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A MEAN-FIELD STOCHASTIC LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEM WITH JUMPS UNDER PARTIAL INFORMATION*

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Abstract. In this article, the stochastic linear-quadratic optimal control problem of mean-field type with jumps under partial information is discussed. The state equation which contains affine terms is a SDE with jumps driven by a multidimensional Brownian motion and a Poisson stochastic martingale measure, and the quadratic cost function contains cross terms. In addition, the state and the control as well as their expectations are contained both in the state equation and the cost functional. This is the so-called optimal control problem of mean-field type. Firstly, the existence and uniqueness of the optimal control is proved. Secondly, the adjoint processes of the state equation is introduced, and by using the duality technique, the optimal control is characterized by the stochastic Hamiltonian system. Thirdly, by applying a decoupling technology, we deduce two integro-differential Riccati equations and get the feedback representation of the optimal control under partial information. Fourthly, the existence and uniqueness of the solutions of two Riccati equations are proved. Finally, we discuss a special case, and establish the corresponding feedback representation of the optimal control by means of filtering technique.

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1. Introduction

The so-called optimal control is to find the optimal control plan among the possible control schemes, so that the control system can achieve the desired goal optimally. The research on optimal control theory has a long history. As early as the early 1950s, Bushaw studied the time-optimal control of servo systems, and then LaSalle developed the time-optimal control theory, the so-called Bang-Bang control theory was proposed. From 1953 to 1957, American scholar Bellman established the theory of dynamic programming and developed the Hamilton-Jacobi theory of variation. From 1956 to 1958, the former Soviet Union scholar Pontryagin and others established the principle of maximum. As we all know, Pontryagin's maximum principle and Bellman's dynamic

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programming principle were the two main methods for solving stochastic optimal control problems, and also the most commonly used methods. With the continuous in-depth research of aerospace, navigation, aviation and guidance technology, the optimization of system has become an important issue. Optimal control theory has also made great progress and become a very important branch of modern control theory.

On the basis of optimal control, when the state equation of the control system is a linear equation and the cost functional is quadratic, the optimal control can be given in the form of linear feedback. Such a problem is called the Linear-Quadratic (LQ) optimal control problem. Getting the feedback representation of optimal control is the most basic task. The LQ optimal control problem was first studied by Bellman-Glicksberg-Gross in 1958. In 1960, Kalman established the state feedback optimal control, and introduced the Riccati equation into the control theory. In 2000, [9] studied the relationship between stochastic control problems and the backward stochastic differential equation (BSDE) for the first time. Later, on the basis of [9], the explicit form of optimal control for LQ problem of BSDE was obtained by Lim and Zhou in [11]. In 2016, Sun *et al.* in [17] investigated the open-loop and closed-loop solvability of LQ optimal control problem.

The LQ optimal control problem is widely used and can be applied to many aspects, but when dealing with unexpected situations in financial problems, it is necessary to use a jump system to characterize. Therefore, the discussion on LQ problem with jumps is also very important. In [2], Boel, Varaiya *et al.* discussed the optimal control problem of the process with jumps for the first time. Tang and Li in [19] studied the essential conditions for optimal control of stochastic systems with random jumps and first discussed the BSDE with Poisson process. In 2003, Wu and Wang studied the stochastic problem of the system driven by Brownian motion and Poisson jump in [20]. In 2014, Meng [13] explored the solutions of the backward stochastic Riccati equations exist and were unique. In 2020, Zhang *et al.* [24] studied the solvability of matrix valued backward stochastic Riccati equations with jumps, which are associated with a stochastic linear quadratic optimal control problem with random coefficients and driven by both Brownian motion and a Poisson processes. In 2021, Moon *et al.* [14] studied the indefinite linear quadratic stochastic optimal control problem for stochastic differential equations with jump diffusions and random coefficients driven by both the Brownian motion and (compensated) Poisson processes.

Nowadays, in the stochastic control theory, the systems with interactions have attracted more and more attentions, because it actually described a phenomenon in reality: Interaction exists in the whole system, so the state of a single individual depends not only on its own state, but also on the mean-value of the whole system. And the model is designed to study such systems is called mean-field (MF) model, then it began to be widely applied in many fields such as statistical physics, biology, financial engineering, social science, etc. In 1956, Kac [8] considered the MF-SDE in the first time. The control problems with mean-field type have become more and more popular after Buckdahn *et al.* [3]. In 2008, Hu and Oksendal [5] researched the LQ problem with random coefficients under partial information driven by a Poisson jumps, and proved that the optimal control has a state feedback representation by a BSDE with jumps. In 2011, Yong [22] dealt with a LQ optimal control problem of MF-SDE with deterministic coefficients driven by Brownian motion in finite horizon, by using variational method, the optimality system was derived. And by using decoupling technology, he got the optimal control in the feedback form. In 2015, Yong *et al.* [6] studied an LQ optimal control problem of MF-SDE in infinite horizon. Aimed at MF-LQ optimal control problem, in 2017, Sun [16] explored the relationship between the open-loop solvability and the (uniformly) convexity of the cost functional. In 2019, Meng and Tang [18] explored the optimal control problem of the mean-field type driven by Brownian motion and a Poisson stochastic martingale measure. The existence and uniqueness of optimal control was obtained by the classical convex variational principle [4], two Riccati equations were deduced, and the state feedback representation of optimal control was also obtained.

Besides, when we make some decisions, it is common that the information we observed is not always complete, thus the research of MF-LQ under partial information is getting deeper and deeper. This kind of problem is related to filtering theory. A systematic introduction to the theory and application of linear filtering can be consulted in Bensoussan [1]. Xiong in 2008 [21] introduced the stochastic filtering theory systematically. Oksendal and Sulem [15] dealt with the case when the state equation is described by a forward-backward SDE (FBSDE) with random jump under partial information. Ma and Liu [12] researched on LQ optimal control problem of

FBSDE with mean-field type under partially observed. For LQ optimal control problem of MF-BSDE, Li *et al.* [10] concluded that the optimal control can be represented by two Riccati equations and a MF-SDE. In 2020, under partial information, Zhang [24] studied the optimal control problem of MF-SDE with terminal constraint. Huang *et al.* [7] in 2020 considered LQ optimal control problem of BSDE under partial information with its application, they also extended BSDE to MF-BSDE, and obtained the corresponding feedback form of optimal control under partial information.

In this paper, we discuss a kind of mean-field stochastic linear-quadratic (MF-LQ) optimal control problem with jumps under partial information. Comparing with the classical LQ optimal control problem, the features of this case are as follows: (1) the state equation contains affine terms is a SDE with jump driven by a multidimensional standard Brownian motion and a Poisson stochastic martingale measure; (2) the quadratic cost function contains cross terms; (3) mean-field type, the state equation and the cost functional also conclude the expectations of state $X(\cdot)$ and the control $u(\cdot)$; (4) under partial information, the state $X(\cdot)$ is $\mathscr{F}_t$-adapted, and the control $u(\cdot)$ is $\mathscr{G}_t$-predictable, where $\mathscr{G}_t \subseteq \mathscr{F}_t$.

The rest of the article is arranged as follows. In Section 1, we introduce some basic notations used throughout the article, state the problem, and show our needed assumptions. Secondly, through the classical convex variation principle, we prove the existence and uniqueness of the optimal control. In Section 3, we introduce the adjoint processes of the state equation, by the duality method, a stochastic Hamiltonian system is derived to characterize the optimal control. Fourthly, by using decoupling technology, we deduce two stochastic integro-differential Riccati equations and the corresponding state feedback representation of optimal control. In Section 5, we prove the existence and uniqueness of the solutions of two Riccati equations. And we discuss a special case of MF-LQ optimal control problem under partial information, by means of filtering technique, and get the corresponding feedback representation of optimal control in Section 6.

1.1. Notations

Throughout this paper, given a fixed $T > 0$, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space equipped with a right-continuous $\mathbb{P}$-complete filtration $\mathbb{F} := \{\mathscr{F}_t |\ t \in [0, T]\}$, to be specified later. Furthermore, we assume that $\mathscr{F}_T = \mathscr{F}$. Denote by $\mathscr{P}$ the predictable σ-algebra on $\Omega \times [0, T]$ associated with $\mathbb{F}$, and by $\mathscr{B}(\Lambda)$ the Borel σ-algebra of any topological space Λ. Let $\{W(t)\}_{0 \le t \le T} = \{W_1(t), W_2(t), \cdots, W_d(t)\}_{0 \le t \le T}$ be a given d-dimensional standard Brownian motion. Let $(\mathbb{Z}, \mathscr{B}(\mathbb{Z}), \nu)$ be a measure space with $\nu(\mathbb{E}) < \infty$ for any $\mathbb{E} \in \mathscr{B}(\mathbb{Z})$, and η be a stationary Poisson point process with the characteristic measure ν. Then, the counting measure induced by η is defined by

$$\mu((0, t] \times A) \triangleq \#\{s; s \le t, \eta(s) \in A\}, \quad \text{for } t > 0,\ A \in \mathscr{B}(\mathbb{Z}),$$

and then $\tilde{\mu}(de, \mathrm{d}t) \triangleq \mu(de, \mathrm{d}t) - \nu(de)\mathrm{d}t$ is the compensated Poisson random martingale measure which is assumed to be independent of the Brownian motion $\{W(t)\}_{0 \le t \le T}$. Moreover, the filtration $\mathbb{F}$ is assumed to be the $\mathbb{P}$-augmentation of the natural filtration generated by the Brownian motion $\{W(t)\}_{0 \le t \le T}$ and the Poisson random measure $\{\mu((0, t] \times A)|\ t \in [0, T], A \in \mathscr{B}(\mathbb{Z})\}$.

Also, under many situations, the full information $\mathscr{F}_t$ is inaccessible for control and ones can only observe a partial information. In this case, the admissible control process $u(\cdot)$ is assumed to be a $\mathscr{G}_t$-prediction process, here $\mathscr{G}_t \subseteq \mathscr{F}_t$. We set $\mathbb{G} = \{\mathscr{G}_t\}_{0 \le t \le T}$ is a given subfiltration which represents the information available to the controller at time t. For example, we can set $\mathscr{G}_t = \mathscr{F}_{t-\delta}$, where $\delta > 0$, is a fixed delay of information.

Let's introduce some basic notations needed throughout this paper.

- $\mathbb{R}^n$: the n-dimensional Euclidean space.
- $\mathbb{R}^{n\times m}$: the space of all $n \times m$ real matrices.
- $M^\top$: the transpose of matrix M.
- N^{-1}: the inverse of matrix N.
- $\langle M, N\rangle = tr(M^\top N)$: the inner product on $\mathbb{R}^{n\times m}$.
- $|M| = \sqrt{tr(M^\top M)}$: the induced norm of M.

- $\mathbb{S}^n \in \mathbb{R}^{n\times n}$: the space of all $n \times n$ symmetric matrices.
- $\mathbb{S}^n_+ \in \mathbb{S}^n$: the space of all $n \times n$ nonnegative definite symmetric matrices.
- $\mathbb{H}$: a given Hilbert space with norm $\|\cdot\|_{\mathbb{H}}$.
- $S^2_{\mathbb{F}}(t,T;\mathbb{H})$: the space of all $\mathbb{H}$-valued and $\mathscr{F}_s$-adapted càdlàg processes $g:[t,T]\times\Omega\to\mathbb{H}$ satisfying

$$\|g\|^2_{S^2_{\mathbb{F}}(0,T;\mathbb{H})} \triangleq \mathbb{E}\Big[\sup_{t\le s\le T}\|g(s)\|^2_{\mathbb{H}}\Big] < +\infty.$$

- $L^2_{\mathbb{F}}(t,T;\mathbb{H})$: the space of all $\mathbb{H}$-valued and $\mathbb{F}$-progressively measurable processes $g:[t,T]\times\Omega\to\mathbb{H}$ satisfying

$$\|g\|^2_{L^2_{\mathbb{F}}(t,T;\mathbb{H})} \triangleq \mathbb{E}\Big[\int_t^T \|g(s)\|^2_{\mathbb{H}}\mathrm{d}s\Big] < \infty.$$

- $L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{H}))$: the space all $\mathbb{H}$-valued and $\mathbb{F}$-progressively measurable processes $g:[t,T]\times\Omega\to\mathbb{H}$ satisfying

$$\mathbb{E}\Big[\int_t^T \|g(s)\|_{\mathbb{H}}\mathrm{d}s\Big]^2 < \infty.$$

- $L^{\nu,2}(\mathbb{Z};\mathbb{H})$: the space of all $\mathbb{H}$-valued measurable function $r:\mathbb{Z}\to\mathbb{H}$ defined on the measure space $(Z,\mathscr{B}(Z);v)$ satisfying

$$\|r\|^2_{L^{\nu,2}(\mathbb{Z};\mathbb{H})} \triangleq \int_{\mathbb{Z}}\|r(\theta)\|^2_{\mathbb{H}}\nu(\mathrm{d}\theta) < \infty.$$

- $L^{\nu,2}_{\mathbb{F}}([t,T]\times\mathbb{Z};\mathbb{H})$: the space of all $L^{\nu,2}(\mathbb{Z};\mathbb{H})$-valued and $\mathscr{F}_s$-predictable processes $r:[t,T]\times\Omega\times\mathbb{Z}\to\mathbb{H}$ satisfying

$$\|r\|^2_{L^{\nu,2}_{\mathbb{F}}([t,T]\times\mathbb{Z};\mathbb{H})} \triangleq \mathbb{E}\Big(\int_t^T \|r(s,\cdot)\|^2_{L^{\nu,2}(\mathbb{Z};\mathbb{H})}\mathrm{d}s\Big) < \infty.$$

1.2. Formulation of the problem

Now we discuss a linear system driven by a d-dimensional standard Brownian motion $W(t)$ and a Poisson random martingale measure $\{\tilde{\mu}(\mathrm{d}t,\mathrm{d}\theta)\}_{0\le t\le T}$, in addition, the system is mean-field type with partial information. Within the time interval $[t,T]$, the state equation is described by the following MF-SDE:

$$\begin{cases} \mathrm{d}X(s) = \big\{A(s)X(s)+\bar{A}(s)\mathbb{E}[X(s)]+B(s)u(s)+\bar{B}(s)\mathbb{E}[u(s)]+b(s)\big\}\mathrm{d}s \\ \qquad\qquad + \displaystyle\sum_{i=1}^{d}\big\{C_i(s)X(s)+\bar{C}_i(s)\mathbb{E}[X(s)]+D_i(s)u(s)+\bar{D}_i(s)\mathbb{E}[u(s)]+\sigma_i(s)\big\}\mathrm{d}W_i(s) \\ \qquad\qquad + \displaystyle\int_{\mathbb{Z}}\big\{E(s,\theta)X(s-)+\bar{E}(s,\theta)\mathbb{E}[X(s-)]+F(s,\theta)u(s)+\bar{F}(s,\theta)\mathbb{E}[u(s)]+h(s,\theta)\big\}\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta),\ s\in[t,T], \\ X(t)=x\in\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $A(\cdot),\bar{A}(\cdot),B(\cdot),\bar{B}(\cdot),C_i(\cdot),\bar{C}_i(\cdot),D_i(\cdot),\bar{D}_i(\cdot),E(\cdot,\cdot),\bar{E}(\cdot,\cdot),F(\cdot,\cdot),\bar{F}(\cdot,\cdot)$ are given deterministic matrix-valued functions; $b(\cdot)$, $\sigma_i(\cdot)$, $h(\cdot,\cdot)$ are vector-valued $\mathscr{F}_t$-predictable processes; The mathematical expectation is denoted by $\mathbb{E}$, $\mathbb{E}[X(s)]$ and $\mathbb{E}[u(s)]$ are called the mean-field terms of equation (1.1). And the solution of (1.1) denoted by $X(\cdot)$ or $X^{x,u}(\cdot)$ valued in $\mathbb{R}^n$, is called the state process; $u(\cdot)$ valued in $\mathbb{R}^m$ is said to be the control process required to be $\mathscr{G}_t$-predictable.

For any $t \in [0,T]$, we introduce the following Hilbert space:

$$\mathcal{A} = \left\{ u : [t,T] \times \Omega \to \mathbb{R}^m |\ u(\cdot)\ is\ \mathscr{G}_t - predictable, \mathbb{E}\left[\int_t^T |u(s)|^2 \mathrm{d}s \right] < \infty \right\}. \tag{1.2}$$

where $u(\cdot) \in \mathcal{A}$ is called the admissible control process, and $(X(\cdot), u(\cdot))$ is called an admissible pair.

Next we consider the quadratic cost functional with cross terms as follows:

$$\begin{aligned} J(t,x;u(\cdot)) =& \mathbb{E}\Big[\langle GX(T), X(T) \rangle + \langle \bar{G}\mathbb{E}[X(T)], \mathbb{E}[X(T)] \rangle + 2\langle g, X(T) \rangle + 2\langle \bar{g}, \mathbb{E}[X(T)] \rangle \Big] \\ &+ \mathbb{E}\Big[\int_t^T \Big(\langle Q(s)X(s), X(s) \rangle + \langle \bar{Q}(s)\mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle + 2\langle S(s)^\top u(s), X(s) \rangle \\ &+ 2\langle \bar{S}(s)^\top \mathbb{E}[u(s)], \mathbb{E}[X(s)] \rangle + \langle R(s)u(s), u(s) \rangle + \langle \bar{R}(s)\mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \\ &+ 2\langle q(s), X(s) \rangle + 2\langle \bar{q}(s), \mathbb{E}[X(s)] \rangle + 2\langle \rho(s), u(s) \rangle + 2\langle \bar{\rho}(s), \mathbb{E}[u(s)] \rangle \Big) \mathrm{d}s \Big], \end{aligned} \tag{1.3}$$

where $G,\ \bar{G}$ are symmetric and nonnegative; $Q(\cdot),\ \bar{Q}(\cdot),\ S(\cdot),\ \bar{S}(\cdot),\ R(\cdot),\ \bar{R}(\cdot)$ satisfying $Q(\cdot)^\top = Q(\cdot)$, $\bar{Q}(\cdot)^\top = \bar{Q}(\cdot)$, $R(\cdot)^\top = R(\cdot)$, $\bar{R}(\cdot)^\top = \bar{R}(\cdot)$ are given deterministic matrix-valued functions; g is an $\mathscr{F}_T$-measurable random variable, $\bar{g}$ is a deterministic vector; $q(\cdot), \rho(\cdot)$ are allowed to be vector-valued $\mathscr{F}_t$-adapted processes, and $\bar{q}(\cdot), \bar{\rho}(\cdot)$ are deterministic vector-valued functions. Then we can formulate the MF-LQ optimal control problem under partial information.

Proposition 1.1. *(Partial Information MF-LQ) Under partial information, for any given $t \in [0,T]$, $x \in \mathbb{R}^n$, in order to minimize the cost functional, to find an admissible control $u^*(\cdot) \in \mathcal{A}$ such that*

$$J(t,x;u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{A}} J(t,x;u(\cdot)) = V(t,x). \tag{1.4}$$

It is generally accepted for the initial pair $(t,x) \in [0,T] \times \mathbb{R}^n$, any $u(\cdot) \in \mathcal{A}$ satisfying the above conditions is called the optimal control of the Problem 1.1, the corresponding solution $X^*(\cdot) = X(\cdot; t, x, u^*(\cdot)) \in S^2_{\mathbb{F}}(t,T;\mathbb{H})$ is called an optimal state process, and the pair $(X^*(\cdot), u^*(\cdot))$ is called an optimal pair. The function $V(\cdot,\cdot)$ is called the value function of Problem 1.1.

