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TOPOLOGY OPTIMIZATION FOR QUASISTATIC ELASTOPLASTICITY

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Abstract. Topology optimization is concerned with the identification of optimal shapes of deformable bodies with respect to given target functionals. The focus of this paper is on a topology optimization problem for a time-evolving elastoplastic medium under kinematic hardening. We adopt a phase-field approach and argue by subsequent approximations, first by discretizing time and then by regularizing the flow rule. Existence of optimal shapes is proved both at the time-discrete and time-continuous level, independently of the regularization. First order optimality conditions are firstly obtained in the regularized time-discrete setting and then proved to pass to the nonregularized time-continuous limit. The phase-field approximation is shown to pass to its sharp-interface limit *via* an evolutive variational convergence argument.

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1. Introduction

The design of a mechanical piece is often driven by an optimization process. The mechanical response of a given shape is tested against a number of criteria, possibly including weight, material and manufacturing costs, topological, and geometrical features. The tenet of Topology Optimization (TO in the following) is that of identifying the optimal shape of a body $E \subset \Omega$ within a given design region $\Omega \subset \mathbb{R}^n$ with respect to a given target functional. This optimality depends on the mechanical response of the body with respect to the imposed actions (boundary displacements, forces, tractions) and is hence a function of E itself. As such, the target functional is minimized with respect to the shape E. This general setting is common to most TO problems and arises ubiquitously, from mechanical engineering, to aerospace and automotive, to architectural engineering, to biomechanics [3].

In this paper, we investigate a TO problem for a linearized elastoplastic medium showing kinematic hardening. The mechanical state of the the system is described by its time-dependent *displacement* $u(x,t) \in \mathbb{R}^n$ and its *plastic strain* $\boldsymbol{p}(x,t) \in \mathbb{R}^{n\times n}_{\rm dev}$ (symmetric deviatoric tensors). We assume that the *total strain* $\mathrm{E}u = (\nabla u + \nabla u^T)/2$

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of the body can be additively decomposed into an elastic part $\boldsymbol{\varepsilon} \in \mathbb{R}^{n\times n}_{\rm sym}$ (symmetric tensors) related to the stress state of the material and the plastic part $\boldsymbol{p}$, namely,

$$\mathrm{E}u = \boldsymbol{\varepsilon} + \boldsymbol{p}. \tag{1.1}$$

In the so-called *sharp-interface* setting, the actual position of the body within the design domain Ω is identified by means of the scalar function $z\colon \Omega \to \{0,1\}$. In particular, the level set $\{z=1\}$ indicates the position of the body to be determined *via* TO. In the following, we shall interpret z as a phase indicator and assume the region not occupied by the body to be filled by a very compliant medium, again of elastoplastic type. This approach is rather classical [1] and allows for a sound mathematical treatment. In particular, by taking the material parameters to be suitably dependent on z, all state quantities will be assumed to be defined in the whole design region Ω. The mechanical problem will be hence addressed in the fixed domain Ω and the actual position of the body to be determined *via* TO is identified *via* z.

In the following, we will mostly leave the classical sharp-interface setting by considering a *phase-field* approach instead. Here, the scalar function z is allowed to take intermediate values $z \in (0,1)$, as well. We refer to z as *phase field* or *phase*, alluding to the interpretation of the material in Ω as a two-phase system. Following this interpretation, the set $\{0<z<1\}$ could be seen as the region where the two materials mix.

We assume linear material response, namely, the stress $\boldsymbol{\sigma}$ of the medium is obtained as $\boldsymbol{\sigma} = \mathbb{C}(z)\boldsymbol{\varepsilon}$ where $\mathbb{C}(z)$ is the positive-definite symmetric *elasticity tensor*. Note that the tensor $\mathbb{C}(z)$ depends on the value of z. All materials parameters are indeed assumed to depend on z, in order to distinguish the different mechanical response of the body to be determined *via* TO and the compliant medium. These dependencies are kept abstract in the paper, in order to possibly accommodate the different phenomenological choices which are in use.

The time-evolution of $\boldsymbol{p}$ is driven by the normality *flow rule*

$$d(z)\partial|\dot{\boldsymbol{p}}| \ni \boldsymbol{\sigma} - \mathbb{H}(z)\boldsymbol{p}\,. \tag{1.2}$$

Here, $d(z)>0$ represents the *yield stress* which activates plasticization and the symbol ∂ stands for the set-valued subdifferential in the sense of Convex Analysis, namely $\partial|\dot{\boldsymbol{p}}| = \dot{\boldsymbol{p}}/|\dot{\boldsymbol{p}}|$ for $\dot{\boldsymbol{p}} \neq 0$ and $\partial|0| = \{\boldsymbol{q} \in \mathbb{R}^{n\times n}_{\rm dev} \ : \ |\boldsymbol{q}| \leq 1\}$. Eventually, $\mathbb{H}(z)\boldsymbol{p}$ represents the *backstress* due to kinematic hardening, here modulated by the positive-definite symmetric *kinematic hardening tensor* $\mathbb{H}(z)$ [12]. The flow rule (1.2) is of course to be complemented by an initial condition for $\boldsymbol{p}$ which we will take as $\boldsymbol{p}(0)=0$ for simplicity.

The body is assumed to be clamped on the portion Γ_D of the boundary $\partial\Omega$ and to evolve quasistatically under the combined effect of the time-dependent body force $\ell(z)f$ and of the time-dependent boundary traction g on the portion Γ_N of $\partial\Omega$. The *quasistatic equilibrium system* hence reads

$$\nabla\cdot\boldsymbol{\sigma} + \ell(z)f(t) = 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \Gamma_{\rm D}, \quad \boldsymbol{\sigma} n = g(t) \ \text{ on } \Gamma_{\rm N} \tag{1.3}$$

where n indicates the outward pointing normal to $\partial\Omega$ and the term $\ell(z)$ corresponds to the *density* of the medium at phase z. Under suitable assumptions on data, see Section 2 below, for each $z \in L^\infty(\Omega)$ one can uniquely identify a trajectory $t \in [0,T] \mapsto (u(t), \boldsymbol{p}(t))$ solving the *quasistatic elastoplastic evolution system* (1.1)–(1.3) in a suitable weak sense, see Definition 3.1 and the comments thereafter.

Our aim is to identify phases z which, together with their associated quasistatic elastoplastic evolutions $t \in [0,T] \mapsto (u(t), \boldsymbol{p}(t))$, minimize the compliance-type functional $\mathcal{J}_\delta(z,u)$ given by

$$\begin{aligned} \mathcal{J}_\delta(z,u) := & \int_\Omega \ell(z)\, f(T)\cdot u(T)\,\mathrm{d}x + \int_{\Gamma_N} g(T)\cdot u(T)\,\mathrm{d}\mathcal{H}^{n-1} \\ & - \int_0^T \int_\Omega \ell(z)\,\dot{f}(\tau)\cdot u(\tau)\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^T \int_{\Gamma_N} \dot{g}(\tau)\cdot u(\tau)\,\mathrm{d}\mathcal{H}^{n-1}\mathrm{d}\tau \\ & + \int_\Omega \frac{\delta}{2}|\nabla z|^2 + \frac{z^2(1-z)^2}{2\delta}\,\mathrm{d}x\,. \end{aligned} \tag{1.4}$$

The first four terms in $\mathcal{J}_\delta$ measure the compliance of the medium, integrated over the time interval $(0,T)$. The last two terms in $\mathcal{J}_\delta$ are the classical Modica-Mortola functional [27]. Under the modulation of the user-defined small parameter $\delta > 0$, the gradient term penalizes changes in z whereas the double-well term favours the values 0 and 1. The combination of the two last terms in $\mathcal{J}_\delta$ expresses the competition between phase separation and minimization of transitions between phases. In the limit $\delta \to 0$ one recovers a sharp-interface situation, where minimizing phases z take exclusively values 0 or 1 and the length of the interface separating the two regions $\{z = 0\}$ and $\{z = 1\}$ is penalized, see Section 4.

Our main TO problem reads

$$\min \left\{ \mathcal{J}_\delta(z,u) \ : \ (u,\boldsymbol{p}) \ \text{ solve } (1.1)\text{–}(1.3) \text{ given } z \right\}. \tag{1.5}$$

The main contribution of this paper is in proving that this TO problem admits solutions, in investigating its discretization and regularization, and in providing first-order optimality conditions.

More precisely, in order to tackle the TO problem (1.5) we proceed by subsequent approximations. At first, we investigate a time-discrete version of (1.5), where continuous-in-time states are replaced by the time-discrete solutions of the incremental elastoplastic problem, see Definition 3.3. The time-discrete TO problem is proved to admit solutions (Prop. 3.4) which converge to solutions of the time-continuous (1.5) as the fineness of the time partition goes to 0 (Cor. 3.6).

The time-discrete TO problem is then regularized by replacing the nonsmooth term $|\dot{\boldsymbol{p}}|$ in the flow rule (1.2) by the smooth function $h_\gamma(\dot{\boldsymbol{p}}) = (|\dot{\boldsymbol{p}}|^2 + \gamma^{-2})^{1/2} - 1/\gamma$ depending on $\gamma > 0$. The corresponding approximate time-discrete TO problem admits solutions (Prop. 3.8) which converge to solutions of the time-discrete TO problem as $\gamma \to +\infty$ (Cor. 3.10). Introducing the regularization *via* h_γ is instrumental to obtain the differentiability of the *control-to-state* map $z \mapsto (u, \boldsymbol{p})$ which is in turn needed in order to derive first-order optimality conditions, see also [2, 10, 16, 34]. This differentiability is tackled in Section 5 in the frame of the approximate time-discrete TO problem (Theorem 5.1) and allows to prove corresponding first-order optimality conditions (Cor. 5.4). The passage to the limit as $\gamma \to +\infty$ first and then as the fineness of the time partition goes to 0 provide the first-order optimality conditions for the time-discrete TO problem (Thm. 6.1) and the time-continuous TO problem (1.5) (Thm. 6.4), which are the main results of the paper.

All the above mentioned results are obtained in the setting of the phase-field approximation $\delta > 0$. Still, the existence and the convergence results are valid in the sharp-interface case $\delta = 0$, where the phase z takes the values 0 or 1 only, and the limit $\delta \to 0$ can be rigorously ascertained. We give some detail in this direction in Section 4 for the time-continuous TO problem (1.5). In particular, we prove that solutions to (1.5) for $\delta > 0$ converge to solutions to (1.5) for $\delta = 0$ as $\delta \to 0$ by means of an evolutive Γ-convergence argument (Prop. 4.2). Let us remark however that, due to the limited regularity of solutions to (1.5) for $\delta = 0$, first-order optimality conditions are available for the case $\delta > 0$ only.

Before moving on, let us comment on the literature and put our work in perspective. The mathematical TO literature in the static *elastic* setting is abundant, see [7, 8, 29] and [4, 5, 9] for a selection of existence results and first-order optimality conditions in different linear and nonlinear settings. Results in the *elastoplastic* setting are available in the two-dimensional case, both in the static [13, 14, 17, 19] and in the evolutive regime [18], but exclusively under the a priori assumption that the unknown optimal shape $\{z = 1\}$ is Lipschitz regular. The beam structure and frame optimization was investigated in [20, 21, 28] from the point of view of the existence of minimizers. First-order optimality conditions in terms of shape derivatives appeared in Chapters 4.8 and 4.9 of [32] for an elastic torsion problem and for the viscoplastic model of Perzyna, see also [6, 23]. To the best of our knowledge, the existence analysis and the study of optimality conditions in the corresponding regularity setting are unprecedented for quasistatic evolution TO problems for elasto-plasticity.

On the other hand, control problem for quasistatic elastoplasticity have already been studied and the reader is referred to the analysis in [33–35], see also the general theory in [30, 31]. Compared with these contributions, where controls usually are modeled as imposed forces, in the frame of TO the action of controls is more involved, for they modify the elastic response *via* material parameters. Correspondingly, our analysis is at specific places more involved than that in the above papers, albeit being inspired by the same general principles.

In our recent paper [2], we have tackled the three-dimensional *static* kinematic-hardening case and analyzed the existence of solutions, the first-order optimality conditions, and the sharp-interface limit. This indeed sets the basis for the current contribution, which however focuses on the quasistatic evolutive case. Moving from static state-problem formulations, based on the minimization of one single functional, to evolutive formulations, based on the time-continuous limits of sequences of time-discretizations in the frame of rate-independent processes [25] is analytically challenging. Remarkably, in order to tackle the various limiting procedures one has to resort to *evolutive* Γ-convergence techniques [26], which are more involved than their static counterparts.

Let us now present the structure of the paper and of our results:

Section 2 is devoted to discussion of the model, notation, and assumptions on data. These assumptions are then assumed to hold throughout the paper, without further mention.

Section 3 brings to statement of the time-continuous TO problem, as well as of its time-discrete and approximate time-discrete versions. Here, we also check existence of optimal solutions and convergence of time-discrete to time-continuous and approximate time-discrete to time-discrete solutions.

Section 4 focuses on the sharp interface limit $\delta \to 0$. In particular, we prove that solutions of time-continuous TO problem (1.5) converge to solutions of the corresponding sharp-interface limiting TO problem for $\delta = 0$. This is based on an evolutive Modica-Mortola argument.

Section 5 contains the investigation of the differentiability of the control-to-state map for the approximate time-discrete TO problem, where $\gamma < +\infty$. Correspondingly, a detailed analysis of first-order optimality conditions in the approximate time-discrete case is presented.

Section 6 eventually leads to first-order optimality conditions for both time-continuous and the time-discrete TO problems. These ensue by passing to the limit in the corresponding ones for the approximate time-discrete TO problem from Section 5.

The Appendix features a technical convergence argument which is used in the study of discrete-to-continuous limits for quasistatic evolutions.

2. Assumptions on data

We devote this section to fixing notation and assumptions on data. In the following, $\mathbb{M}^n$ indicates the space of 2-tensors in n dimensions, indicated in bold face in the following, and $\mathbb{M}^n_S$ is the subspace of symmetric 2-tensors. The symbol $\mathbb{M}^n_D$ indicates symmetric and deviatoric 2-tensors, namely those with vanishing trace. The symbol $\cdot$ indicates contraction with respect to all indices. In particular $\boldsymbol{A} \cdot \boldsymbol{B} = A_{ij}B_{ij}$ and $u \cdot v = u_i v_i$ (summation convention on repeated indices) for all $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{M}^n$, $u, v \in \mathbb{R}$.

The elasticity tensor $\mathbb{C}$ and the kinematic-hardening tensor $\mathbb{H}$ are asked to be *isotropic* for all z. In particular, we ask for

$$\mathbb{C}(z) := 2\mu(z)\mathbb{I} + \lambda(z)(\mathbf{I} \otimes \mathbf{I}), \quad \mathbb{H}(z) := h(z)\mathbb{I} \tag{2.1}$$

where $\lambda(z)$ and $\mu(z)$ are the Lamé coefficients, $h(z)$ is the hardening modulus, and $\mathbb{I}$ and $\mathbf{I}$ denote the identity 4 and 2-tensor, respectively. Isotropy in particular guarantees that $\mathbb{C}$ and $\mathbb{H}$ map $\mathbb{M}^n_D$ to $\mathbb{M}^n_D$.

We assume the material coefficients to be differentiable with respect to z and to be defined in all of $\mathbb{R}$. In particular, we ask

$$\mu,\, \lambda,\, h,\, d \in C^1(\mathbb{R})\,. \tag{2.2}$$

We moreover define them as constant on $\{z \leq 0\}$ and $\{z \geq 1\}$. This last provision allows us to recover the property $z \in [0, 1]$ a posteriori, without the need of enforcing it a-priori as a constraint. The reader is referred to (3.3), (3.10), and (3.17) for additional details.

All material coefficients are asked to be positive and bounded, uniformly with respect to z, namely, we assume that

$$\exists 0<\alpha<\beta<+\infty \ \forall z \in [0,1]: \quad \alpha \le \mu(z)\,,\ \lambda(z),\ h(z),\ d(z) \le \beta\,. \tag{2.3}$$

This in particular implies that $\mathbb{C}$ and $\mathbb{H}$ are uniformly positive definite and bounded, independently of z. Indeed, one can find $0<\alpha_{\mathbb{C}}<\beta_{\mathbb{C}}<+\infty$ and $0<\alpha_{\mathbb{H}}<\beta_{\mathbb{H}}<+\infty$ such that

$$\alpha_{\mathbb{C}}|\mathbf{E}|^2 \le \mathbb{C}(z)\mathbf{E}\cdot\mathbf{E} \le \beta_{\mathbb{C}}|\mathbf{E}|^2 \qquad \text{for every } \mathbf{E}\in\mathbb{M}^n_S\,, \tag{2.4}$$

$$\alpha_{\mathbb{H}}|\mathbf{Q}|^2 \le \mathbb{H}(z)\mathbf{Q}\cdot\mathbf{Q} \le \beta_{\mathbb{H}}|\mathbf{Q}|^2 \qquad \text{for every } \mathbf{Q}\in\mathbb{M}^n_D\,. \tag{2.5}$$

The design domain $\Omega\subset\mathbb{R}^n$ is taken to be open, connected, and with Lipschitz boundary $\partial\Omega$. We also fix two subsets Γ_N,Γ_D of $\partial\Omega$, which from now on will be referred to as *Neumann* and *Dirichlet* part of $\partial\Omega$, respectively. We assume Γ_D, $\Gamma_N\subset\partial\Omega$ to be open in the topology of $\partial\Omega$ with $\Gamma_N\cap\Gamma_D=\emptyset$, $\overline{\Gamma}_N\cup\overline{\Gamma}_D=\partial\Omega$, and where $\overline{\Gamma}_N$ and $\overline{\Gamma}_D$ are closures in $\partial\Omega$. We moreover assume that Γ_D has positive surface measure, namely $\mathcal{H}^{n-1}(\Gamma_D)>0$, where the latter is the $(n-1)$-Hausdorff surface measure in $\mathbb{R}^n$. Furthermore, we suppose that $\Omega\cup\Gamma_N$ is regular *in the sense of Gröger* ([11], Def. 2), that is, for every $x\in\partial\Omega$ there exists an open neighborhood $U_x\subseteq\mathbb{R}^n$ of x and a bi-Lipschitz map $\Psi_x\colon U_x\to\Psi(U_x)$ such that $\Psi_x(U_x\cap(\Omega\cup\Gamma_N))$ coincides with one of the following sets:

$$\begin{aligned}
V_1 &:= \{y\in\mathbb{R}^n:\ |y|\le 1,\ y_n<0\}\,,\\
V_2 &:= \{y\in\mathbb{R}^n:\ |y|\le 1,\ y_n\le 0\}\,,\\
V_3 &:= \{y\in V_2;\ y_n<0 \text{ or } y_1>0\}\,,
\end{aligned}$$

where y_i is the ith component of $y\in\mathbb{R}^n$. This last assumption is crucially used in the proof of Theorem 5.1.

For every $w\in H^1(\Omega;\mathbb{R}^n)$, we define the set of admissible displacements

$$\mathcal{A}(w) := \{(u,\boldsymbol{\varepsilon},\boldsymbol{p})\in H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D):\ \mathrm{E}u=\boldsymbol{\varepsilon}+\boldsymbol{p},\ u=w \text{ on } \Gamma_D\}\,,$$

where $\mathrm{E}u$ denotes the symmetric part of the gradient of u, namely $\mathrm{E}u=(\nabla u+\nabla u)^\top/2$.

As concerns data, we assume the *volume-force density per unit mass* f, the *surface-traction density* g, and the Dirichlet *boundary displacement* w, to satisfy

$$f\in H^1(0,T;L^p(\Omega;\mathbb{R}^n)),\quad g\in H^1(0,T;L^p(\Gamma_N;\mathbb{R}^n)),\quad w\in H^1(0,T;W^{1,p}(\Omega;\mathbb{R}^n)) \tag{2.6}$$

for some given $p\in(2,+\infty)$. Additionally, we assume that

$$f(0)=g(0)=w(0)=0\,. \tag{2.7}$$

This last requirement ensures the compatibility of the initial datum

$$(u(0),\boldsymbol{\varepsilon}(0),\boldsymbol{p}(0))=(0,0,0)\,. \tag{2.8}$$

The assumptions (2.1)–(2.8) of this Section are assumed throughout the paper, without further explicit mention.

3. The topology optimization problem and its approximations

This section is devoted to make the topology optimization problem precise and present its time-discretization and regularization. In particular, we prove the existence of optimal phase-fields z in the various settings, which are then connected *via* variational convergence arguments.

Let us start by defining *quasistatic evolutions* of the elastoplastic system given the phase-field z. We follow here the energetic formulation of quasistatic evolutions [25], in which the elastoplastic system is driven by energy-storage and energy-dissipation mechanism, calling for the definition of the *energy* $\mathcal{E}$ and the *dissipation* $\mathcal{D}$. We define

$$\mathcal{E}(t, z, u, \boldsymbol{\varepsilon}, \boldsymbol{p}) := \frac{1}{2}\int_\Omega \mathbb{C}(z)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathbb{H}(z)\boldsymbol{p}\cdot\boldsymbol{p}\,\mathrm{d}x - \int_\Omega \ell(z)f(t)\cdot u\,\mathrm{d}x - \int_{\Gamma_N} g(t)\cdot u\,\mathrm{d}\mathcal{H}^{n-1}$$

$$\mathcal{D}(z, \boldsymbol{q}) := \int_\Omega d(z)|\boldsymbol{q}|\,\mathrm{d}x$$

for every $z \in L^\infty(\Omega)$, every $(u, \boldsymbol{\varepsilon}, \boldsymbol{p}) \in \mathcal{A}(w(t))$, and every $\boldsymbol{q} \in L^1(\Omega;\mathbb{M}^n_D)$. For given $\boldsymbol{p}\colon [0,T] \to L^2(\Omega;\mathbb{M}^n_D)$ and $z \in L^\infty(\Omega)$, we further define the *total dissipation functional*

$$\mathcal{V}([0,t]; z, \boldsymbol{p}(\cdot)) := \sup\left\{\sum_{t_j\in\mathcal{P}} \mathcal{D}\big(z, \boldsymbol{p}(t_j) - \boldsymbol{p}(t_{j-1})\big) :\ \mathcal{P} \text{ is a partition of } [0,t]\right\}.$$

With these ingredients at hand, we are able to pose the following definition.

Definition 3.1 (Quasistatic evolution given z)**.** Let $z \in L^\infty(\Omega)$ be given. We say that a triple $(u, \boldsymbol{\varepsilon}, \boldsymbol{p})\colon [0,T] \to H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D)$ is a *quasistatic evolution given* z if $(u(0), \boldsymbol{\varepsilon}(0), \boldsymbol{p}(0)) = (0,0,0)$ and the following conditions hold:

(i) for every $t \in [0,T]$ and every $(\hat{u}, \hat{\boldsymbol{\varepsilon}}, \hat{\boldsymbol{p}}) \in \mathcal{A}(w(t))$

$$\mathcal{E}(t, z, u(t), \boldsymbol{\varepsilon}(t), \boldsymbol{p}(t)) \le \mathcal{E}(t, z, \hat{u}, \hat{\boldsymbol{\varepsilon}}, \hat{\boldsymbol{p}}) + \mathcal{D}(z, \hat{\boldsymbol{p}} - \boldsymbol{p}(t))\,; \tag{3.1}$$

(ii) for every $t \in [0,T]$:

$$\begin{aligned}
&\mathcal{E}(t, z, u(t), \boldsymbol{\varepsilon}(t), \boldsymbol{p}(t)) + \mathcal{V}([0,t]; z, \boldsymbol{p}(\cdot)) \\
&\quad= \int_0^t\int_\Omega \mathbb{C}(z)\boldsymbol{\varepsilon}(\tau)\cdot \mathrm{E}\dot{w}(\tau)\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t\int_\Omega \ell(z)\dot{f}(\tau)\cdot u(\tau)\,\mathrm{d}x\,\mathrm{d}\tau \\
&\qquad - \int_0^t\int_\Omega \ell(z)f(\tau)\cdot\dot{w}(\tau)\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t\int_{\Gamma_N} \dot{g}(\tau)\cdot u(\tau)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\tau \\
&\qquad - \int_0^t\int_{\Gamma_N} g(\tau)\cdot\dot{w}(\tau)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\tau\,.
\end{aligned} \tag{3.2}$$

As mentioned, Definition 3.1 falls within the class of *energetic formulations* for rate independent systems [25]. In particular, relation (3.1) is usually referred to as *global stability* and consists in a time dependent variational inequality. The scalar condition (3.2) is the energy balance: for all times $t \in [0,T]$ the sum of energy at time t and dissipated energy on $[0,t]$ (left-hand side of (3.2)) equals the initial energy (which is actualy 0 as we ask for $(u(0), \boldsymbol{\varepsilon}(0), \boldsymbol{p}(0)) = (0,0,0)$) plus the work supplied to the system by external actions (right-hand side in (3.2)).

Formulation (3.1)–(3.2) is particularly convenient when investigating asymptotics and is equivalent to the classical weak formulation of the quasistatic elastoplastic problem (1.1)–(1.3), as the general theory in Section 1.3.3 of [25] ensures. In particular, a trajectory $(u(\cdot), \boldsymbol{\varepsilon}(\cdot), \boldsymbol{p}(\cdot))$ fulfilling the initial condition is a quasistatic evolution in the sense of Definition 3.1 if and only if the following evolutionary variational equality holds

$$\int_\Omega \mathbb{C}(z)(\mathrm{E}u(t) - \boldsymbol{p}(t))\cdot \mathrm{E}v\,\mathrm{d}x = \int_\Omega \ell(z)f(t)\cdot v\,\mathrm{d}x + \int_{\Gamma_N} g(t)\cdot v\,\mathrm{d}\mathcal{H}^{n-1}$$

for all $v \in H^1(\Omega;\mathbb{R}^n)$ with $v = 0$ on Γ_D and all $t \in [0,T]$ and the flow rule (1.2) holds almost everywhere. This is nothing but the classical weak formulation of quasistatic elastoplasticity evolution, which is well-posed ([12], Thm. 7.3, p. 166). This in particular implies that, for every $z \in L^\infty(\Omega)$ there exists unique a quasistatic evolution $(u(\cdot), \boldsymbol{\varepsilon}(\cdot), \boldsymbol{p}(\cdot))$ in the sense of Definition 3.1. In fact, for such trajectory one can also check that $(u(\cdot), \boldsymbol{\varepsilon}(\cdot), \boldsymbol{p}(\cdot)) \in H^1(0,T; H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D))$, so that the dissipative term in the first line of (3.2) can be rewritten as

$$\mathcal{V}([0,T]; z, \boldsymbol{p}(\cdot)) = \int_0^T \int_\Omega d(z)\, |\dot{\boldsymbol{p}}(t)| \,\mathrm{d}x \,\mathrm{d}t\,.$$

We further note that the initial condition $(u(0), \boldsymbol{\varepsilon}(0), \boldsymbol{p}(0)) = (0,0,0)$ has been fixed in such a way that the elastoplastic body Ω is at equilibrium at time $t = 0$.

The TO problem consists in minimizing the compliance-type target functional $\mathcal{J}_\delta(z, u(\cdot))$ from (1.4) under the constraint that $u(\cdot)$ is the first component of the quasistatic evolution given z. In particular, we are interested in the following

$$\min \left\{ \mathcal{J}_\delta(z, u(\cdot)) : \; z \in H^1(\Omega) \text{ and } (u(\cdot), \boldsymbol{\varepsilon}(\cdot), \boldsymbol{p}(\cdot)) \text{ is a quasistatic evolution given } z \right\}. \tag{3.3}$$

The existence of an optimal phase-field z solving (3.3) can be proved by applying the Direct Method as we show in the next proposition.

Proposition 3.2 (Existence)**.** *The* TO *problem* (3.3) *admits a solution. In particular, every solution z satisfies $0 \le z \le 1$ almost everywhere in Ω.*

Proof. Note that $\mathcal{J}_\delta(z, u(\cdot)) > -\infty$ for all $z \in H^1(\Omega)$ and for the corresponding quasistatic evolution $(u(\cdot), \boldsymbol{\varepsilon}(\cdot), \boldsymbol{p}(\cdot))$. Let $z_j \in H^1(\Omega)$ be a minimizing sequence for (3.3). By the assumptions on $\mathbb{C}$, $\mathbb{H}$, d, and ℓ, we may assume without loss of generality that $z_j \in [0,1]$ almost everywhere, so that, up to a not relabeled subsequence, $z_j \rightharpoonup z$ weakly in $H^1(\Omega)$ and $0 \le z \le 1$ almost everywhere. Let us denote by $(u_j(\cdot), \boldsymbol{\varepsilon}_j(\cdot), \boldsymbol{p}_j(\cdot))$ the quasistatic evolution given z_j. In view of the energy balance (3.2) and of the hypotheses (2.3)–(2.5), $(u_j(\cdot), \boldsymbol{\varepsilon}_j(\cdot), \boldsymbol{p}_j(\cdot))$ is bounded in $L^\infty(0,T; H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D))$ and $\dot{\boldsymbol{p}}_j$ is bounded in $L^1(0,T; L^1(\Omega;\mathbb{M}^n_D))$. Therefore, by Helly's Selection Principle, $\boldsymbol{p}_j(t) \rightharpoonup \boldsymbol{p}(t)$ weakly in $L^2(\Omega;\mathbb{M}^n_D)$ for every $t \in [0,T]$ for some $\boldsymbol{p} \in L^\infty(0,T; L^2(\Omega;\mathbb{M}^n_D))$.

