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SMOOTH OUTPUT-TO-STATE STABILITY FOR MULTISTABLE SYSTEMS ON COMPACT MANIFOLDS

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Abstract. Output-to-State Stability (OSS) is a notion of detectability for nonlinear systems that is formulated in the ISS framework. We generalize the notion of OSS for systems which possess a decomposable invariant set and evolve on compact manifolds. Building upon a recent extension of the ISS theory for this very class of systems [D. Angeli and D. Efimov, *IEEE Trans. Autom. Control* **60** (2015) 3242–3256.], the paper provides equivalent characterizations of the OSS property in terms of asymptotic estimates of the state trajectories and, in particular, in terms of existence of smooth Lyapunov-like functions.

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1. Introduction

As a central notion in control theory, detectability corresponds roughly to the property of eventually having small states whenever the outputs are small, regardless of the initial state. The state of any detectable and stabilizable linear system can in fact be driven to zero using only output measurements. For this reason, detectability has proved its usefulness both in state- and output-feedback design, as well as in state estimation. While the concept of detectability is well defined and fully understood in the context of linear systems, the same does not hold for nonlinear systems where several possible formulations are available. One such formulation is obtained by dualizing the notion of Input-to-State Stability (ISS) [15]. Indeed, ISS formalizes the idea of having small states whenever the inputs are small, regardless of the initial state, and thus appears as a natural candidate for detectability by simply replacing inputs with outputs. The resulting notion is called Output-to-State Stability (OSS) [17] and embeds a nonlinear gain function of the infinity norm of the output to ultimately bound the norm of the state from above. Equivalent Lyapunov-like characterizations of OSS have been provided in [6, 16].

At the same time, the analysis of the stability and robustness properties of nonlinear systems exhibiting a variety of non-trivial dynamical behaviors such as multiple equilibria, periodicity, amost-periodicity, chaos – in short, multistable systems – is becoming more and more appealing from the perspective of systems and control theory due to its importance for several scientific disciplines ranging from mechanics and electronics to biology and neuroscience. In a recent work [2], the ISS concept has been generalized for such multistable systems. In

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particular, the authors in [2] have shown that the most natural way of investigating the stability properties of multistable systems by means of Lyapunov-like dissipation inequalities is to relax the Lyapunov stability requirement on the invariant set of the system, and to incorporate instead relatively mild assumptions on the decomposition of such a set.

One may wonder if it is possible to dualize ISS also in this generalized context, so as to obtain an analogous property for OSS by simply using outputs instead of inputs in the definition. Indeed, this paper provides a notion of detectability for multistable systems which portrays the behavior of ultimately having small distance of the states from some invariant set of the dynamics whenever the outputs are small. Specifically, we characterize such generalized OSS in terms of asymptotic estimates via nonlinear gain functions and in terms of a smooth Lyapunov/LaSalle-like dissipation inequality.

Notation: The symbol $\mathfrak{d}[x_1, x_2]$ denotes the Riemannian distance [5] between two points x_1 and x_2 of a Riemannian manifold M. For a set $S \subset M$ define $|\cdot|_S$ as $|x|_S = \inf_{a\in S} \mathfrak{d}[x, a]$. For a compact set $A \subset M$, $\eth(A)$ and int $\{A\}$ respectively denote the boundary and the interior of A, [5]. For a smooth manifold M we denote by TM its tangent bundle, and by $T^\star M$ its cotangent bundle, [1]. As customary, we let T_xM denote the tangent space of M at a point x. We denote the differential ([1], Def. 4.2.5) of a smooth function $V : M \to \mathbb{R}$ by the covector field $\mathrm{d}V : M \to T^\star M$, and then we denote the Lie derivative ([1], Def. 4.2.6) of V along a vector field $f : M \to TM$ at a point $x \in M$ by $\mathcal{L}_f V(x) := \mathrm{d}V(f)(x) = \mathrm{d}V(x) \cdot f(x)$, namely the action of $\mathrm{d}V$ on f at x. Given $x \in M$, notation $\mathrm{d}V(x) = 0$ is intended to be equivalent to $\mathrm{d}V(f) = 0$ for any $f \in T_xM$. Notations $|\cdot|$ and $|\cdot|_\mathfrak{g}$ respectively indicate the standard Euclidean norm and the norm induced by the Riemannian metric $\mathfrak{g}$ on T_xM, *i.e.* $|v|_\mathfrak{g} := \sqrt{\mathfrak{g}_x(v, v)}$ for any $v \in T_xM$. For a measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ define its infinity norm over the time interval $[t_1, t_2]$ as $\left\|d_{[t_1,t_2]}\right\| = \operatorname{ess\,sup}_{t_1\leq t\leq t_2} |d(t)|$, and denote $\|d\| := \left\|d_{[0,+\infty)}\right\|$. Define the infinity norm of $d(\cdot)$ with respect to a compact set S as follows: $\left\|d_{[t_1,t_2]}\right\|_S = \operatorname{ess\,sup}_{t_1\leq t\leq t_2} |d(t)|_S$, and define $\|d\|_S := \left\|d_{[0,+\infty)}\right\|_S$. We define the saturation function as $\mathrm{sat}(x) := \mathrm{sign}(x) \min\{1, |x|\}$.

2. Decomposition modulo infinity

Let M be a compact, geodesically complete, connected, n-dimensional Riemannian manifold without boundary. Let Y be a closed subset of $\mathbb{R}^l$ containing the origin. Consider the autonomous system:

$$\dot{x}(t) = f(x(t)), \quad y(t) = h(x(t)), \tag{2.1}$$

with state x taking values in M, output y taking values in $Y \subseteq \mathbb{R}^l$, and where $f(x) : M \to T_xM$ is a locally Lipschitz continuous vector field, and $h(x) : M \to Y$ is a locally Lipschitz continuous mapping. Denote by $X(t, x)$ (or also $\varphi_t(x)$) the uniquely defined solution of (2.1) at time t fulfilling $X(0, x) = x$ (or, respectively, $\varphi_0(x) = x$). Let y denote the signal $y(\cdot) = h(X(\cdot, x))$.

Notice that system (2.1) is governed by autonomous dynamics. The more general case of systems with exogenous input signals is of great theoretical and practical interest, but is outside the scope of the present manuscript.

In this paper, we will consider a specific compact invariant set[1] for the dynamics (2.1). In particular, we will characterize such invariant set in terms of its decomposition, according to the following definition.

Definition 2.1 (Decomposition for a compact invariant set)**.** A decomposition for a compact invariant set Λ is a finite family of disjoint compact invariant sets $\Lambda_1, \ldots, \Lambda_N$ such that

$$\Lambda = \bigcup_{i=1}^{N} \Lambda_i.$$

The $\Lambda_1, \ldots, \Lambda_N$ are referred to as the *atoms* of the decomposition of Λ.

[1] A closed set Λ is invariant if for any $x \in \Lambda$, it holds $X(t, x) \in \Lambda$ for all $t \in \mathbb{R}$.

Notice that a decomposition is not uniquely defined. In particular, even in the simplest case of Λ comprising finitely many points (equilibria), one could choose atoms Λ_i of different cardinalities, thus yielding a non-uniquely defined total number N of atoms.

Since the Λ_is are a finite number of compact sets, there exist disjoint open neighborhoods $U_i \supset \Lambda_i$, for all $i \in \{1, \dots, N\}$.

Assumption 2.2. Let $\mathcal{Z} \subset M$ be a compact invariant set containing all α- and ω-limit sets of (2.1). We assume the existence of a decomposition of $\mathcal{Z}$ given by $\mathcal{Z}_1, \dots, \mathcal{Z}_{K+1}$. Furthermore, we denote with $\mathcal{W}$ the collection of K atoms $\mathcal{Z}_1, \dots, \mathcal{Z}_K$, which we label $\mathcal{W}_i = \mathcal{Z}_i$ for all $i \in \{1, \dots, K\}$. Denote with $\mathcal{Z}^\infty$ the compact invariant set $\mathcal{Z} \setminus \mathcal{W} = \mathcal{Z}_{K+1}$ and assume that $\mathcal{Z}^\infty$ consists of fixed points only.

Notice that, in Assumption 2.2, the atom $\mathcal{Z}_{K+1}$ is singled out and relabeled as $\mathcal{Z}^\infty$ as it plays a special role in the way solutions may "converge to" or "diverge from" this atom. Intuitively, the considered detectability property implies that solutions corresponding to small outputs are constrained to converge to the $\mathcal{W}$ part of the attractor $\mathcal{Z}$. This will be characterized through a Lyapunov dissipation inequality where strict dissipation occurs outside of $\mathcal{W}$, which, however may be offset by non-zero output values. In this respect, of all ω and α limit sets, $\mathcal{Z}^\infty$ can be identified by looking at where the output function h is vanishing, as clarified in the following section.

Definition 2.3 (Attracting and repulsing subsets)**.** For a compact invariant set Ξ, its attracting and repulsing subsets are defined as follows:

$$\begin{aligned}\mathcal{A}(\Xi) &= \{x \in M : |X(t,x)|_\Xi \to 0 \text{ as } t \to +\infty\}\,, \\ \mathcal{R}(\Xi) &= \{x \in M : |X(t,x)|_\Xi \to 0 \text{ as } t \to -\infty\}\,.\end{aligned}$$

Definition 2.4 (Connecting orbit)**.** Given two invariant and disjoint subsets Ξ_1, Ξ_2, we define the relation $\Xi_1 \prec \Xi_2$ if $\mathcal{A}(\Xi_1) \cap \mathcal{R}(\Xi_2) \neq \emptyset$.

Definition 2.5 (Cycles-modulo-infinity)**.** Let $\mathcal{Z}_1, \dots, \mathcal{Z}_{K+1}$ be a decomposition of a compact invariant set $\mathcal{Z}$.

1. An *r-cycle modulo-infinity* ($r \geq 2$) is an ordered r-tuple of distinct indices $i_1, \dots, i_r \in \{1, \dots, K\}$ such that $\mathcal{Z}_{i_1} \prec \mathcal{Z}_{i_2} \prec \cdots \prec \mathcal{Z}_{i_r} \prec \mathcal{Z}_{i_1}$.
2. A *1-cycle modulo-infinity* is an index $i \in \{1, \dots, K\}$ such that $\mathcal{A}(\mathcal{Z}_i) \cap \mathcal{R}(\mathcal{Z}_i) \setminus \mathcal{Z}_i \neq \emptyset$.
3. A *filtration ordering modulo-infinity* is a numbering of the $\mathcal{Z}_h$s with $h \in \{1, \dots, K\}$ so that $\mathcal{Z}_i \prec \mathcal{Z}_j \Rightarrow i \leq j$ and $i, j \in \{1, \dots, K\}$.

Existence of a 1-cycle modulo-infinity is equivalent to the existence of a homoclinic cycle in one of the atoms $\mathcal{Z}_1, \dots, \mathcal{Z}_K$. Existence of a r-cycle modulo-infinity is equivalent to the existence of a heteroclinic cycle among the atoms $\mathcal{Z}_1, \dots, \mathcal{Z}_K$.

Assumption 2.6. The decomposition of $\mathcal{Z} = \bigcup_{i=1}^{K+1} \mathcal{Z}_i$ has a filtration ordering modulo-infinity.

Note that the existence of a filtration ordering modulo-infinity automatically rules out the existence of 1- and r-cycles modulo-infinity, namely it rules out the existence of cycles among any of the atoms of $\mathcal{W}$, but it does not rule out the existence of cycles among atoms of $\mathcal{Z} = \mathcal{Z}^\infty \cup \mathcal{W}$. While homoclinic and heteroclinic cycles are known to prevent existence of Lyapunov functions which are strictly decreasing along solutions outside the considered invariant set $\mathcal{W}$, we aim here at allowing cycles which involve a specific atom $\mathcal{Z}^\infty$. Their existence is compatible with the dissipation inequality that we will be considering since OSS Lyapunov functions are allowed to increase whenever the output is sufficiently large. Checking existence of the filtration ordering might be difficult, in general, as it entails a global understanding of the layout of solutions of the system "departing from" and "converging to" atoms in $\mathcal{W}$. This is a common difficulty when dealing with Lyapunov characterizations of multistable systems. In this set-up, however, the challenge is mildly relaxed as only cycles within $\mathcal{W}$ need to be ruled out.

3. Definitions and main result

The following assumption and definitions provide characterizations of the asymptotic behavior of any solution of (2.1) with respect to the output map $y = h(x)$. Notice that, due to compactness of the manifold M, solutions are defined for all times.

Assumption 3.1. The output map $h(x)$ vanishes on $\mathcal{W}$ and is non-zero on $\mathcal{Z}^\infty$.

Assumption 3.1 characterizes which atoms of $\mathcal{Z}$ should be seen as part of $\mathcal{Z}^\infty$.

Definition 3.2 (O-pAG)**.** System (2.1) is said to satisfy the *output practical asymptotic gain* (O-pAG) property if there exist η_y of class $\mathcal{K}$ and a positive constant q, such that the following estimate holds for all $t \geq 0$ and all $x \in M$:

$$\limsup_{t\to+\infty} |X(t,x)|_{\mathcal{W}} \leq \eta_y(\|y\|) + q \ . \tag{3.1}$$

having defined $y(t) = h(X(t,x))$. If $q = 0$, then we say that the *output asymptotic gain* (O-AG) property holds.

Definition 3.3 (GATTMO)**.** System (2.1) is said to satisfy the *global attractivity modulo output* (GATTMO) property if there exist a function $\rho \in \mathcal{K}_\infty$ such that the following property holds:

$$\begin{aligned} &\forall x \in M \ \ \forall \varepsilon > 0 \ \ \exists T_{x,\varepsilon} > 0 \ \ : \ \ \forall T \in [T_{x,\varepsilon},\, +\infty) \\ &\text{if } \Big(\forall t \in [0,\, T] \ , \ |X(t,x)|_{\mathcal{W}} \geq \rho\left(|h(X(t,x))|\right) \Big) \\ &\text{then } \Big(\forall t \in [T_{x,\varepsilon},\, T] \ , \ |X(t,x)|_{\mathcal{W}} \leq \varepsilon \Big). \end{aligned} \tag{3.2}$$

Roughly speaking, GATTMO requires that any solution whose output is small with respect to the state for enough time would be attracted to an ε-neighborhood of $\mathcal{W}$.

Definition 3.4 (O-LIM)**.** System (2.1) is said to satisfy the *output limit* (O-LIM) property if there exists μ_y of class $\mathcal{K}$ such that the following estimate holds for all $x \in M$:

$$\inf_{t\geq 0} |X(t,x)|_{\mathcal{W}} \leq \mu_y(\|y\|) \ . \tag{3.3}$$

having defined $y(t) = h(X(t,x))$.

Definition 3.5 (O-pGS)**.** System (2.1) is said to satisfy the *output practical global stability* (O-pGS) property if there exists class-$\mathcal{K}_\infty$ functions β_0, β_y and a constant $q \geq 0$ such that the following estimate holds for all $t \geq 0$ and all $x \in M$:

$$|X(t,x)|_{\mathcal{W}} \leq \max\left\{\beta_0(|x|_{\mathcal{W}}), \beta_y(\left\|y_{[0,t]}\right\|)\right\} + q \ . \tag{3.4}$$

If $q = 0$, then we say that the *output global stability* (O-GS) property holds.

We are now going to consider a characterization of the O-AG property in terms of a specific kind of Lyapunov dissipation inequality.

Definition 3.6 (OSS-Lyapunov function)**.** Any function $V : M \to \mathbb{R}$ is called a *practical OSS-Lyapunov function* for system (2.1) if it satisfies the following two properties:

– there exist a constant $c \geq 0$, and class-$\mathcal{K}_\infty$ functions α_1, α_2 such that:

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{W}} + c) \quad \forall\, x \in M; \tag{3.5}$$

– V is differentiable along trajectories of (2.1), namely $V(X(t,x))$ is differentiable in $t \in \mathbb{R}$ for all $x \in M$. Furthermore, there exist class-$\mathcal{K}_\infty$ functions α, γ_y and a constant $q_0 \geq 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} V\left(X(t,x)\right) \leq -\alpha(|X(t,x)|_{\mathcal{W}}) + \gamma_y(|h(X(t,x))|) + q_0 \tag{3.6}$$

for all $t \geq 0$ and all $x \in M$.

If (3.6) holds for $q_0 = 0$, then V is called an *OSS-Lyapunov function*. If, in addition, the set $\bigcup_{x \in \mathcal{W}_i} \{V(x)\}$ is a singleton for all $i \in \{1, \ldots, K\}$, then V is called an *OSS-Lyapunov function constant on invariant sets*. Moreover, if V is smooth in $\mathcal{W}$ and $\mathrm{d}V(x) = 0$ for all $x \in \mathcal{W}$, then V is called an *OSS-Lyapunov function flat on invariant sets*.

Remark 3.7. For the sake of generality we consider here the case of $V(x)$ differentiable along solutions and for this reasons we formulate condition (3.6) by making use of explicit solutions. If $V(x)$ is differentiable then (3.6) can equivalently be expressed as follows:

$$\mathcal{L}_f V(x) \leq -\alpha(|x|_{\mathcal{W}}) + \gamma_y(|h(x)|) + q_0.$$

Remark 3.8. Note that, in contrast to the classical definition of OSS-Lyapunov function as given in [17], function $V(x)$ is bounded from above by an increasing function of $|x|_{\mathcal{W}}$ which is not necessarily class-$\mathcal{K}_\infty$ (as c may be positive). Setting $c = 0$ would in fact imply classic OSS of the set $\mathcal{W}$ (by standard arguments as in [17]), which cannot be the case for decompositions with multiple connected components as they typically fail to satisfy the Lyapunov stability modulo-output requirement.

Theorem 3.9. *Consider system (2.1) and let Assumptions 2.2, 2.6, and 3.1 be satisfied for the compact invariant set $\mathcal{W}$ containing all α- and ω-limit sets. Then the following properties are equivalent for system (2.1):*

1. *O-AG;*
2. *O-pAG;*
3. *O-LIM and O-pGS;*
4. *GATTMO;*
5. *existence of a lower semicontinuous OSS-Lyapunov function.*
6. *existence of a lower semicontinuous practical OSS-Lyapunov function.*
7. *existence of a smooth OSS-Lyapunov function flat on invariant sets.*

Proof. Implications proving sufficiency, *i.e.* 5 ⇒ 3, 3 ⇒ 2, 7 ⇒ 1, 2 ⇒ 1, and 6 ⇒ 2 are proved in Lemmas 4.1, 4.2, 4.3, 4.4, and 4.5 respectively. Implications 7 ⇒ 5, and 5 ⇒ 6 are trivial. Necessity of the existence of an OSS-Lyapunov function, *i.e.* implications 1 ⇒ 4 and 4 ⇒ 7, is proved throughout Section 5. □

Definition 3.10. System (2.1) is said to be *OSS in the multistable sense* with respect to the set $\mathcal{W}$ if it satisfies the properties of Theorem 3.9.

Example 3.11. Consider the system of equations

$$\begin{array}{rcl} \dot{x}_1 &=& \sin(x_1) \\ \dot{x}_2 &=& \frac{1+\cos(x_1)}{2} - \cos(x_2) \\ y &=& 1 + \cos(x_1) \end{array} \tag{3.7}$$

defined on $M = \mathbb{S}^2$, the two-dimensional torus. Equilibria of (3.7) are in $\{[0,0], [\pi, +\pi/2], [\pi, -\pi/2]\}$. It can be shown that every solution converges forward and backwards in time to one of these equilibria. Hence we may let $\mathcal{Z} = \{[0,0], [\pi, +\pi/2], [\pi, -\pi/2]\}$. Morevover, y is 0 in $[\pi, +\pi/2]$ and $[\pi, -\pi/2]$, hence we denote $\mathcal{Z}^\infty = \{[0,0]\}$, $\mathcal{Z}_1 = \{[\pi, +\pi/2]\}$ and $\mathcal{Z}_2 = \{[\pi, -\pi/2]\}$. It is worth pointing out that a homoclinic cycle exists in M. In fact, for $x_1(0) = 0$, we see that any solution fulfils $x_1(t) = 0$ identically, and $\dot{x}_2(t) = 1 - \cos(x_2(t)) \geq 0$, for all $t \in \mathbb{R}$. Hence $x_2(t)$ is monotonically non-decreasing with respect to time and it fulfils:

$$\lim_{t \to \pm\infty} \cos(x_2(t)) = 1.$$

The set $\{0\} \times \mathbb{S}$ is a homoclinic cycle in M. Due to existence of this cycle, no Lyapunov function can be found fulfilling $\mathcal{L}_f V(x) \leq -\alpha(|x|_{\mathcal{Z})})$. It is, nevertheless possible to find an OSS-Lyapunov function. Choose $V(x)$ as follows:

$$V(x) = 2 + \sin(x_2). \tag{3.8}$$

Notice that $V(x) \geq 1$, for all $x \in M$ and therefore it also fulfils the lower-bound $V(x) \geq \alpha_1(|x|_{\mathcal{W}})$ for some α_1 of class $\mathcal{K}_\infty$. Moreover, taking derivatives along solutions yields:

$$\mathcal{L}_f V(x) = \cos(x_2) \left[\frac{1 + \cos(x_1)}{2} - \cos(x_2) \right]$$

$$\leq -\cos(x_2)^2 + |y|/2 = -\cos^2(x_2) - (1 + \cos(x_1))^2 + \gamma_y(|y|).$$

where we defined $\gamma_y(r) := r^2 + r/2$. Notice that $\cos^2(x_2) + (1 + \cos(x_1))^2$ only vanishes in $\mathcal{W} := \mathcal{Z}_1 \cup \mathcal{Z}_2$. Hence, $\cos(x_2)^2 + (1 + \cos(x_1))^2 \geq \alpha(|x|_{\mathcal{W}})$ for some α of class $\mathcal{K}_\infty$ (due to compactness of M). Hence the following inequality holds:

$$\mathcal{L}_f V(x) \leq -\alpha(|x|_{\mathcal{W}}) + \gamma_y(|h(x)|),$$

which shows that V is an OSS-Lyapunov function.

4. Results on sufficiency part

In this Section, we provide a list of Lemmas which show how the existence of an OSS-Lyapunov function characterizes the asymptotic behavior of the state of system (2.1) with respect to its output signal y and the compact invariant set $\mathcal{W}$.

Lemma 4.1. *If system (2.1) admits an OSS-Lyapunov function then it satisfies the O-LIM and O-pGS properties.*

Proof. We are first going to prove the O-LIM property. Consider the case $x \in M$ such that $\|y\| = 0$. Inequality (3.6) then reads as $\dot{V} \leq -\alpha(|X(t,x)|_{\mathcal{W}})$ which, together with boundedness from below (3.5) and continuity of V along trajectories, satisfies the hyphothesis of LaSalle's invariance principle and thus implies $\lim_{t \to +\infty} |X(t,x)|_{\mathcal{W}} = 0$. It then follows that $\inf_{t \geq 0} |X(t,x)|_{\mathcal{W}} = 0$, thus proving the O-LIM property.

Consider now the case $x \in M$ such that $\|y\| = \bar{y} > 0$. We show by contradiction that the O-LIM property is satisfied with $\mu_y = \alpha^{-1} \circ 2\gamma_y$. It this were not the case, indeed we would have:

$$\gamma_y(|h(X(t,x))|) \leq \gamma_y(\|y\|) = \gamma_y(\bar{y}) < \frac{\alpha(|X(t,x)|_{\mathcal{W}})}{2}, \tag{4.1}$$

for all $t \geq 0$. Combining bound (4.1) with inequality (3.6) yields:

$$\frac{\mathrm{d}}{\mathrm{d}t} V(X(t,x)) \leq -\frac{1}{2}\alpha(|X(t,x)|_{\mathcal{W}}) \ \ \forall t \geq 0.$$

By virtue of LaSalle's invariance principle, we conclude that $\inf_{t\geq 0} |X(t,x)|_{\mathcal{W}} = 0$, which contradicts (4.1).

We are now going to prove the O-pGS property. Observe that $\alpha(|x|_{\mathcal{W}}/2 + c/2) \leq \alpha(|x|_{\mathcal{W}}) + \alpha(c)$, therefore we can write (3.6) as:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V(X(t,x))\bigg|_{t=0} &\leq -\alpha(|x|_{\mathcal{W}}/2 + c/2) + \alpha(c) + \gamma_y(|y|) \\ &\leq -\alpha \circ \frac{1}{2}\alpha_2^{-1}(V(x)) + \alpha(c) + \gamma_y(|y|)\ ,\end{aligned}$$

which, by means of a standard technique (for instance appearing in the proof of Lemma 11 in [17]), leads to the O-pGS property. □

The following lemma is obtained by adapting the arguments of Lemma 3.4 in [3].

Lemma 4.2. *If system (2.1) satisfies the O-LIM and O-pGS, then it satisfies the O-pAG property.*

Proof. Let μ_y, β_0, β_y be as in (3.3) and (3.4). For any $x \in M$ and any $\varepsilon > 0$, by the O-LIM property, there is some $T \geq 0$ such that

$$|X(T,x)|_{\mathcal{W}} \leq \mu_y(\|y\|) + \varepsilon\ .$$

Applying the O-pGS property to the initial state $x_T := X(T,x)$, whose corresponding output signal is $y_T(\cdot, x_T) = y(\cdot + T, x)$, we conclude that

$$\begin{aligned}\sup_{t\geq 0} |X(t,x_T)|_{\mathcal{W}} &\leq \max\{\beta_0(|x_T|_{\mathcal{W}}), \beta_y(\|y_T\|)\} + q \\ &\leq \max\{\beta_0(\mu_y(\|y\|) + \varepsilon), \beta_y(\|y\|)\} + q\ .\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we see that:

$$\limsup_{t\to+\infty} |X(t,x)|_{\mathcal{W}} \leq \eta_y(\|y\|) + q\ ,$$

with $\eta_y(s) := \sup\{\beta_0(\mu_y(s)), \beta_y(s)\}$. □

Lemma 4.3. *If system (2.1) admits an OSS-Lyapunov function constant on invariant sets, then it satisfies the O-AG property.*

Proof. From Lemmas 4.1 and 4.2, the system (2.1) enjoys the O-LIM, O-pGS, and O-pAG properties. Let $\eta_y, \mu_y, \alpha, \gamma_y$ as in (3.1), (3.3), and (3.6). The O-AG property is easily proved for all trajectories $X(t,x)$ such that $\|y\| \geq \varepsilon$. In fact, for all $\varepsilon > 0$ and for all such trajectories, it holds:

$$\eta_y(\|y\|) + q \leq \eta_y(\|y\|) + q\frac{\|y\|}{\varepsilon} \leq \bar{\eta}_y(\|y\|),$$

with $\bar{\eta}_{y,\epsilon}(s) := \eta_y(s) + qs/\varepsilon$. Therefore, it is enough to show that there exists a sufficiently small $\bar{\varepsilon} > 0$ such that O-AG holds for all trajectories such that $\|y\| < \bar{\varepsilon}$. Without loss of generality, assume $\bar{\varepsilon} = 1$ at first. The latter assumption and the O-pAG property imply that $X(t,x)$ enters a compact set $\mathcal{X} := \{x : |x|_{\mathcal{W}} \leq q + \eta_y(1) + 1,\ |h(x)| \leq 1\}$ in finite time. Let F be:

$$F := \max_{x\in\mathcal{X}} \left\{|f(x)|_{\mathfrak{g}}\right\} < +\infty. \tag{4.2}$$

Observe that F is finite by continuity of f and compactness of $\mathcal{X}$. Since $X(t,x)$ enters $\mathcal{X}$ whenever $\|y\| \leq 1$, $|f(X(t,x))|_{\mathfrak{g}} \leq F$ for all sufficiently large $t \geq 0$.

Consider next the minimum distance between the elements of the decomposition:

$$\bar{D} := \min_{\substack{1 \leq i \neq j \leq K,\\ x_a \in \mathcal{W}_i, x_b \in \mathcal{W}_j}} \mathfrak{d}[x_a, x_b] < +\infty \ . \tag{4.3}$$

The minimum exists and it is striclty positive by finiteness of the decomposition and compactness of the $\mathcal{W}_i$s. Pick a $0 < \Delta < 1$ such that $\alpha^{-1}(2\gamma_y(\Delta)) \leq \bar{D}/4$. Define the sets $N_i(\Delta)$ as:

$$N_i(\Delta) := \left\{ x \in M : |x|_{\mathcal{W}_i} \leq \alpha^{-1}(2\gamma_y(\Delta)) \right\} \ .$$

By the O-LIM property, it holds for all $x \in M$ such that $\|y\| \leq \Delta$ that there exists a $t' \geq 0$ such that:

$$X(t', x) \in N_i(\Delta) \ ,$$

for some $1 \leq i \leq K$. By proceeding along the lines of the proof of Claim 4 in [2], it is possible to find an asymptotic gain function $\underline{\eta}(\cdot)$ and a threshold $\bar{\varepsilon} = \Delta_2$ such that O-AG holds with gain $\eta_y(s) := \max\left\{\bar{\eta}_{y,\bar{\varepsilon}}(s), \underline{\eta}(s)\right\}$. □

Lemma 4.4. *If system (2.1) satisfies the O-pAG property, then it satisfies the O-AG property.*

Proof. Consider the class-$\mathcal{K}_\infty$ function η_y as in Definition 3.2. Pick any $x \in M$. If $\|y\| = \sup_{t\geq 0} |h(X(t,x))| = +\infty$, then O-AG trivially follows. If $\|y\| < +\infty$, then, by definition of $\limsup$, it follows that trajectory $X(t,x)$ enters the set $\Omega := \{x \in M \ : \ |x|_{\mathcal{W}} \leq \eta_y(\|y\|) + q + 1\}$ in finite time $0 \leq T < +\infty$. By compactness of $\mathcal{W}$, solutions are bounded for all $t \geq T$. Then, Markus's result in [9] yields $\limsup_{t\to+\infty} |X(t,x)|_{\mathcal{W}} = 0$ which trivially implies the O-AG property. □

Lemma 4.5. *If system (2.1) admits a practical OSS-Lyapunov function, then it satisfies the O-pAG property.*

Proof. Let class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha, \gamma_y$ and constants $c, q_0 \geq 0$ be as in inequalities (3.5) and (3.6). Observe that $\alpha(|w|_{\mathcal{W}}/2 + c/2) \leq \alpha(|w|_{\mathcal{W}}) + \alpha(c)$ and that $|w|_{\mathcal{W}}/2 + c/2 \geq \alpha_2^{-1}(V(w))$ for any $w \in M$. Then, inequality (3.6) reads as:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V(X(t,x))\bigg|_{t=0} &\leq -(\alpha \circ \alpha_2^{-1})\,(V(x)) + \alpha(c) \\ &\quad + \gamma_y\,(|h(x)|) + q.\end{aligned}$$

Let $v_t^\star = (\alpha_2 \circ \alpha^{-1})\left(2\alpha(c) + 2\gamma_y(\|y\|_{[0,t]}) + 2q\right)$. It is possible to prove along the lines of Lemma 11 and 13 in [17] that there exists a class-$\mathcal{KL}$ function $\hat{\beta}$ such that: $V(X(t,x)) \leq \hat{\beta}\,(V(x), t) + v_t^\star$ for all $t \geq 0$. Embedding bounds (3.5) yields:

$$|X(t,x)|_{\mathcal{W}} \leq \beta(|x|_{\mathcal{W}}, t) + \eta_y(\left\|y_{[0,t]}\right\|) + \tilde{q} \tag{4.4}$$

for a properly suited $\mathcal{KL}$ function β, and $\mathcal{K}_\infty$ function η_y, and some constant $\tilde{q} \geq 0$. By taking the $\limsup$ on both sides of (4.4), we obtain the O-pAG property. □

5. Existence of OSS-Lyapunov function

5.1. Preliminary results

Throughout this Section, we discuss the steps of the proof leading to existence of OSS-Lyapunov functions as a consequence of item (1) of Theorem 3.9. Hence, we make the following

Assumption 5.1. System (2.1) satisfies the O-AG property with respect to $\mathcal{W}$ for some class-$\mathcal{K}_\infty$ function η_y.

