$$

where $z(t)=(v(t),p(t),v_t(t),p_t(t))$ is the unique weak solution of the system (1.5)–(1.6) with the initial data $z_0=(v_0,p_0,v_1,p_1)\in\mathcal{H}$. Thus, the pair $(\mathcal{H},S_\mu(t))$ constitutes a dynamical system that will describe the long-time behavior of (1.5)–(1.6).

2. Absorbing set

This section is developed to prove the existence of an absorbing set of the dynamical system $(\mathcal{H},S_\mu(t))$.

Lemma 2.1. *Suppose that* **Assumptions (i) and (ii)** *in (1.11)–(1.14) hold. Then for sufficiently large $T>0$, there exist positive constants $C_1(T),C_2(T),C_3(T)$ (which may depend on μ) such that*

$$\mathcal{E}_\mu(T) \leqslant C_1(T)\Phi(v) + C_2(T)\int_0^T \big(\delta\|v_t(t)\|_2^2 + g_1(v_t(L,t))v_t(L,t) + g_2(p_t(L,t))p_t(L,t)\big)\,\mathrm{d}t + C_3(T) \tag{2.1}$$

where $\Phi(v)=\int_0^T(|v(L,t)|^{\theta+1}+\|v\|_{\theta+1}^{\theta+1})\,\mathrm{d}t$.

Proof. The proof of the theorem is split into three steps.
Step 1. First, multiply the first and second equations in (1.5) by xv_x and xp_x, respectively, and then integrate by parts over $[0,L]\times[0,T]$ to obtain

$$\begin{aligned}\int_0^T E_\mu(t)\,\mathrm{d}t &= -\int_0^L(\rho v_t v_x x + \mu p_t p_x x)\,\mathrm{d}x\Big|_0^T - \int_0^T\int_0^L f(v)v_x x\,\mathrm{d}x\mathrm{d}t - \delta\int_0^T\int_0^L v_t v_x x\,\mathrm{d}x\mathrm{d}t \\ &= \frac{L}{2}\int_0^T\big(\rho|v_t(L,t)|^2 + \mu|p_t(L,t)|^2 + \alpha_1|v_x(L,t)|^2 + \beta|\gamma v_x(L,t)-p_x(L,t)|^2\big)\,\mathrm{d}t.\end{aligned} \tag{2.2}$$

By Hölder's and Young's inequalities,

$$
\begin{aligned}
-\int_0^L (\rho v_t v_x x + \mu p_t p_x x)\,\mathrm{d}x &\leqslant \rho\|v_t\|_2\|v_x x\|_2 + \mu\|p_t\|_2\|p_x x\|_2 \\
&\leqslant (\rho\|v_t\|_2^2 + \mu\|p_t\|_2) + C(\|v_x\|_2^2 + \|p_x\|_2^2) \\
&\leqslant CE_\mu(t)
\end{aligned}
$$

for some constant $C > 0$ independent of T. Hence,

$$
-\int_0^L (\rho v_t v_x x + \mu p_t p_x x)\,\mathrm{d}x\Big|_0^T \leqslant C_1(E_\mu(T) + E_\mu(0)) \tag{2.3}
$$

for some constant $C_1 > 0$ independent for T.

Now, observe that

$$
\begin{aligned}
-\int_0^T\int_0^L f(v)v_x x\,\mathrm{d}x\mathrm{d}t &= -\int_0^T\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}(\hat{f}(v)x)\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_0^L \hat{f}(v)\,\mathrm{d}x\mathrm{d}t, \\
&= L\int_0^T \hat{f}(v(L,t))\,\mathrm{d}t + \int_0^T\int_0^L \hat{f}(v)\,\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

and therefore,

$$
-\int_0^T\int_0^L f(v)v_x x\,\mathrm{d}x \leqslant L\int_0^T |\hat{f}(v(L,t))|\,\mathrm{d}t + \int_0^T\int_0^L |\hat{f}(v)|\,\mathrm{d}x\mathrm{d}t. \tag{2.4}
$$

By the Young's inequality one more time

$$
-\delta\int_0^T\int_0^L v_t v_x x\,\mathrm{d}x\mathrm{d}t \leqslant C\int_0^T \|v_t\|_2^2\,\mathrm{d}t + \frac{1}{2}\int_0^T E_\mu(t)\,\mathrm{d}t. \tag{2.5}
$$

Now, using (1.14) yields

$$
|v_t(L,t)|^2 \leqslant \frac{1}{m_1}g_1(v_t(L,t))v_t(L,t) \quad \text{and} \quad |p_t(L,t)|^2 \leqslant \frac{1}{m_2}g_2(p_t(L,t))p_t(L,t).
$$

Therefore, there exists a constant $C > 0$, independent of T, such that

$$
\int_0^T (\rho|v_t(L,t)|^2 + \mu|p_t(L,t)|^2)\,\mathrm{d}t \leqslant C\int_0^T (g_1(v_t(L,t))v_t(L,t) + g_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t. \tag{2.6}
$$

By (1.6) and the Young's inequality, it is deduced that for any $\epsilon > 0$

$$
\begin{aligned}
&\alpha_1|v_x(L,t)|^2 + \beta|\gamma v_x(L,t) - p_x(L,t)|^2 = -g_1(v_t(L,t))v_x(L,t) - g_2(p_t(L,t))p_x(L,t) \\
&\quad\leqslant C_\epsilon(|g_1(v_t(L,t))|^2 + |g_2(p_t(L,t))|^2) + \epsilon(|v_x(L,t)|^2 + |p_x(L,t)|^2) \\
&\quad\leqslant C_\epsilon(|g_1(v_t(L,t))|^2 + |g_2(p_t(L,t))|^2) + \epsilon\Big((2\gamma^2+1)|v_x(L,t)|_2^2 + 2|\gamma v_x(L,t) - p_x(L,t)|_2^2\Big) \\
&\quad\leqslant C_\epsilon(|g_1(v_t(L,t))|^2 + |g_2(p_t(L,t))|^2) + \epsilon\tilde{c}\Big(\alpha_1|v_x(L,t)|_2^2 + \beta|\gamma v_x(L,t) - p_x(L,t)|_2^2\Big)
\end{aligned} \tag{2.7}
$$

where $\tilde{c} = \max\{(2\gamma^2+1)\alpha_1^{-1}, 2\beta^{-1}\}$. This leads to

$$(1-\epsilon\tilde{c})\Big(\alpha_1|v_x(L,t)|^2 + \beta|\gamma v_x(L,t) - p_x(L,t)|^2\Big) \leqslant C_\epsilon(|g_1(v_t(L,t))|^2 + |g_2(p_t(L,t))|^2).$$