When $b(\cdot), \sigma_i(\cdot), h(\cdot,\cdot), g, \bar{g}, q(\cdot), \bar{q}(\cdot), \rho(\cdot), \bar{\rho}(\cdot) = 0$, we denote the corresponding cost functional by $J^0(t,x;u(\cdot))$, the corresponding value function by $V^0(t,x)$. At the moment, the cost functional can be represented as:

$$\begin{aligned} J^0(t,x;u(\cdot)) =& \mathbb{E}\Big[\langle GX(T), X(T) \rangle + \langle \bar{G}\mathbb{E}[X(T)], \mathbb{E}[X(T)] \rangle \Big] + \mathbb{E}\Big[\int_t^T \Big(\langle Q(s)X(s), X(s) \rangle \\ &+ \langle \bar{Q}(s)\mathbb{E}[X(s)], \mathbb{E}[X(s)] \rangle + 2\langle S(s)^\top u(s), X(s) \rangle + 2\langle \bar{S}(s)^\top \mathbb{E}[u(s)], \mathbb{E}[X(s)] \rangle \\ &+ \langle R(s)u(s), u(s) \rangle + \langle \bar{R}(s)\mathbb{E}[u(s)], \mathbb{E}[u(s)] \rangle \Big) \mathrm{d}s \Big]. \end{aligned}$$

Comparing with the general state equation (1.1), when the initial pair is $(t,0)$, in other words, the initial state is 0 at the initial time t, i.e. $X(t)=0$, then we note $X^{0,u}(\cdot)$ is the adapted solution of the following system:

$$\begin{cases} \mathrm{d}X^{0,u}(s)=\big\{A(s)X^{0,u}(s)+\bar{A}(s)\mathbb{E}[X^{0,u}(s)]+B(s)u(s)+\bar{B}(s)\mathbb{E}[u(s)]\big\}\mathrm{d}s \\ \qquad\qquad +\sum_{i=1}^{d}\big\{C_i(s)X^{0,u}(s)+\bar{C}_i(s)\mathbb{E}[X^{0,u}(s)]+D_i(s)u(s)+\bar{D}_i(s)\mathbb{E}[u(s)]\big\}\mathrm{d}W_i(s) \\ \qquad\qquad +\int_{\mathbb{Z}}\big\{E(s,\theta)X^{0,u}(s-)+\bar{E}(s,\theta)\mathbb{E}[X^{0,u}(s-)]+F(s,\theta)u(s)+\bar{F}(s,\theta)\mathbb{E}[u(s)]\big\}\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta),\ s\in[t,T], \\ X^{0,u}(t)=0, \end{cases} \tag{1.5}$$

and the corresponding cost functional is as follows:

$$\begin{aligned} J^0(t,0;u(\cdot))=&\mathbb{E}\Big[\langle GX^{0,u}(T),X^{0,u}(T)\rangle+\langle\bar{G}\mathbb{E}[X^{0,u}(T)],\mathbb{E}[X^{0,u}(T)]\rangle\Big]+\mathbb{E}\Big[\int_t^T\Big(\langle Q(s)X^{0,u}(s),X^{0,u}(s)\rangle \\ &+\langle\bar{Q}(s)\mathbb{E}[X^{0,u}(s)],\mathbb{E}[X^{0,u}(s)]\rangle+2\langle S(s)^\top u(s),X^{0,u}(s)\rangle+2\langle\bar{S}(s)^\top\mathbb{E}[u(s)],\mathbb{E}[X^{0,u}(s)]\rangle \\ &+\langle R(s)u(s),u(s)\rangle+\langle\bar{R}(s)\mathbb{E}[u(s)],\mathbb{E}[u(s)]\rangle\Big)\mathrm{d}s\Big]. \end{aligned} \tag{1.6}$$

Definition 1.2. For an initial pair $(t,x)\in[0,T]\times R^n$, $u^*(\cdot)\in\mathcal{A}$ is called an open-loop optimal control if

$$J(t,x;u^*(\cdot))\le J(t,x;u(\cdot)),\ \forall u(\cdot)\in\mathcal{A}.$$

Definition 1.3. (1) When we said Problem 1.1 is finite at $(t,x)\in[0,T]\times\mathbb{R}^n$, it means

$$|V(t,x)|<+\infty.$$

(2) When we said Problem 1.1 is solvable at $(t,x)\in[0,T]\times\mathbb{R}^n$, it means exists a control $\bar{u}(\cdot)\in\mathcal{A}$ such that

$$J(t,x;\bar{u}(\cdot))=\inf_{u(\cdot)\in\mathcal{A}}J(t,x;u(\cdot)).$$

For $i=1,2,\dots,d$, then we make some assumptions on the coefficients.

Assumption 1.4. The coefficients of state equation $A(\cdot),\ \bar{A}(\cdot),\ C_i(\cdot),\ \bar{C}_i(\cdot):[t,T]\to\mathbb{R}^{n\times n};\ E(\cdot,\cdot),\ \bar{E}(\cdot,\cdot):[t,T]\to L^{\nu,2}(\mathbb{Z};\mathbb{R}^{n\times n});\ B(\cdot),\ \bar{B}(\cdot),\ D_i(\cdot),\ \bar{D}_i(\cdot):[t,T]\to\mathbb{R}^{n\times m};\ F(\cdot,\cdot),\ \bar{F}(\cdot,\cdot):[t,T]\to L^{\nu,2}(\mathbb{Z};\mathbb{R}^{n\times m})$ are uniformly bounded measurable functions. And $b(\cdot)\in L^2_{\mathbb{F}}(\Omega;L^1(t,T;\mathbb{R}^n));\ \sigma_i(\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n);\ h(\cdot,\cdot)\in L^{\nu,2}_{\mathbb{F}}([t,T]\times\mathbb{Z};\mathbb{R}^n)$.

Assumption 1.5. The coefficients of cost functional $Q(\cdot),\ \bar{Q}(\cdot):[t,T]\to\mathbb{R}^{n\times n};\ S(\cdot),\ \bar{S}(\cdot):[t,T]\to\mathbb{R}^{m\times n};\ R(\cdot),\ \bar{R}(\cdot):[t,T]\to\mathbb{R}^{m\times m}$ are uniformly bounded measurable functions. And $G,\ \bar{G}\in\mathbb{S}^n;\ g\in L^2_{\mathbb{F}}(\Omega;\mathbb{R}^n),\ \bar{g}\in\mathbb{R}^n;\ q(\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^n),\ \bar{q}(\cdot)\in L^2(t,T;\mathbb{R}^n);\ \rho(\cdot)\in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m),\ \bar{\rho}\in L^2(t,T;\mathbb{R}^m)$.

On the other hand, we can rewrite (1.3):

$$
\begin{aligned}
J(t,x;u(\cdot)) =&\mathbb{E}\Big[\langle G(X(T)-\mathbb{E}[X(T)]),X(T)-\mathbb{E}[X(T)]\rangle+\langle[G+\bar G]\mathbb{E}[X(T)],\mathbb{E}[X(T)]\rangle+2\langle g,X(T)\rangle\\
&+\langle\bar g,\mathbb{E}[X(T)]\rangle\Big]+\mathbb{E}\Big[\int_t^T\Big(\langle Q(s)(X(s)-\mathbb{E}[X(s)]),X(s)-\mathbb{E}[X(s)]\rangle+2\langle S(s)(X(s)\\
&-\mathbb{E}[X(s)]),u(s)-\mathbb{E}[u(s)]\rangle+\langle R(s)(u(s)-\mathbb{E}[u(s)]),u(s)-\mathbb{E}[u(s)]\rangle\\
&+\langle[Q(s)+\bar Q(s)]\mathbb{E}[X(s)],\mathbb{E}[X(s)]\rangle+2\langle[S(s)+\bar S(s)]\mathbb{E}[X(s)],\mathbb{E}[u(s)]\rangle\\
&+\langle[R(s)+\bar R(s)]\mathbb{E}[u(s)],\mathbb{E}[u(s)]\rangle+2\langle q(s),X(s)\rangle+2\langle\bar q(s),\mathbb{E}[X(s)]\rangle\\
&+2\langle\rho(s),u(s)\rangle+2\langle\bar\rho(s),\mathbb{E}[u(s)]\rangle\Big)\mathrm{d}s\Big].
\end{aligned}
\tag{1.7}
$$

Note that

$$
\begin{aligned}
&\langle Q(s)(X(s)-\mathbb{E}[X(s)]),X(s)-\mathbb{E}[X(s)]\rangle+2\langle S(s)(X(s)-\mathbb{E}[X(s)]),u(s)-\mathbb{E}[u(s)]\rangle\\
&\quad+\langle R(s)(u(s)-\mathbb{E}[u(s)]),u(s)-\mathbb{E}[u(s)]\rangle\\
=&\big\langle\big[Q(s)-S(s)^\top R(s)^{-1}S(s)\big](X(s)-\mathbb{E}[X(s)]),X(s)-\mathbb{E}[X(s)]\big\rangle+\big\langle R(s)\big\{(u(s)-\mathbb{E}[u(s)])\\
&\quad+R(s)^{-1}S(s)(X(s)-\mathbb{E}[X(s)])\big\},(u(s)-\mathbb{E}[u(s)])+R(s)^{-1}S(s)(X(s)-\mathbb{E}[X(s)])\big\rangle.
\end{aligned}
\tag{1.8}
$$

Similarly, we also have

$$
\begin{aligned}
&\langle[Q(s)+\bar Q(s)]\mathbb{E}[X(s)],\mathbb{E}[X(s)]\rangle+2\langle[S(s)+\bar S(s)]\mathbb{E}[X(s)],\mathbb{E}[u(s)]\rangle+\langle[R(s)+\bar R(s)]\mathbb{E}[u(s)],\mathbb{E}[u(s)]\rangle\\
=&\big\langle\big\{[Q(s)+\bar Q(s)]-[S(s)+\bar S(s)]^\top[R(s)+\bar R(s)]^{-1}[S(s)+\bar S(s)]\big\}\mathbb{E}[X(s)],\mathbb{E}[X(s)]\big\rangle\\
&+\big\langle[R(s)+\bar R(s)]\big\{\mathbb{E}[u(s)]+[R(s)+\bar R(s)]^{-1}[S(s)+\bar S(s)]\mathbb{E}[X(s)]\big\},\mathbb{E}[u(s)]\\
&+[R(s)+\bar R(s)]^{-1}[S(s)+\bar S(s)]\mathbb{E}[X(s)]\big\rangle.
\end{aligned}
\tag{1.9}
$$

In addition, we impose the following assumption:

Assumption 1.6. Assume for $t\in[0,T]$, there is a constant $\delta>0$ such that

$$
\begin{cases}
G,G+\bar G\ge 0,\quad R(s),R(s)+\bar R(s)\ge\delta I,\\
Q(s)-S(s)^\top R(s)^{-1}S(s)\ge 0,\\
[Q(s)+\bar Q(s)]-[S(s)+\bar S(s)]^\top[R(s)+\bar R(s)]^{-1}[S(s)+\bar S(s)]\ge 0,\ \textit{a.e. } s\in[t,T].
\end{cases}
$$

2. Existence and uniqueness of optimal control

Lemma 2.1. *Under Assumptions 1.4–1.6, for any $(t,x)\in[0,T]\times\mathbb{R}^n$, we claim that the cost functional $J(t,x;u(\cdot))$ is continuous over $\mathcal{A}$.*

Proof. By means of Lemma 1.2 in [18], for any admissible control $u(\cdot)\in\mathcal{A}$, the state equation (1.1) has a unique solution $X(\cdot)$. And by applying the Itô's formula to $|X(s)|^2$, Gronwall inequality and B-D-G inequality, we can easily obtain that there is a constant K satisfy

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\le s\le T}|X(s)|^2\Big] \le& K\mathbb{E}\Big[\int_t^T|u(s)|^2\mathrm{d}s+\int_t^T|b(s)|^2\mathrm{d}s+\sum_{i=1}^d\int_t^T|\sigma_i(s)|^2\mathrm{d}s\\&+\int_t^T\Big(\int_{\mathbb{Z}}|h(s,\theta)|^2\nu(\mathrm{d}\theta)\Big)\mathrm{d}s+|x|^2\Big].\end{aligned}\tag{2.1}$$

Suppose that $\bar{X}(\cdot)$ is the state process corresponding to another admissible control $\bar{u}(\cdot)\in\mathcal{A}$, similarly, by applying the Itô's formula to $|X(s)-\bar{X}(s)|^2$, we can conclude:

$$\mathbb{E}\Big[\sup_{t\le s\le T}|X(s)-\bar{X}(s)|^2\Big]\le K\mathbb{E}\Big[\int_t^T|u(s)-\bar{u}(s)|^2\mathrm{d}s\Big].\tag{2.2}$$

Combining with (1.3), and making use of Hölder inequality and the elementary inequality $2xy\le x^2+y^2,\ \forall x,y>0$,

$$\begin{aligned}|J(t,x;u(\cdot))|\le& K\Big\{\mathbb{E}\Big[\sup_{t\le s\le T}|X(s)|^2\Big]+\mathbb{E}\Big[\int_t^T|u(s)|^2\mathrm{d}s\Big]+\mathbb{E}\Big[\int_t^T|q(s)|^2\mathrm{d}s\Big]\\&+\int_t^T|\bar{q}(s)|^2\mathrm{d}s+\mathbb{E}\Big[\int_t^T|\rho(s)|^2\mathrm{d}s\Big]+\int_t^T|\bar{\rho}(s)|^2\mathrm{d}s\Big\}\\\le& K\mathbb{E}\Big\{\mathbb{E}\Big[\int_t^T|u(s)|^2\mathrm{d}s\Big]+\mathbb{E}\Big[\int_t^T|b(s)|^2\mathrm{d}s\Big]+\sum_{i=1}^d\mathbb{E}\Big[\int_t^T|\sigma_i(s)|^2\mathrm{d}s\Big]\\&+\mathbb{E}\Big[\int_t^T\Big(\int_{\mathbb{Z}}|h(s,\theta)|^2\nu(\mathrm{d}\theta)\Big)\mathrm{d}s\Big]+|x|^2+\mathbb{E}\Big[\int_t^T|q(s)|^2\mathrm{d}s\Big]\\&+\int_t^T|\bar{q}(s)|^2\mathrm{d}s+\mathbb{E}\Big[\int_t^T|\rho(s)|^2\mathrm{d}s\Big]+\int_t^T|\bar{\rho}(s)|^2\mathrm{d}s\Big\}\\<&\infty.\end{aligned}\tag{2.3}$$

After applying Hölder inequality, we have the following estimate:

$$\begin{aligned}&\int_t^T\Big[\langle Q(s)X(s),X(s)\rangle-\langle Q(s)\bar{X}(s),\bar{X}(s)\rangle\Big]\mathrm{d}s\\=&\int_t^T\Big[\langle Q(s)(X(s)-\bar{X}(s)),X(s)\rangle+\langle Q(s)\bar{X}(s),(X(s)-\bar{X}(s))\rangle\Big]\mathrm{d}s\\\le& K\int_t^T|X(s)-\bar{X}(s)||X(s)|\mathrm{d}s+K\int_t^T|\bar{X}(s)||X(s)-\bar{X}(s)|\mathrm{d}s,\end{aligned}\tag{2.4}$$

$\langle R(s)u(s),u(s)\rangle-\langle R(s)\bar{u}(s),\bar{u}(s)\rangle$, $\langle S(s)^\top u(s),X(s)\rangle-\langle S(s)^\top\bar{u}(s),\bar{X}(s)\rangle$ can be treated in the same way, and the mean-field type is also applicable, then we have:

$$\begin{aligned}|J(t,x;u(\cdot))-J(t,x;\bar{u}(\cdot))|^2\le& K\mathbb{E}\Big[\int_t^T|u(s)-\bar{u}(s)|^2\mathrm{d}s\Big]\times\mathbb{E}\Big[\int_t^T\Big(|u(s)|^2+|\bar{u}(s)|^2+|b(s)|^2\\&+\sum_{i=1}^d|\sigma_i(s)|^2+\int_{\mathbb{Z}}|h(s,\theta)|^2\nu(\mathrm{d}\theta)+|x|^2\Big)\mathrm{d}s\Big].\end{aligned}\tag{2.5}$$

Thus, we can conclude that

$$J(t,x;u(\cdot))-J(t,x;\bar{u}(\cdot))\to 0,\ \text{as}\ u(\cdot)\to\bar{u}(\cdot),$$

which implies that $J(t,x;u(\cdot))$ is continuous over $\mathcal{A}$. And according to (2.3), we know that $|J(t,x;u(\cdot))|<\infty$, the Problem 1.1 is well-defined. □

Lemma 2.2. *Let Assumptions 1.4–1.6 be hold. Then for any admissible control $u(\cdot),\ v(\cdot)\in\mathcal{A}$, $J(t,x;u(\cdot))$ is Fréchet differentiable at $u(\cdot)$, and the corresponding Fréchet derivative $J'(t,x;u(\cdot))$ at $u(\cdot)$ is given by*

$$\begin{aligned}\langle J'(t,x;u(\cdot)),v(\cdot)\rangle =&2\mathbb{E}\bigg[\langle GX(T)+\bar{G}\mathbb{E}[X(T)]+g+\bar{g},X^{0,v}(T)\rangle+\int_t^T\bigg(\langle Q(s)X(s)+\bar{Q}(s)\mathbb{E}[X(s)]\\&+S(s)^\top u(s)+\bar{S}(s)^\top\mathbb{E}[u(s)]+q(s)+\bar{q}(s),X^{0,v}(s)\rangle+\langle S(s)X(s)+\bar{S}(s)\mathbb{E}[X(s)]\\&+R(s)u(s)+\bar{R}(s)\mathbb{E}[u(s)]+\rho(s)+\bar{\rho}(s),v(s)\rangle\bigg)\mathrm{d}s\bigg],\end{aligned}\tag{2.6}$$

where $X(\cdot)$ or $X^{x,u}(\cdot)$ is the solution of (1.1) corresponding to the admissible control $u(\cdot)$ and $X^{0,v}(\cdot)$ is the solution of (1.5) corresponding to the admissible control $v(\cdot)$.

