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EXACT BOUNDARY SYNCHRONIZATION FOR A KIND OF FIRST ORDER HYPERBOLIC SYSTEM

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Abstract. In recent years there have been many in-depth researches on the boundary controllability and boundary synchronization for coupled systems of wave equations with various types of boundary conditions. In order to extend the study of synchronization from wave equations to a much larger range of hyperbolic systems, in this paper we will define and establish the exact boundary synchronization for the first order linear hyperbolic system based on previous work on its exact boundary controllability. The determination and estimate of exactly synchronizable states and some related problems are also discussed. This work can be applied to a great deal of diverse systems, and a new perspective to study the synchronization problem for the coupled system of wave equations can be also provided.

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1. Introduction

We know that for a coupled hyperbolic system, the exact boundary controllability can always be realized by enough boundary controls in a suitable large control time (see [2, 9, 23, 24, 26, 27, 30, 31] and references therein, etc). When there is a lack of boundary controls, although the exact boundary controllability can not be realized, the system may still possess some weak properties, among which one is the exact boundary synchronization, and the other is the approximate boundary controllability or the approximate boundary synchronization (*cf.* [19] and references therein, etc).

For first order hyperbolic systems, the exact (null) controllability has been considered in [2, 7, 9, 30, 31] for solutions in L^∞, L^2 or C^1. Moreover, there are many studies on the optimal time for the exact (null) controllability for hyperbolic systems in recent years, see for example [1, 3–6] and references therein. For the results that will be frequently used in this paper, readers can refer to [26, 27].

The synchronization is a widespread natural phenomenon, the research on which is of great value and prospect [8, 29, 32–34]. Li and Rao first established the concept and theory of synchronization in PDEs case for a coupled system of wave equations with Dirichlet boundary controls in the framework of weak solutions, and in their study

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on the exact boundary synchronization and the approximate boundary synchronization, they established a close relationship between controllability and synchronization so that the study of controllability can be regarded as a foundation of the study of synchronization [13–15]. Up to now, related researches for a coupled system of wave equations with Dirichlet boundary controls, Neumann boundary controls and coupled Robin boundary controls are almost complete [10–22], etc.

In order to extend the study of synchronization from coupled systems of wave equations to a larger range of systems, we take the first order linear hyperbolic system into consideration. First order hyperbolic systems are widely used to model various systems in real life such as traffic flow, the flow of fluid in gas pipelines and the blood flow in the mammalian vessels, etc. To study the synchronization from coupled systems of wave equations to first order hyperbolic systems is of great significance, since the latter has much wider connotation than the former. For instance, in 1-D case, a wave equation can be always transformed into a first order hyperbolic system with the same number of positive and negative eigenvalues [9]. While, for any given first order hyperbolic system, generally speaking, the number of positive eigenvalues is not necessarily equal to that of negative ones, which often makes trouble in the treatment. Besides, for wave equations with the usual boundary conditions (Dirichlet, Neumann, Robin, even dissipative boundary conditions), the whole system is always time reversible. However, first order hyperbolic systems are not time reversible in general. Hence, first order hyperbolic systems have not only much more abundant connotations and practical applications than wave equations, but also they have their own difficulties, thus it is necessary to carry out a special research on them.

In this paper, we mainly study the one-sided exact boundary synchronization for the 1-D first order linear hyperbolic system (2.3). We first give the definition of synchronization in Section 2, including some preliminaries. Then, based on the results of the exact boundary controllability obtained in our previous works [26, 27], the one-sided exact boundary synchronization for system (2.3) will be established in Section 3, and its two-sided exact boundary synchronization will be discussed in a similar way (see Sect. 8). In Sections 4–5 we study the exactly synchronizable states, including the system satisfied by exactly synchronizable states, their attainable set, their determination and estimates. The necessity of the conditions of compatibility for the coupling matrices will be discussed in Section 6. We give also some discussions on the number of boundary controls necessary for the exact boundary controllability and the exact boundary synchronization in Section 7.

2. Definition and preliminaries

2.1. Definition and notations

As a control problem, the synchronization for first order hyperbolic systems can be similarly realized by two-sided control or one-sided control (*cf.* [2, 9, 26, 27, 30, 31], etc.). This paper mainly discusses the one-sided control case, in which fewer number of boundary controls are required, the two-sided control case can be similarly treated.

Differently from the synchronization for a coupled system of wave equations, the synchronization for the first order hyperbolic system doesn't mean the synchronization of all the components of the state variable. In fact, in practical applications, the components of the state variable for first order hyperbolic systems often have different physical meanings, it makes no sense to require all the components of the state variable to be synchronized each other. Instead, synchronization happens among those components who possess similar properties (in particular, correspond to the same spreading speed), and the synchronization patterns are diverse: sometimes the analogous components synchronize as a whole, sometimes they split and synchronize in even smaller groups. But each component will be affected not only by its analogue, but also by all the other components coming from the whole system. Hence, the synchronization for first order hyperbolic systems should be essentially the synchronization by groups.

Let $N_i(\geqslant 2, i = 1, \dots, n)$ be any given positive integers, and let

$$N = \sum_{i=1}^n N_i, \qquad N^- = \sum_{i=1}^m N_i \qquad \text{and} \qquad N^+ = \sum_{i=m+1}^n N_i. \tag{2.1}$$

Assume that $\Lambda = \text{diag}\{\Lambda^-, \Lambda^+\}$ with

$$\Lambda^- = \text{diag}\{\lambda_1 I_{N_1}, \dots, \lambda_m I_{N_m}\}, \quad \Lambda^+ = \text{diag}\{\lambda_{m+1} I_{N_{m+1}}, \dots, \lambda_n I_{N_n}\}, \tag{2.2}$$

where $\lambda_r < 0 (r = 1, \dots, m)$ and $\lambda_s > 0 (s = m+1, \dots, n)$, and I_{N_i} denotes the unit matrix of order $N_i (i = 1, \dots, n)$. We discuss the one-sided exact boundary synchronization for the following system with boundary controls acting only on $x = L$:

$$\begin{cases} U_t + \Lambda U_x + AU = 0, & t \in (0, +\infty), \quad x \in (0, L), \\ U^+(t, 0) = G_0 U^-(t, 0), & t \in (0, +\infty), \\ U^-(t, L) = G_1 U^+(t, L) + DH(t), & t \in (0, +\infty) \end{cases} \tag{2.3}$$

with the initial data

$$t = 0: \quad U(0, x) = U_0(x), \quad x \in (0, L), \tag{2.4}$$

where $U = U(t, x) : (0, +\infty) \times (0, L) \to \mathbb{R}^N$ denotes the state variable. The state variable U is divided into n groups, including m groups of components corresponding to negative spreading speeds, and $\bar{m}$ groups of components corresponding to positive spreading speeds, where

$$\bar{m} \overset{def.}{=} n - m, \tag{2.5}$$

and the i^{th} group consists of N_i components, corresponding to the same eigenvalue λ_i for $i = 1, \dots, n$. Thus, N^- (resp. N^+) denotes the number of negative (resp. positive) eigenvalues, while m (resp. $\bar{m}$) denotes the number of groups of components corresponding to negative (resp. positive) wave speeds. The coupling matrix $A = (a_{ij})$ is of order N, G_0 and G_1 are of order $N^+ \times N^-$ and $N^- \times N^+$, respectively, the boundary control matrix D is a full column-rank matrix of order $N^- \times M$ with $M \leqslant N^-$. All the matrices mentioned above are with constant elements. The state variable can be written as $U = (U^-, U^+)^T$ with $U^- = (U_1, \dots, U_m)^T$, $U^+ = (U_{m+1}, \dots, U_n)^T$ and $U_i = (u_i^{(1)}, u_i^{(2)}, \dots, u_i^{(N_i)})^T (i = 1, \dots, n)$, while $H(t) = (h_1(t), \dots, h_M(t))^T$ is the boundary control. Let

$$G = \begin{pmatrix} 0 & G_1 \\ G_0 & 0 \end{pmatrix}. \tag{2.6}$$

Definition 2.1. System (2.3) is exactly synchronizable at the time $T > 0$, if for any given initial data $U_0 \in (L^2(0, L))^N$, there exists a boundary control $H \in (L^2(0, T))^M$ and a decomposition (2.1), such that the unique weak solution $U = U(t, x)$ to the mixed problem (2.3)–(2.4) satisfies

$$t \geqslant T: \quad u_i^{(1)}(t, x) \equiv u_i^{(2)}(t, x) \equiv \cdots \equiv u_i^{(N_i)}(t, x) \overset{\text{def.}}{=} \tilde{u}_i(t, x), \quad i = 1, \dots, n. \tag{2.7}$$

And $\tilde{u} = (\tilde{u}_1(t, x), \dots, \tilde{u}_n(t, x))^T$ is the exactly synchronizable state, which is a priori unknown.

System (2.3) is exactly controllable at the time $T > 0$, if for any given initial data $U_0 \in (L^2(0, L))^N$ and final data $U_T \in (L^2(0, L))^N$, there exists a boundary control $H \in (L^2(0, T))^M$, such that the unique weak solution $U = U(t, x)$ to the mixed problem (2.3)–(2.4) satisfies exactly

$$t = T: \quad U(T, x) = U_T(x), \quad 0 < x < L. \tag{2.8}$$

In particular, if $U_T = 0$, then system (2.3) is exactly null controllable at the time $T > 0$.

Remark 2.2. The exact boundary synchronization (2.7) indicates that as $t \geqslant T$, although eliminating the boundary control $H \equiv 0$, the synchronization can remain.

Corresponding to the decomposition (2.1), define C_1 as the following full row-rank $(N-n) \times N$ matrix of synchronization

$$C_1 = \begin{pmatrix} \tilde{C}_1 & & & \\ & \tilde{C}_2 & & \\ & & \ddots & \\ & & & \tilde{C}_n \end{pmatrix} \quad \text{with} \quad \tilde{C}_i = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & & \\ & & & 1 & -1 \end{pmatrix} \tag{2.9}$$

an $(N_i - 1) \times N_i$ full row-rank matrix for $i = 1, \ldots, n$. It is easy to see that

$$\operatorname{Ker}(\tilde{C}_i) = \operatorname{Span}\{\tilde{e}_i\} \quad \text{with} \quad \tilde{e}_i = (\underbrace{1, \ldots, 1}_{N_i})^T, \qquad i = 1, \ldots, n, \tag{2.10}$$

and

$$\operatorname{Ker}(C_1) = \operatorname{Span}\{e_1, e_2, \ldots, e_n\}, \tag{2.11}$$

where $e_i (i = 1, \ldots, n) \in \mathbb{R}^N$ satisfy

$$e_i = (\underbrace{0, \ldots, 0}_{\sum_{j=1}^{i-1} N_j}, \tilde{e}_i^T, \underbrace{0, \ldots, 0}_{\sum_{j=i+1}^{n} N_j})^T, \quad i = 1, \ldots, n. \tag{2.12}$$

Remark 2.3. The exact boundary synchronization (2.7) can be equivalently written as

$$t \geqslant T: \quad U = \textstyle\sum_{i=1}^n \tilde{u}_i e_i \quad \text{or} \quad U_i = \tilde{u}_i \tilde{e}_i \ (i = 1, \ldots, n), \tag{2.13}$$

or, equivalently,

$$t \geqslant T: \quad C_1 U \equiv 0 \quad \text{or} \quad \tilde{C}_i U_i \equiv 0 \ (i = 1, \ldots, n). \tag{2.14}$$

Remark 2.4. When all the eigenvalues of Λ are negative, the corresponding one-sided exact boundary synchronization has been discussed in [25], which mainly deals with the exact boundary synchronization for a kind of first order quasilinear hyperbolic system in the framework of classical solutions.

2.2. Properties of the one-sided control system (2.3)

In [4] the well-posedness in L^∞-norm was given for a more general control system ([4], Lem. 3.2), the extension to L^2-norm is straightforward (see [5]). More precisely, by Theorem 3.1 in [26], we have

Lemma 2.5. *For any given $T > 0$, for any given initial data $U_0 \in (L^2(0, L))^N$ and any given boundary function $H \in (L^2(0, T))^M$, the mixed problem (2.3) with (2.4) admits a unique weak solution $U = U(t, x) \in (L^2(0, T; L^2(0, L)))^N$, satisfying*

$$\|U(T, \cdot)\|_{(L^2(0,L))^N} \leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^M}) \tag{2.15}$$

and

$$\|U(\cdot,0)\|_{(L^2(0,T))^N} + \|U(\cdot,L)\|_{(L^2(0,T))^N} \leqslant c(\|U_0\|_{(L^2(0,L))^N} + \|H\|_{(L^2(0,T))^M}), \tag{2.16}$$

here and hereafter, c denotes a positive constant.

Usually, system (2.3) is not time reversible. However, we have

Lemma 2.6. *([26], Rem. 3.4) Assume that the number of positive eigenvalues is equal to that of negative ones, namely, $N^- = N^+ = \frac{N}{2}$. Assume furthermore that $G_i (i = 0, 1)$ are invertible, namely, $rank(G_0) = rank(G_1) = \frac{N}{2}$. Then system (2.3) is time reversible, namely, the backward problem of system (2.3) is well-posed. Moreover, if $U = \hat{U}(t, x)$ is the weak solution to the forward problem of system (2.3), then it is also the weak solution to the corresponding backward problem of system (2.3) with the final condition $t = T:\ U(T, x) = \hat{U}(T, x),\ x \in (0, L)$, and vice versa.*

The following result on the exact boundary null controllability is a consequence of the standard control theory of hyperbolic system. When the number of boundary controls on $x = L$ is equal to the number of negative eigenvalues, namely, $\text{rank}(D) = N^-$, replacing $DH(t)$ by $\widetilde{H}(t) \in \mathbb{R}^{N^-}$ as the control, the result can be directly derived from the theory of controllability of the hyperbolic system, see for example [4, 5] where the optimal time is also considered.

In this situation, since the system might be not time reversible, it is not necessarily exactly controllable. However, if we assume that the number of positive eigenvalues is not bigger than that of negative eigenvalues, namely, $N^+ \le N^-$, and assume furthermore that the coupling matrix G_0 on $x = 0$ is of full row-rank, then with the same amount of boundary controls acting on $x = L$, the system is in fact exactly controllable (see [26], Lems. 4.2–4.3). We have

Lemma 2.7. *Let $T \geqslant \bar{T}_0 > 0$, where $\bar{T}_0 = L\big(\max_{1\leqslant r\leqslant m} \frac{1}{|\lambda_r|} + \max_{m+1\leqslant s\leqslant n} \frac{1}{\lambda_s}\big)$. If $M = rank(D) = N^-$, then system (2.3) is exactly null controllable with the boundary control $H(t)$ satisfying*

$$\|H\|_{(L^2(0,T))^M} \leqslant c\|U_0\|_{(L^2(0,L))^N}. \tag{2.17}$$

Moreover, assume furthermore that $N^+ \leqslant N^-$ and $rank(G_0) = N^+$. If $M = rank(D) = N^-$, then system (2.3) is exactly controllable.