Let us fix $t \in [0,T]$. By the boundedness of $u_j(t)$ and of $\boldsymbol{\varepsilon}_j(t)$, we may assume that, up to a not relabeled subsequence, $u_j(t) \rightharpoonup u(t)$ weakly in $H^1(\Omega;\mathbb{R}^n)$ and $\boldsymbol{\varepsilon}_j(t) \rightharpoonup \boldsymbol{\varepsilon}(t)$ weakly in $L^2(\Omega;\mathbb{M}^n_S)$. For every $(\hat{u}, \hat{\boldsymbol{\varepsilon}}, \hat{\boldsymbol{p}}) \in \mathcal{A}(w(t))$, we test the equilibrium condition (3.1) for $(u_j(t), \boldsymbol{\varepsilon}_j(t), \boldsymbol{p}_j(t))$ by the triple

$$(\hat{u}_j(t), \hat{\boldsymbol{\varepsilon}}_j(t), \hat{\boldsymbol{p}}_j(t)) := (\hat{u} + u_j(t) - u(t), \hat{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}_j(t) - \boldsymbol{\varepsilon}(t), \hat{\boldsymbol{p}} + \boldsymbol{p}_j(t) - \boldsymbol{p}(t)) \in \mathcal{A}(w(t)).$$

We now exploit the quadratic character of $\mathcal{E}$ in order to pass to the limit as $j \to \infty$. In particular, we have that

$$\begin{aligned}
0 &\le \liminf_{j\to\infty} \big(\mathcal{E}(t, z_j, \hat{u}_j(t), \hat{\boldsymbol{\varepsilon}}_j(t), \hat{\boldsymbol{p}}_j(t)) - \mathcal{E}(t, z_j, u_j(t), \boldsymbol{\varepsilon}_j(t), \boldsymbol{p}_j(t)) + \mathcal{D}(z_j, \hat{\boldsymbol{p}}_j(t) - \boldsymbol{p}_j(t)) \big) \\
&= \liminf_{j\to\infty} \big(\mathcal{E}(t, z_j, \hat{u}_j(t), \hat{\boldsymbol{\varepsilon}}_j(t), \hat{\boldsymbol{p}}_j(t)) - \mathcal{E}(t, z_j, u_j(t), \boldsymbol{\varepsilon}_j(t), \boldsymbol{p}_j(t)) + \mathcal{D}(z_j, \hat{\boldsymbol{p}} - \boldsymbol{p}(t)) \big) \\
&= \liminf_{j\to\infty} \big(\mathcal{E}(t, z_j, \hat{u} + u_j(t) - u(t), \hat{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}_j(t) - \boldsymbol{\varepsilon}(t), \hat{\boldsymbol{p}} + \boldsymbol{p}_j(t) - \boldsymbol{p}(t)) \\
&\qquad - \mathcal{E}(t, z_j, u_j(t), \boldsymbol{\varepsilon}_j(t), \boldsymbol{p}_j(t)) + \mathcal{D}(z_j, \hat{\boldsymbol{p}} - \boldsymbol{p}(t)) \big) \\
&= \mathcal{E}(t, z, \hat{u}, \hat{\boldsymbol{\varepsilon}}, \hat{\boldsymbol{p}}) - \mathcal{E}(t, z, u(t), \boldsymbol{\varepsilon}(t), \boldsymbol{p}(t)) + \mathcal{D}(z, \hat{\boldsymbol{p}} - \boldsymbol{p}(t)).
\end{aligned}$$

This proves that $(u(t), \boldsymbol{\varepsilon}(t), \boldsymbol{p}(t))$ is the unique solution of (3.1). In particular, the whole sequence $(u_j(t), \boldsymbol{\varepsilon}_j(t), \boldsymbol{p}_j(t))$ converges to $(u(t), \boldsymbol{\varepsilon}(t), \boldsymbol{p}(t))$ weakly in $H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D)$. Moreover,

$(u_j, \boldsymbol{\varepsilon}_j, \boldsymbol{p}_j)$ converges weakly* in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))$ to $(u,\boldsymbol{\varepsilon},\boldsymbol{p})$. This last convergence implies that for every $t\in[0,T]$

$$\begin{aligned}\mathcal{E}(t,z,u(t),\boldsymbol{\varepsilon}(t),\boldsymbol{p}(t)) &+ \mathcal{V}([0,t];z,\boldsymbol{p}(\cdot))\\ &\le \int_0^t\int_\Omega \mathbb{C}(z)\boldsymbol{\varepsilon}(\tau)\cdot \mathrm{E}\dot w(\tau)\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t\int_\Omega \ell(z)\dot f(\tau)\cdot u(\tau)\,\mathrm{d}x\,\mathrm{d}\tau\\ &\quad - \int_0^t\int_\Omega \ell(z) f(\tau)\cdot \dot w(\tau)\,\mathrm{d}x\,\mathrm{d}\tau - \int_0^t\int_{\Gamma_N}\dot g(\tau)\cdot u(\tau)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\tau\\ &\quad - \int_0^t\int_{\Gamma_N} g(\tau)\cdot\dot w(\tau)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\tau\,.\end{aligned}$$

The opposite inequality can be recovered by exploiting the equilibrium condition (3.1) by applying ([24], Prop. 5.7). Hence, the triple $(u(\cdot),\boldsymbol{\varepsilon}(\cdot),\boldsymbol{p}(\cdot))$ is the unique quasistatic evolution given $z\in H^1(\Omega;[0,1])$. As the target functional $\mathcal{J}_\delta$ is lower semicontinuous, we deduce that z is a solution of (3.3).

Let us now prove the second part of the statement. Let $z\in H^1(\Omega)$ solve (3.3) and define $\hat z := \min\{1;\max\{z;0\}\}$. As the material-parameter functions on $\mathbb{C}$, $\mathbb{H}$, d, and ℓ are assumed to be constant on $(-\infty,0]$ and $[1,\infty)$, we have that $(u,\boldsymbol{\varepsilon},\boldsymbol{p})$ solves (1.1)–(1.3) given both z and $\hat z$. On the other hand if $z\neq\hat z$ one has that

$$\int_\Omega \frac{\delta}{2}|\nabla\hat z|^2 + \frac{\hat z^2(1-\hat z)^2}{2\delta}\mathrm{d}x < \int_\Omega \frac{\delta}{2}|\nabla z|^2 + \frac{z^2(1-z)^2}{2\delta}\mathrm{d}x,$$

contradicting the fact that z solves (3.3). □

The existence of solutions to (3.3) being proved, in the remainder of this section we focus on their approximation. At first, we discretize the quasistatic evolution constraint in time. Subsequenly, we regularize the flow rule. This will be instrumental to obtaining first-order optimality conditions, which we then tackle in Sections 5–6.

Let us hence start by a time discretization of the quasistatic evolution problem (see also [30, 31, 33]). Precisely, fixed $k\in\mathbb{N}$ and $\tau_k := T/k$, we define for $i=0,\dots,k$ the time nodes $t_i^k := i\tau_k$ and the functions

$$f_i^k := f(t_i^k)\,,\quad g_i^k := g(t_i^k)\,,\quad w_i^k := w(t_i^k)\,. \tag{3.4}$$

For later use, we further set for $t\in[t_{i-1}^k,t_i^k)$

$$\begin{aligned} &f_k(t) := f_{i-1}^k + \frac{t-t_{i-1}^k}{\tau_k}(f_i^k - f_{i-1}^k)\,, \qquad g_k(t) := g_{i-1}^k + \frac{t-t_{i-1}^k}{\tau_k}(g_i^k-g_{i-1}^k)\,,\\ &w_k(t) := w_{i-1}^k + \frac{t-t_{i-1}^k}{\tau_k}(w_i^k - w_{i-1}^k)\,.\end{aligned} \tag{3.5}$$

Notice that (f_k,g_k,w_k) converges to (f,g,w) in $H^1(0,T;L^2(\Omega;\mathbb{R}^n)\times L^2(\Gamma_N)\times H^1(\Omega;\mathbb{R}^n))$. We define the time-discrete energy functional

$$\mathcal{E}_k(t_i^k,z,u,\boldsymbol{\varepsilon},\boldsymbol{p}) := \frac12\int_\Omega \mathbb{C}(z)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\,\mathrm{d}x + \frac12\int_\Omega \mathbb{H}(z)\boldsymbol{p}\cdot\boldsymbol{p}\,\mathrm{d}x - \int_\Omega \ell(z) f_i^k\cdot u\,\mathrm{d}x - \int_{\Gamma_N} g_i^k\cdot u\,\mathrm{d}\mathcal{H}^{n-1}$$

and the discrete target functional

$$\begin{aligned}\mathcal{J}_{k,\delta}(z,(u_i)_{i=0}^k) := &\int_\Omega \ell(z)\, f_k^k \cdot u_k \,\mathrm{d}x + \int_{\Gamma_N} g_k^k \cdot u_k \,\mathrm{d}\mathcal{H}^{n-1} \\ &- \sum_{i=0}^{k-1}\Big(\int_\Omega \ell(z)\big(f_{i+1}^k - f_i^k\big)\cdot u_i\,\mathrm{d}x + \int_{\Gamma_N}\big(g_{i+1}^k - g_i^k\big)\cdot u_i\,\mathrm{d}\mathcal{H}^{n-1}\Big) \\ &+ \int_\Omega \frac{\delta}{2}|\nabla z|^2 + \frac{z^2(1-z)^2}{2\delta}\,\mathrm{d}x\,,\end{aligned} \tag{3.6}$$

for $z \in H^1(\Omega)\cap L^\infty(\Omega)$ and $(u_i)_{i=0}^k \in \big(H^1(\Omega;\mathbb{R}^n)\big)^{k+1}$. In the sequel, we will use a similar notation for $(\boldsymbol{\varepsilon}_i)_{i=0}^k \in \big(L^2(\Omega;\mathbb{M}^n_S)\big)^{k+1}$ and $(\boldsymbol{p}_i)_{i=0}^k \in \big(L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$

In the minimization of the time-discrete target functional $\mathcal{J}_{k,\delta}$ we ask the triple $(u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)_{i=0}^k \in \big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ to be a time-discrete quasistatic evolution given z, whose definition is given here below.

Definition 3.3 (Time-discrete quasistatic evolution given z)**.** Let $z\in L^\infty(\Omega)$ be given and f_i^k, g_i^k, w_i^k be defined as in (3.4). We say that $(u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)_{i=0}^k \in \big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ is a *time-discrete quasistatic evolution given* z if $(u_0,\boldsymbol{\varepsilon}_0,\boldsymbol{p}_0)=(0,0,0)$ and the following holds: for every $i=1,\dots,k$, $(u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)\in\mathcal{A}(w_i^k)$ and

$$\mathcal{E}_k(t_i^k,z,u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)+\mathcal{D}(z,\boldsymbol{p}_i-\boldsymbol{p}_{i-1}) \le \mathcal{E}_k(t_i^k,z,\hat u,\hat{\boldsymbol{\varepsilon}},\hat{\boldsymbol{p}})+\mathcal{D}(z,\hat{\boldsymbol{p}}-\boldsymbol{p}_{i-1}) \tag{3.7}$$

for every $(\hat u,\hat{\boldsymbol{\varepsilon}},\hat{\boldsymbol{p}})\in\mathcal{A}(w_i^k)$.

As a consequence of (3.7) we have that every time-discrete quasistatic evolution $(u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)_{i=0}^k$ satisfies the following energy inequality: for every $i=1,\dots,k$

$$\begin{aligned}&\mathcal{E}_k(t_i^k,z,u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)+\sum_{j=1}^i\mathcal{D}(z,\boldsymbol{p}_j-\boldsymbol{p}_{j-1}) \\ &\le \sum_{j=1}^i \int_\Omega \mathbb{C}(z)\boldsymbol{\varepsilon}_{j-1}\cdot \mathrm{E}(w_j^k-w_{j-1}^k)\,\mathrm{d}x - \int_\Omega \ell(z)(f_j^k-f_{j-1}^k)\cdot u_{j-1}\,\mathrm{d}x \\ &\quad - \int_\Omega \ell(z) f_j\cdot(w_j^k-w_{j-1}^k)\,\mathrm{d}x + \int_{\Gamma_N}(g_j^k-g_{j-1}^k)\cdot u_{j-1}\,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad + \int_{\Gamma_N} g_j^k\cdot(w_j^k-w_{j-1}^k)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Omega \mathbb{C}(z)\mathrm{E}(w_j^k-w_{j-1}^k)\cdot\mathrm{E}(w_j^k-w_{j-1}^k)\,\mathrm{d}x\,.\end{aligned} \tag{3.8}$$

Furthermore, we note that a time-discrete quasistatic evolution can always be constructed by iteratively solving the minimum problems

$$\min\{\mathcal{E}_k(t_i^k,z,u,\boldsymbol{\varepsilon},\boldsymbol{p})+\mathcal{D}(z,\boldsymbol{p}-\boldsymbol{p}_{i-1})\,:\,(u,\boldsymbol{\varepsilon},\boldsymbol{p})\in\mathcal{A}(w_i^k)\}\,, \tag{3.9}$$

for $i\ge 1$, where we have set $(u_0,\boldsymbol{\varepsilon}_0,\boldsymbol{p}_0)=(0,0,0)$. In particular, given the data f, g, and w, the time-discrete quasistatic evolution is unique, as the solution of the minimum problem (3.9) is unique.

The time-discrete TO problem reads as

$$\begin{aligned}\min\{\mathcal{J}_{k,\delta}(z,(u_i)_{i=0}^k) : z\in H^1(\Omega) \text{ and } &(u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)_{i=0}^k \\ &\text{is a time-discrete quasistatic evolution given } z\}\,.\end{aligned} \tag{3.10}$$

Proposition 3.4 (Existence, time-discrete)**.** *The time-discrete* TO *problem* (3.10) *admits a solution. In particular, every solution z satisfies $0 \leq z \leq 1$ almost everywhere in Ω.*

A proof of this proposition can be obtained by the Direct Method, by following the argument of Proposition 3.2. In the interest of shortness, we omit the details.

In the following proposition, we state an auxiliary result regarding the convergence of a sequence of time-discrete quasistatic evolutions to a quasistatic evolution. The proof is provided in Appendix A. Such a result will be used to show that a sequence of minimizers of (3.10) converges to a minimizer of the time-continuous problem (3.3) as the time-step τ_k tends to 0, as well as to obtain suitable first-order optimality conditions for (3.3), starting from those of (3.10) (see Cor. 3.6 and Thms. 6.1 and 6.4, respectively).

Proposition 3.5 (Convergence of time-discrete quasistatic evolutions)**.** *Let $z_k, z \in H^1(\Omega;[0,1])$ be such that $z_k \rightharpoonup z$ weakly in $H^1(\Omega)$. For every k, let $(u^k_i, \boldsymbol{\varepsilon}^k_i, \boldsymbol{p}^k_i)^k_{i=0}$ be the time-discrete quasistatic evolution associated with z_k and let $(u(\cdot), \boldsymbol{\varepsilon}(\cdot), \boldsymbol{p}(\cdot))$ be the quasistatic evolution associated with z according to Definition 3.1. Let us further set*

$$
\begin{aligned}
&u_k(t) := u^k_i + \frac{t - t^k_i}{\tau_k}(u^k_{i+1} - u^k_i)\,, \qquad \boldsymbol{\varepsilon}_k(t) := \boldsymbol{\varepsilon}^k_i + \frac{t - t^k_i}{\tau_k}(\boldsymbol{\varepsilon}^k_{i+1} - \boldsymbol{\varepsilon}^k_i)\,,\\
&\boldsymbol{p}_k(t) := \boldsymbol{p}^k_i + \frac{t - t^k_i}{\tau_k}(\boldsymbol{p}^k_{i+1} - \boldsymbol{p}^k_i)\,.
\end{aligned}
\tag{3.11}
$$

Then, $(u_k, \boldsymbol{\varepsilon}_k, \boldsymbol{p}_k)$ converges to $(u, \boldsymbol{\varepsilon}, \boldsymbol{p})$ in $H^1(0,T; H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D))$.

Proof. See Appendix A. □

As a corollary of Proposition 3.5 we infer the convergence of time-discrete TO minimizers of (3.10) to time-continuous TO minimizers of (3.3).

Corollary 3.6 (Convergence of time-discrete TO minimizers)**.** *Under the assumptions of Proposition 3.5, let $z_k \in H^1(\Omega;[0,1])$ be a sequence of minimizers of* (3.10)*. Then, there exists $z \in H^1(\Omega;[0,1])$ solution of* (3.3) *such that, up to a subsequence, $z_k \rightharpoonup z$ weakly in $H^1(\Omega)$ as the corresponding time step τ_k converges to 0.*

Proof. By inequality (3.8), by the assumptions (2.4)–(2.5), and by the regularity of f, g, and w, the time-discrete evolutions are bounded uniformly with respect to $k \in \mathbb{N}$. Hence, we deduce from minimality (3.9) of z_k that z_k is bounded in $H^1(\Omega)$. Up to a subsequence, $z_k \rightharpoonup z$ weakly in $H^1(\Omega)$ and $z \in H^1(\Omega;[0,1])$. In view of Proposition 3.5, the time-discrete quasistatic evolution associated with z_k converges to $(u, \boldsymbol{\varepsilon}, \boldsymbol{p})$ in $H^1(0,T; H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D))$, where $(u, \boldsymbol{\varepsilon}, \boldsymbol{p})$ is the quasistatic evolution associated with z.

To show the minimality of z, let us fix a competitor $\hat{z} \in H^1(\Omega;[0,1])$ and consider the quasistatic evolution $(\hat{u}, \hat{\boldsymbol{\varepsilon}}, \hat{\boldsymbol{p}})$ associated with $\hat{z}$. For every k, we can construct the time-discrete quasistatic evolution associated with $\hat{z}$ according to Definition 3.3. Let us denote by $(\hat{u}_k, \hat{\boldsymbol{\varepsilon}}_k, \hat{\boldsymbol{p}}_k)$ the piecewise affine functions

$$
\begin{aligned}
&\hat{u}_k(t) := \hat{u}^k_{i-1} + \frac{(t - t^k_{i-1})}{\tau_k}(\hat{u}^k_i - \hat{u}^k_{i-1})\,, \qquad \hat{\boldsymbol{\varepsilon}}_k(t) := \hat{\boldsymbol{\varepsilon}}^k_{i-1} + \frac{(t - t^k_{i-1})}{\tau_k}(\hat{\boldsymbol{\varepsilon}}^k_i - \hat{\boldsymbol{\varepsilon}}^k_{i-1})\,,\\
&\hat{\boldsymbol{p}}_k(t) := \hat{\boldsymbol{p}}^k_{i-1} + \frac{(t - t^k_{i-1})}{\tau_k}(\hat{\boldsymbol{p}}^k_i - \hat{\boldsymbol{p}}^k_{i-1})\,,
\end{aligned}
$$

for $t \in [t^k_{i-1}, t^k_i)$. In view of Proposition 3.5, we have that $(\hat{u}_k, \hat{\boldsymbol{\varepsilon}}_k, \hat{\boldsymbol{p}}_k)$ converges to $(\hat{u}, \hat{\boldsymbol{\varepsilon}}, \hat{\boldsymbol{p}})$ in $H^1(0,T; H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D))$. By the minimality of z_k we have that

$$
\mathcal{J}_{k,\delta}(z_k, (u^k_i)^k_{i=0}) \leq \mathcal{J}_{k,\delta}(\hat{z}, (\hat{u}^k_i)^k_{i=0})\,. \tag{3.12}
$$

Hence, passing to the liminf in (3.12) as $k \to \infty$ we deduce that

$$\mathcal{J}_\delta(z,u) \le \liminf_{k\to\infty} \mathcal{J}_{k,\delta}(z_k,(u_i^k)_{i=0}^k) \le \liminf_{k\to\infty} \mathcal{J}_{k,\delta}(\hat z,(\hat u_i^k)_{i=0}^k) = \mathcal{J}_\delta(\hat z,\hat u)\,.$$

Note that the last equality follows from the above mentioned strong convergence $\hat u_k \to \hat u$ in $H^1(0,T;H^1(\Omega;\mathbb{R}^n))$. We conclude by the arbitrariness of $\hat z \in H^1(\Omega;[0,1])$. □

For the computation of the first-order optimality conditions for (3.3), the time-discrete approximation introduced in (3.6)–(3.10) is still insufficient, as the dissipation term $\mathcal{D}$ is not differentiable. As in [2] (see also [10, 16, 34]), we define the regularized dissipation

$$\mathcal{D}_\gamma(z,\boldsymbol{q}) := \int_\Omega d(z) h_\gamma(\boldsymbol{q})\,\mathrm{d}x\,, \qquad \text{for every } \boldsymbol{q} \in L^2(\Omega;\mathbb{M}^n_D)\,,$$
$$h_\gamma(\mathbf{Q}) := \sqrt{|\mathbf{Q}|^2 + \frac{1}{\gamma^2}} - \frac{1}{\gamma} \qquad \text{for every } \mathbf{Q} \in \mathbb{M}^n_D\,.$$

In particular, $h_\gamma \in C^\infty(\mathbb{M}^n_D)$ is convex and satisfies

$$|h_\gamma(\mathbf{Q}_1) - h_\gamma(\mathbf{Q}_2)| \le |\mathbf{Q}_1 - \mathbf{Q}_2|\,, \tag{3.13}$$
$$|\nabla_{\mathbf{Q}} h_\gamma(\mathbf{Q}_1) - \nabla_{\mathbf{Q}} h_\gamma(\mathbf{Q}_2)| \le 2\gamma|\mathbf{Q}_1 - \mathbf{Q}_2|\,. \tag{3.14}$$

Accordingly, we formulate the concept of approximate time-discrete quasistatic evolution as follows.

Definition 3.7 (Approximate time-discrete quasistatic evolution given z)**.** Let $z \in L^\infty(\Omega)$ be given and let f_i^k, g_i^k, w_i^k be defined as in (3.4). We say that $(u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)_{i=0}^k \in \big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ is an *approximate time-discrete quasistatic evolution* if $(u_0,\boldsymbol{\varepsilon}_0,\boldsymbol{p}_0) = (0,0,0)$ and the following holds: for every $i = 1,\dots,k$, $(u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i) \in \mathcal{A}(w_i^k)$ and

$$\mathcal{E}_k(t_i^k,z,u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i) + \mathcal{D}_\gamma(z,\boldsymbol{p}_i - \boldsymbol{p}_{i-1}) \le \mathcal{E}_k(t_i^k,z,u,\boldsymbol{\varepsilon},\boldsymbol{p}) + \mathcal{D}_\gamma(z,\boldsymbol{p} - \boldsymbol{p}_{i-1}) \tag{3.15}$$

for every $(u,\boldsymbol{\varepsilon},\boldsymbol{p}) \in \mathcal{A}(w_i^k)$.

As for a time-discrete quasistatic evolutions, for every $z \in L^\infty(\Omega)$ and every $k \in \mathbb{N}$ an approximate time-discrete evolution $(u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)_{i=0}^k$ is uniquely determined by iteratively solving the minimum problems

$$\min\,\{\mathcal{E}_k(t_i^k,z,u,\boldsymbol{\varepsilon},\boldsymbol{p}) + \mathcal{D}_\gamma(z,\boldsymbol{p}-\boldsymbol{p}_{i-1}) :\ (u,\boldsymbol{\varepsilon},\boldsymbol{p}) \in \mathcal{A}(w_i^k)\} \tag{3.16}$$

for $i \ge 1$, where we have set $(u_0,\boldsymbol{\varepsilon}_0,\boldsymbol{p}_0) = (0,0,0)$. The approximate time-discrete TO problem reads as

$$\begin{aligned}\min\,\{\mathcal{J}_{k,\delta}(z,(u_i)_{i=0}^k) :\ & z \in H^1(\Omega)\\ & \text{and } (u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)_{i=0}^k \in \big(H^1(\Omega,\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}\\ & \text{is an approximate time-discrete quasistatic evolution given } z\}\,.\end{aligned} \tag{3.17}$$

Proposition 3.8 (Existence, approximate time-discrete)**.** *The approximate time-discrete* TO *problem* (3.17) *admits a solution. In particular, every solution z satisfies $0 \le z \le 1$ almost everywhere in Ω.*

Proof. Repeat the steps of the proof of Proposition 3.4 taking into account minimality (3.16). □

We aim now at showing the convergence of solutions to the approximate time-discrete TO problem 3.17 to solutions of the time-discrete TO problem (3.10). To this end, we first have to discuss the convergence of approximate time-discrete quasistatic evolutions to a time-discrete quasistatic evolution as the regularization parameter γ tends to $+\infty$. This is the subject of the following proposition.

Proposition 3.9 (Convergence of approximate time-discrete quasistatic evolutions)**.** *Let $k \in \mathbb{N}$ be fixed and let $z_\gamma, z \in H^1(\Omega;[0,1])$ be such that $z_\gamma \rightharpoonup z$ weakly in $H^1(\Omega)$ as $\gamma \to +\infty$. Let us denote by $(u_i^\gamma, \boldsymbol{\varepsilon}_i^\gamma, \boldsymbol{p}_i^\gamma)_{i=0}^k \in \big(H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ the approximate time-discrete quasistatic evolution associated with z_γ and by $(u_i, \boldsymbol{\varepsilon}_i, \boldsymbol{p}_i)_{i=0}^k \in \big(H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ the time-discrete quasistatic evolution associated with z. Then, $(u_i^\gamma, \boldsymbol{\varepsilon}_i^\gamma, \boldsymbol{p}_i^\gamma)_{i=0}^k$ converges to $(u_i, \boldsymbol{\varepsilon}_i, \boldsymbol{p}_i)_{i=0}^k$ in $\big(H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ as $\gamma \to +\infty$.*

Proof. By minimality of $(u_i^\gamma, \boldsymbol{\varepsilon}_i^\gamma, \boldsymbol{p}_i^\gamma)$ we have that

$$
\begin{aligned}
&\mathcal{E}_k(t_i^k, z_\gamma, u_i^\gamma, \boldsymbol{\varepsilon}_i^\gamma, \boldsymbol{p}_i^\gamma) + \mathcal{D}_\gamma(z_\gamma, \boldsymbol{p}_i^\gamma - \boldsymbol{p}_{i-1}^\gamma) \\
&\leq \mathcal{E}_k(t_i^k, z_\gamma, u_{i-1}^\gamma + w_i^k - w_{i-1}^k, \boldsymbol{\varepsilon}_{i-1}^\gamma + \mathrm{E}w_i^k - \mathrm{E}w_{i-1}^k, \boldsymbol{p}_{i-1}^\gamma) \\
&= \mathcal{E}_k(t_{i-1}^k, z_\gamma, u_{i-1}^\gamma, \boldsymbol{\varepsilon}_{i-1}^\gamma, \boldsymbol{p}_{i-1}^\gamma) + \int_\Omega \mathbb{C}(z_\gamma)\boldsymbol{\varepsilon}_{i-1}^\gamma \cdot \mathrm{E}(w_i^k - w_{i-1}^k)\,\mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega \mathbb{C}(z_\gamma)\mathrm{E}(w_i^k - w_{i-1}^k)\cdot \mathrm{E}(w_i^k - w_{i-1}^k)\,\mathrm{d}x \\
&\quad - \int_\Omega \ell(z_\gamma)(f_i^k - f_{i-1}^k)\cdot u_{i-1}^\gamma\,\mathrm{d}x - \int_\Omega \ell(z_\gamma) f_i^k \cdot (w_i^k - w_{i-1}^k)\,\mathrm{d}x \\
&\quad - \int_{\Gamma_N} (g_i^k - g_{i-1}^k)\cdot u_{i-1}^\gamma\,\mathrm{d}\mathcal{H}^{n-1} - \int_{\Gamma_N} g_i^k \cdot (w_i^k - w_{i-1}^k)\,\mathrm{d}\mathcal{H}^{n-1}
\end{aligned}
\tag{3.18}
$$

Adding the term $\mathcal{D}_\gamma(z_\gamma, \boldsymbol{p}_{i-1}^\gamma - \boldsymbol{p}_{i-2}^\gamma)$ to both sides of (3.18) and repeating the previous argument for every i, we deduce that

$$
\begin{aligned}
&\mathcal{E}_k(t_i^k, z_\gamma, u_i^\gamma, \boldsymbol{\varepsilon}_i^\gamma, \boldsymbol{p}_i^\gamma) + \sum_{j=1}^i \mathcal{D}_\gamma(z_\gamma, \boldsymbol{p}_j^\gamma - \boldsymbol{p}_{j-1}^\gamma) \\
&\leq \sum_{j=1}^i \int_\Omega \mathbb{C}(z_\gamma)\boldsymbol{\varepsilon}_{j-1}^\gamma \cdot \mathrm{E}(w_j^k - w_{j-1}^k)\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathbb{C}(z_\gamma)\mathrm{E}(w_j^k - w_{j-1}^k)\cdot \mathrm{E}(w_j^k - w_{j-1}^k)\,\mathrm{d}x \\
&\quad - \int_\Omega \ell(z_\gamma)(f_j^k - f_{j-1}^k)\cdot u_{j-1}^\gamma\,\mathrm{d}x - \int_\Omega \ell(z_\gamma) f_j^k \cdot (w_j^k - w_{j-1}^k)\,\mathrm{d}x \\
&\quad - \int_{\Gamma_N} (g_j^k - g_{j-1}^k)\cdot u_{j-1}^\gamma\,\mathrm{d}\mathcal{H}^{n-1} - \int_{\Gamma_N} g_j^k \cdot (w_j^k - w_{j-1}^k)\,\mathrm{d}\mathcal{H}^{n-1}\,,
\end{aligned}
\tag{3.19}
$$

which implies that $(u_i^\gamma, \boldsymbol{\varepsilon}_i^\gamma, \boldsymbol{p}_i^\gamma)$ is bounded in $H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D)$ uniformly w.r.t. i and γ. Arguing as in Proposition 3.4, we can prove recursively that $(u_i^\gamma, \boldsymbol{\varepsilon}_i^\gamma, \boldsymbol{p}_i^\gamma)$ converges to $(u_i, \boldsymbol{\varepsilon}_i, \boldsymbol{p}_i)$ in $H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D)$ as $\gamma \to +\infty$, and $(u_i, \boldsymbol{\varepsilon}_i, \boldsymbol{p}_i)$ satisfies (3.7) for $i = 1, \dots, k$. This concludes the proof of the proposition. □

As a corollary of Proposition 3.9 we obtain the convergence of solutions of the approximate time-discrete TO problem (3.17) to solutions of the time-discrete TO problem (3.10).