Proof. Since the state equation is linear, for $\forall\varepsilon\in(0,1)$, it is easy to check that

$$X^{x,u+\varepsilon v}(s)=X(s)+X^{0,\varepsilon v}(s)=X(s)+\varepsilon X^{0,v}(s).\tag{2.7}$$

After making use of (2.7), and subtracting from (1.3), we obtain

$$\begin{aligned}&J(t,x;u(\cdot)+\varepsilon v(\cdot))-J(t,x;u(\cdot))\\=&\varepsilon^2\mathbb{E}\bigg[\langle GX^{0,v}(T),X^{0,v}(T)\rangle+\langle\bar{G}\mathbb{E}[X^{0,v}(T)],\mathbb{E}[X^{0,v}(T)]\rangle+\int_t^T\bigg(\langle Q(s)X^{0,v}(s),X^{0,v}(s)\rangle\\&+\langle\bar{Q}(s)\mathbb{E}[X^{0,v}(s)],\mathbb{E}[X^{0,v}(s)]\rangle+2\langle S(s)^\top v(s),X^{0,v}(s)\rangle+2\langle\bar{S}(s)^\top\mathbb{E}[v(s)],\mathbb{E}[X^{0,v}(s)]\rangle\\&+\langle R(s)v(s),v(s)\rangle+\langle\bar{R}(s)\mathbb{E}[v(s)],\mathbb{E}[v(s)]\rangle\bigg)\mathrm{d}s\bigg]+2\varepsilon\mathbb{E}\bigg[\langle GX(T)+\bar{G}\mathbb{E}[X(T)]+g+\bar{g},X^{0,v}(T)\rangle\\&+\int_t^T\bigg(\langle Q(s)X(s)+\bar{Q}(s)\mathbb{E}[X(s)]+S(s)^\top u(s)+\bar{S}(s)^\top\mathbb{E}[u(s)]+q(s)+\bar{q}(s),X^{0,v}\rangle\\&+\langle S(s)X(s)+\bar{S}(s)\mathbb{E}[X(s)]+R(s)u(s)+\bar{R}(s)\mathbb{E}[u(s)]+\rho(s)+\bar{\rho}(s),v(s)\rangle\bigg)\mathrm{d}s\bigg],\end{aligned}\tag{2.8}$$

we express the right hand of (2.6) as $\Delta^{u,v}$. Specially, when we set $\varepsilon=1$, combining with (1.6), it's easy to verify that

$$J(t,x;u(\cdot)+v(\cdot))-J(t,x;u(\cdot))=J^0(t,0;v(\cdot))+\Delta^{u,v}.\tag{2.9}$$

On the other hand, according to (2.3), the following estimate holds,

$$J^0(t,0;v(\cdot))\le K\bigg\{\mathbb{E}\bigg[\int_t^T|v(s)|^2\mathrm{d}s\bigg]\bigg\}=K\|v(s)\|^2_{\mathcal{A}}.\tag{2.10}$$

Therefore, by the definition of Fréchet differentiable,

$$\begin{aligned}&\lim_{\|v(s)\|_{\mathcal{A}}\to 0^+}\frac{J(t,x;u(\cdot)+v(\cdot))-J(t,x;u(\cdot))-\Delta^{u,v}}{\|v(s)\|_{\mathcal{A}}}\\=&\lim_{\|v(s)\|_{\mathcal{A}}\to 0^+}\frac{J^0(t,0;v(\cdot))}{\|v(s)\|_{\mathcal{A}}}=0,\end{aligned}\tag{2.11}$$

which implies that the cost functional $J(t,x;u(\cdot))$ has Fréchet derivative $J'(t,x;u(\cdot))$ given by (2.6). □

Corollary 2.3. *Let Assumptions 1.4–1.6 be satisfied. For any $(t,x)\in[0,T]\times\mathbb{R}^n$, the relationship between $J(t,x;u(\cdot)+\varepsilon v(\cdot))$ and $J(t,x;u(\cdot))$ satisfies the following:*

$$J(t,x;u(\cdot)+\varepsilon v(\cdot))=J(t,x;u(\cdot))+\varepsilon^2J^0(t,0;v(\cdot))+\varepsilon\langle J'(t,x;u(\cdot)),v(\cdot)\rangle,\tag{2.12}$$

and the Gâteaux derivative of $J(t,x;u(\cdot))$ at $u(\cdot)$ in the direction $v(\cdot)$ is given by (2.6).

Proof. In the process of proving Lemma 2.2, we have already got (2.8), comparing with (1.6) and (2.6),

$$\begin{aligned}J(t,x;u(\cdot)+\varepsilon v(\cdot))-J(t,x;u(\cdot))=&\varepsilon^2J^0(t,0;v(\cdot))+2\varepsilon\mathbb{E}\Bigg[\langle GX(T)+\bar{G}\mathbb{E}[X(T)]+g+\bar{g},X^{0,v}(T)\rangle\\&+\int_t^T\Big(\langle Q(s)X(s)+\bar{Q}(s)\mathbb{E}[X(s)]+S(s)^\top u(s)+\bar{S}(s)^\top\mathbb{E}[u(s)]\\&+q(s)+\bar{q}(s),X^{0,v}(s)\rangle+\langle S(s)X(s)+\bar{S}(s)\mathbb{E}[X(s)]+R(s)u(s)\\&+\bar{R}(s)\mathbb{E}[u(s)]+\rho(s)+\bar{\rho}(s),v(s)\rangle\Big)\mathrm{d}s\Bigg]\\=&\varepsilon^2J^0(t,0;v(\cdot))+\varepsilon\langle J'(t,x;u(\cdot)),v(\cdot)\rangle.\end{aligned}\tag{2.13}$$

On the other hand, according to (2.13), we can easily obtain that

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\frac{J(t,x;u(\cdot)+\varepsilon v(\cdot))-J(t,x;u(\cdot))}{\varepsilon}\\=&\lim_{\varepsilon\to 0^+}\frac{\varepsilon^2J^0(t,0;v(\cdot))+\varepsilon\langle J'(t,x;u(\cdot)),v(\cdot)\rangle}{\varepsilon}\\=&\langle J'(t,x;u(\cdot)),v(\cdot)\rangle.\end{aligned}\tag{2.14}$$

Thus, we can conclude that since the cost functional $J(t,x;u(\cdot))$ is Fréchet differentiable, then it is also *Gâteaux* differentiable, and the *Gâteaux* derivative is the same as the Fréchet derivative. □

Lemma 2.4. *Under Assumptions 1.4–1.6, we conjecture $J(t,x;u(\cdot))$ is strictly convex and coercive over $\mathcal{A}$.*

Proof. For any $v(\cdot)\in\mathcal{A}$, takeing $u(\cdot)=0$ and $\varepsilon=1$ in (2.12), we get that

$$J(t,x;v(\cdot))=J(t,x;0)+J^0(t,0;v(\cdot))+\langle J'(t,x;0),v(\cdot)\rangle\tag{2.15}$$

It's obvious that $J(t,x;0)+\langle J'(t,x;0),v(\cdot)\rangle$ is convex in $v(\cdot)$, since it is liner with respect with $v(\cdot)$. On the other hand, substituting (1.8) and (1.9) into (1.7), we have

$$\begin{aligned}
&J^0(t,0;v(\cdot))\\
=&\mathbb{E}\Big[\langle G(X^{0,v}(T)-\mathbb{E}[X^{0,v}(T)]),X^{0,v}(T)-\mathbb{E}[X^{0,v}(T)]\rangle+\langle[G+\bar{G}]\mathbb{E}[X^{0,v}(T)],\mathbb{E}[X^{0,v}(T)]\rangle\Big]+\mathbb{E}\Big[\int_t^T\Big(\big\langle[Q(s)\\
&-S(s)^\top R(s)^{-1}S(s)](X^{0,v}(s)-\mathbb{E}[X^{0,v}(s)]),X^{0,v}(s)-\mathbb{E}[X^{0,v}(s)]\big\rangle+\big\langle R(s)\{(v(s)-\mathbb{E}[v(s)])\\
&+R(s)^{-1}S(s)(X^{0,v}(s)-\mathbb{E}[X^{0,v}(s)])\},(v(s)-\mathbb{E}[v(s)])+R(s)^{-1}S(s)(X^{0,v}(s)-\mathbb{E}[X^{0,v}(s)])\big\rangle+\big\langle\{[Q(s)\\
&+\bar{Q}(s)]-[S(s)+\bar{S}(s)]^\top[R(s)+\bar{R}(s)]^{-1}[S(s)+\bar{S}(s)]\}\mathbb{E}[X^{0,v}(s)],\mathbb{E}[X^{0,v}(s)]\big\rangle+\big\langle[R(s)+\bar{R}(s)]\{\mathbb{E}[v(s)]\\
&+[R(s)+\bar{R}(s)]^{-1}[S(s)+\bar{S}(s)]\mathbb{E}[X^{0,v}(s)]\},\mathbb{E}[v(s)]+[R(s)+\bar{R}(s)]^{-1}[S(s)+\bar{S}(s)]\mathbb{E}[X^{0,v}(s)]\big\rangle\Big)\mathrm{d}s\Big].
\end{aligned}\tag{2.16}$$

So under Assumptions 1.4–1.6, $J^0(t,0;v(\cdot))$ is strictly convex in $v(\cdot)\in\mathcal{A}$, since $R(s),R(s)+\bar{R}(s)\geq\delta I$. Therefore, $J(t,x;v(\cdot))=J(t,x;0)+J^0(t,0;v(\cdot))+\langle J'(t,x;0),v(\cdot)\rangle$ is strictly convex in $v(\cdot)\in\mathcal{A}$. And for some positive constant δ, by Assumption 1.6, the following inequality holds:

$$\begin{aligned}
J^0(t,0;u(\cdot))\geq&\mathbb{E}\Big[\int_t^T\big\langle R(s)\{(u(s)-\mathbb{E}[u(s)])+R(s)^{-1}S(s)(X(s)-\mathbb{E}[X(s)])\},(u(s)-\mathbb{E}[u(s)])\\
&+R(s)^{-1}S(s)(X(s)-\mathbb{E}[X(s)])\big\rangle+\big\langle\{[Q(s)+\bar{Q}(s)]-[S(s)+\bar{S}(s)]^\top[R(s)+\bar{R}(s)]^{-1}\\
&\cdot[S(s)+\bar{S}(s)]\}\mathbb{E}[X(s)],\mathbb{E}[X(s)]\big\rangle+\big\langle[R(s)+\bar{R}(s)]\{\mathbb{E}[u(s)]+[R(s)+\bar{R}(s)]^{-1}\\
&\cdot[S(s)+\bar{S}(s)]\mathbb{E}[X(s)]\},\mathbb{E}[u(s)]+[R(s)+\bar{R}(s)]^{-1}[S(s)+\bar{S}(s)]\mathbb{E}[X(s)]\big\rangle\Big]\mathrm{d}s\\
\geq&\delta\mathbb{E}\Big[\int_t^T\Big|(u(s)-\mathbb{E}[u(s)])+R(s)^{-1}S(s)(X(s)-\mathbb{E}[X(s)])\Big|^2\mathrm{d}s\Big]+\delta\mathbb{E}\Big[\int_t^T\Big|\mathbb{E}[u(s)]\\
&+[R(s)+\bar{R}(s)]^{-1}[S(s)+\bar{S}(s)]\mathbb{E}[X(s)]\Big|^2\mathrm{d}s\Big].
\end{aligned}\tag{2.17}$$

Then according to the Lemma 5.1 in [16], there exists a constant $\gamma>0$ such that

$$\begin{aligned}
J^0(t,0;u(\cdot))\geq&\delta\mathbb{E}\Big\{\int_t^T\Big[\Big|(u(s)-\mathbb{E}[u(s)])+R(s)^{-1}S(s)(X(s)-\mathbb{E}[X(s)])\Big|^2+\gamma|\mathbb{E}[u(s)]|^2\Big]\mathrm{d}s\Big\}\\
=&\delta\mathbb{E}\Big\{\int_t^T\Big[\Big|u(s)+R(s)^{-1}S(s)(X(s)-\mathbb{E}[X(s)])\Big|^2-2\langle u(s)+R(s)^{-1}S(s)(X(s)\\
&-\mathbb{E}[X(s)]),\mathbb{E}[u(s)]\rangle+(1+\gamma)|\mathbb{E}[u(s)]|^2\Big]\mathrm{d}s\Big\}\\
\geq&\frac{\delta\gamma^2}{1+\gamma}\mathbb{E}\Big\{\int_t^T|u(s)|^2\mathrm{d}s\Big\}.
\end{aligned}\tag{2.18}$$

Therefore $\lim\limits_{\|u(\cdot)\|_{\mathcal{A}}\to\infty}J(t,x;u(\cdot))=\lim\limits_{\|u(\cdot)\|_{\mathcal{A}}\to\infty}\{J(t,x;0)+J^0(t,0;u(\cdot))+\langle J'(t,x;0),u(\cdot)\rangle\}=\infty.$ □

Theorem 2.5. *Let Assumptions 1.4–1.6 be satisfied. Then we claim that Problem 1.1 has a unique optimal control.*

Proof. According to Lemma 2.1–2.4, we have already illustrated the continuity, coercive, strictly convexity and Fréchet differentiability of $J(t,x;u(\cdot))$. Then we can obtain directly that the optimal control of Problem 1.1 exists and is unique according to the Proposition 2.1.2 of [4]. □

Theorem 2.6. *Under Assumptions 1.4–1.6, for any admissible control $v(\cdot)\in\mathcal{A}$, the necessary and sufficient condition for an admissible control $u(\cdot)\in\mathcal{A}$ to be an optimal control of Problem 1.1 is that*

$$\langle J'(t,x;u(\cdot)),v(\cdot)\rangle=0. \tag{2.19}$$

Proof. (Necessary): If $u^*(\cdot)$ is the optimal control, according to Problem 1.1, we have

$$J(t,x;u^*(\cdot))=\min_{u(\cdot)\in\mathcal{A}}J(t,x;u(\cdot)),$$

so

$$J(t,x;u^*(\cdot)+\varepsilon v(\cdot))-J(t,x;u^*(\cdot))\geq 0,$$

$$J(t,x;u^*(\cdot)-\varepsilon v(\cdot))-J(t,x;u^*(\cdot))\geq 0,$$

from the definition of *Gâteaux* derivative $J'(t,x;u(\cdot))$, we can get

$$\langle J'(t,x;u^*(\cdot)),v(\cdot)\rangle=\lim_{\varepsilon\to 0^+}\frac{J(t,x;u^*(\cdot)+\varepsilon v(\cdot))-J(t,x;u^*(\cdot))}{\varepsilon}\geq 0,$$

$$\langle J'(t,x;u^*(\cdot)),-v(\cdot)\rangle=\lim_{\varepsilon\to 0^+}\frac{J(t,x;u^*(\cdot)-\varepsilon v(\cdot))-J(t,x;u^*(\cdot))}{\varepsilon}\geq 0.$$

However $\langle J'(t,x;u^*(\cdot)),v(\cdot)\rangle$ is linear about $v(\cdot)$, and

$$\langle J'(t,x;u^*(\cdot)),v(\cdot)\rangle=-\langle J'(t,x;u^*(\cdot)),-v(\cdot)\rangle\leq 0,$$

then we deduce that

$$\langle J'(t,x;u^*(\cdot)),v(\cdot)\rangle=0.$$

(Sufficient): If there exists a control $u(\cdot)\in\mathcal{A}$, s.t. $\langle J'(t,x;u(\cdot)),v(\cdot)\rangle=0$ holds, we seek to prove that $u(\cdot)$ is the optimal control. In other words, we just need to prove for any admission control $v(\cdot)\in\mathcal{A}$,

$$J(t,x;v(\cdot))-J(t,x;u(\cdot))\geq 0.$$

Actually, according to (2.12) and (2.18), we have

$$\begin{aligned}J(t,x;v(\cdot))-J(t,x;u(\cdot))&=J^0(t,0;v(\cdot)-u(\cdot))+\langle J'(t,x;u(\cdot)),v(\cdot)-u(\cdot)\rangle\\&\geq 0+\langle J'(t,x;u(\cdot)),v(\cdot)-u(\cdot)\rangle=0.\end{aligned} \tag{2.20}$$

which implies that $u(\cdot)$ is the optimal control. The proof is complete. □

3. Optimal control and stochastic Hamilton system

Now, for any admissible pair $(X(\cdot), u(\cdot))$, $t \in [0,T]$, we define the Hamiltonian function H: $[t,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times L^{\nu,2}(\mathbb{Z};\mathbb{R}^n)$ by

$$\begin{aligned}
&H(s,x,u,\bar{x},\bar{u},y,z,r)\\
=&\langle y, A(s)x+\bar{A}(s)\bar{x}+B(s)u+\bar{B}(s)\bar{u}+b(s)\rangle+\sum_{i=1}^{d}\langle z_i, C_i(s)x+\bar{C}_i(s)\bar{x}+D_i(s)u+\bar{D}_i(s)\bar{u}+\sigma_i(s)\rangle\\
&+\int_{\mathbb{Z}}\langle r(\theta), E(s,\theta)x+\bar{E}(s,\theta)\bar{x}+F(s,\theta)u+\bar{F}(s,\theta)\bar{u}+h(s,\theta)\rangle\nu(\mathrm{d}\theta)+\langle Q(s)x,x\rangle+\langle\bar{Q}(s)\bar{x},\bar{x}\rangle\\
&+2\langle S(s)^\top u,x\rangle+2\langle\bar{S}(s)^\top\bar{u},\bar{x}\rangle+\langle R(s)u,u\rangle+\langle\bar{R}(s)\bar{u},\bar{u}\rangle+2\langle q(s),x\rangle+2\langle\bar{q}(s),\bar{x}\rangle+2\langle\rho(s),u\rangle+2\langle\bar{\rho}(s),\bar{u}\rangle.
\end{aligned}\tag{3.1}$$

Let $(Y(\cdot), Z(\cdot), r(\cdot,\cdot))$ be the adjoint processes corresponding to $(X(\cdot), u(\cdot))$, and can be defined as the unique solution of the MF-BSDE as follows:

$$\begin{cases}
\mathrm{d}Y(s)=-\Big[A(s)^\top Y(s)+\bar{A}(s)^\top\mathbb{E}[Y(s)]+\sum_{i=1}^{d}\{C_i(s)^\top Z_i(s)+\bar{C}_i(s)^\top\mathbb{E}[Z_i(s)]\}+\int_{\mathbb{Z}}\{E(s,\theta)^\top r(s,\theta)\\
\qquad+\bar{E}(s,\theta)^\top\mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta)+2Q(s)X(s)+2\bar{Q}(s)\mathbb{E}[X(s)]+2S(s)^\top u(s)+2\bar{S}(s)^\top\mathbb{E}[u(s)]\\
\qquad+2q(s)+2\bar{q}(s)\Big]\mathrm{d}s+\sum_{i=1}^{d}Z_i(s)\mathrm{d}W_i(s)+\int_{\mathbb{Z}}r(s,\theta)\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta),\ s\in[t,T],\\
Y(T)=2\{GX(T)+\bar{G}\mathbb{E}[X(T)]+g+\bar{g}\}.
\end{cases}\tag{3.2}$$

Finding the partial derivative of (3.1) with respect to x and $\bar{x}$,

$$\begin{aligned}
&H_x(s,X(s),u(s),\bar{X}(s),\bar{u}(s),Y(s),Z(s),r(s,\theta))+\mathbb{E}\big[H_{\bar{x}}(s,X(s),u(s),\bar{X}(s),\bar{u}(s),Y(s),Z(s),r(s,\theta))\big]\\
=&A(s)^\top Y(s)+\bar{A}(s)^\top\mathbb{E}[Y(s)]+\sum_{i=1}^{d}\{C_i(s)^\top Z_i(s)+\bar{C}_i(s)^\top\mathbb{E}[Z_i(s)]\}+\int_{\mathbb{Z}}\{E(s,\theta)^\top r(s,\theta)\\
&+\bar{E}(s,\theta)^\top\mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta)+2Q(s)X(s)+2\bar{Q}(s)\mathbb{E}[X(s)]+2S(s)^\top u(s)+2\bar{S}(s)^\top\mathbb{E}[u(s)]+2q(s)+2\bar{q}(s),
\end{aligned}\tag{3.3}$$

with respect to u and $\bar{u}$,

$$\begin{aligned}
&H_u(s,X(s),u(s),\bar{X}(s),\bar{u}(s),Y(s),Z(s),r(s,\theta))+\mathbb{E}\big[H_{\bar{u}}(s,X(s),u(s),\bar{X}(s),\bar{u}(s),Y(s),Z(s),r(s,\theta))\big]\\
=&B(s)^\top Y(s)+\bar{B}(s)^\top\mathbb{E}[Y(s)]+\sum_{i=1}^{d}\{D_i(s)^\top Z_i(s)+\bar{D}_i(s)^\top\mathbb{E}[Z_i(s)]\}+\int_{\mathbb{Z}}\{F(s,\theta)^\top r(s,\theta)\\
&+\bar{F}(s,\theta)^\top\mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta)+2S(s)X(s)+2\bar{S}(s)\mathbb{E}[X(s)]+2R(s)u(s)+2\bar{R}(s)\mathbb{E}[u(s)]+2\rho(s)+2\bar{\rho}(s).
\end{aligned}\tag{3.4}$$