3. Conditions of C_1-compatibility and related properties

In the study of synchronization for system (2.3), we will assume that the coupling matrices A and G, given by (2.6), satisfy some conditions of C_1-compatibility for a given decomposition (2.1). In this way we can correspondingly select the synchronization pattern (how the synchronizers are grouped), then it becomes simpler to look for the boundary control and to describe and estimate the exactly synchronizable states. Giving a different condition of C_1-compatibility will lead to a different synchronization pattern, but the discussion is similar. On the other hand, assuming that system (2.3) is synchronized for a given synchronization pattern, whether the coupling matrices A and G should satisfy a corresponding conditions of compatibility, a discussion will be given in Section 6.

Definition 3.1. The matrices A and G satisfy the conditions of C_1-compatibility, if there exists a decomposition (2.1) such that, for the corresponding matrix of synchronization C_1 given by (2.9), we have

$$A\text{Ker}(C_1) \subseteq \text{Ker}(C_1), \tag{3.1}$$

$$G\text{Ker}(C_1) \subseteq \text{Ker}(C_1), \quad \text{respectively.} \tag{3.2}$$

We point out that in what follows, we always consider the same decomposition (2.1) and the same matrix of synchronization C_1.

If we express A in the block form

$$A = \begin{pmatrix} A_1^{(1)} & A_2^{(1)} & \cdots & A_n^{(1)} \\ A_1^{(2)} & A_2^{(2)} & \cdots & A_n^{(2)} \\ \vdots & \vdots & & \vdots \\ A_1^{(n)} & A_2^{(n)} & \cdots & A_n^{(n)} \end{pmatrix}, \tag{3.3}$$

where $A_i^{(j)}$are $N_j \times N_i$ matrices $(i, j = 1, \ldots, n)$, then we have

Lemma 3.2. *The condition of C_1-compatibility (3.1) for A is equivalent to*

$$A_i^{(j)} Ker(\tilde{C}_i) \subseteq Ker(\tilde{C}_j), \qquad i, j = 1, \ldots, n, \tag{3.4}$$

which is also equivalent to that there exist constants $\alpha_{ij}(i, j = 1, \ldots, n)$ such that

$$Ae_i = \textstyle\sum_{j=1}^n \alpha_{ij} e_j, \qquad i = 1, \ldots, n, \tag{3.5}$$

where $e_i (i = 1, \ldots, n)$ are given by (2.12). Moreover, we have

$$A_i^{(j)} \tilde{e}_i = \alpha_{ij} \tilde{e}_j, \qquad i, j = 1, \ldots, n, \tag{3.6}$$

where $\tilde{e}_i (i = 1, \ldots, n)$ are given by (2.10), and $\alpha_{ij}(i, j = 1, \ldots, n)$ are given by (3.5).

Proof. Noting (2.9), (2.12) and (3.3), a direct calculation yields that $C_1 A e_i = \begin{pmatrix} \tilde{C}_1 A_i^{(1)} \tilde{e}_i \\ \tilde{C}_2 A_i^{(2)} \tilde{e}_i \\ \vdots \\ \tilde{C}_n A_i^{(n)} \tilde{e}_i \end{pmatrix}$ for $i = 1, \ldots, n$. Thus $C_1 A e_i = 0 (i = 1, \ldots, n)$ is equivalent to $\tilde{C}_j A_i^{(j)} \tilde{e}_i = 0 (i, j = 1, \ldots, n)$. Then, noting (2.10) and (2.11), we have (3.4).

Noting (2.12) and (3.5), we have

$$Ae_i = \textstyle\sum_{j=1}^n \alpha_{ij} e_j = \begin{pmatrix} \alpha_{i1} \tilde{e}_1 \\ \alpha_{i2} \tilde{e}_2 \\ \vdots \\ \alpha_{in} \tilde{e}_n \end{pmatrix}, \quad i = 1, \ldots, n. \tag{3.7}$$

Moreover, noting (2.12) and (3.3), we have $Ae_i = \begin{pmatrix} A_i^{(1)} \tilde{e}_i \\ A_i^{(2)} \tilde{e}_i \\ \vdots \\ A_i^{(n)} \tilde{e}_i \end{pmatrix}$ $(i = 1, \ldots, n)$, comparing which to (3.7), we get (3.6). □

Remark 3.3. (3.6) indicates that the sum of each row of $A_i^{(j)}$ is a common constant α_{ij} for $i, j = 1, \ldots, n$, thus (3.4) can be also called the row-sum condition in block.

For G, the condition of C_1-compatibility (3.2) should be certain conditions of compatibility for G_0 and G_1. In order to show this, let $C_1 = \begin{pmatrix} C_1^- & \\ & C_1^+ \end{pmatrix}$ with

$$C_1^- = \begin{pmatrix} \tilde{C}_1 & & & \\ & \tilde{C}_2 & & \\ & & \ddots & \\ & & & \tilde{C}_m \end{pmatrix} \quad \text{and} \quad C_1^+ = \begin{pmatrix} \tilde{C}_{m+1} & & & \\ & \tilde{C}_{m+2} & & \\ & & \ddots & \\ & & & \tilde{C}_n \end{pmatrix}. \tag{3.8}$$

Let $\epsilon_i (i = 1, \dots, m) \in \mathbb{R}^{N^-}$ and $\epsilon_i (i = m+1, \dots, n) \in \mathbb{R}^{N^+}$ satisfy

$$\epsilon_i = (\underbrace{0, \dots, 0}_{\sum_{j=1}^{i-1} N_j}, \tilde{e}_i^T, \underbrace{0, \dots, 0}_{\sum_{j=i+1}^{m} N_j})^T (i = 1, \dots, m), \quad \epsilon_i = (\underbrace{0, \dots, 0}_{\sum_{j=m+1}^{i-1} N_j}, \tilde{e}_i^T, \underbrace{0, \dots, 0}_{\sum_{j=i+1}^{n} N_j})^T (i = m+1, \dots, n), \tag{3.9}$$

where $\tilde{e}_i (i = 1, \dots, n)$ are given by (2.10). It is easy to see that

$$\mathrm{Ker}(C_1^-) = \mathrm{Span}\{\epsilon_1, \dots, \epsilon_m\}, \qquad \mathrm{Ker}(C_1^+) = \mathrm{Span}\{\epsilon_{m+1}, \dots, \epsilon_n\} \tag{3.10}$$

and

$$e_i = (\epsilon_i^T, 0)^T \quad (i = 1, \dots, m) \quad \text{and} \quad e_i = (0, \epsilon_i^T)^T \quad (i = m+1, \dots, m). \tag{3.11}$$

Remark 3.4. The exact boundary synchronization (2.7) can be equivalently written as

$$t \geqslant T: \quad U^- = \textstyle\sum_{i=1}^m \tilde{u}_i \epsilon_i, \quad U^+ = \sum_{i=m+1}^n \tilde{u}_i \epsilon_i, \quad \text{or, equivalently,} \quad C_1^- U^- \equiv 0, \quad C_1^+ U^+ \equiv 0. \tag{3.12}$$

Lemma 3.5. *The condition of C_1-compatibility (3.2) for G is equivalent to the following row-sum condition in block for G_0 and G_1:*

$$G_0 Ker(C_1^-) \subseteq Ker(C_1^+), \qquad G_1 Ker(C_1^+) \subseteq Ker(C_1^-). \tag{3.13}$$

Thus there exist constants $\beta_{ij} (i, j = 1, \dots, n)$ such that

$$G_0 \epsilon_i = \textstyle\sum_{j=m+1}^n \beta_{ij} \epsilon_j \quad (i = 1, \dots, m), \qquad G_1 \epsilon_i = \sum_{j=1}^m \beta_{ij} \epsilon_j \quad (i = m+1, \dots, n). \tag{3.14}$$

Proof. Noting (3.11), for $i = 1, \dots, m$, we have

$$C_1 G e_i = \begin{pmatrix} C_1^- & \\ & C_1^+ \end{pmatrix} \begin{pmatrix} 0 & G_1 \\ G_0 & 0 \end{pmatrix} e_i = \begin{pmatrix} 0 & C_1^- G_1 \\ C_1^+ G_0 & 0 \end{pmatrix} \begin{pmatrix} \epsilon_i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ C_1^+ G_0 \epsilon_i \end{pmatrix}.$$

Similarly, we have $C_1 G e_i = \begin{pmatrix} C_1^- G_1 \epsilon_i \\ 0 \end{pmatrix} (i = m+1, \dots, n)$. Thus the condition of C_1-compatibility (3.2) is equivalent to $C_1^+ G_0 \epsilon_i = 0 (i = 1, \dots, m)$ and $C_1^- G_1 \epsilon_i = 0 (i = m+1, \dots, n)$, which, noting (3.10), is equivalent to (3.13). □

Moreover, let $G = \begin{pmatrix} 0 & G_1 \\ G_0 & 0 \end{pmatrix}$ with

$$G_0 = \begin{pmatrix} G_1^{(m+1)} & \cdots & G_m^{(m+1)} \\ \vdots & & \vdots \\ G_1^{(n)} & \cdots & G_m^{(n)} \end{pmatrix}, \qquad G_1 = \begin{pmatrix} G_{m+1}^{(1)} & \cdots & G_n^{(1)} \\ \vdots & & \vdots \\ G_{m+1}^{(m)} & \cdots & G_n^{(m)} \end{pmatrix}. \tag{3.15}$$

We have

Lemma 3.6. *The conditions of compatibility* (3.13) *are equivalent to*

$$G_i^{(j)} Ker(\tilde{C}_i) \subseteq Ker(\tilde{C}_j) \tag{3.16}$$

for $i = 1, \ldots, m; j = m+1, \ldots, n$ *and for* $i = m+1, \ldots, n; j = 1, \ldots, m$*, respectively. Moreover, we have*

$$G_i^{(j)} \tilde{e}_i = \beta_{ij} \tilde{e}_j, \tag{3.17}$$

where β_{ij} *are given by* (3.14) *for* $i = 1, \ldots, m; j = m+1, \ldots, n$ *and for* $i = m+1, \ldots, n; j = 1, \ldots, m$*, respectively.*

Proof. (3.16) can be proved in the same manner of Lemma 3.2. The proof of (3.17) is similar to that of (3.6). In fact, for $i = m+1, \ldots, n$, noting (3.9) and (3.14)–(3.15), we have $G_1 \epsilon_i = \begin{pmatrix} G_i^{(1)} \tilde{e}_i \\ G_i^{(2)} \tilde{e}_i \\ \vdots \\ G_i^{(m)} \tilde{e}_i \end{pmatrix}$ and $G_1 \epsilon_i = \sum_{j=1}^m \beta_{ij} \epsilon_j = \begin{pmatrix} \beta_{i1} \tilde{e}_1 \\ \beta_{i2} \tilde{e}_2 \\ \vdots \\ \beta_{im} \tilde{e}_m \end{pmatrix}$, respectively. Therefore, we have $G_i^{(j)} \tilde{e}_i = \beta_{ij} \tilde{e}_j (i = m+1, \ldots, n; j = 1, \ldots, m)$. We can similarly get $G_i^{(j)} \tilde{e}_i = \beta_{ij} \tilde{e}_j$ for $i = 1, \ldots, m$ and $j = m+1, \ldots, n$. □

Remark 3.7. Noting (3.17), the conditions of compatibility (3.13) for G_0 and G_1 are also row-sum conditions in block for $G_i^{(j)}$ for $i = 1, \ldots, m; j = m+1, \ldots, n$ and for $i = m+1, \ldots, n; j = 1, \ldots, m$, respectively.

Finally, by the following lemma, we will give in Corollary 3.9 below the equivalent statements of the conditions of C_1-compatibility (3.1) and (3.2) for A and G, respectively.

Lemma 3.8. *For any given* $N_1 \times N_2$ *matrix* G *and any given full row-rank* $M_1 \times N_1$ *matrix* C_1 *and* $M_2 \times N_2$ *matrix* C_2*, there exists an* $M_1 \times M_2$ *matrix* $\overline{G}$ *such that*

$$C_1 G = \overline{G} C_2 \tag{3.18}$$

if and only if the image of G *on the null space* $Ker(C_2)$ *of* C_2 *is a subspace of the null space* $Ker(C_1)$ *of* C_1*:*

$$G Ker(C_2) \subseteq Ker(C_1). \tag{3.19}$$

Moreover, the matrix $\overline{G}$ *is uniquely determined by* $\overline{G} = C_1 G C_2^\dagger$*, where* $C_2^\dagger = C_2^T (C_2 C_2^T)^{-1}$*is the Moore-Penrose generalized inverse of* C_2 *(see Lem. 3.1 in [22]).*

Proof. By (3.18), we have $C_1GX = \overline{G}C_2X = 0,\ \forall X \in \mathrm{Ker}(C_2)$, which proves (3.19).

Inversely, since C_2 is of full row-rank, $C_2C_2^T$ is invertible. Let $\overline{G} = C_1GC_2^\dagger$ as above. Let Y be the $N_2 \times (N_2 - M_2)$ basis matrix of $\mathrm{Ker}(C_2)$. Since $\mathrm{Ker}(C_2) \oplus \mathrm{Im}(C_2^T) = \mathbb{R}^{N_2}$, noting that C_2^T is a basis matrix of $\mathrm{Im}(C_2^T)$, (Y, C_2^T) is a basis in $\mathbb{R}^{N_2}$. Hence, in order to prove (3.18), it suffices to prove

$$(C_1G - \overline{G}C_2)(Y, C_2^T) = 0. \tag{3.20}$$

On one hand, by (3.19), we have $C_1GY = 0$. Noting $\overline{G}C_2Y = 0$, we have $(C_1G - \overline{G}C_2)Y = 0$. On the other hand, $C_1GC_2^T = C_1GC_2^T(C_2C_2^T)^{-1}(C_2C_2^T) = \overline{G}(C_2C_2^T)$, then $(C_1G - \overline{G}C_2)C_2^T = 0$. Thus we obtain (3.20). At last, multiplying C_2^T to (3.18) from the right, we have $C_1GC_2^T = \overline{G}C_2C_2^T$, from which we can get $\overline{G} = C_1GC_2^T(C_2C_2^T)^{-1}$, thus $\overline{G}$ is uniquely determined. □

By Lemma 3.5 and Lemma 3.8, we have

Corollary 3.9. *The condition of C_1-compatibility (3.1) for A holds if and only if there exists a reduced matrix $\overline{A}$ of order $(N-n)$, such that*

$$C_1A = \overline{A}C_1. \tag{3.21}$$

The condition of C_1-compatibility (3.2) for G holds if and only if there exist $(N^+ - \bar{m}) \times (N^- - m)$ matrix $\overline{G}_0$ and $(N^- - m) \times (N^+ - \bar{m})$ matrix $\overline{G}_1$ such that

$$C_1^+G_0 = \overline{G}_0C_1^-, \qquad C_1^-G_1 = \overline{G}_1C_1^+, \tag{3.22}$$

where N^- and N^+ are given by (2.1), while m and $\bar{m}$ are given by (2.5).