Corollary 3.10 (Convergence of approximate time-discrete TO minimizers)**.** *Let $k \in \mathbb{N}$ and let $z_\gamma \in H^1(\Omega;[0,1])$ be a sequence of solutions of (3.17). Then, there exists $z \in H^1(\Omega;[0,1])$ solution of (3.10) such that, up to a subsequence, $z_\gamma \rightharpoonup z$ weakly in $H^1(\Omega)$.*

Proof. Repeating the argument of (3.19), we infer that the approximate time-discrete quasistatic evolution $(u_i^\gamma, \boldsymbol{\varepsilon}_i^\gamma, \boldsymbol{p}_i^\gamma)_{i=0}^k$ corresponding to z_γ is bounded in $\big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$. By minimality, also z_γ is bounded in $H^1(\Omega)$ and weakly converges to some z in $H^1(\Omega)$, with $0\le z\le 1$ almost everywhere. By Proposition 3.9, we have that $(u_i^\gamma, \boldsymbol{\varepsilon}_i^\gamma, \boldsymbol{p}_i^\gamma)_{i=0}^k \to (u_i, \boldsymbol{\varepsilon}_i, \boldsymbol{p}_i)$ in $H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)$ as $\gamma\to+\infty$, where $(u_i, \boldsymbol{\varepsilon}_i, \boldsymbol{p}_i)_{i=0}^k$ is the time-discrete quasistatic evolution corresponding to z. From the lower semicontinuity of $\mathcal{J}_{k,\delta}$ and from Proposition 3.9 we also deduce that z solves (3.10). □

4. Sharp-interface limit $\delta\to 0$

We prove in this section that the sharp-interface limit $\delta\to 0$ can be rigorously ascertained. This check is performed below in the time-continuous case of quasistatic evolutions. An analogous argument could be developed in the case of time-discrete and approximate time-discrete quasistatic evolutions.

Let us start by recording that the set of quasistatic evolution is closed with respect to the convergence of the phase field.

Proposition 4.1 (Convergence of quasistatic evolutions)**.** *Let $z_m, z\in L^\infty(\Omega;[0,1])$ be such that $z_m\to z$ strongly in $L^1(\Omega)$. For every m, let $(u_m(\cdot), \boldsymbol{\varepsilon}_m(\cdot), \boldsymbol{p}_m(\cdot))$ be the quasistatic evolution associated with z_m and let $(u(\cdot), \boldsymbol{\varepsilon}(\cdot), \boldsymbol{p}(\cdot))$ be the quasistatic evolution associated with z. Then, $(u_m, \boldsymbol{\varepsilon}_m, \boldsymbol{p}_m)$ converges to $(u, \boldsymbol{\varepsilon}, \boldsymbol{p})$ in $H^1(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))$.*

Proof. The argument follows closely the general approximation tool from [26]. The coercivity of the energy, which is independent of z_m, and an application of the Helly Selection principle entails that, up to not relabeled subsequences

$$(u_m(t), \boldsymbol{\varepsilon}_m(t), \boldsymbol{p}_m(t)) \rightharpoonup (u(t), \boldsymbol{\varepsilon}(t), \boldsymbol{p}(t)) \ \text{ in } \ H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)$$

for all times $t\in[0,T]$. This suffices to check that

$$\mathcal{E}(t,z,u(t),\boldsymbol{\varepsilon}(t),\boldsymbol{p}(t)) \le \liminf_{m\to\infty} \mathcal{E}(t,z_m,u_m(t),\boldsymbol{\varepsilon}_m(t),\boldsymbol{p}_m(t))\,, \tag{4.1}$$

$$\mathcal{V}([0,t];z,\boldsymbol{p}(\cdot)) \le \liminf_{m\to\infty} \mathcal{V}([0,t];z_m,\boldsymbol{p}_m(\cdot))\,, \tag{4.2}$$

which follow by lower semicontinuity. In particular, we have used the fact that $\mathbb{C}(z_m)\to\mathbb{C}(z)$ and $\mathbb{H}(z_m)\to\mathbb{H}(z)$ strongly in $L^q(\Omega;\mathbb{R}^{n\times n\times n\times n})$ for all $q<+\infty$. As for the dissipation part, letting $\{0=t_0<t_1<\cdots<t_M=t\}$ be an arbitrary partition of $[0,t]$ we compute

$$\sum_{j=1}^M \mathcal{D}(z,\boldsymbol{p}(t_j)-\boldsymbol{p}(t_{j-1})) \le \liminf_{m\to\infty}\sum_{j=1}^M \mathcal{D}(z_m,\boldsymbol{p}_m(t_j)-\boldsymbol{p}_m(t_{j-1})) \le \liminf_{m\to\infty}\mathcal{V}([0,t];z_m,\boldsymbol{p}_m(\cdot)).$$

We hence conclude for (4.2) by taking the supremum over all partitions of $[0,t]$.

On the other hand, given any $(\hat u, \hat{\boldsymbol{\varepsilon}}, \hat{\boldsymbol{p}})\in\mathcal{A}(w(t))$, by defining the *mutual recovery sequence*

$$(\hat u_m, \hat{\boldsymbol{\varepsilon}}_m, \hat{\boldsymbol{p}}_m) = (\hat u + u(t) - u_m(t), \hat{\boldsymbol{\varepsilon}} + \boldsymbol{\varepsilon}(t) - \boldsymbol{\varepsilon}_m(t), \hat{\boldsymbol{p}} + \boldsymbol{p}(t) - \boldsymbol{p}_m(t))$$

and exploiting the quadratic character of $\mathcal{E}$ one can check that

$$\begin{aligned}&\limsup_{m\to\infty}\Big(\mathcal{E}(t,z_m,\hat u_m,\hat{\boldsymbol{\varepsilon}}_m,\hat{\boldsymbol{p}}_m) - \mathcal{E}(t,z_k,u_m(t),\boldsymbol{\varepsilon}_m(t),\boldsymbol{p}_m(t)) + \mathcal{D}(z_m,\hat{\boldsymbol{p}}_m-\boldsymbol{p}_m(t))\Big) \\ &\quad\le \Big(\mathcal{E}(t,z,\hat u,\hat{\boldsymbol{\varepsilon}},\hat{\boldsymbol{p}}) - \mathcal{E}(t,z,u(t),\boldsymbol{\varepsilon}(t),\boldsymbol{p}(t)) + \mathcal{D}(z,\hat{\boldsymbol{p}}-\boldsymbol{p}(t))\Big)\,.\end{aligned} \tag{4.3}$$

Properties (4.1)–(4.3) allow to apply ([26], Thm. 3.1) ensuring that $(u(\cdot), \boldsymbol{\varepsilon}(\cdot), \boldsymbol{p}(\cdot))$ is the quasistatic evolution associated with z, as well as

$$\mathcal{E}(t, z_m, u_m(t), \boldsymbol{\varepsilon}_m(t), \boldsymbol{p}_m(t)) \to \mathcal{E}(t, z, u(t), \boldsymbol{\varepsilon}(t), \boldsymbol{p}(t))$$

for all times. The latter entails that the pointwise convergence in $H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^n_S) \times L^2(\Omega; \mathbb{M}^n_D)$ is strong. This can be further improved to a strong convergence in $H^1(0, T; H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^n_S) \times L^2(\Omega; \mathbb{M}^n_D))$ by repeating the argument of Proposition 3.5, see Appendix A. $\square$

In order to discuss the sharp-interface limit $\delta \to 0$ we start by defining the *sharp-interface* target functional

$$\begin{aligned}\mathcal{J}_0(z, u) := &\int_\Omega \ell(z)\, f(T) \cdot u(T)\, \mathrm{d}x + \int_{\Gamma_N} g(T) \cdot u(T)\, \mathrm{d}\mathcal{H}^{n-1} \\ &- \int_0^T \int_\Omega \ell(z)\, \dot f(\tau) \cdot u(\tau)\, \mathrm{d}x\, \mathrm{d}\tau - \int_0^T \int_{\Gamma_N} \dot g(\tau) \cdot u(\tau)\, \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}\tau + \frac{1}{6} \mathrm{Per}(\{z = 1\}; \Omega)\end{aligned}$$

where now the phase z is assumed to belong to $\mathrm{BV}(\Omega)$ and take values in $\{0, 1\}$ only. The term $\mathrm{Per}(\{z = 1\}; \Omega)$ is the perimeter in Ω of the set $\{z = 1\}$ and effectively penalizes phases with large boundaries. The constant $1/6$ has no physical relevance and is just chosen to simplify notations. Indeed, setting a different constant here will be possible. Correspondingly, the *sharp-interface* TO problem reads

$$\begin{aligned}\min \big\{ \mathcal{J}_0(z, u(\cdot)) : z \in BV(\Omega; \{0, 1\}), \\ \text{and } (u(\cdot), \boldsymbol{\varepsilon}(\cdot), \boldsymbol{p}(\cdot)) \text{ is a quasistatic evolution given } z \big\}. \end{aligned} \tag{4.4}$$

The main result of this section is the following convergence.

Proposition 4.2 (Sharp-interface limit of TO minimizers)**.** *Let $z_\delta \in H^1(\Omega; [0, 1])$ solve the* TO *problem* (3.3)*. Then, up to a not relabeled subsequence, $z_\delta \to z$ strongly in $L^1(\Omega)$, where z solves the sharp-interface* TO *problem* (4.4).

Proof. The statement follows by combining the stability of Proposition 4.1 with the classical Modica-Mortola construction [27].

Let $(u_\delta, \boldsymbol{\varepsilon}_\delta, \boldsymbol{p}_\delta)$ and $(u^0, \boldsymbol{\varepsilon}^0, \boldsymbol{p}^0)$ be the quasistatic evolutions associated to z_δ and $z = 0$, respectively. From minimality we deduce that

$$\begin{aligned}\mathcal{J}_\delta(z_\delta, u_\delta) \le \mathcal{J}_\delta(0, u^0) = &\int_\Omega \ell(0)\, f(T) \cdot u^0(T)\, \mathrm{d}x + \int_{\Gamma_N} g(T) \cdot u^0(T)\, \mathrm{d}\mathcal{H}^{n-1} \\ &- \int_0^T \int_\Omega \ell(0)\, \dot f(\tau) \cdot u^0(\tau)\, \mathrm{d}x\, \mathrm{d}\tau - \int_0^T \int_{\Gamma_N} \dot g(\tau) \cdot u^0(\tau)\, \mathrm{d}\mathcal{H}^{n-1} \mathrm{d}\tau < +\infty.\end{aligned}$$

As $\sup_\delta \mathcal{J}_\delta(z_\delta, u_\delta) < +\infty$ one can extract a not relabeled subsequence such that $z_\delta \rightharpoonup z$ weakly in $BV(\Omega)$ and strongly in $L^1(\Omega)$. Owing to Proposition 4.1 we hence have that $(u_\delta, \boldsymbol{\varepsilon}_\delta, \boldsymbol{p}_\delta)$ converges to $(u, \boldsymbol{\varepsilon}, \boldsymbol{p})$ strongly in $H^1(0, T; H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^n_S) \times L^2(\Omega; \mathbb{M}^n_D))$ where $(u, \boldsymbol{\varepsilon}, \boldsymbol{p})$ is the quasistatic evolution given z. We can hence use the fact that [27]

$$\frac{1}{6} \mathrm{Per}(\{z = 1\}; \Omega) \le \liminf_{\delta \to 0} \int_\Omega \frac{\delta}{2} |\nabla z_\delta|^2 + \frac{z_\delta^2 (1 - z_\delta)^2}{2\delta}\, \mathrm{d}x$$

in order to check that

$$\mathcal{J}_0(z,u) \le \liminf_{\delta\to 0} \mathcal{J}_\delta(z_\delta,u_\delta)\,. \tag{4.5}$$

In order to prove that z actually solves the sharp-interface TO problem (4.4), let $\hat z \in BV(\Omega;\{0,1\})$ be given and let $\hat z_\delta \in H^1(\Omega)$ be the corresponding Modica-Mortola recovery sequence from [27]. This fulfills

$$\hat z_\delta \to \hat z \;\text{ strongly in }\; L^1(\Omega) \;\text{ and }\; \liminf_{\delta\to 0}\int_\Omega \frac{\delta}{2}|\nabla \hat z_\delta|^2 + \frac{\hat z_\delta^2(1-\hat z_\delta)^2}{2\delta}\,\mathrm{d}x = \frac{1}{6}\mathrm{Per}(\{\hat z = 1\};\Omega)\,. \tag{4.6}$$

Let now $(\hat u_\delta, \hat{\boldsymbol{\varepsilon}}_\delta, \hat{\boldsymbol{p}}_\delta)$ be the quasistatic evolution given $\hat z_\delta$ and use again Proposition 4.1 in order to check that $\hat u_\delta \to \hat u$ in $H^1(0,T;H^1(\Omega;\mathbb{R}^n))$ where $(\hat u, \hat{\boldsymbol{\varepsilon}}, \hat{\boldsymbol{p}})$ is the quasistatic evolution given $\hat z$. We can hence use convergence (4.6) in order to get that

$$\lim_{\delta\to 0} \mathcal{J}_\delta(\hat z_\delta, \hat u_\delta) = \mathcal{J}_0(\hat z, \hat u)\,. \tag{4.7}$$

By combining (4.5) and (4.7) we have that z solves the sharp-interface TO problem. □

5. Differentiability of the state operator for $\gamma < +\infty$

In preparation for obtaining first-order optimality conditions in Section 6, we develop here the analysis of the control-to-state operator $S_{k,\gamma}\colon L^\infty(\Omega) \to \big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega,\mathbb{M}^n_D)\big)^{k+1}$. For fixed $\gamma \in (0,+\infty)$ and $k\in\mathbb{N}$, the operator $S_{k,\gamma}$ maps a control $z\in L^\infty(\Omega)$ in the unique corresponding approximate time-discrete quasistatic evolution $(u_i^k, \boldsymbol{\varepsilon}_i^k, \boldsymbol{p}_i^k)_{i=0}^k$. The differentiability result is stated in Theorem 5.1. For this statement, an auxiliary functional has to be introduced. For every $i = 1,\ldots,k$, every $(v,\boldsymbol{\eta},\boldsymbol{q}) \in H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)$, every $\boldsymbol{p}\in L^2(\Omega;\mathbb{M}^n_D)$, and every $\varphi\in L^\infty(\Omega)$, we define the functional

$$\begin{aligned}\mathcal{F}^{\boldsymbol{p}}_\gamma(t_i^k, z, \varphi, v, \boldsymbol{\eta}, \boldsymbol{q}) := {}& \frac12\int_\Omega \mathbb{C}(z)\boldsymbol{\eta}\cdot\boldsymbol{\eta}\,\mathrm{d}x + \frac12\int_\Omega \mathbb{H}(z)\boldsymbol{q}\cdot\boldsymbol{q}\,\mathrm{d}x\\ &+ \int_\Omega (\mathbb{C}'(z)\varphi)\boldsymbol{\varepsilon}_i^k\cdot\boldsymbol{\eta}\,\mathrm{d}x + \int_\Omega (\mathbb{H}'(z)\varphi)\boldsymbol{p}_i^k\cdot\boldsymbol{q}\,\mathrm{d}x\\ &+ \int_\Omega \varphi\, d'(z)\nabla_{\mathbf{Q}} h_\gamma(\boldsymbol{p}_i^k - \boldsymbol{p}_{i-1}^k)\cdot\boldsymbol{q}\,\mathrm{d}x\\ &+ \frac12\int_\Omega d(z)\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}_i^k - \boldsymbol{p}_{i-1}^k)(\boldsymbol{q}-\boldsymbol{p})\cdot(\boldsymbol{q}-\boldsymbol{p})\,\mathrm{d}x\\ &- \int_\Omega \varphi\,\ell'(z) f_i^k\cdot v\,\mathrm{d}x\,,\end{aligned} \tag{5.1}$$

where we recall that $(u_0, \boldsymbol{\varepsilon}_0, \boldsymbol{p}_0) = (0,0,0)$.

Theorem 5.1 (Differentiability of the control-to-state operator $S_{k,\gamma}$)**.** *Let $\gamma\in(0,+\infty)$. Then, the control-to-state operator $S_{k,\gamma}\colon L^\infty(\Omega)\to \big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega,\mathbb{M}^n_D)\big)^{k+1}$ is Frechét differentiable. Denoting by $(u_i^k,\boldsymbol{\varepsilon}_i^k,\boldsymbol{p}_i^k)_{i=1}^k$ the approximate time-discrete quasistatic evolution associated with $z\in L^\infty(\Omega)$, for every $\varphi\in L^\infty(\Omega)$ the derivative of $S_{k,\gamma}$ in z in the direction φ is given by the vector $(v_i^{k,\varphi}, \boldsymbol{\eta}_i^{k,\varphi}, \boldsymbol{q}_i^{k,\varphi})_{i=0}^k \in \mathcal{A}(0)^{k+1}$ defined recursively as the unique solution of*

$$\min\left\{\mathcal{F}^{\boldsymbol{q}_{i-1}^{k,\varphi}}_\gamma(t_i^k, z,\varphi, v,\boldsymbol{\eta},\boldsymbol{q}) : (v,\boldsymbol{\eta},\boldsymbol{q})\in\mathcal{A}(0)\right\}, \tag{5.2}$$

where we have set $\boldsymbol{q}_{-1}^{k,\varphi} = 0$.

Remark 5.2. Since $f_0^k = g_0^k = w_0^k = 0$ and $\boldsymbol{q}_{-1}^{k,\varphi} = 0$, it is easy to see that $(v_0^{k,\varphi}, \boldsymbol{\eta}_0^{k,\varphi}, \boldsymbol{q}_0^{k,\varphi}) = (0,0,0)$ for every $\varphi \in L^\infty(\Omega)$.

Remark 5.3. Notice that the incremental minimum problems (5.2) define a linear operator from $L^\infty(\Omega)$ to $\big(H^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)\big)^{k+1}$.

As a corollary of Theorem 5.1 we get the first-order optimality conditions for the regularized optimization problem (3.17). This will be the starting point of the analysis of Section 6.

Corollary 5.4 (Optimality conditions for the approximate time-discrete TO problem)**.** *Under the assumptions of Theorem 5.1, if $z \in H^1(\Omega;[0,1])$ is a solution of (3.17) with associated approximate time-discrete quasistatic evolution $(u_i^k, \boldsymbol{\varepsilon}_i^k, \boldsymbol{p}_i^k)_{i=0}^k$, then there exists $(\overline{u}_i^k, \overline{\boldsymbol{\varepsilon}}_i^k, \overline{\boldsymbol{p}}_i^k)_{i=1}^{k+1} \in \mathcal{A}(0)^{k+1}$ such that $(\overline{u}_{k+1}^k, \overline{\boldsymbol{\varepsilon}}_{k+1}^k, \overline{\boldsymbol{p}}_{k+1}^k) = (0,0,0)$ and for every $i = k, \dots, 1$, every $\varphi \in L^\infty(\Omega) \cap H^1(\Omega)$, and every $(v, \boldsymbol{\eta}, \boldsymbol{q}) \in \mathcal{A}(0)$*

$$\begin{aligned}\int_\Omega \mathbb{C}(z)\overline{\boldsymbol{\varepsilon}}_i^k \cdot \boldsymbol{\eta}\,\mathrm{d}x + \int_\Omega \mathbb{H}(z)\overline{\boldsymbol{p}}_i^k \cdot \boldsymbol{q}\,\mathrm{d}x + \int_\Omega d(z)\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}_i^k - \boldsymbol{p}_{i-1}^k)(\overline{\boldsymbol{p}}_i^k - \overline{\boldsymbol{p}}_{i+1}^k) \cdot \boldsymbol{q}\,\mathrm{d}x \\ - \int_\Omega \ell(z) f_i^k \cdot v\,\mathrm{d}x - \int_{\Gamma_N} g_i^k \cdot v\,\mathrm{d}\mathcal{H}^{n-1} = 0\,,\end{aligned} \tag{5.3}$$

$$\begin{aligned}\sum_{j=1}^k \bigg(\bigg(\int_\Omega \varphi\,\ell'(z)(f_j^k - f_{j-1}^k) \cdot (u_j^k + \overline{u}_j^k)\,\mathrm{d}x \\ - \int_\Omega \big(\mathbb{C}'(z)\varphi\big)(\boldsymbol{\varepsilon}_j^k - \boldsymbol{\varepsilon}_{j-1}^k) \cdot \overline{\boldsymbol{\varepsilon}}_j^k\,\mathrm{d}x - \int_\Omega \big(\mathbb{H}'(z)\varphi\big)(\boldsymbol{p}_j^k - \boldsymbol{p}_{j-1}^k) \cdot \overline{\boldsymbol{p}}_j^k\,\mathrm{d}x \bigg) \\ - \int_\Omega \varphi\, d'(z)\nabla_{\mathbf{Q}}\big(h_\gamma(\boldsymbol{p}_j^k - \boldsymbol{p}_{j-1}^k) - \nabla_{\mathbf{Q}} h_\gamma(\boldsymbol{p}_{j-1}^k - \boldsymbol{p}_{j-2}^k)\big) \cdot \overline{\boldsymbol{p}}_j^k\,\mathrm{d}x \bigg) \\ + \int_\Omega \delta \nabla z \cdot \nabla\varphi + \frac{\varphi}{\delta}(z(1-z)^2 - z^2(1-z))\,\mathrm{d}x = 0\,,\end{aligned} \tag{5.4}$$

where $\boldsymbol{p}_{-1}^k := 0$.

Proof. Let $z \in H^1(\Omega;[0,1])$ be a minimizer of $\mathcal{J}_{k,\delta}$ with corresponding approximate time-discrete quasistatic evolution $(u_i^k, \boldsymbol{\varepsilon}_i^k, \boldsymbol{p}_i^k)_{i=0}^k$. Let $\varphi \in L^\infty(\Omega) \cap H^1(\Omega)$ and $t \in \mathbb{R} \setminus \{0\}$. Setting $z_t := z + t\varphi$ and denoting by $(u_{i,t}^k, \boldsymbol{\varepsilon}_{i,t}^k, \boldsymbol{p}_{i,t}^k)_{i=0}^k$ the approximate time-discrete quasistatic evolution corresponding to z_t, we have that $\mathcal{J}_{k,\delta}(z, (u_i^k)_{i=0}^k) \le \mathcal{J}_{k,\delta}(z_t, (u_{i,t}^k))$. Differentiating $\mathcal{J}_{k,\delta}(z_t, (u_{i,t}^k))$ w.r.t. t, we deduce from the minimality of z and from Theorem 5.1 that

$$\begin{aligned}\int_\Omega \varphi\,\ell'(z)\, f_k^k \cdot u_k\,\mathrm{d}x + \int_\Omega \ell(z)\, f_k^k \cdot v_k^{k,\varphi}\,\mathrm{d}x + \int_{\Gamma_N} g_k^k \cdot v_k^{k,\varphi}\,\mathrm{d}\mathcal{H}^{n-1} \\ - \sum_{j=1}^k \bigg(\int_\Omega \varphi\,\ell'(z)(f_j^k - f_{j-1}^k) \cdot u_{j-1}^k\,\mathrm{d}x + \int_\Omega \ell(z)(f_j^k - f_{j-1}^k) \cdot v_{j-1}^{k,\varphi}\,\mathrm{d}x \bigg) \\ - \sum_{j=1}^k \int_\Omega (g_j^k - g_{j-1}^k) \cdot v_{j-1}^{k,\varphi}\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Omega \delta\nabla z \cdot \nabla\varphi + \frac{\varphi}{\delta}(z(1-z)^2 - z^2(1-z))\,\mathrm{d}x = 0\end{aligned} \tag{5.5}$$

for every $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$, where $(v_i^{k,\varphi}, \boldsymbol{\eta}_i^{k,\varphi}, \boldsymbol{q}_i^{k,\varphi})_{i=0}^k \in \mathcal{A}(0)^k$ has been defined in Theorem 5.1. We further set $(v_{-1}^{k,\varphi}, \boldsymbol{\eta}_{-1}^{k,\varphi}, \boldsymbol{q}_{-1}^{k,\varphi}) := (0,0,0)$.

We now define $(\overline{u}_i^k, \overline{\boldsymbol{\varepsilon}}_i^k, \overline{\boldsymbol{p}}_i^k) \in \mathcal{A}(0)$ as the unique solution of the minimum problem

$$\min \Big\{ \frac{1}{2}\int_\Omega \mathbb{C}(z)\boldsymbol{\eta}\cdot\boldsymbol{\eta}\,\mathrm{d}x + \frac{1}{2}\int_\Omega \mathbb{H}(z)\boldsymbol{q}\cdot\boldsymbol{q}\,\mathrm{d}x + \frac{1}{2}\int_\Omega d(z)\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}_i^k - \boldsymbol{p}_{i-1}^k)(\boldsymbol{q} - \overline{\boldsymbol{p}}_{i+1}^k)\cdot(\boldsymbol{q} - \overline{\boldsymbol{p}}_{i+1}^k)\,\mathrm{d}x - \int_\Omega \ell(z) f_i^k \cdot v\,\mathrm{d}x - \int_{\Gamma_N} g_i^k\cdot v\,\mathrm{d}\mathcal{H}^{n-1} : (v,\boldsymbol{\eta},\boldsymbol{q}) \in \mathcal{A}(0)\Big\}$$

for $i = k, \dots, 1$, where we have set $(\overline{u}_{k+1}^k, \overline{\boldsymbol{\varepsilon}}_{k+1}^k, \overline{\boldsymbol{p}}_{k+1}^k) := (0,0,0)$. In particular, $(\overline{u}_i^k, \overline{\boldsymbol{\varepsilon}}_i^k, \overline{\boldsymbol{p}}_i^k)$ satisfies (5.3).

