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CONTROLLABILITY OF A STOKES SYSTEM WITH A DIFFUSIVE BOUNDARY CONDITION

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Abstract. We are interested by the controllability of a fluid-structure interaction system where the fluid is viscous and incompressible and where the structure is elastic and located on a part of the boundary of the fluid domain. In this article, we simplify this system by considering a linearization and by replacing the wave/plate equation for the structure by a heat equation. We show that the corresponding system coupling the Stokes equations with a heat equation at its boundary is null-controllable. The proof is based on Carleman estimates and interpolation inequalities. One of the Carleman estimates corresponds to the case of Ventcel boundary conditions. This work can be seen as a first step to handle the real system where the structure is modeled by the wave or the plate equation.

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1. Introduction

Fluid-structure interaction systems are important systems for many applications such as aerodynamics, medicine (for instance the study of the motion of the blood in veins or in arteries), biology (animal locomotion in a fluid), civil engineering (design of bridges), naval architecture (design of boats and submarines), etc. Moreover, their mathematical studies can be very challenging due to several difficulties: in particular, the complexity of fluid equations such as the Navier-Stokes system, the strong coupling between the fluid system and the structure system and the free-boundary corresponding to the structure displacement.

In this article we consider a simplified fluid–structure interaction system. The corresponding system without simplification has been proposed in [50] as a model for the blood flow in a vessel. It writes as follows: we denote by $\mathcal{I}$ the torus (in order to consider periodic boundary conditions):

$$\mathcal{I} := \mathbb{R}/(2\pi\mathbb{Z}),$$

and for any deformation $\ell : \mathcal{I} \to (-1, \infty)$, we consider the corresponding fluid domain

$$\Omega_\ell = \left\{(x_1, x_2) \in \mathcal{I} \times \mathbb{R} \; ; \; x_2 \in (0, 1 + \ell(x_1))\right\}. \tag{1.1}$$

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Then the equations of motion are

$$\left\{\begin{array}{rl} \partial_t w + (w\cdot\nabla)w - \operatorname{div}\mathbb{T}(w,\pi) = 0 & t>0,\ x\in\Omega_{\ell(t)},\\ \operatorname{div} w = 0 & t>0,\ x\in\Omega_{\ell(t)},\\ w(t,x_1,1+\ell(t,x_1)) = (\partial_t \ell)(t,x_1)e_2 & t>0,\ x_1\in\mathcal{I},\\ w = 0 & t>0,\ x\in\Gamma_0,\\ \partial_{tt}\ell + \alpha_1\partial_{x_1}^4\ell - \alpha_2\partial_{x_1}^2\ell - \delta\partial_t\partial_{x_1}^2\ell = -\widetilde{\mathbb{H}}_\ell(w,\pi) & t>0,\ x_1\in\mathcal{I}, \end{array}\right. \tag{1.2}$$

where

$$\Gamma_0 = \mathcal{I}\times\{0\},\quad \Gamma_1 = \mathcal{I}\times\{1\}.$$

In the above system, we have used the following notations: (e_1, e_2) is the canonical basis of $\mathbb{R}^2$ and

$$\mathbb{T}(w,\pi) = 2D(w) - \pi I_2,\quad D(w) = \frac{1}{2}\left(\nabla w + (\nabla w)^*\right), \tag{1.3}$$

$$\widetilde{\mathbb{H}}_\ell(w,\pi)(t,x_1) = \left[(1+|\partial_{x_1}\ell|^2)^{1/2}\left[\mathbb{T}(w,\pi)n\right](t,x_1,1+\ell(t,x_1))\cdot e_2\right]. \tag{1.4}$$

The two first lines of (1.2) correspond to the Navier-Stokes system for the fluid velocity w and the pressure π. The last line of (1.2) is a beam equation satisfied by the deformation ℓ. We have used the standard no-slip boundary conditions (third and forth equations). To simplify, we assume that the viscosity of the fluid is constant and equal to 1. The vector fields n corresponds to the unit exterior normal to $\Omega_{\ell(t)}$.

This system has been studied by many authors: [16] (existence of weak solutions), [7, 24, 39, 42] (existence of strong solutions), [52] (stabilization of strong solutions), [4] (stabilization of weak solutions around a stationary state). There are also some works in the case $\delta = 0$, that is without damping on the beam equation: the existence of weak solutions is proved in [23] and in [46] (see also [15]). In [25], the existence of local strong solutions is obtained for a structure described by either a wave equation ($\alpha_1 = \delta = 0$ and $\alpha_2 > 0$) or a beam equation with inertia of rotation ($\alpha_1 > 0$, $\alpha_2 = \delta = 0$ and with an additional term $-\partial_{ttss}\ell$). In [5, 6], the authors show the existence and uniqueness of strong solutions in the case $\alpha_1 > 0$, $\alpha_2 \geqslant 0$ and $\delta = 0$. Using similar techniques they also analyze the case of the wave equation ($\alpha_1 = \delta = 0$ and $\alpha_2 > 0$) in [1] showing in particular that the semigroup of the linearized system is analytic. Let us mention also some results for more complex models: [37, 38] (linear elastic Koiter shell), [47] (dynamic pressure boundary conditions), [48, 49] (3D cylindrical domain with nonlinear elastic cylindrical Koiter shell), [58, 59] (nonlinear elastic and thermoelastic plate equations), [41, 43] (compressible fluids), etc.

The advantage of the damping in the beam equation is that the term $-\delta\partial_t\partial_{x_1}^2\ell$ is a structural damping so that the corresponding beam equation becomes a parabolic equation (see, for instance, [18]). In this work, we consider a simplified model associated with (1.2). We neglect the deformation of the fluid domain due to the elastic déformation and we also linearized the Navier-Stokes system by only considering the Stokes system. Moreover, we replace the damped beam equation by a heat equation. By setting

$$\Omega = \mathcal{I}\times(0,1)$$

(see Fig. 1), we are thus considering the following system

$$\left\{\begin{array}{rl} \partial_t w - \Delta w + \nabla\pi = 1_\omega f & \text{in } (0,T)\times\Omega,\\ \operatorname{div} w = 0 & \text{in } (0,T)\times\Omega,\\ w = 0 & \text{on } (0,T)\times\Gamma_0,\\ w = \zeta e_2 & \text{on } (0,T)\times\Gamma_1,\\ \partial_t\zeta - \partial_{x_1x_1}\zeta = -\mathbb{T}(w,\pi)n\cdot e_2 & \text{in } (0,T)\times\mathcal{I},\\ w(0,\cdot) = w^0 \quad \text{in } \Omega, & \zeta(t,0) = \zeta^0 \quad \text{in } \mathcal{I}. \end{array}\right. \tag{1.5}$$

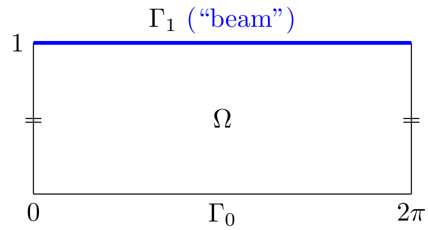

FIGURE 1. Our geometry.