We recall the following preliminary result whose proof is given in Appendix A.

Lemma 5.2. *The O-AG implies the GATTMO property with the class-$\mathcal{K}_\infty$ function ρ being smooth on $(0, +\infty)$ and fulfilling:*

$$\rho(s) > \eta_y(s) \text{ for all } s > 0. \tag{5.1}$$

Select a class-$\mathcal{K}_\infty$ function ρ according to Lemma 5.2. Then, consider the following sets:

$$\begin{aligned}
\mathcal{D} &:= \{x \in M \ : \ |x|_{\mathcal{W}} \leq \rho(|h(x)|)\} \\
\mathcal{E} &:= M \setminus \mathcal{D} \\
\mathcal{E}_0 &:= \{x \in M \ : \ |x|_{\mathcal{W}} \geq \rho(|h(x)|)\} \\
\mathcal{E}_3 &:= \{x \in M \ : \ |x|_{\mathcal{W}} \geq 3\rho(|h(x)|)\} \\
\mathcal{L}_{-\infty,+\infty} &:= \{x \in M \ : \ X(t,x) \in \mathcal{E}_0 \text{ for all } t \in \mathbb{R}\}.
\end{aligned}$$

The set $\mathcal{E}_0$ will be alternatively referred to as the *output set* and represents the region of the state space where the distance from $\mathcal{W}$ dominates the norm of the output.

5.2. Sectors

The following Lemma highlights a crucial property of solutions of systems evolving on a compact manifold.

Lemma 5.3. *M is the disjoint union of the attracting (or repulsing) subsets of the decomposition of $\mathcal{Z}$, namely:*

$$M = \bigcup_{i=1}^{K+1} \mathcal{A}(\mathcal{Z}_i) = \bigcup_{i=1}^{K+1} \mathcal{R}(\mathcal{Z}_i) \tag{5.2}$$

$$\mathcal{R}(\mathcal{Z}_i) \cap \mathcal{R}(\mathcal{Z}_j) = \mathcal{A}(\mathcal{Z}_i) \cap \mathcal{A}(\mathcal{Z}_j) = \emptyset, \quad i \neq j. \tag{5.3}$$

Proof. We recall here the proof of Lemma 2.2 in [13]. We prove the first equality of (5.2) and the second one will follow by considering the backward flow. Since the vector field f is locally Lipschitz continuous and due to the $\mathcal{Z}_i$s being compact and disjoint, we can select open neighborhoods U_i of $\mathcal{Z}_i$ such that

$$\varphi_t(\operatorname{clos} U_{i_1}) \cap \varphi_s(\operatorname{clos} U_{i_2}) = \emptyset \tag{5.4}$$

for all $t, s \in [-1, 1]$ and all $i_1, i_2 \in \{1, \dots, K+1\}$ such that $i \neq j$. By virtue of Assumption 2.2, for any $x \in M$, it holds $\omega(x) \subseteq \mathcal{Z} \subseteq \bigcup_{i=1}^{K+1} U_i$. It thus follows that $\varphi_t(x)$ is contained in $\bigcup_{i=1}^{K+1} U_i$ for all $t > T$, with $T > 0$ large enough. In particular, due to (5.4), the solution $X(t, x)$ is eventually contained in one particular U_i. Choosing the U_is arbitrarily close to the $\mathcal{Z}_i$, we see that $x \in \mathcal{A}(\mathcal{Z}_i)$. □

In the remaining of this Section, let Assumptions 2.2, 2.6, and 5.1 hold true.

Under aforementioned assumptions, the behavior of solutions with respect to the output set is characterized as follows. Define for all $x \in \mathcal{E}_0$ the first hitting times $\lambda(x)$ and $\lambda^\star(x)$ as:

$$\lambda(x) := \inf\{t \geq 0 \;:\; X(t,x) \in \mathcal{D}\}$$
$$\lambda^\star(x) := \inf\{t \geq 0 \;:\; X(-t,x) \in \mathcal{D}\}\,.$$

Lemma 5.4. *$\mathcal{W} \subseteq \mathcal{D}$.*

Proof. $x \in \mathcal{W} \Rightarrow |x|_{\mathcal{W}} = 0 \Rightarrow |x|_{\mathcal{W}} \leq \rho\left(|h(x)|\right) \Rightarrow x \in \mathcal{D}$. □

Lemma 5.5. *$\mathcal{Z}^\infty \cap \mathcal{E}_0 = \emptyset$.*

Proof. Assume by contradiction that there exists $x \in \mathcal{Z}^\infty \cap \mathcal{E}_0$. Assumption 2.2 guarantees that $\{x\}$ is an invariant set, therefore $X(t,x) \in \mathcal{E}_0$ for all $t \geq 0$. By virtue of the GATTMO property, the omega limit set of x satisfies $\omega(x) \subseteq \mathcal{W}$. However, $\lim_{t\to+\infty} |X(t,x)|_{\mathcal{W}} = 0$ contradicts invariance of $\{x\} \subseteq \mathcal{Z}^\infty$. □

Lemma 5.6. *For all $x \in M$ it holds:*

(i) $\lambda(x) < +\infty$ *or* $\lim_{t\to+\infty} |X(t,x)|_{\mathcal{W}} = 0$*;*
(ii) $\lambda^\star(x) < +\infty$ *or* $\lim_{t\to-\infty} |X(t,x)|_{\mathcal{W}} = 0$.

Proof. We first prove statement (5.6). If $x \in \mathcal{D}$ then $\lambda(x) = 0$ trivially. Consider now $x \in \mathcal{E}$ and assume $\lambda(x) = +\infty$. Then, by definition of $\lambda(x)$, it holds that $X(t,x) \notin \mathcal{D}$ for all $t \geq 0$ which implies $|X(t,x)|_{\mathcal{W}} > \rho\left(|h(X(t,x))|\right)$ for all $t \geq 0$. By GATTMO, $\lim_{t\to+\infty} |X(t,x)|_{\mathcal{W}} = 0$.

Second, we prove statement (5.6). Assume $\lim_{t\to-\infty} |X(t,x)|_{\mathcal{W}} \neq 0$. Then, Lemma 5.3 implies that $\lim_{t\to-\infty} |X(t,x)|_{\mathcal{Z}^\infty} = 0$. Since the compact set $\mathcal{Z}^\infty$ is contained in the open set $M \setminus \mathcal{E}_0$ (Lem. 5.5), there must exists a finite $\tau \geq 0$ such that $X(-\tau,x) \in \mathcal{D}$, thus $\lambda^\star(x) < +\infty$. □

The next result is a direct consequence of Lemma 5.6.

Corollary 5.7. *If $\lim_{t\to+\infty} |X(t,x)|_{\mathcal{Z}^\infty} = 0$, then $\lambda(x) < +\infty$ and there exists $\bar{t} > 0$ such that $X(\bar{t},x) \notin \mathcal{E}_0$. If $\lim_{t\to-\infty} |X(t,x)|_{\mathcal{Z}^\infty} = 0$, then $\lambda^\star(x) < +\infty$ and there exists $\bar{t} > 0$ such that $X(-\bar{t},x) \notin \mathcal{E}_0$.*

Consider the following sets:

$$A_i := \begin{cases} \emptyset & i = 0 \\ \bigcup_{j\leq i} \mathcal{R}(\mathcal{W}_j) & i = 1,\dots,K, \end{cases}$$
$$B_i := \begin{cases} \bigcup_{l>i} \mathcal{A}(\mathcal{W}_l) & i = 0,\dots,K-1 \\ \emptyset & i = K. \end{cases} \tag{5.5}$$

with $1 \leq i \leq K$. We furthermore define:

$$C_i := \begin{cases} \mathcal{A}(\mathcal{Z}^\infty) & i = 0 \\ \mathcal{A}(\mathcal{Z}^\infty) \cup \bigcup_{j\leq i} \mathcal{A}(\mathcal{W}_j) & i = 1,\dots,K-1 \\ M & i = K, \end{cases}$$
$$D_i := \begin{cases} M & i = 0 \\ \mathcal{R}(\mathcal{Z}^\infty) \cup \bigcup_{l>i} \mathcal{R}(\mathcal{W}_l) & i = 1,\dots,K-1 \\ \mathcal{R}(\mathcal{Z}^\infty) & i = K. \end{cases} \tag{5.6}$$

In particular, we will focus our attention on the attractivity property modulo-output of the sets:

$$\bar{A}_i := A_i \cap \mathcal{L}_{-\infty,0}$$
$$\bar{B}_i := B_i \cap \mathcal{L}_{0,+\infty}.$$

with $1 \leq i \leq K$, where we have made use of the following definition:

$$\mathcal{L}_{s_1,s_2} := \{x \in \mathcal{E}_0 \text{ such that } \varphi_t(x) \in \mathcal{E}_0,\ \forall t \in [s_1, s_2]\}\,. \tag{5.7}$$

Lemma 5.8. *Let Assumptions 2.2, 2.6, and 5.1 hold true. The following properties hold true for all $i \in \{0, 1, \ldots, K\}$:*

(i) Invariance and compactness*: $\bar{A}_i$ is backward invariant, $\bar{B}_i$ is forward invariant, and both are compact;*
(ii) Complementarity*: $M \setminus B_i = C_i$ and $M \setminus A_i = D_i$;*
(iii) Disjointness*: $A_i \cap B_i = \emptyset$;*
(iv) Inclusions*: $\bar{A}_i \subseteq C_i$ and $\bar{B}_i \subseteq D_i$;*
(v) Attractivity*: for all $x \in C_i$ it holds $\lambda(x) < +\infty$ or $\lim_{t\to+\infty} |X(t, x)|_{\bar{A}_i} = 0$.*
(vi) Repulsion*: for all $x \in D_i$ it holds $\lambda^\star(x) < +\infty$ or $\lim_{t\to-\infty} |X(t, x)|_{\bar{B}_i} = 0$.*

Proof. Backward (respectively forward) invariance of $\bar{A}_i$ (respectively $\bar{B}_i$) trivially follows by backward (respectively forward) invariance of $\mathcal{L}_{-\infty,0}$ and $\mathcal{R}(\mathcal{W}_j)$ (respectively $\mathcal{L}_{0,+\infty}$ and $\mathcal{A}(\mathcal{W}_j)$). We prove in Section E that $\bar{A}_i, \bar{B}_i$ are closed, thus compact by compactness of the manifold M.

Complementarity of B_i, C_i and of A_i, D_i in M follows by Lemma 5.3.

Disjointness is proved by the existence of a filtration ordering modulo-infinity in Assumption 2.6.

Inclusions $\bar{A}_i \subseteq C_i$ and $\bar{B}_i \subseteq D_i$, follow from the disjointness property $A_i \cap B_i = \emptyset$ and from the complementarity property that $A_i \subseteq (M \setminus B_i) = C_i$ and $B_i \subseteq (M \setminus A_i) = D_i$.

The attractivity property is proved as follows. By definition of C_i, for any $x \in C_i$ it holds $\lim_{t\to+\infty} |X(t, x)|_{\bigcup_{j\leq i} \mathcal{W}_j} = 0$ or $\lim_{t\to+\infty} |X(t, x)|_{\mathcal{Z}^\infty} = 0$. In the first case, since $\mathcal{W} \cap \mathcal{E}_0$ and $\bigcup_{j\leq i} \mathcal{W}_j \subseteq A_i$ and all $\mathcal{W}_i$s are invariant, we have that $\bigcup_{j\leq i} \mathcal{W}_j \subseteq \bar{A}_i$, which implies $|X(t, x)|_{\bigcup_{j\leq i} \mathcal{W}_j} \geq |X(t, x)|_{\bar{A}_i}$, and thus $\lim_{t\to+\infty} |X(t, x)|_{\bar{A}_i} = 0$. In the second case, Corollary 5.7 proves that $\lambda(x) < +\infty$.

The repulsion property follows along the same lines of the proof of the attractivity property. □

The definitions of sets $\bar{A}_i, \bar{B}_i$ provide an appropriate partition of the output set $\mathcal{E}_0$, in the following sense.

Lemma 5.9. *Let Assumptions 2.2, 5.1, and 2.6 hold true. For any $x \notin \mathcal{W} \cup \mathcal{Z}^\infty$, there exists an index $i(x) \in \{0, 1, \ldots, K\}$ such that $x \notin \bar{A}_{i(x)} \cup \bar{B}_{i(x)}$. In particular:*

(i) if $x \in \mathcal{A}(\mathcal{W}_h)$ for some $h \in \{1, \ldots, K\}$ then $i(x) = h$;
(ii) if $x \in \mathcal{R}(\mathcal{Z}^\infty) \cap \mathcal{A}(\mathcal{Z}^\infty)$, then $x \notin \bar{A}_{i(x)} \cup \bar{B}_{i(x)}$ for all $i(x) \in \{1, \ldots, K\}$.
(iii) if $x \in \mathcal{R}(\mathcal{W}_j) \cap \mathcal{A}(\mathcal{Z}^\infty)$ for some $j \in \{1, \ldots, K\}$, then $i(x) = j - 1$;

Proof. (i) If $x \in \mathcal{R}(\mathcal{Z}^\infty) \cap \mathcal{A}(\mathcal{W}_h)$, Lemma 5.3 implies $x \notin A_h$. If, on the other hand, $x \in \mathcal{R}(\mathcal{W}_j) \cap \mathcal{A}(\mathcal{W}_h)$ for some $j \in \{1, \ldots, K\}$, then the existence of a filtration ordering modulo-infinity implies that $j > h$, and Lemma 5.3 thus implies $x \notin A_h$;
(ii) the statement easily follows from Lemma 5.3;
(iii) if $x \in \mathcal{R}(\mathcal{W}_j)$ Lemma 5.3 implies $x \notin A_{j-1}$; since $x \in \mathcal{A}(\mathcal{Z}^\infty)$, it follows from Lemma 5.3 that $x \notin B_h$ for any $h \in \{1, \ldots, K\}$ and, in particular, $x \notin B_{j-1}$.

□

Lemma 5.10. *Let Assumptions 2.2, 2.6, and 5.1 hold true. For any $i \in \{0, 1, \ldots, K\}$, there exist:*

- *a closed neighborhood of $\bar{A}_i$, say $\mathcal{A}_i$, with the property $\mathcal{A}_i \cap \mathcal{W}_j = \emptyset$ for all $j > i$*
- *a closed neighborhood of $\bar{B}_i$, say $\mathcal{B}_i$, with the property $\mathcal{B}_i \cap \mathcal{W}_j = \emptyset$ for all $j \leq i$;*

such that $\mathcal{A}_i \cup \mathcal{B}_i = M$. In particular, we select:

- $\mathcal{A}_i = M$ *and* $\mathcal{B}_i = \emptyset$ *if* $i = K$;
- $\mathcal{A}_i = \emptyset$ *and* $\mathcal{B}_i = M$ *if* $i = 0$.

Proof. Since $\bar{A}_i$ and $\bar{B}_i$ are closed and disjoint, it is possible to select a closed set $\mathcal{A}_i \subset M \setminus \bar{B}_i$ and a closed set $\mathcal{B}_i \subset M \setminus \bar{A}_i$ such that $\mathcal{A}_i \cup \mathcal{B}_i = M$. By the definition of $\bar{A}_i$ and due to inclusion $\bar{B}_i \supseteq \bigcup_{j>i} \mathcal{W}_j$, it holds that $\mathcal{A}_i \cap \bar{B}_i = \emptyset$, and thus $\mathcal{A}_i \cap \mathcal{W}_j = \emptyset$, for all $j > i$. Similarly, it follows that $\mathcal{B}_i \cap \mathcal{W}_j = \emptyset$, for all $j \leq i$. □

5.3. OSS-Lyapunov functions in sectors

In this Section, we show that sectors $\bar{A}_i, \bar{B}_i$ as defined in 5.2 satisfy a stability property which is highly reminiscent of the so-called global asymptotic stability modulo-output (GASMO), as defined in [6] and reported in the following.

Definition 5.11 (GASMO)**.** Let $\mathcal{Q}, \mathcal{S}$ be two subsets of M such that $\mathcal{Q}$ is closed and $\mathcal{Q} \in \operatorname{int} \mathcal{S}$. The set $\mathcal{Q}$ is said to satisfy the global asymptotic stability modulo-output property (GASMO) on $\mathcal{S}$ if there exists a class-$\mathcal{KL}$ function β such that $|X(t,x)|_{\mathcal{Q}} \leq \beta(|x|_{\mathcal{Q}}, t)$. for all $x \in \mathcal{S}$ and all $t \in [0, \lambda(x))$.

GASMO has been shown to be equivalent to OSS in the classical framework [6] and instrumental in the construction of a smooth OSS-Lyapunov function. Similarly, the following Lemmas 5.12 and 5.13 prove that sectors $\bar{A}_i, \bar{B}_i$ satisfy a GASMO property with respect to two measures, which will be instrumental in the construction of a smooth OSS-Lyapunov function on $\mathcal{A}_i, \mathcal{B}_i$ respectively.

Let Assumptions 2.2, 2.6, 3.1, and 5.1 hold true. For all $i \in \{0, 1, \ldots, K\}$, select closed sets $\mathcal{A}_i$ and $\mathcal{B}_i$ according to Lemma 5.10. Then, for any $i \in \{0, 1, \ldots, K\}$, the following five Lemmas hold true.

Lemma 5.12. (GASMO with respect to two measures on $\mathcal{A}_i$) *There exists a class-$\mathcal{KL}$ function β such that, for all $t \in [0, \lambda(x))$ and all $x \in (\mathcal{E} \cap \mathcal{A}_i) \cup \bar{A}_i$, it holds:* $|X(t,x)|_{\bar{A}_i} \leq \beta\left(|x|_{A_i \cap \mathcal{L}_{-\infty,+\infty}}, t\right)$.

Proof. See Appendix B. □

Lemma 5.13. (GASMO with respect to two measures on $\mathcal{B}_i$ for the backward flow) *There exists a class $\mathcal{KL}$ function β such that, for all $t \in [0, \lambda^\star(x))$ and all $x \in (\mathcal{E} \cap \mathcal{B}_i) \cup \bar{B}_i$, it holds:* $|X(-t,x)|_{\bar{B}_i} \leq \beta\left(|x|_{B_i \cap \mathcal{L}_{-\infty,+\infty}}, t\right)$.

Proof. See Appendix C. □

We bring to the attention of the reader that the type of stability property being satisfied in $\mathcal{A}_i$ and $\mathcal{B}_i$ respectively allows the construction of classical OSS-Lyapunov functions in $\mathcal{A}_i$ and $\mathcal{B}_i$ which follows along the lines of [16]. Let $\mathcal{E}_3 := \{x \in M \;:\; |x|_{\mathcal{W}} \geq 3\rho(|h(x)|)\}$.

Lemma 5.14. (Existence of an OSS-Lyapunov function in $\mathcal{A}_i$) *There exists a smooth function $V_1 : \mathcal{A}_i \to \mathbb{R}_{\geq 0}$ and class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \beta_1$ such that the following two properties hold:*

$$\alpha_1(|x|_{\bar{A}_i}) \leq V_1(x) \leq \alpha_2(|x|_{\bar{A}_i}) \;\; \forall\, x \in \mathcal{A}_i \tag{5.8}$$

$$\mathcal{L}_f V_1(x) \leq -\beta_1\left(|x|_{\bar{A}_i}\right) \;\; \forall\, x \in (\mathcal{E}_3 \cap \mathcal{A}_i) \cup \bar{A}_i. \tag{5.9}$$

In particular, $\mathrm{d}V_1(x) = 0$ *whenever* $x \in \bar{A}_i$.

Proof. See Appendix D. □

Lemma 5.15. (Existence of an OSS-Lyapunov function in $\mathcal{B}_i$) *There exists a smooth function $V_2 : \mathcal{B}_i \to \mathbb{R}_{\geq 0}$ and of class-$\mathcal{K}_\infty$ functions $\alpha_3, \alpha_4, \beta_2$ such that the following two properties hold:*

$$\alpha_3(|x|_{\bar{B}_i}) \leq V_2(x) \leq \alpha_4(|x|_{\bar{B}_i}) \;\; \forall\, x \in \mathcal{B}_i \tag{5.10}$$

$$\mathcal{L}_f V_2(x) \leq -\beta_2\left(|x|_{\bar{B}_i}\right) \;\; \forall\, x \in (\mathcal{E}_3 \cap \mathcal{B}_i) \cup \bar{B}_i. \tag{5.11}$$

In particular, $\mathrm{d}V_2(x) = 0$ *whenever* $x \in \bar{B}_i$.

Proof. The proof is symmetrical with respect to the proof of Lemma (5.14) and thus follows along the lines of Appendix D. □

Let $G_i := \bar{A}_i \cup \bar{B}_i$.

Lemma 5.16. *There exists a smooth function $L_i : M \to \mathbb{R}_{\geq 0}$ which satisfies the following two properties:*

1. *there exists a class-$\mathcal{K}$ function α_1^i such that:*

$$\alpha_1^i(|x|_{\bar{A}_i}) \leq L_i(x) \quad \forall x \in M. \tag{5.12}$$

In particular, $L_i(x) = 0$ whenever $x \in \bar{A}_i$ and there exists a constant $b^i > 0$ such that $L_i(x) = b^i$ whenever $x \in \bar{B}_i$.

2. *$L_i(X(t,x))$ is continuously differentiable in $t \geq 0$ for all $x \in \mathcal{E}_3 \cup \bar{A}_i \cup \bar{B}_i$. Furthermore, there exists a class-$\mathcal{K}_\infty$ function β^i such that:*

$$\mathcal{L}_f L_i(x) \leq -\beta^i\left(|x|_{G_i}\right), \tag{5.13}$$

for all $x \in \mathcal{E}_3 \cup G_i$. In particular, $dL_i(x) = 0$ whenever $x \in G_i$.

Proof. See Appendix F. □

5.4. An OSS-Lyapunov function wrt $\mathcal{W}$ on $\mathcal{E}_3$

Lemma 5.16 has proved for any $i \in \{0, 1, \ldots, K\}$ the necessity of the existence of functions L_i whenever Assumptions 2.2 and 2.6, and 5.1 hold for system (2.1). In this Subsection, we are going to aggregate the contributions of the L_is in order to construct an OSS-Lyapunov function with respect to the set $\mathcal{W}$ which decreases along the trajectories evolving on $\mathcal{E}_3 \cup \mathcal{W}$.

In particular, we will be focusing on the cases $i = 0$ and $i = K$. Indeed, we have:

$$\begin{aligned}
\bar{A}_K &= (\mathcal{R}(\mathcal{W}_1) \cup \cdots \cup \mathcal{R}(\mathcal{W}_K)) \cap \mathcal{L}_{-\infty,0} \\
\mathcal{A}_K &= M \\
B_K &= \mathcal{B}_K = \emptyset \\
A_0 &= \mathcal{A}_0 = \emptyset \\
\bar{B}_0 &= (\mathcal{A}(\mathcal{W}_1) \cup \cdots \cup \mathcal{A}(\mathcal{W}_K)) \cap \mathcal{L}_{0,+\infty} \\
\mathcal{B}_0 &= M.
\end{aligned}$$

Lemma 5.14 proves the existence of a smooth Lyapunov function $V_K : M \to \mathbb{R}_{\geq 0}$ and of class-$\mathcal{K}_\infty$ functions $\alpha_1^K, \alpha_2^K, \hat{\beta}^K$ such that

$$\alpha_1^K(|x|_{\bar{A}_K}) \leq V_K(x) \leq \alpha_2^K(|x|_{\bar{A}_K}) \tag{5.14}$$

for all $x \in M$ and

$$\mathcal{L}_f V_K(x) \leq -\hat{\beta}^K(|x|_{\bar{A}_K}) \tag{5.15}$$

for all $x \in \mathcal{E}_3 \cup \bar{A}_K$. In particular, $\mathrm{d}V_K(x) = 0$ whenever $x \in \bar{A}_K$.

Lemma 5.15 proves the existence of a smooth Lyapunov function $V_0 : M \to \mathbb{R}_{\geq 0}$ and of class-$\mathcal{K}_\infty$ functions $\alpha_1^0, \alpha_2^0, \hat{\beta}^0$ such that

$$\alpha_1^0(|x|_{\bar{B}_0}) \leq V_0(x) \leq \alpha_2^0(|x|_{\bar{B}_0}) \tag{5.16}$$

for all $x \in M$ and

$$\mathcal{L}_f V_0(x) \leq -\hat{\beta}^0(|x|_{\bar{B}_0}) \tag{5.17}$$

for all $x \in \mathcal{E}_3 \cup \bar{B}_0$. In particular, $\mathrm{d}V_0(x) = 0$ whenever $x \in \bar{B}_0$.

Consider now the case $i = 1$. By virtue of Lemma 5.16, we have that $L_1(x) \geq \alpha_1^1(|x|_{\bar{A}_1})$. Since $\bar{A}_1$ is compact and due to $\mathcal{W}_1 \subseteq \bar{A}_1$, it also holds that $|x|_{\bar{A}_1} \leq |x|_{\mathcal{W}_1} \leq |x|_{\bar{A}_1} + \tilde{\nu}$ for some constant $\tilde{\nu} \geq 0$, and thus $\alpha_1^1(0.5\,|x|_{\mathcal{W}_1}) \leq \alpha_1^1(|x|_{\bar{A}_1}) + \nu$ with $\nu := \alpha_1^1(\tilde{\nu})$.

Let $L : M \to \mathbb{R}_{\geq 0}$ be the Lyapunov function defined as:

$$L(x) := \nu + V_0(x) + V_K(x) + \sum_{i=1}^{K-1} L_i(x). \tag{5.18}$$

It is then immediate to obtain the following lower bound:

$$\begin{aligned} L(x) &\geq \nu + \alpha_1^0(|x|_{\bar{B}_0}) + \alpha_1^K(|x|_{\bar{A}_K}) + \sum_{i=1}^{K-1} \alpha_1^i(|x|_{\bar{A}_i}) \\ &\geq \nu + \alpha_1^1(|x|_{\bar{A}_1}) + \alpha_1^K(|x|_{\bar{A}_K}) \\ &\geq \alpha_1^1(0.5|x|_{\mathcal{W}_1}) + \alpha_1^K(|x|_{\bar{A}_K}) \\ &\geq \alpha_1^1(0.5|x|_{\mathcal{W}}) + \alpha_1^K(|x|_{\bar{A}_K}). \end{aligned}$$

Note that $\alpha_1^1(\cdot)$ is a class-$\mathcal{K}$ function while α_1^K is a class-$\mathcal{K}_\infty$. Furthermore, by compactness of $\mathcal{W}$ and $\bar{A}_K$, it holds that $|x|_{\mathcal{W}} \leq |x|_{\bar{A}_K} + \nu_K$ for some $\nu_K \geq 0$. Therefore, we can select:

$$\alpha_1(s) := \begin{cases} \alpha_1^1(0.5\,s) & \text{if } s \leq \nu_K \\ \alpha_1^1(0.5\,s) + \alpha_1^K(s - \nu_K) & \text{if } s > \nu_K. \end{cases}$$

It follows that α_1 is a class-$\mathcal{K}_\infty$ function which satisfies:

$$\alpha_1(|x|_{\mathcal{W}}) \leq L(x) \quad \forall x \in M. \tag{5.19}$$

By Lemmas 5.16, 5.14, and 5.15, it holds that $\mathrm{d}V_0(x) = 0$ for any $x \in \bar{B}_0 = G_0$, $\mathrm{d}V_K(x) = 0$ for any $x \in \bar{A}_K = G_K$, and $dK_i(x) = 0$ for any $i \in \{1, \ldots, K-1\}$ and any $x \in G_i$. Furthermore, for any $i, j \in \{0, 1, \ldots, K\}$ with $j \neq 0$, it holds $\mathcal{W}_j \in G_i$. For these reasons, we can conclude that $dL(x) = 0$ whenever $x \in \mathcal{W}$, namely L is flat on $\mathcal{W}$.