In order to deduce (5.4), we notice that by Theorem 5.1 and by (5.3) and using that $f_0^k = g_0^k = \overline{u}_{k+1}^k = 0$ and $\overline{\boldsymbol{\varepsilon}}_{k+1}^k = \overline{\boldsymbol{p}}_{k+1}^k = 0$ we have that

$$\begin{aligned}
&\int_\Omega \ell(z)\, f_k^k \cdot v_k^{k,\varphi}\,\mathrm{d}x + \int_{\Gamma_N} g_k^k\cdot v_k^{k,\varphi}\,\mathrm{d}\mathcal{H}^{n-1} \\
&\qquad - \sum_{j=1}^k \Big(\int_\Omega \ell(z)(f_j^k - f_{j-1}^k)\cdot v_{j-1}^{k,\varphi}\,\mathrm{d}x + \int_\Omega (g_j^k - g_{j-1}^k)\cdot v_{j-1}^{k,\varphi}\,\mathrm{d}\mathcal{H}^{n-1}\Big) \\
&= \int_\Omega \mathbb{C}(z)\overline{\boldsymbol{\varepsilon}}_k^k \cdot \mathrm{E}v_k^{k,\varphi}\,\mathrm{d}x - \sum_{j=1}^k \int_\Omega \mathbb{C}(z)(\overline{\boldsymbol{\varepsilon}}_j^k - \overline{\boldsymbol{\varepsilon}}_{j-1}^k)\cdot \mathrm{E}v_{j-1}^{k,\varphi}\,\mathrm{d}x \\
&= \int_\Omega \mathbb{C}(z)\mathrm{E}\overline{u}_k^k\cdot\boldsymbol{\eta}_k^{k,\varphi}\,\mathrm{d}x - \int_\Omega \mathbb{C}(z)\overline{\boldsymbol{p}}_k^k\cdot\boldsymbol{\eta}_k^{k,\varphi}\,\mathrm{d}x + \int_\Omega \mathbb{C}(z)\overline{\boldsymbol{\varepsilon}}_k^k\cdot\boldsymbol{q}_k^{k,\varphi}\,\mathrm{d}x \\
&\qquad - \sum_{j=1}^k \Big(\int_\Omega \mathbb{C}(z)(\mathrm{E}\overline{u}_j^k - \mathrm{E}\overline{u}_{j-1}^k)\cdot\boldsymbol{\eta}_{j-1}^{k,\varphi}\,\mathrm{d}x - \int_\Omega \mathbb{C}(z)(\overline{\boldsymbol{p}}_j^k - \overline{\boldsymbol{p}}_{j-1}^k)\cdot\boldsymbol{\eta}_{j-1}^{k,\varphi}\,\mathrm{d}x \\
&\qquad + \int_\Omega \mathbb{C}(z)(\overline{\boldsymbol{\varepsilon}}_j^k - \overline{\boldsymbol{\varepsilon}}_{j-1}^k)\cdot\boldsymbol{q}_{j-1}^{k,\varphi}\,\mathrm{d}x\Big) \\
&= \sum_{j=1}^k \Big(\int_\Omega \mathbb{C}(z)\mathrm{E}\overline{u}_j^k\cdot(\boldsymbol{\eta}_j^{k,\varphi} - \boldsymbol{\eta}_{j-1}^{k,\varphi})\,\mathrm{d}x - \int_\Omega \mathbb{C}(z)\overline{\boldsymbol{p}}_j^k\cdot(\boldsymbol{\eta}_j^{k,\varphi} - \boldsymbol{\eta}_{j-1}^{k,\varphi})\,\mathrm{d}x \\
&\qquad + \int_\Omega \mathbb{C}(z)\overline{\boldsymbol{\varepsilon}}_j^k\cdot(\boldsymbol{q}_j^{k,\varphi} - \boldsymbol{q}_{j-1}^{k,\varphi})\,\mathrm{d}x\Big) \\
&= \sum_{j=1}^k \Big(\int_\Omega \varphi\,\ell'(z)(f_j^k - f_{j-1}^k)\cdot\overline{u}_j^k\,\mathrm{d}x - \int_\Omega \big(\mathbb{C}'(z)\varphi\big)(\boldsymbol{\varepsilon}_j^k - \boldsymbol{\varepsilon}_{j-1}^k)\cdot\mathrm{E}\overline{u}_j^k\,\mathrm{d}x\Big) \\
&\qquad - \sum_{j=1}^k \Big(\int_\Omega \mathbb{H}(z)(\boldsymbol{q}_j^{k,\varphi} - \boldsymbol{q}_{j-1}^{k,\varphi})\cdot\overline{\boldsymbol{p}}_j^k\,\mathrm{d}x - \int_\Omega \big(\mathbb{C}'(z)\varphi\big)(\boldsymbol{\varepsilon}_j^k - \boldsymbol{\varepsilon}_{j-1}^k)\cdot\overline{\boldsymbol{p}}_j^k\,\mathrm{d}x \\
&\qquad + \int_\Omega \big(\mathbb{H}'(z)\varphi\big)(\boldsymbol{p}_j^k - \boldsymbol{p}_{j-1}^k)\cdot\overline{\boldsymbol{p}}_j^k\,\mathrm{d}x \\
&\qquad + \int_\Omega \varphi\, d'(z)\big(\nabla_{\mathbf{Q}} h_\gamma(\boldsymbol{p}_j^k - \boldsymbol{p}_{j-1}^k) - \nabla_{\mathbf{Q}} h_\gamma(\boldsymbol{p}_{j-1}^k - \boldsymbol{p}_{j-2}^k)\big)\cdot\overline{\boldsymbol{p}}_j^k\,\mathrm{d}x \\
&\qquad + \int_\Omega d(z)\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}_j^k - \boldsymbol{p}_{j-1}^k)(\boldsymbol{q}_j^{k,\varphi} - \boldsymbol{q}_{j-1}^{k,\varphi})\cdot\overline{\boldsymbol{p}}_j^k\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \int_\Omega d(z)\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_{j-1} - \boldsymbol{p}^k_{j-2})(\boldsymbol{q}^{k,\varphi}_{j-1} - \boldsymbol{q}^{k,\varphi}_{j-2}) \cdot \overline{\boldsymbol{p}}^k_j \,\mathrm{d}x\Big) \\
&\qquad + \sum_{j=1}^k \Big(\int_\Omega \mathbb{H}(z) \overline{\boldsymbol{p}}^k_j \cdot (\boldsymbol{q}^{k,\varphi}_j - \boldsymbol{q}^{k,\varphi}_{j-1}) \,\mathrm{d}x \\
&\qquad + \int_\Omega d(z)\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_j - \boldsymbol{p}^k_{j-1})(\overline{\boldsymbol{p}}^k_j - \overline{\boldsymbol{p}}^k_{j+1}) \cdot (\boldsymbol{q}^{k,\varphi}_j - \boldsymbol{q}^{k,\varphi}_{j-1}) \,\mathrm{d}x\Big) \\
&= \sum_{j=1}^k \Big(\int_\Omega \varphi\, \ell'(z)(f^k_j - f^k_{j-1}) \cdot \overline{u}^k_j \,\mathrm{d}x - \int_\Omega \big(\mathbb{C}'(z)\varphi\big)(\boldsymbol{\varepsilon}^k_j - \boldsymbol{\varepsilon}^k_{j-1}) \cdot \overline{\boldsymbol{\varepsilon}}^k_j \,\mathrm{d}x \\
&\qquad - \int_\Omega \big(\mathbb{H}'(z)\varphi\big)(\boldsymbol{p}^k_j - \boldsymbol{p}^k_{j-1}) \cdot \overline{\boldsymbol{p}}^k_j \,\mathrm{d}x \\
&\qquad - \int_\Omega \varphi\, d'(z)\big(\nabla_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_j - \boldsymbol{p}^k_{j-1}) - \nabla_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_{j-1} - \boldsymbol{p}^k_{j-2})\big) \cdot \overline{\boldsymbol{p}}^k_j \,\mathrm{d}x\Big) ,
\end{aligned}
$$

where, in the last equality, we have used the following

$$
\begin{aligned}
&\sum_{j=1}^k \int_\Omega d(z)\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_j - \boldsymbol{p}^k_{j-1})(\overline{\boldsymbol{p}}^k_j - \overline{\boldsymbol{p}}^k_{j+1}) \cdot (\boldsymbol{q}^{k,\varphi}_j - \boldsymbol{q}^{k,\varphi}_{j-1}) \,\mathrm{d}x \\
&\quad = \sum_{j=1}^k \int_\Omega d(z)\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_j - \boldsymbol{p}^k_{j-1})(\boldsymbol{q}^{k,\varphi}_j - \boldsymbol{q}^{k,\varphi}_{j-1}) \cdot \overline{\boldsymbol{p}}^k_j \,\mathrm{d}x \\
&\qquad - \sum_{j=1}^k \int_\Omega d(z)\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_{j-1} - \boldsymbol{p}^k_{j-2})(\boldsymbol{q}^{k,\varphi}_{j-1} - \boldsymbol{q}^{k,\varphi}_{j-2}) \cdot \overline{\boldsymbol{p}}^k_j \,\mathrm{d}x \,.
\end{aligned}
$$

All in all, we have proved that

$$
\begin{aligned}
&\int_\Omega \ell(z)\, f^k_k \cdot v^{k,\varphi}_k \,\mathrm{d}x + \int_{\Gamma_N} g^k_k \cdot v^{k,\varphi}_k \,\mathrm{d}\mathcal{H}^{n-1} \\
&\qquad - \sum_{j=1}^k \Big(\int_\Omega \ell(z)(f^k_j - f^k_{j-1}) \cdot v^{k,\varphi}_{j-1} \,\mathrm{d}x - \int_\Omega (g^k_j - g^k_{j-1}) \cdot v^{k,\varphi}_{j-1} \,\mathrm{d}\mathcal{H}^{n-1}\Big) \\
&= \sum_{j=1}^k \Big(\int_\Omega \varphi\, \ell'(z)(f^k_j - f^k_{j-1}) \cdot \overline{u}^k_j \,\mathrm{d}x - \int_\Omega \big(\mathbb{C}'(z)\varphi\big)(\boldsymbol{\varepsilon}^k_j - \boldsymbol{\varepsilon}^k_{j-1}) \cdot \overline{\boldsymbol{\varepsilon}}^k_j \,\mathrm{d}x \\
&\qquad - \int_\Omega \big(\mathbb{H}'(z)\varphi\big)(\boldsymbol{p}^k_j - \boldsymbol{p}^k_{j-1}) \cdot \overline{\boldsymbol{p}}^k_j \,\mathrm{d}x\Big) \\
&\qquad - \int_\Omega \varphi\, d'(z)\nabla_{\mathbf{Q}}\big(h_\gamma(\boldsymbol{p}^k_j - \boldsymbol{p}^k_{j-1}) - \nabla_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_{j-1} - \boldsymbol{p}^k_{j-2})\big) \cdot \overline{\boldsymbol{p}}^k_j \,\mathrm{d}x\Big) ,
\end{aligned}
$$

which, together with (5.5), implies (5.4). □

The rest of the section is devoted to the proof of Theorem 5.1. The next two lemmas are a reformulation of Lemmas 3.5 and 3.6 in [2], which is needed in order to take care of the term $\boldsymbol{p}^k_{i-1}$ appearing in the minimization problem (3.15) at time t^k_i and which is also varying with the phase field z. We recall that this was not the case in [2], as the problem considered there is static.

Lemma 5.5. *For $z \in [0,1]$, $\gamma \in (0,+\infty)$, and $\boldsymbol{P} \in \mathbb{M}^n_D$, let $F_{z,\gamma,\boldsymbol{P}} \colon \mathbb{M}^n_D \to \mathbb{M}^n_D$ be the map defined by*

$$F_{z,\gamma,\boldsymbol{P}}(\mathbf{Q}) := \mathbb{C}(z)\mathbf{Q} + \mathbb{H}(z)\mathbf{Q} + d(z)\nabla_{\mathbf{Q}} h_\gamma(\mathbf{Q}-\boldsymbol{P}) \qquad \text{for every } \mathbf{Q} \in \mathbb{M}^n_D\,. \tag{5.6}$$

Then, there exist three constants $C_1, C_2, C_3 > 0$ independent of γ and of z and a constant $C_\gamma > 0$ (dependent on γ but not on z) such that for every $z, z_1, z_2 \in [0,1]$ and every $\boldsymbol{P}, \boldsymbol{P}_1, \boldsymbol{P}_2, \mathbf{Q}_1, \mathbf{Q}_2 \in \mathbb{M}^n_D$ the following holds:

$$|F_{z,\gamma,\boldsymbol{P}}(\mathbf{Q}_1) - F_{z,\gamma,\boldsymbol{P}}(\mathbf{Q}_2)| \le C_\gamma |\mathbf{Q}_1 - \mathbf{Q}_2|\,; \tag{5.7}$$

$$\big(F_{z,\gamma,\boldsymbol{P}}(\mathbf{Q}_1) - F_{z,\gamma,\boldsymbol{P}}(\mathbf{Q}_2)\big)\cdot\big(\mathbf{Q}_1 - \mathbf{Q}_2\big) \ge C_1 |\mathbf{Q}_1 - \mathbf{Q}_2|^2\,; \tag{5.8}$$

$$\begin{aligned}\big(F_{z_1,\gamma,\boldsymbol{P}_1}(\mathbf{Q}_1) - F_{z_2,\gamma,\boldsymbol{P}_2}(\mathbf{Q}_2)\big)\cdot\big(\mathbf{Q}_1 - \mathbf{Q}_2\big) &\ge C_1 |\mathbf{Q}_1 - \mathbf{Q}_2|^2 \\ &\quad - C_2(|\mathbf{Q}_2|+1)|z_1 - z_2||\mathbf{Q}_1 - \mathbf{Q}_2| - C_3\gamma|\mathbf{Q}_1 - \mathbf{Q}_2||\boldsymbol{P}_1 - \boldsymbol{P}_2|\,.\end{aligned} \tag{5.9}$$

In particular, $F_{z,\gamma,\boldsymbol{P}}$ is invertible and $F^{-1}_{z,\gamma,\boldsymbol{P}} \colon \mathbb{M}^n_D \to \mathbb{M}^n_D$ satisfies

$$|F^{-1}_{z,\gamma,\boldsymbol{P}}(\mathbf{Q}_1) - F^{-1}_{z,\gamma,\boldsymbol{P}}(\mathbf{Q}_2)| \le \widetilde{C}|\mathbf{Q}_1 - \mathbf{Q}_2|\,. \tag{5.10}$$

for a positive constant $\widetilde{C}$ independent of z, γ, and $\boldsymbol{P}$.

Proof. Inequalities (5.7), (5.8), and (5.10) can be proved repeating the arguments of Lemma 3.5 in [2]. Let us prove (5.9). Let $z_1, z_2 \in [0,1]$ and $\boldsymbol{P}_1, \boldsymbol{P}_2, \mathbf{Q}_1, \mathbf{Q}_2 \in \mathbb{M}^n_D$. By a simple algebraic argument and by using (5.8) we get

$$\begin{aligned}&\big(F_{z_1,\gamma,\boldsymbol{P}_1}(\mathbf{Q}_1) - F_{z_2,\gamma,\boldsymbol{P}_2}(\mathbf{Q}_2)\big)\cdot\big(\mathbf{Q}_1 - \mathbf{Q}_2\big) \\ &\qquad = \big(F_{z_1,\gamma,\boldsymbol{P}_1}(\mathbf{Q}_1) - F_{z_1,\gamma,\boldsymbol{P}_1}(\mathbf{Q}_2)\big)\cdot\big(\mathbf{Q}_1 - \mathbf{Q}_2\big) \\ &\qquad\quad + \big(F_{z_1,\gamma,\boldsymbol{P}_1}(\mathbf{Q}_2) - F_{z_2,\gamma,\boldsymbol{P}_2}(\mathbf{Q}_2)\big)\cdot\big(\mathbf{Q}_1 - \mathbf{Q}_2\big) \\ &\qquad \ge C_1|\mathbf{Q}_1 - \mathbf{Q}_2|^2 + \big(F_{z_1,\gamma,\boldsymbol{P}_1}(\mathbf{Q}_2) - F_{z_2,\gamma,\boldsymbol{P}_2}(\mathbf{Q}_2)\big)\cdot\big(\mathbf{Q}_1 - \mathbf{Q}_2\big)\,.\end{aligned} \tag{5.11}$$

We now estimate the last term on the right-hand side of (5.11) rewritten as

$$\begin{aligned}&\big(F_{z_1,\gamma,\boldsymbol{P}_1}(\mathbf{Q}_2) - F_{z_2,\gamma,\boldsymbol{P}_2}(\mathbf{Q}_2)\big)\cdot\big(\mathbf{Q}_1 - \mathbf{Q}_2\big) \\ &\qquad = \big(\mathbb{C}(z_1) - \mathbb{C}(z_2)\big)\mathbf{Q}_2\cdot\big(\mathbf{Q}_1 - \mathbf{Q}_2\big) + \big(\mathbb{H}(z_1) - \mathbb{H}(z_2)\big)\mathbf{Q}_2\cdot\big(\mathbf{Q}_1 - \mathbf{Q}_2\big) \\ &\qquad\quad + \big(d(z_1)\nabla_{\mathbf{Q}}h_\gamma(\mathbf{Q}_2 - \boldsymbol{P}_1) - d(z_2)\nabla_{\mathbf{Q}}h_\gamma(\mathbf{Q}_2 - \boldsymbol{P}_2)\big)\cdot\big(\mathbf{Q}_1 - \mathbf{Q}_2\big)\,.\end{aligned} \tag{5.12}$$

By the Lipschitz continuity of $\mathbb{C}$, $\mathbb{H}$, and d, and by (3.13)–(3.14), we hence have

$$\begin{aligned}&\big(F_{z_1,\gamma,\boldsymbol{P}_1}(\mathbf{Q}_2) - F_{z_2,\gamma,\boldsymbol{P}_2}(\mathbf{Q}_2)\big)\cdot\big(\mathbf{Q}_1 - \mathbf{Q}_2\big) \\ &\qquad \ge -C_2\big(|\mathbf{Q}_2|+1\big)|z_1 - z_2||\mathbf{Q}_1 - \mathbf{Q}_2| - C_3\gamma|\boldsymbol{P}_1 - \boldsymbol{P}_2||\mathbf{Q}_1 - \mathbf{Q}_2|\,,\end{aligned} \tag{5.13}$$

for some positive constant C_2, C_3 independent of z, γ, $\boldsymbol{P}_1$, and $\boldsymbol{P}_2$. Combining (5.11)–(5.13) we deduce (5.9). Relation (5.8) entails that $F_{z,\gamma,\boldsymbol{P}}$ is invertible and (5.10) follows from (5.8) with $\tilde{C} = C_1^{-1}$. □

Lemma 5.6. *For every $\gamma \in (0,+\infty)$, every $z \in \mathbb{R}$, and every $\boldsymbol{P} \in \mathbb{M}^n_D$, let the map $b_{z,\gamma,\boldsymbol{P}} \colon \mathbb{M}^n_S \to \mathbb{M}^n_S$ be defined as*

$$b_{z,\gamma,\boldsymbol{P}}(\mathbf{E}) := \mathbb{C}(z)\big(\mathbf{E} - F^{-1}_{z,\gamma,\boldsymbol{P}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z)\mathbf{E}))\big) \qquad \text{for every } \mathbf{E} \in \mathbb{M}^n_S, \tag{5.14}$$

where $\Pi_{\mathbb{M}^n_D}\colon \mathbb{M}^n \to \mathbb{M}^n_D$ *denotes the projection operator on* $\mathbb{M}^n_D$*. Then, there exist two positive constants* c_1, c_2 *such that for every* $\gamma \in (0,+\infty)$*, every* $z \in \mathbb{R}$*, every* $\boldsymbol{P} \in \mathbb{M}^n_D$*, and every* $\mathbf{E}_1, \mathbf{E}_2 \in \mathbb{M}^n_S$

$$|b_{z,\gamma,\boldsymbol{P}}(\mathbf{E}_1) - b_{z,\gamma,\boldsymbol{P}}(\mathbf{E}_2)| \leq c_1 |\mathbf{E}_1 - \mathbf{E}_2|\,, \tag{5.15}$$

$$(b_{z,\gamma,\boldsymbol{P}}(\mathbf{E}_1) - b_{z,\gamma,\boldsymbol{P}}(\mathbf{E}_2)) \cdot (\mathbf{E}_1 - \mathbf{E}_2) \geq c_2 |\mathbf{E}_1 - \mathbf{E}_2|^2\,. \tag{5.16}$$

Proof. The lemma can be proved as Lemma 3.6 of [2] by making use of the already established Lemma 5.5. □

We are now in a position to prove an L^p-regularity estimate and a Lipschitz dependence on the phase-field variable for an approximate time-discrete quasistatic evolution. Before stating these results, we introduce the notation

$$\|(u,\boldsymbol{\varepsilon},\boldsymbol{p})\|_{H^1\times L^2\times L^2} := \|u\|_{H^1} + \|\boldsymbol{\varepsilon}\|_2 + \|\boldsymbol{p}\|_2 \tag{5.17}$$

for $(u,\boldsymbol{\varepsilon},\boldsymbol{p}) \in H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)$. The symbol $\|(u,\boldsymbol{\varepsilon},\boldsymbol{p})\|_{W^{1,r}\times L^r\times L^r}$ is used for $(u,\boldsymbol{\varepsilon},\boldsymbol{p}) \in W^{1,r}(\Omega;\mathbb{R}^n)\times L^r(\Omega;\mathbb{M}^n_S)\times L^r(\Omega;\mathbb{M}^n_D)$. Finally, for $(u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)_{i=0}^k \in \big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ the norm $\|(u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)_{i=0}^k\|_{(H^1\times L^2\times L^2)^{k+1}}$ is defined by naturally extending (5.17). The same is done in $\big(W^{1,r}(\Omega;\mathbb{R}^n)\times L^r(\Omega;\mathbb{M}^n_S)\times L^r(\Omega;\mathbb{M}^n_D)\big)^{k+1}$.

Lemma 5.7. *Let* $k \in \mathbb{N}$ *and* $\gamma \in (0,+\infty)$*. Then, there exists* $\tilde p \in (2,p)$ *such that the control-to-state operator* $S_{k,\gamma}\colon L^\infty(\Omega) \to (H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))^{k+1}$ *takes values in* $(W^{1,\tilde p}(\Omega;\mathbb{R}^n)\times L^{\tilde p}(\Omega;\mathbb{M}^n_S)\times L^{\tilde p}(\Omega;\mathbb{M}^n_D))^{k+1}$ *and satisfies*

$$\begin{aligned} &\|S_{k,\gamma}(z)\|_{(W^{1,\tilde p}\times L^{\tilde p}\times L^{\tilde p})^{k+1}} \\ &\quad \leq C\big(1 + \|f\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^n))} + \|g\|_{L^\infty(0,T;L^p(\Gamma_N;\mathbb{R}^n))} + \|w\|_{L^\infty([0,T];W^{1,p}(\Omega;\mathbb{R}^n))}\big) \end{aligned} \tag{5.18}$$

for some positive constant C *independent of* i*,* k*,* γ*, and* z*.*

Furthermore, there exists a positive constant $C_{\gamma,k}$ *depending only on* γ *and* k *such that for every* $z_1, z_2 \in L^\infty(\Omega)$ *and every* $q \in (2,\tilde p]$

$$\begin{aligned} &\|S_{k,\gamma}(z_1) - S_{k,\gamma}(z_2)\|_{(W^{1,q}\times L^q\times L^q)^{k+1}} \\ &\quad \leq C_{\gamma,k}\big(1 + \|f\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^n))} + \|g\|_{L^\infty(0,T;L^p(\Gamma_N;\mathbb{R}^n))} \\ &\qquad + \|w\|_{L^\infty([0,T];W^{1,p}(\Omega;\mathbb{R}^n))}\big)\|z_1 - z_2\|_\infty\,. \end{aligned} \tag{5.19}$$

Proof. The proof of (5.18)–(5.19) follows from an application of Theorem 1.1 in [15]. To apply such result, we first have to recast the Euler-Lagrange equations associated to the equilibrium condition (3.15) in terms of the sole displacement variable u.

Let us fix $\gamma > 0$ and $z \in L^\infty(\Omega)$. For simplicity of notation, let $(u_i,\boldsymbol{\varepsilon}_i,\boldsymbol{p}_i)_{i=0}^k = S_{k,\gamma}(z)$ and $(u^j_i,\boldsymbol{\varepsilon}^j_i,\boldsymbol{p}^j_i)_{i=0}^k = S_{k,\gamma}(z_j)$, $j = 1,2$. We further recall the definition of f^k_i, g^k_i, and w^k_i given in (3.4) and that $(u_0,\boldsymbol{\varepsilon}_0,\boldsymbol{p}_0) = (u^j_0,\boldsymbol{\varepsilon}^j_0,\boldsymbol{p}^j_0) = (0,0,0)$.

From the minimization problem (3.15) we deduce the following Euler-Lagrange equation: for every $(v,\boldsymbol{\eta},\boldsymbol{q}) \in \mathcal{A}(0)$ and every $i = 1,\ldots,k$

$$\begin{aligned} &\int_\Omega \mathbb{C}(z)(\mathrm{E}u_i - \boldsymbol{p}_i)\cdot\boldsymbol{\eta}\,\mathrm{d}x + \int_\Omega \mathbb{H}(z)\boldsymbol{p}_i\cdot\boldsymbol{q}\,\mathrm{d}x + \int_\Omega d(z)\nabla_{\mathbf{Q}}h_\gamma(\boldsymbol{p}_i - \boldsymbol{p}_{i-1})\cdot\boldsymbol{q}\,\mathrm{d}x \\ &\quad - \int_\Omega \ell(z) f^k_i\cdot v\,\mathrm{d}x - \int_{\Gamma_N} g^k_i\cdot v\,\mathrm{d}\mathcal{H}^{n-1} = 0\,. \end{aligned} \tag{5.20}$$

By testing (5.20) with $(0, \boldsymbol{\eta}, -\boldsymbol{\eta}) \in \mathcal{A}(0)$ for $\boldsymbol{\eta} \in L^2(\Omega; \mathbb{M}^n_D)$ we get that

$$\mathbb{C}(z)\boldsymbol{p}_i + \mathbb{H}(z)\boldsymbol{p}_i + d(z)\nabla_{\mathbf{Q}} h_\gamma(\boldsymbol{p}_i - \boldsymbol{p}_{i-1}) = \Pi_{\mathbb{M}^n_D}(\mathbb{C}(z)\mathrm{E}u_i) \qquad \text{a.e. in } \Omega\,. \tag{5.21}$$

In view of the definition (5.6) of $F_{z,\gamma,\boldsymbol{P}}$, we have $F_{z(x),\gamma,\boldsymbol{p}_{i-1}(x)}(\boldsymbol{p}_i(x)) = \Pi_{\mathbb{M}^n_D}\big(\mathbb{C}(z(x))\mathrm{E}u_i(x)\big)$ and $\boldsymbol{p}_i(x) = F^{-1}_{z(x),\gamma,\boldsymbol{p}_{i-1}(x)}\big(\Pi_{\mathbb{M}^n_D}\big(\mathbb{C}(z(x))\mathrm{E}u_i(x)\big)\big)$ for a.e. $x \in \Omega$.

Recalling definition (5.14), we define for $x \in \Omega$, $\mathbf{E} \in \mathbb{M}^n_S$, and $i = 1, \dots, k$,

$$b_{z,\gamma,\boldsymbol{p}_{i-1}}(x, \mathbf{E}) := b_{z(x),\gamma,\boldsymbol{p}_{i-1}(x)}(\mathbf{E}) = \mathbb{C}(z(x))\big(\mathbf{E} - F^{-1}_{z(x),\gamma,\boldsymbol{p}_{i-1}(x)}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z(x))\mathbf{E})\big)\,.$$

From now on, when not explicitly needed, we drop the dependence on the spatial variable $x \in \Omega$ in the definition of $F^{-1}_{z,\gamma,\boldsymbol{p}_{i-1}}$, since all the arguments discussed below are valid uniformly in Ω. We rewrite the Euler-Lagrange equation (5.20) in terms of the sole displacement u_i and for test functions of the form $(\psi, \mathrm{E}\psi, 0) \in \mathcal{A}(0)$ for $\psi \in H^1(\Omega; \mathbb{R}^n)$ with $\psi = 0$ on Γ_D:

$$\int_\Omega b_{z,\gamma,\boldsymbol{p}_{i-1}}(x, \mathrm{E}u_i) \cdot \mathrm{E}\psi \,\mathrm{d}x = \int_\Omega \ell(z) f_i^k \cdot \psi \,\mathrm{d}x + \int_{\Gamma_N} g_i^k \cdot \psi \,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{5.22}$$

In view of Lemma 5.6, the nonlinear operator $B_{z,\gamma,\boldsymbol{p}_{i-1}} \colon W^{1,p}(\Omega; \mathbb{R}^n) \to W^{-1,p}(\Omega; \mathbb{R}^n)$ defined as $B_{z,\gamma,\boldsymbol{p}_{i-1}}(u) := b_{z,\gamma,\boldsymbol{p}_{i-1}}(x, \mathrm{E}u)$ satisfies the hypotheses of Theorem 1.1 in [15]. Since $\Omega \cup \Gamma_N$ is Gröger regular, $p \in (2, +\infty)$, $f_i^k \in L^p(\Omega; \mathbb{R}^n)$, $g_i^k \in L^p(\Gamma_N; \mathbb{R}^n)$, and $w_i^k \in W^{1,p}(\Omega; \mathbb{R}^n)$, we infer from Theorem 1.1 of [15] applied to equation (5.22) that there exist $\tilde{p} \in (2, p)$ and a constant $C > 0$ (both independent of i and k) such that

$$\|u_i\|_{W^{1,q}} \le C(\|f_i^k\|_p + \|g_i^k\|_p + \|w_i^k\|_{W^{1,p}}) \tag{5.23}$$

for every $q \in (2, \tilde{p}]$. In particular, C is independent of $z \in L^\infty(\Omega)$, of $\gamma \in (0, +\infty)$, of $k \in \mathbb{N}$, and of $q \in (2, \tilde{p}]$. Inequality (5.18) can be deduced by combining (5.10) and (5.23). Indeed, we have that

$$\begin{aligned} \|\boldsymbol{p}_i\|_q &= \|F^{-1}_{z,\gamma,\boldsymbol{p}_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z)\mathrm{E}u_i))\|_q \\ &\le \|F^{-1}_{z,\gamma,\boldsymbol{p}_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z)\mathrm{E}u_i)) - F^{-1}_{z,\gamma,\boldsymbol{p}_{i-1}}(0)\|_q + \|F^{-1}_{z,\gamma,\boldsymbol{p}_{i-1}}(0)\|_q \\ &\le \widetilde{C}\|\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z)\mathrm{E}u_i)\|_q + \|F^{-1}_{z,\gamma,\boldsymbol{p}_{i-1}}(0)\|_q \le \overline{C}\|u_i\|_{W^{1,q}} + \|F^{-1}_{z,\gamma,\boldsymbol{p}_{i-1}}(0)\|_q\,. \end{aligned} \tag{5.24}$$

To conclude the estimate, we notice that if $\boldsymbol{q} := F^{-1}_{z,\gamma,\boldsymbol{p}_{i-1}}(0)$, we have that

$$\mathbb{C}(z)\boldsymbol{q} + \mathbb{H}(z)\boldsymbol{q} = -d(z)\nabla_{\mathbf{Q}} h_\gamma(\boldsymbol{q} - \boldsymbol{p}_{i-1}) = -d(z)\frac{\boldsymbol{q} - \boldsymbol{p}_{i-1}}{\sqrt{|\boldsymbol{q} - \boldsymbol{p}_{i-1}|^2 + \frac{1}{\gamma^2}}}\,.$$

Multiplying the previous expression by $\boldsymbol{q}$ and using (2.4)–(2.5) we deduce that

$$(\alpha_{\mathbb{C}} + \alpha_{\mathbb{H}})|\boldsymbol{q}|^2 \le d(z)|\boldsymbol{q}|$$

a.e. in Ω. Hence, $\|F^{-1}_{z,\gamma,\boldsymbol{p}_{i-1}}(0)\|_q$ is bounded uniformly w.r.t. i, k, γ, and z. Thus, combining (5.23)–(5.24) we infer (5.18) by the triangle inequality.