Under conditions of C_1-compatibility (3.1) and (3.2) for the coupling matrices A and G, the one-sided exact boundary synchronization of system (2.3) can be realized by solving a one-sided controllability problem for a reduced system.

Theorem 3.10. *Assume that A and G satisfy conditions of C_1-compatibility (3.1) and (3.2), respectively. If rank$(C_1^-D) = N^- - m$, then system (2.3) is exactly synchronizable.*

Proof. Noting that A and G satisfy conditions of C_1-compatibility (3.1) and (3.2), respectively, by Corollary 3.9, we have (3.21) and (3.22). Moreover, it is easy to check that

$$C_1\Lambda = \overline{\Lambda}C_1, \tag{3.23}$$

where $\overline{\Lambda} = \mathrm{diag}\{\lambda_1 I_{N_1-1}, \lambda_2 I_{N_2-1}, \dots, \lambda_n I_{N_n-1}\}$. Let $W = C_1U$, $W^- = C_1^-U^-$ and $W^+ = C_1^+U^+$, where C_1^- and C_1^+ are given by (3.8). Multiplying C_1 on both sides of (2.3), noting (3.21)–(3.23), we get the following reduced system:

$$\begin{cases} W_t + \overline{\Lambda}W_x + \overline{A}W = 0, & t \in (0, +\infty), \quad x \in (0, L), \\ W^+(t, 0) = \overline{G}_0W^-(t, 0), & t \in (0, +\infty), \\ W^-(t, L) = \overline{G}_1W^+(t, L) + C_1^-DH(t), & t \in (0, +\infty) \end{cases} \tag{3.24}$$

with the initial data

$$t = 0: \quad W(0, x) = C_1U_0(x), \quad x \in (0, L). \tag{3.25}$$

By Lemma 2.7, when $\operatorname{rank}(C_1^- D) = N^- - m$, there exist $T > 0$ and boundary control $H(t)$, such that system (3.24) is exactly null controllable at the time $t = T$. Since the exact boundary synchronization of system (2.3) is equivalent to the exact boundary null controllability of the reduced system (3.24), system (2.3) is exactly synchronizable at the time $t = T$ by the above boundary control $H(t)$. □

Remark 3.11. Under the assumptions of Theorem 3.10, we necessarily have

$$M = \operatorname{rank}(D) \geqslant \operatorname{rank}(C_1^- D) = N^- - m. \tag{3.26}$$

If $M = \operatorname{rank}(D) = \operatorname{rank}(C_1^- D) = N^- - m$, then we have the following continuous dependence:

$$\|H\|_{(L^2(0,T))^M} \leqslant c\|C_1 U_0\|_{(L^2(0,L))^{N-n}} \leqslant c\|U_0\|_{(L^2(0,L))^N}. \tag{3.27}$$

This is a direct corollary of Lemma 2.7. If $M = \operatorname{rank}(D) > \operatorname{rank}(C_1^- D) = N^- - m$, noting the way of constructing the boundary controls in the proof of the exact boundary null controllability by the constructive method (see [26]), we can still get (3.27) by properly choosing the free components of the boundary control, such that their norm can be controlled by $\|U_0\|_{(L^2(0,L))^N}$.

Remark 3.12. Differently from a coupled system of wave equations, since AU as a perturbation in the first formula of system (2.3) is not compact, and the boundary conditions on $x = 0$ and $x = L$ even make it more complicated, the minimum number of boundary controls necessary for the exact boundary synchronization of system (2.3), with or without conditions of C_1-compatibility for A and G, is still an open problem. We will give an example in Section 7 to show that in order to realize the one-sided exact boundary synchronization for system (2.3), it requires $\operatorname{rank}(C_1^- D) \geqslant N^- - m$ in general.

Remark 3.13. By the proof of Theorem 3.10, under conditions of C_1-compatibility of A and G, the optimal time to reach the exact boundary synchronization of system (2.3) is determined entirely by the optimal time for the reduced system (3.24) to reach the null controllability, which is in general smaller than $\bar{T}_0$ in Lemma 2.7. More details can be found in [4, 5] and references therein.

4. Exactly synchronizable states and their determination

4.1. System of exactly synchronizable states

Theorem 4.1. *Assume that A and G satisfy conditions of C_1-compatibility (3.1) and (3.2), respectively. If system (2.3) is exactly synchronizable at the time $T > 0$, then the exactly synchronizable state $(\tilde{u}_1(t,x), \dots, \tilde{u}_n(t,x))^T$ satisfies*

$$\tilde{u}_{it} + \lambda_i \tilde{u}_{ix} + \textstyle\sum_{j=1}^n \alpha_{ji}\tilde{u}_j = 0, \quad i = 1, \dots, n, \quad t \in (T, +\infty), \quad x \in (0, L) \tag{4.1}$$

with the boundary conditions

$$\tilde{u}_i(t,0) = \textstyle\sum_{j=1}^m \beta_{ji}\tilde{u}_j(t,0), \quad i = m+1, \dots, n, \quad t \in (T, +\infty), \tag{4.2}$$

$$\tilde{u}_i(t,L) = \textstyle\sum_{j=m+1}^n \beta_{ji}\tilde{u}_j(t,L), \quad i = 1, \dots, m, \quad t \in (T, +\infty), \tag{4.3}$$

where α_{ij} and $\beta_{ij} (i, j = 1, \dots, n)$ are given by (3.5) and (3.14), respectively.

Proof. As $t \geqslant T$, we write (2.3) with $H(t) \equiv 0$ in the following form:

$$\begin{cases} U_{it} + \lambda_i U_{ix} + \sum_{j=1}^n A_j^{(i)} U_j = 0, & i = 1, \dots, n, \quad t \in (T, +\infty), \quad x \in (0, L), \\ U_i(t,0) = \sum_{j=1}^m G_j^{(i)} U_j(t,0), & i = m+1, \dots, n, \quad t \in (T, +\infty), \\ U_i(t,L) = \sum_{j=m+1}^n G_j^{(i)} U_j(t,L), & i = 1, \dots, m, \quad t \in (T, +\infty), \end{cases} \tag{4.4}$$

where $A_j^{(i)}$ and $G_j^{(i)}(i,j=1,\dots,n)$ are given by (3.3) and (3.15), respectively.

Since system (2.3) is exactly synchronizable, substituting (2.13) into (4.4), for $i=1,\dots,n$, we have

$$\begin{cases} \tilde u_{it}\tilde e_i+\lambda_i\tilde u_{ix}\tilde e_i+\sum_{j=1}^n A_j^{(i)}\tilde u_j\tilde e_j=0, & t\in(T,+\infty),\quad x\in(0,L),\\ \tilde u_i(t,0)\tilde e_i=\sum_{j=1}^m G_j^{(i)}\tilde u_j(t,0)\tilde e_j,\quad i=m+1,\dots,n, & t\in(T,+\infty),\\ \tilde u_i(t,L)\tilde e_i=\sum_{j=m+1}^n G_j^{(i)}\tilde u_j(t,L)\tilde e_j,\quad i=1,\dots,m, & t\in(T,+\infty),\\ t=T:\quad \tilde u_i\tilde e_i=\tilde u_i(T,x)\tilde e_i, & x\in(0,L). \end{cases} \tag{4.5}$$

Since A and G satisfy conditions of C_1-compatibility (3.1) and (3.2), respectively, by Lemma 3.2 and Lemma 3.6, we have $A_j^{(i)}\tilde e_j=\alpha_{ji}\tilde e_i$ and $G_j^{(i)}\tilde e_j=\beta_{ji}\tilde e_i$ for $i,j=1,\dots,n$, where α_{ij} and $\beta_{ij}(i,j=1,\dots,n)$ are given by (3.5) and (3.14), respectively. Thus, (4.5) can be written as

$$\begin{cases} \tilde u_{it}\tilde e_i+\lambda_i\tilde u_{ix}\tilde e_i+\sum_{j=1}^n \alpha_{ji}\tilde u_j\tilde e_i=0,\quad i=1,\dots,n, & t\in(T,+\infty),\quad x\in(0,L),\\ \tilde u_i(t,0)\tilde e_i=\sum_{j=1}^m \beta_{ji}\tilde u_j(t,0)\tilde e_i,\quad i=m+1,\dots,n, & t\in(T,+\infty),\\ \tilde u_i(t,L)\tilde e_i=\sum_{j=m+1}^n \beta_{ji}\tilde u_j(t,L)\tilde e_i,\quad i=1,\dots,m, & t\in(T,+\infty),\\ t=T:\quad \tilde u_i\tilde e_i=\tilde u_i(T,x)\tilde e_i, & x\in(0,L). \end{cases} \tag{4.6}$$

Thus we get (4.1)–(4.3). □

Under the assumptions of Theorem 4.1, the following theorem shows that when system (4.1)–(4.3) of the exactly synchronizable state is time reversible, the attainable set of the exactly synchronizable state at the time $t=T$ can run through the whole space $(L^2(0,L))^n$.

Let

$$\beta_0=\begin{pmatrix}\beta_{1,m+1} & \cdots & \beta_{m,m+1}\\ \vdots & & \vdots\\ \beta_{1,n} & \cdots & \beta_{m,n}\end{pmatrix}\quad\text{and}\quad \beta_1=\begin{pmatrix}\beta_{m+1,1} & \cdots & \beta_{n,1}\\ \vdots & & \vdots\\ \beta_{m+1,m} & \cdots & \beta_{n,m}\end{pmatrix}, \tag{4.7}$$

where $\beta_{ij}(i,j=1,\dots,n)$ are given by (3.14). To guarantee the time reversibility of system (4.1)–(4.3), it requires that the number of positive eigenvalues of system (4.1)–(4.3) is equal to the negative ones, namely, $m=\bar m$, where $\bar m$ is given by (2.5), and the coupling matrices β_0 and β_1 on the boundary should be invertible (see Rem. 3.4 in [26]).

Theorem 4.2. *Assume that system (2.3) is exactly synchronizable at the time $t=T$ under conditions of C_1-compatibility (3.1) and (3.2) for A and G, respectively. If system (4.1)–(4.3) is time reversible, namely, $m=\bar m$, and the matrices β_0 and β_1, given by (4.7), are invertible, then the attainable set of the exactly synchronizable state $(\tilde u_1(t,x),\dots,\tilde u_n(t,x))^T$ at the time $t=T$ is the whole space $(L^2(0,L))^n$.*

Proof. Since matrices β_0 and β_1 are invertible, by Lemma 2.6, for any given $(v_1,\dots,v_n)^T\in(L^2(0,L))^n$, we can solve the backward problem of system (4.1) on the domain $R(T)=\{(t,x)|0<t<T,0<x<L\}$ with the boundary conditions

$$(\tilde u_1,\dots,\tilde u_m)^T(t,0)=\beta_0^{-1}(\tilde u_{m+1},\dots,\tilde u_n)^T(t,0),\quad t\in(0,T), \tag{4.8}$$

$$(\tilde u_{m+1},\dots,\tilde u_n)^T(t,L)=\beta_1^{-1}(\tilde u_1,\dots,\tilde u_m)^T(t,L),\quad t\in(0,T) \tag{4.9}$$

and the final condition $t=T:\ \tilde u_i=v_i(x)\ (i=1,\dots,n),\ x\in(0,L)$. There exists a unique solution $(\tilde u_1(t,x),\dots,\tilde u_n(t,x))^T\in(L^2(0,T;L^2(0,L)))^n$, which is also the weak solution to the forward problem of system (4.1)–(4.3) on $R(T)$. It is then easy to check that under conditions of C_1-compatibility for A and G, $U=U(t,x)=\sum_{i=1}^n\tilde u_i(t,x)e_i\in(L^2(0,T;L^2(0,L)))^N$ is the solution to the forward problem of system

(2.3) on $R(T)$ with $H(t) \equiv 0$ and the final data $t = T$: $U(T,x) = \sum_{i=1}^n v_i e_i$. Hence, when the initial data $U_0 = U(0,x) = \sum_{i=1}^n \tilde{u}_i(0,x)e_i$ varies through the space $(L^2(0,L))^N$, the attainable set of the exactly synchronizable state $(\tilde{u}_1, \dots, \tilde{u}_n)^T(T,x) = (v_1, \dots, v_n)^T$ at the time $t = T$ is the whole space $(L^2(0,L))^n$. □

The following lemma gives the property of G so that matrices β_0 and β_1 are invertible.

Lemma 4.3. *Assume that $m = \bar{m}$, namely, the number of positive eigenvalues of system (4.1)–(4.3) is equal to the negative ones. Then the matrices β_0 and β_1 given by (4.7) are invertible if and only if $Ker(G) \cap Ker(C_1) = \{0\}$, namely,*

$$Ker(G_0) \cap Ker(C_1^-) = \{0\} \quad \text{and} \quad Ker(G_1) \cap Ker(C_1^+) = \{0\}. \tag{4.10}$$

Proof. By (3.14) we have

$$G_0(\epsilon_1, \dots, \epsilon_m) = (\epsilon_{m+1}, \dots, \epsilon_n)\beta_0. \tag{4.11}$$

Assume that (4.10) holds, by the first formula in (4.10) and $\text{Ker}(C_1^-) = \text{Span}\{\epsilon_1, \dots, \epsilon_m\}$, we have $\text{Ker}(G_0) \cap \text{Im}(\epsilon_1, \dots, \epsilon_m) = \{0\}$, which, by Proposition 2.11 in [19], is equivalent to $\text{rank}(G_0(\epsilon_1, \dots, \epsilon_m)) = \text{rank}(\epsilon_1, \dots, \epsilon_m) = m$. Hence, by (4.11), we have $m = \text{rank}(G_0(\epsilon_1, \dots, \epsilon_m)) = \text{rank}((\epsilon_{m+1}, \dots, \epsilon_n)\beta_0) \leqslant \text{rank}(\beta_0)$. Thus, since $m = \bar{m}$, β_0 is invertible. We can similarly prove that β_1 is invertible.