So comparing (3.3) with (3.2), we can rewrite (3.2) in Hamiltonian form,

$$\begin{cases}
\mathrm{d}Y(s)=-\Big\{H_x(s,X(s),u(s),\bar{X}(s),\bar{u}(s),Y(s),Z(s),r(s,\theta))+\mathbb{E}\big[H_{\bar{x}}(s,X(s),u(s),\bar{X}(s),\bar{u}(s),Y(s),\\
\qquad Z(s),r(s,\theta))\big]\Big\}\mathrm{d}s+\sum_{i=1}^{d}Z_i(s)\mathrm{d}W_i(s)+\int_{\mathbb{Z}}r(s,\theta)\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta),\\
Y(T)=2\{GX(T)+\bar{G}\mathbb{E}[X(T)]+g+\bar{g}\}.
\end{cases}\tag{3.5}$$

Theorem 3.1. *Let Assumptions 1.4–1.6 be hold. Then under partial information, a necessary and sufficient condition for an admission pair $(X(\cdot), u(\cdot))$ to be an optimal pair of Problem 1.1 is that the admission control $u(\cdot)$ satisfies*

$$\begin{aligned}&\mathbb{E}\Big[H_u(s, X(s), u(s), \bar{X}(s), \bar{u}(s), Y(s), Z(s), r(s,\theta))\Big|\mathscr{G}_s\Big]\\&\qquad+\mathbb{E}\big[H_{\bar{u}}(s, X(s), u(s), \bar{X}(s), \bar{u}(s), Y(s), Z(s), r(s,\theta))\big]=0.\end{aligned}\tag{3.6}$$

that is

$$\begin{aligned}&B(s)^\top\mathbb{E}[Y(s)|\mathscr{G}_s]+\bar{B}(s)^\top\mathbb{E}[Y(s)]+\sum_{i=1}^{d}\{D_i(s)^\top\mathbb{E}[Z_i(s)|\mathscr{G}_s]+\bar{D}_i(s)^\top\mathbb{E}[Z_i(s)]\}\\&+\int_{\mathbb{Z}}\big\{F(s,\theta)^\top\mathbb{E}[r(s,\theta)|\mathscr{G}_s]+\bar{F}(s,\theta)^\top\mathbb{E}[r(s,\theta)]\big\}\nu(\mathrm{d}\theta)+2S(s)^\top\mathbb{E}[X(s)|\mathscr{G}_s]\\&+2\bar{S}(s)^\top\mathbb{E}[X(s)]+2R(s)u(s)+2\bar{R}(s)\mathbb{E}[u(s)]+2\mathbb{E}[\rho(s)|\mathscr{G}_s]+2\bar{\rho}(s)=0.\quad a.e.s\in[t,T].\end{aligned}\tag{3.7}$$

Proof. Given an admissible control $u(\cdot)\in\mathcal{A}$, by Lemma 2.2 in [13], we know that under Assumption 1.4–1.6, MF-BSDE (3.2) admits a unique adapted solution $(Y(\cdot), Z(\cdot), r(\cdot,\cdot))\in S^2_{\mathbb{F}}(t,T;\mathbb{R}^n)\times L^2_{\mathbb{F}}(t,T;\mathbb{R}^{n\times d})\times L^{\nu,2}_{\mathbb{F}}([t,T]\times Z;\mathbb{R}^n)$. Then making use of (1.5), and applying Itô formula to $\langle X^{0,v}(s), Y(s)\rangle$, we have

$$\begin{aligned}\mathbb{E}\Big[\int_t^T d\langle X^{0,v}(s), Y(s)\rangle\Big]=&\mathbb{E}\Big[\int_t^T\Big(\langle B(s)^\top Y(s)+\bar{B}(s)^\top\mathbb{E}[Y(s)]+\sum_{i=1}^{d}\{D_i(s)^\top Z_i(s)+\bar{D}_i(s)^\top\mathbb{E}[Z_i(s)]\}\\&+\int_{\mathbb{Z}}\{F(s,\theta)^\top r(s,\theta)+\bar{F}(s,\theta)^\top\mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta), v(s)\rangle\Big)\mathrm{d}s\Big]-2\mathbb{E}\Big[\int_t^T\langle Q(s)X(s)\\&+\bar{Q}(s)\mathbb{E}[X(s)]+S(s)^\top u(s)+\bar{S}(s)^\top\mathbb{E}[u(s)]+q(s)+\bar{q}(s), X^{0,v}(s)\rangle\mathrm{d}s\Big].\end{aligned}\tag{3.8}$$

and $\mathbb{E}\Big[\int_t^T d\langle X^{0,v}(s), Y(s)\rangle\Big]$ can also be represented by

$$\begin{aligned}\mathbb{E}\Big[\int_t^T d\langle X^{0,v}(s), Y(s)\rangle\Big]&=\mathbb{E}\langle X^{0,v}(T), Y(T)\rangle-\mathbb{E}\langle X^{0,v}(t), Y(t)\rangle\\&=2\mathbb{E}\langle GX(T)+\bar{G}\mathbb{E}[X(T)]+g+\bar{g}, X^{0,v}(T)\rangle.\end{aligned}\tag{3.9}$$

Combining (3.8) with (3.9), we can get

$$\begin{aligned}&2\mathbb{E}\langle GX(T)+\bar{G}\mathbb{E}[X(T)]+g+\bar{g}, X^{0,v}(T)\rangle+2\mathbb{E}\Big[\int_t^T\langle Q(s)X(s)+\bar{Q}(s)\mathbb{E}[X(s)]\\&\quad+S(s)^\top u(s)+\bar{S}(s)^\top\mathbb{E}[u(s)]+q(s)+\bar{q}(s), X^{0,v}\rangle\mathrm{d}s\Big]\\&=\mathbb{E}\Big[\int_t^T\Big(\langle B(s)^\top Y(s)+\bar{B}(s)^\top\mathbb{E}[Y(s)]+\sum_{i=1}^{d}\{D_i(s)^\top Z_i(s)+\bar{D}_i(s)^\top\mathbb{E}[Z_i(s)]\}\\&\quad+\int_{\mathbb{Z}}\{F(s,\theta)^\top r(s,\theta)+\bar{F}(s,\theta)^\top\mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta), v(s)\rangle\Big)\mathrm{d}s\Big].\end{aligned}\tag{3.10}$$

Through Lemma 2.2, we have already got the representation of the $\langle J'(t,x;u(\cdot)),v(\cdot)\rangle$. Now putting (3.10) into (2.6), finally $\langle J'(t,x;u(\cdot)),v(\cdot)\rangle$ can be expressed as follows,

$$\begin{aligned}\langle J'(t,x;u(\cdot)),v(\cdot)\rangle =&\mathbb{E}\Big[\int_t^T\Big(\langle B(s)^\top Y(s)+\bar{B}(s)^\top\mathbb{E}[Y(s)]+\sum_{i=1}^d\{D_i(s)^\top Z_i(s)+\bar{D}_i(s)^\top\mathbb{E}[Z_i(s)]\}\\&+\int_{\mathbb{Z}}\{F(s,\theta)^\top r(s,\theta)+\bar{F}(s,\theta)^\top\mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta)+2S(s)X(s)+2\bar{S}(s)\mathbb{E}[X(s)]\\&+2R(s)u(s)+2\bar{R}(s)\mathbb{E}[u(s)]+2\rho(s)+2\bar{\rho}(s),v(s)\rangle\Big)\mathrm{d}s\Big].\end{aligned}\tag{3.11}$$

(necessary): According to Theorem 2.6, let $(X(\cdot),u(\cdot))$ be an optimal pair, we have $\langle J'(t,x;u(\cdot)),v(\cdot)\rangle=0$. But now we consider the situation under partial information, $u(\cdot),v(\cdot)$ are $\mathscr{G}_t$-predictable. According to Lemma 5.4 in [21], we consider to use conditional mathematics expectation $\mathbb{E}[\cdot|\mathscr{G}_s]$, that is

$$\begin{aligned}&\mathbb{E}\Big[\int_t^T\Big(\langle B(s)^\top Y(s)+\bar{B}(s)^\top\mathbb{E}[Y(s)]+\sum_{i=1}^d\{D_i(s)^\top Z_i(s)+\bar{D}_i(s)^\top\mathbb{E}[Z_i(s)]\}\\&+\int_{\mathbb{Z}}\{F(s,\theta)^\top r(s,\theta)+\bar{F}(s,\theta)^\top\mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta)+2S(s)X(s)+2\bar{S}(s)\mathbb{E}[X(s)]\\&+2R(s)u(s)+2\bar{R}(s)\mathbb{E}[u(s)]+2\rho(s)+2\bar{\rho}(s),v(s)\rangle\Big)\mathrm{d}s\Big]\\=&\mathbb{E}\Big[\int_t^T\Big(\langle B(s)^\top\mathbb{E}[Y(s)|\mathscr{G}_s]+\bar{B}(s)^\top\mathbb{E}[Y(s)]+\sum_{i=1}^d\{D_i(s)^\top\mathbb{E}[Z_i(s)|\mathscr{G}_s]+\bar{D}_i(s)^\top\mathbb{E}[Z_i(s)]\}\\&+\int_{\mathbb{Z}}\{F(s,\theta)^\top\mathbb{E}[r(s,\theta)|\mathscr{G}_s]+\bar{F}(s,\theta)^\top\mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta)+2S(s)\mathbb{E}[X(s)|\mathscr{G}_s]+2\bar{S}(s)\mathbb{E}[X(s)]\\&+2R(s)u(s)+2\bar{R}(s)\mathbb{E}[u(s)]+2\mathbb{E}[\rho(s)|\mathscr{G}_s]+2\bar{\rho}(s),v(s)\rangle\Big)\mathrm{d}s\Big]\\=&0.\end{aligned}\tag{3.12}$$

And due to the arbitrariness of $v(\cdot)$ and its $\mathscr{G}_s$-measurability, we can obtain

$$\begin{aligned}&B(s)^\top\mathbb{E}[Y(s)|\mathscr{G}_s]+\bar{B}(s)^\top\mathbb{E}[Y(s)]+\sum_{i=1}^d\{D_i(s)^\top\mathbb{E}[Z_i(s)|\mathscr{G}_s]+\bar{D}_i(s)^\top\mathbb{E}[Z_1(s)]\}\\&+\int_{\mathbb{Z}}\{F(s,\theta)^\top\mathbb{E}[r(s,\theta)|\mathscr{G}_s]+\bar{F}(s,\theta)^\top\mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta)+2S(s)\mathbb{E}[X(s)|\mathscr{G}_s]+2\bar{S}(s)\mathbb{E}[X(s)]\\&+2R(s)u(s)+2\bar{R}(s)\mathbb{E}[u(s)]+2\mathbb{E}[\rho(s)|\mathscr{G}_s]+2\bar{\rho}(s)\\=&0.\end{aligned}\tag{3.13}$$

(sufficient): Let $(X(\cdot),u(\cdot))$ be an admission pair satisfying (3.7). Putting (3.7) into (3.11), we have $\langle J'(t,x;u(\cdot)), v(\cdot)\rangle=0$, according to Theorem 2.6, it means $(X(\cdot),u(\cdot))$ is the optimal pair.

Furthermore, combining (3.4) with (3.13), we conclude that (3.6) holds, and (3.7) can also be called the stationarity condition. The proof is complete. □

Remark 3.2. Let Assumptions 1.4–1.6 be hold. Then under complete information, an optimal pair $(X(\cdot), u(\cdot))$ of Problem 1.1 satisfies

$$\begin{aligned}
&B(s)^\top Y(s) + \bar{B}(s)^\top \mathbb{E}[Y(s)] + \sum_{i=1}^{d} \{D_i(s)^\top Z_i(s) + \bar{D}_i(s)^\top \mathbb{E}[Z_i(s)]\} + \int_{\mathbb{Z}} \{F(s,\theta)^\top r(s,\theta) \\
&+ \bar{F}(s,\theta)^\top \mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta) + 2S(s)^\top X(s) + 2\bar{S}(s)^\top \mathbb{E}[X(s)] + 2R(s)u(s) + 2\bar{R}(s)\mathbb{E}[u(s)] \\
&+ 2\rho(s) + 2\bar{\rho}(s) = 0, \quad a.e. s \in [t,T].
\end{aligned} \tag{3.14}$$

Applying expectation to the above, it means

$$\begin{aligned}
\mathbb{E}[u(s)] = &-\frac{1}{2}[R(s)+\bar{R}(s)]^{-1}\bigg[[B(s)+\bar{B}(s)]\mathbb{E}[Y(s)] + \sum_{i=1}^{d}[D_i(s)+\bar{D}_i(s)]^\top \mathbb{E}[Z_i(s)] \\
&+ \int_{\mathbb{Z}}[F(s,\theta)+\bar{F}(s,\theta)]^\top \mathbb{E}[r(s,\theta)]\nu(\mathrm{d}\theta) + 2[S(s)+\bar{S}(s)]^\top \mathbb{E}[X(s)] + 2\mathbb{E}[\rho(s)] + 2\bar{\rho}(s)\bigg],
\end{aligned}$$

and putting the above into (3.14), the optimal control has the following explicit expression as following:

$$\begin{aligned}
u(s) = &-\frac{1}{2}R(s)^{-1}\bigg\{\bigg[B(s)^\top Y(s) + \bar{B}(s)^\top \mathbb{E}[Y(s)] + \sum_{i=1}^{d}\{D_i(s)^\top Z_i(s) + \bar{D}_i(s)^\top \mathbb{E}[Z_i(s)]\} + \int_{\mathbb{Z}}\{F(s,\theta)^\top r(s,\theta) \\
&+ \bar{F}(s,\theta)^\top \mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta) + 2S(s)^\top X(s) + 2\bar{S}(s)^\top \mathbb{E}[X(s)] + 2\rho(s) + 2\bar{\rho}(s)\bigg] - \bar{R}(s)[R(s)+\bar{R}(s)]^{-1} \\
&\cdot \bigg[[B(s)+\bar{B}(s)]\mathbb{E}[Y(s)] + \sum_{i=1}^{d}[D_i(s)+\bar{D}_i(s)]^\top \mathbb{E}[Z_i(s)] + \int_{\mathbb{Z}}[F(s,\theta)+\bar{F}(s,\theta)]^\top \mathbb{E}[r(s,\theta)]\nu(\mathrm{d}\theta) \\
&+ 2[S(s)+\bar{S}(s)]^\top \mathbb{E}[X(s)] + 2\mathbb{E}[\rho(s)] + 2\bar{\rho}(s)\bigg]\bigg\}.
\end{aligned} \tag{3.15}$$

Finally, we introduce the so-called stochastic Hamilton system which consists of the state equation (1.1), adjoint process (3.2) and the representation of optimal control under complete information (3.14):

$$
\begin{cases}
\mathrm{d}X(s)=\big\{A(s)X(s)+\bar{A}(s)\mathbb{E}[X(s)]+B(s)u(s)+\bar{B}(s)\mathbb{E}[u(s)]+b(s)\big\}\mathrm{d}s\\
\qquad+\sum_{i=1}^{d}\big\{C_i(s)X(s)+\bar{C}_i(s)\mathbb{E}[X(s)]+D_i(s)u(s)+\bar{D}_i(s)\mathbb{E}[u(s)]+\sigma_i(s)\big\}\mathrm{d}W_i(s)\\
\qquad+\int_{\mathbb{Z}}\big\{E(s,\theta)X(s-)+\bar{E}(s,\theta)\mathbb{E}[X(s-)]+F(s,\theta)u(s)+\bar{F}(s,\theta)\mathbb{E}[u(s)]+h(s,\theta)\big\}\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta),\\
\mathrm{d}Y(s)=-\Big[A(s)^\top Y(s)+\bar{A}(s)^\top\mathbb{E}[Y(s)]+\sum_{i=1}^{d}\{C_i(s)^\top Z_i(s)+\bar{C}_i(s)^\top\mathbb{E}[Z_i(s)]\}+\int_{\mathbb{Z}}\{E(s,\theta)^\top r(s,\theta)\\
\qquad+\bar{E}(s,\theta)^\top\mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta)+2Q(s)X(s)+2\bar{Q}(s)\mathbb{E}[X(s)]+2S(s)^\top u(s)+2\bar{S}(s)^\top\mathbb{E}[u(s)]\\
\qquad+2q(s)+2\bar{q}(s)\Big]\mathrm{d}s+\sum_{i=1}^{d}Z_i(s)\mathrm{d}W_i(s)+\int_{\mathbb{Z}}r(s,\theta)\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta),\\
X(t)=x\in\mathbb{R}^n,\, Y(T)=2\{GX(T)+\bar{G}\mathbb{E}[X(T)]+g+\bar{g}\},\\
B(s)^\top Y(s)+\bar{B}(s)^\top\mathbb{E}[Y(s)]+\sum_{i=1}^{d}\{D_i(s)^\top Z_i(s)+\bar{D}_i(s)^\top\mathbb{E}[Z_i(s)]\}+\int_{\mathbb{Z}}\{F(s,\theta)^\top r(s,\theta)\\
+\bar{F}(s,\theta)^\top\mathbb{E}[r(s,\theta)]\}\nu(\mathrm{d}\theta)+2S(s)^\top X(s)+2\bar{S}(s)^\top\mathbb{E}[X(s)]+2R(s)u(s)+2\bar{R}(s)\mathbb{E}[u(s)]+2\rho(s)+2\bar{\rho}(s)=0.
\end{cases}
\tag{3.16}
$$

It is obvious that the system is a fully coupled MF-FBSDE, and it's solution consists of $(u(\cdot),X(\cdot),Y(\cdot),Z(\cdot),r(\cdot,\cdot))$. Under the Assumption 1.4–1.6, according to Theorem 2.5 and Theorem 3.1, it is easy to check that the stochastic Hamilton system has a unique solution $(u(\cdot),X(\cdot),Y(\cdot),Z(\cdot),r(\cdot,\cdot))\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)\times S^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\times S^2_{\mathbb{F}}(0,T;\mathbb{R}^n)\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^{n\times d})\times L^{\nu,2}_{\mathbb{F}}([t,T]\times Z;\mathbb{R}^n)$.