In order to prove (5.19), we first rewrite the Euler-Lagrange equation (5.22) satisfied by u_i^2, $i = 1, \dots, k$. Namely, for every $\psi \in W^{1,\tilde{p}'}(\Omega;\mathbb{R}^n)$ with $\psi = 0$ on Γ_D we have, after a simple algebraic manipulation,

$$\begin{aligned} &\int_\Omega B_{z_1,\gamma,\boldsymbol{p}^1_{i-1}}(u_i^2)\cdot \mathrm{E}\psi \,\mathrm{d}x \\ &\quad = \int_\Omega \big(\mathbb{C}(z_1)-\mathbb{C}(z_2)\big)\big(\mathrm{E}u_i^2 - F^{-1}_{z_1,\gamma,\boldsymbol{p}^1_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_1)\mathrm{E}u_i^2))\big)\cdot \mathrm{E}\psi\,\mathrm{d}x \\ &\qquad + \int_\Omega \mathbb{C}(z_2)\big(F^{-1}_{z_2,\gamma,\boldsymbol{p}^2_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_2)\mathrm{E}u_i^2)) - F^{-1}_{z_1,\gamma,\boldsymbol{p}^1_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_1)\mathrm{E}u_i^2))\big)\cdot \mathrm{E}\psi\,\mathrm{d}x \\ &\qquad + \int_\Omega \ell(z_2) f_i^k\cdot\psi\,\mathrm{d}x + \int_{\Gamma_N} g_i^k\cdot\psi\,\mathrm{d}\mathcal{H}^{n-1}\,. \end{aligned} \tag{5.25}$$

Comparing (5.25) with (5.22) written for z_1 and $(u_i^1, \boldsymbol{\varepsilon}_i^1, \boldsymbol{p}_i^1)$, we deduce that u_i^1 and u_i^2 solve the same kind of equation, with a different right-hand side, which however always belongs to $W^{-1,\tilde{p}}(\Omega;\mathbb{R}^n)$. Thus, applying once more ([15], Thm. 1.1), we infer that there exists $C > 0$ independent of z_1, z_2, of γ, of i, and of k, such that for every $q \in (2, \tilde{p}]$

$$\begin{aligned} \|u_i^1 - u_i^2\|_{W^{1,q}} \le C\Big(&\Big\|\big(\mathbb{C}(z_1)-\mathbb{C}(z_2)\big)\big(\mathrm{E}u_i^2 - F^{-1}_{z_2,\gamma,\boldsymbol{p}^2_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_2)\mathrm{E}u_i^2))\big)\Big\|_{W^{-1,q}} \\ &+ \Big\|\mathbb{C}(z_1)\big(F^{-1}_{z_2,\gamma,\boldsymbol{p}^2_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_2)\mathrm{E}u_i^2)) - F^{-1}_{z_1,\gamma,\boldsymbol{p}^1_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_1)\mathrm{E}u_i^2))\big)\Big\|_{W^{-1,q}} \\ &+ \big\|\big(\ell(z_1)-\ell(z_2)\big) f_i^k\big\|_{W^{-1,q}}\Big) =: C(I_1 + I_2 + I_3)\,. \end{aligned} \tag{5.26}$$

By the Lipschitz continuity of $\mathbb{C}$, by the identification $\boldsymbol{p}_i^2 = F^{-1}_{z_2,\gamma,\boldsymbol{p}^2_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_2)\mathrm{E}u_i^2))$, by the Hölder inequality, and by (5.18) we deduce that

$$\begin{aligned} I_1 &\le C\|z_1 - z_2\|_\infty\big(\|u_i^2\|_{W^{1,\tilde{p}}} + \|\boldsymbol{p}_i^2\|_{\tilde{p}}\big) \\ &\le C\|z_1 - z_2\|_\infty\big(1 + \|f\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^n))} + \|g\|_{L^\infty(0,T;L^p(\Gamma_N;\mathbb{R}^n))} \\ &\qquad + \|w\|_{L^\infty([0,T];W^{1,p}(\Omega;\mathbb{R}^n))}\big)\,. \end{aligned} \tag{5.27}$$

Rewriting (5.9) for $\boldsymbol{P}_j = \boldsymbol{p}^j_{i-1}$ and $\mathbf{Q}_j = F^{-1}_{z_j,\gamma,\boldsymbol{p}^j_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_j)\mathrm{E}u_i^2))$ we get that for a.e. $x \in \Omega$

$$\begin{aligned} &C_1|F^{-1}_{z_1,\gamma,\boldsymbol{p}^1_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_1)\mathrm{E}u_i^2)) - F^{-1}_{z_2,\gamma,\boldsymbol{p}^2_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_2)\mathrm{E}u_i^2))| \\ &\qquad \le \mathrm{Lip}(\mathbb{C})|\mathrm{E}u_i^2||z_1 - z_2| + C_2\big(|F^{-1}_{z_2,\gamma,\boldsymbol{p}^2_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_2)\mathrm{E}u_i^2))| + 1\big)\,|z_1 - z_2| \\ &\qquad\quad + C_3\gamma|\boldsymbol{p}^1_{i-1} - \boldsymbol{p}^2_{i-1}|\,. \end{aligned} \tag{5.28}$$

The identification $\boldsymbol{p}_i^2 = F^{-1}_{z_2,\gamma,\boldsymbol{p}^2_{i-1}}(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_2)\mathrm{E}u_i^2))$ and inequalities (5.18) and (5.28) imply that

$$\begin{aligned} I_2 &\le C(\|u_i^2\|_{W^{1,\tilde{p}}} + \|\boldsymbol{p}_i^2\|_{\tilde{p}} + 1)\|z_1 - z_2\|_\infty + C_3\gamma\|\boldsymbol{p}^1_{i-1} - \boldsymbol{p}^2_{i-1}\|_q \\ &\le C(\|f\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^n))} + \|g\|_{L^\infty(0,T;L^p(\Gamma_N;\mathbb{R}^n))} \\ &\qquad + \|w\|_{L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^n))} + 1)\|z_1 - z_2\|_\infty + C_3\gamma\|\boldsymbol{p}^1_{i-1} - \boldsymbol{p}^2_{i-1}\|_q\,. \end{aligned} \tag{5.29}$$

Finally, by the Lipschitz continuity of ℓ we conclude that

$$I_3 \le C\|f\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^n))}\|z_1 - z_2\|_\infty\,. \tag{5.30}$$

Combining inequalities (5.26)–(5.30) we infer that

$$\begin{aligned}\|u_i^1-u_i^2\|_{W^{1,q}} \le C\Big(&\|f\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^n))}+\|g\|_{L^\infty(0,T;L^p(\Gamma_N;\mathbb{R}^n))}\\ &+\|w\|_{L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^n))}+1\Big)\|z_1-z_2\|_\infty + C_3\gamma\|\boldsymbol{p}_{i-1}^1-\boldsymbol{p}_{i-1}^2\|_q\,.\end{aligned} \tag{5.31}$$

We notice that inequality (5.9) tested with

$$\mathbf{Q}_j = F^{-1}_{z_j(x),\gamma,\boldsymbol{p}^j_{i-1}(x)}\big(\Pi_{\mathbb{M}^n_D}(\mathbb{C}(z_j(x))\mathrm{E}u_i^j(x))\big)=\boldsymbol{p}_i^j(x)\qquad \text{for a.e. } x\in\Omega$$

and integrated over Ω implies

$$\begin{aligned}\|\boldsymbol{p}_i^1-\boldsymbol{p}_i^2\|_q &\le C\big(\|u_i^1-u_i^2\|_{W^{1,q}}+(\|u_i^2\|_{W^{1,\tilde p}}+\|\boldsymbol{p}_i^2\|_{L^{\tilde p}}+1)\|z_1-z_2\|_\infty\big)\\ &\quad + C_3\gamma\|\boldsymbol{p}_{i-1}^1-\boldsymbol{p}_{i-1}^2\|_q\\ &\le C\|u_i^1-u_i^2\|_{W^{1,q}}+C_3\gamma\|\boldsymbol{p}_{i-1}^1-\boldsymbol{p}_{i-1}^2\|_q\\ &\quad + C(\|f\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^n))}+\|g\|_{L^\infty(0,T;L^p(\Gamma_N;\mathbb{R}^n))}\\ &\quad +\|w\|_{L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^n))}+1)\|z_1-z_2\|_\infty\\ &\le C(\|f\|_{L^\infty(0,T;L^p(\Omega;\mathbb{R}^n))}+\|g\|_{L^\infty(0,T;L^p(\Gamma_N;\mathbb{R}^n))}\\ &\quad +\|w\|_{L^\infty(0,T;W^{1,p}(\Omega;\mathbb{R}^n))}+1)\|z_1-z_2\|_\infty + C\gamma\|\boldsymbol{p}_{i-1}^1-\boldsymbol{p}_{i-1}^2\|_q\,.\end{aligned} \tag{5.32}$$

By the triangle inequality, an estimate similar to (5.32) holds for $\boldsymbol{\varepsilon}_i^1-\boldsymbol{\varepsilon}_i^2$, for every $i=1,\dots,k$. Iterating the inequalities (5.31)–(5.32) for $l=1,\dots,i$ and taking into account that $(u_0^1,\boldsymbol{\varepsilon}_0^1,\boldsymbol{p}_0^1)=(u_0^2,\boldsymbol{\varepsilon}_0^2,\boldsymbol{p}_0^2)=(0,0,0)$, we obtain (5.19). This concludes the proof of the lemma. □

We are now ready to prove Theorem 5.1. The proof follows the lines of the proofs of Theorem 3.1 in [2] and of Theorem 3.3 in [5]. The main difference is that, as in [34], the forward problem is now time dependent and not static.

Proof of Theorem 5.1. Let us fix $k\in\mathbb{N}$, $\gamma\in(0,+\infty)$, and $z,\varphi\in L^\infty(\Omega)$. For $t\in\mathbb{R}$, let $z_t:=z+t\varphi$, $(u^k_{i,t},\boldsymbol{\varepsilon}^k_{i,t},\boldsymbol{p}^k_{i,t})_{i=1}^k:=S_{k,\gamma}(z_t)$. The solution for $t=0$ will be simply denoted by $(u_i^k,\boldsymbol{\varepsilon}_i^k,\boldsymbol{p}_i^k)_{i=1}^k$. Moreover, let $(v_i^{k,\varphi},\boldsymbol{\eta}_i^{k,\varphi},\boldsymbol{q}_i^{k,\varphi})$ be the solution of the recursive minimization problem (5.2) and set

$$\overline{v}^k_{i,t}:=u^k_{i,t}-u^k_i-tv_i^{k,\varphi}\,,\qquad \overline{\boldsymbol{\eta}}^k_{i,t}:=\boldsymbol{\varepsilon}^k_{i,t}-\boldsymbol{\varepsilon}^k_i-t\boldsymbol{\eta}_i^{k,\varphi}\,,\qquad \overline{\boldsymbol{q}}^k_{i,t}:=\boldsymbol{p}^k_{i,t}-\boldsymbol{p}^k_i-t\boldsymbol{q}_i^{k,\varphi}\,.$$

We want to show that

$$\|(\overline{v}^k_{i,t},\overline{\boldsymbol{\eta}}^k_{i,t},\overline{\boldsymbol{q}}^k_{i,t})\|_{H^1\times L^2\times L^2}=o(t)\,, \tag{5.33}$$

uniformly w.r.t. $i=1,\dots,k$. In particular, (5.33) implies the Frechét differentiability of the control-to-state map $S_{k,\gamma}$.

We prove (5.33) by induction on i. For $i=1$, (5.33) follows from Theorem 3.1 in [2], as the initial value is $(u_0^k,\boldsymbol{\varepsilon}_0^k,\boldsymbol{p}_0^k)=(0,0,0)$ by the assumptions on the data $f(0)=g(0)=w(0)=0$. For $i>1$, assume that $\|(\overline{v}^k_{i-1,t},\overline{\boldsymbol{\eta}}^k_{i-1,t},\overline{\boldsymbol{q}}^k_{i-1,t})\|_{H^1\times L^2\times L^2}=o(t)$. Writing the Euler-Lagrange equations satisfied by $(u^k_{i,t},\boldsymbol{\varepsilon}^k_{i,t},\boldsymbol{p}^k_{i,t})$, $(u_i^k,\boldsymbol{\varepsilon}_i^k,\boldsymbol{p}_i^k)$, and $(v_i^{k,\varphi},\boldsymbol{\eta}_i^{k,\varphi},\boldsymbol{q}_i^{k,\varphi})$ and subtracting the second and the third from the first one, we obtain, for every $(v,\boldsymbol{\eta},\boldsymbol{q})\in\mathcal{A}(0)$,

$$\begin{aligned}\int_\Omega \mathbb{C}(z_t)\boldsymbol{\varepsilon}^k_{i,t}\cdot\boldsymbol{\eta}\,\mathrm{d}x &- \int_\Omega \mathbb{C}(z)\boldsymbol{\varepsilon}^k_i\cdot\boldsymbol{\eta}\,\mathrm{d}x - t\int_\Omega \mathbb{C}(z)\boldsymbol{\eta}^{k,\varphi}_i\cdot\boldsymbol{\eta}\,\mathrm{d}x - t\int_\Omega (\mathbb{C}'(z)\varphi)\boldsymbol{\varepsilon}^k_i\cdot\boldsymbol{\eta}\,\mathrm{d}x\\
&+\int_\Omega \mathbb{H}(z_t)\boldsymbol{p}^k_{i,t}\cdot\boldsymbol{q}\,\mathrm{d}x - \int_\Omega \mathbb{H}(z)\boldsymbol{p}^k_i\cdot\boldsymbol{q}\,\mathrm{d}x - t\int_\Omega \mathbb{H}(z)\boldsymbol{q}^{k,\varphi}_i\cdot\boldsymbol{q}\,\mathrm{d}x\\
&- t\int_\Omega(\mathbb{H}'(z)\varphi)\boldsymbol{p}^k_i\cdot\boldsymbol{q}\,\mathrm{d}x + \int_\Omega d(z_t)\nabla_{\mathbf{Q}}h_\gamma(\boldsymbol{p}^k_{i,t}-\boldsymbol{p}^k_{i-1,t})\cdot\boldsymbol{q}\,\mathrm{d}x\\
&- \int_\Omega d(z)\nabla_{\mathbf{Q}}h_\gamma(\boldsymbol{p}^k_i-\boldsymbol{p}^k_{i-1})\cdot\boldsymbol{q}\,\mathrm{d}x - t\int_\Omega \varphi\, d'(z)\nabla_{\mathbf{Q}}h_\gamma(\boldsymbol{p}^k_i-\boldsymbol{p}^k_{i-1})\cdot\boldsymbol{q}\,\mathrm{d}x\\
&- t\int_\Omega d(z)\nabla^2_{\mathbf{Q}}h_\gamma(\boldsymbol{p}^k_i-\boldsymbol{p}^k_{i-1})(\boldsymbol{q}^{k,\varphi}_i-\boldsymbol{q}^{k,\varphi}_{i-1})\cdot\boldsymbol{q}\,\mathrm{d}x - \int_\Omega \ell(z_t)f^k_i\cdot v\,\mathrm{d}x\\
&+\int_\Omega \ell(z)f^k_i\cdot v\,\mathrm{d}x + t\int_\Omega \varphi\,\ell'(z)f^k_i\cdot v\,\mathrm{d}x = 0\,.\end{aligned}$$

By a simple algebraic manipulation, we rewrite the previous equality as

$$\begin{aligned}0 = &\Big(\int_\Omega \mathbb{C}(z_t)\boldsymbol{\varepsilon}^k_{i,t}\cdot\boldsymbol{\eta}\,\mathrm{d}x - \int_\Omega \mathbb{C}(z)\boldsymbol{\varepsilon}^k_i\cdot\boldsymbol{\eta}\,\mathrm{d}x - t\int_\Omega \mathbb{C}(z)\boldsymbol{\eta}^{k,\varphi}_i\cdot\boldsymbol{\eta}\,\mathrm{d}x\\
&- t\int_\Omega(\mathbb{C}'(z)\varphi)\boldsymbol{\varepsilon}^k_i\cdot\boldsymbol{\eta}\,\mathrm{d}x\Big)\\
&+\Big(\int_\Omega \mathbb{H}(z_t)\boldsymbol{p}^k_{i,t}\cdot\boldsymbol{q}\,\mathrm{d}x - \int_\Omega \mathbb{H}(z)\boldsymbol{p}^k_i\cdot\boldsymbol{q}\,\mathrm{d}x - t\int_\Omega \mathbb{H}(z)\boldsymbol{q}^{k,\varphi}_i\cdot\boldsymbol{q}\,\mathrm{d}x\\
&- t\int_\Omega(\mathbb{H}'(z)\varphi)\boldsymbol{p}^k_i\cdot\boldsymbol{q}\,\mathrm{d}x\Big)\\
&+\Big(\int_\Omega \big(d(z_t)-d(z)-t\varphi\, d'(z)\big)\nabla_{\mathbf{Q}}h_\gamma(\boldsymbol{p}^k_i-\boldsymbol{p}^k_{i-1})\cdot\boldsymbol{q}\,\mathrm{d}x\\
&+\int_\Omega\big(d(z_t)-d(z)\big)\big(\nabla_{\mathbf{Q}}h_\gamma(\boldsymbol{p}^k_{i,t}-\boldsymbol{p}^k_{i-1,t})-\nabla_{\mathbf{Q}}h_\gamma(\boldsymbol{p}^k_i-\boldsymbol{p}^k_{i-1})\big)\cdot\boldsymbol{q}\,\mathrm{d}x\Big)\\
&+\Big(\int_\Omega d(z)\big(\nabla_{\mathbf{Q}}h_\gamma(\boldsymbol{p}^k_{i,t}-\boldsymbol{p}^k_{i-1,t})-\nabla_{\mathbf{Q}}h_\gamma(\boldsymbol{p}^k_i-\boldsymbol{p}^k_{i-1})\\
&- t\nabla^2_{\mathbf{Q}}h_\gamma(\boldsymbol{p}^k_i-\boldsymbol{p}^k_{i-1})(\boldsymbol{q}^{k,\varphi}_i-\boldsymbol{q}^{k,\varphi}_{i-1})\big)\cdot\boldsymbol{q}\,\mathrm{d}x\Big)\\
&-\Big(\int_\Omega\big(\ell(z_t)-\ell(z)-t\ell'(z)\varphi\big)f^k_i\cdot v\,\mathrm{d}x\Big)\\
=:{}& I_{t,1}+I_{t,2}+I_{t,3}+I_{t,4}+I_{t,5}\,.\end{aligned}\tag{5.34}$$

Let us now rewrite $I_{t,1}$, $I_{t,2}$, and $I_{t,4}$ from (5.34). For $I_{t,1}$ we have that

$$I_{t,1} = \int_\Omega \mathbb{C}(z)\overline{\boldsymbol{\eta}}^k_{i,t}\cdot\boldsymbol{\eta}\,\mathrm{d}x + \int_\Omega(\mathbb{C}(z_t)-\mathbb{C}(z))(\boldsymbol{\varepsilon}^k_{i,t}-\boldsymbol{\varepsilon}^k_i)\cdot\boldsymbol{\eta}\,\mathrm{d}x + \int_\Omega\big(\mathbb{C}(z_t)-\mathbb{C}(z)-t(\mathbb{C}'(z)\varphi)\big)\boldsymbol{\varepsilon}^k_i\cdot\boldsymbol{\eta}\,\mathrm{d}x.$$

In a similar way, we have that

$$I_{t,2} = \int_\Omega \mathbb{H}(z)\overline{\boldsymbol{q}}^k_{i,t}\cdot\boldsymbol{q}\,\mathrm{d}x + \int_\Omega(\mathbb{H}(z_t)-\mathbb{H}(z))(\boldsymbol{p}^k_{i,t}-\boldsymbol{p}^k_i)\cdot\boldsymbol{q}\,\mathrm{d}x + \int_\Omega\big(\mathbb{H}(z_t)-\mathbb{H}(z)-t(\mathbb{H}'(z)\varphi)\big)\boldsymbol{p}^k_i\cdot\boldsymbol{q}\,\mathrm{d}x.$$

As for $I_{t,4}$, since $h_\gamma \in C^\infty(\mathbb{M}^n_D)$, for every $t>0$ there exists $\boldsymbol{\xi}_t$ on the segment $[\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1}, \boldsymbol{p}^k_{i,t} - \boldsymbol{p}^k_{i-1,t}]$ such that

$$\begin{aligned} I_{t,4} &= \int_\Omega d(z)\big(\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{\xi}_t)(\boldsymbol{p}^k_{i,t} - \boldsymbol{p}^k_{i-1,t} - \boldsymbol{p}^k_i + \boldsymbol{p}^k_{i-1}) - t\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1})(\boldsymbol{q}^{k,\varphi}_i - \boldsymbol{q}^{k,\varphi}_{i-1})\big)\cdot \boldsymbol{q}\,\mathrm{d}x \\ &= \int_\Omega d(z)\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1})(\overline{\boldsymbol{q}}^k_{i,t} - \overline{\boldsymbol{q}}^k_{i-1,t})\cdot \boldsymbol{q}\,\mathrm{d}x \\ &\quad + \int_\Omega d(z)\big(\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{\xi}_t) - \nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1})\big)(\boldsymbol{p}^k_{i,t} - \boldsymbol{p}^k_{i-1,t} - \boldsymbol{p}^k_i + \boldsymbol{p}^k_{i-1})\cdot \boldsymbol{q}\,\mathrm{d}x\,. \end{aligned}$$

Inserting the previous equalities in (5.34), choosing the test function $(v,\boldsymbol{\eta},\boldsymbol{q}) = (\overline{v}^k_{i,t}, \overline{\boldsymbol{\eta}}^k_{i,t}, \overline{\boldsymbol{q}}^k_{i,t}) \in \mathcal{A}(0)$, using (2.4)–(2.5), the Lipschitz continuity of $\mathbb{C}(\cdot)$, $\mathbb{H}(\cdot)$, $d(\cdot)$, and $\nabla_{\mathbf{Q}} h_\gamma$, the convexity of h_γ, and Lemma 5.7, we obtain the estimate

$$\begin{aligned} \|(\overline{v}^k_{i,t},\overline{\boldsymbol{\eta}}^k_{i,t},\overline{\boldsymbol{q}}^k_{i,t})\|^2_{H^1\times L^2\times L^2} \leq\ & C_{\gamma,k}\, t^2\|\varphi\|^2_\infty \|(\overline{v}^k_{i,t},\overline{\boldsymbol{\eta}}^k_{i,t},\overline{\boldsymbol{q}}^k_{i,t})\|_{H^1\times L^2\times L^2} \\ & - \int_\Omega \big(\mathbb{C}(z_t) - \mathbb{C}(z) - t(\mathbb{C}'(z)\varphi)\big)\boldsymbol{\varepsilon}^k_i \cdot \overline{\boldsymbol{\eta}}^k_{i,t}\,\mathrm{d}x \\ & - \int_\Omega \big(\mathbb{H}(z_t) - \mathbb{H}(z) - t(\mathbb{H}'(z)\varphi)\big)\boldsymbol{p}^k_i \cdot \overline{\boldsymbol{q}}^k_{i,t}\,\mathrm{d}x \\ & - \int_\Omega \big(d(z_t) - d(z) - t\varphi\, d'(z)\big)\nabla_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1})\cdot \overline{\boldsymbol{q}}^k_{i,t}\,\mathrm{d}x \\ & + \int_\Omega d(z)\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1})\overline{\boldsymbol{q}}^k_{i-1,t}\cdot \overline{\boldsymbol{q}}^k_{i,t}\,\mathrm{d}x \\ & - \int_\Omega d(z)\big(\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{\xi}_t) - \nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1})\big)(\boldsymbol{p}^k_{i,t} - \boldsymbol{p}^k_{i-1,t} - \boldsymbol{p}^k_i + \boldsymbol{p}^k_{i-1})\cdot \overline{\boldsymbol{q}}^k_{i,t}\,\mathrm{d}x \\ & + \int_\Omega \big(\ell(z_t) - \ell(z) - t\varphi\,\ell'(z)\big) f^k_i \cdot \overline{v}^k_{i,t}\,\mathrm{d}x\,, \end{aligned} \tag{5.35}$$

for some positive constant $C_{\gamma,k}$ dependent on $\gamma \in (0,+\infty)$ and $k \in \mathbb{N}$. In view of the regularity of $\mathbb{C}(\cdot)$, $\mathbb{H}(\cdot)$, $d(\cdot)$, and $\ell(\cdot)$, of the bounds $|\nabla_{\mathbf{Q}} h_\gamma(\mathbf{Q})| \leq 1$ and $|\nabla^2_{\mathbf{Q}} h_\gamma(\mathbf{Q})| \leq 2\gamma$, and of Lemma 5.7, we can continue in (5.35) with

$$\begin{aligned} & \|(\overline{v}^k_{i,t},\overline{\boldsymbol{\eta}}^k_{i,t},\overline{\boldsymbol{q}}^k_{i,t})\|^2_{H^1\times L^2\times L^2} \\ & \leq \widetilde{C}_{\gamma,k}\, t^2\|\varphi\|^2_\infty\big(\|(u^k_i,\boldsymbol{\varepsilon}^k_i,\boldsymbol{p}^k_i)\|_{H^1\times L^2\times L^2} + \|f^k_i\|_2 + 1\big)\|(\overline{v}^k_{i,t},\overline{\boldsymbol{\eta}}^k_{i,t},\overline{\boldsymbol{q}}^k_{i,t})\|_{H^1\times L^2\times L^2} \\ & \qquad + \widetilde{C}_\gamma \|(\overline{v}^k_{i-1,t},\overline{\boldsymbol{\eta}}^k_{i-1,t},\overline{\boldsymbol{q}}^k_{i-1,t})\|_{H^1\times L^2\times L^2}\|(\overline{v}^k_{i,t},\overline{\boldsymbol{\eta}}^k_{i,t},\overline{\boldsymbol{q}}^k_{i,t})\|_{H^1\times L^2\times L^2} \\ & \qquad - \int_\Omega d(z)\big(\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{\xi}_t) - \nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1})\big)(\boldsymbol{p}^k_{i,t} - \boldsymbol{p}^k_i)\cdot \overline{\boldsymbol{q}}^k_{i,t}\,\mathrm{d}x \\ & \qquad + \int_\Omega d(z)\big(\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{\xi}_t) - \nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1})\big)(\boldsymbol{p}^k_{i-1,t} - \boldsymbol{p}^k_{i-1})\cdot \overline{\boldsymbol{q}}^k_{i,t}\,\mathrm{d}x \\ & \leq \overline{C}_{\gamma,k}\, t\|\varphi\|_\infty\Big(t\|\varphi\|_\infty\big(\|(u^k_i,\boldsymbol{\varepsilon}^k_i,\boldsymbol{p}^k_i)\|_{H^1\times L^2\times L^2} + \|f^k_i\|_2 + 1\big) \\ & \qquad + \|\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{\xi}_t) - \nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1})\|_\nu\Big)\|(\overline{v}^k_{i,t},\overline{\boldsymbol{\eta}}^k_{i,t},\overline{\boldsymbol{q}}^k_{i,t})\|_{H^1\times L^2\times L^2} \\ & \qquad + \widetilde{C}_\gamma \|(\overline{v}^k_{i-1,t},\overline{\boldsymbol{\eta}}^k_{i-1,t},\overline{\boldsymbol{q}}^k_{i-1,t})\|_{H^1\times L^2\times L^2}\|(\overline{v}^k_{i,t},\overline{\boldsymbol{\eta}}^k_{i,t},\overline{\boldsymbol{q}}^k_{i,t})\|_{H^1\times L^2\times L^2} \end{aligned} \tag{5.36}$$

for some positive constants $\widetilde{C}_{\gamma,k}, \overline{C}_{\gamma,k}$ depending on γ and k, for $\widetilde{C}_\gamma$ depending only on γ, and for some $\nu \in (1,+\infty)$. In order to conclude for (5.33) we are left to show that

$$\lim_{t\to 0} \|\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{\xi}_t) - \nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1})\|_\nu = 0\,. \tag{5.37}$$

Arguing as in Proposition 3.4 we get that $\boldsymbol{p}^k_{j,t} \to \boldsymbol{p}^k_j$ in $L^2(\Omega;\mathbb{M}^n_D)$ as $t\to 0$ for every $j = 1,\dots,k$, Hence, up to a subsequence we may assume that $\boldsymbol{p}^k_{j,t} \to \boldsymbol{p}^k_j$ a.e. in Ω for $j=1,\dots,k$, which implies that $\boldsymbol{\xi}_t \to \boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1}$ and $\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{\xi}_t) \to \nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1})$ a.e. in Ω. In view of the bound $|\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{\xi}_t)| \le 2\gamma$ in Ω by the Dominated Convergence Theorem we get (5.37). This, together with (5.36), concludes the proof of (5.33). In particular, estimate (5.33) can be made uniform in i as we have to control a finite number of norms $\|(\overline{v}^k_{i,t}, \overline{\boldsymbol{\eta}}^k_{i,t}, \overline{\boldsymbol{q}}^k_{i,t})\|_{H^1\times L^2\times L^2}$ for $i = 1,\dots,k$. □

6. Optimality conditions

The aim of this section is to provide first-order optimality conditions for the TO problem (3.3), see Theorem 6.4. This will be obtained by passing to the limit in the corresponding optimality conditions for the time-discrete TO problem (3.10). Since we believe this to be of independent interest, also in view of a possible numerical implementation of this TO perspective, we analyse the time-discrete problem in detail in Section 6.1.