Inversely, assume that β_0 is invertible, then, by (4.11), we have

$$\text{rank}(G_0(\epsilon_1, \dots, \epsilon_m)) = \text{rank}((\epsilon_{m+1}, \dots, \epsilon_n)\beta_0) = \text{rank}(\epsilon_{m+1}, \dots, \epsilon_n) = \bar{m} = m,$$

thus $\text{rank}(G_0(\epsilon_1, \dots, \epsilon_m)) = m = \text{rank}(\epsilon_1, \dots, \epsilon_m)$, which, by Proposition 2.11 in [19], is equivalent to $\text{Ker}(G_0) \cap \text{Im}(\epsilon_1, \dots, \epsilon_m) = \text{Ker}(G_0) \cap \text{Ker}(C_1^-) = \{0\}$. Similarly, we have $\text{Ker}(G_1) \cap \text{Ker}(C_1^+) = \{0\}$. □

4.2. Determination of exactly synchronizable states

In general, the exactly synchronizable states are a priori unknown, which depend not only on the initial data, but also on applied boundary controls. In this section, we will discuss the determination of exactly synchronizable states.

Throughout this section, let $\varepsilon_i \in \mathbb{R}^{N^-} (i = 1, \dots, m)$, $\varepsilon_i \in \mathbb{R}^{N^+} (i = m+1, \dots, n)$, and let

$$E_i = \begin{pmatrix} \varepsilon_i \\ 0 \end{pmatrix} \in \mathbb{R}^N \ (i = 1, \dots, m), \qquad E_i = \begin{pmatrix} 0 \\ \varepsilon_i \end{pmatrix} \in \mathbb{R}^N \ (i = m+1, \dots, n) \tag{4.12}$$

be linearly independent such that $\text{Span}\{E_1, \dots, E_n\}$ and $\text{Ker}(C_1) = \text{Span}\{e_1, \dots, e_n\}$ are bi-orthonormal.

By (3.11), it is easy to check that $(E_i, e_j) = (\varepsilon_i, \epsilon_j)$ for $i, j = 1, \dots, m$ (resp. $i, j = m+1, \dots, n$), thus, since $\text{Span}\{E_1, \dots, E_n\}$ and $\text{Ker}(C_1)$ are bi-orthonormal, we have $V = \text{Span}\{\varepsilon_1, \dots, \varepsilon_m\}$ and $\text{Ker}(C_1^-) = \text{Span}\{\epsilon_1, \dots, \epsilon_m\}$ are bi-orthonormal, and $W = \text{Span}\{\varepsilon_{m+1}, \dots, \varepsilon_n\}$ and $\text{Ker}(C_1^+) = \text{Span}\{\epsilon_{m+1}, \dots, \epsilon_n\}$ are bi-orthonormal, respectively.

We first show that in some special cases the exactly synchronizable state will be determined only by the initial data, but not depend on applied boundary controls. The idea is to project the system to the subspace $\text{Span}\{E_1, \dots, E_n\}$ bi-orthonormal to $\text{Ker}(C_1)$, so that by properly choosing the boundary control matrix D, the projection system can be independent of any given boundary controls, but the solution to this system coincides with the exactly synchronizable state of the original system as $t \geqslant T$. However, in order to obtain a self-closed projection system that is independent of boundary controls, matrices Λ, A^T and G^T should possess a common invariant subspace, which, in this situation, must be $\text{Span}\{E_1, \dots, E_n\}$. To assume that $E_i (i = 1, \dots, n)$ are eigenvectors of Λ in the following theorem is to keep the first-order part of diagonal form in the projection system.

Theorem 4.4. *Assume that A and G satisfy conditions of C_1-compatibility (3.1) and (3.2), respectively. Assume furthermore that $E_i (i = 1, \dots, n)$ given by (4.12) are eigenvectors of Λ, and Span$\{E_1, \dots, E_n\}$ is a common invariant subspace of A^T and G^T. Then there exists a boundary control matrix D with*

$$rank(C_1^- D) = rank(D) = N^- - m, \tag{4.13}$$

such that system (2.3) is exactly synchronizable, and the exactly synchronizable states are independent of applied boundary controls.

Proof. Let D be defined by

$$\text{Ker}(D^T) = V = \text{Span}\{\varepsilon_1, \dots, \varepsilon_m\}. \tag{4.14}$$

Noting that V and $\text{Ker}(C_1^-)$ are bi-orthonormal, by Proposition 2.5 and Proposition 2.11 in [19], we immediately get (4.13). Then by Theorem 3.10, system (2.3) is exactly synchronizable.

Noting that $E_i (i = 1, \dots, n)$ are eigenvectors of Λ, without loss of generality, we may assume that

$$\Lambda E_i = \lambda_i E_i, \qquad i = 1, \dots, n. \tag{4.15}$$

Noting that Span$\{E_1, \dots, E_n\}$ is invariant for A^T, there exist constants a_{ij} $(i, j = 1, \dots, n)$ such that

$$A^T E_i = \sum_{j=1}^n a_{ij} E_j, \qquad i = 1, \dots, n. \tag{4.16}$$

Noting that Span$\{E_1, \dots, E_n\}$ is invariant for G^T, there exist constants b_{ij} $(i, j = 1, \dots, n)$ such that for $i = 1, \dots, m$ we have

$$G^T E_i = \begin{pmatrix} 0 & G_0^T \\ G_1^T & 0 \end{pmatrix} \begin{pmatrix} \varepsilon_i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ G_1^T \varepsilon_i \end{pmatrix} = \sum_{j=1}^n b_{ij} E_j = \begin{pmatrix} \sum_{j=1}^m b_{ij}\varepsilon_j \\ \sum_{j=m+1}^n b_{ij}\varepsilon_j \end{pmatrix}, \tag{4.17}$$

thus we have $b_{ij} = 0 (i, j = 1, \dots, m)$ and

$$G_1^T \varepsilon_i = \sum_{j=m+1}^n b_{ij}\varepsilon_j, \quad i = 1, \dots, m. \tag{4.18}$$

Similarly, we have

$$G_0^T \varepsilon_i = \sum_{j=1}^m b_{ij}\varepsilon_j, \quad i = m+1, \dots, n. \tag{4.19}$$

Let $\phi_i = (U, E_i)(i = 1, \dots, n)$. Noting $U = (U^-, U^+)^T$ and (4.12), it is easy to see that

$$\phi_i = (U^-, \varepsilon_i) \ (i = 1, \dots, m), \qquad \phi_i = (U^+, \varepsilon_i) \ (i = m+1, \dots, n). \tag{4.20}$$

Noting (4.15)–(4.16), by multiplying E_i $(i = 1, \dots, n)$ on the equations in system (2.3), we have ϕ_i $(i = 1, \dots, n)$ satisfy

$$\phi_{it} + \lambda_i \phi_{ix} + \sum_{j=1}^n a_{ij}\phi_j = 0, \quad t \in (0, +\infty), \quad x \in (0, L). \tag{4.21}$$

Noting (4.14), (4.18) and (4.20), by the boundary conditions on $x = L$ in system (2.3), we have

$$\phi_i(t, L) = \big(U^+(t, L), G_1^T \varepsilon_i\big) + (H, D^T\varepsilon_i) = \big(U^+(t, L), \sum_{j=m+1}^n b_{ij}\varepsilon_j\big) = \sum_{j=m+1}^n b_{ij}\phi_j(t, L) \tag{4.22}$$

for $i = 1, \dots, m$. Thus, we have

$$\phi_i(t, L) = \sum_{j=m+1}^{n} b_{ij}\phi_j(t, L), \quad i = 1, \dots, m, \quad t > 0. \tag{4.23}$$

Similarly, we have

$$\phi_i(t, 0) = \sum_{j=1}^{m} b_{ij}\phi_j(t, 0), \quad i = m+1, \dots, n, \quad t > 0. \tag{4.24}$$

On the other hand, noting that $\mathrm{Span}\{E_1, \dots, E_n\}$ and $\mathrm{Ker}(C_1) = \mathrm{Span}\{e_1, \dots, e_n\}$ are bi-orthonormal, as $t \geqslant T$ we have

$$t \geqslant T: \quad \phi_i = (U, E_i) = (\sum_{j=1}^{n} \tilde{u}_j e_j, E_i) = \tilde{u}_i, \quad i = 1, \dots, n, \tag{4.25}$$

thus, the exactly synchronizable state $(\tilde{u}_1(t,x), \dots, \tilde{u}_n(t,x))^T$ is entirely determined by the solution to system (4.21) and (4.23)–(4.24) with the initial data

$$t = 0: \quad \phi_i = (U_0, E_i), \quad i = 1, \dots, n, \tag{4.26}$$

then is independent of applied boundary control H. Thus, in order to obtain the exactly synchronizable state $(\tilde{u}_1(t,x), \dots, \tilde{u}_n(t,x))^T$, we only need to solve the mixed problem of the homogeneous system (4.21) and (4.23)–(4.24), the solution of which is determined only by the initial data (4.26). □

Lemma 4.5. *Under the condition of C_1-compatibility (3.2) for G, if $\mathrm{Span}\{E_1, \dots, E_n\}$ with $E_i (i = 1, \dots, n)$ given by (4.12) is an invariant subspace of G^T, then*

$$G_0^T \varepsilon_i = \sum_{j=1}^{m} \beta_{ji}\varepsilon_j \ (i = m+1, \dots, n), \qquad G_1^T \varepsilon_i = \sum_{j=m+1}^{n} \beta_{ji}\varepsilon_j \ (i = 1, \dots, m), \tag{4.27}$$

where $\beta_{ij} (i, j = 1, \dots, n)$ are given by (3.14).

Proof. We only prove the second formula in (4.27). By the proof of Theorem 4.4, $\mathrm{Span}\{E_1, \dots, E_n\}$ is an invariant subspace of G^T if and only if (4.18)–(4.19) hold. Noting that $V = \mathrm{Span}\{\varepsilon_1, \dots, \varepsilon_m\}$ and $\mathrm{Ker}(C_1^-) = \mathrm{Span}\{\epsilon_1, \dots, \epsilon_m\}$ are bi-orthonormal, we have

$$(G_1^T \varepsilon_i, \epsilon_k) = (\sum_{j=m+1}^{n} b_{ij}\varepsilon_j, \epsilon_k) = b_{ik}, \quad i = 1, \dots, m; k = m+1, \dots, n. \tag{4.28}$$

On the other hand, under the condition of C_1-compatibility (3.14) for G, we have

$$(G_1^T \varepsilon_i, \epsilon_k) = (\varepsilon_i, G_1 \epsilon_k) = (\varepsilon_i, \sum_{j=1}^{m} \beta_{kj}\epsilon_j) = \beta_{ki}, \quad i = 1, \dots, m; k = m+1, \dots, n. \tag{4.29}$$

Comparing (4.28) and (4.29) we have $b_{ik} = \beta_{ki} (i = 1, \dots, m; k = m+1, \dots, n)$. The proof is complete. □

Remark 4.6. Similarly, under the condition of C_1-compatibility (3.1) for A, if $\mathrm{Span}\{E_1, \dots, E_n\}$ with $E_i (i = 1, \dots, n)$ given by (4.12) is an invariant subspace for A^T, then

$$A^T E_i = \sum_{j=1}^{n} \alpha_{ji} E_j, \quad i = 1, \dots, n, \tag{4.30}$$

where $\alpha_{ij} (i, j = 1, \dots, n)$ are given by (3.5).

To obtain the reverse result of Theorem 4.4, we first give the following

Lemma 4.7. *Assume that A and G satisfy conditions of C_1-compatibility* (3.1) *and* (3.2)*, respectively. Let $H \in (L^2(0,T))^M$ be the boundary control with which system* (2.3) *realizes the exact boundary synchronization at the time $t = T$, where $T > 0$ is large enough. Then, for a small $t_0 > 0$, the value of H on $(0, t_0)$ can be chosen arbitrarily.*

Proof. At first, there exists $T_0 > 0$ independent of the initial data, such that, as $T > T_0$, the reduced problem (3.24) is exactly null controllable at the time $t = T$. Taking $t_0 > 0$, such that $T - t_0 > T_0$, and arbitrarily giving $\widehat{H} \in (L^2(0,t_0))^M$, we solve the forward problem (3.24)–(3.25) with $H = \widehat{H}$ in $R(t_0) = \{(t,x) | 0 < t < t_0, 0 < x < L\}$, and get the solution $\widehat{W} \in (L^2(0,t_0; L^2(0,L)))^{N-n}$. Since $T - t_0 > T_0$, the mixed problem (3.24) is still exactly null controllable in (t_0, T), we can find a boundary control $\widetilde{H} \in (L^2(t_0,T))^M$, such that the corresponding solution $\widetilde{W} = \widetilde{W}(t,x) \in (L^2(t_0,T; L^2(0,L)))^{N-n}$ satisfies exactly the initial condition $t = t_0: \ \widetilde{W} = \widehat{W}(t_0, x)$ and the final condition $t = T: \ \widetilde{W} \equiv 0$. Let

$$H = \begin{cases} \widehat{H}, & t \in (0,t_0), \\ \widetilde{H}, & t \in (t_0,T), \end{cases} \qquad W = \begin{cases} \widehat{W}, & t \in (0,t_0), \\ \widetilde{W}, & t \in (t_0,T). \end{cases} \tag{4.31}$$

It is easy to check that W given by (4.31) is the solution to the mixed problem (3.24), and it is exactly null controllable at the time $t = T$ under the boundary control H. Thus system (2.3) is exactly synchronizable at the time $t = T$ under the boundary control H, the value of which is arbitrarily chosen on $(0, t_0)$. □

We still project system (2.3) to the subspace $\text{Span}\{E_1, \ldots, E_n\}$ which is bi-orthonormal to $\text{Ker}(C_1)$. If the projection system is independent of applied boundary controls, and if the reduced system of (2.3) is exactly controllable, then we can obtain the properties mentioned in Theorem 4.4 for the corresponding matrices Λ, A, G_0, G_1 and D for system (2.3). To guarantee the exact boundary controllability of the reduced system, one has to assume that the number of positive eigenvalues of the reduced system is not bigger than that of negative ones, and the corresponding coupling matrix $\overline{G}_0$ on $x = 0$ is of full row-rank, then Lemma 2.7 can be applied.