By virtue of property 1. in Lemma 5.16 and inequality (5.8), the L_is and V_K vanish on the $\bar{A}_i$s and $\bar{A}_K$ respectively, and, moreover, the L_is take constant values on the B_is. It then follows that the L_is and V_K take constant values on $\mathcal{W}_j$, and we thus conclude that L is constant on any $\mathcal{W}_j$.

Moreover, the definition of $G_i := \bar{A}_i \cup B_i$ implies that $G_K = \bar{A}_K$ and, in particular, that $G_i \supseteq \mathcal{W}$ for any $i \in \{1, \ldots, K\}$. Then, by making use of property 2) in Lemma 5.16 and dissipation (5.15), we conclude that:

$$\mathcal{L}_f L(x) = 0 \quad \forall x \in \mathcal{W}. \tag{5.20}$$

Lemma 5.9 has proved that, if $x \notin \mathcal{W}$, there exists some $i \in \{0, 1, \ldots, K\}$ such that $|x|_{G_i} > 0$. Viceversa, for any $i \in \{0, 1, \ldots, K\}$, inclusion $\mathcal{W} \subseteq G_i$ implies $|x|_{G_i} \leq |x|_{\mathcal{W}}$, which in turn implies that $x \notin \mathcal{W}$ whenever $x \notin G_i$.

Therefore, there exist a continuous positive definite function ϖ such that

$$\begin{aligned}&\hat{\beta}^0\left(|x|_{G_0}\right)+\hat{\beta}^K\left(|x|_{G_K}\right)+\sum_{i=1}^{K-1}\beta^i\left(|x|_{G_i}\right)\\&\geq\varpi(|x|_{\mathcal{W}})\end{aligned}$$

for all $x\in M$, where last inequality embeds the property $|x|\leq|x|_{\mathcal{W}}+\nu_W$ with $\nu_W\geq 0$. In order to obtain a class-$\mathcal{K}_\infty$ dissipation function for the time derivative of L, consider the property $|x|_{G_K}\leq|x|_{\mathcal{W}}+\nu_K$ with $\nu_K\geq 0$ and define the following $\mathcal{K}_\infty$ function:

$$\beta(s):=\begin{cases}\frac{s}{1+s}\inf_{s\leq r\leq\nu_K}\varpi(r) & \text{if } s\leq\nu_K\\ \frac{\nu_K}{1+\nu_K}\varpi(c)+\hat{\beta}^K(s-\nu_K) & \text{if } s>\nu_K.\end{cases}$$

Then, it holds that:

$$\mathcal{L}_fL(x)\leq-\beta(|x|_{\mathcal{W}})\ \ ,\ \ \forall x\in\mathcal{E}_3\cup\mathcal{W}. \tag{5.21}$$

5.5. Extending the dissipation to the rest of the state space

Observe that $L(x)$ is the composition of functions which are smooth everywhere on M and, in particular, along the trajectories of system (2.1), namely $L(X(t,x))$ is smooth in t. For this reason and due to $\mathcal{L}_fL(\mathcal{W})=0$, we can always find a class-$\mathcal{K}_\infty$ function σ_0 such that:

$$\mathcal{L}_fL(x)\leq\sigma_0(|x|_{\mathcal{W}}) \tag{5.22}$$

for all x such that $|x|_{\mathcal{W}}\leq 3\rho(|h(x)|)$. Let:

$$\sigma(r):=\sigma_0(3r)+\beta(3r).$$

It then follows from (5.21) and (5.22) that

$$\mathcal{L}_fL(x)\leq-\beta(|x|_{\mathcal{W}})+\sigma(\rho(|h(x)|))\ \ ,\ \forall x\in M.$$

We have thus proved that L is a smooth OSS-Lyapunov function constant on invariant sets, as in Definition 3.6.

6. Conclusions

Classical OSS [17] results from the dualization of the ISS property by simply replacing inputs with outputs in the definition. In analogous way, dualizing the generalized ISS notion in [2] yields a notion of detectability for multistable systems, which is still called OSS but it does not require the invariant set of the system in consideration to be Lyapunov stable, while instead requiring absence of homoclinic orbits and heteroclinic cycles between the atoms of its decomposition. The resulting generalized OSS property portrays the behavior of ultimately having small distance of the states from the invariant set of the system whenever the outputs are small. This paper has provided characterizations of such property in terms of asymptotic estimates via nonlinear gain functions and in terms of smooth Lyapunov/LaSalle-like dissipation inequalities.

We bring to the attention of the reader that our converse OSS-Lyapunov theorem in Section 5 encompasses interesting results, such as attractivity of an invariant set $\mathcal{K}$ and absence of homoclinic orbits to $\mathcal{K}$ being a necessary and sufficient condition for Lyapunov stability of $\mathcal{K}$ on $\mathcal{C}$.

Appendix A. GATTMO

This Appendix addresses the proof of Lemma 5.2.

Proof. If $\|y\| = 0$, the O-AG property yields $\limsup_{t\to+\infty} |X(t,x)|_{\mathcal{W}} = 0$. By definition of lim sup, for all $x \in M$ and all $\varepsilon > 0$ there exists $T_{x,\varepsilon} > 0$, such that (3.2) is satisfied for any choice of $\rho \in \mathcal{K}_\infty$.

Consider now the case $0 < \|y\| < +\infty$. Select ρ as any smooth $\mathcal{K}_\infty$ function such that

$$\rho(s) > \eta_y(s) \text{ for all } s > 0.$$

By contradiction, we assume that there exist $x \in M$ and a constant $\varepsilon > 0$ such that, for all $T_{x,\varepsilon} > 0$, there exists $T \in [T_{x,\varepsilon}, +\infty)$ which verifies the following two conditions:

$$\Big(\forall t \in [0\, T]\ ,\ |X(t,x)|_{\mathcal{W}} \geq \rho\left(|h(X(t,x))|\right) \Big) \text{ and}$$
$$\Big(\exists \bar{t} \in [T_{x,\varepsilon}\, T]\ ,\ |X(\bar{t},x)|_{\mathcal{W}} > \varepsilon \Big).$$

The previous contradiction assumption can equivalently be formulated as follows: there exist $x \in M$, a constant $\varepsilon > 0$, and two sequences $\{T_n\}_{n\in\mathbb{N}} > 0$ and $\{\bar{t}_n\}_{n\in\mathbb{N}} > 0$ which verify the following three conditions

$$\Big(n \leq \bar{t}_n \leq T_n\Big)$$
$$\text{and } \Big(\forall t \in [0\, T_n]\ ,\ |X(t,x)|_{\mathcal{W}} \geq \rho\left(|h(X(t,x))|\right) \Big) \tag{A.1}$$
$$\text{and } \Big(|X(\bar{t}_n,x)|_{\mathcal{W}} > \varepsilon \Big). \tag{A.2}$$

for all $n \in \mathbb{N}$. Since (A.1) holds for all $n \in \mathbb{N}$ and T_n is diverging, we can take the lim sup on both sides of (A.1) so as to obtain:

$$\limsup_{t\to+\infty} |X(t,x)|_{\mathcal{W}} \geq \rho\left(\limsup_{t\to+\infty} |h(X(t,x))|\right). \tag{A.3}$$

Moreover, since (A.2) holds for all $n \in \mathbb{N}$ and $\bar{t}_n$ is diverging, we have that

$$\limsup_{t\to+\infty} |X(t,x)|_{\mathcal{W}} = \bar{\varepsilon} \tag{A.4}$$

for some $\bar{\varepsilon} \in (0, +\infty]$. In particular, $\bar{\varepsilon} \in (0, +\infty)$ due to $\|y\| < +\infty$ and O-AG.

Claim A.1. *The O-AG property implies that*

$$\limsup_{t\to+\infty} |X(t,x)|_{\mathcal{W}} \leq \eta_y\left(\limsup_{t\to+\infty} |h(X(t,x))|\right). \tag{A.5}$$

Proof. Pick $T > 0$. Define $x_T := X(T,x)$ and recall that $\|y(\cdot + T)\| = \sup_{t\geq T} |h(t,x)|$. Then, applying the O-AG property to $X(\cdot, x_T)$ yields:

$$\limsup_{t\to+\infty} |X(t,x)|_{\mathcal{W}} = \limsup_{t\to+\infty} |X(t,x_T)|_{\mathcal{W}} \leq \eta_y\left(\|y(\cdot + T)\|\right).$$

Since T is arbitrary and $\lim_{T\to+\infty} \eta_y\left(\|y(\cdot + T)\|\right) = \eta_y\left(\limsup_{t\to+\infty} |h(t,x)|\right)$, we obtain property (A.5). □

By comparing (A.3), (A.4), and (A.5), we obtain:

$$\rho\left(\limsup_{t\to+\infty}|h(X(t,x))|\right)\leq\bar{\varepsilon}\leq\eta_y\left(\limsup_{t\to+\infty}|h(X(t,x))|\right),$$

which contradicts (5.1). □

Appendix B. GASMO with respect to two measures on $\mathcal{A}_i$ for the forward flow

We proceed to prove *global asymptotic stability modulo-output with respect to two measures* of the set $\bar{A}_i$ in $\mathcal{A}_i$, that is existence of a class-$\mathcal{KL}$ function β such that:

$$|X(t,x)|_{\bar{A}_i}\leq\beta\left(|x|_{A_i\cap\mathcal{L}_{-\infty,+\infty}},t\right),$$

for all $t\in[0,\lambda(x))$ and all $x\in\mathcal{A}_i$. Throughout this Section, let Assumptions 2.2, 2.6, and 5.1, hold true.

First, we recall the following definition:

$$\mathcal{L}_{s_1,s_2}:=\{x\in\mathcal{E}_0\text{ such that }\varphi_t(x)\in\mathcal{E}_0,\ \forall t\in[s_1,s_2]\}\,. \tag{B.1}$$

Claim B.1. *The set $\mathcal{L}_{s_1,s_2}$ satisfies the following two properties:*

1. *$\varphi_t(\mathcal{L}_{s_1,s_2})=\mathcal{L}_{s_1-t,s_2-t}$, for all $t,s_1,s_2\in\mathbb{R}$ with $s_1\leq s_2$;*
2. *$\mathcal{L}_{a_1,b_1}\subseteq\mathcal{L}_{a_2,b_2}$, whenever the two intervals $I_1:=[a_1,b_1],I_2:=[a_2,b_2]$ in $\mathbb{R}$ satisfy $I_1\supseteq I_2$.*

Proof. It easily follows from the definition (B.1). □

Claim B.2. *For any $s_1,s_2\in\mathbb{R}$ with $s_1\leq s_2$, the set $\mathcal{L}_{s_1,s_2}$ is closed.*

Proof. Consider a sequence $\{x_n\}_{n\in\mathbb{N}}\in\mathcal{L}_{s_1,s_2}$ so that $\lim_{n\to+\infty}x_n=x$. Pick any $t\in[s_1,s_2]$. By continuity of solutions, we obtain $\varphi_t(x)=\varphi_t(\lim_{n\to+\infty}x_n)=\lim_{n\to+\infty}\varphi_t(x_n)$. Since the sequence $\varphi_t(x_n)$ converges to $\varphi_t(x)$ and belongs to $\mathcal{E}_0$ by definition of $\mathcal{L}_{s_1,s_2}$, and due to $\mathcal{E}_0$ being a closed set, we conclude $\varphi_t(x)\in\mathcal{E}_0$. Since the last statement holds for all $t\in[s_1,s_2]$, we conclude that $x\in\mathcal{L}_{s_1,s_2}$. □

Lemma B.3. (Existence of an arbitrary small forward invariant neighborhood modulo output on $\mathcal{A}_i$) *For all compact neighborhoods Q of $\bar{A}_i$ in $\mathcal{A}_i$, there exists a compact neighborhood V of $A_i\cap\mathcal{L}_{-\infty,+\infty}$ such that:*

$$\varphi_t\left(V\cap\mathcal{L}_{0,t}\right)\subseteq Q\textit{ for all }t\geq0. \tag{B.2}$$

Proof. Pick a compact neighborhood Q of $\bar{A}_i$. The proof is then adapted from Lemma 6 in [11] and consists of four steps:

Step 1: Consider the following definitions:

$$A_r^0:=\bigcap_{t\in[0,r]}\varphi_t(Q)\,,\quad A_r:=A_r^0\cap\mathcal{L}_{-r,0}.$$

Claim B.4. $\bigcap_{r\geq0}A_r=\bar{A}_i$

Proof. Indeed, we have that $x\in\bigcap_{r\geq0}A_r$ if and only if

$$\forall r\geq0\ \ \forall t\in[0,r]\ \ \varphi_{-t}(x)\in Q\cap\mathcal{E}_0. \tag{B.3}$$

Clearly, a necessary condition for x to be an element of $\bigcap_{r\geq 0} A_r$ is $x \in Q$. Furthermore, only the following scenarios may arise:

– $x \in Q \setminus (A_i \cup \mathcal{R}(\mathcal{Z}^\infty))$. In this case, Lemma 5.3 implies that $x \in \bigcup_{j>i} \mathcal{R}(\mathcal{W}_j)$. Therefore, since $X(t,x)$ is attracted backward in time to $\mathcal{W}_j$ with $j > i$, and due to the fact that $Q \cap \bigcup_{j>i} \mathcal{W}_j \subseteq \mathcal{A}_i \cap \bigcup_{j>i} \mathcal{W}_j = \emptyset$ with Q being a closed set, there exists $\bar{t} > 0$ such that $\varphi_{-\bar{t}}(x) \notin Q$, which in turn contradicts condition (B.3) for $r = t = \bar{t}$.
– $x \in Q \setminus \mathcal{L}_{-\infty,0}$. In this case, there exists $\bar{t} \leq 0$ such that $\varphi_{\bar{t}}(x) \notin \mathcal{E}_0$, which contradicts condition (B.3) by selecting $r = |\bar{t}|$.
– $x \in Q \cap \mathcal{L}_{-\infty,0} \cap A_i$. Then, $x \in \bar{A}_i$ and condition (B.3) is trivially satisfied.
– $x \in Q \cap \mathcal{L}_{-\infty,0} \cap \mathcal{R}(\mathcal{Z}^\infty)$. Then, due to Lemma 5.5 and since $\lim_{t\to-\infty} |X(t,x)|_{\mathcal{Z}^\infty} = 0$, there exists $\bar{t} < 0$ such that $X(t,x) \notin \mathcal{E}_0$ for all $t < \bar{t}$, and thus $x \notin \mathcal{L}_{-\infty,0}$.

□

Since a locally Lipschitz vector field f generates a flow φ_t which maps any compact neighborhood of a backward invariant set to a compact neighborhood of the same set whenever $t \geq 0$, we have that A_r^0 is a compact neighborhood of $\bar{A}_i$. Furthermore, recall that the sets $\bar{A}_i$ and $\mathcal{L}_{-r,0}$ are compact (Lem. E.7 and Claim B.2). It thus follows that A_r is compact. Moreover, since the sequences $\{A_r^0\}_{r\in\mathbb{R}_{\geq 0}}$ and $\{\mathcal{L}_{-r,0}\}_{r\in\mathbb{R}_{\geq 0}}$ are monotone non-increasing in r (Claim B.1), we conclude that $\{A_r\}_{r\in\mathbb{R}_{\geq 0}}$ is also monotone non-increasing and, in particular, is a nested sequence of compact sets converging to $\bar{A}_i$.
Step 2: We claim that, for sufficiently large r and sufficiently small $\bar{T}$, $\varphi_T(A_r) \subseteq A_1^0$, for all $T \in [0, \bar{T}]$. Indeed, let D denote the minimum distance of the boundary $\eth A_1^0$ from $\bar{A}_i$, that is $D := \inf_{a\in\eth A_1^0} \mathfrak{d}(a, \bar{A}_i)$. Due to A_1^0 being a compact neighborhood of $\bar{A}_i$, we have that $D > 0$. Let F denote the maximum velocity of the flux $\varphi_t(\cdot)$ in A_1^0, namely $F := \sup_{x\in A_1^0} |f(x)|_{\mathfrak{g}}$. By virtue of Lemma G.2, for the neighborhood $N := \left\{x \in A_1^0 : |x|_{\bar{A}_i} \leq \frac{\mathrm{d}}{2}\right\}$ there exists $R > 0$ such that, for all $r \geq R$, it holds $A_r \subseteq N$. Therefore, for all such rs and for all $y \in A_r$, it takes at least $\frac{\mathrm{d}}{2F}$ time units for the flux $\varphi_t(y)$ to reach the border $\eth A_1^0$. It follows that $\varphi_T(A_r) \subseteq A_1^0$ for all $r \geq R$ and all $T \in [0, \bar{T}]$ with $\bar{T} := \min\left\{1, \frac{\mathrm{d}}{2F}\right\}$.
Step 3: We claim that, for all $r \geq R$ and all $T \in [0, \bar{T}]$, $\varphi_T(A_r) \subseteq A_r^0$. As a preliminary result, we note that $\varphi_T(A_r^0) = \bigcap_{T\leq t\leq r+T} \varphi_t(Q)$ and this is easily proved by comparing the following definitions:

(i) $x \in \varphi_T(A_r^0)$ if and only if $\varphi_{-T-t}(x) \in Q$ for all $t \in [0, r]$;
(ii) $x \in \bigcap_{T\leq t\leq r+T} \varphi_t(Q)$ if and only if $\varphi_{-t}(x) \in Q$ for all $t \in [T, r+T]$.

Then, the following passages hold true for all $r \geq R$ and all $T \in [0, \bar{T}]$ with $\bar{T} = \min\left\{1, \frac{\mathrm{d}}{2F}\right\}$:

$$\begin{aligned}
\varphi_T(A_r) &= \varphi_T(A_r) \cap A_1^0 \\
&\subseteq \varphi_T(A_r^0) \cap A_1^0 \\
&= \left(\bigcap_{T\leq t\leq r+T} \varphi_t(Q)\right) \cap \left(\bigcap_{0\leq t\leq 1} \varphi_t(Q)\right) \\
&= \left(\bigcap_{0\leq t\leq r+T} \varphi_t(Q)\right) \subseteq A_r^0.
\end{aligned}$$

Step 4: By making use of the flow property 1) in Claim B.1, it holds for all $T \in [0, \bar{T}]$ with $\bar{T} := \min\left\{1, \frac{\mathrm{d}}{2F}\right\}$, all $\hat{T} \geq T$, and all $r \geq R$ that

$$\varphi_T\left(A_r^0 \cap \mathcal{L}_{-r,\hat{T}}\right) = \varphi_T\left(A_r^0 \cap \mathcal{L}_{-r,0} \cap \mathcal{L}_{-r,\hat{T}}\right)$$

$$\begin{aligned} &= \varphi_T\left(A_r \cap \mathcal{L}_{-r,\hat{T}}\right) \\ &= \varphi_T(A_r) \cap \varphi_T(\mathcal{L}_{-r,\hat{T}}) \\ &\subseteq A_r^0 \cap \mathcal{L}_{-r,\hat{T}-T} \end{aligned} \tag{B.4}$$

and clearly $A_r^0 \cap \mathcal{L}_{-r,\hat{T}-T} \subseteq A_r^0$.

We are now going to show that, for all $t \geq 0$ and all $r \geq R$, it holds: $\varphi_t(A_r^0 \cap \mathcal{L}_{-r,t}) \subseteq A_r^0$. To this end, pick any $t \geq 0$ and choose an integer $n \in \mathbb{N}$ so that $\frac{t}{n} \leq \bar{T}$. Then, by virtue of inclusion (B.4), we obtain:

$$\begin{aligned} \varphi_t\left(A_r^0 \cap \mathcal{L}_{-r,t}\right) &= \underbrace{\left(\varphi_{\frac{t}{n}} \circ \cdots \circ \varphi_{\frac{t}{n}}\right)}_{n \text{ times}} \left(A_r^0 \cap \mathcal{L}_{-r,t}\right) \\ &\subseteq \underbrace{\left(\varphi_{\frac{t}{n}} \circ \cdots \circ \varphi_{\frac{t}{n}}\right)}_{n-1 \text{ times}} \left(A_r^0 \cap \mathcal{L}_{-r,t-\frac{t}{n}}\right) \\ &\subseteq \varphi_{\frac{t}{n}}\left(A_r^0 \cap \mathcal{L}_{-r,t-\frac{n-1}{n}t})\right) \\ &\subseteq A_r^0. \end{aligned} \tag{B.5}$$

Consider now any $r \geq R$. Let V be defined as:

$$V := \bigcap_{t \in [-r,0]} \varphi_t(A_r^0).$$

Since $\bar{A}_i$ is generally not forward invariant, V is not a compact neighborhood of $\bar{A}_i$. However, it can be proved along the lines of Section E that $A_i \cap \mathcal{L}_{-\infty,+\infty}$ is compact and invariant. Therefore, V qualifies as a compact neighborhood of $A_i \cap \mathcal{L}_{-\infty,+\infty}$ and its definition implies:

$$\varphi_t(V) \subseteq A_r^0 \text{ for all } t \in [0, r]. \tag{B.6}$$

By virtue of (B.6), we have $\varphi_t(V \cap \mathcal{L}_{0,t}) \subseteq \bar{A}_r^0$ for all $t \in [0, r]$. Conversely, if $t > r$, inclusion (B.5) and the flow property 1) in Claim B.1 imply:

$$\begin{aligned} \varphi_t(V \cap \mathcal{L}_{0,t}) &= (\varphi_{t-r} \circ \varphi_r)(V \cap \mathcal{L}_{0,t}) \\ &\subseteq \varphi_{t-r}(\varphi_r(V) \cap \varphi_r(\mathcal{L}_{0,t})) \\ &\subseteq \varphi_{t-r}(A_r^0 \cap \mathcal{L}_{-r,t-r}) \\ &= \varphi_{t'}(A_r^0 \cap \mathcal{L}_{-r,t'}) \\ &\subseteq A_r^0 \\ &\subseteq Q, \end{aligned}$$

with $t' := t - r > 0$. We have thus concluded (B.2). □

Lemma B.5. (Arbitrary boundedness till hitting time on $\mathcal{A}_i$) *For all compact neighborhoods Q of $\bar{A}_i$, there exists a compact neighborhood V of $A_i \cap \mathcal{L}_{-\infty,+\infty}$ such that $V \subseteq Q$ and, for all $x \in V$, it holds:*

$$\varphi_t(x) \in Q \textit{ for all } t \in [0, \lambda(x)]. \tag{B.7}$$

Proof. Pick an arbitrary compact neighborhood Q of $\bar{A}_i$. Select a compact neighborhood V of $A_i \cap \mathcal{L}_{-\infty,+\infty}$ according to Lemma B.3. Then, statement (B.7) follows immediately by observing that, by definition of $\lambda(\cdot)$, the property $x \in \mathcal{L}_{0,\lambda(x)}$ holds for all $x \in M$ and, in particular, for all $x \in V$. □

For ease of presentation, denote $\hat{A}_i = A_i \cap \mathcal{L}_{-\infty,+\infty}$ and $\mathcal{N}_S(r) := \{x \in M \,:\, |x|_S \leq r\}$.

Corollary B.6. (Lyapunov stability modulo output on $\mathcal{A}_i$) *The following property holds true:*

$$\begin{aligned}&\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall T \geq 0 \;\; \forall t \in [0, T]\\ &\textit{if } x \in \mathcal{L}_{0,T} \cap \mathcal{N}_{\hat{A}_i}(\delta) \textit{ then } X(t,x) \in \mathcal{N}_{\bar{A}_i}(\varepsilon).\end{aligned} \tag{B.8}$$

Moreover, there exists a class-$\mathcal{K}_\infty$ function ϕ such that:

$$|X(t,x)|_{\bar{A}_i} \leq \phi(|x|_{\hat{A}_i}) \;\; \forall t \in [0, \lambda(x)]. \tag{B.9}$$

Proof. Pick $\varepsilon > 0$. Set $Q = \mathcal{N}_{\bar{A}_i}(\varepsilon)$.

We first prove proposition (B.8). By virtue of Lemma B.3, there exists a compact neighborhood $V \subseteq Q$ of $\hat{A}_i$ such that $X(t,x) \in Q$ for all $x \in V \cap \mathcal{L}_{0,T}$, all $T \geq 0$, and all $t \in [0,T]$. Since V is a compact neighborhood of $\hat{A}_i$, we can select $\delta > 0$ so that $\mathcal{N}_{\hat{A}_i}(\delta) \subseteq V$, thus proving (B.8).

We are now going to prove proposition (B.9). We first claim that for all $\varepsilon > 0$ there exists $\delta > 0$ such that:

$$|x|_{\hat{A}_i} \leq \delta \;\Rightarrow\; |X(t,x)|_{\bar{A}_i} \leq \varepsilon \;\; \forall t \in [0, \lambda(x)]. \tag{B.10}$$

Indeed, by virtue of Lemma B.5, there exists a compact neighborhood V of $\hat{A}_i$ such that $\varphi_t(x) \in Q$ for all $t \in [0, \lambda(x)]$ and all $x \in V_2$. Being V a compact neighborhood of $\hat{A}_i$, we can select $\delta > 0$ so that $\mathcal{N}_{\hat{A}_i}(\delta) \subseteq V$ and we will thus have (B.10). Being δ a function of ε, we can then select $\phi = \delta^{-1}$. □

Lemma B.7. (Uniform attraction modulo output on $\mathcal{A}_i$) *The following property holds true:*

$$\begin{aligned}&\forall \mathcal{K} \subseteq \mathcal{A}_i \textit{ compact} \;\; \forall \varepsilon > 0 \;\; \exists T_{\mathcal{K},\varepsilon} \geq 0 \;\; \forall T \geq T_{\mathcal{K},\varepsilon} \;:\\ &x \in \mathcal{K} \cap \mathcal{L}_{0,T} \;\Rightarrow\; |X(t,x)|_{\bar{A}_i} \leq \varepsilon \textit{ for all } t \in [T_{\mathcal{K},\varepsilon}, T].\end{aligned} \tag{B.11}$$

Moreover, it also holds that:

$$\begin{aligned}&\forall \bar{\mathcal{K}} \subseteq \mathcal{A}_i \textit{ compact} \;\; \forall \bar{\varepsilon} > 0 \;\; \exists \bar{T}_{\bar{\mathcal{K}},\bar{\varepsilon}} \geq 0 \;:\\ &\big(x \in \bar{\mathcal{K}} \textit{ and } \bar{T}_{\bar{\mathcal{K}},\bar{\varepsilon}} \leq \lambda(x)\big) \;\Rightarrow\; |X(t,x)|_{\bar{A}_i} \leq \bar{\varepsilon}\\ &\textit{for all } t \in [\bar{T}_{\bar{\mathcal{K}},\bar{\varepsilon}}, \lambda(x)].\end{aligned} \tag{B.12}$$

Proof. We first prove (B.11). By contradiction, assume that there exists a compact subset $\mathcal{K} \subseteq \mathcal{A}_i$ and a constant $\varepsilon > 0$ such that, for all $T_{\mathcal{K},\varepsilon} \geq 0$ and some $T \geq T_{\mathcal{K},\varepsilon}$ we have

$$\Big(\; x \in \mathcal{K} \cap \mathcal{L}_{0,T} \;\Big) \text{ and } \Big(\; |X(\bar{t},x)|_{\bar{A}_i} > \varepsilon\Big),$$

for some x and $\bar{t} \in [T_{\mathcal{K},\varepsilon}, T]$. The previous contradiction assumption can equivalently be formulated as follows: there exists a compact subset $\mathcal{K} \subseteq \mathcal{A}_i$, a constant $\varepsilon > 0$, three sequences $\{T_n\}_{n\in\mathbb{N}} > 0$, $\{\bar{t}_n\}_{n\in\mathbb{N}} > 0$, and $\{x_n\}_{n\in\mathbb{N}} \in \mathcal{K}$ such that $x_n \in \mathcal{L}_{0,T_n}$ and

$$\Big(n \leq \bar{t}_n \leq T_n\Big) \text{ and } \Big(|X(\bar{t}_n, x_n)|_{\bar{A}_i} > \varepsilon\Big) \;\; \forall n \in \mathbb{N}. \tag{B.13}$$

By virtue of Corollary B.6 (Lyapunov stability modulo output in $\mathcal{A}_i$), there exists $\delta > 0$ such that, for all $T \geq 0$ and all $t \in [0, T]$, $|X(t,x)|_{\bar{A}_i} \leq \varepsilon$ whenever $x \in \mathcal{L}_{0,T} \cap \mathcal{N}_{\hat{A}_i}(\delta)$. By Bolzano-Weierstrass, we can consider a subsequence $\{x_n\}_{n\in\mathbb{N}} \in \mathcal{K}$ such that $\lim_{n\to+\infty} x_n =: x_\infty \in \mathcal{K}$.

Claim B.8. $x_\infty \in \mathcal{L}_{0,+\infty}$.

Proof. Pick any $T_\infty > 0$. There exists $N \in \mathbb{N}$ such that $T_\infty \leq T_n$, for all $n \in \mathbb{N}_{\geq N}$. Recall that $x_n \in \mathcal{K} \cap \mathcal{L}_{0,T_n}$. By monotonicity (Claim B.1), $x_n \in \mathcal{K} \cap \mathcal{L}_{0,T_\infty}$ for all $n \in \mathbb{N}_{\geq N}$. By closedness of $\mathcal{K} \cap \mathcal{L}_{0,T_\infty}$ (Claim B.2), it follows that $x_\infty \in \mathcal{K} \cap \mathcal{L}_{0,T_\infty}$. Since the latter property holds for all $T_\infty > 0$, we conclude that $x_\infty \in \mathcal{K} \cap \mathcal{L}_{0,+\infty}$. □

Claim B.9. $x_\infty \in C_i$.