6.1. Optimality of the time-discrete problem

In the following we give the first-order optimality conditions for the time-discrete problem (3.10) by passing to the limit as $\gamma \to +\infty$ in (5.3)–(5.4). We start by proving a uniform bound for the adjoint variables $(\overline{u}^k_{i,\gamma}, \overline{\boldsymbol{\varepsilon}}^k_{i,\gamma}, \overline{\boldsymbol{p}}^k_{i,\gamma})^{k+1}_{i=0} \in \big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ satisfying (5.3)–(5.4). From now on, we will use the notation

$$\|\boldsymbol{\varepsilon}\|^2_{\mathbb{C}(z)} := \int_\Omega \mathbb{C}(z)\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}\,\mathrm{d}x \qquad \|\boldsymbol{p}\|^2_{\mathbb{H}(z)} := \int_\Omega \mathbb{H}(z)\boldsymbol{p}\cdot\boldsymbol{p}\,\mathrm{d}x \tag{6.1}$$

for every $z \in L^\infty(\Omega)$, every $\boldsymbol{\varepsilon} \in L^2(\Omega;\mathbb{M}^n_S)$, and every $\boldsymbol{p} \in L^2(\Omega;\mathbb{M}^n_D)$. In view of (2.4)–(2.5), $\|\cdot\|_{\mathbb{C}(z)}$ and $\|\cdot\|_{\mathbb{H}(z)}$ are two norms in $L^2(\Omega;\mathbb{M}^n_S)$ and $L^2(\Omega;\mathbb{M}^n_D)$, respectively, and are both equivalent to the usual L^2-norm, uniformly w.r.t. $z \in L^\infty(\Omega)$.

We now state the main result of this section.

Theorem 6.1 (Optimality for the time-discrete TO problem)**.** *Let $k \in \mathbb{N}$ and f^k_i, g^k_i, w^k_i be defined as in (3.4). For $\gamma \in (0,+\infty)$, let $z_{k,\gamma} \in H^1(\Omega;[0,1])$ be a solution of the approximate time-discrete* TO *problem (3.17). Assume that $z_{k,\gamma} \rightharpoonup z_k$ weakly in $H^1(\Omega)$ as $\gamma \to +\infty$. Then, $z_k \in H^1(\Omega;[0,1])$ solves (3.10) and, denoted with $(u^k_i, \boldsymbol{\varepsilon}^k_i, \boldsymbol{p}^k_i)^k_{i=0}$ the corresponding time-discrete quasistatic evolution, there exist $(\boldsymbol{\rho}^k_i)^k_{i=0}, (\boldsymbol{\pi}^k_i)^{k+1}_{i=1} \in \big(L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$, and $(\overline{u}^k_i, \overline{\boldsymbol{\varepsilon}}^k_i, \overline{\boldsymbol{p}}^k_i)^{k+1}_{i=1} \in \big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ such that for every every $i = 0,\dots,k$, every $(v,\boldsymbol{\eta},\boldsymbol{q}) \in \mathcal{A}(0)$, and $\varphi \in H^1(\Omega)\cap L^\infty(\Omega)$:*

$$\int_\Omega \mathbb{C}(z_k)\boldsymbol{\varepsilon}^k_i\cdot\boldsymbol{\eta}\,\mathrm{d}x + \int_\Omega \mathbb{H}(z_k)\boldsymbol{p}^k_i\cdot\boldsymbol{q}\,\mathrm{d}x + \int_\Omega \boldsymbol{\rho}^k_i\cdot\boldsymbol{q}\,\mathrm{d}x - \int_\Omega \ell(z_k) f^k_i\cdot v\,\mathrm{d}x - \int_{\Gamma_N} g^k_i\cdot v\,\mathrm{d}\mathcal{H}^{n-1} = 0\,, \tag{6.2}$$

$$\boldsymbol{\rho}^k_i\cdot(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1}) = d(z)|\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1}| \quad \textit{in } \Omega\,, \qquad \boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1} = 0 \quad \textit{in } \{|\boldsymbol{\rho}^k_i| < d(z)\}\,, \tag{6.3}$$

$$\int_\Omega \mathbb{C}(z_k)\overline{\boldsymbol{\varepsilon}}_i^k\cdot\boldsymbol{\eta}\,\mathrm{d}x+\int_\Omega \mathbb{H}(z_k)\overline{\boldsymbol{p}}_i^k\cdot\boldsymbol{q}\,\mathrm{d}x + \int_\Omega \boldsymbol{\pi}_i^k\cdot\boldsymbol{q}\,\mathrm{d}x-\int_\Omega \ell(z_k)f_i^k\cdot v\,\mathrm{d}x-\int_{\Gamma_N} g_i^k\cdot v\,\mathrm{d}\mathcal{H}^{n-1}=0\,, \tag{6.4}$$

$$\sum_{j=1}^k\Big(\int_\Omega \varphi\,\ell'(z_k)(f_j^k-f_{j-1}^k)\cdot(u_j^k+\overline{u}_j^k)\,\mathrm{d}x-\int_\Omega \big(\mathbb{C}'(z_k)\varphi\big)(\boldsymbol{\varepsilon}_j^k-\boldsymbol{\varepsilon}_{j-1}^k)\cdot\overline{\boldsymbol{\varepsilon}}_j^k\,\mathrm{d}x - \int_\Omega \big(\mathbb{H}'(z_k)\varphi\big)(\boldsymbol{p}_j^k-\boldsymbol{p}_{j-1}^k)\cdot\overline{\boldsymbol{p}}_j^k\,\mathrm{d}x-\int_\Omega\varphi\,\frac{d'(z_k)}{d'(z_k)}\,(\boldsymbol{\rho}_j^k-\boldsymbol{\rho}_{j-1}^k)\cdot\overline{\boldsymbol{p}}_j^k\,\mathrm{d}x\Big)\,, + \int_\Omega \delta\nabla z_k\cdot\nabla\varphi+\frac{\varphi}{\delta}(z_k(1-z_k)^2-z_k^2(1-z_k))\,\mathrm{d}x=0\,. \tag{6.5}$$

$$\boldsymbol{\pi}_i^k\cdot(\boldsymbol{p}_i^k-\boldsymbol{p}_{i-1}^k)=0 \qquad \text{in } \Omega\,, \tag{6.6}$$

$$\overline{\boldsymbol{p}}_i^k-\overline{\boldsymbol{p}}_{i+1}^k=0 \qquad \text{in } \{|\boldsymbol{\rho}_i^k|<d(z_k)\} \tag{6.7}$$

In order to prove Theorem 6.1, we need to establish some uniform bounds for the adjoint system (5.4) of Corollary 5.4. This is the content of the following proposition.

Proposition 6.2 (Uniform bounds)**.** *For every $\gamma\in(0,+\infty)$ and $k\in\mathbb{N}$, let $z_k^\gamma\in H^1(\Omega;[0,1])$ be a solution of the approximate time-discrete* TO *problem* (3.17) *with corresponding approximate time-discrete quasistatic evolution $(u_i^{k,\gamma},\boldsymbol{\varepsilon}_i^{k,\gamma},\boldsymbol{p}_i^{k,\gamma})_{i=0}^{k+1}\in\big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$. Furthermore, let $(\overline{u}_i^{k,\gamma},\overline{\boldsymbol{\varepsilon}}_i^{k,\gamma},\overline{\boldsymbol{p}}_i^{k,\gamma})_{i=1}^{k+1}\in\big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ be the adjoint variables introduced in Corollary 5.4. Then, $(\overline{u}_i^{k,\gamma},\overline{\boldsymbol{\varepsilon}}_i^{k,\gamma},\overline{\boldsymbol{p}}_i^{k,\gamma})$ is bounded in $H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D)$ uniformly w.r.t. i, k, and γ.*

Proof. We test the equation (5.3) for $i=k,\dots,1$ with the triple $(\overline{u}_i^{k,\gamma}-\overline{u}_{i+1}^{k,\gamma},\overline{\boldsymbol{\varepsilon}}_i^{k,\gamma}-\overline{\boldsymbol{\varepsilon}}_{i+1}^{k,\gamma},\overline{\boldsymbol{p}}_i^{k,\gamma}-\overline{\boldsymbol{p}}_{i+1}^{k,\gamma})\in\mathcal{A}(0)$. Since the function h_γ is convex, we have that

$$\int_\Omega \mathbb{C}(z_k^\gamma)\overline{\boldsymbol{\varepsilon}}_i^{k,\gamma}\cdot(\overline{\boldsymbol{\varepsilon}}_i^{k,\gamma}-\overline{\boldsymbol{\varepsilon}}_{i+1}^{k,\gamma})\,\mathrm{d}x+\int_\Omega \mathbb{H}(z_k^\gamma)\overline{\boldsymbol{p}}_i^{k,\gamma}\cdot(\overline{\boldsymbol{p}}_i^{k,\gamma}-\overline{\boldsymbol{p}}_{i+1}^{k,\gamma})\,\mathrm{d}x - \int_\Omega \ell(z_k^\gamma)f_i^k\cdot(\overline{u}_i^{k,\gamma}-\overline{u}_{i+1}^{k,\gamma})\,\mathrm{d}x-\int_{\Gamma_N} g_i^k\cdot(\overline{u}_i^{k,\gamma}-\overline{u}_{i+1}^{k,\gamma})\,\mathrm{d}\mathcal{H}^{n-1}\le 0\,. \tag{6.8}$$

We rewrite the first term in (6.8) as

$$\int_\Omega \mathbb{C}(z_k^\gamma)\overline{\boldsymbol{\varepsilon}}_i^{k,\gamma}\cdot(\overline{\boldsymbol{\varepsilon}}_i^{k,\gamma}-\overline{\boldsymbol{\varepsilon}}_{i+1}^{k,\gamma})\,\mathrm{d}x=\frac12\|\overline{\boldsymbol{\varepsilon}}_i^{k,\gamma}\|^2_{\mathbb{C}(z_k^\gamma)}-\frac12\|\overline{\boldsymbol{\varepsilon}}_{i+1}^{k,\gamma}\|^2_{\mathbb{C}(z_k^\gamma)}+\frac12\|\overline{\boldsymbol{\varepsilon}}_i^{k,\gamma}-\overline{\boldsymbol{\varepsilon}}_{i+1}^{k,\gamma}\|^2_{\mathbb{C}(z_k^\gamma)}\,.$$

In a similar way we can rewrite the second term in (6.8), obtaining

$$\frac12\|\overline{\boldsymbol{\varepsilon}}_i^{k,\gamma}\|^2_{\mathbb{C}(z_k^\gamma)}-\frac12\|\overline{\boldsymbol{\varepsilon}}_{i+1}^{k,\gamma}\|^2_{\mathbb{C}(z_k^\gamma)}+\frac12\|\overline{\boldsymbol{\varepsilon}}_i^{k,\gamma}-\overline{\boldsymbol{\varepsilon}}_{i+1}^{k,\gamma}\|^2_{\mathbb{C}(z_k^\gamma)} + \frac12\|\overline{\boldsymbol{p}}_i^{k,\gamma}\|^2_{\mathbb{H}(z_k^\gamma),2}-\frac12\|\overline{\boldsymbol{p}}_{i+1}^{k,\gamma}\|^2_{\mathbb{H}(z_k^\gamma),2}+\frac12\|\overline{\boldsymbol{p}}_i^{k,\gamma}-\overline{\boldsymbol{p}}_{i+1}^{k,\gamma}\|^2_{\mathbb{H}(z_k^\gamma),2} - \int_\Omega \ell(z_k^\gamma)f_i^k\cdot(\overline{u}_i^{k,\gamma}-\overline{u}_{i+1}^{k,\gamma})\,\mathrm{d}x-\int_{\Gamma_N} g_i^k\cdot(\overline{u}_i^{k,\gamma}-\overline{u}_{i+1}^{k,\gamma})\,\mathrm{d}\mathcal{H}^{n-1}\le 0\,. \tag{6.9}$$

For every $j \in \{1, \dots, k\}$ we sum up (6.9) over $i = k, \dots, j$ and use that $\overline{u}^k_{k+1,\gamma} = 0$, so that

$$
\begin{aligned}
&\frac{1}{2}\|\overline{\boldsymbol{\varepsilon}}^{k,\gamma}_j\|^2_{\mathbb{C}(z^\gamma_k)} + \frac{1}{2}\|\overline{\boldsymbol{p}}^{k,\gamma}_j\|^2_{\mathbb{H}(z^\gamma_k)} \\
&\qquad \le \sum_{i=k}^{j+1} \bigg(\int_\Omega \ell(z^\gamma_k)(f^k_i - f^k_{i-1}) \cdot \overline{u}^{k,\gamma}_i \,\mathrm{d}x + \int_{\Gamma_N} (g^k_i - g^k_{i-1}) \cdot \overline{u}^{k,\gamma}_i \,\mathrm{d}\mathcal{H}^{n-1} \bigg) \\
&\qquad\quad + \int_\Omega \ell(z^\gamma_k) f^k_j \cdot \overline{u}^{k,\gamma}_j \,\mathrm{d}x + \int_{\Gamma_N} g^k_j \cdot \overline{u}^{k,\gamma}_j \,\mathrm{d}\mathcal{H}^{n-1} \,.
\end{aligned} \tag{6.10}
$$

By Cauchy inequality and by the regularity of f and g we deduce that

$$
\begin{aligned}
&\sup_{i=1,\dots,k} \big(\|\overline{\boldsymbol{\varepsilon}}^{k,\gamma}_i\|_{\mathbb{C}(z^\gamma_k)} + \|\overline{\boldsymbol{p}}^{k,\gamma}_i\|_{\mathbb{H}(z^\gamma_k)} \big)^2 \\
&\le C \sup_{i=1,\dots,k} \big(\|\overline{\boldsymbol{\varepsilon}}^{k,\gamma}_i\|_{\mathbb{C}(z^\gamma_k)} + \|\overline{\boldsymbol{p}}^{k,\gamma}_i\|_{\mathbb{H}(z^\gamma_k)} \big) \bigg(\int_0^T (\|\dot{f}_k(t)\|_2 + \|\dot{g}_k(t)\|_2) \,\mathrm{d}t + \sup_{i=1,\dots,k} (\|f^k_i\|_2 + \|g^k_i\|_2) \bigg) \\
&\le C \sup_{i=1,\dots,k} \big(\|\overline{\boldsymbol{\varepsilon}}^{k,\gamma}_i\|_{\mathbb{C}(z^\gamma_k)} + \|\overline{\boldsymbol{p}}^{k,\gamma}_i\|_{\mathbb{H}(z^\gamma_k)} \big) \big(\|f\|_{H^1(0,T;H^1(\Omega;\mathbb{R}^n))} + \|g\|_{H^1(0,T;L^2(\Gamma_N;\mathbb{R}^n))} \big) \,,
\end{aligned}
$$

for some positive constant C independent of i, k, and γ. The above inequality implies the boundedness of $(\overline{u}^{k,\gamma}_i, \overline{\boldsymbol{\varepsilon}}^{k,\gamma}_i, \overline{\boldsymbol{p}}^{k,\gamma}_i)$ in $H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D)$ uniformly w.r.t. i, k, and γ. □

We now prove Theorem 6.1.

Proof of Theorem 6.1. Let $z^\gamma_k, z_k \in H^1(\Omega;[0,1])$ be as in the statement of the Theorem, and let $(u^{k,\gamma}_i, \boldsymbol{\varepsilon}^{k,\gamma}_i, \boldsymbol{p}^{k,\gamma}_i)^k_{i=0}, (u^k_i, \boldsymbol{\varepsilon}^k_i, \boldsymbol{p}^k_i)^k_{i=0} \in \big(H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ be the corresponding approximate time-discrete and time-discrete evolutions, respectively. By Proposition 3.9, we know that $(u^{k,\gamma}_i, \boldsymbol{\varepsilon}^{k,\gamma}_i, \boldsymbol{p}^{k,\gamma}_i)^k_{i=0}$ converges to $(u^k_i, \boldsymbol{\varepsilon}^k_i, \boldsymbol{p}^k_i)^k_{i=0}$ in $\big(H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ as $\gamma \to +\infty$.

Equations (6.2)–(6.3) are equivalent to the equilibrium condition (3.7) of Definition 3.3. In particular, we have that

$$
\boldsymbol{\rho}^k_i = \Pi_{\mathbb{M}^n_D}\big(\mathbb{C}(z_k)\boldsymbol{\varepsilon}^k_i\big) - \mathbb{H}(z_k)\boldsymbol{p}^k_i \qquad \text{in } L^2(\Omega;\mathbb{M}^n_D)\,.
$$

Furthermore, setting $\boldsymbol{\rho}^{k,\gamma}_i := d(z^\gamma_k)\nabla_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^{k,\gamma}_i - \boldsymbol{p}^{k,\gamma}_{i-1})$ we have that $\boldsymbol{\rho}^{k,\gamma}_i = \Pi_{\mathbb{M}^n_D}\big(\mathbb{C}(z^\gamma_k)\boldsymbol{\varepsilon}^{k,\gamma}_i\big) - \mathbb{H}(z_k)\boldsymbol{p}^{k,\gamma}_i$ and $\boldsymbol{\rho}^{k,\gamma}_i \to \boldsymbol{\rho}^k_i$ in $L^2(\Omega;\mathbb{M}^n_D)$ as $\gamma \to +\infty$ for every i and k.

Denoting by $(\overline{u}^{k,\gamma}_i, \overline{\boldsymbol{\varepsilon}}^{k,\gamma}_i, \overline{\boldsymbol{p}}^{k,\gamma}_i)^{k+1}_{i=1} \in \mathcal{A}(0)^{k+1}$ the adjoint variables introduced in Corollary 5.4, we have by Proposition 6.2 that $(\overline{u}^{k,\gamma}_i, \overline{\boldsymbol{\varepsilon}}^{k,\gamma}_i, \overline{\boldsymbol{p}}^{k,\gamma}_i)$ are bounded in $H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D)$ uniformly w.r.t. i, k, and γ. Thus, we may assume that, up to a subsequence, $(\overline{u}^{k,\gamma}_i, \overline{\boldsymbol{\varepsilon}}^{k,\gamma}_i, \overline{\boldsymbol{p}}^{k,\gamma}_i) \rightharpoonup (\overline{u}^k_i, \overline{\boldsymbol{\varepsilon}}^k_i, \overline{\boldsymbol{p}}^k_i)$ weakly in $H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D)$ as $\gamma \to +\infty$ for every i and k.

In order to prove that $(\overline{u}^k_i, \overline{\boldsymbol{\varepsilon}}^k_i, \overline{\boldsymbol{p}}^k_i)$ satisfies (6.4)–(6.5), we first rewrite the optimality conditions (5.3)–(5.4) for $\gamma \in (0, +\infty)$ in a form similar to (6.4). To this aim, we define for $i = 1, \dots, k$

$$
\boldsymbol{\pi}^{k,\gamma}_i := d(z^\gamma_k)\nabla^2_{\mathbf{Q}} h_\gamma(\boldsymbol{p}^{k,\gamma}_i - \boldsymbol{p}^{k,\gamma}_{i-1})(\overline{\boldsymbol{p}}^{k,\gamma}_i - \overline{\boldsymbol{p}}^{k,\gamma}_{i+1}) \qquad \text{in } L^2(\Omega;\mathbb{M}^n_D)\,.
$$

Hence, we rewrite (5.3) as

$$
\begin{aligned}
&\int_\Omega \mathbb{C}(z^\gamma_k)\overline{\boldsymbol{\varepsilon}}^{k,\gamma}_i \cdot \boldsymbol{\eta} \,\mathrm{d}x + \int_\Omega \mathbb{H}(z^\gamma_k)\overline{\boldsymbol{p}}^{k,\gamma}_i \cdot \boldsymbol{q} \,\mathrm{d}x + \int_\Omega \boldsymbol{\pi}^{k,\gamma}_i \cdot \boldsymbol{q} \,\mathrm{d}x \\
&\qquad - \int_\Omega \ell(z^\gamma_k) f^k_i \cdot v \,\mathrm{d}x - \int_{\Gamma_N} g^k_i \cdot v \,\mathrm{d}\mathcal{H}^{n-1} = 0\,,
\end{aligned} \tag{6.11}
$$

for $(v,\boldsymbol{\eta},\boldsymbol{q})\in\mathcal{A}(0)$. From (6.11) tested against $(0,-\boldsymbol{q},\boldsymbol{q})$ with $\boldsymbol{q}\in L^2(\Omega;\mathbb{M}^n_D)$ we deduce that

$$\boldsymbol{\pi}_i^{k,\gamma}=\Pi_{\mathbb{M}^n_D}\big(\mathbb{C}(z_k^\gamma)\overline{\boldsymbol{\varepsilon}}_i^{k,\gamma}\big)-\mathbb{H}(z_k^\gamma)\overline{\boldsymbol{p}}_i^{k,\gamma}\,.$$

Thus, setting

$$\boldsymbol{\pi}_i^{k}:=\Pi_{\mathbb{M}^n_D}\big(\mathbb{C}(z_k)\overline{\boldsymbol{\varepsilon}}_i^{k}\big)-\mathbb{H}(z_k)\overline{\boldsymbol{p}}_i^{k}\qquad\text{for } i=1,\dots,k+1$$

we infer that $\boldsymbol{\pi}_i^{k,\gamma}\rightharpoonup\boldsymbol{\pi}_i^k$ weakly in $L^2(\Omega;\mathbb{M}^n_D)$ as $\gamma\to+\infty$, for every k and i. Passing to the limit as $\gamma\to+\infty$ in (6.11) we deduce (6.4).

As for (6.5), we rewrite (5.4) as

$$\begin{aligned}\sum_{j=1}^{k}\bigg(&\int_\Omega\varphi\,\ell'(z_k^\gamma)(f_j^k-f_{j-1}^k)\cdot(u_j^{k,\gamma}+\overline{u}_j^{k,\gamma})\,\mathrm{d}x-\int_\Omega\big(\mathbb{C}'(z_k^\gamma)\varphi\big)(\boldsymbol{\varepsilon}_j^{k,\gamma}-\boldsymbol{\varepsilon}_{j-1}^{k,\gamma})\cdot\overline{\boldsymbol{\varepsilon}}_j^{k,\gamma}\,\mathrm{d}x\\&-\int_\Omega\big(\mathbb{H}'(z_k^\gamma)\varphi\big)(\boldsymbol{p}_j^{k,\gamma}-\boldsymbol{p}_{j-1}^{k,\gamma})\cdot\overline{\boldsymbol{p}}_j^{k,\gamma}\,\mathrm{d}x-\int_\Omega\varphi\,\frac{d'(z_k^\gamma)}{d(z_k^\gamma)}\,(\boldsymbol{\rho}_j^{k,\gamma}-\boldsymbol{\rho}_{j-1}^{k,\gamma})\cdot\overline{\boldsymbol{p}}_j^{k,\gamma}\,\mathrm{d}x\bigg)\\&+\int_\Omega\delta\nabla z_k^\gamma\cdot\nabla\varphi+\frac{\varphi}{\delta}(z_k^\gamma(1-z_k^\gamma)^2-(z_k^\gamma)^2(1-z_k^\gamma))\,\mathrm{d}x=0\,.\end{aligned}\tag{6.12}$$

Owing to the convergences discussed above, we again infer (6.5) by passing to the limit in (6.12) as $\gamma\to+\infty$.

Finally, the proof of (6.6)–(6.7) can be obtained by repeating step by step the proof of formula (4.5), Theorem 4.1 in [2]. ☐

Remark 6.3. We notice that $(\overline{u}_i^k,\overline{\boldsymbol{\varepsilon}}_i^k,\overline{\boldsymbol{p}}_i^k)$ is bounded in $H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega,\mathbb{M}^n_D)$ uniformly w.r.t. i and k as a consequence of Proposition 6.2.

6.2. Optimality of the time-continuous problem

We conclude with the first-order optimality conditions for the TO problem (3.3). Most of the conditions follow directly from those computed in Theorem 6.1 by passing to the limit as the time step τ_k tends to 0. The only difficulty is to find the time-continuous condition corresponding to (6.7), since the adjoint variable $\overline{\boldsymbol{p}}_i^k$ can be bounded in $L^2(\Omega;\mathbb{M}^n_D)$ uniformly w.r.t. i and k (see Rem. 6.3), but no time regularity is expected.

Theorem 6.4 (Optimality for the TO problem)**.** *Let $z_k\in H^1(\Omega;[0,1])$ be the sequence of solutions of the time-discrete* TO *problem (3.10) found in Theorem 6.1. Then, there exists $z\in H^1(\Omega;[0,1])$ solving (3.3) such that, up to a subsequence, $z_k\rightharpoonup z$ weakly in $H^1(\Omega)$. Denoting by $(u(\cdot),\boldsymbol{\varepsilon}(\cdot),\boldsymbol{p}(\cdot))$ the quasistatic evolution corresponding to z, there exists $\boldsymbol{\rho}\in H^1(0,T;L^2(\Omega;\mathbb{M}^n_D))$ such that for every $(v,\boldsymbol{\eta},\boldsymbol{q})\in\mathcal{A}(0)$ and every $t\in[0,T]$ the following holds:*

$$\begin{aligned}\int_\Omega\mathbb{C}(z)\boldsymbol{\varepsilon}(t)\cdot\boldsymbol{\eta}\,\mathrm{d}x+\int_\Omega\mathbb{H}(z)\boldsymbol{p}(t)\cdot\boldsymbol{q}\,\mathrm{d}x+\int_\Omega\boldsymbol{\rho}(t)\cdot\boldsymbol{q}\,\mathrm{d}x&\\-\int_\Omega\ell(z)f(t)\cdot v\,\mathrm{d}x-\int_{\Gamma_N}g(t)\cdot v\,\mathrm{d}x=0\,,&\end{aligned}\tag{6.13}$$

$$\boldsymbol{\rho}(t)\cdot\dot{\boldsymbol{p}}(t)=d(z)|\dot{\boldsymbol{p}}(t)|\quad\textit{in }\Omega\,,\qquad|\dot{\boldsymbol{p}}(t)|=0\quad\textit{in }\{|\boldsymbol{\rho}(t)|<d(z)\}\,.\tag{6.14}$$

Furthermore, there exist the adjoint variables $\overline{\boldsymbol{p}}_0\in L^2(\Omega;\mathbb{M}^n_D)$, $\boldsymbol{\pi}\in L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_D))$, and $(\overline{u},\overline{\boldsymbol{\varepsilon}},\overline{\boldsymbol{p}})\in L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))$, such that for every $(v,\boldsymbol{\eta},\boldsymbol{q})\in\mathcal{A}(0)$, for every $\varphi\in H^1(\Omega)\cap$

$L^\infty(\Omega)$, *and for a.e.* $t \in [0,T]$ *we have*

$$\begin{aligned}\int_\Omega \mathbb{C}(z)\overline{\boldsymbol\varepsilon}(t)\cdot\boldsymbol\eta\,\mathrm{d}x + \int_\Omega \mathbb{H}(z)\overline{\boldsymbol p}(t)\cdot\boldsymbol q,\mathrm{d}x + \int_\Omega \boldsymbol\pi(t)\cdot\boldsymbol q\,\mathrm{d}x \\ - \int_\Omega \ell(z)f(t)\cdot v\,\mathrm{d}x - \int_{\Gamma_N} g(t)\cdot v\,\mathrm{d}\mathcal{H}^{n-1} = 0\,,\end{aligned} \tag{6.15}$$

$$\begin{aligned}\int_0^T \Big(\int_\Omega \varphi\,\ell'(z)\dot f(t)\cdot(u(t)+\overline{u}(t))\,\mathrm{d}x - \int_\Omega \big(\mathbb{C}'(z)\varphi\big)\dot{\boldsymbol\varepsilon}(t)\cdot\overline{\boldsymbol\varepsilon}(t)\,\mathrm{d}x \\ - \int_\Omega \big(\mathbb{H}'(z)\varphi\big)\dot{\boldsymbol p}(t)\cdot\overline{\boldsymbol p}(t)\,\mathrm{d}x - \int_\Omega \varphi\,\frac{d'(z)}{d(z)}\,\dot{\boldsymbol\rho}(t)\cdot\overline{\boldsymbol p}(t)\,\mathrm{d}x\Big)\mathrm{d}t \\ + \int_\Omega \delta\nabla z\cdot\nabla\varphi + \frac{\varphi}{\delta}(z(1-z)^2 - z^2(1-z))\,\mathrm{d}x = 0\,.\end{aligned} \tag{6.16}$$

$$\boldsymbol\pi(t)\cdot\dot{\boldsymbol p}(t) = 0 \quad \textit{in } \Omega\,, \tag{6.17}$$

$$\int_\Omega d^2(z)\,\overline{\boldsymbol p}_0\,\mathrm{d}x - 2\int_0^T\int_\Omega (\boldsymbol\rho(t)\cdot\dot{\boldsymbol\rho}(t))\,\overline{\boldsymbol p}(t)\,\mathrm{d}x\,\mathrm{d}t = 0\,. \tag{6.18}$$

Proof. Conditions (6.13)–(6.14) are a direct consequence of Definition 3.1. In particular, $\boldsymbol\rho(t) = \Pi_{\mathbb{M}^n_D}\big(\mathbb{C}(z)\boldsymbol\varepsilon(t)\big) - \mathbb{H}(z)\boldsymbol p(t)$ and $\boldsymbol\rho(t) \in d(z)\partial|\cdot|(\dot{\boldsymbol p}(t))$ for a.e. $t\in[0,T]$.