Next, choose $\epsilon = \frac{1}{2\tilde{c}}$ so that

$$\alpha_1|v_x(L,t)|^2 + \beta|\gamma v_x(L,t) - p_x(L,t)|^2 \leqslant 2C_\epsilon(|g_1(v_t(L,t))|^2 + |g_2(p_t(L,t))|^2), \tag{2.8}$$

and, in virtue of (1.14)

$$|g_1(v_t(L,t))|^2 \leqslant \frac{M_1^2}{m_1} g_1(v_t(L,t))v_t(L,t), \tag{2.9}$$

$$|g_2(p_t(L,t))|^2 \leqslant \frac{M_2^2}{m_2} g_2(p_t(L,t))p_t(L,t). \tag{2.10}$$

Combining the estimates (2.8)–(2.10), it is concluded that there exists a constant $C > 0$, independent of T, such that

$$\begin{aligned}\int_0^T &(\alpha_1|v_x(L,t)|^2 + \beta|\gamma v_x(L,t) - p_x(L,t)|^2)\,\mathrm{d}t \\ &\leqslant C\int_0^T (g_1(v_t(L,t))v_t(L,t) + g_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t.\end{aligned} \tag{2.11}$$

Finally, substituting estimates (2.3)–(2.6) and (2.11) in (2.2) yields

$$\begin{aligned}\int_0^T E_\mu(t)\,\mathrm{d}t &\leqslant C(E_\mu(T) + E_\mu(0)) + L\int_0^T |\hat{f}(v(L,t))|\,\mathrm{d}t + \int_0^T\int_0^L |\hat{f}(v)|\,\mathrm{d}x\mathrm{d}t \\ &+ C\int_0^T \big(\delta\|v_t(t)\|_2^2 + g_1(v_t(L,t))v_t(L,t) + g_2(p_t(L,t))p_t(L,t)\big)\,\mathrm{d}t\end{aligned} \tag{2.12}$$

for some $C > 0$, independent of T and μ.
Step 2. The following inequalities

$$\begin{aligned}\int_0^T\int_0^L |\hat{f}(v)|\,\mathrm{d}x\mathrm{d}t &\leqslant C\int_0^T (1 + \|v\|_{\theta+1}^{\theta+1})\,\mathrm{d}t, \\ \int_0^T |\hat{f}(v(L,t))|\,\mathrm{d}x\mathrm{d}t &\leqslant C\int_0^T (1 + |v(L,t)|^{\theta+1})\,\mathrm{d}t\end{aligned} \tag{2.13}$$

are immediate by (1.11). Substituting these estimates in (2.12) leads to

$$\begin{aligned}\int_0^T E_\mu(t)\,\mathrm{d}t &\leqslant C(E_\mu(T) + E_\mu(0)) + C\Phi(v) \\ &+ C\int_0^T \big(\delta\|v_t(t)\|_2^2 + g_1(v_t(L,t))v_t(L,t) + g_2(p_t(L,t))p_t(L,t)\big)\,\mathrm{d}t + CT.\end{aligned} \tag{2.14}$$

Step 3. It follows from the the inequality (1.29) that

$$\int_0^T E_\mu(t)\,\mathrm{d}t \leqslant C(\mathcal{E}_\mu(T)+\mathcal{E}_\mu(0)+Lm_f)+C\Phi(v) \\ +C\int_0^T \big(\delta\|v_t(t)\|_2^2+g_1(v_t(L,t))v_t(L,t)+g_2(p_t(L,t))p_t(L,t)\big)\,\mathrm{d}t+CT. \tag{2.15}$$

Now, by the definition of $\mathcal{E}_\mu(t)$ and the estimate (2.13), it is concluded that there exist $C_1, C_2, C_3 > 0$, independent of T and μ, such that

$$\int_0^T \mathcal{E}_\mu(t)\,\mathrm{d}t \leqslant C_1(\mathcal{E}_\mu(T)+\mathcal{E}_\mu(0))+C_1\Phi(v) \\ +C_2\int_0^T (\delta\|v_t(t)\|_2^2+g_1(v_t(L,t))v_t(L,t)+g_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t+C_3T. \tag{2.16}$$

In view of (1.28),

$$\mathcal{E}_\mu(0)=\mathcal{E}_\mu(T)+\int_0^T (\delta\|v_t(t)\|_2^2+g_1(v_t(L,t))v_t(L,t)+g_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t. \tag{2.17}$$

Next, substitute the last equality into (2.16) and use $\mathcal{E}_\mu(T)\leqslant\mathcal{E}_\mu(t)$ in the left-hand side integral

$$T\mathcal{E}_\mu(T)\leqslant 2C_1\mathcal{E}_\mu(T)+C_1\Phi(v) \\ +C_2\int_0^T (\delta\|v_t(t)\|_2^2+g_1(v_t(L,t))v_t(L,t)+g_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t+C_3T. \tag{2.18}$$

Finally, choosing $T>2C_1$ results in

$$\mathcal{E}_\mu(T)\leqslant C_1(T)\Phi(v)+C_2(T)\int_0^T (\delta\|v_t(t)\|_2^2+g_1(v_t(L,t))v_t(L,t)+g_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t+C_3(T). \tag{2.19}$$

The proof is complete. □

Lemma 2.2 (Absorption of the lower order terms)**.** *Suppose that* **Assumptions (i)** *and* **(ii)** *in* (1.11)–(1.14) *hold. Then, for any $T>0$, there exist positive constants $C(T),\tilde{C}(T)>0$ (which may depend on μ) such that*

$$\Phi(v)\leqslant C(T)\int_0^T (\delta\|v_t(t)\|_2^2+g_1(v_t(L,t))v_t(L,t)+g_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t+\tilde{C}(T) \tag{2.20}$$

for any solution $z=(v,p,v_t,p_t)$ with the initial conditions $z_0=(v_1,p_0,v_1,p_1)\in\mathcal{H}$ satisfying $E_\mu(0)\leqslant d$.