In the above system, ζ corresponds to the displacement velocity $\partial_t \ell$ in (1.2) and we do not consider anymore the displacement position ℓ. We have added a control f localized in the fluid domain, in an arbitrary small nonempty open set ω of Ω. Our goal is to show the null-controllability of this simplified system and to do this, as it is standard (see, for instance, [60], Thm. 11.2.1, p. 357), we prove an observability inequality on the adjoint system:

$$\left\{\begin{array}{rl}
\partial_t u - \Delta u + \nabla p = 0 & \text{in } (0,T)\times\Omega,\\
\operatorname{div} u = 0 & \text{in } (0,T)\times\Omega,\\
u = 0 & \text{on } (0,T)\times\Gamma_0,\\
u = \eta e_2 & \text{on } (0,T)\times\Gamma_1,\\
\partial_t \eta - \partial_{x_1x_1}\eta = -\mathbb{T}(u,p)n_{|\Gamma_1}\cdot e_2 & \text{in } (0,T)\times\mathcal{I},\\
u(0,\cdot) = u^0 \quad \text{in } \Omega, & \eta(0,\cdot) = \eta^0 \quad \text{in } \mathcal{I}.
\end{array}\right. \tag{1.6}$$

We set

$$L^2_0(\mathcal{I}) := \left\{ f \in L^2(\mathcal{I}) \ ; \ \int_0^{2\pi} f(x_1)\ \mathrm{d}x_1 = 0 \right\}$$

and we define the space

$$\mathcal{H} := \left\{ [u,\eta] \in L^2(\Omega)\times L^2_0(\mathcal{I}) \ ; \ u_2 = 0 \text{ on } \Gamma_0, \quad u_2 = \eta \text{ on } \Gamma_1, \ \operatorname{div} u = 0 \text{ in } \Omega \right\}. \tag{1.7}$$

Remark 1.1. Using the particular geometry considered here, we can simplify the adjoint system. First on Γ_1, $n = e_2$ and using (1.3), we deduce

$$-\mathbb{T}(u,p)n\cdot e_2 = -2\partial_{x_2}u_2 + p = 2\partial_{x_1}u_1 + p = p, \tag{1.8}$$

since $u_1(x_1,1) = 0$ for $x_1 \in \mathcal{I}$.

Moreover, using the incompressibility of the fluid and the boundary conditions, we deduce that

$$0 = \int_\Omega \operatorname{div} u \ \mathrm{d}x = \int_{\mathcal{I}} \eta \ \mathrm{d}x_1.$$

Using this condition on the heat equation on the boundary and (1.8) yields

$$\int_{\mathcal{I}} p(x_1,1)\ \mathrm{d}x_1 = 0. \tag{1.9}$$

In particular, in contrast with the standard Stokes system, the pressure is not determined up to a constant.

Our main result states as follows:

Theorem 1.2. *Let $\gamma > 1$ and let ω be a nonempty open set of Ω. Then, there exists $C_0 > 0$ such that for any $T \in (0,1)$ and for any $[u^0, \eta^0] \in \mathcal{H}$, the solution $[u, \eta]$ of (1.6) satisfies*

$$\|[u(T,\cdot), \eta(T,\cdot)]\|_{\mathcal{H}}^2 \leqslant C_0 \exp\left(\frac{C_0}{T^\gamma}\right) \iint_{(0,T)\times\omega} |u|^2 \, \mathrm{d}t\mathrm{d}x.$$

In particular, for any $[w^0, \zeta^0] \in \mathcal{H}$ and for any $T > 0$, there exists a control $f \in L^2((0,T)\times\omega)$ such that the solution $[w, \zeta]$ of (1.5) satisfies

$$w(T,\cdot) = 0, \quad \zeta(T,\cdot) = 0.$$

Remark 1.3. In comparison to the Stokes system with the Dirichlet boundary condition, one should expect that the above result remains true for $\gamma = 1$ (see [17]). However, our method is based on a spectral inequality and on a general result to pass from such a spectral inequality to an observability inequality. In this general result, stated in Theorem 1.8, we have the same restriction on γ. Note that the approach used here was already considered in [35, 36] where the same restriction on γ also appears.

Remark 1.4. As explained above, Theorem 1.2 can be seen as a first control result on a simplified model. We expect to extend some of the tools developed here to handle the control properties of the system (1.2) in future works. The controllability properties of fluid-structure interaction systems have been tackled mainly in the case where the structure is a rigid body (see, [10, 11, 19–21, 26, 40, 51, 54], etc.). In [45], the author shows an observability inequality for the adjoint of a linearized and simplified fluid-structure interaction system in the case of a compressible viscous fluid and of a damped beam. Note that the corresponding control problem involves two controls, one for the fluid and one for the structure. For the stabilization of fluid-structure interaction systems, one can quote some results: [4, 52] (for the case of a damped beam), [2, 3, 55, 57] (for the case of a rigid body).

Remark 1.5. The method proposed here to control a system involving the Stokes equations is quite different from the method used in a large part of the literature for the controllability of fluid systems. In general, the method is based on "global Carleman inequalities" (see, for instance [22, 27]). Here, we follow another strategy as in [17, 36]. Such a method is based on local Carleman inequalities for an "augmented" elliptic operator, from which one deduces a spectral inequality, in the spirit of [28, 36]. We then use an adaptation of the original strategy of [35, 36] in our context. This type of spectral inequality has already been used in the context of fluids in [14]. Controllability for coupled parabolic systems through the derivation of Carleman estimates using microlocal analysis near boundaries and interfaces has been widely studied, one can cite for instance [29, 34] (see also [8, 9, 32, 33] and the recent books [30, 31] for elliptic counterparts).