Proof. Observe that $x_\infty \in \mathcal{K} \subseteq \mathcal{A}_i$. Now, recall that that $\mathcal{A}_i \subseteq M \setminus \bar{B}_i$. The definition $\bar{B}_i := B_i \cap \mathcal{L}_{0,+\infty}$ implies that $x_\infty \in M \setminus \bar{B}_i = (M \setminus B_i) \cup (M \setminus \mathcal{L}_{0,+\infty}) = C_i \cup (M \setminus \mathcal{L}_{0,+\infty})$. By virtue of Claim B.8, we conclude that $x_\infty \in C_i$. □

By virtue of Claims B.8 and B.9 and the GATTMO property (Def. 3.3), there exists $T_{x_\infty, \frac{\delta}{2}}$ such that $|X(t, x_\infty)|_{\bigcup_{j\leq i} \mathcal{W}_j} \leq \frac{\delta}{2}$ for all $t \geq T_{x_\infty,\frac{\delta}{2}}$. Then, inclusion $\hat{A}_i \supseteq \bigcup_{j\leq i} \mathcal{W}_j$ yields $|X(t,x_\infty)|_{\hat{A}_i} \leq \frac{\delta}{2}$ for all $t \geq T_{x_\infty,\frac{\delta}{2}}$.

By Lipschitz continuity of trajectories with initial condition in $\mathcal{K}$ and local Lipschitz continuity of $|\cdot|_{\hat{A}_i}$, there exists an index $Q_1 \in \mathbb{N}$ such that, for all $q \in \mathbb{N}_{\geq Q_1}$, it holds:

$$|X\left(T_{x_\infty,\frac{\delta}{2}}, x_q\right)|_{\hat{A}_i} \leq \delta. \tag{B.14}$$

By Lyapunov stability modulo output, for all such qs, all $T \geq 0$, and all $t \in [0,T]$,

$$|X\left(t\,,\, X\left(T_{x_\infty,\frac{\delta}{2}}\,,\, x_q\right)\right)|_{\bar{A}_i} \leq \varepsilon \tag{B.15}$$

whenever $x_q \in \mathcal{L}_{0,T_{x_\infty,\frac{\delta}{2}}+T}$. However, we can always select $Q_2 \in \mathbb{N}$ so as to have $Q_2 \geq Q_1$ and $q \geq T_{x_\infty,\frac{\delta}{2}}$ for all $q \in \mathbb{N}_{\geq Q_2}$. Since $T_q \geq q$ and $x_q \in \mathcal{L}_{0,T_q}$ for all $q \geq Q_2$, we have $|X(t,x_q)|_{\bar{A}_i} \leq \varepsilon$ for all $t \in [q, T_q]$, which contradicts (B.13).

We are now going to prove (B.12). By contradiction, assume that there exists a compact subset $\bar{\mathcal{K}} \subseteq \mathcal{A}_i$ and a constant $\bar{\varepsilon} > 0$ such that, for all $\bar{T}_{\bar{\mathcal{K}},\bar{\varepsilon}} \geq 0$, there exists $x \in \bar{\mathcal{K}}$ and $\bar{t} \in [\bar{T}_{\bar{\mathcal{K}},\bar{\varepsilon}}, \lambda(x)]$ satisfying the property $|X(\bar{t},x)|_{\bar{A}_i} > \bar{\varepsilon}$. The previous contradiction assumption can equivalently be formulated as follows: there exists a compact subset $\bar{\mathcal{K}} \subseteq \mathcal{A}_i$, a constant $\bar{\varepsilon} > 0$, two sequences $\{t_n\}_{n\in\mathbb{N}} > 0$ and $\{x_n\}_{n\in\mathbb{N}} \in \bar{\mathcal{K}}$, such that:

$$\Big(n \leq t_n \leq \lambda(x_n)\Big) \text{ and } \Big(|X(t_n,x_n)|_{\bar{A}_i} > \bar{\varepsilon}\Big) \;\; \forall n \in \mathbb{N}. \tag{B.16}$$

By virtue of (B.11) with $\mathcal{K} = \bar{\mathcal{K}}$ and $\varepsilon = \bar{\varepsilon}$, it holds that:

$$\begin{aligned} &\exists T_{\bar{\mathcal{K}},\bar{\varepsilon}} \geq 0 \text{ such that } \forall T \geq T_{\bar{\mathcal{K}},\bar{\varepsilon}} \text{ if } x \in \bar{\mathcal{K}} \cap \mathcal{L}_{0,T} \\ &\text{then } |X(t,x)|_{\bar{A}_i} \leq \bar{\varepsilon} \text{ for all } t \in [T_{\bar{\mathcal{K}},\bar{\varepsilon}}\,,\, T]. \end{aligned} \tag{B.17}$$

Pick $\bar{n} \in \mathbb{N}$ such that $\bar{n} \geq T_{\bar{\mathcal{K}},\bar{\varepsilon}}$. By the definitions of $\lambda(\cdot)$ and $\mathcal{L}_{\cdot,\cdot}$, it holds for all $w \in M$ that $w \in \mathcal{L}_{0,\lambda(w)}$. In particular, we have that $x_{\bar{n}} \in \bar{\mathcal{K}} \cap \mathcal{L}_{0,\lambda(x_{\bar{n}})}$. Since $T_{\bar{\mathcal{K}},\bar{\varepsilon}} \leq \bar{n} \leq \lambda(x_{\bar{n}})$, it holds from (B.17) that $|X(t,x)|_{\bar{A}_i} \leq \bar{\varepsilon}$ for all $t \in [\bar{n}\,,\, \lambda(x_{\bar{n}})] \subseteq [T_{\bar{\mathcal{K}},\bar{\varepsilon}}\,,\, \lambda(x_{\bar{n}})]$ which contradicts (B.16). □

Corollary B.10. (Existence of a family of mappings $\{T_r\}_{r>0}$ on $\mathcal{A}_i$) *Lyapunov stability modulo-output (Corollary B.6) and uniform attraction modulo-output (Lem. B.7) on $\mathcal{A}_i$ imply the existence of a family of mappings $\{T_r\}_{r>0}$ with:*

1. *for each fixed $r > 0$, $T_r : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is continuous and is strictly decreasing;*

2. for each fixed $\varepsilon > 0$, $T_r(\varepsilon)$ is strictly increasing as r increases and $\lim_{r\to\infty} T_r(\varepsilon) = \infty$;

such that:

$$\forall x \in \mathcal{N}_{\hat{A}_i}(r) \cap \mathcal{A}_i \;\; \textit{if } T_r(\varepsilon) \leq \lambda(x) \\ \textit{then } |X(t,x)|_{\bar{A}_i} < \varepsilon \textit{ for all } t \in [T_r(\varepsilon), \lambda(x)]$$

Furthermore, there exists a class-$\mathcal{KL}$ function β such that

$$|X(t,x)|_{\bar{A}_i} \leq \beta\left(|x|_{A_i \cap \mathcal{L}_{-\infty,+\infty}}, t\right), \tag{B.18}$$

for all $t \in [0, \lambda(x))$ and all $x \in \mathcal{A}_i$.

Proof. For all $r > 0$ we can identify a compact set $\mathcal{N}_{\hat{A}_i}(r) \cap \mathcal{A}_i$ which plays the role of $\bar{\mathcal{K}}$ in Lemma B.7. Therefore, for all $r, \varepsilon > 0$ there exists $T_{r,\varepsilon} \geq 0$ such that $|X(t,x)|_{\bar{A}_i} \leq \varepsilon$ whenever $x \in \mathcal{N}_{\hat{A}_i}(r) \cap \mathcal{A}_i$ and $t \in [T_{r,\varepsilon}, \lambda(x)]$. For any $r, \varepsilon > 0$, define

$$A_{r,\varepsilon} := \{T \geq 0 \;:\; \forall x \in \mathcal{N}_{\hat{A}_i}(r) \cap \mathcal{A}_i \;\; \forall t \in [T, \lambda(x)] \\ \text{it holds } |X(t,x)|_{\bar{A}_i} < \varepsilon\}.$$

Now consider the following definitions:

$$\bar{T}_{r,\varepsilon} := \inf A_{r,\varepsilon} \\ \tilde{T}_{r,\varepsilon} := \frac{2}{\varepsilon}\int_{\varepsilon/2}^{\varepsilon} \bar{T}_r(s)\,\mathrm{d}s \\ T_{r,\varepsilon} := \tilde{T}_r(\varepsilon) + \frac{r}{\varepsilon}.$$

The proof of Lemma 3.1 in [8] shows that $T_{r,\varepsilon}$ indeed satisfies properties (i) and (ii) and, moreover, $T_{r,\varepsilon} \geq \tilde{T}_{r,\varepsilon} \geq \bar{T}_{r,\varepsilon}$. It is then clear from the definition of $A_{r,\varepsilon}$ that $|X(t,x)|_{\bar{A}_i} < \varepsilon$ whenever $x \in \mathcal{N}_{\hat{A}_i}(r) \cap \mathcal{A}_i$ and $t \in [T_{r,\varepsilon}, \lambda(x)]$.

The equivalence between existence of family of mappings $\{T_r\}_{r>0}$ and the existence of a class-$\mathcal{KL}$ function satistying (B.18) follows along the lines of Proposition 2.5 in [8] and for this reason we omit the proof here. □

Appendix C. GASMO with respect to two measures on sector $\mathcal{B}_i$ for the backward flow

We proceed to prove *global asymptotic stability modulo-output with respect to two measures* of the set $\bar{B}_i$ in $\mathcal{B}_i$, that is existence of a class-$\mathcal{KL}$ function β such that:

$$|X(-t,x)|_{\bar{B}_i} \leq \beta\left(|x|_{B_i \cap \mathcal{L}_{-\infty,+\infty}}, t\right).$$

for all $t \in [0, \lambda^\star(x))$ and all $x \in \mathcal{B}_i$. Throughout this Section, let Assumptions 2.2, 2.6, and 5.1, hold true.

For ease of presentation, we consider the backward flow:

$$\dot{x}(t) = f^\star(x(t)) = -f(x(t)), \tag{C.1}$$

with f as in (2.1). Let $X^\star(t,x)$ and $\varphi_t^\star(x)$ equivalently denote the uniquely defined solution of (C.1) for all $t \in \mathbb{R}$ with initial condition $x \in M$. Clearly, it holds $X^\star(t,x) = X(-t,x)$ and $\varphi_t^\star(x) = \varphi_{-t}(x)$. In a similar way, we can define:

$$\mathcal{A}^\star(\Lambda) := \{x \in M : |X^\star(t,x;0)|_\Lambda \to 0 \text{ as } t \to +\infty\}\,,$$
$$\mathcal{R}^\star(\Lambda) := \{x \in M : |X^\star(t,x;0)|_\Lambda \to 0 \text{ as } t \to -\infty\}\,, \tag{C.2}$$

with Λ being a closed set, and then observe that $D_i = \mathcal{A}^\star(\infty) \cup \bigcup_{i<l} \mathcal{A}^\star(\mathcal{W}_l)$ and $B_i = \bigcup_{i<l} \mathcal{R}^\star(\mathcal{W}_l)$.

Definitions (C.1) and (C.2) allow us to provide an equivalent formulatation of the repulsion property in Lemma 5.8 as the following attractivity property for all $x \in \mathcal{B}_i$:

$$\lim_{t\to+\infty} |X^\star(t,x)|_{\mathcal{Z}^\infty} = 0 \quad \text{or} \quad \lim_{t\to+\infty} |X^\star(t,x)|_{\bar{B}_i} = 0. \tag{C.3}$$

Lemma C.1. (Existence of an arbitrary small backward invariant neighborhood modulo output on $\mathcal{B}_i$) *Let Assumptions 2.2, 2.6, and 5.1 hold true. Then, for all compact neighborhoods Q of $\bar{B}_i$ in $M \setminus \operatorname{clos} A_i$, there exists a compact neighborhood V of $B_i \cap \mathcal{L}_{-\infty,+\infty}$ such that:*

$$\varphi_t^\star\left(V \cap \mathcal{L}_{-t,0}\right) \subseteq Q \text{ for all } t \geq 0. \tag{C.4}$$

Proof. Pick a compact neighborhood Q of $\bar{B}_i$. The proof is then adapted from Lemma 6 in [11] and consists of four steps:
Step 1: Consider the following definitions:

$$A_r^0 := \bigcap_{t\in[0,r]} \varphi_t^\star(Q)\,, \quad A_r := A_r^0 \cap \mathcal{L}_{0,r}.$$

Claim C.2. $\bigcap_{r\geq 0} A_r = \bar{B}_i$

Proof. Indeed, we have that $x \in \bigcap_{r\geq 0} A_r$ if and only if

$$\forall r \geq 0 \ \ \forall t \in [0,r] \ \ \varphi_{-t}^\star(x) \in Q \cap \mathcal{E}_0\Big). \tag{C.5}$$

Clearly, a necessary condition for x to be an element of $\bigcap_{r\geq 0} A_r$ is $x \in Q$. Furthermore, only the following scenarios may arise:

- $x \in Q \setminus (B_i \cup \mathcal{R}^\star(\mathcal{Z}^\infty))$. In this case, Lemma 5.3 implies that $x \in \bigcup_{j\leq i} \mathcal{R}^\star(\mathcal{W}_j)$. Therefore, since $X(t,x)$ is attracted forward in time to $\mathcal{W}_j$ with $j \leq i$, and due to the fact that $Q \cap \bigcup_{j\leq i} \mathcal{W}_j \subseteq \mathcal{B}_i \cap \bigcup_{j\leq i} \mathcal{W}_j = \emptyset$ with Q being a closed set, there exists $\bar{t} > 0$ such that $\varphi_{-\bar{t}}^\star(x) \notin Q$, which in turn contradicts condition (C.5) for $r = t = \bar{t}$.
- $x \in Q \setminus \mathcal{L}_{0,+\infty}$. In this case, there exists $\bar{t} \leq 0$ such that $\varphi_{\bar{t}}^\star(x) \notin \mathcal{E}_0$, which contradicts condition (C.5) by selecting $r = |\bar{t}|$.
- $x \in Q \cap \mathcal{L}_{0,+\infty} \cap B_i$. Then, $x \in \bar{B}_i$ and condition (C.5) is trivially satisfied.
- $x \in Q \cap \mathcal{L}_{0,+\infty} \cap \mathcal{R}^\star(\mathcal{Z}^\infty)$. Then, due to Lemma 5.5 and since $\lim_{t\to+\infty} |X(t,x)|_{\mathcal{Z}^\infty} = 0$, there exists $\bar{t} > 0$ such that $X(t,x) \notin \mathcal{E}_0$ for all $t > \bar{t}$, and thus $x \notin \mathcal{L}_{0,+\infty}$.

□

Since a locally Lipschitz vector field f generates a flow φ_t which maps any compact neighborhood of a forward invariant set to a compact neighborhood of the same set whenever $t \leq 0$, we have that A_r^0 is a compact neighborhood of $\bar{B}_i$. Furthermore, recall that the sets $\bar{B}_i$ and $\mathcal{L}_{0,r}$ are compact (Lem. E.7 and Claim B.2). It thus follows that A_r is compact. Moreover, since the sequences $\left\{A_r^0\right\}_{r\in\mathbb{R}_{\geq 0}}$ and $\{\mathcal{L}_{0,r}\}_{r\in\mathbb{R}_{\geq 0}}$ are monotone non-increasing in r (Claim B.1), we conclude that $\{A_r\}_{r\in\mathbb{R}_{\geq 0}}$ is also monotone non-increasing and, in particular, is a nested sequence of compact sets converging to $\bar{B}_i$.

Step 2: We claim that, for sufficiently large r and sufficiently small $\bar{T}$, $\varphi^\star_T(A_r) \subseteq A^0_1$, for all $T \in [0, \bar{T}]$. Indeed, let D denote the minimum distance of the boundary $\eth A^0_1$ from $\bar{B}_i$, that is $D := \inf_{a\in\eth A^0_1} \delta(a, \bar{B}_i)$. Due to A^0_1 being a compact neighborhood of $\bar{B}_i$, we have that $D > 0$. Let F denote the maximum velocity of the flux $\varphi^\star_t(\cdot)$ in A^0_1, namely $F := \sup_{x\in A^0_1} |f(x)|_{\mathfrak{g}}$. By virtue of Lemma G.2, for the neighborhood $N := \left\{x \in A^0_1 : |x|_{\bar{B}_i} \leq \frac{\mathrm{d}}{2}\right\}$ there exists $R > 0$ such that, for all $r \geq R$, it holds $A_r \subseteq N$. Therefore, for all such rs and for all $y \in A_r$, it takes at least $\frac{\mathrm{d}}{2F}$ time units for the flux $\varphi^\star_t(y)$ to reach the boundary $\eth A^0_1$. It follows that $\varphi^\star_T(A_r) \subseteq A^0_1$ for all $r \geq R$ and all $T \in [0, \bar{T}]$ with $\bar{T} := \min\left\{1, \frac{\mathrm{d}}{2F}\right\}$.

Step 3: We claim that, for all $r \geq R$ and all $T \in [0, \bar{T}]$, $\varphi^\star_T(A_r) \subseteq A^0_r$. As a preliminary result, we note that $\varphi^\star_T(A^0_r) = \bigcap_{T\leq t\leq r+T} \varphi^\star_t(Q)$ and this is easily proved by comparing the following definitions:

(i) $x \in \varphi^\star_T(A^0_r)$ if and only if $\varphi^\star_{-T-t}(x) \in Q$ for all $t \in [0, r]$;
(ii) $x \in \bigcap_{T\leq t\leq r+T} \varphi^\star_t(Q)$ if and only if $\varphi^\star_{-t}(x) \in Q$ for all $t \in [T, r + T]$.

Then, the following passages hold true for all $r \geq R$ and all $T \in [0, \bar{T}]$ with $\bar{T} = \min\left\{1, \frac{\mathrm{d}}{2F}\right\}$:

$$\begin{aligned}
\varphi^\star_T(A_r) &= \varphi^\star_T(A_r) \cap A^0_1 \\
&\subseteq \varphi^\star_T(A^0_r) \cap A^0_1 \\
&= \left(\bigcap_{T\leq t\leq r+T} \varphi^\star_t(Q)\right) \cap \left(\bigcap_{0\leq t\leq 1} \varphi^\star_t(Q)\right) \\
&= \left(\bigcap_{0\leq t\leq r+T} \varphi^\star_t(Q)\right) \subseteq A^0_r.
\end{aligned}$$

Step 4: By making use of the flow property 1) in Claim B.1, it holds for all $T \in [0, \bar{T}]$ with $\bar{T} := \min\left\{1, \frac{\mathrm{d}}{2F}\right\}$, all $\hat{T} \geq T$, and all $r \geq R$ that

$$\begin{aligned}
\varphi^\star_T\left(A^0_r \cap \mathcal{L}_{-\hat{T},r}\right) &= \varphi^\star_T\left(A^0_r \cap \mathcal{L}_{0,r} \cap \mathcal{L}_{-\hat{T},r}\right) \\
&\subseteq \varphi^\star_t\left(A^0_r \cap \mathcal{L}_{0,r} \cap \mathcal{L}_{-\hat{T},r}\right) \\
&= \varphi^\star_T\left(A_r \cap \mathcal{L}_{-\hat{T},r}\right) \\
&= \varphi^\star_T(A_r) \cap \varphi^\star_T(\mathcal{L}_{-\hat{T},r}) \\
&\subseteq A^0_r \cap \mathcal{L}_{-\hat{T}+T,r}
\end{aligned} \tag{C.6}$$

and clearly $A^0_r \cap \mathcal{L}_{-\hat{T}+T,r} \subseteq A^0_r$. We are now going to show that, for all $t \geq 0$ and all $r \geq R$, it holds: $\varphi^\star_t(A^0_r \cap \mathcal{L}_{-t,r}) \subseteq A^0_r$. To this end, pick any $t \geq 0$ and choose an integer $n \in \mathbb{N}$ so that $\frac{t}{n} \leq \bar{T}$. Then, by virtue of inclusion (B.4), we obtain:

$$\begin{aligned}
\varphi^\star_t\left(A^0_r \cap \mathcal{L}_{-t,r}\right) &= \underbrace{\left(\varphi^\star_{\frac{t}{n}} \circ \cdots \circ \varphi^\star_{\frac{t}{n}}\right)}_{n \text{ times}} \left(A^0_r \cap \mathcal{L}_{-t,r}\right) \\
&\subseteq \underbrace{\left(\varphi^\star_{\frac{t}{n}} \circ \cdots \circ \varphi^\star_{\frac{t}{n}}\right)}_{n-1 \text{ times}} \left(A^0_r \cap \mathcal{L}_{-t+\frac{t}{n},r}\right)
\end{aligned}$$

$$\subseteq \varphi^\star_{\frac{t}{n}}\left(A^0_r \cap \mathcal{L}_{-t+\frac{n-1}{n}t,r}\right)$$
$$\subseteq A^0_r. \tag{C.7}$$

Consider now any $r \geq R$. Let V be defined as:

$$V := \bigcap_{t\in[-r,0]} \varphi^\star_t(A^0_r).$$

Since $\bar{B}_i$ is generally not backward invariant, V is not a compact neighborhood of $\bar{B}_i$. However, it can be proved along the lines of Section E that that $B_i \cap \mathcal{L}_{-\infty,+\infty}$ is compact and invariant. Therefore, V qualifies as a compact neighborhood of $B_i \cap \mathcal{L}_{-\infty,+\infty}$ and its definition implies:

$$\varphi^\star_t(V) \subseteq A^0_r \text{ for all } t \in [0, r]. \tag{C.8}$$

By virtue of (C.8), we have $\varphi^\star_t(V \cap \mathcal{L}_{-t,0}) \subseteq \bar{A}^0_r$ for all $t \in [0, r]$. Conversely, if $t > r$, inclusion (C.7) and the flow property 1) in Claim B.1 imply:

$$\begin{aligned}
\varphi^\star_t(V \cap \mathcal{L}_{-t,0}) &= (\varphi^\star_{t-r} \circ \varphi_r)(V \cap \mathcal{L}_{-t,0})\\
&\subseteq \varphi^\star_{t-r}(\varphi^\star_r(V) \cap \varphi^\star_r(\mathcal{L}_{-t,0}))\\
&\subseteq \varphi^\star_{t-r}(A^0_r \cap \mathcal{L}_{-t+r,r})\\
&= \varphi^\star_{t'}(A^0_r \cap \mathcal{L}_{-t',r})\\
&\subseteq A^0_r\\
&\subseteq Q,
\end{aligned}$$

with $t' := t - r > 0$. We have thus concluded (C.4). □

Lemma C.3. (Arbitrary boundedness till hitting time on $\mathcal{B}_i$) *For all compact neighborhoods Q of $\bar{B}_i$, there exists a compact neighborhood V of $B_i \cap \mathcal{L}_{-\infty,+\infty}$ such that $V \subseteq Q$ and, for all $x \in V$, it holds:*

$$\varphi^\star_t(x) \in Q \textit{ for all } t \in [0, \lambda^\star(x)]. \tag{C.9}$$

Proof. Pick an arbitrary compact neighborhood Q of $\bar{B}_i$. Select a compact neighborhood V of $B_i \cap \mathcal{L}_{-\infty,+\infty}$ according to Lemma C.1. Then, statement (C.9) follows immediately by observing that, by definition of $\lambda^\star(\cdot)$, the property $x \in \mathcal{L}_{-\lambda^\star(x),0}$ holds for all $x \in M$ and, in particular, for all $x \in V$. □

For ease of presentation, denote $\hat{B}_i = B_i \cap \mathcal{L}_{-\infty,+\infty}$. and $\mathcal{N}_S(r) := \{x \in M : |x|_S \leq r\}$.

Corollary C.4. (Lyapunov stability modulo output on $\mathcal{B}_i$) *The following property holds true:*

$$\begin{aligned}
&\forall\varepsilon > 0\ \ \exists\delta > 0\ \ \forall T \geq 0\ \ \forall t \in [0, T]\\
&\textit{if } x \in \mathcal{L}_{-T,0} \cap \mathcal{N}_{\hat{B}_i}(\delta) \textit{ then } X^\star(t, x) \in \mathcal{N}_{\bar{B}_i}(\varepsilon).
\end{aligned} \tag{C.10}$$

Moreover, there exists a class-$\mathcal{K}_\infty$ function ϕ such that:

$$|X^\star(t, x)|_{\bar{B}_i} \leq \phi(|x|_{\hat{B}_i})\ \ \forall t \in [0, \lambda^\star(x)]. \tag{C.11}$$

Proof. It follows along the lines of the proof of Corollary B.6. □

Lemma C.5. (Uniform attraction modulo output in $\mathcal{B}_i$) *The following property holds true:*

$$\begin{aligned} &\forall \mathcal{K} \subseteq \mathcal{B}_i \text{ compact } \ \forall \varepsilon > 0 \ \exists T_{\mathcal{K},\varepsilon} \geq 0 \ \forall T \geq T_{\mathcal{K},\varepsilon} \ : \\ &x \in \mathcal{K} \cap \mathcal{L}_{-T,0} \ \Rightarrow \ |X^\star(t,x)|_{\bar{B}_i} \leq \varepsilon \text{ for all } t \in [T_{\mathcal{K},\varepsilon}, T]. \end{aligned} \tag{C.12}$$

Moreover, it also holds that:

$$\begin{aligned} &\forall \bar{\mathcal{K}} \subseteq \mathcal{B}_i \text{ compact } \ \forall \bar{\varepsilon} > 0 \ \exists \bar{T}_{\bar{\mathcal{K}},\bar{\varepsilon}} \geq 0 \ : \\ &\big(x \in \bar{\mathcal{K}} \text{ and } \bar{T}_{\bar{\mathcal{K}},\bar{\varepsilon}} \leq \lambda^\star(x)\big) \ \Rightarrow \ |X^\star(t,x)|_{\bar{B}_i} \leq \bar{\varepsilon} \\ &\text{for all } t \in [\bar{T}_{\bar{\mathcal{K}},\bar{\varepsilon}}, \lambda^\star(x)]. \end{aligned} \tag{C.13}$$

Proof. It follows along the lines of the proof of Lemma B.7. □

Corollary C.6. (Existence of a family of mappings $\{T_r\}_{r>0}$ on $\mathcal{B}_i$) *Lyapunov stability modulo-output (Cor. C.4) and uniform attraction modulo-output (Lem. C.5) imply the existence of a family of mappings $\{T_r\}_{r>0}$ with:*

1. *for each fixed $r > 0$, $T_r : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is continuous and is strictly decreasing;*
2. *for each fixed $\varepsilon > 0$, $T_r(\varepsilon)$ is strictly increasing as r increases and $\lim_{r\to+\infty} T_r(\varepsilon) = +\infty$;*

such that:

$$\begin{aligned} &\forall x \in \mathcal{N}_{\hat{B}_i}(r) \cap \mathcal{B}_i \ \text{ if } T_r(\varepsilon) \leq \lambda^\star(x) \\ &\text{then } |X^\star(t,x)|_{\bar{B}_i} < \varepsilon \text{ for all } t \in [T_r(\varepsilon), \lambda^\star(x)] \end{aligned}$$

Furthermore, there exists a class-$\mathcal{KL}$ function β such that

$$|X^\star(t,x)|_{\bar{B}_i} \leq \beta\left(|x|_{B_i \cap \mathcal{L}_{-\infty,+\infty}}, t\right), \tag{C.14}$$

for all $t \in [0, \lambda^\star(x))$ and all $x \in \mathcal{B}_i$.

Proof. It follows along the lines of the proof of Corollary B.10. □

Appendix D. OSS-Lyapunov function in $\mathcal{A}_i$ for the forward flow

D.1 Preliminary definitions and auxiliary systems

This Section addresses the construction of a smooth Lyapunov function on $\mathcal{A}_i$ which shows particular dissipation properties on a subset of $M \setminus (\mathcal{D} \cup \mathcal{B}) \subseteq \mathcal{E}$, where $\mathcal{D}, \mathcal{E}$ have been defined in Section 5.1 and:

$$\mathcal{B} := \{\xi \in M : \rho(|h(\xi)|) \leq |\xi|_{\mathcal{W}} \leq 1.5\rho(|h(\xi)|)\}.$$

Instrumental in the construction of a smooth Lyapunov function on $\mathcal{A}_i$ is the introduction of an auxiliary system $\hat{\Sigma}$ which slows down the motions of the original one:

$$\hat{\Sigma} \ : \ \dot{z} = \hat{f}(z) = \frac{1}{1 + |f(z)|_{\mathfrak{g}}^2 + \kappa(z)} f(z), \tag{D.1}$$

where κ is any smooth function $M \to [0, \infty)$ with the property that

$$\kappa(\xi) \geq 2\, |\mathcal{L}_f \rho(|h(\xi)|)| \tag{D.2}$$

whenever $|h(\xi)| \geq 1$. Denote by $Z(t, z)$ the uniquely defined solution of (D.1) at time t fulfilling $Z(0, z) = z$. It has been proven in Section 4.1 of [6] that each solution $Z(t, z)$ of (D.1) satisfies the following property:

$$X(t, x) = Z\left(\sigma_x(t), x\right), \tag{D.3}$$

with $\sigma_x : [0, t_{\max}(x)) \to \mathbb{R}_{\geq 0}$ defined as:

$$\sigma_x(t) = \int_0^t \left[1 + |f(X(s, x))|^2 + \kappa(X(s, x))\right] \mathrm{d}s \tag{D.4}$$

Moreover, it holds $\sigma(t_{\max}(x)) = +\infty$ for all $x \in M$. For each initial state $z \in M$, we can define:

$$\theta(z) := \inf\left\{t \geq 0 \ : \ Z(t, z) \in \mathcal{D}\right\}.$$

where $\theta(z) = +\infty$ if $Z(t, z) \notin \mathcal{D}$ for all $t \geq 0$. Clearly, $\lambda(z), \theta(z) > 0$ for all $z \in \mathcal{E}$. Observe that:

$$\theta(z) = \sigma_z(\lambda(z)). \tag{D.5}$$

Claim D.1. *The set $\bar{A}_i$ satisfies the GASMO property with respect to two measures on $\mathcal{A}_i$ along the trajectories of system $\hat{\Sigma}$, i.e. there exists a class-$\mathcal{KL}$ function $\hat{\beta}$ such that*

$$|Z(t, z)|_{\bar{A}_i} \leq \hat{\beta}\left(|z|_{A_i \cap \mathcal{L}_{-\infty,+\infty}}, t\right), \tag{D.6}$$

for all $t \in [0, \theta(z))$ and all $z \in \mathcal{A}_i$.