Let us consider the time-discrete quasistatic evolution $(u^k_i, \boldsymbol\varepsilon^k_i, \boldsymbol p^k_i)_{i=0}^k \in \big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega,\mathbb{M}^n_D)\big)^{k+1}$ associated with z_k and let $(\overline{u}^k_i, \overline{\boldsymbol\varepsilon}^k_i, \overline{\boldsymbol p}^k_i)_{i=1}^{k+1} \in \big(H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega,\mathbb{M}^n_D)\big)^{k+1}$, $(\boldsymbol\rho^k_i)_{i=0}^k, (\boldsymbol\pi^k_i)_{i=1}^{k+1} \in \big(L^2(\Omega;\mathbb{M}^n_D)\big)^{k+1}$ be the corresponding adjoint variables introduced in Theorem 6.1. We further define the interpolation functions

$$\begin{aligned}&\overline{u}_k(t) := \overline{u}^k_i\,, && \overline{\boldsymbol\varepsilon}_k(t) := \overline{\boldsymbol\varepsilon}^k_i\,, && \overline{\boldsymbol p}_k(t) := \overline{\boldsymbol p}^k_i\,,\\ &\tilde u_k(t) := u^k_i\,, && \tilde{\boldsymbol\varepsilon}_k(t) := \boldsymbol\varepsilon^k_i\,, && \tilde{\boldsymbol p}_k(t) := \boldsymbol p^k_i\,,\\ &\tilde{\boldsymbol\rho}_k(t) := \boldsymbol\rho^k_i\,, && \tilde{\boldsymbol\pi}_k(t) := \boldsymbol\pi^k_i\,, && \\ &\tilde f_k(t) := f^k_i\,, && \tilde g_k(t) := g^k_i\,, && \tilde w_k(t) := w^k_i\,,\\ &\boldsymbol\rho_k(t) := \boldsymbol\rho^k_{i-1} + \frac{(t-t^k_{i-1})}{\tau_k}\,(\boldsymbol\rho^k_i - \boldsymbol\rho^k_{i-1})\end{aligned}$$

for $t\in(t^k_{i-1}, t^k_i]$. We recall that the piecewise affine interpolation functions f_k, g_k, w_k, u_k, $\boldsymbol\varepsilon_k$, and $\boldsymbol p_k$ have been introduced in (3.5) and (3.11). As a consequence of Proposition 3.5, we have that $(\tilde u_k, \tilde{\boldsymbol\varepsilon}_k, \tilde{\boldsymbol p}_k) \to (u, \boldsymbol\varepsilon, \boldsymbol p)$ in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))$ and $\boldsymbol\rho_k \to \boldsymbol\rho$ in $H^1(0,T;L^2(\Omega;\mathbb{M}^n_D))$ (see also Lemma. A.1). By the equilibrium conditions (6.2)–(6.3) we also infer that $\tilde{\boldsymbol\rho}_k(t) \in d(z_k)\partial|\cdot|(\dot{\boldsymbol p}_k(t))$, which implies that $\tilde{\boldsymbol\rho}_k, \boldsymbol\rho_k$ are bounded in $L^\infty(0,T;L^\infty(\Omega;\mathbb{M}^n_D))$. Moreover, by Proposition 6.2 we have that $\tilde{\boldsymbol\pi}_k$ and $(\overline{u}_k, \overline{\boldsymbol\varepsilon}_k, \overline{\boldsymbol p}_k)$ are bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_D))$ and in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))$, respectively. Therefore, we may assume that, up to a subsequence, $\tilde{\boldsymbol\pi} \rightharpoonup^* \boldsymbol\pi$ weakly* in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_D))$ and $(\overline{u}_k, \overline{\boldsymbol\varepsilon}_k, \overline{\boldsymbol p}_k) \rightharpoonup^* (\overline{u}, \overline{\boldsymbol\varepsilon}, \overline{\boldsymbol p})$ weakly* in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))$.

Let us show that $\boldsymbol\pi$ and $(\overline{u}, \overline{\boldsymbol\varepsilon}, \overline{\boldsymbol p})$ satisfy (6.15)–(6.18). We start with (6.15). Let us fix an at most countable and dense subset D of $\mathcal{A}(0)$. For every $(v, \boldsymbol\eta, \boldsymbol q)\in D$, every $\psi\in C^\infty_c(0,T)$, and every $t\in[0,T]$, we consider the test function $(\psi(t)v, \psi(t)\boldsymbol{\eta}, \psi(t)\boldsymbol{p}) \in \mathcal{A}(0)$ and rewrite the optimality condition (6.4) as

$$\begin{aligned}\int_\Omega \mathbb{C}(z_k)\overline{\boldsymbol{\varepsilon}}_k(t)\cdot\psi(t)\boldsymbol{\eta}\,\mathrm{d}x + \int_\Omega \mathbb{H}(z_k)\overline{\boldsymbol{p}}_k(t)\cdot\psi(t)\boldsymbol{q}\,\mathrm{d}x + \int_\Omega \tilde{\boldsymbol{\pi}}_k(t)\cdot\psi(t)\boldsymbol{q}\,\mathrm{d}x \\ - \int_\Omega \ell(z_k)\tilde{f}_k(t)\cdot\psi(t)v\,\mathrm{d}x - \int_{\Gamma_N}\tilde{g}_k(t)\cdot\psi(t)v\,\mathrm{d}\mathcal{H}^{n-1} = 0\,.\end{aligned} \tag{6.19}$$

We integrate (6.19) over $[0,T]$ and pass to the limit as $k\to\infty$. In view of the above convergences, we infer that for every $\psi \in C_c^\infty(0,T)$

$$\begin{aligned}\int_0^T \psi(t)\Big(\int_\Omega \mathbb{C}(z)\overline{\boldsymbol{\varepsilon}}(t)\cdot\boldsymbol{\eta}\,\mathrm{d}x + \int_\Omega \mathbb{H}(z)\overline{\boldsymbol{p}}(t)\cdot\boldsymbol{q}\,\mathrm{d}x + \int_\Omega \boldsymbol{\pi}(t)\cdot\boldsymbol{q}\,\mathrm{d}x \\ - \int_\Omega \ell(z)f(t)\cdot v\,\mathrm{d}x - \int_{\Gamma_N} g(t)\cdot v\,\mathrm{d}\mathcal{H}^{n-1}\Big)\,\mathrm{d}t = 0\,.\end{aligned} \tag{6.20}$$

Since D is at most coutable, we deduce from (6.20) that (6.15) holds for a.e. $t\in[0,T]$ and for every $(v,\boldsymbol{\eta},\boldsymbol{q})\in D$. By density we extend the equality to $\mathcal{A}(0)$.

Arguing in the same way, we can also prove that (6.17) holds for a.e. $t\in[0,T]$, as the corresponding time-discrete condition (6.6) holds for every $t\in[0,T]$ and only the time derivative $\dot{\boldsymbol{p}}_k$ is involved, which converges to $\dot{\boldsymbol{p}}$ in $L^2(0,T;L^2(\Omega;\mathbb{M}^n_D))$.

As for (6.16), for every $\varphi\in H^1(\Omega)\cap L^\infty(\Omega)$ we rewrite (6.5) as

$$\begin{aligned}\int_0^T \bigg(\int_\Omega \varphi\,\ell'(z_k)\dot{f}_k(t)\cdot(\tilde{u}_k(t)+\overline{u}_k(t))\,\mathrm{d}x - \int_\Omega \big(\mathbb{C}'(z_k)\varphi\big)\dot{\boldsymbol{\varepsilon}}_k(t)\cdot\overline{\boldsymbol{\varepsilon}}_k(t)\,\mathrm{d}x \\ - \int_\Omega \big(\mathbb{H}'(z_k)\varphi\big)\dot{\boldsymbol{p}}_k(t)\cdot\overline{\boldsymbol{p}}_k(t)\,\mathrm{d}x - \int_\Omega \varphi\,\frac{d'(z_k)}{d(z_k)}\,\dot{\boldsymbol{\rho}}_k(t)\cdot\overline{\boldsymbol{p}}_k(t)\,\mathrm{d}x\bigg)\mathrm{d}t \\ + \int_\Omega \delta\nabla z_k\cdot\nabla\varphi + \frac{\varphi}{\delta}(z_k(1-z_k)^2 - z_k^2(1-z_k))\,\mathrm{d}x = 0\,.\end{aligned} \tag{6.21}$$

Thus, condition (6.16) is obtained by passing to the limit in (6.21) as $k\to\infty$ relying on the continuity of ℓ', d', $\mathbb{C}'$, and $\mathbb{H}'$, and on the convergences discussed above.

We conclude with (6.18). First we notice that, thanks to (6.3), (6.7) can be equivalently expressed as

$$\sum_{i=1}^k \int_\Omega (d^2(z_k) - |\boldsymbol{\rho}_i^k|^2)|\overline{\boldsymbol{p}}_i^k - \overline{\boldsymbol{p}}_{i+1}^k|\,\mathrm{d}x = 0\,, \tag{6.22}$$

which, owing to the fact that $|\boldsymbol{\rho}_i^k| \le d(z_i^k)$, implies

$$\sum_{i=1}^k \int_\Omega (d^2(z_k) - |\boldsymbol{\rho}_i^k|^2)(\overline{\boldsymbol{p}}_i^k - \overline{\boldsymbol{p}}_{i+1}^k)\,\mathrm{d}x = 0\,. \tag{6.23}$$

Recalling that $\overline{\boldsymbol{p}}_{k+1}^k = 0$, we rewrite (6.23) as follows:

$$\begin{aligned}0 &= \int_\Omega (d^2(z_k) - |\boldsymbol{\rho}_1^k|^2)\,\overline{\boldsymbol{p}}_1^k\,\mathrm{d}x - \sum_{i=2}^k \int_\Omega \big((\boldsymbol{\rho}_{i-1}^k + \boldsymbol{\rho}_i^k)\cdot(\boldsymbol{\rho}_i^k - \boldsymbol{\rho}_{i-1}^k)\big)\,\overline{\boldsymbol{p}}_i^k\,\mathrm{d}x \\ &= \int_\Omega (d^2(z_k) - |\boldsymbol{\rho}_1^k|^2)\,\overline{\boldsymbol{p}}_1^k\,\mathrm{d}x - \int_{t_1^k}^T \int_\Omega \big((\tilde{\boldsymbol{\rho}}_k(t) + \tilde{\boldsymbol{\rho}}_k(t-\tau_k))\cdot\dot{\boldsymbol{\rho}}_k(t)\big)\,\overline{\boldsymbol{p}}_k(t)\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned} \tag{6.24}$$

Since $\overline{\boldsymbol{p}}_1^k$ is bounded in $L^2(\Omega;\mathbb{M}^n_D)$, there exists $\overline{\boldsymbol{p}}_0 \in L^2(\Omega;\mathbb{M}^n_D)$ such that, up to a subsequence, $\overline{\boldsymbol{p}}_1^k \rightharpoonup \overline{\boldsymbol{p}}_0$ weakly in $L^2(\Omega;\mathbb{M}^n_D)$. Since $\boldsymbol{\rho}_k \to \boldsymbol{\rho}$ in $H^1(0,T;L^2(\Omega;\mathbb{M}^n_D))$, both interpolating functions $\tilde{\boldsymbol{\rho}}_k$ and $\tilde{\boldsymbol{\rho}}_k(\cdot-\tau_k)$ converge to $\boldsymbol{\rho}$ in $L^2(0,T;L^2(\Omega;\mathbb{M}^n_D))$ and $\boldsymbol{\rho}_1^k \to \boldsymbol{\rho}(0) = 0$ in $L^2(\Omega;\mathbb{M}^n_D)$. Moreover, $\tilde{\boldsymbol{\rho}}_k$ is bounded in $L^\infty(0,T;L^\infty(\Omega;\mathbb{M}^n_D))$. Thus, $\boldsymbol{\rho}_1^k \to 0$ also in $L^r(\Omega;\mathbb{M}^n_D)$ for every $r \in (1,+\infty)$ and

$$\lim_{k\to\infty}\int_\Omega (d^2(z_k) - |\boldsymbol{\rho}_1^k|^2)\,\overline{\boldsymbol{p}}_1^k \,\mathrm{d}x = \int_\Omega d^2(z)\overline{\boldsymbol{p}}_0\,\mathrm{d}x\,.$$

As $\dot{\tilde{\boldsymbol{\rho}}}_k$ converges to $\dot{\tilde{\boldsymbol{\rho}}}$ in $L^2(0,T;L^2(\Omega;\mathbb{M}^n_D))$ and $\tilde{\boldsymbol{\rho}}_k$ is bounded in $L^\infty(0,T;L^\infty(\Omega;\mathbb{M}^n_D))$, we have that $(\tilde{\boldsymbol{\rho}}_k + \tilde{\boldsymbol{\rho}}_k(\cdot-\tau_k))\cdot\dot{\boldsymbol{\rho}}_k$ converges to $2\boldsymbol{\rho}\cdot\dot{\boldsymbol{\rho}}$ in $L^2(0,T;L^2(\Omega))$. Since $\overline{\boldsymbol{p}}_k \rightharpoonup^* \overline{\boldsymbol{p}}$ weakly* in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_D))$, we obtain that

$$\lim_{k\to\infty}\int_{t_1^k}^T\int_\Omega \big((\tilde{\boldsymbol{\rho}}_k(t)+\tilde{\boldsymbol{\rho}}_k(t-\tau_k))\cdot\dot{\boldsymbol{\rho}}_k(t)\big)\,\overline{\boldsymbol{p}}_k(t)\,\mathrm{d}x\,\mathrm{d}t = 2\int_0^T\int_\Omega (\boldsymbol{\rho}(t)\cdot\dot{\boldsymbol{\rho}}(t))\overline{\boldsymbol{p}}(t)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Hence, passing to the limit in (6.24) and deduce (6.18) □

Appendix A. Proof or Proposition 3.5

We start by recalling that by the definition of quasistatic evolution (see Def. 3.1), there exists $\boldsymbol{\rho} \in L^\infty((0,T)\times\Omega;\mathbb{M}^n_D))$ such that $\boldsymbol{\rho}(t) \in d(z)\partial|\cdot|(\dot{\boldsymbol{p}}(t))$ almost everywhere (see, *e.g.*, [12]), where the symbol ∂ denotes here the subdifferential, and such that for $t\in[0,T]$ the equilibrium condition (3.1) is equivalent to

$$\begin{aligned}\int_\Omega \mathbb{C}(z)\boldsymbol{\varepsilon}(t)\cdot\boldsymbol{\eta}\,\mathrm{d}x + \int_\Omega \mathbb{H}(z)\boldsymbol{p}(t)\cdot\boldsymbol{q}\,\mathrm{d}x + \int_\Omega \boldsymbol{\rho}(t)\cdot\boldsymbol{q}\,\mathrm{d}x&\\ - \int_\Omega \ell(z)f(t)\cdot v\,\mathrm{d}x - \int_{\Gamma_N} g(t)\cdot v\,\mathrm{d}\mathcal{H}^{n-1} = 0&\end{aligned} \tag{A.1}$$

for every $(v,\boldsymbol{\eta},\boldsymbol{q}) \in \mathcal{A}(0)$.

In the next lemma we prove that the piecewise affine functions defined in (3.11) converge in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))$ to a quasistatic evolution.

Lemma A.1. *Let $z_k, z \in H^1(\Omega;[0,1])$, $(u_k,\boldsymbol{\varepsilon}_k,\boldsymbol{p}_k)$, and $(u(\cdot),\boldsymbol{\varepsilon}(\cdot),\boldsymbol{p}(\cdot))$ be as in Proposition 3.5. Then, $(u_k,\boldsymbol{\varepsilon}_k,\boldsymbol{p}_k)$ converges to $(u(\cdot),\boldsymbol{\varepsilon}(\cdot),\boldsymbol{p}(\cdot))$ in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))$.*

Proof. We first show that $(u_k,\boldsymbol{\varepsilon}_k,\boldsymbol{p}_k)$ is bounded in $H^1(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))$. We notice that the L^∞-boundedness is a consequence of the energy inequality (3.8).

By the uniform convexity of the functional $\mathcal{E}_k(z_k,t_i^k,\cdot,\cdot,\cdot)+\mathcal{D}(z_k,\cdot-\boldsymbol{p}_{i-1}^k)$ and by testing the minimality of $(u_i^k,\boldsymbol{\varepsilon}_i^k,\boldsymbol{p}_i^k)$ at time t_i^k with $(u_{i-1}^k + w_i^k - w_{i-1}^k, \boldsymbol{\varepsilon}_{i-1}^k + \mathrm{E}w_i^k - \mathrm{E}w_{i-1}^k, \boldsymbol{p}_{i-1}^k) \in \mathcal{A}(w_i^k)$ and the minimality of $(u_{i-1}^k,\boldsymbol{\varepsilon}_{i-1}^k,\boldsymbol{p}_{i-1}^k)$ at time t_{i-1}^k with $(u_i^k - w_i^k + w_{i-1}^k, \boldsymbol{\varepsilon}_i^k - \mathrm{E}w_i^k + \mathrm{E}w_{i-1}^k, \boldsymbol{p}_i^k) \in \mathcal{A}(w_{i-1}^k)$, we have that

$$\begin{aligned}
c\big(\|\boldsymbol{\varepsilon}_i^k &- (\boldsymbol{\varepsilon}_{i-1}^k - \mathrm{E}w_{i-1}^k + \mathrm{E}w_i^k)\|_2^2 + \|\boldsymbol{p}_i^k - \boldsymbol{p}_{i-1}^k\|_2^2\big)\\
&\le \int_\Omega \mathbb{C}(z_k)(\mathrm{E}w_i^k - \mathrm{E}w_{i-1}^k)\cdot(\mathrm{E}w_i^k - \mathrm{E}w_{i-1}^k)\,\mathrm{d}x\\
&\quad - \int_\Omega \mathbb{C}(z_k)(\boldsymbol{\varepsilon}_i^k - \boldsymbol{\varepsilon}_{i-1}^k)\cdot(\mathrm{E}w_i^k - \mathrm{E}w_{i-1}^k)\,\mathrm{d}x\\
&\quad + \int_\Omega \ell(z_k)(f_i^k - f_{i-1}^k)\cdot(u_i^k - (u_{i-1}^k - w_{i-1}^k + w_i^k))\,\mathrm{d}x\\
&\quad + \int_{\Gamma_N} (g_i^k - g_{i-1}^k)\cdot(u_i^k - (u_{i-1}^k - w_{i-1}^k + w_i^k))\,\mathrm{d}\mathcal{H}^{n-1}
\end{aligned}$$

$$
\begin{aligned}
&+ \mathcal{D}(z_k, \boldsymbol{p}_i^k - \boldsymbol{p}_{i-2}^k) - \mathcal{D}(z_k, \boldsymbol{p}_{i-1}^k - \boldsymbol{p}_{i-2}^k) - \mathcal{D}(z_k, \boldsymbol{p}_i^k - \boldsymbol{p}_{i-1}^k) \\
&\le C\big(\|\boldsymbol{\varepsilon}_i^k - (\boldsymbol{\varepsilon}_{i-1}^k - \mathrm{E}w_{i-1}^k + \mathrm{E}w_i^k)\|_2 + \|u_i^k - (u_{i-1}^k - w_{i-1}^k + w_i^k)\|_{H^1}\big)\big(\|f_i^k - f_{i-1}^k\|_2 \\
&+ \|g_i^k - g_{i-1}^k\|_2 + \|w_i^k - w_{i-1}^k\|_{H^1}\big)\,,
\end{aligned}
$$

from which we deduce the bound in $H^1(0,T; H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D))$. In particular, this implies that $(u_k, \boldsymbol{\varepsilon}_k, \boldsymbol{p}_k)$ converges to $(u, \boldsymbol{\varepsilon}, \boldsymbol{p})$ weakly in $H^1(0,T; H^1(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{M}^n_S) \times L^2(\Omega;\mathbb{M}^n_D))$, and the derivatives exist a.e. in $(0,T)$.

Defining $\boldsymbol{\rho}_i^k := \Pi_{\mathbb{M}^n_D}\big(\mathbb{C}(z_k)\boldsymbol{\varepsilon}_i^k\big) - \mathbb{H}(z_k)\boldsymbol{p}_i^k$, we have that the stability condition (6.2) holds. Setting

$$
\boldsymbol{\rho}_k(t) := \boldsymbol{\rho}_i^k + \frac{(t - t_i^k)}{\tau_k}\,(\boldsymbol{\rho}_{i+1}^k - \boldsymbol{\rho}_i^k) \qquad \text{for } t \in [t_i^k, t_{i+1}^k)
$$

we have that $\boldsymbol{\rho}_k$ is bounded in $H^1(0,T; L^2(\Omega;\mathbb{M}^n_D))$ as well.

We proceed now by proving the uniform convergence. To this end, we need to introduce the piecewise constant interpolants

$$
\begin{aligned}
&\tilde{u}_k(t) := u_i^k\,, \qquad \tilde{\boldsymbol{\varepsilon}}_k(t) := \boldsymbol{\varepsilon}_i^k\,, \qquad \tilde{\boldsymbol{p}}_k(t) := \boldsymbol{p}_i^k\,, \qquad \tilde{\boldsymbol{\rho}}_k(t) := \boldsymbol{\rho}_i^k, \\
&\tilde{f}_k(t) := f_i^k\,, \qquad \tilde{g}_k(t) := g_i^k\,, \qquad \tilde{w}_k(t) := w_i^k
\end{aligned}
$$

for $t \in (t_{i-1}^k, t_i^k]$. In particular, $\|\tilde{u}_k(t) - u_k(t)\|_{H^1} \le \tau_k \|\dot{u}_k(t)\|_{H^1}$, and similar inequalities hold for $\tilde{\boldsymbol{\varepsilon}}_k$ and $\tilde{\boldsymbol{p}}_k$ for the L^2-norm.

For a.e. $t \in (t_{i-1}^k, t_i^k]$, we test the equilibrium conditions (6.2) and (A.1) with the triple

$$
(\dot{u}(t) - \dot{u}_k(t) + \dot{w}_k(t) - \dot{w}(t), \dot{\boldsymbol{\varepsilon}}(t) - \dot{\boldsymbol{\varepsilon}}_k(t) + \mathrm{E}\dot{w}_k(t) - \mathrm{E}\dot{w}(t), \dot{\boldsymbol{p}}(t) - \dot{\boldsymbol{p}}_k(t)) \in \mathcal{A}(0)
$$

and we subtract one from the other, obtaining

$$
\begin{aligned}
&\int_\Omega \mathbb{C}(z_k)\tilde{\boldsymbol{\varepsilon}}_k(t)\cdot(\dot{\boldsymbol{\varepsilon}}(t) - \dot{\boldsymbol{\varepsilon}}_k(t) + \mathrm{E}\dot{w}_k(t) - \mathrm{E}\dot{w}(t))\,\mathrm{d}x \\
&\quad - \int_\Omega \mathbb{C}(z)\boldsymbol{\varepsilon}(t)\cdot(\dot{\boldsymbol{\varepsilon}}(t) - \dot{\boldsymbol{\varepsilon}}_k(t) + \mathrm{E}\dot{w}_k(t) - \mathrm{E}\dot{w}(t))\,\mathrm{d}x \\
&\quad + \int_\Omega \mathbb{H}(z_k)\tilde{\boldsymbol{p}}_k(t)\cdot(\dot{\boldsymbol{p}}(t) - \dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x - \int_\Omega \mathbb{H}(z)\boldsymbol{p}(t)\cdot(\dot{\boldsymbol{p}}(t) - \dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x \\
&\quad + \int_\Omega \tilde{\boldsymbol{\rho}}_k(t)\cdot(\dot{\boldsymbol{p}}(t) - \dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x - \int_\Omega \boldsymbol{\rho}(t)\cdot(\dot{\boldsymbol{p}}(t) - \dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x \\
&\quad - \int_\Omega \ell(z_k)\tilde{f}_k(t)\cdot(\dot{u}(t) - \dot{u}_k(t) + \dot{w}_k(t) - \dot{w}(t))\,\mathrm{d}x \\
&\quad + \int_\Omega \ell(z) f(t)\cdot(\dot{u}(t) - \dot{u}_k(t) + \dot{w}_k(t) - \dot{w}(t))\,\mathrm{d}x \\
&\quad - \int_{\Gamma_N} (\tilde{g}_k(t) - g(t))\cdot(\dot{u}(t) - \dot{u}_k(t) + \dot{w}_k(t) - \dot{w}(t))\,\mathrm{d}\mathcal{H}^{n-1} = 0\,.
\end{aligned}
\tag{A.2}
$$

We notice that, being $\boldsymbol{\rho}(t) \in d(z)\partial|\cdot|(\dot{\boldsymbol{p}}(t))$ and $\tilde{\boldsymbol{\rho}}_k(t) \in d(z_k)\partial|\cdot|(\dot{\boldsymbol{p}}_k(t))$ almost everywhere in Ω, it holds

$$
\int_\Omega \left(\frac{d(z_k)}{d(z)}\,\boldsymbol{\rho}(t) - \tilde{\boldsymbol{\rho}}_k(t)\right)\cdot(\dot{\boldsymbol{p}}(t) - \dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x \ge 0\,. \tag{A.3}
$$

Hence, adding and subtracting in (A.2) the terms

$$\int_\Omega \mathbb{C}(z_k)(\boldsymbol{\varepsilon}_k(t)+\boldsymbol{\varepsilon}(t))\cdot(\dot{\boldsymbol{\varepsilon}}(t)-\dot{\boldsymbol{\varepsilon}}_k(t)+\mathrm{E}\dot{w}_k(t)-\mathrm{E}\dot{w}(t))\,\mathrm{d}x\,,$$
$$\int_\Omega \mathbb{H}(z_k)(\boldsymbol{p}_k(t)+\boldsymbol{p}(t))\cdot(\dot{\boldsymbol{p}}(t)-\dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x\,,$$
$$\int_\Omega \frac{d(z_k)}{d(z)}\,\boldsymbol{\rho}(t)\cdot(\dot{\boldsymbol{p}}(t)-\dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x\,,$$

and using (A.3), we obtain, after a simple algebraic manipulation,

$$\begin{aligned}
&\int_\Omega \mathbb{C}(z_k)(\boldsymbol{\varepsilon}(t)-\boldsymbol{\varepsilon}_k(t))\cdot(\dot{\boldsymbol{\varepsilon}}(t)-\dot{\boldsymbol{\varepsilon}}_k(t))\,\mathrm{d}x+\int_\Omega \mathbb{H}(z_k)(\boldsymbol{p}(t)-\boldsymbol{p}_k(t))\cdot(\dot{\boldsymbol{p}}(t)-\dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x \\
&\le \int_\Omega \mathbb{C}(z_k)(\boldsymbol{\varepsilon}_k(t)-\boldsymbol{\varepsilon}(t))\cdot(\mathrm{E}\dot{w}_k(t)-\mathrm{E}\dot{w}(t))\,\mathrm{d}x \\
&\quad+\int_\Omega \mathbb{C}(z_k)(\tilde{\boldsymbol{\varepsilon}}_k(t)-\boldsymbol{\varepsilon}_k(t))\cdot(\dot{\boldsymbol{\varepsilon}}(t)-\dot{\boldsymbol{\varepsilon}}_k(t)+\mathrm{E}\dot{w}_k(t)-\mathrm{E}\dot{w}(t))\,\mathrm{d}x \\
&\quad+\int_\Omega (\mathbb{C}(z_k)-\mathbb{C}(z))\boldsymbol{\varepsilon}(t)\cdot(\dot{\boldsymbol{\varepsilon}}(t)-\dot{\boldsymbol{\varepsilon}}_k(t)+\mathrm{E}\dot{w}_k(t)-\mathrm{E}\dot{w}(t))\,\mathrm{d}x \\
&\quad+\int_\Omega \mathbb{H}(z_k)(\tilde{\boldsymbol{p}}_k(t)-\boldsymbol{p}_k(t))\cdot(\dot{\boldsymbol{p}}(t)-\dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x \\
&\quad+\int_\Omega (\mathbb{H}(z_k)-\mathbb{H}(z))\boldsymbol{p}(t)\cdot(\dot{\boldsymbol{p}}(t)-\dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x \\
&\quad+\int_\Omega \frac{d(z_k)-d(z)}{d(z)}\,\boldsymbol{\rho}(t)\cdot(\dot{\boldsymbol{p}}(t)-\dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x \\
&\quad-\int_\Omega (\ell(z_k)\tilde{f}_k(t)-\ell(z)f(t))\cdot(\dot{u}(t)-\dot{u}_k(t)+\dot{w}_k(t)-\dot{w}(t))\,\mathrm{d}x \\
&\quad-\int_{\Gamma_N} (\tilde{g}_k(t)-g(t))\cdot(\dot{u}(t)-\dot{u}_k(t)+\dot{w}_k(t)-\dot{w}(t))\,\mathrm{d}\mathcal{H}^{n-1}\,.
\end{aligned} \tag{A.4}$$

Integrating (A.4) w.r.t. t on the interval $[0,s]$, for $s\in[0,T]$, recalling (2.3)–(2.5) and that $u_k(0)=u(0)=0$ and $\boldsymbol{\varepsilon}_k(0)=\boldsymbol{\varepsilon}(0)=\boldsymbol{p}_k(0)=\boldsymbol{p}(0)=0$, we further estimate

$$\begin{aligned}
&\frac{\alpha_{\mathbb{C}}}{2}\|\boldsymbol{\varepsilon}(s)-\boldsymbol{\varepsilon}_k(s)\|_2^2+\frac{\alpha_{\mathbb{H}}}{2}\|\boldsymbol{p}(s)-\boldsymbol{p}_k(s)\|_2^2 \\
&\quad\le \beta_{\mathbb{C}}\Big(\|\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_k\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_S))}+\tau_k\|\dot{\boldsymbol{\varepsilon}}_k\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_S))} \\
&\qquad+\|(\mathbb{C}(z_k)-\mathbb{C}(z))\boldsymbol{\varepsilon}\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_S))}\Big)\|w_k-w\|_{H^1(0,T;H^1(\Omega;\mathbb{R}^n))} \\
&\qquad+\Big(\beta_{\mathbb{C}}\tau_k\|\dot{\boldsymbol{\varepsilon}}_k\|_{L^2([0,T];L^2(\Omega;\mathbb{M}^n_S))} \\
&\qquad+\|(\mathbb{C}(z_k)-\mathbb{C}(z))\boldsymbol{\varepsilon}\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_S))}\Big)\|\dot{\boldsymbol{\varepsilon}}-\dot{\boldsymbol{\varepsilon}}_k\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_S))} \\
&\qquad+\beta_{\mathbb{H}}\Big(\tau_k\|\dot{\boldsymbol{p}}_k\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_D))}+\|(\mathbb{H}(z_k)-\mathbb{H}(z))\boldsymbol{p}\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_D))} \\
&\qquad+\frac{1}{\alpha}\|(d(z_k)-d(z))\boldsymbol{\rho}\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_D))}\Big)\|\dot{\boldsymbol{p}}-\dot{\boldsymbol{p}}_k\|_{L^2(0,T;L^2(\Omega;\mathbb{M}^n_D))} \\
&\qquad+C\Big(\|\ell(z_k)\tilde{f}_k-\ell(z)f\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^n))} \\
&\qquad+\|\tilde{g}_k-g\|_{L^2(0,T;L^2(\Gamma_N;\mathbb{R}^n))}\Big)\|u-u_k+w_k-w\|_{H^1(0,T;H^1(\Omega;\mathbb{R}^n))}\,,
\end{aligned} \tag{A.5}$$

for some positive constant C independent of k. Since $(u_k, \boldsymbol{\varepsilon}_k, \boldsymbol{p}_k)$ is bounded in $H^1(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))$, $\tilde f_k \to f$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$, $\tilde g_k \to g$ in $L^2(0,T;L^2(\Gamma_N;\mathbb{R}^n))$, and $z_k \rightharpoonup z$ in $H^1(\Omega)$ with $0\le z_k, z\le 1$ almost everywhere, we deduce from (A.5) that $\boldsymbol{\varepsilon}_k \to \boldsymbol{\varepsilon}$ in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_S))$ and $\boldsymbol{p}_k\to\boldsymbol{p}$ in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^n_D))$. By Korn's inequality and by the convergence of w_k to w in $H^1(0,T;H^1(\Omega;\mathbb{R}^n))$, we infer that $u_k\to u$ in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))$. This concludes the proof of the lemma. □

We are now in a position to conclude the proof of Proposition 3.5. We follow here the lines of Theorem 3.3 in [33].