Remark 1.6. In a large part of the works devoted to the controllability of fluid systems, a first step consists in removing the pressure by taking the rotational operator of the Stokes system (or another operator). As observed in [17], the unique continuation property does not hold for the augmented operator in the direction of the additional variable, due to the pressure. Here, due to the presence of the pressure in the structure equation, we choose, as in [22] to obtain a Carleman estimate of the fluid velocity by considering the pressure as a source term. Then, we obtain a Carleman estimate for the pressure by using that it is an harmonic function. However, due to the lack of boundary condition for the pressure, such an estimate contains boundary terms and one can see that we only need the high frequencies of the boundary value of the pressure. In order to estimate these high frequency terms, we derive a priori estimates of the system. Note that in this part, we use another important feature of our work which consists in considering two different weight functions to separate the characteristic manifolds of the concerned operators and perform proper elliptic estimates.

Remark 1.7. Let us point out that one can use Theorem 1.2 to handle nonlinear controllability issues by applying the general method proposed in [40]. For instance, one could replace in (1.5) the Stokes system by the Navier-Stokes system and by using a fixed-point argument, we would obtain the null-controllability of the corresponding system for small initial conditions

$$\left[w^0, \zeta^0\right] \in \mathcal{H} \cap \left(H^1(\Omega) \times H^1(\mathcal{I})\right).$$

An important ingredient in the proof of Theorem 1.2 is a general result showing that one can deduce the final-state observability (and thus the null-controllability) of a parabolic system from a spectral inequality. Similar results can be found in [35, 36] but here, the main differences is that we assume that our operator is self-adjoint and that we do not assume the Weyl asymptotic formula that are needed in the previous references. The proof is similar to the proofs in [35, 36], but for sake of completeness, we state below the result and prove it in the next section.

Let us consider a general Hilbert space $\mathcal{H}$ (not necessarily defined by (1.7)) and a nonnegative self-adjoint operator $A : \mathcal{D}(A) \to \mathcal{H}$ with compact resolvents. We denote by (λ_j) the nondecreasing sequence of eigenvalues and by (w_j) an orthonormal basis of $\mathcal{H}$ composed by eigenvectors of A: $Aw_j = \lambda_j w_j$ for $j \geqslant 1$. We also consider a control operator $B \in \mathcal{L}(\mathcal{U}, \mathcal{H})$.

Theorem 1.8. *Assume the above hypotheses. Assume moreover the existence of $S_0 > 0$, $C > 0$ and $\kappa \in C_0^\infty(0, S_0)$ such that that for any $\Lambda > 0$, and for any $(a_j)_j \in \mathbb{C}^{\mathbb{N}}$,*

$$\sum_{\lambda_j \leqslant \Lambda} |a_j|^2 \leqslant C e^{C\sqrt{\Lambda}} \int_0^{S_0} \kappa^2(s) \left\| \sum_{\lambda_j \leqslant \Lambda} a_j \cosh(s\sqrt{\lambda_j}) B^* w^{(j)} \right\|_{\mathcal{U}}^2 \, \mathrm{d}s. \tag{1.10}$$

Then for all $\gamma > 1$, there exists $C_0 > 0$ such that for any $T \in (0,1)$ and for any $z^0 \in \mathcal{H}$,

$$\left\| e^{-TA} z^0 \right\|_{\mathcal{H}}^2 \leqslant C_0 \exp\left(\frac{C_0}{T^\gamma}\right) \int_0^T \left\| B^* e^{-tA} z^0 \right\|_{\mathcal{U}}^2 \, \mathrm{d}t. \tag{1.11}$$

We recall that relation (1.11) implies the null-controllability of the system

$$\begin{cases} \dfrac{\mathrm{d}\theta}{\mathrm{d}t} + A\theta = Bg \quad \text{in } (0,T), \\ \theta(0) = \theta^0 \in \mathcal{H}. \end{cases} \tag{1.12}$$

The outline of the article is as follows: in Section 2, we prove Theorem 1.8 stated above. Then using this general result, we are reduced to show a spectral inequality that we state in Section 3 along with the functional framework. The spectral inequality is itself the consequence of an interpolation inequality that we obtain in Section 5. One of the main difficulties to obtain such an inequality comes from the fact that we need to estimate the pressure. Section 4 is devoted to such an estimate which is one of the main parts of this article. The proof of the spectral inequality and thus of Theorem 1.2 is obtained at the end of Section 5. In the appendix, we show an interpolation estimate for the Ventcel boundary condition that is mainly a consequence of a Carleman estimate obtained in [13].

Notation 1.9. *In the whole paper, we use C as a generic positive constant that does not depend on the other terms of the inequality. The value of the constant C may change from one appearance to another. We also use the notation $X \lesssim Y$ if there exists a constant $C > 0$ such that we have the inequality $X \leqslant CY$. In what follows, norms in the interior and at the boundary are denoted respectively by $\|\cdot\|$ and $|\cdot|$.*

2. From a spectral inequality to the null-controllability

This section is devoted to the proof of Theorem 1.8. The proof follows closely the proof in [35, 36].

2.1. Controllability of the first modes

We define

$$\mathcal{H}_\Lambda = \operatorname{span}\{w_j,\ \lambda_j \leqslant \Lambda\}, \quad \Pi_\Lambda : \mathcal{H} \to \mathcal{H}_\Lambda \text{ the orthogonal projection.}$$

We are interested here by the control problem

$$\begin{cases} \dfrac{\mathrm{d}\theta}{\mathrm{d}t} + A\theta = \Pi_\Lambda B g \quad \text{in } (0,\tau), \\ \theta(0) = \theta^0 \in \mathcal{H}_\Lambda, \end{cases} \tag{2.1}$$

for some $\tau > 0$. We first prove an observability inequality associated with the above control problem, with an estimate of the cost of the control with respect to τ and Λ:

Proposition 2.1. *Let $\gamma > 1$. There exists $C_1 > 0$ such that for all $\tau > 0$ and for all $\Lambda > 1$*

$$\left\|\Pi_\Lambda e^{-\tau A} z^0\right\|_{\mathcal{H}}^2 \leqslant C_1 \exp\left(C_1\left(\frac{1}{\tau^\gamma} + \sqrt{\Lambda} + (\Lambda\tau)^{\gamma/(2\gamma-1)}\right)\right)\int_0^\tau \left\|B^*\Pi_\Lambda e^{-tA} z^0\right\|_{\mathcal{U}}^2 \ \mathrm{d}t \tag{2.2}$$

for all $z_0 \in \mathcal{H}_\Lambda$.