Proof. The standard compactness-uniqueness approach is followed together with arguing by contradiction (see [18, 24]). Fix $T>0$. Supposing that for any $\tilde{C}>0$, there exist $d>0$ and a sequence of initial data

$$z^n(0)=(v^n(0),p^n(0),v_t^n(0),p_t^n(0))\in\mathcal{H}$$

such that

$$E_\mu^n(0)\leqslant d \tag{2.21}$$

and the corresponding weak solutions $z^n = (v^n, p^n, v_t^n, p_t^n)$ satisfy

$$\Phi(v^n) > n\left(\int_0^T (\delta\|v_t^n(t)\|_2^2 + g_1(v_t^n(L,t))v_t^n(L,t) + g_2(p_t^n(L,t))p_t^n(L,t))\,\mathrm{d}t\right) + \tilde{C}, \quad \forall n \in \mathbb{N}. \tag{2.22}$$

Using (1.29) and (2.21), it follows that

$$E_\mu^n(t) \leqslant C_d, \quad \forall t \geqslant 0. \tag{2.23}$$

Therefore, $\Phi(v^n) \leqslant C_d$, and (2.22) implies that

$$\lim_{n\to\infty} \int_0^T (\delta\|v_t^n(t)\|_2^2 + g_1(v_t^n(L,t))v_t^n(L,t) + g_2(p_t^n(L,t))p_t^n(L,t))\,\mathrm{d}t = 0. \tag{2.24}$$

It follows from (2.23) that

$$\begin{aligned} &\{v^n\}, \{p^n\} \quad \text{are bounded in} \quad L^\infty(0,T; H_*^1(0,L)),\\ &\{v_t^n\}, \{p_t^n\} \quad \text{are bounded in} \quad L^\infty(0,T; L^2(0,L)), \end{aligned}$$

from which, there exist subsequences, reindexed again by n, such that

$$(v^n, p^n) \to (v, p) \quad \text{weakly}^* \text{ in } \ L^\infty(0,T; H_*^1(0,L)), \tag{2.25}$$

$$(v_t^n, p_t^n) \to (v_t, p_t) \quad \text{weakly}^* \text{ in } \ L^\infty(0,T; L^2(0,L)). \tag{2.26}$$

Observe that the imbedding $H_*^1(0,L) \hookrightarrow H^{1-\epsilon}(0,L)$ is compact for any $0 < \epsilon < 1$, and therefore, there exists a subsequence by the Aubin's Compactness Theorem (see [37]) such that

$$(v^n, p^n) \to (v, p) \quad \text{strongly in } \ C(0,T; (H^{1-\epsilon}(0,L))^2), \tag{2.27}$$

and thus

$$\lim_{n\to\infty} \Phi(v^n) = \Phi(v). \tag{2.28}$$

Notice that the functions (v^n, p^n) satisfy (in distributional sense) the following system

$$\begin{cases} \rho v_{tt}^n - \alpha v_{xx}^n + \gamma\beta p_{xx}^n + \delta v_t^n + f(v^n) = 0, & (x,t) \in (0,L) \times \mathbb{R}^+,\\ \mu p_{tt}^n - \beta p_{xx}^n + \gamma\beta v_{xx}^n = 0, & (x,t) \in (0,L) \times \mathbb{R}^+, \end{cases} \tag{2.29}$$

$$\begin{cases} v^n(0,t) = \alpha v_x^n(L,t) - \gamma\beta p_x^n(L,t) + g_1\big(v_t^n(L,t)\big) = 0, \ t \in \mathbb{R}^+\\ p^n(0,t) = \beta p_x^n(L,t) - \gamma\beta v_x^n(L,t) + g_2\big(p_t^n(L,t)\big) = 0, \ t \in \mathbb{R}^+. \end{cases} \tag{2.30}$$

Now, we pass the limit above as $n \to \infty$ to infinity. Observe that (2.24) implies

$$v_t = 0, \quad \text{in} \quad (0,L) \times (0,T), \tag{2.31}$$

and $v(x,t) = v(x)$ in $(0,L)$. By (1.15),

$$\int_0^T |v_t^n(L,t)|^2\,\mathrm{d}t \leqslant \frac{1}{m_1}\int_0^T g_1(v_t^n(L,t))v_t^n(L,t)\,\mathrm{d}t,$$

and it is implied by (2.24) that

$$v_t^n(L) \to 0, \quad \text{strongly in } L^2(0,T). \tag{2.32}$$

Analogously,

$$p_t^n(L) \to 0, \quad \text{strongly in } L^2(0,T). \tag{2.33}$$

By the assumption on the damping in (1.14),

$$\int_0^T |g_1(v_t^n(L,t))|^2 \,\mathrm{d}t \leqslant \frac{M_1^2}{m_1} \int_0^T g_1(v_t^n(L,t)) v_t^n(L,t) \,\mathrm{d}x\mathrm{d}t$$

and together with (2.24) lead to

$$g_1(v_t^n(L)) \to 0, \quad \text{strongly in } L^2(0,T), \tag{2.34}$$

and

$$g_2(p_t^n(L)) \to 0, \quad \text{strongly in } L^2(0,T). \tag{2.35}$$

By (2.27) and (1.11), it is concluded that

$$f(v^n) \to f(v), \quad \text{strongly in } L^2(0,T;L^2(0,L)). \tag{2.36}$$

Passing to the limit in (2.29)–(2.30) as $n \to \infty$, and by (2.31)–(2.36),

$$\begin{cases} -\alpha v_{xx} + \gamma\beta p_{xx} + f(v) = 0, \ (x,t) \in (0,L) \times (0,T), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \quad (x,t) \in (0,L) \times (0,T), \end{cases} \tag{2.37}$$

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) = 0, \ t \in (0,T), \\ p(0,t) = \beta p_x(L,t) - \gamma\beta v_x(L,t) = 0, \ t \in (0,T). \end{cases} \tag{2.38}$$

Next, differentiate the first equation in (2.37) and the second one in (2.38) with respect to t, and define $u := p_t$ to get

$$\begin{cases} u_{xx} = 0 & \text{in} \quad (0,L) \times (0,T), \\ u(0,t) = u_x(L,t) = 0 & \text{in} \quad (0,T). \end{cases} \tag{2.39}$$

which results in $u \equiv 0$ in $(0,L) \times (0,T)$. Thus, $p(x,t) = p(x)$ in $(0,L)$. Therefore, the limit $z = (v,p,0,0)$ is a solution to the stationary system

$$\begin{cases} -\alpha v_{xx} + \gamma\beta p_{xx} + f(v) = 0 & \text{in} \quad (0,L), \\ -\beta p_{xx} + \gamma\beta v_{xx} = 0 & \text{in} \quad (0,L), \end{cases} \tag{2.40}$$

$$\begin{cases} v(0) = \alpha v_x(L) - \gamma\beta p_x(L) = 0, \\ p(0) = \beta p_x(L) - \gamma\beta v_x(L) = 0. \end{cases} \tag{2.41}$$

Now, multiply the first and second equations in (2.40) by v and p, respectively, and integrate by parts the result over $(0, L)$ to obtain

$$\alpha_1\|v_x\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2 = -\int_0^L f(v)v\,\mathrm{d}x. \tag{2.42}$$