4. Riccati equations and state feedback representation of optimal control

4.1. Derivation of Riccati equations

In the last equation of (3.16), $(Y(\cdot),Z(\cdot),r(\cdot,\cdot))$ and $X(\cdot)$ are completely coupled. In order to decouple, we need to introduce two Riccati equations. The following derivation is formal. Here we can set $X(\cdot)$ and $Y(\cdot)$ have the following relationship:

$$
Y(s)=2\Big[P(s)(X(s)-\mathbb{E}[X(s)])+\Pi(s)\mathbb{E}[X(s)]+\eta_1(s)+\eta_2(s)\Big],
\tag{4.1}
$$

where $P(s),\Pi(s)$ taking values in $\mathbb{S}^n_+$, are deterministic differentiable functions such that

$$
P(T)=G,\ \Pi(T)=G+\bar{G}.
\tag{4.2}
$$

We set $\eta_1(s)=\eta(s)-\mathbb{E}[\eta(s)]$, $\eta_2(s)=\bar{\eta}(s)+\mathbb{E}[\eta(s)]$, with $\eta(s)$ being an undetermined stochastic process to be determined and $\bar{\eta}(s)$ being an deterministic function to be determined. Assume that $\eta_1(s)$ satisfies the following BSDE:

$$
\begin{cases}
\mathrm{d}\eta_1(s)=f(s)\mathrm{d}s+\sum_{i=1}^{d}\xi_i(s)\mathrm{d}W_i(s)+\int_{\mathbb{Z}}\zeta(s,\theta)\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta),\\
\eta_1(T)=g-\mathbb{E}(g),
\end{cases}
\tag{4.3}
$$

and $\eta_2(s)$ satisfies the following ODE:

$$
\begin{cases}
\mathrm{d}\eta_2(s)=\bar{f}(s)\mathrm{d}s,\\
\eta_2(T)=\mathbb{E}(g)+\bar{g}.
\end{cases}
\tag{4.4}
$$

Hence, taking expectation on the both sides of (4.1), we have the following relationship:

$$\begin{cases} \mathbb{E}[Y(s)] = 2\{\Pi(s)\mathbb{E}[X(s)] + \eta_2(s)\}, \\ Y(s) - \mathbb{E}[Y(s)] = 2P(s)\{(X(s) - \mathbb{E}[X(s)]) + \eta_1(s)\}. \end{cases} \tag{4.5}$$

Making full use of (1.1), we can get the expressions of $\mathrm{d}\mathbb{E}[X(s)]$, $d(X(s) - \mathbb{E}[X(s)])$,

$$\begin{cases} \mathrm{d}\mathbb{E}[X(s)] = \Big[[A(s) + \bar{A}(s)]\mathbb{E}[X(s)] + [B(s) + \bar{B}(s)]\mathbb{E}[u(s)] + \mathbb{E}[b(s)]\Big]\mathrm{d}s, \\ d(X(s) - \mathbb{E}[X(s)]) = \Big[A(s)(X(s) - \mathbb{E}[X(s)]) + B(s)(u(s) - \mathbb{E}[u(s)]) + (b(s) - \mathbb{E}[b(s)])\Big]\mathrm{d}s + \sum_{i=1}^{d}\Big[C_i(s)(X(s) \\ \qquad -\mathbb{E}[X(s)]) + D_i(s)(u(s) - \mathbb{E}[u(s)]) + [C_i(s) + \bar{C}_i(s)]\mathbb{E}[X(s)] + [D_i(s) + \bar{D}_i(s)]\mathbb{E}[u(s)] \\ \qquad +\sigma_i(s)\Big]\mathrm{d}W_i(s) + \int_{\mathbb{Z}}\Big[E(s,\theta)(X(s) - \mathbb{E}[X(s)]) + F(s,\theta)(u(s) - \mathbb{E}[u(s)]) + [E(s,\theta) \\ \qquad +\bar{E}(s,\theta)]\mathbb{E}[X(s)] + [F(s,\theta) + \bar{F}(s,\theta)]\mathbb{E}[u(s)] + h(s,\theta)\Big]\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta), \\ \mathbb{E}[X(t)] = x, \quad X(t) - \mathbb{E}[X(t)] = 0. \end{cases} \tag{4.6}$$

Then applying the Itô's formula to $Y(s)$ in (4.1), and combining with (4.6), we have

$$\begin{aligned} \mathrm{d}Y(s) =& 2\Big[[\dot{P}(s) + P(s)A(s)](X(s) - \mathbb{E}[X(s)]) + P(s)B(s)(u(s) - \mathbb{E}[u(s)]) + \{\dot{\Pi}(s) + \Pi(s)[A(s) \\ & + \bar{A}(s)]\}\mathbb{E}[X(s)] + \Pi(s)[B(s) + \bar{B}(s)]\mathbb{E}[u(s)] + P(s)(b(s) - \mathbb{E}[b(s)]) + \Pi(s)\mathbb{E}[b(s)] + f(s) + \bar{f}(s)\Big]\mathrm{d}s \\ & + 2\sum_{i=1}^{d}\Big\{P(s)\Big[C_i(s)(X(s) - \mathbb{E}[X(s)]) + D_i(s)(u(s) - \mathbb{E}[u(s)]) + [C_i(s) + \bar{C}_i(s)]\mathbb{E}[X(s)] + [D_i(s) \\ & + \bar{D}_i(s)]\mathbb{E}[u(s)] + \sigma_i(s)\Big] + \xi_i(s)\Big\}\mathrm{d}W_i(s) + 2\int_{\mathbb{Z}}\Big\{P(s)\Big[E(s,\theta)(X(s) - \mathbb{E}[X(s)]) + F(s,\theta)(u(s) \\ & - \mathbb{E}[u(s)]) + [E(s,\theta) + \bar{E}(s,\theta)]\mathbb{E}[X(s)] + [F(s,\theta) + \bar{F}(s,\theta)]\mathbb{E}[u(s)] + h(s,\theta)\Big] + \zeta(s,\theta)\Big\}\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta). \end{aligned} \tag{4.7}$$

Comparing the diffusion terms with the adjoint equation (3.2), we can get

$$\begin{cases} Z_i(s) = & 2P(s)\Big[C_i(s)(X(s) - \mathbb{E}[X(s)]) + D_i(s)(u(s) - \mathbb{E}[u(s)]) + [C_i(s) + \bar{C}_i(s)]\mathbb{E}[X(s)] \\ & +[D_i(s) + \bar{D}_i(s)]\mathbb{E}[u(s)] + \sigma_i(s)\Big] + 2\xi_i(s), \\ r(s,\theta) = & 2P(s)\Big[E(s,\theta)(X(s) - \mathbb{E}[X(s)]) + F(s,\theta)(u(s) - \mathbb{E}[u(s)]) + [E(s,\theta) + \bar{E}(s,\theta)]\mathbb{E}[X(s)] \\ & +[F(s,\theta) + \bar{F}(s,\theta)]\mathbb{E}[u(s)] + h(s,\theta)\Big] + 2\zeta(s,\theta), \end{cases} \tag{4.8}$$

therefore

$$\begin{cases} \mathbb{E}[Z_i(s)] = 2P(s)\big\{[C_i(s)+\bar{C}_i(s)]\mathbb{E}[X(s)]+[D_i(s)+\bar{D}_i(s)]\mathbb{E}[u(s)]+\mathbb{E}[\sigma_i(s)]\big\}+2\mathbb{E}[\xi_i(s)],\\ \mathbb{E}[r(s,\theta)] = 2P(s)\big\{[E(s,\theta)+\bar{E}(s,\theta)]\mathbb{E}[X(s)]+[F(s,\theta)+\bar{F}(s,\theta)]\mathbb{E}[u(s)]+\mathbb{E}[h(s,\theta)]\big\}+2\mathbb{E}[\zeta(s,\theta)],\\ Z_i(s)-\mathbb{E}[Z_i(s)] = 2P(s)\big\{C_i(s)(X(s)-\mathbb{E}[X(s)])+D_i(s)(u(s)-\mathbb{E}[u(s)])+(\sigma_i(s)-\mathbb{E}[\sigma_i(s)])\big\}\\ \qquad +2(\xi_i(s)-\mathbb{E}[\xi_i(s)]),\\ r(s,\theta)-\mathbb{E}[r(s,\theta)] = 2P(s)\big\{E(s,\theta)(X(s)-\mathbb{E}[X(s)])+F(s,\theta)(u(s)-\mathbb{E}[u(s)])+(h(s,\theta)-\mathbb{E}[h(s,\theta)])\big\}\\ \qquad +2(\zeta(s,\theta)-\mathbb{E}[\zeta(s,\theta)]). \end{cases} \tag{4.9}$$

On the other hand, (3.14) can also be reformulated as:

$$\begin{aligned} &B(s)^\top(Y(s)-\mathbb{E}[Y(s)])+[B(s)+\bar{B}(s)]^\top\mathbb{E}[Y(s)]+\sum_{i=1}^d\big\{D_i(s)^\top(Z_i(s)-\mathbb{E}[Z_i(s)])+[D_i(s)+\bar{D}_i(s)]^\top\mathbb{E}[Z_i(s)]\big\}\\ &+\int_{\mathbb{Z}}\big\{F(s,\theta)^\top(r(s,\theta)-\mathbb{E}[r(s,\theta)])+[F(s,\theta)+\bar{F}(s,\theta)]^\top\mathbb{E}[r(s,\theta)]\big\}\nu(\mathrm{d}\theta)+2S(s)(X(s)-\mathbb{E}[X(s)])\\ &+2[S(s)+\bar{S}(s)]\mathbb{E}[X(s)]+2R(s)(u(s)-\mathbb{E}[u(s)])+2[R(s)+\bar{R}(s)]\mathbb{E}[u(s)]+2[\rho(s)+\bar{\rho}(s)]=0, \end{aligned} \tag{4.10}$$

and putting (4.5), (4.6), (4.9) into (4.10), we get

$$M(s)(X(s)-\mathbb{E}[X(s)])+N(s)(u(s)-\mathbb{E}[u(s)])+\bar{M}(s)\mathbb{E}[X(s)]+\bar{N}(s)\mathbb{E}[u(s)]+\Sigma_1(s)+\Sigma_2(s)=0, \tag{4.11}$$

where

$$\begin{cases} M(s) = B(s)^\top P(s)+\sum_{i=1}^d D_i(s)^\top P(s)C_i(s)+\int_{\mathbb{Z}}F(s,\theta)^\top P(s)E(s,\theta)\nu(\mathrm{d}\theta)+S(s),\\ N(s) = R(s)+\sum_{i=1}^d D_i(s)^\top P(s)D_i(s)+\int_{\mathbb{Z}}F(s,\theta)^\top P(s)F(s,\theta)\nu(\mathrm{d}\theta),\\ \Sigma_1(s) = B(s)^\top\eta_1(s)+\sum_{i=1}^d D_i(s)^\top\{P(s)(\sigma_i(s)-\mathbb{E}[\sigma_i(s)])+(\xi_i(s)-\mathbb{E}[\xi_i(s)])\}\\ \qquad +\int_{\mathbb{Z}}F(s,\theta)^\top\big\{P(s)(h(s,\theta)-\mathbb{E}[h(s,\theta)])+(\zeta(s,\theta)-\mathbb{E}[\zeta(s,\theta)])\big\}\nu(\mathrm{d}\theta)+(\rho(s)-\mathbb{E}[\rho(s)]),\\ \Sigma_2(s) = [B(s)+\bar{B}(s)]^\top\eta_2(s)+\sum_{i=1}^d[D_i(s)+\bar{D}_i(s)]^\top\{P(s)\mathbb{E}[\sigma_i(s)]+\mathbb{E}[\xi_i(s)]\}\\ \qquad +\int_{\mathbb{Z}}[F(s,\theta)+\bar{F}(s,\theta)]^\top\{P(s)\mathbb{E}[h(s,\theta)]+\mathbb{E}[\zeta(s,\theta)]\}\nu(\mathrm{d}\theta)+[\mathbb{E}[\rho(s)]+\bar{\rho}(s)],\\ \bar{M}(s) = [B(s)+\bar{B}(s)]^\top\Pi(s)+\sum_{i=1}^d[D_i(s)+\bar{D}_i(s)]^\top P(s)[C_i(s)+\bar{C}_i(s)]\\ \qquad +\int_{\mathbb{Z}}[F(s,\theta)+\bar{F}(s,\theta)]^\top P(s)[E(s,\theta)+\bar{E}(s,\theta)]\nu(\mathrm{d}\theta)+[S(s)+\bar{S}(s)], \end{cases}$$

$$\begin{cases} \bar{N}(s) = [R(s)+\bar{R}(s)] + \sum_{i=1}^{d}[D_i(s)+\bar{D}_i(s)]^\top P(s)[D_i(s)+\bar{D}_i(s)] \\ \qquad + \int_{\mathbb{Z}}[F(s,\theta)+\bar{F}(s,\theta)]^\top P(s)[F(s,\theta)+\bar{F}(s,\theta)]\nu(\mathrm{d}\theta), \\ \bar{\Sigma}_1(s) = A(s)^\top \eta_1(s) + \sum_{i=1}^{d} C_i(s)^\top \big\{P(s)(\sigma_i(s)-\mathbb{E}[\sigma_i(s)]) + (\xi_i(s)-\mathbb{E}[\xi_i(s)])\big\} + \int_{\mathbb{Z}} E(s,\theta)^\top \big\{P(s)(h(s,\theta) \\ \qquad -\mathbb{E}[h(s,\theta)]) + (\zeta(s,\theta)-\mathbb{E}[\zeta(s,\theta)])\big\}\nu(\mathrm{d}\theta) + P(s)(b(s)-\mathbb{E}[b(s)]) + (q(s)-\mathbb{E}[q(s)]) + f(s), \\ \bar{\Sigma}_2(s) = [A(s)+\bar{A}(s)]^\top \eta_2(s) + \sum_{i=1}^{d}[C_i(s)+\bar{C}_i(s)]^\top \{P(s)\mathbb{E}[\sigma_i(s)] + \mathbb{E}[\xi_i(s)]\} \\ \qquad + \int_{\mathbb{Z}}[E(s,\theta)+\bar{E}(s,\theta)]^\top \{P(s)\mathbb{E}[h(s,\theta)] + \mathbb{E}[\zeta(s,\theta)]\}\nu(\mathrm{d}\theta) + [\mathbb{E}[q(s)]+\bar{q}(s)] + \Pi(s)\mathbb{E}[b(s)] + \bar{f}(s). \end{cases} \tag{4.12}$$

Taking expectation $\mathbb{E}$ on (4.11), we can obtain:

$$\bar{M}(s)\mathbb{E}[X(s)] + \bar{N}(s)\mathbb{E}[u(s)] + \Sigma_2(s) = 0, \tag{4.13}$$

which implies that

$$M(s)(X(s)-\mathbb{E}[X(s)]) + N(s)(u(s)-\mathbb{E}[u(s)]) + \Sigma_1(s) = 0. \tag{4.14}$$

Under Assumptions 1.4–1.6, and as we set $P(\cdot) \geq 0$, it's easy to verify $N(s), \bar{N}(s)$ are strictly positive definite and are invertible, then we have

$$\begin{cases} u(s)-\mathbb{E}[u(s)] = -N(s)^{-1}M(s)(X(s)-\mathbb{E}[X(s)]) - N(s)^{-1}\Sigma_1(s), \\ \mathbb{E}[u(s)] = -\bar{N}(s)^{-1}\bar{M}(s)\mathbb{E}[X(s)] - \bar{N}(s)^{-1}\Sigma_2(s), \end{cases} \tag{4.15}$$

so

$$\begin{aligned} u(s) =& \mathbb{E}[u(s)] - N(s)^{-1}M(s)(X(s)-\mathbb{E}[X(s)]) - N(s)^{-1}\Sigma_1(s) \\ =& - N(s)^{-1}M(s)(X(s)-\mathbb{E}[X(s)]) - N(s)^{-1}\Sigma_1(s) - \bar{N}(s)^{-1}\bar{M}(s)\mathbb{E}[X(s)] - \bar{N}(s)^{-1}\Sigma_2(s). \end{aligned} \tag{4.16}$$

Next combining (4.5), (4.9) with (3.2), and comparing the drift terms with (4.7),

$$\begin{aligned} &[\dot{P}(s)+P(s)A(s)](X(s)-\mathbb{E}[X(s)]) + P(s)B(s)(u(s)-\mathbb{E}[u(s)]) + (\dot{\Pi}(s)+\Pi(s)[A(s)+\bar{A}(s)])\mathbb{E}[X(s)] \\ &+ \Pi(s)[B(s)+\bar{B}(s)]\mathbb{E}[u(s)] + P(s)(b(s)-\mathbb{E}[b(s)]) + f(s) + \Pi(s)\mathbb{E}[b(s)] + \bar{f}(s) \\ =& -\Big[A(s)^\top P(s) + \sum_{i=1}^{d} C_i(s)^\top P(s)C_i(s) + \int_{\mathbb{Z}} E(s,\theta)^\top P(s)E(s,\theta)\nu(\mathrm{d}\theta) + Q(s)\Big](X(s)-\mathbb{E}[X(s)]) \\ &- \{M(s)^\top - P(s)B(s)\}(u(s)-\mathbb{E}[u(s)]) - \Big[[A(s)+\bar{A}(s)]^\top \Pi(s) + \sum_{i=1}^{d}[C_i(s)+\bar{C}_i(s)]^\top P(s)[C_i(s)+\bar{C}_i(s)] \\ &+ \int_{\mathbb{Z}}[E(s,\theta)+\bar{E}(s,\theta)]^\top P(s)[E(s,\theta)+\bar{E}(s,\theta)]\nu(\mathrm{d}\theta) + [Q(s)+\bar{Q}(s)]\Big]\mathbb{E}[X(s)] - \{\bar{M}(s)^\top - \Pi(s)[B(s) \\ &+ \bar{B}(s)]\}\mathbb{E}[u(s)] - \{\bar{\Sigma}_1(s) - P(s)(b(s)-\mathbb{E}[b(s)]) - f(s)\} - \{\bar{\Sigma}_2(s) - \Pi(s)\mathbb{E}[b(s)] - \bar{f}(s)\}, \end{aligned} \tag{4.17}$$

merging items with the same coefficient, and making full use of (4.15),

$$\begin{aligned}
&\bigg[\dot{P}(s)+P(s)A(s)+A(s)^\top P(s)+\sum_{i=1}^{d}C_i(s)^\top P(s)C_i(s)+\int_{\mathbb{Z}}E(s,\theta)^\top P(s)E(s,\theta)\nu(\mathrm{d}\theta)+Q(s)\\
&-M(s)^\top N(s)^{-1}M(s)\bigg]\cdot(X(s)-\mathbb{E}[X(s)])+\bigg[\dot{\Pi}(s)+\Pi(s)[A(s)+\bar{A}(s)]+[A(s)+\bar{A}(s)]^\top\Pi(s)\\
&+\sum_{i=1}^{d}[C_i(s)+\bar{C}_i(s)]^\top P(s)[C_i(s)+\bar{C}_i(s)]+\int_{\mathbb{Z}}[E(s,\theta)+\bar{E}(s,\theta)]^\top P(s)[E(s,\theta)+\bar{E}(s,\theta)]\nu(\mathrm{d}\theta)\\
&+[Q(s)+\bar{Q}(s)]-\bar{M}(s)^\top\bar{N}(s)^{-1}\bar{M}(s)\bigg]\cdot\mathbb{E}[X(s)]\\
=&-[\bar{\Sigma}_1(s)+\bar{\Sigma}_2(s)-M(s)^\top N(s)^{-1}\Sigma_1(s)-\bar{M}(s)^\top\bar{N}(s)^{-1}\Sigma_2(s)].
\end{aligned}\tag{4.18}$$