Proof. We follow [35, 36]: the proof is based on the use of a Gramian matrix G_Λ that we conjugate with an adequate matrix. Using the Paley-Wiener theorem, we construct and estimate a control for (2.1) so that $\theta(\tau) = 0$.

Let us define the linear operator

$$G_\Lambda := \int_0^{S_0} \kappa^2(s)\cosh(s\sqrt{A})\Pi_\Lambda B B^* \cosh(s\sqrt{A})\Pi_\Lambda \ \mathrm{d}s.$$

From (1.10), $G_\Lambda \in \mathcal{L}(\mathcal{H}_\Lambda)$ is symmetric, positive and invertible with

$$\left\|G_\Lambda^{-1}\right\|_{\mathcal{L}(\mathcal{H}_\Lambda)} \leqslant C e^{C\sqrt{\Lambda}}.$$

We set

$$\sigma := 2 - \frac{1}{\gamma} \in (1,2).$$

From Lemma A.1 of [35], there exists $e \in C^\infty(\mathbb{R})$ such that for some constants c_j

$$\operatorname{supp} e = [0,1], \tag{2.3}$$

$$|\widehat{e}(z)| \leqslant c_1 e^{-c_2|z|^{1/\sigma}} \quad \text{if } \operatorname{Im}(z) \leqslant 0, \tag{2.4}$$

$$|\widehat{e}(z)| \geqslant c_3 e^{-c_4|z|^{1/\sigma}} \quad \text{if } z \in i\mathbb{R}^-. \tag{2.5}$$

From (2.5), we have that $\widehat{e}(-i\tau A) \in \mathcal{L}(\mathcal{H}_\Lambda)$ is invertible and

$$\left\|\widehat{e}(-i\tau A)^{-1}\right\|_{\mathcal{L}(\mathcal{H}_\Lambda)} \leqslant \frac{1}{c_3} e^{c_4(\Lambda\tau)^{1/\sigma}}. \tag{2.6}$$

Then we define

$$h_\Lambda(s) := -\frac{1}{2}\kappa^2(s)B^*\cosh(s\sqrt{A})G_\Lambda^{-1}\widehat{e}(-i\tau A)^{-1}e^{-\tau A}\theta^0 \quad (s\in\mathbb{R}).$$

We have that $h_\Lambda \in C_0^\infty(\mathbb{R},\mathcal{U})$ with $\operatorname{supp} h_\Lambda \subset (0,S_0)$ and

$$\|h_\Lambda\|_{L^\infty(\mathbb{R},\mathcal{U})} \leqslant Ce^{C\sqrt{\Lambda}+c_4(\Lambda\tau)^{1/\sigma}} \left\|\theta^0\right\|_{\mathcal{H}}. \tag{2.7}$$

Thus, $\widehat{h}_\Lambda \in \mathrm{Hol}(\mathbb{C};\mathcal{U})$ and

$$\left\|\widehat{h}_\Lambda(z)\right\|_{\mathcal{U}} \leqslant Ce^{C\sqrt{\Lambda}+c_4(\Lambda\tau)^{1/\sigma}} e^{S_0|\mathrm{Im}(z)|}\left\|\theta^0\right\|_{\mathcal{H}}. \tag{2.8}$$

As in [56], we introduce $Q_\Lambda \in \mathrm{Hol}(\mathbb{C};\mathcal{U})$ such that

$$Q_\Lambda(-iz^2) = \widehat{h}_\Lambda(iz) + \widehat{h}_\Lambda(-iz) \quad (z\in\mathbb{C}).$$

We deduce from the above relation and (2.8) that

$$\|Q_\Lambda(z)\|_{\mathcal{U}} \leqslant Ce^{C\sqrt{\Lambda}+c_4(\Lambda\tau)^{1/\sigma}} e^{S_0\sqrt{|z|}}\left\|\theta^0\right\|_{\mathcal{H}}. \tag{2.9}$$

We define $\mathfrak{g}_\Lambda \in \mathrm{Hol}(\mathbb{C};\mathcal{U})$ by

$$\mathfrak{g}_\Lambda(z) := \widehat{e}(\tau z)Q_\Lambda(z).$$

From (2.3), (2.4) and (2.9), we have

$$\|\mathfrak{g}_\Lambda(z)\|_{\mathcal{U}} \leqslant Ce^{C\sqrt{\Lambda}+c_4(\Lambda\tau)^{1/\sigma}} e^{S_0\sqrt{|z|}}e^{\tau|\operatorname{Im} z|}\left\|\theta^0\right\|_{\mathcal{H}} \quad (z\in\mathbb{C}) \tag{2.10}$$

and

$$\|\mathfrak{g}_\Lambda(z)\|_{\mathcal{U}} \leqslant Ce^{C\sqrt{\Lambda}+c_4(\Lambda\tau)^{1/\sigma}} e^{S_0\sqrt{|z|}}e^{-c_2\tau^{1/\sigma}|z|^{1/\sigma}}\left\|\theta^0\right\|_{\mathcal{H}} \quad \text{if } \operatorname{Im} z\leqslant 0. \tag{2.11}$$

Since $\sigma<2$, we can use a Paley-Wiener type theorem (see [35], Prop. A.3) and deduce the existence of $g_\Lambda \in C_0^\infty((0,\tau);\mathcal{U})$ such that

$$\widehat{g}_\Lambda(z) = \mathfrak{g}_\Lambda(z).$$

In particular, from (2.11) and the Laplace method, for all $t\in(0,\tau)$,

$$\|g_\Lambda(t)\|_{\mathcal{U}} \leqslant \|\mathfrak{g}_\Lambda\|_{L^1(\mathbb{R};\mathcal{U})} \leqslant Ce^{C\sqrt{\Lambda}+c_4(\Lambda\tau)^{1/\sigma}+\frac{C}{\tau^{1/(2-\sigma)}}}\left\|\theta^0\right\|_{\mathcal{H}}. \tag{2.12}$$