By (1.12) and (1.13), the following inequality is obtained

$$-\int_0^L f(v)v\,\mathrm{d}x \leqslant 2k_1\|v\|_2^2 + 2m_f \leqslant \frac{2k_1}{\lambda_1\alpha_1}\big(\alpha_1\|v_x\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2\big) + 2Lm_f. \tag{2.43}$$

Combining (2.42), (2.43) and (1.30) yields

$$C_0\big(\alpha_1\|v_x\|_2^2 + \beta\|\gamma v_x - p_x\|_2^2\big) \leqslant 2Lm_f. \tag{2.44}$$

Therefore, there exists a constant $K > 0$ such that

$$\Phi(v) < K. \tag{2.45}$$

On the other hand, choosing $\tilde{C} = K$, $\Phi(v) \geqslant K$ follows from (2.22) and (2.28), which contradicts by (2.45). The proof is complete. □

Theorem 2.3 (Absorbing set)**.** *Suppose that* **Assumptions (i)** *and* **(ii)** *in* (1.11)–(1.14) *hold. Then, there exists a bounded absorbing set B_0 (which may depend on μ).*

Proof. First, combine Lemma 2.1 and Lemma 2.2 to obtain

$$\mathcal{E}_\mu(T) \leqslant C(T)\int_0^T (\delta\|v_t(t)\|_2^2 + g_1(v_t(L,t))v_t(L,t) + g_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t + K(T),$$

and therefore, by the energy equality (2.17)

$$\mathcal{E}_\mu(T) \leqslant C(T)(\mathcal{E}_\mu(0) - \mathcal{E}_\mu(T)) + K(T).$$

Letting $\lambda_T = \frac{C(T)}{1+C(T)}$ and $R_T = \frac{K(T)}{1+C(T)}$ leads to

$$\mathcal{E}_\mu(T) \leqslant \lambda_T\mathcal{E}_\mu(0) + R_T. \tag{2.46}$$

By iterating the estimate on intervals $[mT, (m+1)T]$, $m \in \mathbb{N}$, and using that $\lambda_T < 1$,

$$\mathcal{E}_\mu((m+1)T) \leqslant \lambda_T\mathcal{E}_\mu(mT) + R_T \leqslant \lambda_T^m\mathcal{E}_\mu(0) + \left(\sum_{i=0}^m \lambda_T^i\right) R_T \leqslant \lambda_T^m\mathcal{E}_\mu(0) + \frac{R_T}{1-\lambda_T}.$$

Thus, given $t \geqslant 0$, there exist $m \in \mathbb{N}$ and $r \in [0, T)$ such that $t = mT + r$, and

$$\mathcal{E}_\mu(t) \leqslant \mathcal{E}_\mu(mT) \leqslant \lambda_T^m\mathcal{E}_\mu(0) + \frac{R_T}{1-\lambda_T} = \lambda_T^{-1}\lambda_T^m\mathcal{E}_\mu(0) + \frac{R_T}{1-\lambda_T} \leqslant \lambda_T^{-1}\lambda_T^{\frac{t}{T}}\mathcal{E}_\mu(0) + \frac{R_T}{1-\lambda_T}.$$

Now, choose $\lambda = \lambda_T^{-1}$ and $\omega = -\ln(\lambda_T)/T$ to get

$$\mathcal{E}_\mu(t) \leqslant \lambda \mathcal{E}_\mu(0) e^{-\omega t} + \frac{R_T}{1-\lambda_T}, \quad \forall t \geqslant 0. \tag{2.47}$$

Finally, combining the last estimate and (1.29),

$$\|S_\mu(t) z_0\|_{\mathcal{H}}^2 \leqslant \frac{2\lambda}{C_0} \mathcal{E}_\mu(0) e^{-\omega t} + \frac{2L m_f R_T}{C_0(1-\lambda_T)}, \quad \forall t \geqslant 0.$$

Therefore, the closed ball $B_0 = B^{\mathcal{H}}(0, R_0)$ in $\mathcal{H}$ centered at zero and radius R_0 with $R_0^2 = 1 + \frac{2Lm_f R_T}{C_0(1-\lambda_T)}$ is a bounded absorbing. The proof is complete. ☐

Corollary 2.4 (Exponential Stability)**.** *Under the assumptions of Theorem 2.3 with $f(v) = 0$, the system (1.5)–(1.6) is exponentially stable. More precisely, there exist constants $\kappa_1 > 0$ and $\kappa_2 > 0$ (which may depend on μ) such that the energy $E(t)$ defined in (1.27) satisfies*

$$E_\mu(t) \leqslant \kappa_1 E_\mu(0) e^{-\kappa_1 t}, \quad \forall t \geqslant 0.$$

Proof. Since $f(v) = 0$, $E_\mu(t) = \mathcal{E}_\mu(t)$ and the constant $R_T = 0$ in (2.47). Thus,

$$E_\mu(t) \leqslant \lambda E_\mu(0) e^{-\omega t}, \quad \forall t \geqslant 0.$$

The proof is complete. ☐

3. Global attractors

The main result for the long-time dynamics is given by the following theorem whose proof will be provided at the end of this section.

Theorem 3.1. *Suppose that* **Assumptions (i)** *and* **(ii)** *in (1.11)–(1.14) hold, and $\mu \in [0,1]$. Then,*

(i) The dynamical system $(\mathcal{H}, S_\mu(t))$ is quasi-stable on any bounded positively invariant set $B \subset \mathcal{H}$.
(ii) The dynamical system $(\mathcal{H}, S_\mu(t))$ possesses a unique compact global attractor $\mathcal{A}_\mu \subset \mathcal{H}$, which is characterized by the unstable manifold $\mathcal{A}_\mu = \mathbb{M}_+(\mathcal{N})$ of the set of stationary solutions

$$\mathcal{N} = \left\{ (v,p,0,0) \in \mathcal{H} \;\middle|\; \begin{array}{c} -\alpha v_{xx} + \gamma\beta p_{xx} + f(v) = 0 \\ -\beta p_{xx} + \gamma\beta v_{xx} = 0 \end{array} \right\}.$$

(iii) The attractor $\mathcal{A}_\mu$ has finite fractal and Hausdorff dimension $\dim_{\mathcal{H}}^f \mathcal{A}_\mu$.
(iv) The global attractor $\mathcal{A}_\mu$ is bounded in

$$\mathcal{H}_1 = (H^2(0,L) \cap H_*^1(0,L)) \times (H^2(0,L) \cap H_*^1(0,L)) \times H_*^1(0,L) \times H_*^1(0,L).$$

Moreover, every trajectory $z = (v, p, v_t, p_t)$ in $\mathcal{A}_\mu$ satisfies

$$\|(v,p)\|_{(H^2 \cap H_*^1)^2}^2 + \|(v_t, p_t)\|_{H_*^1 \times H_*^1}^2 + \|(v_{tt}, \sqrt{\mu} p_{tt})\|_{L^2 \times L^2}^2 \leqslant R_1^2, \tag{3.1}$$

for some constant $R_1 > 0$ dependent of $\mathcal{A}_\mu$.