Therefore we should let $P(\cdot)$, $\Pi(\cdot)$ be the solutions to the following integro-differential Riccati equations, respectively:

$$\begin{cases}
\dot{P}(s)+P(s)A(s)+A(s)^\top P(s)+\sum_{i=1}^{d}C_i(s)^\top P(s)C_i(s)+\int_{\mathbb{Z}}E(s,\theta)^\top P(s)E(s,\theta)\nu(\mathrm{d}\theta)\\
\qquad +Q(s)-M(s)^\top N(s)^{-1}M(s)=0,\\
P(T)=G,
\end{cases}\tag{4.19}$$

$$\begin{cases}
\dot{\Pi}(s)+\Pi(s)[A(s)+\bar{A}(s)]+[A(s)+\bar{A}(s)]^\top\Pi(s)+\sum_{i=1}^{d}[C_i(s)+\bar{C}_i(s)]^\top P(s)[C_i(s)+\bar{C}_i(s)]\\
\qquad +\int_{\mathbb{Z}}[E(s,\theta)+\bar{E}(s,\theta)]^\top P(s)[E(s,\theta)+\bar{E}(s,\theta)]\nu(\mathrm{d}\theta)+[Q(s)+\bar{Q}(s)]-\bar{M}(s)^\top\bar{N}(s)^{-1}\bar{M}(s)=0,\\
\Pi(T)=G+\bar{G}.
\end{cases}\tag{4.20}$$

The existence and uniqueness result of the solutions of (4.19) and (4.20) will be given in the fourth part of this paper. At this moment, $f(s)$ in (4.3) satisfies:

$$\begin{aligned}
f(s)=&-\bigg[\{A(s)^\top-M(s)^\top N(s)^{-1}B(s)^\top\}\eta_1(s)+\sum_{i=1}^{d}\{C_i(s)^\top-M(s)^\top N(s)^{-1}D_i(s)^\top\}\{P(s)(\sigma_i(s)-\mathbb{E}[\sigma_i(s)])\\
&+(\xi_i(s)-\mathbb{E}[\xi_i(s)])\}+\int_{\mathbb{Z}}\{E(s,\theta)^\top-M(s)^\top N(s)^{-1}F(s,\theta)^\top\}\{P(s)(h(s,\theta)-\mathbb{E}[h(s,\theta)])+(\zeta(s,\theta)\\
&-\mathbb{E}[\zeta(s,\theta)])\}\nu(\mathrm{d}\theta)-M(s)^\top N(s)^{-1}(\rho(s)-\mathbb{E}[\rho(s)])+P(s)(b(s)-\mathbb{E}[b(s)])+(q(s)-\mathbb{E}[q(s)])\bigg],
\end{aligned}\tag{4.21}$$

and $\bar{f}(s)$ in (4.4) satisfies:

$$\begin{aligned}\bar{f}(s) = &-\Bigg[\{[A(s)+\bar{A}(s)]^\top - \bar{M}(s)^\top \bar{N}(s)^{-1}[B(s)+\bar{B}(s)]^\top\}\eta_2(s) + \sum_{i=1}^{d}\{[C_i(s)+\bar{C}_i(s)]^\top - \bar{M}(s)^\top \bar{N}(s)^{-1}[D_i(s)\\ &+\bar{D}_i(s)]^\top\{P(s)\mathbb{E}[\sigma_i(s)]+\mathbb{E}[\xi_i(s)]\} + \int_{\mathbb{Z}}\{[E(s,\theta)+\bar{E}(s,\theta)]^\top - \bar{M}(s)^\top \bar{N}(s)^{-1}[F(s,\theta)+\bar{F}(s,\theta)]^\top\}\\ &\cdot\{P(s)\mathbb{E}[h(s,\theta)]+\mathbb{E}[\zeta(s,\theta)]\}\nu(\mathrm{d}\theta) - \bar{M}(s)^\top \bar{N}(s)^{-1}[\mathbb{E}[\rho(s)]+\bar{\rho}(s)] + \pi(s)\mathbb{E}[b(s)] + [\mathbb{E}[q(s)]+\bar{q}(s)]\Bigg].\end{aligned} \tag{4.22}$$

For $\forall s\in[t,T]$, $(\eta_1(\cdot),\xi(\cdot),\zeta(\cdot,\cdot))$ satisfies the following BSDE:

$$\begin{cases}\mathrm{d}\eta_1(s) = -\Bigg[\{A(s)^\top - M(s)^\top N(s)^{-1}B(s)^\top\}\eta_1(s) + \displaystyle\sum_{i=1}^{d}\{C_i(s)^\top - M(s)^\top N(s)^{-1}D_i(s)^\top\}\{P(s)(\sigma_i(s)\\ \qquad -\mathbb{E}[\sigma_i(s)]) + (\xi_i(s)-\mathbb{E}[\xi_i(s)])\} + \displaystyle\int_{\mathbb{Z}}\{E(s,\theta)^\top - M(s)^\top N(s)^{-1}F(s,\theta)^\top\}\{P(s)(h(s,\theta)-\mathbb{E}[h(s,\theta)])\\ \qquad +(\zeta(s,\theta)-\mathbb{E}[\zeta(s,\theta)])\}\nu(\mathrm{d}\theta) - M(s)^\top N(s)^{-1}(\rho(s)-\mathbb{E}[\rho(s)]) + P(s)(b(s)-\mathbb{E}[b(s)])\\ \qquad +(q(s)-\mathbb{E}[q(s)])\Bigg]\mathrm{d}s + \displaystyle\sum_{i=1}^{d}\xi_i(s)\mathrm{d}W_i(s) + \int_{\mathbb{Z}}\zeta(s,\theta)\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta),\\ \eta_1(T) = g-\mathbb{E}(g),\end{cases} \tag{4.23}$$

and $\eta_2(\cdot)$ satisfies the following ODE:

$$\begin{cases}\dot{\eta}_2(s) + \{[A(s)+\bar{A}(s)]^\top - \bar{M}(s)^\top \bar{N}(s)^{-1}[B(s)+\bar{B}(s)]^\top\}\eta_2(s) + \displaystyle\sum_{i=1}^{d}\{[C_i(s)+\bar{C}_i(s)]^\top - \bar{M}(s)^\top \bar{N}(s)^{-1}\\ \quad [D_i(s)+\bar{D}_i(s)]^\top\}\{P(s)\mathbb{E}[\sigma_i(s)]+\mathbb{E}[\xi_i(s)]\} + \displaystyle\int_{\mathbb{Z}}\{[E(s,\theta)+\bar{E}(s,\theta)]^\top - \bar{M}(s)^\top \bar{N}(s)^{-1}[F(s,\theta)+\bar{F}(s,\theta)]^\top\}\\ \qquad \{P(s)\mathbb{E}[h(s,\theta)]+\mathbb{E}[\zeta(s,\theta)]\}\nu(\mathrm{d}\theta) - \bar{M}(s)^\top \bar{N}(s)^{-1}[\mathbb{E}[\rho(s)]+\bar{\rho}(s)] + \pi(s)\mathbb{E}[b(s)] + [\mathbb{E}[q(s)]+\bar{q}(s)] = 0,\\ \eta_2(T) = \mathbb{E}[g]+\bar{g}.\end{cases} \tag{4.24}$$

4.2. The state feedback representation of optimal control

Theorem 4.1. *Under Assumptions 1.4–1.6, let* $P(\cdot)$ *and* $\Pi(\cdot)$ *be the unique solution of the Riccati equation* (4.19) *and* (4.20) *respectively. Then the Problem 1.1 under complete information is uniquely open-loop solvable, the state feedback form of the optimal control* $u^*(\cdot)$ *can be expressed as follows:*

$$u^*(s) = -\Bigg[N(s)^{-1}M(s)(X^*(s)-\mathbb{E}[X^*(s)]) + N(s)^{-1}\Sigma_1(s) + \bar{N}(s)^{-1}\bar{M}(s)\mathbb{E}[X^*(s)] + \bar{N}(s)^{-1}\Sigma_2(s)\Bigg], \tag{4.25}$$

where $M(s), N(s), \Sigma_1(s), \Sigma_2(s), \bar{M}(s), \bar{N}(s)$ are represented by (4.12), with $(\eta_1(\cdot), \xi(\cdot), \zeta(\cdot,\cdot))$ being the adapted solution of (4.23), $\eta_2(\cdot)$ being the solution of (4.24). And $X^*(\cdot)$ is the solution of the following MF-SDE,

$$
\begin{cases}
\mathrm{d}X^*(s) = \big\{A(s)X^*(s) + \bar{A}(s)\mathbb{E}[X^*(s)] + B(s)u^*(s) + \bar{B}(s)\mathbb{E}[u^*(s)] + b(s)\big\}\mathrm{d}s \\
\qquad\qquad + \sum_{i=1}^{d} \big\{C_i(s)X^*(s) + \bar{C}_i(s)\mathbb{E}[X^*(s)] + D_i(s)u^*(s) + \bar{D}_i(s)\mathbb{E}[u^*(s)] + \sigma_i(s)\big\}\mathrm{d}W_i(s) \\
\qquad\qquad + \int_{\mathbb{Z}} \big\{E(s,\theta)X^*(s-) + \bar{E}(s,\theta)\mathbb{E}[X^*(s-)] + F(s,\theta)u^*(s) + \bar{F}(s,\theta)\mathbb{E}[u^*(s)] + h(s,\theta)\big\}\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta), \\
X^*(t) = x \in \mathbb{R}^n.
\end{cases}
\tag{4.26}
$$

Furthermore,

$$
\begin{aligned}
V(t,x) =& \mathbb{E}\langle \Pi(t)x + 2\eta_2(t), x\rangle + \mathbb{E}\bigg[\int_t^T \bigg(\sum_{i=1}^{d} \langle P(s)\sigma_i(s) + \xi_i(s), \sigma_i(s)\rangle + \int_Z \langle P(s)h(s,\theta) \\
&+ \zeta(s,\theta), h(s,\theta)\rangle \nu(\mathrm{d}\theta) + 2\langle \eta_1(s), (b(s) - \mathbb{E}[b(s)])\rangle + 2\langle \eta_2(s), \mathbb{E}[b(s)]\rangle \\
&- \langle \Sigma_1(s), N(s)^{-1}\Sigma_1(s)\rangle - \langle \Sigma_2(s), \bar{N}(s)^{-1}\Sigma_2(s)\rangle\bigg)\mathrm{d}s\bigg].
\end{aligned}
\tag{4.27}
$$

Proof. Let $P(\cdot)$ and $\Pi(\cdot)$ be the unique solution of the Riccati equation (4.19) and (4.20) respectively. We define

$$
\begin{cases}
Y^*(s) = 2[P(s)(X^*(s) - \mathbb{E}[X^*(s)]) + \Pi(s)\mathbb{E}[X^*(s)] + \eta_1(s) + \eta_2(s)], \\
Z_i^*(s) = 2P(s)\Big[\{C_i(s) - D_i(s)N(s)^{-1}M(s)\}(X^*(s) - \mathbb{E}[X^*(s)]) + \{[C_i(s) + \bar{C}_i(s)] - [D_i(s) + \bar{D}_i(s)] \\
\qquad \bar{N}(s)^{-1}\bar{M}(s)\}\mathbb{E}[X^*(s)] - D_i(s)N(s)^{-1}\Sigma_1(s) - [D_i(s) + \bar{D}_i(s)]\bar{N}(s)^{-1}\Sigma_2(s) + \sigma_i(s)\Big] + 2\xi_i(s), \\
r^*(s,\theta) = 2P(s)\Big[\{E(s,\theta) - F(s,\theta)N(s)^{-1}M(s)\}(X^*(s) - \mathbb{E}[X^*(s)]) + \{[E(s,\theta) + \bar{E}(s,\theta)] - [F(s,\theta) \\
+\bar{F}(s,\theta)]\bar{N}(s)^{-1}\bar{M}(s)\}\mathbb{E}[X^*(s)] - F(s,\theta)N(s)^{-1}\Sigma_1(s) - [F(s,\theta) + \bar{F}(s,\theta)]\bar{N}(s)^{-1}\Sigma_2(s) + h(s,\theta)\Big] + 2\zeta(s,\theta),
\end{cases}
\tag{4.28}
$$

then by applying Itô's formula to $Y^*(\cdot)$, we can get $(Y^*(\cdot), Z^*(\cdot), r^*(\cdot,\cdot))$ satisfies the following MF-BSDE:

$$
\begin{cases}
\mathrm{d}Y^*(s) = -\Big[A(s)^\top Y^*(s) + \bar{A}(s)^\top \mathbb{E}[Y^*(s)] + \sum_{i=1}^{d}\{C_i(s)^\top Z_i^*(s) + \bar{C}_i(s)\mathbb{E}[Z_i^*(s)]\} + \int_{\mathbb{Z}}\{E(s,\theta)^\top r^*(s,\theta) \\
\qquad +\bar{E}(s,\theta)^\top \mathbb{E}[r^*(s,\theta)]\}\nu(\mathrm{d}\theta) + 2Q(s)X^*(s) + 2\bar{Q}(s)\mathbb{E}[X^*(s)] + 2S(s)^\top u^*(s) + 2\bar{S}(s)^\top \mathbb{E}[u^*(s)] \\
\qquad +2q(s) + 2\bar{q}(s)\Big]\mathrm{d}s + \sum_{i=1}^{d} Z_i^*(s)\mathrm{d}W_i(s) + \int_{\mathbb{Z}} r^*(s,\theta)\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta), s \in [t,T], \\
Y^*(T) = 2[GX^*(T) + \bar{G}\mathbb{E}[X^*(T)] + g + \bar{g}].
\end{cases}
\tag{4.29}
$$

Actually, (4.29) is the adjoint equation of (4.26) associated with $(u^*(\cdot), X^*(\cdot))$. And we can check easily that

$$
\begin{aligned}
&2R(s)u^*(s) + 2\bar{R}(s)\mathbb{E}[u^*(s)] + B(s)^\top Y^*(s) + \bar{B}(s)^\top \mathbb{E}[Y^*(s)] + \sum_{i=1}^{d}\{D_i(s)^\top Z_i^*(s) + \bar{D}_i(s)^\top \mathbb{E}[Z_i^*(s)]\} \\
&+ \int_{\mathbb{Z}}\{F(s,\theta)^\top r^*(s,\theta) + \bar{F}(s,\theta)^\top \mathbb{E}[r^*(s,\theta)]\}\nu(\mathrm{d}\theta) + 2S(s)X^*(s) + 2\bar{S}(s)\mathbb{E}[X^*(s)] + 2\rho(s) + 2\bar{\rho}(s) = 0.
\end{aligned}
\tag{4.30}
$$

Consequently, according to Theorem 3.1, we come to the conclusion that $u^*(\cdot)$ is the optimal control satisfies (4.25) and the optimal state $X^*(\cdot)$ satisfies (4.26). To prove (4.27), we split the cost functional (1.3) into two parts:

$$
\begin{aligned}
J(t,x;u(\cdot)) =&\mathbb{E}\Big[\langle G\bar{X}(T)+2g,\bar{X}(T)\rangle+\int_t^T\Big(\langle Q(s)\bar{X}(s),\bar{X}(s)\rangle+2\langle S(s)^\top\bar{X}(s),\bar{u}(s)\rangle+\langle R(s)\bar{u}(s),\bar{u}(s)\rangle\\
&+2\langle q(s),\bar{X}(s)\rangle+2\langle \rho(s),\bar{u}(s)\rangle\Big)\mathrm{d}s\Big]+\mathbb{E}\Big[\langle [G+\bar{G}]\mathbb{E}[X(T)]+2[g+\bar{g}],\mathbb{E}[X(T)]\rangle+\int_t^T\Big(\langle[Q(s)\\
&+\bar{Q}(s)]\mathbb{E}[X(s)],\mathbb{E}[X(s)]\rangle+2\langle[S(s)+\bar{S}(s)]^\top\mathbb{E}[X(s)],\mathbb{E}[u(s)]\rangle+\langle[R(s)+\bar{R}(s)]\mathbb{E}[u(s)],\mathbb{E}[u(s)]\rangle\\
&+2\langle[\mathbb{E}[q(s)]+\bar{q}(s)],\mathbb{E}[X(s)]\rangle+2\langle[\mathbb{E}[\rho(s)]+\bar{\rho}(s)],\mathbb{E}[u(s)]\rangle\Big)\mathrm{d}s\Big]\\
=&J_1(t,x;u(\cdot))+J_2(t,x;u(\cdot)).
\end{aligned}
\tag{4.31}
$$

where we denote $\bar{X}(s)=X(s)-\mathbb{E}[X(s)],\ \bar{u}(s)=u(s)-\mathbb{E}[u(s)]$. After applying Itô's formula to $\langle P(s)\bar{X}(s)+2\eta_1(s),\bar{X}(s)\rangle$ and $\langle\Pi(s)\mathbb{E}[X(s)]+2\eta_2(s),\mathbb{E}[X(s)]\rangle$, we have

$$
\begin{aligned}
&J_1(t,x;u(\cdot))\\
=&\mathbb{E}\Big[\int_t^T\Big(\langle M(s)^\top N(s)^{-1}M(s)\bar{X}(s),\bar{X}(s)\rangle+2\langle\bar{u}(s),M(s)\bar{X}(s)\rangle+\langle N(s)\bar{u}(s),\bar{u}(s)\rangle+2\langle\Sigma_1(s),\bar{u}(s)\\
&+N(s)^{-1}M(s)\bar{X}(s)\rangle+\Big\langle\Big\{\sum_{i=1}^d[C_i(s)+\bar{C}_i(s)]^\top P(s)[C_i(s)+\bar{C}_i(s)]+\int_Z[E(s,\theta)+\bar{E}(s,\theta)]^\top P(s)[E(s,\theta)\\
&+\bar{E}(s,\theta)]\nu(\mathrm{d}\theta)\Big\}\mathbb{E}[X(s)],\mathbb{E}[X(s)]\Big\rangle+2\Big\langle\Big\{\sum_{i=1}^d[D_i(s)+\bar{D}_i(s)]^\top P(s)[C_i(s)+\bar{C}_i(s)]+\int_Z[F(s,\theta)+\bar{F}(s,\theta)]^\top\\
&\cdot P(s)[E(s,\theta)+\bar{E}(s,\theta)]\nu(\mathrm{d}\theta)\Big\}\mathbb{E}[X(s)],\mathbb{E}[u(s)]\Big\rangle+\Big\langle\Big\{\sum_{i=1}^d[D_i(s)+\bar{D}_i(s)]^\top P(s)[D_i(s)+\bar{D}_i(s)]+\int_Z[F(s,\theta)\\
&+\bar{F}(s,\theta)]^\top P(s)[F(s,\theta)+\bar{F}(s,\theta)]\nu(\mathrm{d}\theta)\Big\}\mathbb{E}[u(s)],\mathbb{E}[u(s)]\Big\rangle+2\Big\langle\sum_{i=1}^d[C_i(s)+\bar{C}_i(s)]^\top\{P(s)\mathbb{E}[\sigma_i(s)]+\mathbb{E}[\xi_i(s)]\}\\
&+\int_Z[E(s,\theta)+\bar{E}(s,\theta)]^\top\{P(s)\mathbb{E}[h(s)]+\mathbb{E}[\zeta(s,\theta)]\}\nu(\mathrm{d}\theta),\mathbb{E}[X(s)]\Big\rangle+2\sum_{i=1}^d\Big\langle[D_i(s)+\bar{D}_i(s)]^\top\{P(s)\mathbb{E}[\sigma_i(s)]\\
&+\mathbb{E}[\xi_i(s)]\}+\int_Z[F(s,\theta)+\bar{F}(s,\theta)]^\top\{P(s)\mathbb{E}[h(s,\theta)]+\mathbb{E}[\zeta(s,\theta)]\}\nu(\mathrm{d}\theta),\mathbb{E}[u(s)]\Big\rangle+\sum_{i=1}^d\langle P(s)\sigma_i(s)+\xi_i(s),\sigma_i(s)\rangle\\
&+\int_Z\langle P(s)h(s,\theta)+\zeta(s,\theta),h(s,\theta)\rangle\nu(\mathrm{d}\theta)+2\langle\eta_1(s),(b(s)-\mathbb{E}[b(s)])\rangle\Big)\mathrm{d}s\Big],
\end{aligned}
\tag{4.32}
$$

and

$$
\begin{aligned}
&J_2(t,x;u(\cdot))\\
=&\mathbb{E}\langle\Pi(t)x+2\eta_2(t),x\rangle+\mathbb{E}\bigg[\int_t^T\bigg(\langle\{\dot{\Pi}(s)+\Pi(s)[A(s)+\bar{A}(s)]+[A(s)+\bar{A}(s)]^\top\Pi(s)+[Q(s)+\bar{Q}(s)]\}\mathbb{E}[X(s)],\\
&\cdot\mathbb{E}[X(s)]\rangle+2\langle\{\Pi(s)[B(s)+\bar{B}(s)]+[S(s)+\bar{S}(s)]^\top\}\mathbb{E}[u(s)],\mathbb{E}[X(s)]\rangle+\langle[R(s)+\bar{R}(s)]\mathbb{E}[u(s)],\mathbb{E}[u(s)]\rangle\\
&+2\langle\dot{\eta}_2(s)+[A(s)+\bar{A}(s)]^\top\eta_2(s)+\Pi(s)\mathbb{E}[b(s)]+[\mathbb{E}[q(s)]+\bar{q}(s)],\mathbb{E}[X(s)]\rangle+2\langle[B(s)+\bar{B}(s)]^\top\eta_2(s)\\
&+[\mathbb{E}[\rho(s)]+\bar{\rho}(s)],\mathbb{E}[u(s)]\rangle+2\langle\eta_2(s),\mathbb{E}[b(s)]\rangle\bigg)\mathrm{d}s\bigg].
\end{aligned}
\tag{4.33}
$$