Now, for any j such that $\lambda_j\leqslant\Lambda$,

$$\begin{aligned}\left(\int_0^\tau e^{-(\tau-t)A}Bg_\Lambda(\tau-t)\ \mathrm{d}t, w_j\right)_{\mathcal{H}} &= (B\widehat{g}_\Lambda(-i\lambda_j), w_j)_{\mathcal{H}}\\ &= \left(\widehat{e}(-i\tau A)B\left(\widehat{h}_\Lambda(i\sqrt{\lambda_j})+\widehat{h}_\Lambda(-i\sqrt{\lambda_j})\right), w_j\right)_{\mathcal{H}}\end{aligned}$$

$$= \left(\widehat{e}(-i\tau A) B \int_0^{S_0} h_\Lambda(s) 2\cosh(s\sqrt{\lambda_j})\, \mathrm{d}s, w_j \right)_{\mathcal{H}} = -\left(e^{-\tau A}\theta^0, w_j\right)_{\mathcal{H}}$$

so that the solution θ of (2.1) with the control $g(t) = g_\Lambda(\tau - t)$ satisfies $\theta(\tau) = 0$.

By a duality argument and (2.12), this implies that (2.2). □

2.2. Proof of Theorem 1.8

We are now in a position to prove Theorem 1.8, adapting the method of [44]. The proof is based on Theorem 2.1 and on the exponential decay of the semigroup associated with A for high frequencies. This allows us to obtain an observability estimate between two times and then we apply this estimate between $T/2^{k+1}$ and $T/2^k$, $k \geqslant 0$.

Proof of Theorem 1.8. We set

$$z(t) = e^{-tA} z^0.$$

Assume

$$T^{(1)} > 0, \quad \tau > 0, \quad \text{and} \quad T^{(2)} = T^{(1)} + \tau.$$

From (2.2)

$$\left\| \Pi_\Lambda z\left(T^{(2)}\right) \right\|_{\mathcal{H}}^2 \leqslant C_1 \exp\left(C_1 \left(\frac{1}{(\varepsilon\tau)^\gamma} + \sqrt{\Lambda} + (\Lambda\varepsilon\tau)^{\gamma/(2\gamma-1)} \right) \right) \int_{T^{(2)}-\varepsilon\tau}^{T^{(2)}} \left\| B^* \Pi_\Lambda z(t) \right\|_{\mathcal{U}}^2 \, \mathrm{d}t. \tag{2.13}$$

We set

$$\Lambda = \frac{1}{(\varepsilon\tau)^{1+\gamma}} \tag{2.14}$$

so that for

$$\tau, \varepsilon \in (0,1),$$

(2.13) becomes

$$2\rho(\tau) \left\| \Pi_\Lambda z\left(T^{(2)}\right) \right\|_{\mathcal{H}}^2 \leqslant \int_{T^{(2)}-\varepsilon\tau}^{T^{(2)}} \left\| B^* \Pi_\Lambda z(t) \right\|_{\mathcal{U}}^2 \, \mathrm{d}t \tag{2.15}$$

with

$$\rho(\tau) := \frac{1}{2C_1} \exp\left(-\frac{3C_1}{(\varepsilon\tau)^\gamma} \right). \tag{2.16}$$

Then from (2.15), we deduce

$$\rho(\tau) \left\| z\left(T^{(2)}\right) \right\|_{\mathcal{H}}^2 \leqslant \int_{T^{(2)}-\varepsilon\tau}^{T^{(2)}} \left\| B^* z(t) \right\|_{\mathcal{U}}^2 \, \mathrm{d}t + C \left\| z\left(T^{(1)}\right) \right\|_{\mathcal{H}}^2 \varepsilon\tau e^{-2\Lambda\tau(1-\varepsilon)} + \rho(\tau) \left\| z\left(T^{(1)}\right) \right\|_{\mathcal{H}}^2 e^{-2\Lambda\tau} \tag{2.17}$$

For $\varepsilon > 0$ small enough, the above relation yields

$$\rho(\tau)\left\|z\left(T^{(2)}\right)\right\|_{\mathcal{H}}^{2} \leqslant \int_{T^{(1)}}^{T^{(2)}} \|B^* z(t)\|_{\mathcal{U}}^2 \, \mathrm{d}t + \rho(\tau/2)\left\|z\left(T^{(1)}\right)\right\|_{\mathcal{H}}^{2} \tag{2.18}$$

Assume

$$T \in (0,1).$$

Then for all $k \geqslant 0$, (2.18) implies

$$\rho\left(\frac{T}{2^{k+1}}\right)\left\|z\left(\frac{T}{2^{k}}\right)\right\|_{\mathcal{H}}^{2} \leqslant \int_{\frac{T}{2^{k+1}}}^{\frac{T}{2^{k}}} \|B^* z(t)\|_{\mathcal{U}}^2 \, \mathrm{d}t + \rho\left(\frac{T}{2^{k+2}}\right)\left\|z\left(\frac{T}{2^{k+1}}\right)\right\|_{\mathcal{H}}^{2} \tag{2.19}$$

and thus

$$\rho\left(\frac{T}{2}\right)\|z(T)\|_{\mathcal{H}}^{2} \leqslant \int_{0}^{T} \|B^* z(t)\|_{\mathcal{U}}^2 \, \mathrm{d}t. \tag{2.20}$$

Thus for some constant $C_2 > 0$,

$$\|z(T)\|_{\mathcal{H}}^{2} \leqslant C_2 \exp\left(\frac{C_2}{T^{\gamma}}\right)\int_{0}^{T} \|B^* z(t)\|_{\mathcal{U}}^2 \, \mathrm{d}t. \tag{2.21}$$

□

3. Functional framework and spectral inequality

In order to prove Theorem 1.2, we are going to apply Theorem 1.8. In this section, we first give the functional framework associated with (1.6). Then we write the spectral inequality that will be proven in the remaining part of the article.