Proof. By virtue of Corollary B.10, there exists a class-$\mathcal{KL}$ function $\beta(s, r)$ such that $|X(t, x)|_{\bar{A}_i} \leq \beta(|x|_{A_i \cap \mathcal{L}_{-\infty,+\infty}}, t)$ for all $t \in [0, \lambda(x)]$ and all $x \in \mathcal{A}_i$, and thus $|Z(t, z)|_{\bar{A}_i} \leq \beta(|z|_{A_i \cap \mathcal{L}_{-\infty,+\infty}}, \sigma_z^{-1}(t))$ for all $t \in [0, \theta(z)]$ and all $z \in \mathcal{A}_i$. Observe that, by taking $M_r := \max_{|\xi|_{A_i \cap \mathcal{L}_{-\infty,+\infty}} \leq r} \left\{1 + |f(\xi)|^2 + \kappa(\xi)\right\}$, it holds that $\sigma_\xi(t) \leq M_r t$ for all ξ such that $|\xi|_{A_i \cap \mathcal{L}_{-\infty,+\infty}} \leq r$, and thus $\sigma_\xi^{-1}(t) \geq t/M_r$ whenever $|\xi|_{A_i \cap \mathcal{L}_{-\infty,+\infty}} \leq r$. Setting $\hat{\beta} := \beta(s, t/M_s)$ will then yield (D.6). □

In order for the sought Lyapunov function to yield desired dissipation properties over $M \setminus (\mathcal{D} \cup \mathcal{B})$, we first define $\phi : M \setminus \mathcal{W} \to [0, 1]$ as any smooth function such that:

$$\phi(\xi) := \begin{cases} 1, & \xi \in \mathcal{D} \\ 0, & \xi \in M \setminus (\mathcal{D} \cup \mathcal{B}). \end{cases}$$

and $\varphi : \mathbb{R}_{\geq 0} \to [0, 1]$ as any smooth nondecreasing saturation function such that

$$\varphi(s) := \begin{cases} 1, & s \geq 1 \\ 0, & s = 0 \\ \text{any value in } (0, 1), & s \in (0, 1). \end{cases}$$

Then, we introduce another system on the state space $M \setminus \mathcal{W}$:

$$\tilde{\Sigma} \;:\; \dot{z} = \tilde{f}\,(z, v) = \hat{f}(z) + 2\phi(z) f_0(z) \left(v_0 \hat{f}(z) + \varphi(|z|_{\bar{A}_i}) v_1 \right), \tag{D.7}$$

with $f_0(z) := |\hat{f}(z)|_{\mathfrak{g}}$ and controls $v := (v_0, v_1) \in V$, where $V := [-1,1] \times \{w \in T_x M \;:\; |w|_{\mathfrak{g}} \leq 1\}$. Denote by $Z(t, z; v)$ the uniquely defined solution of (D.1) at time t with initial condition $Z(0, z) = z$ and input $v \in \mathbb{V}$. Here we denote with $\mathbb{V}$ the set of so-called *auxiliary controls*, that is measurable functions of time, taking values in V equipped with the weak convergence topology ([6]., Sects. 4.2 and 4.3) System $\tilde{\Sigma}$ satisfies the following properties:

- $\tilde{\Sigma}$ is affine in v on $M \setminus \mathcal{W}$;
- since $|\tilde{f}(z, v)|_{\mathfrak{g}} \leq 5$ for all $z \in M \setminus \mathcal{W}$ and all $v \in V$, system $\tilde{\Sigma}$ is complete;
- $\tilde{f}(z, v) = \hat{f}(z)$ for all $z \notin \mathcal{D} \cup \mathcal{B}$ and all $v \in V$; then, due to the GATTMO property, for any $z \notin \mathcal{D} \cup \mathcal{B}$ and any $v \in \mathbb{V}$, there exists a $t_0 > 0$ such that $Z(t, z; v) \notin \mathcal{D} \cup \mathcal{B}$ and $Z(t, z; v) = Z(t, z)$ for all $t \in [0, t_0]$.

D.2 Continuity

Define for any $\xi \in \mathcal{E}$ and any $v \in \mathbb{V}$:

$$\theta(\xi, v) := \inf \{t \geq 0 : Z(t, \xi; v) \in \mathcal{D}\}.$$

Lemma D.2. *For every neighborhood U of $\bar{A}_i$, there is a constant c such that for any $\xi \in U \setminus \bar{A}_i$, any $v \in \mathbb{V}$, and any $t \in [0, \theta(\xi, v)]$, we have a lower bound*

$$|Z(t, \xi; v)|_{\bar{A}_i} > \frac{1}{2} \, |\xi|_{\bar{A}_i} \, e^{-ct}. \tag{D.8}$$

Proof. By virtue of Lemma I.1, for any $\xi \in U \cap \mathcal{E} \setminus \bar{A}_i$ and $v \in V$, it holds

$$\left| \mathfrak{L}_{\tilde{f}(\xi, v)} |\xi|_{\bar{A}_i} \right| \leq c |\xi|_{\bar{A}_i}.$$

From this point onwards, the proof follows along the lines of Lemma 4.1 in [6] and we recall it here. Fix $\xi \in U \cap \mathcal{E} \setminus \bar{A}_i$ and $v \in \mathbb{V}$. Note that inequality (D.8) holds for $t = 0$, and thus it holds for all small enough $t > 0$. Suppose that (D.8) fails at some $t_2 \in (0, \theta(\xi, v)]$, so that:

$$|Z(t_2, \xi; v)|_{\bar{A}_i} \leq \frac{1}{2} |\xi|_{\bar{A}_i} e^{-ct_2} < |\xi|_{\bar{A}_i}. \tag{D.9}$$

Then there exists a $t_1 < t_2$ such that $|Z(t_1, \xi; v)|_{\bar{A}_i} = |\xi|_{\bar{A}_i}$ and $|Z(t, \xi; v)|_{\bar{A}_i} \leq |\xi|_{\bar{A}_i}$ for all $t \in [t_1, t_2]$. Let

$$w(t) := \frac{1}{2} |Z(t, z; v)|^2_{\bar{A}_i}.$$

Then, for almost all $t \in [t_1, t_2]$, it holds that:

$$\begin{aligned}
|\dot{w}(t)| &= \left| |Z(t, z; v)|_{\bar{A}_i} \frac{\mathrm{d}}{\mathrm{d}t} |Z(t, z; v)|_{\bar{A}_i} \right| \\
&= \left| |Z(t, z; v)|_{\bar{A}_i} \; \mathfrak{L}_{\tilde{f}(Z(t,z;v), v(t))} |Z(t, z; v)|_{\bar{A}_i} \right| \\
&\leq \left| c \, |Z(t, z; v)|^2_{\bar{A}_i} \right|
\end{aligned}$$

$$= 2cw(t).$$

It then follows that $\dot{w} + 2cw(t) \geq 0$ for almost all $t \in [t_1, t_2]$. In particular, for all such ts, we have

$$0 \leq e^{2ct} \left(\dot{w}(t) + 2cw(t)\right) = \frac{d(e^{2ct}w(t))}{\mathrm{d}t},$$

implying that $e^{2ct}w(t) \geq e^{2ct_1}w(t_1)$ for all $t \in [t_1, t_2]$. Thus,

$$\begin{aligned}\frac{1}{2}|Z(t_2, \xi; v)|^2_{\bar{A}_i} = w(t_2) \geq e^{2ct_1}w(t_1)e^{-2ct_2} &= \frac{1}{2}|Z(t_1, \xi; v)|^2_{\bar{A}_i} e^{-2c(t_2 - t_1)} \\ &\geq \frac{1}{2}|\xi|^2_{\bar{A}_i} e^{-2ct_2},\end{aligned}$$

so that $|Z(t_2, \xi; v)|_{\bar{A}_i} \geq e^{-ct_2}|\xi|_{\bar{A}_i}$, contradicting (D.9). □

Lemma D.3. *For every $r > 0$ and $T > 0$, there is a $\sigma = \sigma(r, T) > 0$ such that $|Z(t, z; v)|_{\bar{A}_i} \geq \sigma$ for all $v \in \mathbb{V}$, all $z \in M$ with $|z|_{\bar{A}_i} \geq r$, and all $t \in [0, \min\{\theta(z, v), T\}]$.*

Proof. It directly follows from Lemma D.2. □

Lemma D.4. *For all $\xi \in \mathcal{A}_i \setminus \bar{A}_i$, all $v \in \mathbb{V}$, and all $T \geq 0$, if $\xi_k \to \xi$ in $M \setminus \bar{A}_i$ and $v_k \to v$, then $Z(t, \xi_k; v_k)$ converges to $Z(t, \xi; v)$ uniformly on $[0, T]$.*

Proof. The proof follows along the lines of Lemma 4.3 of [6] and we adapt it here to the manifold case. Assume without loss of generality that $\xi_k \in U$ with

$$U := \left\{\eta \in \mathcal{A}_i \;:\; \mathfrak{d}[\eta, \xi] \leq |\xi|_{\bar{A}_i}/2\right\}.$$

Observe that, due to $|\tilde{f}(\cdot, \cdot)|_{\mathfrak{g}} \leq 5$ and Lemma D.3, the reachable set $\mathcal{R}^{\leq T}(U)$ is a subset of

$$\left\{\eta \in \mathcal{A}_i \;:\; \sigma(|\xi|_{\bar{A}_i}/2, T) \leq |\eta|_{\bar{A}_i} \leq 1.5|\xi|_{\bar{A}_i} + 5T\right\},$$

with σ as in Lemma D.3. Therefore, by recalling Remark H.3, we can select Lipschitz constants M_1, M_2, and M_3 for $\hat{f}(\cdot)$, $2\phi(\cdot)f_0(\cdot)\hat{f}(\cdot)$, and $2\phi(\cdot)f_0(\cdot)\varphi(|\cdot|_{\bar{A}_i})$ respectively.

Now, select an integer $N \in \mathbb{N}$, a finite sequence of times $T_1, \ldots, T_N \in [0, T]$ and a finite sequence of open sets $H_1, \ldots, H_N \subseteq \mathcal{R}^{\leq T}(U)$ such that:

- $T_1 := 0$, $T_N := T$, and $T_1 < T_2 < \cdots < T_N$;
- for any $i \in \{1, \ldots, N-1\}$, set H_i is a neighborhood of $\{Z(t, \xi; T_i)\} \cup \{Z(t, \xi; T_{i+1})\}$ and belongs to a single coordinate chart of the manifold M;
- for any $i \in \{1, \ldots, N-1\}$, it holds $H_i \cap H_{i+1} \neq \emptyset$.

Assume by induction that for some $i \in \{1, \ldots, N-1\}$, $Z(T_i, \xi_k; v_k)$ is converging to $Z(T_i, \xi, v)$, namely $\lim_{k\to+\infty} \mathfrak{d}[Z(T_i, \xi_k; v_k), Z(T_i, \xi; v)] = 0$. We are now going to prove that, for any $\varepsilon > 0$, there exists an index $K \in \mathbb{N}$ such that, for any $k \in \mathbb{N}_{\geq K}$ and any $t \in [0, T_{i+1} - T_i]$, it holds $\mathfrak{d}[Z(T_i + t, \xi_k; v_k), Z(T_i + t, \xi; v)] \leq \varepsilon$, and thus $\lim_{k\to+\infty} \mathfrak{d}[Z(T_{i+1}, \xi_k; v_k), Z(T_{i+1}, \xi; v)] = 0$. Assume without loss of generality that the orbit $\{Z(T_i + t, \xi_k, v_k),\ t \in [0, T_{i+1} - T_i]\}$ enters a single coordinate chart H_i for all ks large enough. Denote $Z(t) := Z(T_i + t, \xi; v)$ and $Z_k(t) := Z(T_i + t, \xi_k; v_k)$. Then, in local coordinates, we have:

$$|Z_k(t) - Z(t)| = \left|\xi_k - \xi + \int_0^t \left(\tilde{f}(Z_k(s), v_k(s)) - \tilde{f}(Z(s), v(s))\right) \mathrm{d}s\right|$$

$$\leq |\xi_k - \xi| + \left| \int_0^t \left(\tilde{f}(Z(s), v_k(s)) - \tilde{f}(Z(s), v(s)) \right) \mathrm{d}s \right|$$
$$\left| \int_0^t \left(\tilde{f}(Z_k(s), v_k(s)) - \tilde{f}(Z(s), v_k(s)) \right) \mathrm{d}s \right|$$
$$\leq |\xi_k - \xi| + 2 \left| \int_0^t \phi(Z(s)) f_0(Z(s)) \hat{f}(Z(s)) \left(v_{0k}(s) - v_0(s)\right) \mathrm{d}s \right|$$
$$+ 2 \left| \int_0^t \phi(Z(s)) f_0(Z(s)) \varphi(|Z(s)|_{\bar{A}_i}) \left(v_{1k}(s) - v_1(s)\right) \mathrm{d}s \right|$$
$$\int_0^t (M_1 + M_2 + M_3) \left| Z_k(s) - Z(s) \right| \mathrm{d}s,$$

for all $t \in [0, T_{i+1} - T_i]$. Due to the weak convergence of v_k to v, the first two integrals tend to 0, namely for any $\varepsilon > 0$ there exists a $K_1 \in \mathbb{N}$ with $K_1 > K_0$ such that, for all $k \in \mathbb{N}_{\geq K_1}$,

$$|\xi_k - \xi| + 2 \left| \int_0^t \phi(Z(s)) f_0(Z(s)) \hat{f}(Z(s)) \left(v_{0k}(s) - v_0(s)\right) \mathrm{d}s \right|$$
$$+ 2 \left| \int_0^t \phi(Z(s)) f_0(Z(s)) \varphi(|Z(s)|_{\bar{A}_i}) \left(v_{1k}(s) - v_1(s)\right) \mathrm{d}s \right|$$
$$\leq \varepsilon \, e^{-(M_1+M_2+M_3)(T_{i+1}-T_i)}.$$

Then, for all such ks and all $t \in [0, T_{i+1} - T_i]$, we have, by the Gronwall inequality,

$$|Z(t) - Z_k(t)| \leq \varepsilon e^{-(M_1+M_2+M_3)(T_{i+1}-T_i)} e^{(M_1+M_2+M_3)(T_{i+1}-T_i)} \leq \varepsilon.$$

The Lemma is thus proved by letting $\varepsilon \to 0$. □

Lemma D.5. *Let $\xi \in \mathcal{A}_i \cap \eth\mathcal{D} \setminus \bar{A}_i$. Then, for each $\tau > 0$, there exists a neighborhood U of ξ, such that for any $\eta \in U$ there is some control $v \in \mathbb{V}$, and some $0 \leq t_1 \leq \tau$, such that $Z(t_1, \eta; v) = \xi$ and $\mathfrak{d}[Z(t, \eta; v), \xi] \leq \mathfrak{d}[\eta, \xi]$ for all $0 \leq t \leq t_1$.*

Proof. Since $\xi \in \mathcal{D} \setminus \bar{A}_i$, it holds that $\phi(\xi) f_0(\xi) \varphi(|\xi|_{\bar{A}_i}) \neq 0$ and thus, by continuity of $\phi, f_0, \varphi(|\cdot|_{\bar{A}_i})$, there exist a neighborhood U_1 of ξ and some constant $c_1 > 0$ such that:

$$2\phi(\eta) f_0(\eta) \varphi(|\eta|_{\bar{A}_i}) > c_1, \quad \forall \eta \in U_1. \tag{D.10}$$

Since $\xi \in \eth\mathcal{D}$, it holds that $\phi(\xi) = 1$ and thus, by continuity, there exists a neighborhood U_2 of ξ such that:

$$\phi(\eta) > \frac{2}{3}, \quad \forall \eta \in U_2. \tag{D.11}$$

Consider now the geodesic ball:

$$B := \left\{ \eta \in M \ : \ \mathfrak{d}[\eta, \xi] \leq \tau c_1 \right\}. \tag{D.12}$$

Observe that, for any $\eta \in B$, there exists a unique vector $r(\eta) \in T_\eta M$ such that the geodesic $\gamma_{r(\eta)}(t) : \mathbb{R}_{\geq 0} \to M$ satisfies

$$\gamma_{r(\eta)}(0) = \eta, \ \dot{\gamma}_{r(\eta)}(0) = r(\eta), \ \left| \dot{\gamma}_{r(\eta)}(t) \right|_{\mathfrak{g}} = c_1 \text{ for all } t \geq 0, \ \ \gamma_{r(\eta)}(t_1) = \xi \tag{D.13}$$

for some $t_1 \leq \tau$. Furthermore it can be verified, for all $\eta \in B$, that:

$$\mathfrak{d}[\eta,\xi] = L\left[\gamma_{r(\eta)}\right]\Big|_0^{t_1} = \int_0^{t_1} \left|\dot{\gamma}_{r(\eta)}(u)\right|_{\mathfrak{g}} \mathrm{d}u = t_1 c_1 \leq \tau c_1.$$

Now select U as the largest geodesic ball contained in $U_1 \cap U_2 \cap B$ and let $\eta \in U$. We are now going to construct the open-loop auxiliary control v fullfilling the statement of the Lemma. For all $t \geq 0$, let $v(t)$ be the following open-loop auxiliary control:

$$v_0(t) = -\frac{1}{2\phi(\gamma_{r(\eta)}(t))}, \quad v_1(t) = \frac{\dot{\gamma}_{r(\eta)}(t)}{2\phi(\gamma_{r(\eta)}(t))f_0(\gamma_{r(\eta)}(t))\varphi(|\gamma_{r(\eta)}(t)|_{\mathcal{W}})}. \tag{D.14}$$

It follows from (D.11) that $|v_0(t)|_{\mathfrak{g}} \leq 3/4$ for all $\eta \in U$. Since $|\dot{\gamma}_{r(\eta)}(t)|_{\mathfrak{g}} = c_1$ for all $t \geq 0$ and due to condition (D.10), it follows that $|v_1(t)|_{\mathfrak{g}} \leq 1$. Furthermore, injecting the open-loop control (D.14) in system (D.7) with initial condition at η yields $\dot{z} = \dot{\gamma}_{r(\eta)}(t)$ and thus, by virtue of (D.13), $Z(t_1,\eta;v) = \xi$ and $\mathfrak{d}[Z(t,\eta;v),\xi] \leq \mathfrak{d}[\eta,\xi]$ for all $t \in [0,t_1]$. □

Lemma D.6. *The map $(\xi,v) \mapsto \theta(\xi,v)$ is lower semicontinuous on $(\mathcal{A}_i \cap \mathcal{E} \setminus \bar{A}_i) \times \mathbb{V}$. The map $\xi \mapsto \theta(\xi)$ is lower semicontinuous on $\mathcal{E}$.*

Proof. The proof follows along the lines of Lemma 4.5 in [6] and we recall it here. Pick two sequences $\{\xi_k\}_{k\in\mathbb{N}} \in \mathcal{E} \setminus \bar{A}_i$ and open loop auxiliary controls $\{v_k\}_{k\in\mathbb{N}} \in \mathbb{V}$ such that $\xi_k \to \xi$ and $v_k \to v$ for some $\xi \in \mathcal{E} \setminus \bar{A}_i$ and $v \in \mathbb{V}$. We need to show that:

$$\theta(\xi,v) \leq \liminf_{k\to+\infty} \theta(\xi_k,v_k).$$

Let $\theta_k := \theta(\xi_k,v_k)$. Without loss of generality, we may assume that $\liminf_{k\to+\infty}\theta_k =: \theta_0 < +\infty$. Passing to a subsequence if necessary, we assume that $\theta_k \to \theta_0$. Thus, there exists some $K \in \mathbb{N}$ such that $\theta_k \leq \theta_0 + 1$ for all $k \geq K$. By virtue of Lemma D.4, $Z(t,\xi;v)$ converges uniformly on $[0,\theta_0+1]$, and thus $Z(\theta_0,\xi;v) = \lim_{k\to+\infty} Z(\theta_k,\xi_k;v_k)$. Since $\mathcal{D}$ is closed and $Z(\theta_k,\xi_k,v_k) \in \mathcal{D}$ for each k, we know that $Z(\theta_0,\xi;v) \in \mathcal{D}$, and hence $\theta(\xi,v) \leq \theta_0$. □

By virtue of Proposition 7 in [14], there exists two class-$\mathcal{K}_\infty$ functions Ξ, μ_2 such that the following upper bound holds true for the class-$\mathcal{KL}$ function $\hat{\beta}$ of Claim D.1:

$$\hat{\beta}(r,t) \leq \Xi^{-1}\left(\mu_2(r)e^{-t}\right) \quad \forall r,t \geq 0.$$

We are now going to use function Ξ in the definition of a provisional candidate Lyapunov function for $\tilde{\Sigma}$. Define, for $\xi \in (\mathcal{A}_i \cap \mathcal{E}) \cup \bar{A}_i$ and $v \in \mathbb{V}$,

$$V(\xi,v) := \int_0^{\theta(\xi,v)} \Xi(|Z(t,\xi;v)|_{\bar{A}_i})\,\mathrm{d}t. \tag{D.15}$$

We also define for $\xi \in (\mathcal{A}_i \cap \mathcal{E}) \cup \bar{A}_i$:

$$V(\xi) := \inf_{v\in\mathbb{V}} V(\xi,v). \tag{D.16}$$

Clearly, it holds that $V(\xi) \leq V(\xi, \mathbf{0})$ where we have denoted with $\mathbf{0}$ the auxiliary control identically equal to 0. Furthermore, since $\bigcup_{j\leq i} \mathcal{W}_j \subseteq A_i \cap \mathcal{L}_{-\infty,+\infty}$, observe that

$$\begin{aligned} V(\xi, \mathbf{0}) &= \int_0^{\theta(\xi,\mathbf{0})} \Xi(|Z(t,z;\mathbf{0})|_{\bar{A}_i})\,\mathrm{d}t \\ &\leq \int_0^{+\infty} \mu_2(|z|_{A_i\cap\mathcal{L}_{-\infty,+\infty}})e^{-t}\,\mathrm{d}t \\ &\leq \mu_2(|z|_{A_i\cap\mathcal{L}_{-\infty,+\infty}}) \\ &\leq \mu_2(|z|_{\bigcup_{j\leq i}\mathcal{W}_j}), \end{aligned} \tag{D.17}$$

for all $z \in \mathcal{E} \cap \mathcal{A}_i$.

Lemma D.7. *The map $(\xi, v) \mapsto V(\xi, v)$ is lower semicontinuous on $(\mathcal{A}_i \cap \mathcal{E} \setminus \bar{A}_i) \times \mathbb{V}$.*

Proof. The proof follows along the lines of Lemma 4.6 in [6] and we recall it here. Pick two sequences $\{\xi_k\}_{k\in\mathbb{N}} \in \mathcal{A}_i \cap \mathcal{E} \setminus \bar{A}_i$ and $\{v_k\}_{k\in\mathbb{N}} \in \mathbb{V}$ such that $\xi_k \to \xi$ and $v_k \to v$ for some $\xi \in \mathcal{A}_i \cap \mathcal{E} \setminus \bar{A}_i$ and $v \in \mathbb{V}$. We need to show that:

$$V(\xi, v) \leq \liminf_{k\to+\infty} V(\xi_k, v_k).$$

Case $V(\xi, v) < +\infty$. The GATTMO property (D.6) guarantees that, for any $\varepsilon > 0$, there exists some $T \in (0, \theta(\xi, v))$ such that

$$V(\xi, v) = \int_0^{\theta(\xi,v)} \Sigma(|Z(t,\xi;v)|_{\bar{A}_i})\,\mathrm{d}t \leq \int_0^T \Sigma(|Z(t,\xi;v)|_{\bar{A}_i})\,\mathrm{d}t + \varepsilon.$$

Without loss of generality we can assume that the ξ_k are within the unit distance from ξ. Recall that the reachable set from th eunit ball around ξ is bounded. By virtue of Lemma D.4, $Z(t, \xi_k; v_k)$ converges to $Z(t, \xi; v)$ uniformly on $[0, T]$. Since Ξ is uniformly continuous on compacts, it then follows that there exists some $K \in \mathbb{N}$ such that

$$\left|\Xi(|Z(t,\xi_k;v_k)|_{\bar{A}_i}) - \Xi(|Z(t,\xi;v)|_{\bar{A}_i})\right| \leq \frac{\varepsilon}{1+T} \quad \forall k \in \mathbb{N}_{\geq K},\ \forall t \in [0, T].$$

This implies that:

$$\int_0^T \Xi(|Z(t,\xi_k;v_k)|_{\bar{A}_i})\,\mathrm{d}t \geq \int_0^T \Xi(|Z(t,\xi;v)|_{\bar{A}_i})\,\mathrm{d}t - \varepsilon \tag{D.18}$$

for all $k \in \mathbb{N}_{\geq K}$. By Lemma D.6, there exists some $K_1 \in \mathbb{N}$ such that $K_1 \geq K$ and $\theta(\xi, v_k) > T$ for all $k \in \mathbb{N}_{\geq K_1}$. Thus, for all $k \geq K_1$,

$$\begin{aligned} V(\xi_k, v_k) &\geq \int_0^T \Xi(|Z(t,\xi_k;v_k)|_{\bar{A}_i})\,\mathrm{d}t \\ &\geq \int_0^T \Xi(|Z(t,\xi;v)|_{\bar{A}_i})\,\mathrm{d}t - \varepsilon \geq V(\xi, v) - 2\varepsilon. \end{aligned}$$

As $\varepsilon \to 0$, we obtain $V(\xi, v) \leq \liminf_{k\to+\infty} V(\xi_k, v_k)$.

Case $V(\xi, v) = +\infty$. It easily follows that $\theta(\xi, v) = +\infty$. For any integer $k \geq 0$, there exists a time $T_k \geq 0$ such that

$$\int_0^{T_k} \Xi(|Z(t,\xi;v)|_{\bar{A}_i})\,\mathrm{d}t \geq k.$$

Repeating the same argument used above, one sees that

$$\int_0^{T_k} \Xi(|Z(t,\xi_h;v_h)|_{\bar{A}_i})\,\mathrm{d}t \geq k-1.$$

for all $h \in \mathbb{N}_{\geq H_0}$ with $H_0 \in \mathbb{N}$. By Lemma D.6, there is some $H_1 \in \mathbb{N}$ such that $H_1 \geq H_0$ and $\theta(\xi_h, v_h) \geq T_k$ for all $h \in \mathbb{N}_{\geq H_1}$. Consequently, for all such hs,

$$V(\xi_h, v_h) \geq \int_0^{T_k} \Xi(|Z(t,\xi_h;v_h)|_{\bar{A}_i})\,\mathrm{d}t \geq k-1.$$

Since $k > 0$ can be picked arbitrarily, it follows that

$$\liminf_{h\to+\infty} V(\xi_h, v_h) = +\infty,$$

and thus $\liminf_{h\to+\infty} V(\xi_h, v_h) \geq V(\xi, v)$. □

Lemma D.8. *For every $\xi \in \mathcal{A}_i \cap \mathcal{E} \setminus \bar{A}_i$, there exists a control $\bar{v}$ such that $V(\xi,\bar{v}) = V(\xi)$.*

Proof. The proof follows along the lines of Lemma 4.7 in [6] and we recall it here. Let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence of open loop auxiliary controls such that $V(\xi, v_k) \to V(\xi)$ as $k \to +\infty$. Without loss of generality we are assuming that all these controls are defined for all positive t. Extract from $\{v_k\}_{k\in\mathbb{N}}$ a subsequence such that, without relabeling, the v_ks converge to some $\bar{v} \in \mathbb{V}$. By Lemma D.7,

$$V(\xi,\bar{v}) \leq \lim_{k\to+\infty} V(\xi, v_k) = V(\xi).$$

Since $V(\xi) \leq V(\xi, v)$ for all $v \in \mathbb{V}$, it follows that $\bar{v}$ satisfies $V(\xi) = V(\xi, \bar{v})$. □

Corollary D.9. *For any $\xi \in \mathcal{A}_i \cap \mathcal{E} \setminus \bar{A}_i$, $v \in \mathbb{V}$, and $T \in [0, \theta(\xi, v))$, it holds that*

$$V(\xi) \leq \int_0^{T} \Xi(|Z(t,\xi;v)|_{\bar{A}_i})\,\mathrm{d}s + V(Z(T,\xi;v)). \tag{D.19}$$

Proof. The proof follows along the lines of Corollary 4.8 in [6] and we recall it here. Assume by contradiction that there exists $\xi \in \mathcal{A}_i \cap \mathcal{E} \setminus \bar{A}_i$, $v \in \mathbb{V}$, and $T \in [0, \theta(\xi, v))$ such that (D.19) fails. By Lemma D.8, there exists a control $v_1 \in \mathbb{V}$ such that $V(Z(T,\xi;v)) = V(Z(T,\xi;v), v_1)$. Define $\bar{v}$ to be the concatenation of v and v_1. Then, letting $\theta := \theta(Z(T,\xi;v), v_1)$ and noticing that $\theta(\xi,\bar{v}) = \theta + T$, we get, by out assumption,

$$\begin{aligned} V(\xi) &> \int_0^{T} \Xi(|Z(t,\xi;v)|_{\bar{A}_i})\,\mathrm{d}t + V(Z(T,\xi;v)) \\ &= \int_0^{T} \Xi(|Z(t,\xi;v)|_{\bar{A}_i})\,\mathrm{d}t + \int_0^{\theta} \Xi(|Z(t,Z(T,\xi;v);v_1)|_{\bar{A}_i})\,\mathrm{d}t \\ &= \int_0^{\theta+T} \Xi(|Z(t,z;\bar{v})|_{\bar{A}_i})\,\mathrm{d}t, \end{aligned} \tag{D.20}$$

which contradict the minimality of $V(\xi)$. □

Lemma D.10. *For any $\xi \in \mathcal{A}_i \cap \eth\mathcal{D} \setminus \bar{A}_i$, it holds $\lim_{\eta\to\xi} V(\eta) = 0$.*

Proof. The proof follows along the lines of Lemma 4.9 in [6] and we adapt it here with minor modifications. Pick any $\xi \in \mathcal{A}_i \cap \eth\mathcal{D} \setminus \bar{A}_i$. Pick any $\varepsilon > 0$. Let $\tau := \varepsilon/\Xi(|\xi|_{\bar{A}_i} + 1)$. Let $U \subset M$ be the neighborhood of ξ as provided by Lemma D.5. Without loss of generality, assume $\mathfrak{d}[\eta, \xi] \le 1$ for all $\eta \in U$. Thus, by virtue of Lemma D.5, for any $\eta \in U$ there exists a control v_η and a time $t_1 \in [0, \tau]$ such that $Z(t_1, \eta; v_\eta) = \xi$ and $\mathfrak{d}[Z(s, \eta; v_\eta), \xi] \le 1$ for all $s \in [0, t_1]$.