Theorem 4.8. *Assume that A and G satisfy conditions of C_1-compatibility* (3.1) *and* (3.2)*, respectively. Assume furthermore that the number of positive eigenvalues of $\overline{\Lambda}$, given by* (3.23)*, is not bigger than that of negative ones, and $\overline{G}_0$, given by* (3.22)*, is of full row-rank, namely,*

$$N^+ - \bar{m} \leqslant N^- - m \quad \textit{and} \quad \textit{rank}(\overline{G}_0) = N^+ - \bar{m}, \tag{4.32}$$

where N^- and N^+ are given by (2.1)*, and $\bar{m}$ is given by* (2.5)*. Assume finally that system* (2.3) *is exactly synchronizable under condition* (3.26)*. Let U be the solution which realizes the exact boundary synchronization at the time $t = T$. If the projection functions $\phi_i = (E_i, U) (i = 1, \ldots, n)$ with $E_i (i = 1, \ldots, n)$ given by* (4.12) *are independent of applied boundary controls H, then $E_i (i = 1, \ldots, n)$ are eigenvectors of Λ, $\textit{Span}\{E_1, \ldots, E_n\}$ is a common invariant subspace of A^T and G^T, and $\varepsilon_i \in \textit{Ker}(D^T) (i = 1, \ldots, m)$. In particular, if D satisfies* (4.13)*, then we have $\textit{Ker}(D^T) = \textit{Span}\{\varepsilon_1, \ldots, \varepsilon_m\}$.*

Proof. Let $U_0 = 0$ in (2.4). By the well-posedness of problem (2.3) and (2.4), the linear mapping $F: \ H \to U = (U^-, U^+)^T$ is continuous from $(L^2_{loc}(0,+\infty))^M$ to $(L^2_{loc}(0,+\infty; L^2(0,L)))^N$, where H has compact support in $(0,T)$. Let F' denote the Fréchet derivative of the mapping F. For any given $H \in (L^2_{loc}(0,+\infty))^M$, we define $\widehat{U} = (\widehat{U}^-, \widehat{U}^+)^T = F'(0)H$. By linearity, $\widehat{U}$ satisfies a system similar to that of U:

$$\begin{cases} \widehat{U}_t + \Lambda \widehat{U}_x + A\widehat{U} = 0, & t \in (0,+\infty), \quad x \in (0,L), \\ \widehat{U}^+(t,0) = G_0 \widehat{U}^-(t,0), & t \in (0,+\infty), \\ \widehat{U}^-(t,L) = G_1 \widehat{U}^+(t,L) + DH, & t \in (0,+\infty), \\ t = 0: \quad \widehat{U} = 0, & x \in (0,L). \end{cases} \tag{4.33}$$

Since the projection functions ϕ_i $(i=1,\ldots,n)$ are independent of applied boundary control H, we have

$$(E_i,\widehat{U})\equiv 0(i=1,\ldots,n),\quad \forall H\in (L^2_{loc}(0,+\infty))^M. \tag{4.34}$$

Noting that $\mathrm{Span}\{E_1,\ldots,E_n\}$ and $\mathrm{Ker}(C_1)=\mathrm{Span}\{e_1,\ldots,e_n\}$ are bi-orthonormal, we have $E_i\notin \mathrm{Im}(C_1^T)$ $(i=1,\ldots,n)$. Otherwise, there exists $P_i\in\mathbb{R}^{N-n}$, such that $E_i=C_1^TP_i$, then $1=(E_i,e_i)=(C_1^TP_i,e_i)=(P_i,C_1e_i)=0$. Then $(E_1,\ldots,E_n,C_1^T)$ constitutes a set of basis in $\mathbb{R}^N$. Thus, there exist constants $\mu_{ij}(i,j=1,\ldots,n)$, $a_{ij}(i,j=1,\ldots,n)$ and vectors $P_i,Q_i\in\mathbb{R}^{N-n}(i=1,\ldots,n)$, such that

$$\Lambda E_i=\textstyle\sum_{j=1}^n\mu_{ij}E_j+C_1^TP_i,\qquad A^TE_i=\sum_{j=1}^n a_{ij}E_j+C_1^TQ_i,\qquad i=1,\ldots,n. \tag{4.35}$$

Taking the inner product with $E_i(i=1,\ldots,n)$ on the equations in system (4.33), it follows from (4.34)–(4.35) that

$$0=(\Lambda\widehat{U}_x,E_i)+(A\widehat{U},E_i)=(\widehat{U}_x,C_1^TP_i)+(\widehat{U},C_1^TQ_i)=(C_1\widehat{U}_x,P_i)+(C_1\widehat{U},Q_i),\quad i=1,\ldots,n. \tag{4.36}$$

Under conditions of C_1-compatibility (3.1) and (3.2) for A and G, respectively, by multiplying corresponding C_1 on (4.33) and setting $W=C_1\widehat{U}$, we get the reduced system (3.24). Noting (3.26) and (4.32), by Lemma 2.7, the reduced system (3.24) is exactly controllable, then the value $C_1\widehat{U}$ of its solution at the time $t=T$ can be chosen arbitrarily. Specially taking $C_1\widehat{U}_x=0$ with $C_1\widehat{U}$ being arbitrarily chosen in (4.36), we get $Q_i=0(i=1,\ldots,n)$. Then, by arbitrarily choosing $C_1\widehat{U}_x$, we have $P_i=0(i=1,\ldots,n)$. Thus, we have

$$\Lambda E_i=\textstyle\sum_{j=1}^n\mu_{ij}E_j,\qquad A^TE_i=\sum_{j=1}^n a_{ij}E_j,\qquad i=1,\ldots,n. \tag{4.37}$$

Then $\mathrm{Span}\{E_1,\ldots,E_n\}$ is an invariant subspace of A^T. Taking the inner product with $e_i(i=1,\ldots,n)$ on the first formula in (4.37), and noting that $\mathrm{Span}\{E_1,\ldots,E_n\}$ and $\mathrm{Ker}(C_1)=\mathrm{Span}\{e_1,\ldots,e_n\}$ are bi-orthonormal, we have

$$(\Lambda E_i,e_k)=(\textstyle\sum_{j=1}^n\mu_{ij}E_j,e_k)=\sum_{j=1}^n\mu_{ij}(E_j,e_k)=\mu_{ik},\quad i,k=1,\ldots,n, \tag{4.38}$$

while

$$(\Lambda E_i,e_k)=(E_i,\Lambda e_k)=(E_i,\lambda_k e_k)=\begin{cases}\lambda_i, & k=i,\\ 0, & k\neq i,\end{cases}\qquad i=1,\ldots,n. \tag{4.39}$$

Therefore, comparing (4.38) and (4.39), we get $\mu_{ii}=\lambda_i$ and $\mu_{ik}=0$ $(k\neq i)$ for $i=1,\ldots,n$. Thus, $E_i(i=1,\ldots,n)$ are eigenvectors of Λ.

On the other hand, noting that $W=\mathrm{Span}\{\varepsilon_{m+1},\ldots,\varepsilon_n\}$ and $\mathrm{Ker}(C_1^+)=\mathrm{Span}\{\epsilon_{m+1},\ldots,\epsilon_n\}$ are bi-orthonormal, similarly, we can prove that $(\varepsilon_{m+1},\ldots,\varepsilon_n,(C_1^+)^T)$ constitutes a set of basis in $\mathbb{R}^{N^+}$. Thus, there exist constants $b_{ij}(i,j=1,\ldots,n)$ and vectors $\Theta_i\in\mathbb{R}^{N^+-\bar{m}}(i=1,\ldots,m)$, such that

$$G_1^T\varepsilon_i=\textstyle\sum_{j=m+1}^n b_{ij}\varepsilon_j+(C_1^+)^T\Theta_i,\quad i=1,\ldots,m. \tag{4.40}$$

Taking the inner product with $\varepsilon_i(i=1,\ldots,m)$ on the boundary condition on $x=L$ in (4.33), noting (4.34), (4.40), and that H has a compact support in $(0,T)$, for $i=1,\ldots,m$, as $t\geqslant T$ we have

$$\begin{aligned}0&=(\widehat{U}(t,L),E_i)=(\widehat{U}^-(t,L),\varepsilon_i)=(G_1\widehat{U}^+(t,L),\varepsilon_i)=(\widehat{U}^+(t,L),\textstyle\sum_{j=m+1}^n b_{ij}\varepsilon_j)+(\widehat{U}^+(t,L),(C_1^+)^T\Theta_i)\\&=\textstyle\sum_{j=m+1}^n b_{ij}(\widehat{U}(t,L),E_j)+(C_1^+\widehat{U}^+(t,L),\Theta_i)=(C_1^+\widehat{U}^+(t,L),\Theta_i).\end{aligned} \tag{4.41}$$

Similarly, by the exact boundary controllability of the reduced system (3.24), the value of $C_1^+\widehat{U}^+$ at the time $t = T$ can be arbitrarily chosen, then we have $\Theta_i = 0 (i = 1, \dots, m)$, thus, we get

$$G_1^T\varepsilon_i = \textstyle\sum_{j=m+1}^n b_{ij}\varepsilon_j, \quad i = 1, \dots, m. \tag{4.42}$$

We can similarly prove (4.19). Thus $\text{Span}\{E_1, \dots, E_n\}$ is an invariant subspace of G^T.

On the other hand, by taking the inner product with $\varepsilon_i (i = 1, \dots, m)$ on the boundary condition on $x = L$ in (4.33), noting (4.12), (4.34) and (4.42), as $0 < t < T$, for $i = 1, \dots, m$ we have

$$\begin{aligned} 0 &= (\widehat{U}(t,L), E_i) = (\widehat{U}^-(t,L), \varepsilon_i) = (G_1\widehat{U}^+(t,L), \varepsilon_i) + (DH, \varepsilon_i) = (\widehat{U}^+(t,L), \textstyle\sum_{j=m+1}^n b_{ij}\varepsilon_j) + (H, D^T\varepsilon_i) \\ &= \textstyle\sum_{j=m+1}^n b_{ij}(\widehat{U}^+(t,L), \varepsilon_j) + (H, D^T\varepsilon_i) = \textstyle\sum_{j=m+1}^n b_{ij}(\widehat{U}(t,L), E_j) + (H, D^T\varepsilon_i) = (H, D^T\varepsilon_i). \end{aligned} \tag{4.43}$$

Since, by Lemma 4.7, the value of H on $(0, t_0)$ can be chosen arbitrarily for $t_0 > 0$ small enough, we have $D^T\varepsilon_i = 0 (i = 1, \dots, m)$, thus $\varepsilon_i \in \text{Ker}(D^T) (i = 1, \dots, m)$. The proof is complete. □

We then discuss the properties of G_0 which imply that the matrix $\overline{G}_0$ is of full row-rank.

Lemma 4.9. *Let C be an $M \times N$ matrix and D be an $N \times L$ matrix. Then rank(CD) = rank(C) holds if and only if we have Ker$(D^T) \cap$ Im$(C^T) = \{0\}$.*

Proof. Noting that $\text{rank}(D^TC^T) = \text{rank}(CD) = \text{rank}(C) = \text{rank}(C^T)$, Lemma 4.9 comes from Proposition 2.11 in [19]. □

Corollary 4.10. *Assume that the number of positive eigenvalues of $\overline{\Lambda}$, given by (3.23), is not bigger than that of negative ones, namely, $N^+ - \bar{m} \leqslant N^- - m$, where N^- and N^+ are given by (2.1), while $\bar{m}$ is given by (2.5). Then*

$$\textit{Ker}(G_0^T) \cap \textit{Im}((C_1^+)^T) = \{0\} \tag{4.44}$$

if and only if $\overline{G}_0$, given by (3.22), is of full row-rank, namely, rank$(\overline{G}_0) = N^+ - \bar{m}$. In particular, if G_0 is of full row-rank, then $\overline{G}_0$ is of full row-rank.

Proof. By Lemma 4.9, (4.44) is equivalent to $\text{rank}(C_1^+G_0) = \text{rank}(C_1^+) = N^+ - \bar{m}$. Noting (3.22), we have $N^+ - \bar{m} = \text{rank}(C_1^+G_0) = \text{rank}(\overline{G}_0C_1^-) \leqslant \text{rank}(\overline{G}_0)$, then, noting $N^+ - \bar{m} \leqslant N^- - m$ and that $\overline{G}_0$ is an $(N^+ - \bar{m}) \times (N^- - m)$ matrix, $\overline{G}_0$ is of full row-rank.

Inversely, assume that $\text{rank}(\overline{G}_0) = N^+ - \bar{m}$. Since C_1^- is of full row-rank, we have $\text{Ker}((C_1^-)^T) = \{0\}$. Thus $\text{Ker}((C_1^-)^T) \cap \text{Im}(\overline{G}_0^T) = \{0\}$, by Lemma 4.9, $\text{rank}(\overline{G}_0C_1^-) = \text{rank}(\overline{G}_0)$. Hence, noting (3.22), we have $N^+ - \bar{m} = \text{rank}(\overline{G}_0) = \text{rank}(\overline{G}_0C_1^-) = \text{rank}(C_1^+G_0) = \text{rank}(C_1^+)$, then, by Lemma 4.9 we get (4.44).

In particular, if G_0 is of full row-rank, then $\text{Ker}(G_0^T) = \{0\}$, and (4.44) holds. □

5. Estimate of exactly synchronizable states

In general, the exactly synchronizable state depends on applied boundary controls. We can not obtain precisely the exactly synchronizable state simply by solving a mixed problem of a homogeneous system as in Theorem 4.4. However, we can estimate the exactly synchronizable state by the solution to such problem of homogeneous system as follows.