3.1. Functional framework

We recall that $\mathcal{H}$ is defined by (1.7). We also define

$$\mathcal{V} := \left\{[u,\eta] \in \left(H^1(\Omega)\times H^1(\mathcal{I})\right)\cap\mathcal{H}\ ;\ u_1 = 0 \text{ on } \partial\Omega\right\}.$$

We denote by P_0 the orthogonal projection from $L^2(\Omega)\times L^2_0(\mathcal{I})$ onto $\mathcal{H}$. We now define the linear operator $A_0 : \mathcal{D}(A_0) \subset \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{D}(A_0) := \mathcal{V} \cap \left[H^2(\Omega)\times H^2(\mathcal{I})\right], \tag{3.1}$$

and for $\left[u,\eta\right] \in \mathcal{D}(A_0)$, we set

$$A_0\begin{bmatrix} u \\ \eta \end{bmatrix} := P_0\begin{bmatrix} \Delta u \\ \partial_{x_1}^2 \eta \end{bmatrix}. \tag{3.2}$$

Then one can check that (1.6) writes

$$\left\{\begin{array}{l}\dfrac{\mathrm{d}z}{\mathrm{d}t} = A_0 z \quad \text{in } (0,T),\\ z(0) = z^0,\end{array}\right.$$

with $z = [u, \eta]$, $z^0 = [u^0, \eta^0]$. In the next proposition, we show in particular that A_0 is the infinitesimal generator of a semigroup so that $z(t) = e^{tA_0} z^0$ for $t \geqslant 0$.

Proposition 3.1. *The operator A_0 defined by (3.1)–(3.2) has compact resolvents, and is self-adjoint negative on $\mathcal{H}$. For all $k \geqslant 1$, there exists a constant C such that*

$$\|[u,\eta]\|_{H^{2k}(\Omega)\times H^{2k}(\mathcal{I})} \leqslant C \left\|A_0^k[u,\eta]\right\|_{\mathcal{H}}. \tag{3.3}$$

Proof. By definition of A_0, we have for any $[u, \eta], [v, \zeta] \in \mathcal{D}(A_0)$,

$$\left\langle A_0 \begin{bmatrix} u \\ \eta \end{bmatrix}, \begin{bmatrix} v \\ \zeta \end{bmatrix} \right\rangle_{\mathcal{H}} = \int_\Omega \Delta u \cdot v \ \mathrm{d}x + \int_0^{2\pi} \left(\partial_{x_1}^2 \eta\right) \zeta \ \mathrm{d}x_1 = -\int_\Omega \nabla u : \nabla v \ \mathrm{d}x - \int_0^{2\pi} \left(\partial_{x_1}\eta\right)\left(\partial_{x_1}\zeta\right) \ \mathrm{d}x_1.$$

Thus A_0 is symmetric and negative (by using the Poincaré inequalities).

In order to show that A_0 is self-adjoint it is sufficient to show that it is onto. Assume $[f, g] \in \mathcal{H}$ and let us solve the equation

$$-A_0 \begin{bmatrix} u \\ \eta \end{bmatrix} = [f, g]. \tag{3.4}$$

Multiplying the above equation by $[v, \zeta] \in \mathcal{V}$ leads to the weak formulation

$$\int_\Omega \nabla u : \nabla v \ \mathrm{d}x + \int_0^{2\pi} \left(\partial_{x_1}\eta\right)\left(\partial_{x_1}\zeta\right) \ \mathrm{d}x_1 = \int_\Omega f \cdot v \ \mathrm{d}x + \int_0^{2\pi} g\zeta \ \mathrm{d}x_1 \quad ([v,\zeta] \in \mathcal{V}). \tag{3.5}$$

Using the Poincaré inequalities, we see that we can apply the Riesz theorem and deduce the existence and uniqueness of $[u, \eta] \in \mathcal{V}$ solution of (3.5). Then if $v \in C_c^\infty(\Omega)$ with $\operatorname{div} v = 0$ and $\zeta = 0$ in (3.5), we obtain that

$$\int_\Omega \nabla u : \nabla v \ \mathrm{d}x = \int_\Omega f \cdot v \ \mathrm{d}x \quad (v \in C_c^\infty(\Omega), \ \operatorname{div} v = 0). \tag{3.6}$$

Using the De Rham theorem, we deduce the existence of p such that

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega,\\ \operatorname{div} u = 0 & \text{in } \Omega,\\ u = 0 & \text{on } \Gamma_0\\ u = \eta e_2 & \text{on } \Gamma_1. \end{cases} \tag{3.7}$$

Using the elliptic regularity of the Stokes system, we deduce that $(u, p) \in H^{3/2}(\Omega) \times H^{1/2}(\Omega)$. Multiplying the first above equation by v, with $[v, \zeta] \in \mathcal{V}$, we deduce that for any $\zeta \in H^1(\mathcal{I}) \cap L^2_0(\mathcal{I})$,

$$\int_0^{2\pi} \left(\partial_{x_1}\eta\right)\left(\partial_{x_1}\zeta\right) \ \mathrm{d}x_1 = -\langle p_{|\Gamma_1}, \zeta\rangle + \int_0^{2\pi} g\zeta \ \mathrm{d}x_1. \tag{3.8}$$

Since $p_{|\Gamma_1} = ([pI_3 - \nabla u]\, n)_{|\Gamma_1} \in H^{-1/2}(\mathcal{I})$, we deduce that $\eta \in H^{3/2}(\mathcal{I})$ and from (3.7) that $(u, p) \in H^2(\Omega) \times H^1(\Omega)$. Thus $p_{|\Gamma_1} \in H^{1/2}(\mathcal{I})$, and from (3.8), we deduce that $\eta \in H^2(\mathcal{I})$. We conclude that $[u, \eta] \in \mathcal{D}(A_0)$ and satisfies (3.4). We also obtain relation (3.3) for $k = 1$ and with a similar argument we can also obtain relation (3.3) for any $k \geqslant 1$.