Claim D.11. *For any closed set $\mathcal{U}$ and any $x_1, x_2 \in M$, $|x_1|_{\mathcal{U}} \le |x_2|_{\mathcal{U}} + \mathfrak{d}[x_1, x_2]$.*

Proof. By closedness of $\mathcal{U}$, we can find $w_2 := \arg\inf_{a\in\mathcal{U}} \mathfrak{d}\,[a, x_2] \in M$. Then, the minimizing property of geodesics yields $\mathfrak{d}\,[x_1, w_2] \le |x_2|_{\mathcal{U}} + \mathfrak{d}[x_1, x_2]$. The definition of norm $|\cdot|_{\mathcal{U}}$ implies that $|x_1|_{\mathcal{U}} \le \mathfrak{d}\,[x_1, w_2]$, which proves the Claim. □

By virtue of Claim D.11 and closedness of $\bar{A}_i$, it follows that $|Z(s, \eta; v_\eta)|_{\bar{A}_i} \le |\xi|_{\bar{A}_i} + \mathfrak{d}[Z(s, \eta; v_\eta), \xi] \le |\xi|_{\bar{A}_i} + 1$ for all $s \in [0, t_1]$, and thus:

$$V(\eta, v_\eta) \le \int_0^{t_1} \Xi(|Z(s, \eta; v_\eta)|_{\bar{A}_i})\,\mathrm{d}s \le \tau\,\Xi(|\xi|_{\bar{A}_i} + 1) \le \varepsilon,$$

from which it follows that $V(\eta) \le \varepsilon$. □

Lemma D.12. *Function $V(\xi)$ is continuous at $\bar{A}_i$, i.e. $\lim_{|\xi|_{\bar{A}_i}\to 0} V(\xi) = 0$.*

Proof. Assume by contradiction that:

$$\exists \varepsilon > 0 \;\; \forall r > 0 \;\; \exists \xi_r \in \mathcal{A}_i \;:\; |\xi_r|_{\bar{A}_i} \le r \text{ and } V(\xi_r) > \varepsilon.$$

In other words, we are assuming that:

$$\exists \varepsilon > 0 \;\; \exists \{\xi_k\}_{k\in\mathbb{N}} \in \mathcal{A}_i \;:\; \lim_{k\to+\infty} |\xi_k|_{\bar{A}_i} = 0 \text{ and } V(\xi_k) > \varepsilon. \tag{D.21}$$

By compactness of $\bar{A}_i$ and the Bolzano-Weierstrass theorem, we can select a subsequence $\{\xi_k\}_{k\in\mathbb{N}} \in \mathcal{A}_i$ with no relabeling such that $\lim_{k\to+\infty} \xi_k = \bar{\xi} \in \bar{A}_i$.

Case $\bar{\xi} \in A_i \cap \mathcal{L}_{-\infty,+\infty}$. Since $V(\xi_k) \le V(\xi, \mathbf{0})$ for all $k \in \mathbb{N}$ and due to (D.17), it follows that $0 \le V(\xi_k) \le V(\xi, \mathbf{0}) \le \mu_2(|\xi_k|_{A_i\cap\mathcal{L}_{-\infty,+\infty}})$ and thus there exists $K_1 \in \mathbb{N}$ such that $V(\xi_k) \le \varepsilon$ for all $k \in \mathbb{N}_{\ge K_1}$. The latter estimate contradicts (D.21).

Case $\bar{\xi} \in \bar{A}_i \setminus (A_i \cap \mathcal{L}_{-\infty,+\infty})$. Since $\bar{\xi} \in A_i \cap \mathcal{L}_{-\infty,0} \setminus \mathcal{L}_{-\infty,+\infty}$, there exists $\bar{t} \ge 0$ such that $Z(\bar{t}, \bar{\xi}) \in \operatorname{int}\mathcal{D}$. Since $\operatorname{int}\mathcal{D}$ is open, continuity of trajectories implies the existence of an index $K_0 \in \mathbb{N}$ such that, for all $k \in \mathbb{N}_{\ge K_0}$, $Z(\bar{t}, \bar{\xi}_k) \in \operatorname{int}\mathcal{D}$. Note that $\bar{\xi} \in \bar{A}_i$, which implies $0 \le V(\bar{\xi}) \le V(\bar{\xi}, \mathbf{0}) = 0$. Therefore, by continuity of trajectories over a finite time interval $[0, t_1]$, we can find an index $K_1 \in \mathbb{N}$ such that $K_1 > K_0$ and, for all $k \in \mathbb{N}_{\ge K_0}$,

$$V(\xi_k, \mathbf{0}) \le \int_0^{t_1} \Xi\left(|X(s, \xi_k; \mathbf{0})|_{\bar{A}_i}\right)\,\mathrm{d}s \le \varepsilon.$$

It thus follows that $0 \le V(\xi_k) \le V(\xi_k, \mathbf{0}) \le \varepsilon$ contradicts (D.21). □

In order to prove the continuity of $V(\cdot)$, we also need the following result.

Lemma D.13. *Assume that* $V(\xi, v) < +\infty$ *for some* $\xi \in \mathcal{A}_i \cap \mathcal{E} \setminus \bar{A}_i$ *and some* $v \in \mathbb{V}$. *Then there exists* $\xi_0 \in \mathcal{A}_i \cap \eth\mathcal{D} \setminus \bar{A}_i$ *such that*

$$\lim_{t \to \theta(\xi, v)} Z(t, \xi; v) = \xi_0$$

or $\lim_{t\to+\infty} |Z(t, \xi, v)|_{\bar{A}_i} = 0$.

Proof. The proof follows along the lines of Lemma 4.10 in [6] and we adapt it here with minor modifications. If $\theta(\xi, v) < +\infty$, then $\xi_0 = Z(\theta(\xi, v), \xi; v) \in \mathcal{A}_i \cap \eth\mathcal{D}$ by continuity of trajectories. By Lemma D.3, since $\xi \in \mathcal{A}_i \cap \mathcal{E} \setminus \bar{A}_i$, trajectory $Z(\cdot, \xi; v)$ does not converge to $\bar{A}_i$ in finite time, hence $\xi_0 \in \mathcal{A}_i \cap \eth\mathcal{D} \setminus \bar{A}_i$.

Assume now that $\theta(\xi, v) = +\infty$. Since

$$\int_0^{+\infty} \Xi(|Z(s, \xi; v)|_{\bar{A}_i}) \, \mathrm{d}s < +\infty$$

by assumption, it follows that

$$\int_t^{+\infty} \Xi(|Z(s, \xi; v)|_{\bar{A}_i}) \, \mathrm{d}s \to 0$$

as $t \to +\infty$. Consider the family of functions $\{x_t(\cdot),\ t > 0\}$, defined by $x_t(s) := Z(t + s, \xi; v)$. By Lemma D.3, no trajectory converges to an invariant set in finite time, thus $Z(s, \xi; v)$ does not reach $\bar{A}_i \subseteq A_i$ in finite time. Therefore, there exists a positive, strictly decreasing function φ such that $\varphi(s) < |Z(s, \xi; v)|_{\bar{A}_i}$ for all $s \geq 0$. Find a $\mathcal{K}_\infty$ function κ such that

$$\kappa(\varphi(s)) > \int_t^{+\infty} \Xi(|Z(s, \xi; v)|_{\bar{A}_i}) \, \mathrm{d}s = \int_0^{+\infty} \Xi(|x_t(s)|_{\bar{A}_i}) \, \mathrm{d}s \tag{D.22}$$

Then the family $\{x_t(\cdot),\ t > 0\}$ satisfies all the conditions of Proposition 3.9 in [6], and thus, for $r := |\xi|_{\bar{A}_i}$ and for any $\varepsilon > 0$, there exists $T_{r,\varepsilon} \geq 0$ such that $|Z(t, \xi; v)|_{\bar{A}_i} < \varepsilon$. Therefore $\lim_{t\to+\infty} |Z(t, \xi, v)|_{\bar{A}_i} = 0$. ☐

Lemma D.14. *Function* $V : (\mathcal{A}_i \cap \mathcal{E}) \cup \bar{A}_i \to \mathbb{R}_{\geq 0}$ *is continuous.*

Proof. The proof follows along the lines of Proposition 4.11 in [6] and we adapt it here with minor modifications and some additions. Lemma D.12 together with $V(\bar{A}_i) = 0$ proves continuity at $\bar{A}_i$. Fix $\xi \in \mathcal{A}_i \cap \mathcal{E} \setminus \bar{A}_i$. Suppose $\xi_k \to \xi$, where $\xi_k \in \mathcal{E}$. Let $\{k_j\}$ be a subsequence of $\{k\}$ such that

$$\lim_{j\to+\infty} V(\xi_{k_j}) = \liminf_{k\to+\infty} V(\xi_k). \tag{D.23}$$

For each k, Lemma D.8 provides the auxiliary control $v_k \in \mathbb{V}$ such that $V(\xi_k) = V(\xi_k, v_k)$. Notice that $\lim_{j\to+\infty} V(\xi_{k_j}, v_{k_j})$ exists because of (D.23). By sequential compactness of $\mathbb{V}$, there exists a subsequence of $\{v_{k_j}\}$ converging to some $\bar{v} \in \mathbb{V}$. Without relabeling, we assume that $v_{k_j} \to \bar{v}$. It then follows from Lemma D.7 that

$$V(\xi, \bar{v}) \leq \lim_{j\to+\infty} V(\xi_{k_j}, v_{k_j}) = \liminf_{k\to+\infty} V(\xi_k).$$

Consequently, by definition of $V(\xi)$, it follows that $V(\xi) \leq \liminf_{k\to+\infty} V(\xi_k)$. To complete the proof, we will show that

$$V(\xi) \geq \limsup_{k\to+\infty} V(\xi_k). \tag{D.24}$$

To this end, pick $\varepsilon > 0$. By making use of Lemma D.8 again, let $v \in \mathbb{V}$ be a control such that $V(x) = V(\xi, v)$. Observe that, because of $V(\xi) \leq V(\xi, \mathbf{0})$ and (D.17), $V(\xi)$ takes finite value. Then, by virtue of Lemma D.13, $\lim_{t\to+\infty} |Z(t,\xi;v)|_{\bar{A}_i} = 0$ or there exists $\xi_0 \in \mathcal{A}_i \cap \eth\mathcal{D} \setminus \bar{A}_i$ such that $\lim_{t\to\theta(\xi,v)} Z(t,\xi;v) = \xi_0$.

Case $\lim_{t\to\theta(\xi,v)} Z(t,\xi;v) = \xi_0$ *for some* $\xi_0 \in \mathcal{A}_i \cap \eth\mathcal{D} \setminus \bar{A}_i$. By virtue of Lemma D.10, there exists a neighborhood U of ξ_0 such that

$$V(\eta) < \frac{\varepsilon}{4} \;\; \forall \eta \in U \cap \mathcal{E}. \tag{D.25}$$

Case $\lim_{t\to+\infty} |Z(t,\xi;v)|_{\bar{A}_i} = 0$. By virtue of Lemma (D.12), there exists a neighborhood U of $\bar{A}_i$ such that (D.25) holds.

For both aforementioned cases, the proof continues as follows. Let $T \in (0, \theta(\xi, v))$ be such that $Z(T,\xi;v) \in U$. Then, since $\{Z(t,\xi_k;v)\}$ converges to $Z(t,\xi;v)$ uniformly on $[0,T]$, it follows that $Z(T,\xi_k;v) \in U$ for $k \geq K_1$ for some K_1. By Lemma D.6, one may assume $T < \theta(\xi_k, v)$ which implies $\eta = Z(T,\xi_k;v) \in \mathcal{A}_i \cap \mathcal{E}$. By making use of inequality (D.25), we then obtain:

$$V(Z(T,\xi_k;v) < \frac{\varepsilon}{4} \;\; \forall k \geq K_1.$$

By the uniform convergence property of $\{Z(t,\xi_k;v)\}$, there exists a compact set $\mathcal{K}$ such that $Z(t,\xi_k;v) \in \mathcal{K}$ for all k and all $t \in [0,T]$. Using also the uniform continuity of $\Xi(\cdot)$ on compacts, one sees that there is some $K_2 \geq K_1$ such that

$$\int_0^T \Xi(|Z(s,\xi_k;v)|_{\bar{A}_i})\,\mathrm{d}s \leq \int_0^T \Xi(|Z(s,\xi;v)|_{\bar{A}_i})\,\mathrm{d}s + \frac{\varepsilon}{2}$$

for all $k \geq K_2$. By virtue of Corollary D.9, it holds:

$$\begin{aligned} V(\xi_k) &\leq \int_0^T \Xi(|Z(s,\xi_k;v)|_{\bar{A}_i})\,\mathrm{d}s + V(Z(T,\xi_k;v)) \\ &\leq V(\xi,v) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = V(\xi) + \varepsilon \end{aligned}$$

for all $k \geq K_2$. From this, it follows that

$$V(\xi) \geq \limsup_{k\to+\infty} V(\xi_k) - \varepsilon.$$

Letting $\varepsilon \to 0$, one proves (D.24). □

D.3 Lower and upper bound

Lemma D.15. *Without loss of generality, select* $\rho(s) \geq s$ *for all* $s \geq 0$. *and Assume* $p(\cdot)$ *is any smooth class-*$\mathcal{K}_\infty$ *function defined as* $p(s) = b\rho(s)$ *for all* $s > 0$ *and some* $b > 1$. *Then, for system (D.1), it holds that* $|\mathfrak{L}_{\hat{f}}p(|h(\xi)|)| \leq 1$ *for all* ξ *with* $|h(\xi)| \geq 1$. *Furthermore, pick any constant* $a > 0$ *and define* $K_0 = p(1) + (2 + a)/a + 1$. *Then for each* $\xi \in \mathcal{A}_i$ *such that*

$$|\xi|_{\mathcal{W}} \geq (1+a)p(|h(\xi)|) \text{ and } |\xi|_{\mathcal{W}} \geq K_0,$$

it holds that $|Z(t,\xi)|_{\mathcal{W}} > p(|h(Z(t,\xi))|)$ *for all* $t \in [0,1)$ *and all* $\xi \in \mathcal{A}_i$.

Proof. The proof follows along the lines of Lemma 4.14 in [6] and we recall it here. Fix a and ξ as in the formulation of the Lemma, and define

$$\theta := \min\{t \ : \ |Z(t,\xi)|_{\mathcal{W}} \leq p(|h(Z(t,\xi))|)\}$$

with the understanding that $\theta = +\infty$ if the inequality never holds for $t \geq 0$. Assume by contradiction that the Lemma is false, so that $\theta < 1$.

Let $\eta := Z(\theta,\xi)$. By virtue of Claim D.11, since $\mathcal{W}$ is closed and $|\hat{f}(\xi)|_{\mathfrak{g}} \leq 1$ for all $\xi \in M$, it holds that $|\eta|_{\mathcal{W}} \geq |\xi|_{\mathcal{W}} - \theta \geq K_0 - 1$. By the definitions of η, θ, and K_0, one has

$$p(|h(\eta)|) = |\eta|_{\mathcal{W}} \geq K_0 - 1, \tag{D.26}$$

so also $|h(\eta)| \geq p^{-1}(K_0-1) > 1$. By continuity, $|h(Z(s,\eta))| > 1$ for all s near zero.

Claim D.16. *$|h(Z(s,\eta))| > 1$ for all $s \in [-1,0]$.*

Proof. Assume by contradiction that there exists some $s_0 \in [-1,0)$ so that

$$s_0 = \max\{s \leq 0 \ : \ |h(Z(s,\eta))| \leq 1\}.$$

It follows from the definition that $|h(Z(s,\eta))| > 1$ for all $s \in (s_0, 0]$. Observe that $\mathfrak{L}_{\hat{f}} p(|h(\xi)|) \leq 1$ for all ξ with $|h(\xi)| \geq 1$. Thus

$$\left| \frac{\mathrm{d}}{\mathrm{d}s} p\left(|h(Z(s,\eta))|\right) \right| \leq 1 \ \ s \in (s_0, 0].$$

This, in turn, implies that

$$p\left(|h(Z(s_0,\eta))|\right) \geq p(|h(\eta)|) + s_0 \geq K_0 + s_0 - 1 > p(1),$$

and so, since p is strictly increasing, $|h(Z(s_0,\eta))| > 1$, thus contradicting the definition of s_0. □

It follows from the Claim that $|h(Z(s,\xi))| > 1$ for all $s \in [0,\theta]$. Moreover, by making use of $\mathfrak{L}_{\hat{f}} p(|h(\xi)|) \leq 1$ for all ξ with $|h(\xi)| \geq 1$, it holds that $p(|h(\xi)|) \geq p(|h(\eta)|) - \theta$. Thus, by hypothesis,

$$\begin{aligned} p(|h(\eta)|) &= |\eta|_{\mathcal{W}} \geq |\xi|_{\mathcal{W}} - \theta \\ &\geq (1+a)p(|h(\xi)|) - \theta \geq (1+a)p(|h(\eta)|) - (1+a)\theta - \theta. \end{aligned}$$

It follows that $p(|h(\eta)|) \leq (2+a)\theta/a$, so from (D.26) we know that:

$$K_0 \leq 1 + p(|h(\eta)|) \leq 1 + \frac{2+a}{a}\theta,$$

thus contradicting the choice of K_0. This shows that it is impossible to have $\theta < 1$. □

Let:

$$\mathcal{E}_1 := \{z \in M \ : |z|_{\mathcal{W}} \geq 2\rho(|h(z)|)\}, \tag{D.27}$$

Lemma D.17. *There exists a $\mathcal{K}_\infty$ function $\underline{\alpha}$ such that*

$$\underline{\alpha}(|\xi|_{\bar{A}_i}) \leq V(\xi), \tag{D.28}$$

for all $\xi \in \bar{A}_i \cup (\mathcal{A}_i \cap \mathcal{E}_1)$.

Proof. The proof follows along the lines of Lemma 4.13 in [6] and we recall it here. Notice that if $|\xi|_{\mathcal{W}} \geq 1.6\rho(|h(\xi)|)$ and $|\xi|_{\mathcal{W}} \neq 0$, then $\xi \in \mathcal{E}$, because $|\xi|_{\mathcal{W}} > |\xi|_{\mathcal{W}}/1.6 \geq \rho(|h(\xi)|)$. By the definition of $\tilde{\Sigma}$ in (D.7), if $\xi \notin \mathcal{D} \cup \mathcal{B}$, then $\tilde{f}(\xi, v) = \hat{f}(\xi)$ for any $v \in V$, so that

$$\mathfrak{L}_{\tilde{f}}\left((1.5\rho \circ |h|)(\xi)\right) = \frac{1.5\mathfrak{L}_f\left((\rho \circ |h|)(\xi)\right)}{1 + |f(\xi)|_{\mathfrak{g}}^2 + \kappa(\xi)} \leq 1,$$

with κ as in (D.2). By virtue of Lemma D.15, we can find a constant $K_0 > 0$ such that, if $|\xi|_{\mathcal{W}} > K_0$ and $|\xi|_{\mathcal{W}} \geq 1.6\rho(|h(\xi)|)$, then $Z(t, \xi; v) \notin \mathcal{D} \cup \mathcal{B}$ for all $t \in [0, 1)$ and all $v \in \mathbb{V}$. In particular, by virtue of Claim D.11, closedness of $\bar{A}_i$ and $|\hat{f}(w)|_{\mathfrak{g}} \leq 1$ for all $w \in M$, it holds $|\xi|_{\bar{A}_i} \leq |Z(t, \xi; v)|_{\bar{A}_i} + \mathfrak{d}[\xi, Z(t, \xi; v)] < |Z(t, \xi; v)|_{\bar{A}_i} + 1$ and thus

$$\Xi(|Z(t, \xi; v)|_{\bar{A}_i}) > \Xi(|\xi|_{\bar{A}_i} - 1) \tag{D.29}$$

for all $t \in [0, 1)$, all $v \in \mathbb{V}$, and all $\xi \in \mathcal{A}_i$ such that $|\xi|_{\mathcal{W}} > \max\{K_0 + 1, 1.6\rho(|h(\xi)|)\}$. Therefore, for all ξ such that $|\xi|_{\mathcal{W}} > \max\{K_0 + 1, 1.6\rho(|h(\xi)|)\}$ the following estimate holds for any $\bar{v} \in \mathbb{V}$:

$$\begin{aligned} V(\xi) = \inf_{v \in \mathbb{V}} V(\xi, v) &= \int_0^1 \Xi(|Z(s, \xi; \bar{v})|_{\bar{A}_i})\,\mathrm{d}s + V(Z(1, \xi)) \\ &\geq \int_0^1 \Xi(|Z(s, \xi; \bar{v})|_{\bar{A}_i})\,\mathrm{d}s \geq \Xi(|\xi|_{\bar{A}_i} - 1). \end{aligned} \tag{D.30}$$

Next, we claim that V is strictly positive on $\mathcal{E} \setminus \bar{A}_i$. Indeed, since $|\tilde{f}(w, u)|_{\mathfrak{g}} \leq 5$ for any $w \in M$ and any $u \in V$, it holds:

$$\mathfrak{d}[Z(s, \xi; v), \xi] \leq \frac{1}{2}|\xi|_{\mathcal{D} \cup \bar{A}_i} \quad \forall s \leq \frac{1}{10}|\xi|_{\mathcal{D} \cup \bar{A}_i},$$

for any $\xi \in \mathcal{E} \setminus \bar{A}_i$ and any $v \in \mathbb{V}$. Now consider, for all such ξs and vs, function $\theta(\xi, v)$. Due to $\left|\tilde{f}(\cdot, \cdot)\right|_{\mathfrak{g}} \leq 5$, it holds:

$$\begin{aligned} |\xi|_{\mathcal{D} \cup \bar{A}_i} \leq |\xi|_{\mathcal{D}} &\leq \mathfrak{d}[Z(\theta(\xi, v), \xi; v),\, \xi] \\ &\leq 5\theta(\xi, v) \leq 10\,\theta(\xi, v). \end{aligned}$$

Therefore, for all such ξs and vs, it holds that $\theta(\xi, v) \geq |\xi|_{\mathcal{D}}/10$ and, by Claim D.11, $|Z(t, \xi; v)|_{\bar{A}_i} \geq |\xi|_{\bar{A}_i} - |\xi|_{\mathcal{D} \cup \bar{A}_i}/2$ for all $s \leq |\xi|_{\mathcal{D} \cup \bar{A}_i}/10$. It follows:

$$\begin{aligned} V(\xi) = V(\xi, \bar{v}) &\geq \int_0^{|\xi|_{\mathcal{D} \cup \bar{A}_i}/10} \Xi(|Z(s, \xi; \bar{v})|_{\bar{A}_i})\,\mathrm{d}s \\ &\geq \frac{1}{10}|\xi|_{\mathcal{D} \cup \bar{A}_i}\,\Xi\left(|\xi|_{\bar{A}_i} - \frac{1}{2}|\xi|_{\mathcal{D} \cup \bar{A}_i}\right) \\ &\geq \frac{1}{10}|\xi|_{\mathcal{D} \cup \bar{A}_i}\,\Xi(|\xi|_{\bar{A}_i}/2), \end{aligned} \tag{D.31}$$

for any $\xi \in \mathcal{E} \setminus \bar{A}_i$ and the auxiliary control $\bar{v} \in \mathbb{V}$ as provided by Lemma D.8.

Inequality (D.30) shows that V is proper in $\mathcal{E}_1$, whereas inequality (D.31) shows that V is positive on $\mathcal{E} \setminus \bar{A}_i$. We are now going to use (D.30) and (D.31) to prove existence of lower bound (D.28) on $\mathcal{E}_1$. Lemma D.14 has proved that $V(\cdot)$ is continuous, and thus it attains its minimum on any compact set. For each positive l define

$$r_l := \frac{K_0+2}{l} \text{ and } m_l := \inf\left\{V(z) \;:\; z \in \mathcal{E}_1, r_l \le |z|_{\bar{A}_i} \le r_1\right\}.$$

Note that Ξ is a class-$\mathcal{K}_\infty$ function and that the sequence $\{m_l\}_{l\in\mathbb{R}_{\ge 1}}$ is nonincreasing and positive. Therefore, we can find a class-$\mathcal{K}_\infty$ function $\underline{\alpha}$ such that $\underline{\alpha}(s) < m_l$ for all $s \in [r_l, r_{l-1}]$ for all $l \ge 2$ and $\underline{\alpha}(s) < \Xi(s-1)$ for all $s > K_0 + 2$. □

Lemma D.18. *There exists a $\mathcal{K}_\infty$ function $\bar{\alpha}$ such that*

$$V(\xi) \le \bar{\alpha}(|\xi|_{\bar{A}_i}), \tag{D.32}$$

for all $\xi \in \bar{A}_i \cup (\mathcal{A}_i \cap \mathcal{E}_1)$.

Proof. The Lemma follows from continuity of V in Lemma D.14 and bound (D.17). □

D.4 Dissipation

Lemma D.19. *For any $\xi \in \mathcal{A}_i \setminus (\mathcal{D} \cup \mathcal{B})$, and any t_0 such that $Z(t,\xi) \notin (\mathcal{D} \cup \mathcal{B})$, the following dissipation inequality holds for the trajectories of system (D.1):*

$$V(Z(t_0,\xi)) - V(\xi) \le -\int_0^{t_0} \Xi(|Z(s,\xi)|_{\bar{A}_i})\,\mathrm{d}s.$$

Proof. The proof follows along the lines of Lemma 4.15 in [6] and we recall it here. Fix $\xi \in \mathcal{A}_i \setminus (\mathcal{D} \cup \mathcal{B})$ and any positive t_0 as in the formulation of the Lemma. Let $v_1 \in \mathbb{V}$ be a control such that $V(Z(t_0,\xi)) = V(Z(t_0,\xi), v_1)$. Let $\tilde{v} \in \mathbb{V}$ be defined by

$$\tilde{v} = \begin{cases} 0 & \text{if } 0 \le t \le t_0 \\ v_1(t-t_0) & \text{if } t > t_0. \end{cases}$$

Then $Z(t, Z(t_0,\xi); v_1) = Z(t+t_0, \xi; \tilde{v})$ for all $t \ge 0$. By assumption, $Z(t,\xi) \notin (\mathcal{D} \cup \mathcal{B})$ for all $t \in [0,t_0]$, and therefore we have $Z(t,\xi) = Z(t,\xi;v)$ for all $t \in [0,t_0]$ and any $v \in \mathbb{V}$. It thus follows that $\theta(\xi,v) > t_0$ for all $v \in \mathbb{V}$ and

$$\begin{aligned} V(\xi) &= \min_{v\in\mathbb{V}} \int_0^{\theta(\xi,v)} \Xi(|Z(s,\xi;v)|_{\bar{A}_i})\,\mathrm{d}s \\ &= \int_0^{t_0} \Xi(|Z(s,\xi)|_{\bar{A}_i})\,\mathrm{d}s + \min_{v\in\mathbb{V}} \int_{t_0}^{\theta(\xi,v)} \Xi(|Z(s,\xi;v)|_{\bar{A}_i})\,\mathrm{d}s \\ &= \int_0^{t_0} \Xi(|Z(s,\xi)|_{\bar{A}_i})\,\mathrm{d}s + \min_{v\in\mathbb{V}} \int_0^{\theta(Z(t_0,\xi),v)} \Xi(|Z(s,Z(t_0,\xi);v)|_{\bar{A}_i})\,\mathrm{d}s \\ &= \int_0^{t_0} \Xi(|Z(s,\xi)|_{\bar{A}_i})\,\mathrm{d}s + V(Z(t_0,\xi)). \end{aligned}$$

We have thus proved the Lemma. □

We have proven that the OSS dissipation inequality holds for $V(\cdot)$ along the trajectories of the slower system $\hat{\Sigma}$ in (D.1) which are contained in $\mathcal{A}_i \setminus (\mathcal{D} \cup \mathcal{B})$. It follows immediately that the same estimate holds along the trajectories of the original system Σ.