Theorem 5.1. *Assume that A and G satisfy conditions of C_1-compatibility (3.1) and (3.2), respectively. Let $E_i (i = 1, \dots, n)$ given by (4.12) be eigenvectors of Λ, namely, (4.15) holds. If system (2.3) is* *exactly synchronizable under condition* (3.26) *with* H *satisfying* (3.27), *then the exactly synchronizable state* $(\tilde u_1(t,x),\dots,\tilde u_n(t,x))^T$ *satisfies the following estimate:*

$$\|\tilde u_i(T)-\phi_i(T)\|_{L^2(0,L)}\leqslant c\|C_1U_0\|_{(L^2(0,L))^{N-n}},\quad i=1,\dots,n,\tag{5.1}$$

where $(\phi_1(t,x),\dots,\phi_n(t,x))^T$ *is the solution to the following problem:*

$$\begin{cases}\phi_{it}+\lambda_i\phi_{ix}+\sum_{j=1}^n\alpha_{ji}\phi_j=0,\quad i=1,\dots,n, & t\in(0,+\infty),\ x\in(0,L),\\ \phi_i(t,0)=\sum_{j=1}^m\beta_{ji}\phi_j(t,0),\quad i=m+1,\dots,n, & t\in(0,+\infty),\\ \phi_i(t,L)=\sum_{j=m+1}^n\beta_{ji}\phi_j(t,L),\quad i=1,\dots,m, & t\in(0,+\infty),\\ t=0:\quad \phi_i=(E_i,U_0),\quad i=1,\dots,n, & x\in(0,L),\end{cases}\tag{5.2}$$

where α_{ij} *and* $\beta_{ij}(i,j=1,\dots,n)$ *are given by* (3.5) *and* (3.14), *respectively.*

Proof. Let U be the solution to problem (2.3) and (2.4), which realizes the exact boundary synchronization at the time $t=T$, and let

$$z_i=(E_i,U)=\begin{cases}(\varepsilon_i,U^-), & i=1,\dots,m,\\ (\varepsilon_i,U^+), & i=m+1,\dots,n.\end{cases}$$

For $i=1,\dots,n$ we have

$$(E_i,AU)=(A^TE_i,U)=(\textstyle\sum_{j=1}^n\alpha_{ji}E_j,U)+(A^TE_i-\sum_{j=1}^n\alpha_{ji}E_j,U)=\sum_{j=1}^n\alpha_{ji}z_j+(A^TE_i-\sum_{j=1}^n\alpha_{ji}E_j,U).\tag{5.3}$$

Since $\mathrm{Span}\{E_1,\dots,E_n\}$ and $\mathrm{Ker}(C_1)=\mathrm{Span}\{e_1,\dots,e_n\}$ are bi-orthonormal, noting (3.5), we have

$$(A^TE_i-\textstyle\sum_{j=1}^n\alpha_{ji}E_j,e_k)=(E_i,Ae_k)-\alpha_{ki}=(E_i,\sum_{j=1}^n\alpha_{kj}e_j)-\alpha_{ki}=\alpha_{ki}-\alpha_{ki}=0\tag{5.4}$$

for $i,k=1,\dots,n$, hence, noting (2.11), we have $A^TE_i-\sum_{j=1}^n\alpha_{ji}E_j\in\{\mathrm{Ker}(C_1)\}^\perp=\mathrm{Im}(C_1^T)$ $(i=1,\dots,n)$, then there exist $P_i\in\mathbb{R}^{N-n}$ $(i=1,\dots,n)$ such that $A^TE_i-\sum_{j=1}^n\alpha_{ji}E_j=C_1^TP_i$ $(i=1,\dots,n)$. Thus, by (5.3), we have

$$(E_i,AU)=\textstyle\sum_{j=1}^n\alpha_{ji}z_j+(C_1^TP_i,U)=\sum_{j=1}^n\alpha_{ji}z_j+(P_i,C_1U),\quad i=1,\dots,n.\tag{5.5}$$

On the other hand, for $i=1,\dots,m$ we have

$$\begin{aligned}(\varepsilon_i,G_1U^+(t,L))&=(G_1^T\varepsilon_i,U^+(t,L))=(\textstyle\sum_{j=m+1}^n\beta_{ji}\varepsilon_j,U^+(t,L))+(G_1^T\varepsilon_i-\sum_{j=m+1}^n\beta_{ji}\varepsilon_j,U^+(t,L))\\&=\textstyle\sum_{j=m+1}^n\beta_{ji}z_j(t,L)+(G_1^T\varepsilon_i-\sum_{j=m+1}^n\beta_{ji}\varepsilon_j,U^+(t,L)).\end{aligned}\tag{5.6}$$

Since $V=\mathrm{Span}\{\varepsilon_1,\dots,\varepsilon_m\}$ and $\mathrm{Ker}(C_1^-)=\mathrm{Span}\{\epsilon_1,\dots,\epsilon_m\}$ are bi-orthonormal, noting (3.14), for $i=1,\dots,m$ and $k=m+1,\dots,n$ we have

$$(G_1^T\varepsilon_i-\textstyle\sum_{j=m+1}^n\beta_{ji}\varepsilon_j,\epsilon_k)=(\varepsilon_i,G_1\epsilon_k)-\beta_{ki}=(\varepsilon_i,\sum_{j=1}^m\beta_{kj}\epsilon_j)-\beta_{ki}=\beta_{ki}-\beta_{ki}=0.$$

Hence, noting (3.10), we have $G_1^T\varepsilon_i-\sum_{j=m+1}^n\beta_{ji}\varepsilon_j\in\{\mathrm{Ker}(C_1^+)\}^\perp=\mathrm{Im}((C_1^+)^T)$ $(i=1,\dots,m)$, then there exist $Q_i\in\mathbb{R}^{N^+-\bar m}$ $(i=1,\dots,m)$, such that $G_1^T\varepsilon_i-\sum_{j=m+1}^n\beta_{ji}\varepsilon_j=(C_1^+)^TQ_i$ $(i=1,\dots,m)$. Thus, by (5.6), we have

$$(\varepsilon_i,G_1U^+(t,L))=\textstyle\sum_{j=m+1}^n\beta_{ji}z_j(t,L)+(Q_i,C_1^+U^+(t,L)),\quad i=1,\dots,m.\tag{5.7}$$

Similarly, there exist $R_i \in \mathbb{R}^{N^- - m} (i = m+1, \ldots, n)$, such that

$$(\varepsilon_i, G_0 U^-(t,0)) = \textstyle\sum_{j=1}^m \beta_{ji} z_j(t,0) + (R_i, C_1^- U^-(t,0)), \quad i = m+1, \ldots, n. \tag{5.8}$$

Taking the inner product on both sides of problem (2.3) and (2.4) with E_i, and noting (4.15), (5.5) and (5.7)–(5.8), we have

$$\begin{cases} z_{it} + \lambda_i z_{ix} + \sum_{j=1}^n \alpha_{ji} z_j = -(P_i, C_1 U), \quad i = 1, \ldots, n, \quad t \in (0, +\infty), \quad x \in (0, L), \\ z_i(t,0) = \sum_{j=1}^m \beta_{ji} z_j(t,0) + (R_i, C_1^- U^-(t,0)), \quad i = m+1, \ldots, n, \quad t \in (0, +\infty), \\ z_i(t,L) = \sum_{j=m+1}^n \beta_{ji} z_j(t,L) + (Q_i, C_1^+ U^+(t,L)) + (DH, \varepsilon_i), \quad i = 1, \ldots, m, \ t \in (0, +\infty), \\ t = 0: \ z_i = (E_i, U_0), \quad i = 1, \ldots, n, \quad x \in (0, L). \end{cases} \tag{5.9}$$

Let $y_i = z_i - \phi_i$ $(i = 1, \ldots, n)$, where $(\phi_1(t,x), \ldots, \phi_n(t,x))^T$ is the solution to problem (5.2). We have

$$\begin{cases} y_{it} + \lambda_i y_{ix} + \sum_{j=1}^n \alpha_{ji} y_j = -(P_i, C_1 U), \quad i = 1, \ldots, n, \quad t \in (0, +\infty), \quad x \in (0, L), \\ y_i(t,0) = \sum_{j=1}^m \beta_{ji} y_j(t,0) + (R_i, C_1^- U^-(t,0)), \quad i = m+1, \ldots, n, \quad t \in (0, +\infty), \\ y_i(t,L) = \sum_{j=m+1}^n \beta_{ji} y_j(t,L) + (Q_i, C_1^+ U^+(t,L)) + (DH, \varepsilon_i), \quad i = 1, \ldots, m, \ t \in (0, +\infty), \\ t = 0: \ y_i = 0, \quad i = 1, \ldots, n, \quad x \in (0, L). \end{cases} \tag{5.10}$$

Then, by the well-posedness Theorem 3.3 in [26], we have

$$\begin{aligned} \|(z_i - \phi_i)(T)\|_{L^2(0,L)} = \|y_i(T)\|_{L^2(0,L)} \leqslant c(\|(P_i, C_1 U)\|_{L^2(0,T;L^2(0,L))} \\ + \|(R_i, C_1^- U^-(t,0))\|_{L^2(0,T)} + \|(Q_i, C_1^+ U^+(t,L))\|_{L^2(0,T)} + \|(DH, \varepsilon_i)\|_{L^2(0,T)}). \end{aligned} \tag{5.11}$$

Since $C_1 U = \begin{pmatrix} C_1^- U^- \\ C_1^+ U^+ \end{pmatrix}$ is the solution to the reduced system (3.24), noting (2.15)–(2.16) and (3.27), the right-hand side of (5.11)

$$\leqslant c(\|C_1 U_0\|_{(L^2(0,L))^{N-n}} + \|H\|_{(L^2(0,T))^M}) \leqslant c\|C_1 U_0\|_{(L^2(0,L))^{N-n}}. \tag{5.12}$$

On the other hand, noting (2.13), we have

$$t \geqslant T: \quad z_i = (E_i, U) = (E_i, \textstyle\sum_{j=1}^n \tilde{u}_j e_j) = \tilde{u}_i, \quad i = 1, \ldots, n. \tag{5.13}$$

Substituting (5.13) into (5.11)–(5.12), we get (5.1). □

Remark 5.2. Particularly defining D by (4.14), by the proof of Theorem 4.4, we have (4.13), then system (2.3) is exactly synchronizable, where, by Remark 3.11, we can properly choose the boundary control H to satisfy (3.27), thus we have estimate (5.1). In general, since boundary control might act arbitrarily at the beginning, the cost of the control might be arbitrarily large (see Lem. 4.7).

6. Necessity of the conditions of C_1-compatibility

The conditions of C_1-compatibility are important issues in the study of synchronization for first order hyperbolic systems. Under the conditions of C_1-compatibility for the coupling matrices, we can easily obtain the exact boundary synchronization for the system, and the behavior of exactly synchronizable states can be better described. However, the necessity of the conditions of C_1-compatibility are not always certain, because the exact boundary synchronization may happen without these conditions as long as there are enough boundary controls. Readers can refer to [10, 12, 17, 28] to see more about the discussion on the conditions of compatibility for a coupled system of wave equations.

6.1. Condition of C_1-compatibility for the coupling matrix A

In this section we will discuss the necessity of the condition of C_1-compatibility (3.1) for the coupling matrices A for the exact boundary synchronization of system (2.3).

Assume that system (2.3) is exactly synchronizable at the time $T > 0$, namely, there exists a decomposition (2.1) such that, for the corresponding matrix of synchronization C_1 given by (2.9), we have (2.14). Then, by multiplying C_1 on the equations in system (2.3) and noting (2.14), we have

$$t \geqslant T: \quad C_1 \Lambda U_x + C_1 A U = 0, \tag{6.1}$$

in which, noting (2.2), (2.9) and (2.14), by a direct calculation we have $C_1 \Lambda U_x = 0$, hence

$$t \geqslant T: \quad C_1 A U = C_1 A \textstyle\sum_{i=1}^n \tilde{u}_i e_i = \sum_{i=1}^n C_1 A e_i \tilde{u}_i = 0. \tag{6.2}$$

If $C_1 A e_i = 0 (i = 1, \dots, n)$, then, noting (2.11), we have the condition of C_1-compatibility (3.1). Otherwise, without loss of generality, we may assume that there exist constants $\delta_i (i = 1, \dots, n-1)$ such that

$$\tilde{u}_n = \textstyle\sum_{i=1}^{n-1} \delta_i \tilde{u}_i. \tag{6.3}$$

Let

$$\overline{U} = \left(\begin{array}{cccc:c} I_{N_1} & & & & \mathbf{0} \\ & I_{N_2} & & & \mathbf{0} \\ & & \ddots & & \vdots \\ & & & I_{N_{n-1}} & \mathbf{0} \\ \hdashline \frac{\delta_1}{N_1}\tilde{e}_1^T & \frac{\delta_2}{N_2}\tilde{e}_2^T & \cdots & \frac{\delta_{n-1}}{N_{n-1}}\tilde{e}_{n-1}^T & \\ \vdots & \vdots & & \vdots & -I_{N_n} \\ \frac{\delta_1}{N_1}\tilde{e}_1^T & \frac{\delta_2}{N_2}\tilde{e}_2^T & \cdots & \frac{\delta_{n-1}}{N_{n-1}}\tilde{e}_{n-1}^T & \end{array}\right) U = \begin{pmatrix} U_1 \\ \vdots \\ U_{n-1} \\ \left(\sum_{i=1}^{n-1} \frac{\delta_i}{N_i} \tilde{e}_i^T U_i\right)\tilde{e}_n - U_n \end{pmatrix},$$

where $\tilde{e}_i (i = 1, \dots, n)$ are given by (2.10). Noting (2.13) and (6.3), as $t \geqslant T$, we have

$$\textstyle\sum_{i=1}^{n-1} \frac{\delta_i}{N_i} \tilde{e}_i^T U_i = \sum_{i=1}^{n-1} \frac{\delta_i}{N_i} \tilde{e}_i^T \tilde{e}_i \tilde{u}_i = \sum_{i=1}^{n-1} \delta_i \tilde{u}_i = \tilde{u}_n,$$

thus $\overline{U} = (\tilde{u}_1 \tilde{e}_1, \dots, \tilde{u}_{n-1} \tilde{e}_{n-1}, 0 \tilde{e}_n)^T$, namely, one group of the components is in fact exactly null controllable, while the other groups are exactly synchronizable. Thus we have

Theorem 6.1. *If system (2.3) is exactly synchronizable, and under any linear invertible transformation, any given group is not exactly null controllable, then A must satisfy the corresponding condition of C_1-compatibility.*

6.2. Condition of C_1-compatibility for the boundary coupling matrix G

We now discuss the condition of C_1-compatibility for the boundary coupling matrix G on the boundary.

As $t \geqslant T$, noting (3.12), multiplying C_1^+ and C_1^- on the boundary conditions of system (2.3) on $x = 0$ and $x = L$ with $H \equiv 0$, respectively, we have

$$0 = C_1^+ U^+(t, 0) = C_1^+ G_0 U^-(t, 0) = \textstyle\sum_{i=1}^m C_1^+ G_0 \epsilon_i \tilde{u}_i \tag{6.4}$$

and

$$0 = C_1^- U^-(t, L) = C_1^- G_1 U^+(t, L) = \textstyle\sum_{i=m+1}^n C_1^- G_1 \epsilon_i \tilde{u}_i. \tag{6.5}$$

If $C_1^+G_0\epsilon_i = 0$ $(i = 1, \dots, m)$ and $C_1^-G_1\epsilon_i = 0$ $(i = m+1, \dots, n)$, then, noting (3.10) and Lemma 3.5, we have the condition of C_1-compatibility (3.2). Otherwise, components of the exactly synchronizable state corresponding to negative spreading speeds are linearly dependent on $x = 0$, or those corresponding to positive spreading speeds are linearly dependent on $x = L$. However, since the properties on the boundary cannot be extended to the internal domain, we can not obtain similar results for the boundary coupling matrix G as in Theorem 6.1. We will show in what follows that for system with two groups of synchronized components, the condition of C_1-compatibility for G is necessary.