(v) *The dynamical system* $(\mathcal{H}, S_\mu(t))$ *possesses a generalized fractal exponential attractor. More precisely, for any* $\delta \in (0,1]$, *there exists a generalized exponential attractor* $\mathcal{A}^{\exp}_{\mu,\delta} \subset \mathcal{H}$, *with finite fractal dimension in a extended space* $\widetilde{\mathcal{H}}_{-\delta}$, *and it is defined by interpolation of*

$$\widetilde{\mathcal{H}}_0 := \mathcal{H}, \quad \text{and} \quad \widetilde{\mathcal{H}}_{-1} := L^2(0,L) \times L^2(0,L) \times (H^1_*(0,L))^* \times (H^1_*(0,L))^*$$

where $(H^1_*(0,L))^*$ *denotes the dual space of* $H^1_*(0,L)$.

3.1. Quasi-stability estimate uniform in μ

The following theorem plays an important role to prove the existence of a global attractor and its properties. It is usually called the stabilizability inequality (see [8]).

Theorem 3.2. *Suppose that* **Assumptions (i)** *and* **(ii)** *in* (1.11)–(1.14) *hold. Let* B *be a bounded positively invariant set in* $\mathcal{H}$, *and let* $S_\mu(t)z^i = (v^i, p^i, v^i_t, p^i_t)$ *be the weak solution of* (1.5)–(1.6) *with* $z^i \in B$, $i = 1, 2$. *Then, there exist constants* $\vartheta, \eta, C_B > 0$, *depending on* B *yet independent of* μ, *such that*

$$E_\mu(t) \leqslant \vartheta e^{-\eta t} E_\mu(0) + C_B \sup_{s\in[0,t]} \|v(s)\|^2_{2\theta} \tag{3.2}$$

where $v = v^1 - v^2$ *and* $p = p^1 - p^2$.

Proof. For $v = v^1 - v^2$ and $p = p^1 - p^2$, the following notation is adopted

$$F(v) = f(v^1) - f(v^2), \quad G_1(v_t) = g_1(v^1_t) - g_2(v^2_t), \quad G_2(p_t) = g_1(p^1_t) - g_2(p^2_t).$$

Then, (v, p, v_t, p_t) solves the system

$$\begin{cases} \rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} + \delta v_t = -F(v), \\ \mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0, \end{cases} \tag{3.3}$$

with the boundary and initial conditions

$$\begin{cases} v(0,t) = \alpha v_x(L,t) - \gamma\beta p_x(L,t) + G_1\big(v_t(L,t)\big) = 0, \ t \in \mathbb{R}^+, \\ p(0,t) = \beta p_x(L,t) - \gamma\beta v_x(L,t) + G_2\big(p_t(L,t)\big) = 0, \ t \in \mathbb{R}^+, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ x \in (0,L), \\ p(x,0) = p_0(x), \ p_t(x,0) = p_1(x), \ x \in (0,L). \end{cases} \tag{3.4}$$

Now, multiply the first and second equations in (3.3) by xv_x and xp_x, respectively, and integrate by parts over $[0,L] \times [0,T]$ to obtain

$$\begin{aligned} &\int_0^L (\rho v_t v_x x + \mu p_t p_x x)\,\mathrm{d}x\Big|_0^T - \frac{1}{2}\int_0^T\int_0^L \frac{\mathrm{d}}{\mathrm{d}x}\big(\rho|v_t|^2 + \mu|p_t|^2 + \alpha_1|v_x|^2 + \beta|\gamma v_x - p_x|^2\big)x\,\mathrm{d}x\mathrm{d}t \\ &\quad + \delta\int_0^L v_t v_x x\,\mathrm{d}x + \int_0^T\int_0^L F(v)v_x x\,\mathrm{d}x\mathrm{d}t = 0. \end{aligned} \tag{3.5}$$

Therefore,

$$\begin{aligned}\int_0^T E_\mu(t)\,\mathrm{d}t = &-\int_0^L (\rho v_t v_x x + \mu p_t p_x x)\,\mathrm{d}x\Big|_0^T - \int_0^T\int_0^L F(v)v_x x\,\mathrm{d}x\mathrm{d}t - \delta\int_0^L v_t v_x x\,\mathrm{d}x\\ &+\frac{L}{2}\int_0^T \big(\rho|v_t(L,t)|^2+\mu|p_t(L,t)|^2+\alpha_1|v_x(L,t)|^2+\beta|\gamma v_x(L,t)-p_x(L,t)|^2\big)\,\mathrm{d}t.\end{aligned} \tag{3.6}$$

Step 1. First, use the Hölder's and Young's inequalities to get

$$-\int_0^L (\rho v_t v_x x + \mu p_t p_x x)\,\mathrm{d}x\Big|_0^T \leqslant C(E_\mu(T)+E_\mu(0)) \tag{3.7}$$

for some constant $C>0$ independent from T and μ. Next, use (1.11) and the continuous embedding $H^1_*(0,L)\hookrightarrow L^s(0,L)$ for all $s\geqslant 1$ to obtain

$$\int_0^T\int_0^L F(v)v_x x\,\mathrm{d}x\mathrm{d}t \leqslant C_B\int_0^T \|v\|_{2\theta}\|v_x\|_2\,\mathrm{d}t \leqslant C_B\int_0^T \|v\|_{2\theta}^2\,\mathrm{d}t + \epsilon\int_0^T E_\mu(t)\,\mathrm{d}t.$$

From the Young's inequality,

$$-\delta\int_0^T\int_0^L v_t v_x x\,\mathrm{d}x\mathrm{d}t \leqslant C\int_0^T \|v_t\|_2^2\,\mathrm{d}t + \epsilon\int_0^T E_\mu(t)\,\mathrm{d}t. \tag{3.8}$$

By the assumption (1.15), it is deduced that

$$\int_0^T (\rho|v_t(L,t)|^2+\mu|p_t(L,t)|^2)\,\mathrm{d}t \leqslant C\int_0^T (G_1(v_t(L,t))v_t(L,t)+G_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t. \tag{3.9}$$

Analogous to (2.11), the following inequality is immediate

$$\begin{aligned}&\int_0^T (\alpha_1|v_x(L,t)|^2+\beta|\gamma v_x(L,t)-p_x(L,t)|^2)\,\mathrm{d}t\\ \leqslant &C\int_0^T (G_1(v_t(L,t))v_t(L,t)+G_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t.\end{aligned} \tag{3.10}$$

Substituting the estimates (3.7)–(3.10) into (3.6) with $\epsilon>0$ small enough leads to

$$\begin{aligned}\int_0^T E_\mu(t)\,\mathrm{d}t \leqslant &C(E_\mu(T)+E_\mu(0))\\ &+C\int_0^T (\delta\|v_t(t)\|_2^2+G_1(v_t(L,t))v_t(L,t)+G_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t\\ &+C_B\int_0^T \|v(t)\|_{2\theta}^2\,\mathrm{d}t\end{aligned} \tag{3.11}$$

for some constant $C>0$ independent from T and μ.