Corollary D.20. *For any $x \in \mathcal{A}_i \setminus (\mathcal{D} \cup \mathcal{B})$, and any t_0 such that $X(t,x) \notin (\mathcal{D} \cup \mathcal{B})$, the following dissipation inequality holds for the trajectories of system (2.1):*

$$V(X(t_0,x)) - V(x) \leq -\int_0^{t_0} \Xi(|X(s,x)|_{\bar{A}_i})\,\mathrm{d}s. \tag{D.33}$$

Proof. The proof follows along the lines of Lemma 4.15 in [6] and we recall it here. Fix $x \in \mathcal{A}_i \setminus (\mathcal{D} \cup \mathcal{B})$, and any t_0 such that $X(t,x) \notin (\mathcal{D} \cup \mathcal{B})$ for all $t \in [0,t_0]$. Then, by virtue of the time rescaling property (D.3), it holds:

$$\begin{aligned}
V(X(t_0,x)) - V(x) &= V(Z(\sigma_x(t_0),x)) - V(x) \\
&\leq -\int_0^{\sigma_x(t_0)} \Xi(|Z(s,x)|_{\bar{A}_i})\,\mathrm{d}s \\
&= -\int_0^{t_0} \Xi(|Z(\sigma_x(s),x)|_{\bar{A}_i})\,d\sigma_x(s) \\
&= -\int_0^{t_0} \Xi(|X(s,x)|_{\bar{A}_i})\,\frac{\mathrm{d}}{\mathrm{d}s}\sigma_x(s)\,\mathrm{d}s \\
&= -\int_0^{t_0} \Xi(|X(s,x)|_{\bar{A}_i})\,\Big[1 + |f(X(s,x))|_{\mathfrak{g}}^2 + \kappa(X(s,x))\Big]\,\mathrm{d}s \\
&\leq -\int_0^{t_0} \Xi(|X(s,x)|_{\bar{A}_i})\,\mathrm{d}s.
\end{aligned}$$

□

D.5 Extension and smoothing

In this Section, we extend the definition of V to the whole sector $\mathcal{A}_i$ and we show how some classical results concerning the smoothing of Lyapunov functions can be generalized for systems evolving on manifolds. It turns out that most proofs cannot be adapted to the manifold case in a straightforward fashion, unless few assumptions and facts are established beforehand, as in the following.

(i) Manifold M is assumed to be geodesically complete. By virtue of the Hopf-Rinow theorem, closed and bounded subsets of M are then compact. Then, any set of the form $\big\{x \in M \;:\; |x|_{\bar{A}_i} \leq r, r \geq 0\big\}$ is compact.
(ii) Vector field f is assumed to be a locally Lipschitz mapping in the sense of Definition H.1.
(iii) Gronwall lemma applies to systems evolving on manifolds as estabilished in Corollary H.5.

Definition D.21 and the proofs of Lemmas D.22, D.23, D.24, and D.25 follow along the lines of Definition 4.17 and Lemmas 4.18, 4.19, 4.21, 4.22 in [6], and make implicit use of properties (D.5), (D.5), and (D.5).

Definition D.21. Let $x \in M$. A covector $\zeta \in T_x^\star M$ is a *proximal subgradient* of the function $V : M \to (-\infty, +\infty]$ at x if there exists some positive σ such that, for all x' in some neighborhood of x,

$$V(x') \geq V(x) + \zeta(\exp_x^{-1}(x')) - \sigma\mathfrak{d}[x,x']^2.$$

The (possible empty) set of all proximal subgradients of V at x is called the *proximal subdifferential* and is denoted $d_P V(x)$.

Lemma D.22. *Consider any system of type (2.1), some $x \in M$, and some function $V : M \to \mathbb{R}$. Then, if there exist some continuous $\alpha_x(t) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\varepsilon > 0$ such that the following inequality holds for all $\tau < \varepsilon$,*

$$V(X(\tau, x)) - V(x) \leq \int_0^\tau \alpha_x(t)\,\mathrm{d}t, \tag{D.34}$$

then for any $\zeta \in d_P V(x)$ the proximal form of (D.34) holds:

$$\zeta(f(x)) \leq \alpha_x(0).$$

Lemma D.23. *Consider any system of type (2.1) with $x \in M$ and $f(\cdot)$ being a locally Lipschitz vector field. Assume we are given*

- *an open subset $\mathcal{O} \subseteq M$;*
- *a continuous, nonnegative function $V : \mathcal{O} \to \mathbb{R}$, satisfying*

$$\zeta(f(x)) \leq \Theta(x) \quad \forall x \in \mathcal{O}, \;\; \zeta \in d_P V(x),$$

  with some continuous function $\Theta : \mathcal{O} \to \mathbb{R}$;
- *two positive, continuous function Υ_1 and Υ_2 on $\mathcal{O}$.*

Then there exists a function $\tilde{V} : \mathcal{O} \to \mathbb{R}$, locally Lipschitz con $\mathcal{O}$, such that

$$0 \leq V(x) - \tilde{V}(x) \leq \Upsilon_1(x) \;\; \forall x \in \mathcal{O}$$

and

$$\mathfrak{L}_f \tilde{V}(x) \leq \Theta(x) + \Upsilon_2(x) \;\; \textit{for a.a. } x \in \mathcal{O}.$$

Lemma D.24. *Let $\mathcal{O}$ be an open subset of M and assume the following as given:*

- *a locally Lipschitz function $\Phi : \mathcal{O} \to \mathbb{R}$;*
- *a locally Lipschitz vector field $f : M \to TM$;*
- *a continuous function $\alpha : \mathcal{O} \to \mathbb{R}$ and continuous functions $\mu, \nu : \mathcal{O} \to \mathbb{R}_{>0}$*

such that

$$\mathfrak{L}_f \Phi(x) \leq \alpha(x)$$

for almost all $x \in \mathcal{O}$. Then, there exists a smooth function $\Psi : \mathcal{O} \to \mathbb{R}$ such that

$$|\Phi(x) - \Psi(x)| < \mu(x) \quad \forall x \in \mathcal{O},$$
$$\mathfrak{L}_f \Psi(x) \leq \alpha(x) + \nu(x) \quad \forall x \in \mathcal{O}.$$

Lemma D.25. *Under the assumptions of Lemma D.23, there also exists a smooth function $\hat{V}$ on $\mathcal{O}$, satisfying inequalities:*

$$\left| V(x) - \hat{V}(x) \right| \leq \Upsilon_1(x) \quad \forall x \in \mathcal{O}$$
$$\mathfrak{L}_f \hat{V}(x) \leq \Theta(x) + \Upsilon_2(x) \quad \forall x \in \mathcal{O}.$$

We are now ready to state our main result.

Corollary D.26. *There exist a smooth function $V_3 : \mathcal{A}_i \to \mathbb{R}_{>0}$ and class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \alpha_3$ satisfying the following two properties:*

$$\alpha_1(|x|_{\bar{A}_i}) \le V_3(x) \le \alpha_2(|x|_{\bar{A}_i}) \quad \forall x \in \mathcal{A}_i, \tag{D.35}$$
$$\mathfrak{L}_f V_3(x) \le -\alpha_3(|x|_{\bar{A}_i}) \quad \forall x \in (\mathcal{A}_i \cap \mathcal{E}_3) \cup \bar{A}_i. \tag{D.36}$$

Moreover, $\mathrm{d}V_3(x) = 0$ *for any* $x \in \bar{A}_i$.

Proof. First, we construct a function V_1

- defined on $\operatorname{int}\{\mathcal{A}_i \cap \mathcal{E}_1\} \setminus \bar{A}_i$;
- smooth on $\operatorname{int}\{\mathcal{A}_i \cap \mathcal{E}\} \setminus \bar{A}_i$;
- satisfying some sought bounds on $\operatorname{int}\{\mathcal{A}_i \cap \mathcal{E}_1\} \setminus \bar{A}_i$;
- satisfying some sought dissipation properties on $\operatorname{int}\{\mathcal{A}_i \cap \mathcal{E}_1\} \setminus \bar{A}_i$.

Second, we construct a function V_2

- defined on $\mathcal{A}_i \setminus \bar{A}_i$;
- smooth on $\mathcal{A}_i \setminus \bar{A}_i$;
- satisfying some sought bounds on $\mathcal{A}_i \setminus \bar{A}_i$;
- satisfying some sought dissipation properties on $(\mathcal{A}_i \cap \mathcal{E}_3) \setminus \bar{A}_i$, with $\mathcal{E}_3$ as defined in Section 5.1.

Third, we construct a function V_3

- defined on $\mathcal{A}_i$;
- smooth everywhere on $\mathcal{A}_i$;
- satisfying some sought bounds on $\mathcal{A}_i$;
- satisfying some sought dissipation properties on $(\mathcal{A}_i \cap \mathcal{E}_3) \cup \bar{A}_i$.

By Lemmas D.17 and D.18, bounds (D.28) and (D.32) hold true for function V on the open set $\operatorname{int}\{\mathcal{A}_i \cup \mathcal{E}_1\} \setminus \bar{A}_i$. By Corollary D.20, dissipation inequality (D.33) also holds true on $\operatorname{int}\{\mathcal{A}_i \cup \mathcal{E}_1\} \setminus \bar{A}_i$. Then, Lemma D.22 proves that

$$\zeta\,(f(x)) \le -\Xi(|x|_{\bar{A}_i}), \tag{D.37}$$

for all $\zeta \in d_P(V(x))$ and all $x \in \operatorname{int}\mathcal{A}_i \cup \mathcal{E}_1$. Since the assumptions of Lemma D.25 are satisfied with $\mathcal{O} := \operatorname{int}\{\mathcal{E}_1 \cap \mathcal{A}_i\}$, $\Upsilon_1(\cdot) := \underline{\alpha}(|\cdot|_{\bar{A}_i})/2$, $\Upsilon_2(\cdot) := \Xi(|\cdot|_{\bar{A}_i})/2$, $\Theta(\cdot) := -\Xi(|\cdot|_{\bar{A}_i})$, one can find a smooth function $V_1 : \operatorname{int}\{\mathcal{E}_1 \cap \mathcal{A}_i\} \setminus \bar{A}_i \to \mathbb{R}_{>0}$ satisfying

$$\frac{\underline{\alpha}(|x|_{\bar{A}_i})}{2} \le V_1(x) \le \bar{\alpha}(|x|_{\bar{A}_i}) + \frac{\underline{\alpha}(|x|_{\bar{A}_i})}{2} \tag{D.38}$$
$$\mathfrak{L}_f V_1(x) \le -\Xi_1(|x|_{\bar{A}_i}), \tag{D.39}$$

for all $x \in \operatorname{int}\{\mathcal{E}_1 \cap \mathcal{A}_i\} \setminus \bar{A}_i$.

Let $\phi : \mathcal{A}_i \setminus \bar{A}_i \to [0,1]$ be any smooth function with the property

$$\phi(x) = \begin{cases} 1 & \text{if } x \in \mathcal{E}_3 \\ 0 & \text{if } |x|_{\mathcal{W}} \le 2\rho(|h(x)|), \end{cases}$$

and ϕ is nonzero elsewhere. Let $V_2 : \mathcal{A}_i \setminus \bar{A}_i \to \mathbb{R}_{\ge 0}$ be defined as:

$$V_2(x) := \phi(x)V_1(x) + (1-\phi(x))|x|^2_{\bar{A}_i}.$$

It is then easy to show that:

$$\breve{\alpha}_1(|x|_{\bar{A}_i}) \le V_2(x) \le \breve{\alpha}_2(|x|_{\bar{A}_i}) \;\; \forall x \in \mathcal{A}_i \setminus \bar{A}_i \tag{D.40}$$

$$\mathfrak{L}_f V_2(x) \le -\Xi_1(|x|_{\bar{A}_i}) \;\; \forall x \in (\mathcal{A}_i \cap \mathcal{E}_3) \setminus \bar{A}_i, \tag{D.41}$$

with $\breve{\alpha}_1(r) := \min\left\{\underline{\alpha}(r)/2, r^2\right\}$ and $\breve{\alpha}_2(\cdot) := \min\left\{\bar{\alpha}(r) + \underline{\alpha}(r)/2, r^2\right\}$ for all $r \ge 0$.

Due to bounds (D.40), the domain of function V_2 can be extended to $\bar{A}_i$ preserving continuity at $\bar{A}_i$ and properties (D.40) and (D.41) on $\bar{A}_i$ as well. By virtue of Lemma 4.25 in [6], it is then possible to get a class-$\mathcal{K}_\infty$ function β_1 such that $V_3 := \beta_1 \circ V_2$ is smooth everywhere on $\mathcal{A}_i$ and, in particular, satisfies $\mathrm{d}V_3(x) = 0$ whenever $x \in \bar{A}_i$. Such function satisfies properties (D.35) and (D.36) with $\alpha_1 = \beta_1 \circ \breve{\alpha}_1$, $\alpha_2 = \beta_1 \circ \breve{\alpha}_2$, and $\alpha_3 = \beta_1 \circ \breve{\alpha}_1$. □

Appendix E. Compactness of $\bar{A}_i$ and $\bar{B}_i$

Let Assumptions 2.2, 2.6, and 5.1 hold true. This Section then addresses the proof of compactness of the set $A_i \cap \mathcal{L}_{-\infty,+\infty}$ for any $i \in \{1, \dots, K\}$. Compactness of $B_i \cap \mathcal{L}_{-\infty,+\infty}$ simply follows by applying the same arguments to the backward flow.

Claim E.1. *$\mathcal{L}_{-\infty,0}$ is a closed set.*

Proof. Consider a sequence $\{x_n\}_{n\in\mathbb{N}} \in \mathcal{L}_{-\infty,0}$ such that $\lim_{n\to+\infty} x_n = x$ for some $x \in M$. We shall prove that $x \in \mathcal{L}_{-\infty,0}$. Specifically, we prove that $X(t,x) \in \mathcal{E}_0$ for all $t \le 0$. To this end, select any $t \le 0$. It holds:

$$\begin{aligned} X(t,x) &= X(t, \lim_{n\to+\infty} x_n) \\ &= \lim_{n\to+\infty} X(t, x_n) \\ &= \lim_{n\to+\infty} X_n, \end{aligned}$$

where the first inequality holds by definition of x, the second one holds by continuity of trajectories, and the third one by definition of $X_n := X(t, x_n)$. By definition of $\mathcal{L}_{-\infty,0}$, it holds $X_n \in \mathcal{E}_0$. Since $\mathcal{E}_0$ is closed and the sequence $\{X_n\}_{n\in\mathbb{N}}$ is converging to $X(t,x)$, it follows that $X(t,x) \in \mathcal{E}_0$. □

Lemma E.2. *If* $\operatorname{clos}[\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}]$ *intersects* $\mathfrak{R}(\mathcal{Z}_j)$ *then it intersects* $\mathcal{Z}_j$.

Proof. The proof follows along the lines of [11].

Claim E.3. *The set* $\operatorname{clos}[\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}]$ *is backward invariant.*

Proof. For any set E, if $x \in \operatorname{clos} E$ then there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \in E$ such that $x_n \to x$ as $n \to +\infty$. For this reason, if we select $x \in \operatorname{clos}[\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}]$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \in \mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}$ converging to x. Backward invariance of $\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}$ implies $X(\tau, x_n) \in \mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}$ for any $\tau \le 0$. Pick any $t \le 0$. By continuity of trajectories, it holds:

$$X(t,x) = X(t, \lim_{n\to+\infty} x_n) = \lim_{n\to+\infty} X(t, x_n).$$

Since $X(t,x_n) \in \mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}$ for all $n \in \mathbb{N}$ and $X(t,x_n) \to X(t,x)$ for any $n \to +\infty$, it follows that $X(t,x_n) \in \operatorname{clos}\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}$. □

Consider now a point $x \in \operatorname{clos}[\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}] \cap \mathfrak{R}(\mathcal{Z}_j)$. By backward invariance and closedness of $\operatorname{clos}[\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}]$, we have that $\alpha(x) \subseteq \operatorname{clos}[\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}]$. By definition of $\mathfrak{R}(\mathcal{Z}_j)$, $\alpha(x) \subseteq \mathcal{Z}_j$. It then follows that $\alpha(x) \subseteq \operatorname{clos}[\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}] \cap \mathcal{Z}_j$ which is then non-empty. □

Lemma E.4. *If $i \neq j$ and* $\operatorname{clos}[\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}]$ *intersects* $\mathcal{Z}_j$, *then it intersect* $\mathfrak{A}(\mathcal{Z}_j) \setminus \mathcal{Z}_j$.

Proof. Since the vector field f is locally Lipschitz continuous and due to the $\mathcal{Z}_i$s being compact and disjoint, we can select open neighborhoods U_i of $\mathcal{Z}_i$ such that

$$\varphi_t(\operatorname{clos} U_{i_1}) \cap \varphi_s(\operatorname{clos} U_{i_2}) = \emptyset,$$

for all $t, s \in [-1, 1]$ and all $i_1 \neq i_2$. Define the sets $L_j := \varphi_{-1}(\operatorname{clos} U_j \setminus U_j)$. By definition, L_j is compact. Consider now $y \in \operatorname{clos}[\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}] \cap \mathcal{Z}_j$. There exists a sequence $\{y_n\}_{n\in\mathbb{N}} \in \mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}$ converging to y. Without loss of generality, we may assume $y_n \in U_j$. We can then pick $T_n < 0$ so that $X(T_n, y_n) \notin U_j$ but $X(p, y_n) \in U_j$ for all $p \in (T_n, 0]$. It holds that $X_n := X(T_n, y_n) \in L_j$ for all $n \in \mathbb{N}$. By compactness of L_j, $X_n \to \bar{X} \in L_j$. By backward invariance and closedness of $\operatorname{clos}[\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}]$, we have that $\bar{X} \in \operatorname{clos}[\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}]$. Now, let $t > 0$ and consider $X(t, \bar{X})$. By continuity of trajectories, it holds:

$$X(t, \bar{X}) = \lim_{n\to+\infty} X(t, X_n) = \lim_{n\to+\infty} X(T_n + t, y_n).$$

For all $t \geq 0$, we can always find N so that $\forall n > N$, we obtain $T_n + t < 0$, and thus $X(T_n + t, y_n) \in U_j$, which in turn implies $X(t, \bar{X}) \in \operatorname{clos} U_j$ for all $t > 0$. Ultimate boundedness in $\operatorname{clos} U_j$ (Lem. 5.3) and invariance of $\mathcal{Z}_j$ imply $X(t, \bar{X}) \to \mathcal{Z}_j$ as $t \to +\infty$, and thus $\bar{X} \in \mathfrak{A}(\mathcal{Z}_j)$. Moreover, definition of $\bar{X}$ implies that $\bar{X} \in L_j$, therefore $\bar{X} \notin U_j$, hence $\bar{X} \notin \mathcal{Z}_j$. It follows that $\bar{X} \in \operatorname{clos}[\mathfrak{R}(\mathcal{Z}_i) \cap \mathcal{L}_{-\infty,0}] \cap \mathfrak{A}(\mathcal{Z}_j) \setminus \mathcal{Z}_j$. □

Lemma E.5. *If $i \neq j$ and $\operatorname{clos}[\mathfrak{R}(\mathcal{W}_i) \cap \mathcal{L}_{-\infty,0}]$ intersects $\mathcal{W}_j$, then $i > j$.*

Proof. By virtue of Lemma E.4, pick $x_1 \in \operatorname{clos}[\mathfrak{R}(\mathcal{W}_i) \cap \mathcal{L}_{-\infty,0}] \cap \mathfrak{A}(\mathcal{W}_j) \setminus \mathcal{W}_j$. By virtue of Lemmas 5.3 and 5.5, $x_1 \in \mathfrak{R}(\mathcal{W}_{l_1})$ for some $l_1 \in \{1, \ldots, K\}$. Since $x_1 \in \mathfrak{R}(\mathcal{W}_{l_1}) \cap \mathfrak{A}(\mathcal{W}_j)$ and due to the existence of a filtration ordering modulo-infinity (Assumption 2.6), we have $j < l_1$. Since $x_1 \in \operatorname{clos}[\mathfrak{R}(\mathcal{W}_i) \cap \mathcal{L}_{-\infty,0}] \cap \mathfrak{R}(\mathcal{W}_{l_1})$, it follows from Lemmas E.2 and E.4 that there exists $x_2 \in \operatorname{clos}[\mathfrak{R}(\mathcal{W}_i) \cap \mathcal{L}_{-\infty,0}] \cap \mathfrak{A}(\mathcal{W}_{l_2}) \setminus \mathcal{W}_{l_2}$ for some $l_2 \in \{1, \ldots, K\}$. Due to the existence of a filtration ordering modulo-infinity, we have $l_1 < l_2$. By iterating this procedure, we obtain an infinite sequence of indexes l_k, which contradicts finiteness of the $\mathcal{W}_k$s. □

Lemma E.6. $\operatorname{clos}[\mathfrak{R}(\mathcal{W}_i) \cap \mathcal{L}_{-\infty,0}] \subseteq \bigcup_{j\leq i} \mathfrak{R}(\mathcal{W}_j) \cap \mathcal{L}_{-\infty,0}$.

Proof. Select $x \in \operatorname{clos}[\mathfrak{R}(\mathcal{W}_i) \cap \mathcal{L}_{-\infty,0}]$. By virtue of Lemmas 5.3 and 5.5, $\alpha(x) \subseteq \mathcal{W}_j$. By virtue of Lemma E.5, $j \leq i$, and thus $x \in \mathfrak{R}(\mathcal{W}_j)$ for some $j \leq i$. □

Corollary E.7. *The set $\bar{A}_i = A_i \cap \mathcal{L}_{-\infty,0}$ is closed.*

Proof. We are going to show that $\operatorname{clos}\{A_i \cap \mathcal{L}_{-\infty,0}\} \subseteq A_i \cap \mathcal{L}_{-\infty,0}$. Indeed, by making use of Lemma E.6, we obtain:

$$\begin{aligned}
&\operatorname{clos}\{A_i \cap \mathcal{L}_{-\infty,0}\} \\
&\quad= \bigcup_{j\leq i} \operatorname{clos}\{\mathfrak{R}(\mathcal{W}_j) \cap \mathcal{L}_{-\infty,0}\} \\
&\quad\subseteq \bigcup_{j\leq i} \bigcup_{l\leq j} \mathfrak{R}(\mathcal{W}_l) \cap \mathcal{L}_{-\infty,0} \\
&\quad= \bigcup_{j\leq i} \mathfrak{R}(\mathcal{W}_j) \cap \mathcal{L}_{-\infty,0} \\
&\quad= A_i \cap \mathcal{L}_{-\infty,0}.
\end{aligned}$$

□

Appendix F. OSS-Lyapunov function L_i defined on M

This Appendix Section focuses on the proof of Lemma 5.16 and provides a list of additional interesting properties for function L_i defined in the Lemma. The underlying assumptions in all subsequent results of this Section is the fulfillment of the hypotheses in Lemmas 5.14 and 5.15.

Let $\mathcal{A}_i$ and $\mathcal{B}_i$ be defined according to Lemma 5.10. Recall that $\bar{A}_i$ and $\bar{B}_i$ are compact and disjoint (Lem. 5.8), therefore we can define $\rho^i_{max} := \inf_{a\in\bar{A}_i, b\in\bar{B}_i} \delta[a,b]$ with the property $\rho^i_{max} > 0$. Choose $\rho_1, \rho_2 \in \mathbb{R}$ so that $0 < \rho_1 < \rho_2 < \rho^i_{max}$.

By virtue of Lemmas 5.14 and 5.15, it follows the existence of two smooth functions $V_1 : \mathcal{A}_i \to \mathbb{R}_{\geq 0}$ and $V_2 : \mathcal{B}_i \to \mathbb{R}_{\geq 0}$ and class-$\mathcal{K}_\infty$ functions $\alpha_1, \alpha_2, \beta_1, \alpha_3, \alpha_4, \beta_2$ satisfying properties (5.8), (5.9), (5.10), and (5.11). Compactness of M implies the existence of a constant $L > 0$ such that $V_2(x) \leq \alpha_4(|x|_{\bar{B}_i}) \leq \alpha_4(L)$ for all $x \in \mathcal{B}_i$. Hence, we can select an arbitrary constant $v_2 \in (0,1)$ so that function

$$\bar{V}_2(x) := \frac{v_2}{\alpha_4(L)} V_2(x) \tag{F.1}$$

satisfies $\bar{V}_2(x) \leq v_2$ for all $x \in \mathcal{B}_i$. Denote with Υ' and Υ the following two sets:

$$\Upsilon' := \{x \in M \ : \ \alpha_1(\rho_1) \leq V_1(x) \leq \alpha_1(\rho_2)\}$$
$$\Upsilon := \left\{x \in M \ : \ (\alpha_2^{-1} \circ \alpha_1)(\rho_1) \leq |x|_{\bar{A}_i} \leq \rho_2\right\}.$$

It can be easily proven that $\Upsilon' \subseteq \Upsilon$ and that $\alpha_2(\rho_2) = \sup_{x\in\Upsilon} V_1(x)$. Define $v_1 := \alpha_2(\rho_2)$. Define the following smooth activation function $\phi : \mathbb{R}_{\geq 0} \to [0,1]$ as follows:

$$\phi(s) := \begin{cases} 0 & s \leq \alpha_1(\rho_1) \\ 1 & s \geq \alpha_1(\rho_2). \end{cases}$$

Furthermore, function $\phi(s)$ is required to take values in $(0,1)$ and to have positive gradient over the open interval $(\alpha_1(\rho_1), \alpha_1(\rho_2))$. Let $L_i : M \to \mathbb{R}_{\geq 0}$ be the Lyapunov function defined as:

$$L_i(x) := v_4 \left(1 - \phi(V_1(x))\right) V_1(x) + \phi(V_1(x)) \left(1 - \bar{V}_2(x)\right). \tag{F.2}$$

The following properties can be easily proven for L_i:

1. the domain of L_i is indeed $\mathcal{A}_i \cup \mathcal{B}_i = M$;
2. if $V_1(x) \leq \alpha_1(\rho_1)$, then $L_i(x) = v_4 V_1(x)$ and $\mathcal{L}_f L_i(x) = v_4 \mathcal{L}_f V_1(x)$;
3. if $V_1(x) \geq \alpha_1(\rho_2)$, then $L_i(x) = 1 - V_2(x)$ and $\mathcal{L}_f L_i(x) = -\mathcal{L}_f \bar{V}_2(x)$;
4. $L_i(x) = 0$ if and only if $x \in \bar{A}_i$;
5. there exists a constant $b^i > 0$ such that $L_i(x) = b^i$ whenever $x \in \bar{B}_i$;
6. $\bar{B}_i \subseteq \{x \in M \ : \ L_i(x) = 1\}$;
7. $\mathcal{L}_f L_i(x) = 0$ whenever $x \in G_i := \bar{A}_i \cup \bar{B}_i$;
8. there exists a class-$\mathcal{K}$ function α^i_1 such that $\alpha^i_1(|x|_{\bar{A}_i}) \leq L_i$ for all $x \in M$.

The dissipation of L_i along the trajectories of system (2.1) is given by:

$$\begin{aligned} \mathcal{L}_f L_i(x) \ = \ & \left.\frac{\partial \phi(s)}{\partial s}\right|_{s=V_1(x)} \left(-v_4 V_1(x) + 1 - \bar{V}_2(x)\right) \ \mathcal{L}_f V_1(x) \\ & + \left(1 - \phi(V_1(x))\right) v_4 \, \mathcal{L}_f V_1(x) \\ & + \phi(V_1(x)) \left(-\mathcal{L}_f V_2(x)\right). \end{aligned} \tag{F.3}$$

Furthermore, it holds for all $x \in \Upsilon'$ that:

$$\begin{aligned}-v_4 V_1(x) + 1 - \bar{V}_2(x) &\geq -v_4 v_1 + 1 - v_2 \\ &> (1 - v_2) v_1^{-1} v_1 + 1 - v2 = 0.\end{aligned}$$

Define $c_0 := -v_4 v_1 + 1 - v_2 > 0$. By virtue of (5.9), (5.11), and (F.3) and by virtue of the definition of $\phi(\,\cdot\,)$, the following dissipation inequality holds for any $x \in \Upsilon' \cap \mathcal{E}_3$:

$$\begin{aligned}&\mathcal{L}_f L_i(x) \\ &\leq \left[\left.\frac{\partial \phi(s)}{\partial s}\right|_{s=V_1(x)} c_0 + (1 - \phi(V_1(x)))\, v_4 \right] \left(-\beta_1(|x|_{\bar{A}_i})\right) \\ &\quad + \phi(V_1(x)) \frac{v_2}{\alpha_4(L)} \left(-\beta_2(|x|_{\bar{B}_i})\right) \\ &\leq -c_1.\end{aligned} \tag{F.4}$$

for some $c_1 > 0$. Property (F.4) together with previously mentioned properties 2) and 3) yields the existence of a class-$\mathcal{K}_\infty$ function β^i such that:

$$\mathcal{L}_f L_i(x) \leq -\beta^i \left(|x|_{G_i} \right), \tag{F.5}$$

for all $x \in \mathcal{E}_3 \cup \bar{A}_i \cup \bar{B}_i$.

Appendix G. Limit of sequences of nested sets

In this section, we recall basic results on sequences of nested sets, *e.g.* Cantor's intersection theorem ([12], Chap. 2).