For given N_1 and N_2, let

$$\Lambda = \mathrm{diag}\{\lambda_1 I_{N_1}, \lambda_2 I_{N_2}\} \quad \text{with} \quad \lambda_1 < 0 \quad \text{and} \quad \lambda_2 > 0. \tag{6.6}$$

For system (2.3) with $n = 2$, the exact boundary synchronization (2.7) means

$$u_1^{(1)}(t,x) \equiv \cdots \equiv u_1^{(N_1)}(t,x) \overset{\text{def}}{=} u(t,x), \quad u_2^{(1)}(t,x) \equiv \cdots \equiv u_2^{(N_2)}(t,x) \overset{\text{def}}{=} v(t,x) \tag{6.7}$$

as $t \geqslant T$, where $(u,v)^T$ is the exactly synchronizable state. (6.7) is equivalent to

$$t \geqslant T: \quad U = ue_1 + ve_2 \quad \text{or} \quad U^- = U_1 = u\tilde{e}_1,\ U^+ = U_2 = v\tilde{e}_2, \tag{6.8}$$

where $e_i (i = 1,2)$ are given by (2.12), and $\tilde{e}_i(i = 1,2)$ are given by (2.10).

Theorem 6.2. *Under assumption* (6.6)*, if system* (2.3) *is exactly synchronizable as in* (6.7)*, but any one of the two groups is not exactly null controllable, then G must satisfy the corresponding condition of C_1-compatibility.*

Proof. Since system (2.3) is exactly synchronizable, as $t \geqslant T$, the solution $U = U(t,x)$ satisfies (6.8) and we have $H(t) \equiv 0$ for $t \geqslant T$, then the boundary conditions on $x = L$ in system (2.3) gives

$$x = L: \quad u(t,L)\tilde{e}_1 = G_1 v(t,L)\tilde{e}_2, \quad t \geqslant T. \tag{6.9}$$

Multiplying $\tilde{C}_1$, given by (2.9), on (6.9), noting (2.10) we have

$$x = L: \quad v(t,L)\tilde{C}_1 G_1 \tilde{e}_2 = 0, \quad t \geqslant T. \tag{6.10}$$

We claim that $v(t,L) \not\equiv 0$ for $t \geqslant T$ at least for an initial data $U_0(x)$, therefore, $\tilde{C}_1 G_1 \tilde{e}_2 = 0$, then, noting $C_1^- = \tilde{C}_1$, $C_1^+ = \tilde{C}_2$ and (2.10), we have $G_1\mathrm{Ker}(C_1^+) \subseteq \mathrm{Ker}(C_1^-)$. Otherwise, assume $v(t,L) \equiv 0$ for $t \geqslant T$, by (6.9) we have $x = L:\ u(t,L)\tilde{e}_1 \equiv 0$ $(t \geqslant T)$, thus, by (6.8), we have $x = L:\ U(t,L) \equiv 0$ $(t \geqslant T)$. Then, by the well-posedness of the leftward problem of system (2.3) (see (2.56) of Thm. 3.1 in [26]), there exists $\tilde{T} \geqslant T$, such that $U \equiv 0$ $(t \geqslant \tilde{T})$, thus system (2.3) is in fact exactly boundary null controllable, which contradicts the assumption.

Similarly, we have $G_0\mathrm{Ker}(C_1^-) \subseteq \mathrm{Ker}(C_1^+)$. Then the condition of C_1-compatibility (3.2) follows from Lemma 3.5. □

7. Non-exact boundary controllability and synchronization

Similarly to the case of wave equations, the exact boundary controllability and synchronization can not be realized when the number of boundary controls is not enough, but so far there is no conclusion about the minimum number of boundary controls for all the systems. In this section, we will present some systems (2.3) for which the one-sided exact boundary controllability can not be realized if $\mathrm{rank}(D) < N^-$, and the one-sided exact boundary synchronization can not be realized if $\mathrm{rank}(C_1^- D) < N^- - m$.

Here and hereafter, let $\xi_k \in \mathbb{R}^{N^-}$ and $\eta_k \in \mathbb{R}^{N^+}(k = 1, \dots, q)$ be linearly independent, and let

$$\overline{E}_k = \begin{pmatrix} \xi_k \\ 0 \end{pmatrix}, \quad \underline{E}_k = \begin{pmatrix} 0 \\ \eta_k \end{pmatrix} \in \mathbb{R}^N, \quad k = 1, \dots, q \tag{7.1}$$

be linearly independent. Denote

$$\mathfrak{E} = \text{Span}\{\overline{E}_1, \dots, \overline{E}_q, \underline{E}_1, \dots, \underline{E}_q\}, \quad \overline{V} = \text{Span}\{\xi_1, \dots, \xi_q\} \quad \text{and} \quad \overline{W} = \text{Span}\{\eta_1, \dots, \eta_q\}. \tag{7.2}$$

7.1. Non-exact boundary controllability

Theorem 7.1. *Assume that $\overline{E}_k$ and $\underline{E}_k(k = 1, \dots, q)$ are eigenvectors of Λ, $\mathfrak{E}$ is a common invariant subspace of A^T and G^T, and*

$$\overline{V} = Span\{\xi_1, \dots, \xi_q\} \subseteq Ker(D^T). \tag{7.3}$$

Assume furthermore that $Ker(G^T) \cap \mathfrak{E} = \{0\}$, namely,

$$Ker(G_0^T) \cap \overline{W} = \{0\} \quad and \quad Ker(G_1^T) \cap \overline{V} = \{0\}. \tag{7.4}$$

Then we have $rank(D) < N^-$ and system (2.3) is not exactly null controllable.

Proof. Noting (7.3), we have $\text{rank}(D) < N^-$.

Since $\overline{E}_k$ and $\underline{E}_k(k = 1, \dots, q)$ are eigenvectors of Λ, assuming that

$$\Lambda \overline{E}_k = \tilde{\lambda}_k \overline{E}_k \quad \text{and} \quad \Lambda \underline{E}_k = \tilde{\tilde{\lambda}}_k \underline{E}_k, \quad k = 1, \dots, q, \tag{7.5}$$

it follows by a direct calculation that $\tilde{\lambda}_k < 0$ and $\tilde{\tilde{\lambda}}_k > 0$ for $k = 1, \dots, q$.

By assumptions, $\mathfrak{E} = \text{Span}\{\overline{E}_1, \dots, \overline{E}_q, \underline{E}_1, \dots, \underline{E}_q\}$ is a common invariant subspace of A^T and G^T, then we may assume that

$$A^T \overline{E}_k = \textstyle\sum_{l=1}^q a_{kl} \overline{E}_l + \sum_{l=1}^q b_{kl} \underline{E}_l, \qquad A^T \underline{E}_k = \sum_{l=1}^q c_{kl} \overline{E}_l + \sum_{l=1}^q d_{kl} \underline{E}_l \tag{7.6}$$

and

$$G_0^T \eta_k = \textstyle\sum_{l=1}^q g_{kl}^0 \xi_l, \qquad G_1^T \xi_k = \sum_{l=1}^q g_{kl}^1 \eta_l \tag{7.7}$$

for $k = 1, \dots, q$, where $a_{kl}, b_{kl}, c_{kl}, d_{kl}, g_{kl}^0$ and $g_{kl}^1(k, l = 1, \dots, q)$ are constants.

Assume by contradiction that system (2.3) is exactly null controllable at the time $t = T$ for any given initial data $U_0 \in (L^2(0, L))^N$. Let U be the solution to problem (2.3) and (2.4), which realizes the exact boundary null controllability, and let

$$\phi_k = (\overline{E}_k, U) = (\xi_k, U^-), \qquad \psi_k = (\underline{E}_k, U) = (\eta_k, U^+), \quad k = 1, \dots, q. \tag{7.8}$$

Noting (7.3) and (7.5)–(7.7), for $k = 1, \dots, q$ we have

$$\begin{cases} \phi_{kt} + \tilde{\lambda}_k \phi_{kx} + \sum_{l=1}^q a_{kl} \phi_l + \sum_{l=1}^q b_{kl} \psi_l = 0, & t \in (0, T), \quad x \in (0, L), \\ \psi_{kt} + \tilde{\tilde{\lambda}}_k \psi_{kx} + \sum_{l=1}^q c_{kl} \phi_l + \sum_{l=1}^q d_{rs} \psi_l = 0, & t \in (0, T), \quad x \in (0, L), \\ \psi_k(t, 0) = \sum_{l=1}^q g_{kl}^0 \phi_l(t, 0), & t \in (0, T), \\ \phi_k(t, L) = \sum_{l=1}^q g_{kl}^1 \psi_l(t, L), & t \in (0, T) \end{cases} \tag{7.9}$$

with

$$t = 0: \quad \phi_k = (\overline{E}_k, U_0), \quad \psi_k = (\underline{E}_k, U_0) \tag{7.10}$$

and

$$t = T: \quad \phi_k = (\overline{E}_k, U) \equiv 0, \quad \psi_k = (\underline{E}_k, U) \equiv 0. \tag{7.11}$$

We claim that matrices $\hat{G}_0 = (g^0_{kl})$ and $\hat{G}_1 = (g^1_{kl})$ of order q are invertible, then by Lemma 2.6, we can solve the backward problem of system (7.9) with (7.11) on $R(T)$, the solution $\phi_k = \psi_k \equiv 0 (k = 1, \dots, q)$ of which is also the solution to the forward problem of system (7.9), by the uniqueness of the solution, we have $(\overline{E}_k, U_0) = (\underline{E}_k, U_0) \equiv 0 (k = 1, \dots, q)$, which contradicts the fact that the initial data U_0 varies through the whole space $(L^2(0, L))^N$.

At last, we prove that the matrices $\hat{G}_0 = (g^0_{kl})$ and $\hat{G}_1 = (g^1_{kl})$ are invertible. In fact, the first formula in (7.7) implies

$$G_0^T (\eta_1, \dots, \eta_q) = (\xi_1, \dots, \xi_q) \hat{G}_0^T. \tag{7.12}$$

Let $\eta = (\eta_1, \dots, \eta_q)$ be an $N^+ \times q$ full column-rank matrix and $\xi = (\xi_1, \dots, \xi_q)$ an $N^- \times q$ full column-rank matrix. By (7.4), we have $\mathrm{Ker}(G_0^T) \cap \mathrm{Im}(\eta) = \{0\}$, then by Proposition 2.11 in [19], we have $\mathrm{rank}(G_0^T \eta) = \mathrm{rank}(\eta) = q$. Thus (7.12) indicates that $q = \mathrm{rank}(G_0^T \eta) = \mathrm{rank}(\xi \hat{G}_0^T) \leqslant \mathrm{rank}(\hat{G}_0^T)$, then, $\hat{G}_0$ is invertible. We can similarly prove that $\hat{G}_1$ is invertible. □

Example 7.2. Consider the following system:

$$\begin{cases} u_{1t} + \lambda_1 u_{1x} + u_1 + 5v = 0, & t \in (0, +\infty), \quad x \in (0, L), \\ u_{2t} + \lambda_2 u_{2x} + 3u_1 + 4u_2 - v = 0, & t \in (0, +\infty), \quad x \in (0, L), \\ v_t + \lambda_3 v_x + 2u_1 + v = 0, & t \in (0, +\infty), \quad x \in (0, L), \\ x = 0: \quad v = 2u_1, & t \in (0, +\infty), \\ x = L: \quad u_1 = v, \quad u_2 = 2v + 3h(t), & t \in (0, +\infty), \end{cases} \tag{7.13}$$

in which λ_1, $\lambda_2 < 0$ and $\lambda_3 > 0$. Let $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\}$,

$$A = \begin{pmatrix} 1 & 0 & 5 \\ 3 & 4 & -1 \\ 2 & 0 & 1 \end{pmatrix}, \ G_0 = \begin{pmatrix} 2 & 0 \end{pmatrix}, \ G_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \ D = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, E_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \ \eta = 1.$$

It is easy to check that E_1 and E_2 are eigenvectors of Λ, $\mathfrak{E} = \mathrm{Span}\{E_1, E_2\}$ is a common invariant subspace of A^T and G^T, $\overline{V} = \mathrm{Span}\{\xi\} \subseteq \mathrm{Ker}(D^T)$, and $\mathrm{Ker}(G^T) \cap \mathfrak{E} = \{0\}$. By Theorem 7.1, system (7.13) is not exactly null controllable by any given boundary control h. In fact, system (7.13) can be divided into two subsystems: the one corresponding to u_2 is controllable by boundary control h; while the one corresponding to u_1 and v is a closed system independent of u_2 and h, thus is not null controllable by any given boundary control h.

7.2. Non-exact boundary synchronization

In order to study the minimum number of boundary controls for the synchronization, instead of projecting the system to a subspace that is bi-orthonormal to $\mathrm{Ker}(C_1)$ as in Section 4, we now make the projection to a subspace of $\mathrm{Im}(C_1^T)$.

Let $\overline{E}_k$, $\underline{E}_k(k=1,\dots,q)$, $\mathfrak{E}$ and $\overline{V}$ be defined by (7.1)–(7.2). We further require that both $\overline{E}_k$ and $\underline{E}_k \in \mathrm{Im}(C_1^T)$ $(k=1,\dots,q)$, which, by a direct computation, is equivalent to

$$\overline{V}=\mathrm{Span}\{\xi_1,\dots,\xi_q\}\subseteq \mathrm{Im}((C_1^-)^T) \text{ and } \overline{W}=\mathrm{Span}\{\eta_1,\dots,\eta_q\}\subseteq \mathrm{Im}((C_1^+)^T). \tag{7.14}$$

Theorem 7.3. *Assume that $\overline{E}_k$ and $\underline{E}_k(k=1,\dots,q)$ belong to $Im(C_1^T)$ and are eigenvectors of Λ, $\mathfrak{E}$ is a common invariant subspace of A^T and G^T, and (7.3) holds. Assume furthermore that $Ker(G^T)\cap\mathfrak{E}=\{0\}$, namely, (7.4) holds. Then we have $rank(C_1^-D)<N^--m$, and system (2.3) is not exactly synchronizable.*

Proof. Noting (7.3) and (7.14), we have $\overline{V}\subseteq \mathrm{Ker}(D^T)\cap\mathrm{Im}((C_1^-)^T)$. By Proposition 2.11 in [19], we get

$$\mathrm{rank}(C_1^-D)=\mathrm{rank}((C_1^-D)^T)=\mathrm{rank}(D^T(C_1^-)^T)<\mathrm{rank}((C_1^-)^T)=N^--m, \tag{7.15}$$

then we have $\mathrm{rank}(C_1^-D)<N^--m$.

By assumptions, similarly, we have (7.5)–(7.7).

Assume by contradiction that system (2.3) is exactly synchronizable at the time $t=T$. Let U be the solution to problem (2.3) and (2.4), which realizes the exact boundary synchronization and let ϕ_k and ψ_k be defined by (7.8) for $k=1,\dots,q$. Similarly, ϕ_k and $\psi_k(k=1,\dots,q)$ satisfy (7.9) with (7.10).