Step 2. Next, multiply the first and second equations in (3.3) by v_t and p_t, and integrate by parts over $[0,L]\times[s,T]$ so that

$$\begin{aligned}&\int_s^T (\delta\|v_t(t)\|_2^2 + G_1(v_t(L,t))v_t(L,t) + G_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t\\ &\quad= E_\mu(s) - E_\mu(T) - \int_s^T\int_0^L f(v)v_t\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.12}$$

For any $\epsilon > 0$,

$$\int_0^T\int_0^L f(v)v_t\,\mathrm{d}x\mathrm{d}t \leqslant C_B\int_0^T \|v\|_{2\theta}\|v_t\|_2\,\mathrm{d}t \leqslant C_B\int_0^T \|v\|_{2\theta}^2\,\mathrm{d}t + \epsilon\int_0^T E_\mu(t)\,\mathrm{d}t. \tag{3.13}$$

Now, use (3.12) and (3.13) to get

$$\begin{aligned}&\int_0^T (\delta\|v_t(t)\|_2^2 + G_1(v_t(L,t))v_t(L,t) + G_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t\\ &\quad\leqslant E_\mu(0) + E_\mu(T) + \epsilon\int_0^T E_\mu(t)\,\mathrm{d}t + C_B\int_0^T \|v(t)\|_{2\theta}^2\,\mathrm{d}t.\end{aligned} \tag{3.14}$$

Next, combine the estimates (3.11) and (3.14) for $\epsilon > 0$ small enough to obtain

$$\int_0^T E_\mu(t)\,\mathrm{d}t \leqslant C(E_\mu(T) + E_\mu(0)) + C_B\int_0^T \|v(t)\|_{2\theta}^2\,\mathrm{d}t. \tag{3.15}$$

Step 3. Integrate the energy equality (3.12) with respect to s so that

$$\begin{aligned}TE_\mu(T) = &\int_0^T E_\mu(t)\,\mathrm{d}t - \int_0^T\int_s^T (\delta\|v_t(t)\|_2^2 + G_1(v_t(L,t))v_t(L,t) + G_2(p_t(L,t))p_t(L,t))\,\mathrm{d}t\,\mathrm{d}s\\ &- \int_0^T\int_s^T\int_0^L f(v)v_t\,\mathrm{d}x\mathrm{d}t\mathrm{d}s.\end{aligned}$$

By the estimate (3.13) and the fact that $\delta\|v_t(t)\|_2^2 + G_1(v_t(L,t))v_t(L,t) + G_2(p_t(L,t))p_t(L,t) \geqslant 0$, the following inequality is immediate

$$TE_\mu(T) \leqslant 2\int_0^T E_\mu(t)\,\mathrm{d}t + C_{B,T}\int_0^T \|v(t)\|_{2\theta}^2\,\mathrm{d}t. \tag{3.16}$$

By substituting the estimate (3.15) in (3.16),

$$TE_\mu(T) \leqslant C(E_\mu(T) + E_\mu(0)) + C_{B,T}\int_0^T \|v(t)\|_{2\theta}^2\,\mathrm{d}t,$$

and choosing $T > 2C$ to deduce that

$$E_\mu(T) \leqslant \lambda_T E_\mu(0) + C_{B,T}\int_0^T \|v(t)\|_{2\theta}^2\,\mathrm{d}t,$$

where $\lambda_T = \frac{C}{T-C} < 1$. Finally, proceed in a similar fashion as in the proof of Theorem 2.3 to conclude that there exist $\vartheta, \eta, C_B > 0$ such that

$$E_\mu(t) \leqslant \vartheta e^{-\eta t} E_\mu(0) + C_B \sup_{s\in[0,t]} \|v(s)\|_{2\theta}^2.$$

Therefore, (3.2) holds, and the proof is complete. □

3.2. Proof of Theorem 3.1

(i) Consider a bounded positively invariant set $B \subset \mathcal{H}$ and denote $S_\mu(t)z^i = (v^i(t), p^i(t), v_t^i(t), p_t^i(t))$ for $z^i \in B$, $i = 1, 2$. Set also $v = v^1 - v^2, p = p^1 - p^2$, as before. It follows from (1.34) that

$$\|S_\mu(t)z^1 - S_\mu(t)z^2\|_{\mathcal{H}} \le a(t)\|z^1 - z^2\|_{\mathcal{H}} \tag{3.17}$$

with $a(t) = e^{C_0 T}$. Now let $X = H_*^1(0, L)$, and define the semi-norm $n_X(v) := \|v\|_{2\theta}$. Since the embedding $H_*^1(0, L) \hookrightarrow L^{2\theta}(0, L)$ is compact, it is known that n_X is a compact semi-norm on $H_*^1(0, L)$.

By Lemma 3.2, (3.17) leads to

$$\|S_\mu(t)z^1 - S_\mu(t)z^2\|_{\mathcal{H}}^2 \le b(t)\|z_1 - z_2\|_{\mathcal{H}}^2 + c(t) \sup_{s\in[0,t]} \left[n_X(v^1(s) - v^2(s))\right]^2 \tag{3.18}$$

where $b(t) = \vartheta e^{-\eta t}$ and $c(t) = C_B$. Clearly,

$$b(t) \in L^1(\mathbb{R}^+) \ \text{ and } \ \lim_{t\to\infty} b(t) = 0.$$

Since $B \subset \mathcal{H}$ is bounded, $c(t)$ is locally bounded on $[0, \infty)$, and therefore, the dynamical system $(\mathcal{H}, S_\mu(t))$ is quasi-stable on any bounded positively invariant set $B \subset \mathcal{H}$ by Definition 7.9.2 of [8].