Lemma G.1 (Cantor's intersection theorem)**.** *Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{R}^n$ such that $A_k \supseteq A_{k+1}$ for all $k \in \mathbb{N}$ and $\mathcal{K} = \bigcap_{k\in\mathbb{N}} A_k$. Then, if A_k is non-empty for all $k \in \mathbb{N}$, $\mathcal{K}$ is non-empty.*

Lemma G.2 (Compact case)**.** *Let $\{A_k\}_{k\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{R}^n$ such that $A_k \supseteq A_{k+1}$ for all $k \in \mathbb{N}$ and $\mathcal{K} = \bigcap_{k\in\mathbb{N}} A_k$. Then, for all open neighborhoods U of $\mathcal{K}$, there exists $k_0 \in \mathbb{N}$ such that, for all $k \geq k_0$, we have $A_k \subseteq U$.*

Proof. Being U a neighborhood of $\mathcal{K}$, it holds:

$$\emptyset = \mathcal{K} \setminus U = \left(\bigcap_{k\in\mathbb{N}} A_k \right) \setminus U = \bigcap_{k\in\mathbb{N}} (A_k \setminus U), \tag{G.1}$$

where last equality follows be recalling that $E_1 \setminus E_2 = E_1 \cap (\mathbb{R}^n \setminus E_2)$. Since U is an open set and A_k is a compact set for all $k \in \mathbb{N}$, it follows that $A_k \setminus U$ is a compact set for all $k \in \mathbb{N}$. Therefore, expression (G.1) shows that $\{A_k \setminus U\}_{k\in\mathbb{N}}$ is a non-increasing sequence of compact subsets of $\mathbb{R}^n$ whose intersection is empty. By virtue of Lemma G.1, it follows that $A_{k_0} \setminus U = \emptyset$ for some $k_0 \in \mathbb{N}$. By monotonicity of $\{A_k \setminus U\}_{k\in\mathbb{N}}$, we have $A_k \setminus U = \emptyset$ for all $k \geq k_0$, and thus $A_k \subseteq U$ for all such ks. □

Appendix H. Gronwall estimate for systems on manifolds

Notation: Let M be a geodesically complete, connected, n-dimensional, C^2 Riemannian manifold without boundary, with Riemannian metric $\mathfrak{g}$. Let $\nabla : T_x M \times T_x M \to T_x M$ denote the Levi-Civita connection on M and $\exp_x : T_x M \to M$ the exponential map at $x \in M$. Denote by $\mathcal{T}_{\gamma,t_1,t_2}$ the parallel transport along a smooth curve $\gamma : \mathbb{R} \to M$ from the frame at $\gamma(t_1)$ to the frame at $\gamma(t_2)$. Let $L(T_xM)$ denote the vector space of linear endomorphisms on a tangent space T_xM. Let V be a compact subset of $\mathbb{R}^{n+1}$.

Definition H.1. A vector field $F : M \to TM$ is said to be *locally Lipschitz continuous uniformly on* V if, for all compact sets $\mathcal{K} \subset M$, there exists a constant $C_{\mathcal{K}}$ such that, for all $x \in \mathcal{K}$, there exists a neighborhood $U \subset T_xM$ of 0 such that $\exp_x : T_xM \to M$ is bijective on U and for all $r \in U$ it holds:

$$\left|\mathcal{T}_{\exp_x(tv),1,0} F(\exp_x(r)) \;-\; F(x)\right|_{\mathfrak{g}} \le C_{\mathcal{K}} |r|_{\mathfrak{g}}. \tag{H.1}$$

The constant $C_{\mathcal{K}}$ is said to be the Lipschitz constant of F in $C_{\mathcal{K}}$.

This definition of uniform local Lipschitz continuity implies uniform local Lipschitz continuity in local coordinates, therefore we can conclude that the set of points $x \in \Omega(F(\cdot))$ where $F(\cdot)$ is not differentiable has zero measure. For any $x \in M \setminus \Omega(F(\cdot))$, and any $r \in T_xM$, the covariant derivative of $F(x)$ along r, can be defined by

$$\bar{\nabla}_r F(x) := \lim_{h \to 0^+} \frac{\mathcal{T}_{\gamma,h,0}(F(\gamma(t))) - F(x)}{h} \tag{H.2}$$

for a smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = x$, $\gamma'(0) = r$. Observe that, for any fixed $x \in M \setminus \Omega(F)$, we obtain a well defined map $\bar{\nabla} F(x) : T_xM \to T_xM$ given by $r \mapsto \bar{\nabla}_r F(x)$. To make the latter statement more precise, we bring to the attention of the reader that $\bar{\nabla}_r F(x) \in T_xM$ while $\bar{\nabla} F(x) \in L(T_xM)$. We would like to extend this definition to arbitrary points $x \in M$, therefore we introduce a set-valued generalization of the covariant derivative for nonsmooth vector fields:

Definition H.2. Let $F : M \times V \to TM$ be a locally Lipschitz vector field on M, uniformly on V. Then, the *generalized covariant derivative of* F *at* $x \in M$ *along* $r \in T_xM$ is defined as:

$$\nabla_r F(x) = \lim_{h \to 0^+} \bar{\nabla}_r F(\exp_x(hr)).$$

The *generalized covariant derivative* of F at $x \in M$ is defined as:

$$\nabla F(x) = \mathrm{Conv} \Big\{ A \in L(T_xM) \;:\; \exists (x_k) \subset M \setminus \Omega(F)\ , \\ x_k \to x, A = \lim_{k \to \infty} \bar{\nabla} F(x_k) \Big\},$$

where Conv denotes the convex hull of a subset of $L(T_xM)$.

Remark H.3. For a Lipschitz continuous vector field F uniformly on V as in Definition H.1, formula (H.1) is implies that, for any compact set $\mathcal{K} \subset M$, there exists a constant $C_{\mathcal{K}}$ such that

$$Ar \le C_{\mathcal{K}} r, \quad \forall x \in \mathcal{K},\ \forall r \in U,\ \forall A \in \nabla F(x). \tag{H.3}$$

Equivalently, one could replace (H.3) with

$$|\nabla_r F(x)|_{\mathfrak{g}} \le C_{\mathcal{K}}, \quad \forall x \in \mathcal{K},\ \forall r \in U. \tag{H.4}$$

Observe that the converse is not true, *i.e.* (H.4) may not imply (H.1) as $F(\cdot)$ may fail to be continuous at x whereas the generalized covariant derivative may exist nonetheless.

The following Lemmas provide a local Gronwall estimate for dynamical systems evolving on Riemannian manifolds. Consider the following dynamical system:

$$\dot{x} = F(x). \tag{H.5}$$

with states $x \in M$, and F being a locally Lipschitz continuous vector field. Let $X(t,x)$ denote the solution trajectory of (H.5) at time t with initial condition at x.

Lemma H.4 (Gronwall estimate for system (H.5))**.** *Let $c_0 : [a,b] \to M, \tau \mapsto c_0(\tau)$ be a C^2 curve on M. Let $F : M \to T_xM$ be a locally Lipschitz continuous vector field on M. Set $c(t,\tau) := X(t, c_0(\tau))$. Choose T such that $X(\cdot,\cdot)$ is defined on $[0,T] \times c_0([a,b])$. Then, denoting by $l(t)$ the length of $\tau \mapsto c(t,\tau)$, we have*

$$l(t) \le l(0)e^{C_{U,T}t} \quad \forall t \in [0,T]$$

where $C_{U,T}$ is the Lipschitz constant in $U := \{c(t,\tau) \in M \,:\, t \in [0,T], \tau \in [a,b]\}$ according to Definition H.1.

Proof. The proof follows along the lines of Proposition 1.1 in [7] with minor modifications. Without loss of generality, let $\tau \mapsto c(0,\tau)$ be parameterized by arc length, $\tau \in [0, l(0)]$. For ease of presentation, we denote $\partial_\tau := \partial/\partial_\tau$ and $\partial_t := \partial/\partial_t$.

From the theory of ODEs, due to F being locally Lipschitz continuous on M, it follows that $t \mapsto c(t,\tau)$ is absolutely continuous and has a Lipschitz continuous first derivative with respect to τ, *i.e.* $F(c(t,\tau))$ is continuous in τ for any fixed t and $|\nabla_{\partial_\tau} F(c(t,\tau))|_{\mathfrak{g}} \le C_{U,T}$ for all $(t,\tau) \in [0,T] \times [a,b]$. On the other hand, note that $F(c(t,\tau))$ may fail to be continuous and $\left|\bar{\nabla}_{\partial_t} F(c(t,\tau))\right|_{\mathfrak{g}}$ may fail to exist on some subsets of $[0,T]$ with zero measure. However, one may observe that $|\nabla_{\partial_t} F(c(t,\tau))|_{\mathfrak{g}}$ is bounded from above for almost all $t \in [0,T]$. Under the very same conditions, Theorem 9 in [10] has proven equality of mixed partial derivatives, namely that:

$$\bar{\nabla}_{\partial_t} \partial_\tau c(\tau,t) = \bar{\nabla}_{\partial_\tau} \partial_t c(\tau,t), \tag{H.6}$$

for all (t,τ) on a subset $E \subset [0,T] \times [a,b]$ such that $([0,T] \times [a,b]) \setminus E$ has zero measure.

Then, the following chain of inequalities holds true:

$$\begin{aligned}
l(s) - l(0) &= \int_0^s \partial_t l(t)\,\mathrm{d}t = \int_0^s \partial_t \int_0^{l(0)} |\partial_\tau c(t,\tau)|_{\mathfrak{g}}\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_0^s \int_0^{l(0)} \frac{\partial_t\, g\left(\partial_\tau c(t,\tau), \partial_\tau c(t,\tau)\right)}{2|\partial_\tau c(t,\tau)|_{\mathfrak{g}}} \\
&= \int_0^s \int_0^{l(0)} \frac{g\left(\nabla_{\partial_t} \partial_\tau c(t,\tau), \partial_\tau c(t,\tau)\right)}{|\partial_\tau c(t,\tau)|_{\mathfrak{g}}} \\
&= \int_0^s \int_0^{l(0)} \frac{g\left(\nabla_{\partial_\tau} \partial_t c(t,\tau), \partial_\tau c(t,\tau)\right)}{|\partial_\tau c(t,\tau)|_{\mathfrak{g}}} \\
&\le \int_0^s \int_0^{l(0)} |\nabla_{\partial_\tau} \partial_t c(t,\tau)|_{\mathfrak{g}}\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \int_0^s \int_0^{l(0)} \left|\nabla_{\partial_\tau c(t,\tau)} F(c(t,\tau))\right|_{\mathfrak{g}}\,\mathrm{d}\tau\,\mathrm{d}t \\
&\le C_U \int_0^s \int_0^{l(0)} |\partial_\tau c(t,\tau)|_{\mathfrak{g}}\,\mathrm{d}\tau\,\mathrm{d}t
\end{aligned}$$

$$= C_U \int_0^s l(t)\,\mathrm{d}t.$$

The Claim follows by applying Gronwall's inequality. ☐

Corollary H.5 (Gronwall inequality for systems with locally Lipschitz vector fields and evolving on manifolds)**.** *Let $p_0, q_0 \in M$. Let $F : M \to T_x M$ be a locally Lipschitz continuous vector field on M. Let $p(t) := X(t, p_0)$, $q(t) := X(t, q_0)$. Consider $c(t, \cdot) : [a, b] \to M, \tau \mapsto c(t, \tau)$ a family of geodesic curves parameterized by $t \in [0, T]$ whose arc length $l(t) := \int_a^b |\partial_\tau c(t, \tau)|_{\mathfrak{g}}\,\mathrm{d}\tau$ equals $\mathfrak{d}[p(t), q(t)]$. Suppose that $C_{U,T}$ is the Lipschitz constant in $U := \{c(t, \tau) \in M \;:\; t \in [0, T], \tau \in [a, b]\}$ according to Definition H.1. Then:*

$$\mathfrak{d}[p(t), q(t)] \leq \mathfrak{d}[p_0, q_0]\, e^{C_{U,T}\, t} \;\; \forall t \in [0, T] \tag{H.7}$$

Furthermore, if $p(t), q(t)$ evolve on a bounded set $U \subset M$ for all $t \geq 0$, there exists a constant C_U such that (H.7) holds with $T = +\infty$.

Proof. As in Theorem 1.2 of [7]. ☐

Corollary H.6 (upper bound)**.** *Consider the nonautonomous system (H.5) and the flow $X(t, x; v)$ at time $t \in \mathbb{R}$ with initial condition $x \in M$. Assume that Λ is a compact invariant set for (H.5). Then, for all neighborhoods U of Λ and all $T > 0$, there is a constant $\hat{C}_{U,T} > 0$ such that, for all $x \in U$, and all $t \in [0, T]$, the following upper bound holds:*

$$|X(t, x)|_\Lambda \leq |x|_\Lambda\, e^{\hat{C}_{U,T}\, t}. \tag{H.8}$$

Furthermore, if Λ is robustly stable on a compact neighborhood U in the sense of [8, Definition 2.2], then inequality (H.8) holds for all $t \geq 0$, all $x \in U$, and with a exponential constant $\hat{C}_U$ which does not depend upon T.

Proof. We first highlight that Proposition 5.1 in [8] can also be applied to systems evolving on manifolds. Indeed, the proof in [8] makes use of the following two facts:

1. closed and bounded sets are compact, and this is indeed true for geodesically complete manifolds;
2. Gronwall inequality holds (Corollary H.5).

It then follows that the reachable set $\mathcal{R}^{\leq T}(U) =: Q$ is bounded and we can find the Lipschitz constant $\hat{C}_{U,T}$ of Corollary H.5. By definition of $|\cdot|_\Lambda$ and compactness of Λ, it follows that there exists some $x_0 \in \Lambda$ such that $|x|_\Lambda = \mathfrak{d}\,[x, x_0]$. Moreover, by invariance of Λ under the flow of (2.1), we have that $X(t, x_0) \in \Lambda$ for all $t \in \mathbb{R}$. Therefore it holds, for all $t \geq 0$, that:

$$|X(t, x)|_\Lambda \leq \mathfrak{d}\,[X(t, x)\,,\, X(t, x_0)] \leq \mathfrak{d}\,[x, x_0]\;\, e^{\hat{C}_{Q,T}\, \bar{T}} = |x|_\Lambda\, e^{\hat{C}_{Q,T}\, \bar{T}},$$

where the first inequality follows by definition of $|\cdot|_\Lambda$. ☐

Lemma H.7. *Assume Λ is a compact invariant set for (H.5). Then, for all neighborhoods U of Λ, there is a constant $C_U > 0$ such that, for all $x \in U \setminus \Lambda$, all $x_0 \in \Lambda$, and all $t \geq 0$, the following lower bound holds:*

$$\mathfrak{d}\,[X(t, x), X(t, x_0)] > \frac{1}{2}\,|x|_\Lambda\, e^{-C_U\, t}. \tag{H.9}$$

Proof. We observe that $X(t, x_0) \in \Lambda$ for all $t \in \mathbb{R}$. Moreover, note that inequality (H.9) holds for $t = 0$ and thus it holds for all small enough $t > 0$. By contradiction, suppose there exists some neighborhood U of Λ such that, for all constants $C_U > 0$, there exists $x \in U \setminus \Lambda$ and $x_0 \in \Lambda$ and $t_2 > 0$ so that:

$$\delta(t_2) \leq \frac{1}{2}\,|x|_\Lambda\, e^{-C_U\, t_2}, \tag{H.10}$$

where, for ease of notation, we denoted $\delta(\cdot) := \mathfrak{d}\,[X(\cdot,x), X(\cdot,x_0)]$. Therefore there exists a time $t_1 \in [0,t_2)$ defined as follows:

$$t_1 := \sup\{\tau \in [0,t_2) \;:\; \delta(\tau) \geq |x|_\Lambda\}$$

Clearly, $\delta(t_1) = |x|_\Lambda$ and $\delta(t) \leq |x|_\Lambda$ for all $t \in [t_1,t_2]$. We now consider the Dini derivative of $\delta(t)$, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}^{+}\delta(t) := \limsup_{h\to 0^+} \frac{1}{h}\,(\delta(t+h)-\delta(t)).$$

Claim H.8. *There exists a constant $\bar{C}_U > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}^{+}\delta(t) \geq -\bar{C}_U\,\delta(t) \tag{H.11}$$

for all $t \in [t_1,t_2]$.

Proof. Due to compactness of Λ and since $\delta(t) \leq |z|_\Lambda$ for all $t \in [t_1,t_2]$, we have that $X(t,x)$ stays in a bounded set $\bar{U} \subset M$ for all $t \in [t_1,t_2]$. By virtue of Corollary H.5, we can then apply the Gronwall inequality for integral curves on Riemannian manifolds and Lipschitz continuous vector fields. In particular, by reasoning on trajectories of the backward flow, it holds:

$$\delta(t) \leq \delta(t+h)\, e^{\bar{C}_U\, h} \tag{H.12}$$

for all $t \in [t_1,t_2]$ and all positive small enough h. We can then multiply both sides of (H.12) by $e^{-\bar{C}_U h}$, so that to obtain:

$$\begin{aligned}
& \delta(t)e^{-\bar{C}_U h} \leq \delta(t+h) \\
\Rightarrow\quad & \delta(t)\frac{e^{-\bar{C}_U h}-1}{h} \leq \frac{\delta(t+h)-\delta(t)}{h} \\
\Rightarrow\quad & \limsup_{h\to 0^+}\ \delta(t)\frac{e^{-\bar{C}_U h}-1}{h} \leq \limsup_{h\to 0^+} \frac{\delta(t+h)-\delta(t)}{h} \\
\Rightarrow\quad & -\bar{C}_U\,\delta(t) \leq \frac{\mathrm{d}}{\mathrm{d}t}^{+}\delta(t).
\end{aligned}$$

□

We can then expand inequality (H.11) so that to obtain:

$$0 \leq e^{\bar{C}_U\, t}\left(\frac{\mathrm{d}}{\mathrm{d}t}^{+}\delta(t) + \bar{C}_U\,\delta(t)\right) = \frac{\mathrm{d}}{\mathrm{d}t}^{+}\left(e^{\bar{C}_U\, t}\delta(t)\right).$$

Last inequality implies that:

$$\delta(t_2)\, e^{\bar{C}_U\, t_2} \geq \delta(t_1)\, e^{\bar{C}_U\, t_1}$$

$$\Rightarrow \quad \delta(t_2) \geq \delta(t_1)e^{-\bar{C}_U\,(t_2-t_1)} \geq \delta(t_1)e^{-\bar{C}_U\,t_2} \geq |x|_\Lambda e^{-\bar{C}_U\,t_2}$$

which contradicts (H.10) with $C_U = \bar{C}_U$. □

Corollary H.9 (Exponential lower bound for systems with locally Lipschitz vector fields)**.** *Assume Λ is a compact invariant set for (H.5). Then, for all neighborhoods U of Λ, there is a constant $C_U > 0$ such that, for all $x \in U \setminus \Lambda$, and all $t \geq 0$, the following lower bound holds:*

$$|X(t,x)|_\Lambda > \frac{1}{2}\,|x|_\Lambda\, e^{-C_U\,t}. \tag{H.13}$$

Proof. By definition of $|\cdot|_\Lambda$ and compactness of Λ, there exists $\bar{x} \in \Lambda$ such that $|X(t,x)|_\Lambda = \mathfrak{d}\,[X(t,x)\,,\,\bar{x}]$. By invariance of Λ under the flow of (H.5), we can then select $x_0 \in \Lambda$ such that $X(t,x_0) = \bar{x}$. Then, by virtue of Lemma H.7, we obtain:

$$|X(t,x)|_\Lambda = \mathfrak{d}\,[X(t,x)\,,\,\bar{x}] = \mathfrak{d}\,[X(t,x)\,,\,X(t,x_0)] > \frac{1}{2}\,|x|_\Lambda\, e^{-C_U\,t}.$$

□

Appendix I. Growth of the distance from $\bar{A}_i$ along trajectories

Lemma I.1. *Consider sytsem (D.7). For any small enough neighborhood U of $\bar{A}_i$ with $U \subseteq \mathcal{A}_i$, there exists a constant $c > 0$ such that, for all $\xi \in U \cap \mathcal{E} \setminus \bar{A}_i$ and all $v \in V$, it holds:*

$$\left|\mathcal{L}_{\tilde{f}(\xi,v)}|\xi|_{\bar{A}_i}\right| \leq c|\xi|_{\bar{A}_i}.$$

Proof. For any $\xi \notin \bar{A}_i$ and any $v \in V$, the Lie derivative of the distance function from $\mathcal{W}$ along the vector field $\tilde{f}$ as in (D.7) is computed as

$$\mathcal{L}_{\tilde{f}(\xi,v)}|\xi|_{\bar{A}_i} = d|\xi|_{\bar{A}_i}\cdot \tilde{f}(\xi,v), \tag{I.1}$$

namely the action of differential $\mathrm{d}|\xi|_{\bar{A}_i} \in T^\star_\xi M$ on vector $\tilde{f}(\xi,v) \in T_\xi M$. Such action is linear in its argument, and thus:

$$\mathrm{d}|\xi|_{\bar{A}_i}\cdot \tilde{f}(\xi,v) = \mathrm{d}|\xi|_{\bar{A}_i}\cdot \hat{f}(\xi) + 2\phi(\xi)f_0(\xi)\left[v_0\left(\mathrm{d}|\xi|_{\bar{A}_i}\cdot \hat{f}(\xi)\right) + \varphi(|\xi|_{\bar{A}_i})\left(\mathrm{d}|\xi|_{\bar{A}_i}\cdot v_1\right)\right],$$

from which it follows that:

$$\left|\mathcal{L}_{\tilde{f}(\xi,v)}|\xi|_{\bar{A}_i}\right| \leq 3\left|\mathrm{d}|\xi|_{\bar{A}_i}\cdot \hat{f}(\xi)\right| + 2\varphi(|\xi|_{\bar{A}_i})\left|\mathrm{d}|\xi|_{\bar{A}_i}\cdot v_1\right| \tag{I.2}$$

for all $\xi \in M \setminus \bar{A}_i$ and all $v \in V$. Since the metric $\mathfrak{g}$ is smooth in M, the differential of any distance function is a covector whose norm is bounded over bounded set, and thus there exists $\underline{c} > 0$ such that $|d|\xi|_{\bar{A}_i}|_\mathfrak{g} \leq \underline{c}$ for all $\xi \in U \setminus \bar{A}_i$ (with a slight abuse of notation, we have evaluated the $\mathfrak{g}$-norm on a covector instead of a vector). The definition of (D.7) and (**??**) also implies that $|v_1|_\mathfrak{g} \leq \bar{c}$ for some $\bar{c} > 0$, and we thus conclude that the second term of sum (I.2) is bounded in $U \setminus \bar{A}_i$, *i.e.*

$$\exists \hat{c} > 0 \quad \left|\mathrm{d}|\xi|_{\bar{A}_i}\cdot v_1\right| \leq \hat{c}. \tag{I.3}$$

In regard to the first term of sum (I.2), we recall here the definition of the Bouligand tangent cone [4] to a manifold S at a point $\xi_0 \in S$, as follows:

$$T_S^B(\xi_0) := \left\{ v \in T_{\xi_0}M \;:\; \frac{\mathrm{d}}{\mathrm{d}t} |\exp_{\xi_0}(tv)|_S \Big|_{t=0} = 0 \right\}.$$

Observe the following two facts:

- $\bar{A}_i$ is closed and backward invariant for (D.7);
- any solution $Z(t,\xi;v)$ with $\xi \in \operatorname{int} \bar{A}_i$ stays in $\bar{A}_i$ for all small enough $t \geq 0$.

Therefore, we have that $\hat{f}(\xi_0) \in T^B_{\bar{A}_i}(\xi_0)$ for any $\xi_0 \in \bar{A}_i \cap \mathcal{E}$ (see [4]), and thus

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big| \exp_{\xi_0}(t\hat{f}(\xi_0)) \Big|_{\bar{A}_i} \Big|_{t=0} = 0 \quad \forall \xi_0 \in \bar{A}_i. \tag{I.4}$$

Furthermore, we recall that the definition of the exponential map implies that:

$$\hat{f}(\xi) = \frac{\mathrm{d}}{\mathrm{d}t} Z(t,\xi) \Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \exp_\xi(t\hat{f}(\xi)) \Big|_{t=0} \quad \forall \xi \in M. \tag{I.5}$$

Observe that, by virtue of (I.5) and the chain rule of differentiation,

$$\begin{aligned} &\forall\, \xi \in M \quad \exists\, \zeta \in d_P|\xi|_{\bar{A}_i} \text{ such that} \\ &\frac{\mathrm{d}}{\mathrm{d}t} \Big| \exp_\xi(t\hat{f}(\xi)) \Big|_{\bar{A}_i} \Big|_{t=0} = \zeta \cdot \frac{\mathrm{d}}{\mathrm{d}t} \exp_\xi(t\hat{f}(\xi)) \Big|_{t=0} \\ &\qquad\qquad\qquad\qquad\qquad\quad = \zeta \cdot \hat{f}(\xi), \end{aligned} \tag{I.6}$$

where $d_P|\xi|_{\bar{A}_i}$ denotes the proximal subdifferential of $|\xi|_{\bar{A}_i}$ at ξ, according to Definition D.21. The use of proximal subdifferential here is consistent with the fact that function $|\cdot|_{\bar{A}_i}$ is smooth on $M \setminus \bar{A}_i$ and locally Lipschitz continuity everywhere on M. It then follows from (I.4) and (I.6) that

$$\forall\, \xi_0 \in \bar{A}_i \cap \mathcal{E} \quad \exists \zeta \in d_P|\xi_0|_{\bar{A}_i} \text{ such that } \zeta \cdot \hat{f}(\xi_0) = 0. \tag{I.7}$$

Now denote with $\cdot^\star : T_\xi M \to T^\star_\xi M$ the operator which transforms a vector in $T_\xi M$ to its corresponding covector in $T^\star_\xi M$. Denote with $\mathcal{T}_z^{z_0} : T_z M \to T_{z_0} M$ the parallel transport operator of a vector in $T_z M$ to a vector in $T_{z_0} M$ along the geodesic from z to z_0. We then define the parallel transport operator on covectors ${\mathcal{T}^\star}_z^{z_0} : T^\star_z M \to T^\star_{z_0} M$ as follows:

$${\mathcal{T}^\star}_z^{z_0}(v) := \left(\mathcal{T}_z^{z_0} \left(v^{\star -1} \right) \right)^\star \tag{I.8}$$

It can be shown that the action of a covector on a vector is invariant under parallel transport, and therefore, by making use of (I.7), the following chain of inequalities holds true for any $\xi \in U \cap \mathcal{E} \setminus \bar{A}_i$ and any $\xi_0 \in \bar{A}_i$ such that $|\xi|_{\bar{A}_i} = \mathfrak{d}[\xi, \xi_0]$, and some $\zeta \in d_P|\xi_0|_{\bar{A}_i}$ as provided by (I.7):

$$\begin{aligned} \Big| \mathrm{d}|\xi|_{\bar{A}_i} \cdot \hat{f}(\xi) \Big| &= \Big| {\mathcal{T}^\star}_\xi^{\xi_0}(\mathrm{d}|\xi|_{\bar{A}_i}) \cdot \mathcal{T}_\xi^{\xi_0}(\hat{f}(\xi)) \Big| \\ &= \Big| {\mathcal{T}^\star}_\xi^{\xi_0}(\mathrm{d}|\xi|_{\bar{A}_i}) \cdot \mathcal{T}_\xi^{\xi_0}(\hat{f}(\xi)) - \zeta \cdot \hat{f}(\xi_0) + \zeta \cdot \hat{f}(\xi_0) \Big| \end{aligned}$$

$$
\begin{aligned}
&= \left| \mathcal{T}^{\star}{}_{\xi}^{\xi_0}(\mathrm{d}|\xi|_{\bar{A}_i}) \cdot \mathcal{T}_{\xi}^{\xi_0}(\hat{f}(\xi)) - \zeta \cdot \hat{f}(\xi_0) \right| \\
&\leq c\mathfrak{d}[\xi, \xi_0] \\
&\leq c|\xi|_{\bar{A}_i},
\end{aligned} \tag{I.9}
$$

for some c which only depends upon U. The previous inequalities follow from local Lipschitz continuity of functions $|\cdot|_{\bar{A}_i}$ and $\hat{f}(\cdot)$, where the definition of local Lipschitz continuity for real-valued functions on M is adapted from Definition H.1. Finally, injecting (I.3) and (I.9) in (I.2) proves the Lemma. □

References

[1] R. Abraham, J. Marsden and T. Ratiu, Manifolds, Tensor Analysis, and Applications. Springer-Verlag, New York, USA (1988).

[2] D. Angeli and D. Efimov, Characterizations of input-to-state stability for systems with multiple invariant sets. *IEEE Trans. Autom. Control* **60** (2015) 3242–3256.

[3] D. Angeli, B. Ingalls, E. Sontag and Y. Wang, Separation principles for input-output and integral-input to state stability. *SIAM J. Control Optim.* **43** (2004) 256–276.

[4] F.H. Clarke, Y.S. Ledyaev, R.J. Stern and P.R. Wolenski, Nonsmooth Analysis and Control Theory. Springer-Verlag New York, Inc., Secaucus, NJ, USA (1998).

[5] M.P. do Carmo, Riemannian Geometry. Birkhäuser, Birkhäuser Basel (1992).

[6] M. Krichman, E. Sontag and Y. Wang, Input-output-to-state stability. *SIAM J. Control Optim.* **39** (2001) 1874–1928.

[7] M. Kunzinger, H. Schichl, R. Steinbauer and J.A. Vickers, Global Gronwall estimates for integral curves on riemannian manifolds. *Rev. Mate. Complut.* **19** (2006) 133–137.

[8] Y. Lin, E.D. Sontag and Y. Wang, A smooth converse lyapunov theorem for robust stability. *SIAM J. Control Optim.* **34** (1996) 124–160.

[9] L. Markus, Asymptotically autonomous differential systems. Contributions to the Theory of Nonlinear Oscillations III. *S. Lefschetz* **36** (1956) 17–29.

[10] E. Minguzzi, The equality of mixed partial derivatives under weak differentiability conditions. *Real Anal. Exchange* **40** (2015) 81–98.

[11] Z. Nitecki and M. Shub, Filtrations, decompositions, and explosions. *Am. J. Math.* **97** (1975) 1029–1047.

[12] W. Rudin, Principles of mathematical analysis. International series in pure and applied mathematics, McGraw-Hill (1964).

[13] M. Shub, Global Stability of Dynamical Systems. Springer-Verlag, New York, NY (1987).

[14] E.D. Sontag, Comments on integral variants of ISS. *Syst. Control Lett.* **34** (1998) 93–100.

[15] E.D. Sontag, Input to state stability: Basic concepts and results, in Nonlinear and Optimal Control Theory. In vol. 1932 of *Lecture Notes in Mathematics*. Springer (2006) 163–220.

[16] E.D. Sontag and Y. Wang, Detectability of nonlinear systems, in Proc. Conf. on Information Science and Systems (CISS 96), Princeton, NJ (1996) 1031–1036.

[17] E.D. Sontag and Y. Wang, Output-to-state stability and detectability of nonlinear systems. *Syst. Control Lett.* **29** (1997) 279–290.