The fact that A_0 has compact resolvents is coming from the compact embedding of H^2 into L^2 for bounded domains. □

In particular, the eigenvalues $\lambda_j > 0$ of $-A_0$ satisfy $\lambda_j \to \infty$ and there exists

$$\left(\begin{bmatrix} u^{(j)} \\ \eta^{(j)} \end{bmatrix} \right)_j \quad \text{orthonormal basis of} \quad \mathcal{H} \tag{3.9}$$

composed by eigenvectors of A_0:

$$-A_0 \begin{bmatrix} u^{(j)} \\ \eta^{(j)} \end{bmatrix} = \lambda_j \begin{bmatrix} u^{(j)} \\ \eta^{(j)} \end{bmatrix}. \tag{3.10}$$

The above system can be written as

$$\begin{cases} -\Delta u^{(j)} + \nabla p^{(j)} & = \lambda_j u^{(j)} & \text{in } \Omega, \\ \operatorname{div} u^{(j)} & = 0 & \text{in } \Omega, \\ u^{(j)} & = 0 & \text{on } \Gamma_0 \\ u^{(j)} & = \eta^{(j)} e_2 & \text{on } \Gamma_1 \\ -\partial_{x_1}^2 \eta^{(j)} - p^{(j)} & = \lambda_j \eta^{(j)} & \text{in } \mathcal{I} \end{cases} \tag{3.11}$$

and more precisely as

$$\begin{cases} -\lambda_j u_1^{(j)} - (\partial_{x_1}^2 + \partial_{x_2}^2) u_1^{(j)} + \partial_{x_1} p^{(j)} & = 0 & \text{in } \Omega, \\ -\lambda_j u_2^{(j)} - (\partial_{x_1}^2 + \partial_{x_2}^2) u_2^{(j)} + \partial_{x_2} p^{(j)} & = 0 & \text{in } \Omega, \\ \partial_{x_1} u_1^{(j)} + \partial_{x_2} u_2^{(j)} & = 0 & \text{in } \Omega, \\ u_1^{(j)} & = 0 & \text{on } \partial\Omega \\ u_2^{(j)} & = 0 & \text{on } \Gamma_0 \\ -\lambda_j u_2^{(j)} - \partial_{x_1}^2 u_2^{(j)} & = p^{(j)} & \text{on } \Gamma_1. \end{cases} \tag{3.12}$$

Note that, as explained in Theorem 1.1, for all $j \geqslant 1$, the pressure p^j satisfies the relation $\int_{\mathcal{I}} p^{(j)}(x_1, 1)\, dx_1 = 0$ and in particular, p^j is uniquely determined. This differs from the standard Stokes system where the pressure is determined up to constant.

3.2. Spectral inequality

We are now in a position to state the spectral inequality for the operator A_0 defined in the previous section.

Theorem 3.2. *Let ω_0 be a nonempty open subset of Ω and $S_0 > 0$. There exist $C > 0$ and $\kappa \in C_0^\infty(0, S_0)$ such that for any $\Lambda > 0$, and for any $(a_j)_j \in \mathbb{C}^{\mathbb{N}}$,*

$$\sum_{\lambda_j \leqslant \Lambda} |a_j|^2 \leqslant C e^{C\sqrt{\Lambda}} \int_0^{S_0} \kappa^2(s) \left\| \sum_{\lambda_j \leqslant \Lambda} a_j \cosh(s\sqrt{\lambda_j}) u^{(j)} \right\|_{L^2(\omega)}^2 \, ds. \tag{3.13}$$

In order to prove Theorem 3.2, we define for $s \in (0, S_0)$ and $x \in \Omega$,

$$U(s,x) := \sum_{\lambda_j \leqslant \Lambda} a_j \cosh(\sqrt{\lambda_j} s) u^{(j)}(x), \quad P(s,x) := \sum_{\lambda_j \leqslant \Lambda} a_j \cosh(\sqrt{\lambda_j} s) p^{(j)}(x) + c_P(s) \tag{3.14}$$

and the domains

$$Z := (0, S_0) \times \Omega = (0, S_0) \times \mathcal{I} \times (0,1), \quad J_i := (0, S_0) \times \Gamma_i = (0, S_0) \times \mathcal{I} \times \{i\} \quad (i = 0, 1). \tag{3.15}$$

From (3.12), we deduce that

$$\begin{cases} -\partial_s^2 U_1 - (\partial_{x_1}^2 + \partial_{x_2}^2) U_1 + \partial_{x_1} P = 0 \quad \text{in } Z, \\ -\partial_s^2 U_2 - (\partial_{x_1}^2 + \partial_{x_2}^2) U_2 + \partial_{x_2} P = 0 \quad \text{in } Z, \\ \partial_{x_1} U_1 + \partial_{x_2} U_2 = 0 \quad \text{in } Z, \\ U_1 = 0 \quad \text{on } J_0 \cup J_1 \\ U_2 = 0 \quad \text{on } J_0, \\ -\partial_s^2 U_2 - \partial_{x_1}^2 U_2 = P - m_{\mathcal{I}}(P) \quad \text{on } J_1. \end{cases} \tag{3.16}$$

In the above system, we write

$$m_{\mathcal{I}}(P) := \frac{1}{2\pi} \int_0^{2\pi} P(x_1, 1) \, \mathrm{d}x_1$$

and by using this notation in the last equation of (3.16), we can replace the pressure that should satisfies a relation of the form (1.9) by the pressure P defined up to a function c_P of s. In that way, we can, in what follows, impose another condition on P (typically that its mean on an open set is zero).

To show Theorem 3.2, we first truncate U and P in a neighborhood of $\{s = s_0\}$, with

$$s_0 := \frac{S_0}{2}.$$

We thus consider $\chi \in C_0^\infty((0, S_0))$, satisfying $0 \leqslant \chi \leqslant 1$ and

$$\chi(s) = \begin{cases} 1 & \text{if } |s - s_0| \leqslant S_0/8, \\ 0 & \text{if } |s - s_0| \geqslant S_0/6. \end{cases} \tag{3.17}$$

We work with the following localized solutions

$$u(s, x_1, x_2) := \chi(s) U(s, x_1, x_2), \quad p(s, x_1, x_2) := \chi(s) P(s, x_1, x_2) \tag{3.18}$$

that satisfy

$$\begin{cases} -\partial_s^2 u_1 - (\partial_{x_1}^2 + \partial_{x_2}^2) u_1 + \partial_{x_1} p = f_1 \quad \text{in } Z \\ -\partial_s^2 u_2 - (\partial_{x_1}^2 + \partial_{x_2}^2) u_2 + \partial_{x_2} p = f_2 \quad \text{in } Z \\ \partial_{x_1} u_1 + \partial_{x_2} u_2 = 0 \quad \text{in } Z \\ u_1 = 0 \quad \text{on } J_0 \cup J_1 \\ u_2 = 0 \quad \text{on } J_0 \\ -\partial_s^2 u_2 - \partial_{x_1}^2 u_2 = f_3 + p - m_{\mathcal{I}}(p) \quad \text{on } J_1, \end{cases} \tag{3.19}$$

where

$$f_1 := -\chi'' U_1 - 2\chi' \partial_s U_1, \quad f_2 := -\chi'' U_2 - 2\chi' \partial_s U_2, \qquad f_3 := -\chi''(U_2)_{|J_1} - 2\chi'(\partial_s U_2)_{|J_1}. \tag{3.20}$$