Proof of Proposition 3.5. In view of Lemma A.1, it remains to show that $(\dot u_k, \dot{\boldsymbol{\varepsilon}}_k, \dot{\boldsymbol{p}}_k)$ converges to $(\dot u, \dot{\boldsymbol{\varepsilon}}, \dot{\boldsymbol{p}})$ in $L^2(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))$. To this end, we define the auxiliary triples

$$\begin{aligned}(\omega^k_i, \boldsymbol{\xi}^k_i, \boldsymbol{\theta}^k_i) = \operatorname{argmin}\Big\{&\frac12\int_\Omega \mathbb{C}(z_k)\Big(\boldsymbol{\varepsilon} + \frac{\mathrm{E}w^k_i - \mathrm{E}w^k_{i-1}}{\tau_k}\Big)\cdot\Big(\boldsymbol{\varepsilon} + \frac{\mathrm{E}w^k_i - \mathrm{E}w^k_{i-1}}{\tau_k}\Big)\,\mathrm{d}x\\ &+\frac12\int_\omega \mathbb{H}(z_k)\boldsymbol{p}\cdot\boldsymbol{p}\,\mathrm{d}x - \int_\Omega \ell(z_k)\frac{f^k_i - f^k_{i-1}}{\tau_k}\cdot u\,\mathrm{d}x\\ &-\int_{\Gamma_N}\frac{g^k_i - g^k_{i-1}}{\tau_k}\cdot u\,\mathrm{d}\mathcal{H}^{n-1} : (u,\boldsymbol{\varepsilon},\boldsymbol{p})\in\mathcal{A}(0)\Big\}\end{aligned} \tag{A.6}$$

$$\begin{aligned}(\omega(t), \boldsymbol{\xi}(t), \boldsymbol{\theta}(t)) = \operatorname{argmin}\Big\{&\frac12\int_\Omega \mathbb{C}(z)(\boldsymbol{\varepsilon}+\mathrm{E}\dot w(t))\cdot(\boldsymbol{\varepsilon}+\mathrm{E}\dot w(t))\,\mathrm{d}x + \frac12\int_\omega \mathbb{H}(z)\boldsymbol{p}\cdot\boldsymbol{p}\,\mathrm{d}x\\ &-\int_\Omega \ell(z)\dot f(t)\cdot u\,\mathrm{d}x - \int_{\Gamma_N}\dot g(t)\cdot u\,\mathrm{d}\mathcal{H}^{n-1} : (u,\boldsymbol{\varepsilon},\boldsymbol{p})\in\mathcal{A}(0)\Big\}.\end{aligned} \tag{A.7}$$

Since (f_k, g_k, w_k) converges to (f,g,w) in $H^1(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Gamma_N;\mathbb{R}^n)\times H^1(\Omega;\mathbb{R}^n))$, we deduce that the piecewise constant function

$$(\omega_k(t), \boldsymbol{\xi}_k(t), \boldsymbol{\theta}_k(t)) := (\omega^k_i, \boldsymbol{\xi}^k_i, \boldsymbol{\theta}^k_i) \qquad \text{for } t\in(t^k_{i-1}, t^k_i]$$

converges to $(\omega, \boldsymbol{\xi}, \boldsymbol{\theta})$ in $L^2(0,T;H^1(\Omega;\mathbb{R}^n)\times L^2(\Omega;\mathbb{M}^n_S)\times L^2(\Omega;\mathbb{M}^n_D))$.

By the minimality of $(u^k_i, \boldsymbol{\varepsilon}^k_i, \boldsymbol{p}^k_i)$ in (3.7), we have that

$$\begin{aligned}&\int_\Omega \mathbb{C}(z_k)\boldsymbol{\varepsilon}^k_i\cdot\boldsymbol{\eta}\,\mathrm{d}x + \int_\Omega \mathbb{H}(z_k)\boldsymbol{p}^k_i\cdot\boldsymbol{q}\,\mathrm{d}x + \int_\Omega d(z_k)|\boldsymbol{q} - (\boldsymbol{p}^k_{i-1} - \boldsymbol{p}^k_i)|\,\mathrm{d}x\\ &\quad - \int_\Omega d(z_k)|\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1}|\,\mathrm{d}x - \int_\Omega \ell(z_k)f^k_i\cdot v\,\mathrm{d}x - \int_{\Gamma_N} g^k_i\cdot v\,\mathrm{d}\mathcal{H}^{n-1}\ge 0\end{aligned} \tag{A.8}$$

for every $(v,\boldsymbol{\eta},\boldsymbol{q})\in\mathcal{A}(0)$. Testing (A.8) with the triple

$$(u^k_{i-1} - u^k_i - w^k_{i-1} + w^k_i, \boldsymbol{\varepsilon}^k_{i-1} - \boldsymbol{\varepsilon}^k_i - \mathrm{E}w^k_{i-1} + \mathrm{E}w^k_i, \boldsymbol{p}^k_{i-1} - \boldsymbol{p}^k_i)\in\mathcal{A}(0)$$

combined with the equilibrium condition (at time t^k_{i-1})

$$\begin{aligned}-\int_\Omega d(z_k)|\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1}|\,\mathrm{d}x \le &\int_\Omega \mathbb{C}(z_k)\boldsymbol{\varepsilon}^k_{i-1}\cdot(\boldsymbol{\varepsilon}^k_i - \boldsymbol{\varepsilon}^k_{i-1} - \mathrm{E}w^k_i + \mathrm{E}w^k_{i-1})\,\mathrm{d}x\\ &+\int_\Omega \mathbb{H}(z_k)\boldsymbol{p}^k_{i-1}\cdot(\boldsymbol{p}^k_i - \boldsymbol{p}^k_{i-1})\,\mathrm{d}x\\ &-\int_\Omega \ell(z_k)f^k_{i-1}\cdot(u^k_i - u^k_{i-1} - w^k_i + w^k_{i-1})\,\mathrm{d}x\\ &-\int_{\Gamma_N} g^k_{i-1}\cdot(u^k_i - u^k_{i-1} - w^k_i + w^k_{i-1})\,\mathrm{d}\mathcal{H}^{n-1},\end{aligned}$$

we deduce that

$$\int_\Omega \mathbb{C}(z_k)\dot{\boldsymbol{\varepsilon}}_k(t)\cdot(\dot{\boldsymbol{\varepsilon}}_k(t)-\mathrm{E}\dot{w}_k(t))\,\mathrm{d}x+\int_\Omega \mathbb{H}(z_k)\dot{\boldsymbol{p}}_k(t)\cdot\dot{\boldsymbol{p}}_k(t)\,\mathrm{d}x \\ -\int_\Omega \ell(z_k)\dot{f}_k(t)\cdot(\dot{u}_k(t)-\dot{w}_k(t))\,\mathrm{d}x-\int_{\Gamma_N}\dot{g}_k(t)\cdot(\dot{u}_k(t)-\dot{w}_k(t))\,\mathrm{d}\mathcal{H}^{n-1}\le 0\,. \tag{A.9}$$

Testing the Euler-Lagrange equation of (A.6) with the test $(\dot{u}_k(t)-\dot{w}_k(t),\dot{\boldsymbol{\varepsilon}}_k(t)-\mathrm{E}\dot{w}_k(t),\dot{\boldsymbol{p}}_k(t))\in\mathcal{A}(0)$ we also get

$$\int_\Omega \mathbb{C}(z_k)(\boldsymbol{\xi}_k(t)+\mathrm{E}\dot{w}_k(t))\cdot(\dot{\boldsymbol{\varepsilon}}_k(t)-\mathrm{E}\dot{w}_k(t))\,\mathrm{d}x+\int_\Omega \mathbb{H}(z_k)\boldsymbol{\theta}_k(t)\cdot\dot{\boldsymbol{p}}_k(t)\,\mathrm{d}x \\ -\int_\Omega \ell(z_k)\dot{f}_k(t)\cdot(\dot{u}_k(t)-\dot{w}_k(t))\,\mathrm{d}x-\int_{\Gamma_N}\dot{g}_k(t)\cdot(\dot{u}_k(t)-\dot{w}_k(t))\,\mathrm{d}\mathcal{H}^{n-1}=0\,. \tag{A.10}$$

We subtract (A.10) from (A.9) and obtain the inequality

$$\int_\Omega \mathbb{C}(z_k)\big((\dot{\boldsymbol{\varepsilon}}_k(t)-\mathrm{E}\dot{w}_k(t))-\boldsymbol{\xi}_k(t)\big)\cdot(\dot{\boldsymbol{\varepsilon}}_k(t)-\mathrm{E}\dot{w}_k(t))\,\mathrm{d}x \\ +\int_\Omega \mathbb{H}(z_k)(\dot{\boldsymbol{p}}_k(t)-\boldsymbol{\theta}_k(t))\cdot\dot{\boldsymbol{p}}_k(t)\,\mathrm{d}x\le 0\,,$$

which in turn implies

$$\int_\Omega \mathbb{C}(z_k)\big(\boldsymbol{\xi}_k(t)-2(\dot{\boldsymbol{\varepsilon}}_k(t)-\mathrm{E}\dot{w}_k(t))\big)\cdot\big(\boldsymbol{\xi}_k(t)-2(\dot{\boldsymbol{\varepsilon}}_k(t)-\mathrm{E}\dot{w}_k(t))\big)\,\mathrm{d}x \\ +\int_\Omega \mathbb{H}(z_k)(\boldsymbol{\theta}_k(t)-2\dot{\boldsymbol{p}}_k(t))\cdot(\boldsymbol{\theta}_k(t)-2\dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x \\ \le\int_\Omega \mathbb{C}(z_k)\boldsymbol{\xi}_k(t)\cdot\boldsymbol{\xi}_k(t)\,\mathrm{d}x+\int_\Omega \mathbb{H}(z_k)\boldsymbol{\theta}_k(t)\cdot\boldsymbol{\theta}_k(t)\,\mathrm{d}x\,. \tag{A.11}$$

By the equilibrium condition (3.1) of $(u(t),\boldsymbol{\varepsilon}(t),\boldsymbol{p}(t))$ and by the energy balance (3.2), we have that for a.e. $t\in[0,T]$

$$\int_\Omega \mathbb{C}(z)\boldsymbol{\varepsilon}(t)\cdot(\dot{\boldsymbol{\varepsilon}}(t)-\mathrm{E}\dot{w}(t))\,\mathrm{d}x+\int_\Omega \mathbb{H}(z)\boldsymbol{p}(t)\cdot\dot{\boldsymbol{p}}(t)\,\mathrm{d}x+\int_\Omega d(z)\boldsymbol{\rho}(t)\cdot\dot{\boldsymbol{p}}(t)\,\mathrm{d}x \\ -\int_\Omega \ell(z)f(t)\cdot(\dot{u}(t)-\dot{w}(t))\,\mathrm{d}x-\int_{\Gamma_N} g(t)\cdot(\dot{u}(t)-\dot{w}(t))\,\mathrm{d}x=0\,. \tag{A.12}$$

Since $\boldsymbol{\rho}(t)\cdot\dot{\boldsymbol{p}}(t)=d(z)|\dot{\boldsymbol{p}}(t)|$ almost everywhere in Ω and, by the equilibrium (3.1) at time $t+h$,

$$\int_\Omega d(z)|\dot{\boldsymbol{p}}(t)|\,\mathrm{d}x\ge\int_\Omega \mathbb{C}(z)\boldsymbol{\varepsilon}(t+h)\cdot(\mathrm{E}\dot{w}(t)-\dot{\boldsymbol{\varepsilon}}(t))\,\mathrm{d}x-\int_\Omega \mathbb{H}(z)\boldsymbol{p}(t+h)\cdot\dot{\boldsymbol{p}}(t)\,\mathrm{d}x \\ +\int_\Omega \ell(z)f(t+h)\cdot(\dot{u}(t)-\dot{w}(t))\,\mathrm{d}x+\int_{\Gamma_N} g(t+h)\cdot(\dot{u}(t)-\dot{w}(t))\,\mathrm{d}x\,,$$

we infer from (A.12) that for $h \in \mathbb{R} \setminus \{0\}$,

$$\int_\Omega \mathbb{C}(z)(\boldsymbol{\varepsilon}(t+h) - \boldsymbol{\varepsilon}(t)) \cdot (\dot{\boldsymbol{\varepsilon}}(t) - \mathrm{E}\dot{w}(t))\,\mathrm{d}x + \int_\Omega \mathbb{H}(z)(\boldsymbol{p}(t+h) - \boldsymbol{p}(t)) \cdot \dot{\boldsymbol{p}}(t)\,\mathrm{d}x - \int_\Omega \ell(z)(f(t+h) - f(t)) \cdot (\dot{u}(t) - \dot{w}(t))\,\mathrm{d}x - \int_{\Gamma_N} (g(t+h) - g(t)) \cdot (\dot{u}(t) - \dot{w}(t))\,\mathrm{d}x \geq 0\,. \tag{A.13}$$

Dividing (A.13) by h (positive or negative) and passing to the limit as $h \to 0$, we deduce that for a.e. $t \in [0, T]$ there holds

$$\int_\Omega \mathbb{C}(z)\dot{\boldsymbol{\varepsilon}}(t) \cdot (\dot{\boldsymbol{\varepsilon}}(t) - \mathrm{E}\dot{w}(t))\,\mathrm{d}x + \int_\Omega \mathbb{H}(z)\dot{\boldsymbol{p}}(t) \cdot \dot{\boldsymbol{p}}(t)\,\mathrm{d}x - \int_\Omega \ell(z)\dot{f}(t) \cdot (\dot{u}(t) - \dot{w}(t))\,\mathrm{d}x - \int_{\Gamma_N} \dot{g}(t) \cdot (\dot{u}(t) - \dot{w}(t))\,\mathrm{d}x = 0\,. \tag{A.14}$$

Testing the Euler-Lagrange equation relative to (A.7) with the triple $(\dot{u}(t) - \dot{w}(t), \dot{\boldsymbol{\varepsilon}}(t) - \mathrm{E}\dot{w}(t), \dot{\boldsymbol{p}}(t)) \in \mathcal{A}(0)$ we get

$$\int_\Omega \mathbb{C}(z)(\boldsymbol{\xi}(t) + \mathrm{E}\dot{w}(t)) \cdot (\dot{\boldsymbol{\varepsilon}}(t) - \mathrm{E}\dot{w}(t))\,\mathrm{d}x + \int_\Omega \mathbb{H}(z)\boldsymbol{\theta}(t) \cdot \dot{\boldsymbol{p}}(t)\,\mathrm{d}x = \int_\Omega \ell(z)\dot{f}(t) \cdot (\dot{u}(t) - \dot{w}(t))\,\mathrm{d}x + \int_{\Gamma_N} \dot{g}(t) \cdot (\dot{u}(t) - \dot{w}(t))\,\mathrm{d}\mathcal{H}^{n-1}\,. \tag{A.15}$$

Subtracting (A.15) from (A.14) and arguing as in (A.11) we finally obtain that

$$\int_\Omega \mathbb{C}(z)\big(\boldsymbol{\xi}(t) - 2(\dot{\boldsymbol{\varepsilon}}(t)) - \mathrm{E}\dot{w}(t))\big) \cdot \big(\boldsymbol{\xi}(t) - 2(\dot{\boldsymbol{\varepsilon}}(t)) - \mathrm{E}\dot{w}(t))\big)\,\mathrm{d}x + \int_\Omega \mathbb{H}(z)(\boldsymbol{\theta}(t) - 2\dot{\boldsymbol{p}}(t)) \cdot (\boldsymbol{\theta}(t) - 2\dot{\boldsymbol{p}}(t))\,\mathrm{d}x = \int_\Omega \mathbb{C}(z)\boldsymbol{\xi}(t) \cdot \boldsymbol{\xi}(t)\,\mathrm{d}x + \int_\Omega \mathbb{H}(z)\boldsymbol{\theta}(t) \cdot \boldsymbol{\theta}(t)\,\mathrm{d}x\,. \tag{A.16}$$

Let us now set

$$r_k(t) := \int_\Omega \mathbb{C}(z_k)\boldsymbol{\xi}_k(t) \cdot \boldsymbol{\xi}_k(t)\,\mathrm{d}x + \int_\Omega \mathbb{H}(z_k)\boldsymbol{\theta}_k(t) \cdot \boldsymbol{\theta}_k(t)\,\mathrm{d}x\,,$$
$$r(t) := \int_\Omega \mathbb{C}(z)\boldsymbol{\xi}(t) \cdot \boldsymbol{\xi}(t)\,\mathrm{d}x + \int_\Omega \mathbb{H}(z)\boldsymbol{\theta}(t) \cdot \boldsymbol{\theta}(t)\,\mathrm{d}x\,.$$

By the convergence of

$$(\omega_k, \boldsymbol{\xi}_k, \boldsymbol{\theta}_k) \to (\omega, \boldsymbol{\xi}, \boldsymbol{\theta}) \;\text{ in }\; L^2(0, T; H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^n_S) \times L^2(\Omega; \mathbb{M}^n_D))$$

we have that $r_k \to r$ in $L^1(0, T)$. In view of (A.11) and (A.16) we estimate

$$\limsup_{k\to\infty}\bigg(\int_0^T\int_\Omega\Big(\mathbb{C}(z_k)\big(\boldsymbol{\xi}_k(t)-2(\dot{\varepsilon}_k(t)-\mathrm{E}\dot{w}_k(t))\big)-\mathbb{C}(z)\big(\boldsymbol{\xi}(t)$$
$$-2(\dot{\varepsilon}(t)-\mathrm{E}\dot{w}(t))\big)\Big)\cdot\big(\boldsymbol{\xi}_k(t)-2(\dot{\varepsilon}_k(t)-\mathrm{E}\dot{w}_k(t))-\boldsymbol{\xi}(t)+2(\dot{\varepsilon}(t)-\mathrm{E}\dot{w}(t))\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_\Omega\Big(\mathbb{H}(z_k)(\boldsymbol{\theta}_k(t)-2\dot{\boldsymbol{p}}_k(t))-\mathbb{H}(z)(\boldsymbol{\theta}(t)-2\dot{\boldsymbol{p}}(t))\Big)\cdot$$
$$\big(\boldsymbol{\theta}_k(t)-2\dot{\boldsymbol{p}}_k(t)-\boldsymbol{\theta}(t)+2\dot{\boldsymbol{p}}(t)\big)\,\mathrm{d}x\,\mathrm{d}t\bigg)$$
$$=\limsup_{k\to\infty}\bigg(\int_0^T\int_\Omega\mathbb{C}(z_k)\big(\boldsymbol{\xi}_k(t)-2(\dot{\varepsilon}_k(t)-\mathrm{E}\dot{w}_k(t))\big)\cdot$$
$$\cdot\big(\boldsymbol{\xi}_k(t)-2(\dot{\varepsilon}_k(t)-\mathrm{E}\dot{w}_k(t))\big)\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_\Omega\mathbb{H}(z_k)(\boldsymbol{\theta}_k(t)-2\dot{\boldsymbol{p}}_k(t))\cdot(\boldsymbol{\theta}_k(t)-2\dot{\boldsymbol{p}}_k(t))\,\mathrm{d}x\,\mathrm{d}t\bigg)$$
$$-\int_0^T\int_\Omega\mathbb{C}(z)\big(\boldsymbol{\xi}(t)-2(\dot{\varepsilon}(t)-\mathrm{E}\dot{w}(t))\big)\cdot\big(\boldsymbol{\xi}(t)-2(\dot{\varepsilon}(t)-\mathrm{E}\dot{w}(t))\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_0^T\int_\Omega\mathbb{H}(z)(\boldsymbol{\theta}(t)-2\dot{\boldsymbol{p}}(t))\cdot(\boldsymbol{\theta}(t)-2\dot{\boldsymbol{p}}(t))\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq\limsup_{k\to\infty}\int_0^T(r_k(t)-r(t))\,\mathrm{d}t=0\,. \tag{A.17}$$

Using (A.17), we further estimate

$$\limsup_{k\to\infty}\bigg(\int_0^T\int_\Omega\mathbb{C}(z_k)\big(\boldsymbol{\xi}_k(t)-2(\dot{\varepsilon}_k(t)-\mathrm{E}\dot{w}_k(t))-\boldsymbol{\xi}(t)+2(\dot{\varepsilon}(t)-\mathrm{E}\dot{w}(t))\big)\cdot$$
$$\cdot\big(\boldsymbol{\xi}_k(t)-2(\dot{\varepsilon}_k(t)-\mathrm{E}\dot{w}_k(t))-\boldsymbol{\xi}(t)+2(\dot{\varepsilon}(t)-\mathrm{E}\dot{w}(t))\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_\Omega\mathbb{H}(z_k)(\boldsymbol{\theta}_k(t)-2\dot{\boldsymbol{p}}_k(t)-\boldsymbol{\theta}(t)+2\dot{\boldsymbol{p}}(t))\cdot(\boldsymbol{\theta}_k(t)-2\dot{\boldsymbol{p}}_k(t)$$
$$-\boldsymbol{\theta}(t)+2\dot{\boldsymbol{p}}(t))\,\mathrm{d}x\,\mathrm{d}t\bigg)$$
$$=\limsup_{k\to\infty}\bigg(\int_0^T\int_\Omega\mathbb{C}(z_k)\big(\boldsymbol{\xi}_k(t)-2(\dot{\varepsilon}_k(t)-\mathrm{E}\dot{w}_k(t))\big)\cdot\big(\boldsymbol{\xi}_k(t)-2(\dot{\varepsilon}_k(t)-\mathrm{E}\dot{w}_k(t))$$
$$-\boldsymbol{\xi}(t)+2(\dot{\varepsilon}(t)-\mathrm{E}\dot{w}(t))\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_0^T\int_\Omega\mathbb{C}(z)\big(\boldsymbol{\xi}(t)-2(\dot{\varepsilon}(t)-\mathrm{E}\dot{w}(t))\big)\cdot\big(\boldsymbol{\xi}_k(t)-2(\dot{\varepsilon}_k(t)-\mathrm{E}\dot{w}_k(t))$$
$$-\boldsymbol{\xi}(t)+2(\dot{\varepsilon}(t)-\mathrm{E}\dot{w}(t))\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_\Omega\big(\mathbb{C}(z)-\mathbb{C}(z_k)\big)\big(\boldsymbol{\xi}(t)-2(\dot{\varepsilon}(t)-\mathrm{E}\dot{w}(t))\big)\cdot\big(\boldsymbol{\xi}_k(t)-2(\dot{\varepsilon}_k(t)-\mathrm{E}\dot{w}_k(t))$$
$$-\boldsymbol{\xi}(t)+2(\dot{\varepsilon}(t)-\mathrm{E}\dot{w}(t))\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\int_\Omega\mathbb{H}(z_k)(\boldsymbol{\theta}_k(t)-2\dot{\boldsymbol{p}}_k(t))\cdot(\boldsymbol{\theta}_k(t)-2\dot{\boldsymbol{p}}_k(t)-\boldsymbol{\theta}(t)+2\dot{\boldsymbol{p}}(t))\,\mathrm{d}x$$
$$-\int_0^T\int_\Omega\mathbb{H}(z)(\boldsymbol{\theta}(t)-2\dot{\boldsymbol{p}}(t))\cdot(\boldsymbol{\theta}_k(t)-2\dot{\boldsymbol{p}}_k(t)-\boldsymbol{\theta}(t)+2\dot{\boldsymbol{p}}(t))\,\mathrm{d}x\,\mathrm{d}t \tag{A.18}$$

$$
\begin{aligned}
&+ \int_0^T \int_\Omega \big(\mathbb{H}(z) - \mathbb{H}(z_k)\big)\big(\boldsymbol{\theta}(t) - 2\dot{\boldsymbol{p}}(t)\big) \cdot \big(\boldsymbol{\theta}_k(t) - 2\dot{\boldsymbol{p}}_k(t) - \boldsymbol{\theta}(t) + 2\dot{\boldsymbol{p}}(t)\big)\, \mathrm{d}x\, \mathrm{d}t\Big) \\
&\le \limsup_{k\to\infty} \Big(\int_0^T \int_\Omega \big(\mathbb{C}(z) - \mathbb{C}(z_k)\big)\big(\boldsymbol{\xi}(t) - 2(\dot{\boldsymbol{\varepsilon}}(t) - \mathrm{E}\dot{w}(t))\big) \cdot \big(\boldsymbol{\xi}_k(t) \\
&\quad - 2(\dot{\boldsymbol{\varepsilon}}_k(t) - \mathrm{E}\dot{w}_k(t))\big)\, \mathrm{d}x\, \mathrm{d}t \\
&+ \int_0^T \int_\Omega \big(\mathbb{H}(z) - \mathbb{H}(z_k)\big)\big(\boldsymbol{\theta}(t) - 2\dot{\boldsymbol{p}}(t)\big) \cdot \big(\boldsymbol{\theta}_k(t) - 2\dot{\boldsymbol{p}}_k(t) - \boldsymbol{\theta}(t) + 2\dot{\boldsymbol{p}}(t)\big)\, \mathrm{d}x\, \mathrm{d}t\Big) = 0\,.
\end{aligned}
$$

From (2.4)–(2.5) and (A.18) we infer that

$$
\boldsymbol{\xi}_k - 2(\dot{\boldsymbol{\varepsilon}}_k - \mathrm{E}\dot{w}_k) \to \boldsymbol{\xi} - 2(\dot{\boldsymbol{\varepsilon}} - \mathrm{E}\dot{w}) \qquad \text{and} \qquad \boldsymbol{\theta}_k - 2\dot{\boldsymbol{p}}_k \to \boldsymbol{\theta} - 2\dot{\boldsymbol{p}}
$$

in $L^2(0,T;L^2(\Omega;\mathbb{M}^n))$. Since $\boldsymbol{\xi}_k \to \boldsymbol{\xi}$ and $\boldsymbol{\theta}_k \to \boldsymbol{\theta}$ in $L^2(0,T;L^2(\Omega;\mathbb{M}^n))$, we immediately deduce that $\dot{\boldsymbol{\varepsilon}}_k \to \dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{p}}_k \to \dot{\boldsymbol{p}}$ in $L^2(0,T;L^2(\Omega;\mathbb{M}^n))$. Finally, the convergence of $\dot{u}_k$ to $\dot{u}$ in $L^2(0,T;H^1(\Omega;\mathbb{R}^n))$ is a consequence of the convergences of $\dot{\boldsymbol{\varepsilon}}_k$, $\dot{\boldsymbol{p}}_k$, and $\dot{w}_k$, and of Korn's inequality. This concludes the proof of Proposition 3.5. □

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