Substituting (4.32), (4.33) into (4.31), we obtain

$$
\begin{aligned}
&J(t,x;u(\cdot))\\
=&\mathbb{E}\langle\Pi(t)x+2\eta_2(t),x\rangle+\mathbb{E}\bigg[\int_t^T\bigg(\langle M(s)^\top N(s)^{-1}M(s)\bar{X}(s),\bar{X}(s)\rangle+2\langle\bar{u}(s),M(s)\bar{X}(s)\rangle+\langle N(s)\bar{u}(s),\bar{u}(s)\rangle\\
&+2\langle\Sigma_1(s),\bar{u}(s)+N(s)^{-1}M(s)\bar{X}(s)\rangle+\langle\bar{M}(s)^\top\bar{N}(s)^{-1}\bar{M}(s)\mathbb{E}[X(s)],\mathbb{E}[X(s)]\rangle+2\langle\bar{M}(s)\mathbb{E}[X(s)],\mathbb{E}[u(s)]\rangle\\
&+\langle\bar{N}(s)\mathbb{E}[u(s)],\mathbb{E}[u(s)]\rangle+2\langle\Sigma_2(s),\mathbb{E}[u(s)]+\bar{N}(s)^{-1}\bar{M}(s)\mathbb{E}[X(s)]\rangle+\sum_{i=1}^d\langle P(s)\sigma_i(s)+\xi_i(s),\sigma_i(s)\rangle\\
&+\int_Z\langle P(s)h(s,\theta)+\zeta(s,\theta),h(s,\theta)\rangle\nu(\mathrm{d}\theta)+2\langle\eta_1(s),(b(s)-\mathbb{E}[b(s)])\rangle+2\langle\eta_2(s),\mathbb{E}[b(s)]\rangle\bigg)\mathrm{d}s\bigg]\\
=&\mathbb{E}\langle\Pi(t)x+2\eta_2(t),x\rangle+\mathbb{E}\bigg[\int_t^T\bigg(\Big|\bar{u}(s)+N(s)^{-1}M(s)\bar{X}(s)+N(s)^{-1}\Sigma_1(s)\Big|^2_{N(s)}+\Big|\mathbb{E}[u(s)]+\bar{N}(s)^{-1}\bar{M}(s)\mathbb{E}[X(s)]\\
&+\bar{N}(s)^{-1}\Sigma_2(s)\Big|^2_{\bar{N}(s)}-\langle\Sigma_1(s),N(s)^{-1}\Sigma_1(s)\rangle-\langle\Sigma_2(s),\bar{N}(s)^{-1}\Sigma_2(s)\rangle+\sum_{i=1}^d\langle P(s)\sigma_i(s)+\xi_i(s),\sigma_i(s)\rangle\\
&+\int_Z\langle P(s)h(s,\theta)+\zeta(s,\theta),h(s,\theta)\rangle\nu(\mathrm{d}\theta)+2\langle\eta_1(s),(b(s)-\mathbb{E}[b(s)])\rangle+2\langle\eta_2(s),\mathbb{E}[b(s)]\rangle\bigg)\mathrm{d}s\bigg].
\end{aligned}
\tag{4.34}
$$

Now putting the feedback representation of optimal control (4.25) into (4.34), it can be proved that the value function satisfies (4.27). Moreover, when $b(\cdot),\sigma_i(\cdot),h(\cdot,\cdot),g,\bar{g},p(\cdot),\bar{p}(\cdot),\rho(\cdot),\bar{\rho}(\cdot)=0$, we denote the corresponding value function as $V^0(t,x)$,

$$V^0(t,x)=\mathbb{E}\langle\Pi(t)x,x\rangle. \tag{4.35}$$

The proof is complete. □

Theorem 4.2. *Under Assumptions 1.4–1.6, let $P(\cdot)$ and $\Pi(\cdot)$ be the unique solution of the Riccati equation (4.19) and (4.20) respectively. Then we claim that the Problem 1.1 under partial information is uniquely open-loop solvable, the state feedback form of the optimal control $u(\cdot)$ is given by:*

$$
\begin{aligned}
u(s)=-\bigg[&N(s)^{-1}M(s)\mathbb{E}[(X(s)-\mathbb{E}[X(s)])|\mathscr{G}_s]+\bar{N}(s)^{-1}\bar{M}(s)\mathbb{E}[X(s)]\\
&+N(s)^{-1}\mathbb{E}[\Sigma_1(s)|\mathscr{G}_s]+\bar{N}(s)^{-1}\mathbb{E}[\Sigma_2(s)|\mathscr{G}_s]\bigg].
\end{aligned}
\tag{4.36}
$$

where $M(s), N(s), \Sigma_1(s),\ \Sigma_2(s), \bar{M}(s), \bar{N}(s)$ *are represented by* (4.12).

Proof. Under partial information, we just need to apply conditional mathematics expectation $\mathbb{E}[\cdot|\mathscr{G}_s]$ to (4.25). □

5. Existence and uniqueness of Riccati equations

Under Assumptions 1.4–1.6, according to the Theorem 4.1, we know that the solvability of Problem 1.1 is equivalent to the solvability of the two integro-differential Riccati equations (4.19) and (4.20). This part we will discuss the existence and uniqueness the solutions of Riccati equations (4.19) and (4.20) under standard Assumptions 1.4–1.6.

Theorem 5.1. *Let Assumptions 1.4–1.6 be satisfied. Then the Riccati equation* (4.19) *and* (4.20) *admit a unique solution* $P(\cdot)$ *and* $\Pi(\cdot) \in C([t,T];\mathbb{S}^n_+)$ *respectively.*

To prove this theorem, we need the following lemma.

Lemma 5.2. *Consider the following differential equation:*

$$\begin{cases} \dot{P}(s) + P(s)\hat{A}(s) + \hat{A}(s)^\top P(s) + \sum_{i=1}^{d} \hat{C}_i(s)^\top P(s)\hat{C}_i(s) + \int_{\mathbb{Z}} \hat{E}(s,\theta)^\top P(s)\hat{E}(s,\theta)\nu(\mathrm{d}\theta) + \hat{Q}(s) = 0, \\ P(T) = \hat{G},\ \ s\in[t,T], \end{cases} \tag{5.1}$$

where $\hat{A}(\cdot), \hat{C}_i(\cdot) \in L^\infty(t,T;\mathbb{R}^{n\times n}),\ \hat{E}(\cdot,\cdot) \in L^{\nu,2}_{\mathbb{F}([t,T]\times\mathbb{Z};\mathbb{R}^n)},\ \hat{G}\in\mathbb{S}^n_+,\ \hat{Q}(\cdot)\in L^\infty(t,T;\mathbb{S}^n_+)$ *are uniformly bounded and measurable, then the equation* (5.1) *admits a unique solution* $P(\cdot)\in C([t,T];\mathbb{S}^n_+)$.

Proof. Because the equation (5.1) is linear and all the coefficients are uniformly bounded, it admits a unique solution $P(\cdot)\in C([t,T];\mathbb{S}^n)$. Now, for any given $x\in\mathbb{R}^n$, let $\phi(\cdot)$ be the solution of the following equation:

$$\begin{cases} d\phi(s) = \hat{A}(s)\phi(s)\mathrm{d}s + \sum_{i=1}^{d} \hat{C}_i(s)\phi(s)\mathrm{d}W_i(s) + \int_{\mathbb{Z}} \hat{E}(s,\theta)\phi(s)\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta), \\ \phi(t) = x,\ \ t\in[0,T]. \end{cases} \tag{5.2}$$

Clearly, this equation has a unique solution. Then by applying Itô's formula to $\langle P(s)\phi(s),\phi(s)\rangle$, we have

$$\begin{aligned} & d\langle P(s)\phi(s),\phi(s)\rangle \\ =& \big\langle \big[\dot{P}(s) + P(s)\hat{A}(s) + \hat{A}(s)^\top P(s) + \sum_{i=1}^{d}\hat{C}_i(s)^\top P(s)\hat{C}_i(s) + \int_{\mathbb{Z}}\hat{E}(s,\theta)^\top P(s)\hat{E}(s,\theta)\nu(\mathrm{d}\theta)\big]\phi(s),\phi(s)\big\rangle\mathrm{d}s \\ &+\sum_{i=1}^{d}\big\langle\big[\hat{C}_i(s)^\top P(s) + P(s)\hat{C}_i(s)\big]\phi(s),\phi(s)\big\rangle\mathrm{d}W_i(s) + \int_{\mathbb{Z}}\big\langle\big[\hat{E}(s,\theta)^\top P(s) + P(s)\hat{E}(s,\theta)\big]\phi(s),\phi(s)\big\rangle\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta), \end{aligned} \tag{5.3}$$

by integrating and taking $\mathbb{E}$ on the both sides,

$$\langle P(t)x,x\rangle = \mathbb{E}\bigg[\langle\hat{G}\phi(T),\phi(T)\rangle + \int_t^T\langle\hat{Q}(s)\phi(s),\phi(s)\rangle\mathrm{d}s\bigg]. \tag{5.4}$$

Since $\hat{G}\geq 0$, and $\hat{Q}(s)\geq 0$, we have $P(t)\geq 0$. This prove Lemma 5.2. □

Since the integro-differential Riccati equation (4.19) is completely nonlinear, we can first make it liner in formal by Quasi-linearization and then get the existence and uniqueness of the solution by the principle of compression mapping and quasi-linearization method. To this end, we need the following proposition:

Proposition 5.3. *The Riccati equation* (4.19) *is equivalent to the following integro-differential equation*

$$\begin{cases} \dot{P}(s) + P(s)\hat{A}(s) + \hat{A}(s)^\top P(s) + \sum_{i=1}^{d} \hat{C}_i(s)^\top P(s)\hat{C}_i(s) + \int_{\mathbb{Z}} \hat{E}(s,\theta)^\top P(s)\hat{E}(s,\theta)\nu(\mathrm{d}\theta) + \hat{Q}(s) = 0, \\ P(T) = G, \end{cases} \tag{5.5}$$

where

$$\begin{cases} \hat{A}(s) = A(s) - B(s)\varphi(s),\ \hat{C}_i(s) = C_i(s) - D_i(s)\varphi(s),\ \hat{E}(s,\theta) = E(s,\theta) - F(s,\theta)\varphi(s), \\ \hat{Q}(s) = Q(s) + [\varphi(s) - R(s)^{-1}S(s)]^\top R(s)[\varphi(s) - R(s)^{-1}S(s)] - S(s)^\top R(s)^{-1}S(s), \\ \varphi(s) = N(s)^{-1}M(s). \end{cases} \tag{5.6}$$

Proof. According to the above definition of $\varphi(s)$, we have

$$P(s)B(s) + \sum_{i=1}^{d} C_i(s)^\top P(s)D_i(s) + \int_{\mathbb{Z}} E(s,\theta)^\top P(s)F(s,\theta)\nu(\mathrm{d}\theta) + S(s) = (N(s)\varphi(s))^\top = \varphi(s)^\top N(s). \tag{5.7}$$

Here we combine (5.6) with the Riccati equation (4.19),

$$\begin{aligned} &\dot{P}(s) + P(s)\hat{A}(s) + \hat{A}(s)^\top P(s) + \sum_{i=1}^{d} \hat{C}_i(s)^\top P(s)\hat{C}_i(s) + \int_{\mathbb{Z}} \hat{E}(s,\theta)^\top P(s)\hat{E}(s,\theta)\nu(\mathrm{d}\theta) + \hat{Q}(s) \\ &+ \big[P(s)B(s) + \sum_{i=1}^{d} \hat{C}_i(s)^\top P(s)D_i(s) + \int_{\mathbb{Z}} \hat{E}(s,\theta)^\top P(s)F(s,\theta)\nu(\mathrm{d}\theta) + S(s)^\top\big]\varphi(s) + \varphi(s)^\top\big[B(s)^\top P(s) \\ &+ \sum_{i=1}^{d} D_i(s)^\top P(s)\hat{C}_i(s) + \int_{\mathbb{Z}} F(s,\theta)^\top P(s)\hat{E}(s,\theta)\nu(\mathrm{d}\theta) + S(s)\big] - 2\varphi(s)^\top R(s)\varphi(s) = 0, \end{aligned} \tag{5.8}$$

note that

$$\begin{aligned} &\big[P(s)B(s) + \sum_{i=1}^{d} \hat{C}_i(s)^\top P(s)D_i(s) + \int_{\mathbb{Z}} \hat{E}(s,\theta)^\top P(s)F(s,\theta)\nu(\mathrm{d}\theta) + S(s)^\top\big]\varphi(s) \\ =&\big\{M(s)^\top - \varphi(s)^\top[N(s) - R(s)]\big\}\varphi(s) \\ =&\varphi(s)^\top N(s)^\top\varphi(s) - \varphi(s)^\top[N(s) - R(s)]\varphi(s), \end{aligned} \tag{5.9}$$

then putting the above into (5.8), hence we obtain (5.5). □

Proof of Theorem 5.1

(Existence): For illustrating the existence of solution to (4.19), we need to construct a iterative scheme. For $j = 0, 1, 2, \cdots$, we set

$$\begin{cases} \hat{A}_j(s) = A(s) - B(s)\varphi_j(s),\ \hat{C}_{i_j}(s) = C_i(s) - D_i(s)\varphi_j(s), \\ \hat{E}_j(s,\theta) = E(s,\theta) - F(s,\theta)\varphi_j(s),\ \varphi_j(s) = N_j(s)^{-1}M_j(s), \\ \hat{Q}_j(s) = Q(s) + [\varphi_j(s) - R(s)^{-1}S(s)]^\top R(s)[\varphi_j(s) - R(s)^{-1}S(s)] - S(s)^\top R(s)^{-1}S(s), \end{cases} \tag{5.10}$$

where

$$\begin{cases} M_j(s) = B(s)^\top P_j(s) + \sum_{i=1}^d D_i(s)^\top P_j(s) C_i(s) + \int_{\mathbb{Z}} E(s,\theta)^\top P_j(s) F(s,\theta)\nu(\mathrm{d}\theta) + S(s), \\ N_j(s) = R(s) + \sum_{i=1}^d D_i(s)^\top P_j(s) D_i(s) + \int_{\mathbb{Z}} F(s,\theta)^\top P_j(s) F(s,\theta)\nu(\mathrm{d}\theta). \end{cases}$$

Let $P_{j+1}(s)$ be the solution of the following:

$$\begin{cases} \dot{P}_{j+1}(s) + P_{j+1}(s)\hat{A}_j(s) + \hat{A}_j(s)^\top P_{j+1}(s) + \sum_{i=1}^d \hat{C}_{i_j}(s)^\top P_{j+1}(s)\hat{C}_{i_j}(s) \\ \qquad + \int_{\mathbb{Z}} \hat{E}_j(s,\theta)^\top P_{j+1}(s)\hat{E}_j(s,\theta)\nu(\mathrm{d}\theta) + \hat{Q}_j(s) = 0, \\ P_{j+1}(T) = G. \end{cases} \tag{5.11}$$

We define $\Delta_j(s) = P_j(s) - P_{j+1}(s)$, $\Lambda_j(s) = \varphi_j(s) - \varphi_{j-1}(s)$, and according to (5.10), we can get

$$\begin{cases} \hat{A}_{j-1}(s) - \hat{A}_j(s) = B(s)[\varphi_j(s) - \varphi_{j-1}(s)] = B(s)\Lambda_j(s), \\ \hat{C}_{i_{j-1}}(s) - \hat{C}_{i_j}(s) = D_i(s)[\varphi_j(s) - \varphi_{j-1}(s)] = D_i(s)\Lambda_j(s), \\ \hat{E}_{j-1}(s,\theta) - \hat{E}_j(s,\theta) = F(s,\theta)[\varphi_j(s) - \varphi_{j-1}(s)] = F(s,\theta)\Lambda_j(s), \\ \hat{Q}_{j-1}(s) - \hat{Q}_j(s) = \Lambda_j(s)^\top R(s)\Lambda_j(s) - \varphi_j(s)^\top R(s)\Lambda_j(s) - \Lambda_j(s)^\top R(s)\varphi_j(s) + \Lambda_j(s)^\top S(s) + S(s)^\top \Lambda_j(s). \end{cases} \tag{5.12}$$

So

$$\begin{aligned} -\dot{\Delta}_j(s) =& \dot{P}_{j+1}(s) - \dot{P}_j(s) \\ =& \Delta_j(s)\hat{A}_j(s) + \hat{A}_j(s)^\top \Delta_j(s) + \sum_{i=1}^d \hat{C}_{i_j}(s)^\top \Delta_j(s)\hat{C}_{i_j}(s) + \int_{\mathbb{Z}} \hat{E}_j(s,\theta)^\top \Delta_j(s)\hat{E}_j(s,\theta)\nu(\mathrm{d}\theta) \\ & + P_j(s)B(s)\Lambda_j(s) + \Lambda_j(s)^\top B(s)^\top P_j(s) + \sum_{i=1}^d \left[\hat{C}_{i_{j-1}}(s)^\top P_j(s)\hat{C}_{i_{j-1}}(s) - \hat{C}_{i_j}(s)^\top P_j(s)\hat{C}_{i_j}(s)\right] \\ & + \int_{\mathbb{Z}} \hat{E}_{j-1}(s,\theta)^\top P_j(s)\hat{E}_{j-1}(s,\theta)\nu(\mathrm{d}\theta) - \int_{\mathbb{Z}} \hat{E}_j(s,\theta)^\top P_j(s)\hat{E}_j(s,\theta)\nu(\mathrm{d}\theta) + \hat{Q}_{j-1}(s) - \hat{Q}_j(s). \end{aligned} \tag{5.13}$$