On the other hand, noting that both $\overline{E}_k$ and $\underline{E}_k\in\mathrm{Im}(C_1^T)$ $(k=1,\dots,q)$, there exist P_k and $Q_k\in\mathbb{R}^{N-n}(k=1,\dots,q)$, such that $\overline{E}_k=C_1^TP_k$ and $\underline{E}_k=C_1^TQ_k$ for $k=1,\dots,q$. Thus, noting (2.14), as $t=T$, for $k=1,\dots,q$, we have

$$\phi_k=(\overline{E}_k,U(T,x))=(C_1^TP_k,U(T,x))=(P_k,C_1U(T,x))\equiv 0,\quad x\in(0,L), \tag{7.16}$$
$$\psi_k=(\underline{E}_k,U(T,x))=(C_1^TQ_k,U(T,x))=(Q_k,C_1U(T,x))\equiv 0,\quad x\in(0,L). \tag{7.17}$$

By the proof of Theorem 7.1, we have $\hat{G}_0=(g_{kl}^0)$ and $\hat{G}_1=(g_{kl}^1)$ are invertible, then by Lemma 2.6, we can solve the backward problem of system (7.9) with (7.16)–(7.17) on $R(T)$, and the solution $\phi_k=\psi_k\equiv 0(k=1,\dots,q)$. Similarly, we have $(\overline{E}_k,U_0)=(\underline{E}_k,U_0)\equiv 0(k=1,\dots,q)$ in (7.10), which contradicts the fact that the initial data U_0 varies through the whole space $(L^2(0,L))^N$. □

Example 7.4. Consider the following system:

$$\begin{cases} u_{1t}+\lambda_1u_{1x}+u_1-u_2+u_3=0, & t\in(0,+\infty),\quad x\in(0,L),\\ u_{2t}+\lambda_1u_{2x}-u_1+u_2+u_4=0, & t\in(0,+\infty),\quad x\in(0,L),\\ u_{3t}+\lambda_2u_{3x}+u_1+u_3-u_4=0, & t\in(0,+\infty),\quad x\in(0,L),\\ u_{4t}+\lambda_2u_{4x}+u_2-u_3+u_4=0, & t\in(0,+\infty),\quad x\in(0,L),\\ x=0:\quad u_3=-u_1+u_2,\quad u_4=u_1-u_2, & t\in(0,+\infty),\\ x=L:\quad u_1=u_3+2u_4+h(t),\quad u_2=2u_3+u_4+h(t), & t\in(0,+\infty), \end{cases} \tag{7.18}$$

in which $\lambda_1<0$ and $\lambda_2>0$. Let $\Lambda=\mathrm{diag}\{\lambda_1,\lambda_1,\lambda_2,\lambda_2\}$,

$$A=\begin{pmatrix}1&-1&1&0\\-1&1&0&1\\1&0&1&-1\\0&1&-1&1\end{pmatrix},\quad G_0=\begin{pmatrix}-1&1\\1&-1\end{pmatrix},\quad G_1=\begin{pmatrix}1&2\\2&1\end{pmatrix},\quad D=\begin{pmatrix}1\\1\end{pmatrix},$$

$$C_1 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \xi = \eta = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

It is easy to check that A and G, given by (2.6), satisfy the conditions of C_1-compatibility (3.1) and (3.2), respectively. $E_1, E_2 \in \mathrm{Im}(C_1^T)$, E_1 and E_2 are eigenvectors of Λ, $\mathfrak{E} = \mathrm{Span}\{E_1, E_2\}$ is a common invariant subspace of A^T and G^T, $\overline{V} = \mathrm{Span}\{\xi\} \subseteq \mathrm{Ker}(D^T)$, and $\mathrm{Ker}(G^T) \cap \mathfrak{E} = \{0\}$. By Theorem 7.3, system (7.18) is not exactly synchronizable by any given boundary control h. In fact, noting that $C_1^- = (1, -1)$, then $C_1^- D = 0$, the reduced system of system (7.18) is not exactly null controllable, thus system (7.18) is not exactly synchronizable.

8. Two-sided control case

We can similarly study the exact boundary synchronization for the following system with boundary controls acting on both sides of the boundary:

$$\begin{cases} U_t + \Lambda U_x + AU = 0, & t \in (0, +\infty), \quad x \in (0, L), \\ U^+(t, 0) = G_0 U^-(t, 0) + D_0 H^+(t), & t \in (0, +\infty), \\ U^-(t, L) = G_1 U^+(t, L) + D_1 H^-(t), & t \in (0, +\infty), \end{cases} \tag{8.1}$$

where U, Λ, A, G_0 and G_1 are as mentioned before, the boundary control matrices D_0 and D_1 are full column-rank matrices of order $N^+ \times M_0$ and $N^- \times M_1$ with $M_0 \leqslant N^+$ and $M_1 \leqslant N^-$, respectively. The boundary control is given by $H = (H^-, H^+)^T$ with $H^- = (h_1, \ldots, h_{M_1})^T$, $H^+ = (h_{M_1+1}, \ldots, h_M)^T$ and $M = M_0 + M_1$. Let N^- and N^+ be given by (2.1), and m and $\bar{m}$ be given by (2.5) as before.

Theorem 8.1. *Assume that A and G satisfy conditions of C_1-compatibility (3.1) and (3.2), respectively. If*

$$\mathit{rank}(C_1^+ D_0) = N^+ - \bar{m} \quad \mathit{and} \quad \mathit{rank}(C_1^- D_1) = N^- - m, \tag{8.2}$$

then system (8.1) is exactly synchronizable with (3.27).

Remark 8.2. Theorem 4.1 and Theorem 4.2 are also true for system (8.1).

Theorem 8.3. *Assume that A and G satisfy conditions of C_1-compatibility (3.1) and (3.2), respectively. Assume furthermore that $E_i (i = 1, \ldots, n)$ given by (4.12) are eigenvectors of Λ, and Span$\{E_1, \ldots, E_n\}$ is a common invariant subspace of A^T and G^T. Then there exist boundary control matrices D_0 and D_1 satisfying*

$$\mathit{rank}(C_1^+ D_0) = \mathit{rank}(D_0) = N^+ - \bar{m} \quad \mathit{and} \quad \mathit{rank}(C_1^- D_1) = \mathit{rank}(D_1) = N^- - m, \tag{8.3}$$

such that system (8.1) is exactly synchronizable, and the exactly synchronizable states are independent of applied boundary controls.

Remark 8.4. The boundary control matrices D_0 and D_1 satisfying (8.3) can be defined by

$$\mathrm{Ker}(D_0^T) = \mathrm{Span}\{\varepsilon_{m+1}, \ldots, \varepsilon_n\} \quad \text{and} \quad \mathrm{Ker}(D_1^T) = \mathrm{Span}\{\varepsilon_1, \ldots, \varepsilon_m\}, \tag{8.4}$$

where $\varepsilon_i (i = 1, \ldots, n)$ are given by (4.12).

For the reverse side, the two sided control case is different from the one-sided control case that, under condition (8.3), the reduced system of system (8.1) is in fact exactly controllable (*cf.* Thm. 2.1 in [27]), thus condition (4.32) in Theorem 4.8 is no longer required.

Theorem 8.5. *Assume that A and G satisfy conditions of C_1-compatibility* (3.1) *and* (3.2)*, respectively. Assume furthermore that system* (8.1) *is exactly synchronizable under condition* (8.3)*. Let U be the solution which realizes the exact boundary synchronization at the time $t = T$. If the projection functions $\phi_i = (E_i, U)$ with E_i given by* (4.12) *for $i = 1, \dots, n$ are independent of applied boundary controls H, then $E_i (i = 1, \dots, n)$ are eigenvectors of Λ, Span$\{E_1, \dots, E_n\}$ is a common invariant subspace of A^T and G^T, and* (8.4) *holds.*

In general, we can still estimate the exactly synchronizable state by the solution to problem (5.2).

Theorem 8.6. *Assume that A and G satisfy conditions of C_1-compatibility* (3.1) *and* (3.2)*, respectively. Assume furthermore that $E_i (i = 1, \dots, n)$ given by* (4.12) *are eigenvectors of Λ, namely,* (4.15) *holds. If system* (8.1) *is exactly synchronizable under condition* (8.2) *with H satisfying* (3.27)*, then the exactly synchronizable state $(\tilde{u}_1(t, x), \dots, \tilde{u}_n(t, x))^T$ satisfies estimate* (5.1).

Remark 8.7. Theorems 6.1–6.2 on the necessity of the conditions of C_1-compatibility also hold true in the two-sided control case.

References

[1] J. Auriol and F. Di Meglio, Minimum time control of heterodirectional linear coupled hyperbolic PDEs. *Autom. J. IFAC* **71** (2016) 300–307.

[2] G. Bastin and J.-M. Coron, Stability and boundary stabilization of 1-D hyperbolic systems, *Progress in Nonlinear Differential Equations and their Applications*, Vol. 88. Birkhäuser/Springer [Cham] (2016).

[3] J.-M. Coron, L. Hu and G. Olive, Finite-time boundary stabilization of general linear hyperbolic balance laws via Fredholm backstepping transformation. *Autom. J. IFAC* **84** (2017) 95–100.

[4] J.-M. Coron and H.-M. Nguyen, Optimal time for the controllability of linear hyperbolic systems in one-dimensional space. *SIAM J. Control Optim.* **57** (2019) 1127–1156.

[5] J.-M. Coron and H.-M. Nguyen, Null-controllability of linear hyperbolic systems in one dimensional space. *Syst. Control Lett.* **148** (2021) 104851.

[6] L. Hu, Sharp time estimates for exact boundary controllability of quasilinear hyperbolic systems. *SIAM J. Control Optim.* **53** (2015) 3383–3410.

[7] L. Hu, F. Di Meglio, R. Vazquez and M. Krstic, Control of homodirectional and general heterodirectional linear coupled hyperbolic PDEs. *IEEE Trans. Automat. Control* **61** (2016) 3301–3314.

[8] G. Kozyreff, A.G. Vladimirov and P. Mandel, Global coupling with time delay in an array of semiconductor laser. *Phys. Rev. Lett.* **85** (2000) 3809–3812.

[9] T. Li, Vol. 3 of Controllability and Observability for Quasilinear Hyperbolic Systems. *AIMS Series on Applied Mathematics.* American Institute of Mathematical Sciences & Higher Education Press (2010).

[10] T. Li, X. Lu and B. Rao, Exact boundary synchronization for a coupled system of wave equations with Neumann boundary controls. *Chin. Ann. Math. Ser. B* **39** (2018) 233–252.

[11] T. Li, X. Lu and B. Rao, Approximate boundary null controllability and approximate boundary synchronization for a coupled system of wave equations with Neumann boundary controls. Contemporary Computational Mathematics — a Celebration of the 80th Birthday of Ian Sloan (edited by J. Dick, F.Y. Kuo, H. Woźniakowski). Springer-Verlag (2018) 837–868.

[12] T. Li, X. Lu and B. Rao, Exact boundary controllability and exact boundary synchronization for a coupled system of wave equations with coupled Robin boundary controls. *ESAIM: COCV* **27** (2021) S7.

[13] T. Li and B. Rao, Synchronisation exacte d'un système couplé d'équations des ondes par des contrôles frontières de Dirichlet. *C.R. Math. Acad. Sci. Paris* **350** (2012) 767–772.

[14] T. Li and B. Rao, Exact synchronization for a coupled system of wave equation with Dirichlet boundary controls. *Chin. Ann. Math. Ser. B* **34** (2013) 139–160.

[15] T. Li and B. Rao, Asymptotic controllability and asymptotic synchronization for a coupled system of wave equations with Dirichlet boundary controls. *Asymptot. Anal.* **86** (2014) 199–226.

[16] T. Li and B. Rao, A note on the exact synchronization by groups for a coupled system of wave equations. *Math. Methods Appl. Sci.* **38** (2015) 241–246.

[17] T. Li and B. Rao, Exact synchronization by groups for a coupled system of wave equations with Dirichlet boundary control. *J. Math. Pures Appl.* **105** (2016) 86–101.

[18] T. Li and B. Rao, On the approximate boundary synchronization for a coupled system of wave equations: direct and indirect boundary controls. *ESAIM: COCV* **24** (2019) 1675–1704.

[19] T. Li and B. Rao, Boundary Synchronization for Hyperbolic Systems, Progress in Nonlinear Differential Equations and their Applications. Subseries in Control, 94. Birkhäuser (2019).

[20] T. Li and B. Rao, Approximate boundary synchronization by groups for a couples system of wave equations with coupled Robin boundary conditions. *ESAIM: COCV* **27** (2021) 10.

[21] T. Li, B. Rao and L. Hu, Exact boundary synchronization for a coupled system of 1-D wave equations. *ESAIM: COCV* **20** (2014) 339-361.

[22] T. Li, B. Rao and Y. Wei, Generalized exact boundary synchronization for a coupled system of wave equations. *Discrete Contin. Dyn. Syst.* **34** (2014) 2893–2905.

[23] J.-L. Lions, Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués, Vol. 1, Masson, Paris (1988).

[24] J.-L. Lions, Exact controllability, stabilization and perturbations for distributed systems. *SIAM Rev.* **30** (1988) 1–68.

[25] X. Lu, Local exact boundary synchronization for a kind of first order quasilinear hyperbolic systems. *Chin. Ann. Math. Ser. B* **40** (2019) 79–96.

[26] X. Lu and T. Li, Exact boundary controllability of weak solutions for a kind of first order hyperbolic system — the constructive method. *Chin. Ann. Math. Ser. B* **42** (2021) 643–676.

[27] X. Lu and T. Li, Exact boundary controllability of weak solutions for a kind of first order hyperbolic system — The HUM method. *Chin. Ann. Math. Ser. B* **43** (2022) 1–16.

[28] X. Lu, T. Li and B. Rao, Exact boundary synchronization by groups for a coupled system of wave equations with coupled Robin boundary controls on a general bounded domain. *SIAM J. Control Optim.* **59** (2021) 4457–4480.

[29] A. Pikovsky, M. Rosenblum and J. Kurths, Synchronization: A Universal Concept in Nonlinear Sciences. Cambridge Univ. Press, Cambridge, UK (2002).

[30] J. Rauch and M. Taylor, Exponential decay of solutions to hyperbolic equations in bounded domains. *Indiana Univ. Math. J.* **24** (1974) 79–86.

[31] D.L. Russell, Controllability and stabilization theory for linear partial differential equations: recent progress and open questions. *SIAM Rev.* **20** (1978) 639–739.

[32] S.H. Strogatz and I. Stewart, Coupled oscillators and biological synchronization. *Sci. Am.* **269** (1993) 102–109.

[33] F. Varela, J. Lachaux, E. Rodriguez and J. Martinerie, The brainweb: phase synchronization and large-scale integration. *Nat. Rev. Neurosci.* **2** (2001) 229–239.

[34] N. Wiener, Cybernetics, or Control and Communication in the Animal and the Machine, 2nd ed. The M.I.T. Press, Cambridge, Mass., John Wiley & Sons, Inc., New York–London (1961).