(ii) Since the system $(\mathcal{H}, S_\mu(t))$ is quasi-stable, $(\mathcal{H}, S_\mu(t))$ is asymptotically smooth by a direct application of the result of Proposition 7.9.4 in [8]. Recalling that it is already shown in Corollary 2.3 that the dynamical system has an absorbing ball, the existence of a compact global attractor $\mathcal{A}_\mu$ is established by Corollary 7.9.5 of [8]. Moreover, the assumption (1.14) and the energy identity (1.28) imply that the energy $\mathcal{E}(t)$ is a strict Lyapunov function Ψ for $(\mathcal{H}, S_\mu(t))$. Hence, it is implied by Theorem 7.5.6 of [8] that $\mathcal{A}_\mu = \mathbb{M}_+(\mathcal{N})$.

(iii) From the discussion above, $(\mathcal{H}, S_\mu(t))$ is quasi-stable on the attractor $\mathcal{A}_\mu$. Thus, the attractor $\mathcal{A}_\mu$ has finite fractal dimension $\dim_{\mathcal{H}}^f \mathcal{A}_\mu$ by Theorem 7.9.6 of [8].

(iv) Since the system $(\mathcal{H}, S_\mu(t))$ is quasi-stable on the attractor $\mathcal{A}_\mu$ with $c_\infty = \sup_{t\in\mathbb{R}^+} c(t) = C_{\mathcal{A}_\mu} < \infty$, it follows from Theorem 7.9.8 of [8] that any complete trajectory $z = (v, p, v_t, p_t)$ in $\mathcal{A}_\mu$ has the following regularities

$$v_t, p_t \in L^\infty(\mathbb{R}, H_*^1(0, L)) \cap C(\mathbb{R}, L^2(0, L)),$$

and

$$v_{tt}, p_{tt} \in L^\infty(\mathbb{R}, L^2(0, L)).$$

In particular, there exists $R > 0$ such that

$$\|(v_t, p_t)\|_{H_*^1\times H_*^1}^2 + \|(v_{tt}, \sqrt{\mu} p_{tt})\|_{L^2\times L^2}^2 \leqslant R^2.$$

Next, using (1.5) and noting that the nonlinear terms are continuous, it is concluded that, there exists a constant $R' > 0$ such that

$$\|(v,p)\|^2_{H^2\cap H^1_*} \leqslant R'.$$

Therefore, (3.1) holds. Since the global attractors $\mathcal{A}_\mu$ are also characterized by

$$\mathcal{A}_\mu = \{z(0) : z \text{ is a bounded full trajectory of } S_\mu(t)\},$$

$\mathcal{A}_\mu$ is bounded in $\mathcal{H}_1$.

(v) Let B_0 be the absorbing ball given by Theorem 2.3. For the solution $z(t)$ with initial data $z_0 = z(0) \in B_0$, there exists $C_{B_0} > 0$ such that for any $0 \leqslant t \leqslant T$, $\|z_t(t)\|_{\widetilde{\mathcal{H}}_{-1}} \leqslant C_{B_0}$ which leads to

$$\|S_\mu(t_1)z_0 - S_\mu(t_2)z_0\|_{\widetilde{\mathcal{H}}_{-1}} \leqslant \int_{t_1}^{t_2} \|z_t(\tau)\|_{\widetilde{\mathcal{H}}_{-1}}\,\mathrm{d}\tau \leqslant C_{B_0}|t_1 - t_2|, \tag{3.19}$$

for any $0 \leqslant t_1 < t_2 \leqslant T$. From (3.19), it is concluded that for any $z_0 \in B_0$, the map $t \mapsto S_\mu(t)z_0$ is Hölder continuous in the extended space $\widetilde{\mathcal{H}}$ with the exponent $\delta = 1$. Then, the existence of a generalized exponential attractor, whose fractal dimension is finite, is immediate in $\widetilde{\mathcal{H}}_{-1}$. Following the similar arguments in Theorem 5.1 of [25], the existence of exponential attractors is obtained in $\widetilde{\mathcal{H}}_{-\delta}$ with $\delta \in (0,1)$. This completes the proof of Theorem 3.1. □

4. Upper semi-continuity

In this section, we prove that the global attractor $\mathcal{A}_\mu$ for the piezoelectric beam model (1.5)–(1.6) converges upper-semicontinuously to the global attractor $\mathcal{A}_0$ of the electrostatic (or quasi-static) equation (1.7)–(1.8) as $\mu \to 0$.

Lemma 4.1. *Under assumption of Theorem 3.1, the global attractors $\mathcal{A}_\mu$ are uniformly bounded with respect to μ.*

Proof. Since $(\mathcal{H}, S_\mu(t))$ is gradient with a strict Lyapunov function Ψ,

$$\sup_{z\in\mathcal{A}_\mu} \Psi(z) \leqslant \sup_{z\in\mathcal{N}} \Psi(z).$$

by Remark 7.5.8 of [8]. Then, by the definition of Ψ and (1.29), it is deduced that

$$\sup_{z\in\mathcal{A}_\mu} \|z\|^2_{\mathcal{H}} \leqslant C\Big(\sup_{z\in\mathcal{A}_\mu} \Psi(z) + m_f\Big) \leqslant C\Big(\sup_{z\in\mathcal{N}} \Psi(z) + m_f\Big) \leqslant C\Big(\sup_{z\in\mathcal{N}} \|z\|^{\theta+1}_{\mathcal{H}} + 1\Big).$$

Thus, there exists a constant $R > 0$ independent of μ such that

$$\sup_{z\in\mathcal{A}_\mu} \|z\|^2_{\mathcal{H}} \leqslant R$$

by (2.44). The proof is now complete. □

The well-posedness analysis of the electrostatic equation (1.7)–(1.8) is studied on the Hilbert space

$$\mathcal{H}_0 = H^1_*(0,L) \times L^2(0,L) \times H^1_*(0,L).$$

By the same arguments, as in the proof of Theorem 3.1, the dynamical system $(\mathcal{H}_0, S_0(t))$ generated by the electrostatic equation (1.7)–(1.8) has the global attractor $\mathcal{A}_0 \subset \mathcal{H}_0$. We are now ready to formulate and prove the upper-semicontinuity result of global attractors.