It is easy to check that

$$\begin{aligned} & \hat{C}_{i_{j-1}}(s)^\top P_j(s)\hat{C}_{i_{j-1}}(s) - \hat{C}_{i_j}(s)^\top P_j(s)\hat{C}_{i_j}(s) \\ =& [\hat{C}_{i_{j-1}}(s) - \hat{C}_{i_j}(s)]^\top P_j(s)\hat{C}_{i_{j-1}}(s) + \hat{C}_{i_j}(s)^\top P_j(s)[\hat{C}_{i_j}(s) - \hat{C}_{i_j}(s)] \\ =& [D_i(s)\Lambda_j(s)]^\top P_j(s)[\hat{C}_{i_{j-1}}(s) - \hat{C}_{i_j}(s)] + \hat{C}_{i_j}(s)^\top P_j(s)[D_i(s)\Lambda_j(s)] + [D_i(s)\Lambda_j(s)]^\top P_j(s)\hat{C}_{i_j}(s) \\ =& \Lambda_j(s)^\top D_i(s)^\top P_j(s)D_i(s)\Lambda_j(s) + \hat{C}_{i_j}(s)^\top P_j(s)D_i(s)\Lambda_j(s) + \Lambda_j(s)^\top D_i(s)^\top P_j(s)\hat{C}_{i_j}(s), \end{aligned} \tag{5.14}$$

similarly, we can get that

$$\begin{aligned}
&\int_{\mathbb{Z}} \hat{E}_{j-1}(s,\theta)^\top P_j(s)\hat{E}_{j-1}(s,\theta)\nu(\mathrm{d}\theta) - \int_{\mathbb{Z}} \hat{E}_j(s,\theta)^\top P_j(s)\hat{E}_j(s,\theta)\nu(\mathrm{d}\theta)\\
=&\Lambda_j(s)^\top \int_{\mathbb{Z}} F(s,\theta)^\top P_j(s)F(s,\theta)\nu(\mathrm{d}\theta)\Lambda_j(s) + \Lambda_j(s)^\top \int_{\mathbb{Z}} F(s,\theta)^\top P_j(s)\hat{E}_j(s,\theta)\nu(\mathrm{d}\theta)\\
&+ \int_{\mathbb{Z}} \hat{E}_j(s,\theta)^\top P_j(s)F(s,\theta)\nu(\mathrm{d}\theta)\Lambda_j(s).
\end{aligned} \tag{5.15}$$

Finally, (5.13) can be simplified into:

$$\begin{aligned}
-\dot{\Delta}_j(s) =&\Delta_j(s)\hat{A}_j(s) + \hat{A}_j(s)^\top \Delta_j(s) + \sum_{i=1}^{d} \hat{C}_{i_j}(s)^\top \Delta_j(s)\hat{C}_{i_j}(s)\\
&+ \int_{\mathbb{Z}} \hat{E}_j(s,\theta)^\top \Delta_j(s)\hat{E}_j(s,\theta)\nu(\mathrm{d}\theta) + \Lambda_j(s)^\top N_j(s)\Lambda_j(s),
\end{aligned} \tag{5.16}$$

the above is equivalent to the following:

$$\begin{aligned}
&-\big[\dot{\Delta}_j(s) + \Delta_j(s)\hat{A}_j(s) + \hat{A}_j(s)^\top \Delta_j(s) + \sum_{i=1}^{d} \hat{C}_{i_j}(s)^\top \Delta_j(s)\hat{C}_{i_j}(s) + \int_{\mathbb{Z}} \hat{E}_j(s,\theta)^\top \Delta_j(s)\hat{E}_j(s,\theta)\nu(\mathrm{d}\theta)\big]\\
=&\Lambda_j(s)^\top N_j(s)\Lambda_j(s) \geq 0.
\end{aligned} \tag{5.17}$$

Thus we can conclude that $P_j(s) \geq P_{j+1}(s)$, the $P_j(s)$ is a decreasing sequence. Under the Assumption 1.6 and the Proposition 5.3, the sequence $\{P_j(\cdot)\}_{j=1}^{\infty}$ is uniformly bounded. Also, according to Lemma 5.2, $\Delta_j(s) \geq 0$. Therefore, $P_j(\cdot) \in C([t,T];\mathbb{S}_+^n)$ has a limit, denoted as $P(\cdot)$. Consequently, $P(\cdot)$ is the solution of (4.19). Similar to the proof for the Riccati equation (4.19), and by a simple correction, the result of the Riccati equation (4.20) also admits a solution $\Pi(\cdot)$. The above can illustrate the existence.

(uniqueness): The general method to prove the uniqueness, we always suppose exist another solution $\tilde{P}(\cdot)$ of the Riccati Equation, and then certify $\tilde{P}(\cdot) = P(\cdot)$. Here we set $P'(\cdot) = P(\cdot) - \tilde{P}(\cdot)$, then we have

$$\begin{cases}
\dot{P}'(s) + P'(s)A(s) + A(s)^\top P'(s) + \sum_{i=1}^{d} C_i(s)^\top P'(s)C_i(s) + \int_{\mathbb{Z}} E(s,\theta)^\top P'(s)E(s,\theta)\nu(\mathrm{d}\theta)\\
\qquad +M(s)^\top N(s)^{-1}M(s) - \tilde{M}(s)^\top \tilde{N}(s)^{-1}\tilde{M}(s) = 0,\\
P'(T) = 0,
\end{cases} \tag{5.18}$$

where $M(s), N(s)$ are represented by (4.12), and we create the symbols $\tilde{M}(s), \tilde{N}(s)$ as follows,

$$\begin{cases}
\tilde{M}(s) = \Big[B(s)^\top \tilde{P}(s) + \sum_{i=1}^{d} D_i(s)^\top \tilde{P}(s)C_i(s) + \int_{\mathbb{Z}} F(s,\theta)^\top \tilde{P}(s)E(s,\theta)\nu(\mathrm{d}\theta) + S(s)\Big],\\
\tilde{N}(s) = \Big[R(s) + \sum_{i=1}^{d} D_i(s)^\top \tilde{P}(s)D_i(s) + \int_{\mathbb{Z}} F(s,\theta)^\top \tilde{P}(s)F(s,\theta)\nu(\mathrm{d}\theta)\Big].
\end{cases}$$

Then we set $M'(s)=M(s)-\tilde{M}(s)$, $N'(s)=N(s)-\tilde{N}(s)$,

$$\begin{cases} M'(s)=B(s)^\top P'(s)+\sum_{i=1}^d D_i(s)^\top P'(s)C_i(s)+\int_{\mathbb{Z}}F(s,\theta)^\top P'(s)E(s,\theta)\nu(\mathrm{d}\theta),\\ N'(s)=\sum_{i=1}^d D_i(s)^\top P'(s)D_i(s)+\int_{\mathbb{Z}}F(s,\theta)^\top P'(s)F(s,\theta)\nu(\mathrm{d}\theta). \end{cases}$$

To process (5.18), we note that

$$\begin{aligned} N(s)^{-1}-\tilde{N}(s)^{-1}=&N(s)^{-1}[\tilde{N}(s)-N(s)]\tilde{N}(s)^{-1}\\ =&-N(s)^{-1}N'(s)\tilde{N}(s)^{-1}. \end{aligned} \tag{5.19}$$

In that way,

$$\begin{array}{rl} & M(s)^\top N(s)^{-1}M(s)-\tilde{M}(s)^\top\tilde{N}(s)^{-1}\tilde{M}(s)\\ = & \big[M(s)^\top N(s)^{-1}M(s)-\tilde{M}(s)^\top N(s)^{-1}M(s)\big]+\tilde{M}(s)^\top N(s)^{-1}M(s)\\ & +\big[\tilde{M}(s)^\top\tilde{N}(s)^{-1}M(s)-\tilde{M}(s)^\top\tilde{N}(s)^{-1}\tilde{M}(s)\big]-\tilde{M}(s)^\top\tilde{N}(s)^{-1}M(s)\\ = & M'(s)^\top N(s)^{-1}M(s)+\tilde{M}(s)^\top\tilde{N}(s)^{-1}M'(s)+\tilde{M}(s)^\top[N(s)^{-1}-\tilde{N}(s)^{-1}]M(s),\\ = & M'(s)^\top N(s)^{-1}M(s)+\tilde{M}(s)^\top\tilde{N}(s)^{-1}M'(s)-\tilde{M}(s)^\top N(s)^{-1}N'(s)\tilde{N}(s)^{-1}M(s). \end{array} \tag{5.20}$$

Finally, putting (5.20) into (5.18), we have

$$\begin{cases} \dot{P}'(s)+P'(s)A(s)+A(s)^\top P'(s)+\sum_{i=1}^d C_i(s)^\top P'(s)C_i(s)+\int_{\mathbb{Z}}E(s,\theta)^\top P'(s)E(s,\theta)\nu(\mathrm{d}\theta)\\ \qquad +M'(s)^\top N(s)^{-1}M(s)+\tilde{M}(s)\tilde{N}(s)^{-1}M'(s)-\tilde{M}(s)^\top N(s)^{-1}N'(s)\tilde{N}(s)^{-1}M(s)=0,\\ P'(T)=0. \end{cases} \tag{5.21}$$

Due to their continuity we can get $|\bar{N}(s)^{-1}|$ and $|\tilde{N}(s)^{-1}|$ are uniformly bounded. Next apply Gronwall's inequality to obtain $P'(\cdot)=P(\cdot)-\tilde{P}(\cdot)=0$. Similarly, we can proof the solution of the Riccati equation (4.20) is also unique. The proof is complete.

6. A special case

In the previous part of the article, we establish the stochastic Hamilton system (3.16), get the representation of optimal control as a feedback form and deduce two Riccati equations (4.19) and (4.20). But the form of $\mathbb{E}[(X(s)-\mathbb{E}[X(s)])|\mathscr{G}_s]$ is unknown, so this section, in order to obtain the filtering equation, we discuss a special case when the coefficients of the state equation (1.1): $C_i(\cdot)$, $\bar{C}_i(\cdot)$, $D_i(\cdot)$, $\bar{D}_i(\cdot)$, $E(\cdot,\cdot)$, $\bar{E}(\cdot,\cdot)$, $F(\cdot,\cdot)$, $\bar{F}(\cdot,\cdot)=0$, and $b(\cdot)$, $\rho(s)$, $q(s)$ are vector-valued $\mathscr{G}_t$-measurable processes. In this case, the stochastic Hamilton system (3.16) turns into:

$$\begin{cases} \mathrm{d}X(s)=\big\{A(s)X(s)+\bar{A}(s)\mathbb{E}[X(s)]+B(s)u(s)+\bar{B}(s)\mathbb{E}[u(s)]+b(s)\big\}\mathrm{d}s+\sum_{i=1}^{d}\sigma_i(s)\mathrm{d}W_i(s)+\int_{\mathbb{Z}}h(s,\theta)\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta),\\ \mathrm{d}Y(s)=-\Big[A(s)^\top Y(s)+\bar{A}(s)^\top\mathbb{E}[Y(s)]+2Q(s)X(s)+2\bar{Q}(s)\mathbb{E}[X(s)]+2S(s)^\top u(s)+2\bar{S}(s)^\top\mathbb{E}[u(s)]\Big]\mathrm{d}s\\ \qquad\qquad+\sum_{i=1}^{d}Z_i(s)\mathrm{d}W_i(s)+\int_{\mathbb{Z}}r(s,\theta)\tilde{\mu}(\mathrm{d}s,\mathrm{d}\theta),\\ X(t)=x,\ Y(T)=2[GX(T)+\bar{G}\mathbb{E}[X(T)]+g+\bar{g}],\\ B(s)^\top\mathbb{E}(Y(s)|\mathscr{G}_s)+\bar{B}(s)^\top\mathbb{E}[Y(s)]+2S(s)^\top\mathbb{E}(X(s)|\mathscr{G}_s)+2\bar{S}(s)^\top\mathbb{E}[X(s)]+2R(s)u(s)+2\bar{R}(s)\mathbb{E}[u(s)]\\ \qquad+2\rho(s)+2\bar{\rho}(s)=0,\ s\in[t,T], \end{cases} \tag{6.1}$$

where $(u(\cdot),X(\cdot),\ Y(\cdot),\ Z_1(\cdot),\ Z_2(\cdot),\ r(\cdot,\cdot))$ is the unique adapted solution of Hamiltonian system (6.1).

And more specially, we set $\mathscr{G}_t=\sigma\{W_2(t),0\le t\le T\}$, the control $u(\cdot)$ is $\mathscr{G}_t$-adapted. And we set the observation equation as

$$y(s)=W_2(s),$$

in order to get a more explicit expression of optimal control, we have to apply filtering theory. The aim of filtering theory is to estimate $X(\cdot)$ based on the partial information $\mathscr{G}_t$. First, we denote

$$\hat{X}(s)=\mathbb{E}(X(s)|\mathscr{G}_s),\ \hat{Y}(s)=\mathbb{E}(Y(s)|\mathscr{G}_s),\ \hat{Z}_2(s)=\mathbb{E}(Z_2(s)|\mathscr{G}_s),\ t\le s\le T,$$

then from Lemma 5.4 in Xiong [21], we have the following proposition.

Proposition 6.1. *Let Assumptions 1.4–1.6 hold. For any $u\in\mathcal{A}$, the optimal filtering $(u(\cdot),\ \hat{X}(\cdot),\ \hat{Y}(\cdot),\ \hat{Z}_2(\cdot))$ of the solution $(u(\cdot),\ X(\cdot),\ Y(\cdot),\ Z(\cdot),\ r(\cdot,\cdot))$ to (6.1) with respect to $\mathscr{G}_t$ satisfies*

$$\begin{cases} d\hat{X}(s)=\big\{A(s)\hat{X}(s)+\bar{A}(s)\mathbb{E}[X(s)]+B(s)u(s)+\bar{B}(s)\mathbb{E}[u(s)]+b(s)\big\}\mathrm{d}s+\sigma_2(s)\mathrm{d}W_2(s),\\ d\hat{Y}(s)=-\Big[A(s)^\top\hat{Y}(s)+\bar{A}(s)^\top\mathbb{E}[Y(s)]+2Q(s)\hat{X}(s)+2\bar{Q}(s)\mathbb{E}[X(s)]+2S(s)^\top u(s)+2\bar{S}(s)^\top\mathbb{E}[u(s)]\\ \qquad\qquad+2q(s)+2\bar{q}(s)\Big]\mathrm{d}s+\hat{Z}_2(s)\mathrm{d}W_2(s),s\in[t,T]\\ \hat{X}(t)=x\in\mathbb{R}^n,\ \hat{Y}(T)=2[G\hat{X}(T)+\bar{G}\mathbb{E}[X(T)]+g+\bar{g}],\\ B(s)^\top\hat{Y}(s)+\bar{B}(s)^\top\mathbb{E}[Y(s)]+2S(s)\hat{X}(s)+2\bar{S}(s)\mathbb{E}[X(s)]+2R(s)u(s)+2\bar{R}(s)\mathbb{E}[u(s)]+2\rho(s)+2\bar{\rho}(s)=0. \end{cases} \tag{6.2}$$

Furthermore, we have the corresponding state feedback representation of optimal control,

$$\begin{aligned} u(s)=-\Big[&R(s)^{-1}[B(s)^\top\hat{P}(s)+S(s)](\hat{X}(s)-\mathbb{E}[X(s)])+[R(s)+\bar{R}(s)]^{-1}\big\{[B(s)+\bar{B}(s)]^\top\hat{\Pi}(s)\\ &+[S(s)+\bar{S}(s)]\big\}\mathbb{E}[X(s)]+R(s)^{-1}\big\{B(s)^\top\hat{\eta}_1(s)+(\rho(s)-\mathbb{E}[\rho(s)])\big\}\\ &+[R(s)+\bar{R}(s)]^{-1}\big\{[B(s)+\bar{B}(s)]^\top\hat{\eta}_2(s)+[\mathbb{E}[\rho(s)]+\bar{\rho}(s)]\big\}\Big], \end{aligned} \tag{6.3}$$

where $\hat{P}(\cdot)$ and $\hat{\Pi}(\cdot)$ are the solutions to the following two Riccati equations associated with this special case,

$$\begin{cases} \dot{\hat{P}}(s) + \hat{P}(s)A(s) + A(s)^\top \hat{P}(s) + Q(s) - [B(s)^\top \hat{P}(s) + S(s)]^\top R(s)^{-1}[B(s)^\top \hat{P}(s) + S(s)] = 0, \\ \hat{P}(T) = G, \end{cases} \tag{6.4}$$

and

$$\begin{cases} \dot{\hat{\Pi}}(s) + \hat{\Pi}(s)[A(s) + \bar{A}(s)] + [A(s) + \bar{A}(s)]^\top \hat{\Pi}(s) + [Q(s) + \bar{Q}(s)]^\top - \big\{[B(s) + \bar{B}(s)]^\top \hat{\Pi}(s) \\ \qquad + [S(s) + \bar{S}(s)]\big\}^\top [R(s) + \bar{R}(s)]^{-1}\big\{[B(s) + \bar{B}(s)]^\top \hat{\Pi}(s) + [S(s) + \bar{S}(s)]\big\} = 0, \\ \hat{\Pi}(T) = G + \bar{G}. \end{cases} \tag{6.5}$$

$(\hat{\eta}_1(\cdot), \hat{\xi}_2(\cdot))$ and $\hat{\eta}_2(\cdot)$ satisfy the following respectively:

$$\begin{cases} d\hat{\eta}_1(s) = -\Big[\big\{A(s)^\top - [B(s)^\top \hat{P}(s) + S(s)]^\top R(s)^{-1}B(s)^\top\big\}\hat{\eta}_1(s) - [B(s)^\top \hat{P}(s) + S(s)]^\top R(s)^{-1}(\rho(s) - \mathbb{E}[\rho(s)]) \\ \qquad + \hat{P}(s)(b(s) - \mathbb{E}[b(s)]) + (q(s) - \mathbb{E}[q(s)])\Big]\mathrm{d}s + \hat{\xi}_2(s)\mathrm{d}W_2(s), \\ \hat{\eta}_1(T) = g, \end{cases} \tag{6.6}$$

and

$$\begin{cases} \dot{\hat{\eta}}_2(s) + \Big[[A(s) + \bar{A}(s)]^\top - \big\{[B(s) + \bar{B}(s)]^\top \hat{\Pi}(s) + [S(s) + \bar{S}(s)]\big\}^\top [R(s) + \bar{R}(s)]^{-1}[B(s) + \bar{B}(s)]^\top\Big]\hat{\eta}_2(s) \\ + \big\{[B(s) + \bar{B}(s)]^\top \hat{\Pi}(s) + [S(s) + \bar{S}(s)]\big\}^\top [R(s) + \bar{R}(s)]^{-1}[\mathbb{E}[\rho(s)] + \bar{\rho}(s)] + \hat{\Pi}(s)\mathbb{E}[b(s)] + [\mathbb{E}[q(s)] + \bar{q}(s)] = 0, \\ \hat{\eta}_2(T) = \mathbb{E}[g] + \bar{g}. \end{cases} \tag{6.7}$$

7. Conclusion

This paper mainly studies a kind of MF-LQ optimal control problem with jumps under partial information. Through the adjoint processes, Hamiltonian system and filtering technique, we deduce two integro-differential Riccati equations, shown that the optimal control can be represented as a feedback form under complete and partial information. And by means of discuss the existence and uniqueness of Riccati equations, we conclude the feedback representation of optimal control of MF-LQ problem is unique. Finally, we explore a special case, get the corresponding Hamiltonian system, the representation of optimal control and Riccati equations.

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