Theorem 4.2. *Suppose that* **Assumptions (i)** *and* **(ii)** *in* (1.11)–(1.14) *hold. Then, the family of attractors* $\mathcal{A}_\mu$ *is upper-semicontinuous at* $\mu = 0$*, i.e.,*

$$\lim_{\mu\to 0} \mathrm{dist}_{\mathcal{H}_0}(\mathcal{P}\mathcal{A}_\mu, \mathcal{A}_0) = 0 \tag{4.1}$$

where $\mathcal{P} : \mathcal{H} \to \mathcal{H}_0$ *denotes the projection map defined by* $\mathcal{P}(v, p, v', p') = (v, v', p)$*.*

Proof. The arguments of the proof mimics the ones in [15, 17]). First, it is proceeded by a contradiction argument as in [25]. Suppose that (4.1) does not hold. Then, there exist an $\epsilon > 0$ and a sequence $\mu_n \to 0$ such that $\mathrm{dist}_{\mathcal{H}_0}(\mathcal{P}\mathcal{A}_{\mu_n}, \mathcal{A}_0) \geqslant \epsilon > 0, \quad \forall n \in \mathbb{N}$. Thus, by the compactness of $\mathcal{A}_\mu$, there exists a sequence $\{z_0^n\} \in \mathcal{A}_{\mu_n}$ such that

$$\mathrm{dist}_{\mathcal{H}_0}(\mathcal{P}z_0^n, \mathcal{A}_0) \geqslant \epsilon > 0, \quad \forall n. \tag{4.2}$$

Let $z^n(t) = (v^n(t), p^n(t), v_t^n(t), p_t^n(t))$ be a full trajectory from the attractor $\mathcal{A}_{\mu_n}$ such that $z^n(0) = z_0^n$. It follows from (3.1) and the Aubin-Lions theorem (see [37]) that there exist a subsequence, not renumbered, $\{z^n\}$ and $z = (v, p, v_t, p_t) \in C([-T, T]; \mathcal{H})$ such that

$$\lim_{k\to\infty} \max_{t\in[-T,T]} \|z^n(t) - z(t)\|_{\mathcal{H}} = 0. \tag{4.3}$$

Since $\mathcal{P} : \mathcal{H} \to \mathcal{H}_0$ is continuous

$$\lim_{k\to\infty} \max_{t\in[-T,T]} \|\mathcal{P}z^n(t) - \mathcal{P}z(t)\|_{\mathcal{H}_0} = 0. \tag{4.4}$$

Defining $\widehat{z} = \mathcal{P}z = (v, v_t, p)$ and using (3.1) lead to

$$\sup_{t\in\mathbb{R}} \|\widehat{z}(t)\|_{\mathcal{H}_0} < \infty. \tag{4.5}$$

Since $z^n = (v^n, p^n, v_t^n, p_t^n)$ is a strong solution, the following variational identity is satisfied

$$\begin{aligned} &\rho \frac{\mathrm{d}}{\mathrm{d}t}\langle v_t^n, \xi\rangle + \mu_n \frac{\mathrm{d}}{\mathrm{d}t}\langle p_t^n, \zeta\rangle + \alpha_1\langle v_x^n, \xi_x\rangle + \beta\big\langle \gamma v_x^n - p_x^n, \gamma\xi_x - \zeta_x\big\rangle + \langle f(v^n), \xi\rangle + \delta\langle v_t^n, \xi\rangle \\ &\quad + \Big(g_1(v_t^n(L,t))\xi(L) + g_2(p_t^n(L,t))\zeta(L)\Big) = 0 \end{aligned} \tag{4.6}$$

for all $\xi, \zeta \in H_*^1(0, L)$. Now, it is concluded by (4.4) and (1.11) that

$$\lim_{n\to\infty} \int_0^L (f(v^n) - f(v))\,\xi\,\mathrm{d}x = 0. \tag{4.7}$$

Moreover,

$$\sup_{t\in\mathbb{R}}(\mu_n\|p_{tt}^n(t)\|_2) = \sqrt{\mu_n}\sup_{t\in\mathbb{R}}\Big(\sqrt{\mu_n}\|p_{tt}^n(t)\|_2\Big) \leqslant \sqrt{\mu_n}R_1 \to 0 \quad \text{as} \quad n\to\infty \tag{4.8}$$

by the estimate (3.1). Therefore, (4.3) and (4.7) together with (4.8) yield the following

$$\begin{aligned}\rho\frac{\mathrm{d}}{\mathrm{d}t}\langle v_t,\xi\rangle+\alpha_1\langle v_x,\xi_x\rangle+\beta\big\langle\gamma v_x-p_x,\gamma\xi_x-\zeta_x\big\rangle+\langle f(v),\xi\rangle+\delta\langle v_t,\xi\rangle\\+\Big(g_1(v_t(L,t))\xi(L)+g_2(p_t(L,t))\zeta(L)\Big)=0\end{aligned}\tag{4.9}$$

for all $\xi,\zeta\in H^1_*(0,L)$. Therefore, $\widehat{z}(t)=(v(t),v_t(t),p(t))$ is a weak solution of the electrostatic equation (1.7)–(1.8). In view of (4.5), $\widehat{z}(t)$ is a bounded full trajectory for the limiting semi-flow $S_0(t)$. Finally, (4.4) implies that

$$\mathcal{P}z_0^n\to Pz(0)=\widehat{z}(0)\in\mathcal{A}_0\ \text{ in }\ \mathcal{H}_0,$$

which contradicts with (4.2). The proof is complete now. □

5. Conclusions and future work

In this paper, it is proved that solutions of fully-dynamic piezoelectric beam equations even with nonlinear state feedback and nonlinear external sources converge to ones of the electrostatic (quasi-static) equations as the magnetic permeability coefficient $\mu\to 0$.

The existence of a global attractor in finite fractal dimension is shown in addition to the upper semi-continuity of attractors of (1.5)–(1.6) to the ones of (1.7)–(1.8) as $\mu\to 0$. In fact, the hyperbolic dynamics for the system in (1.5)–(1.6) leads to a significantly more difficult class of problems, and this was never considered rigorously for piezoelectric beams in the literature. The comparison of the attractors of the fully-dynamic and electrostatic closed-loop equations is also established here. Since the fully-dynamic model better describes the controllability dynamics for certain classes of piezoelectric acoustic devices, the analysis established here is much needed in the literature.

An immediate future work is already under investigation for multi-layer elastic [16] or piezoelectric [32] laminates, where the hyperbolic (longitudinal vibrations) and parabolic (bending vibrations) interactions across layers as well as the non-compact coupling through the equations make the analysis slightly more challenging with the addition of nonlinear distributed/boundary feedback.

A numerical investigation for the linear model (1.5)–(1.6) is already under investigation [29], where the system is discretized by the filtered Finite Differences [30]. The major discrete energy estimates are proved by discrete multipliers and discrete Ingham-type theorems. Since spectral theory is not a good route for the full nonlinear model (1.5)–(1.6), a thorough analysis by multipliers is much needed to obtain the energy estimates.

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