We also have that

$$u = 0 \quad \text{and} \quad p = 0 \quad \text{if } s \notin \left[\frac{1}{3}S_0, \frac{2}{3}S_0\right]. \tag{3.21}$$

As usual, we can use the three first equations to obtain the following equation for the pressure:

$$-\Delta p = -(\partial_{x_1}^2 + \partial_{x_2}^2)p = \partial_{x_1} f_1 + \partial_{x_2} f_2 = 0. \tag{3.22}$$

4. A global observability estimate on the pressure

In this section, we prove a global estimate on the pressure. We first introduce our weight and the corresponding conjugated operators. We then state our main result, that is Theorem 4.1. Then we show a first estimate on the pressure involving high frequency pressure terms at the boundary. Such terms are then estimated by showing some a priori estimates and this allows us to prove Theorem 4.1.

4.1. Choice of the weight and conjugated operators

Let us consider a nonempty open set ω_0 such that $\overline{\omega_0} \subset \omega$. Let

$$\lambda > 0, \tau > 0.$$

Then we consider $\widetilde{\psi} \in C^\infty(\overline{\Omega}; \mathbb{R}^+)$, such that

$$\widetilde{\psi}(x_1, x_2) = 1 - x_2 \text{ in a neighborhood of } \{x_2 = 1\}, \text{ and } \widetilde{\psi}(x_1, x_2) = x_2 \text{ in a neighborhood of } \{x_2 = 0\} \tag{4.1}$$

and such that all its critical points belong to ω_0:

$$\nabla \widetilde{\psi}(x) = 0 \implies x \in \omega_0. \tag{4.2}$$

We set

$$\varphi(s, x) := e^{\lambda \widetilde{\psi}(x) - (s - s_0)^2}, \quad \varphi_0(s) := e^{-(s-s_0)^2}. \tag{4.3}$$

Note that with our above choices,

$$\varphi_0(s) = \varphi(s, \cdot, 0) = \varphi(s, \cdot, 1) = \min_{x \in \overline{\Omega}} \varphi(s, x).$$

We recall that we define (u, p) from (U, P) by (3.18) (truncation in s) and that the source f_i are defined by (3.20). We then define

$$v := e^{\tau\varphi} u, \quad q := e^{\tau\varphi} p, \quad g_i := e^{\tau\varphi} f_i \quad (i \in \{1, \dots, 3\}). \tag{4.4}$$

In order to take into account the dependence in s, we write

$$z=(s,x)=(s,x_1,x_2)\in Z,\quad \nabla_z=\begin{bmatrix}\partial_s\\ \partial_{x_1}\\ \partial_{x_2}\end{bmatrix},\quad \Delta_z=\partial_s^2+\partial_{x_1}^2+\partial_{x_2}^2,$$

and their tangential counterparts

$$\nabla_{s,x_1}=\begin{bmatrix}\partial_s\\ \partial_{x_1}\end{bmatrix},\quad \Delta_{s,x_1}=\partial_s^2+\partial_{x_1}^2.$$

We keep our previous notation

$$\nabla=\begin{bmatrix}\partial_{x_1}\\ \partial_{x_2}\end{bmatrix},\quad \Delta=\partial_{x_1}^2+\partial_{x_2}^2.$$

The equations satisfied by v and q can be written with the introduction of the following conjugated operators:

$$Q_\varphi:=-e^{\tau\varphi}\Delta_z e^{-\tau\varphi}=-\Delta_z+2\tau\nabla_z\varphi\cdot\nabla_z-\tau^2|\nabla_z\varphi|^2+\tau(\Delta_z\varphi), \tag{4.5}$$

$$D_\varphi:=-e^{\tau\varphi}\Delta e^{-\tau\varphi}=-\Delta+2\tau\,(\nabla\varphi)\cdot\nabla-\tau^2|\nabla\varphi|^2+\tau(\Delta\varphi), \tag{4.6}$$

$$S_\varphi:=-e^{\tau\varphi_0}\Delta_{s,x_1}e^{-\tau\varphi_0}=-\Delta_{s,x_1}+2\tau(\partial_s\varphi_0)\partial_s-\tau^2(\partial_s\varphi_0)^2+\tau(\partial_s^2\varphi_0). \tag{4.7}$$

Then we deduce from (3.19) and (3.22) the following conjugated system:

$$\begin{cases} Q_\varphi v_1+e^{\tau\varphi}\partial_{x_1}p=g_1 & \text{in } Z,\\ Q_\varphi v_2+e^{\tau\varphi}\partial_{x_2}p=g_2 & \text{in } Z,\\ D_\varphi q=0 \quad \text{in } Z,\\ v_1=0 \quad \text{on } J_0\cup J_1,\\ v_2=0 \quad \text{on } J_0,\\ S_\varphi v_2=g_3+q-m_{\mathcal{I}}(q) & \text{on } J_1.\end{cases} \tag{4.8}$$

We define h_i by

$$h_i:=e^{\tau\varphi_0}f_i \quad (i=1,2,3). \tag{4.9}$$

The main result of this section is the following result.

Theorem 4.1. *There exist $\lambda_0=\lambda_0(\widetilde{\psi},S_0)>0$ and $\tau_0=\tau_0(\widetilde{\psi},S_0)>0$ such that for any $\lambda\geqslant\lambda_0$ and $\tau\geqslant\tau_0$, there exists $C=C(\lambda,\widetilde{\psi},S_0)>0$ such that for any $\Lambda>0$ and for any $(a_j)_j\in\mathbb{C}^{\mathbb{N}}$, the function q defined by* (3.14)*,* (3.18) *and* (4.4) *satisfies*

$$\begin{aligned}&\tau^3\|q\|^2_{L^2(Z)}+\tau\|\nabla q\|^2_{L^2(Z)}+\tau^3\left|q-m_{\mathcal{I}}(q)\right|^2_{L^2(J_1)}+\tau\left|\nabla q\right|^2_{L^2((0,S_0)\times\partial\Omega)}\\ &\quad\leqslant C\left(\tau^3\|q\|^2_{L^2((0,S_0)\times\omega_0)}+\tau\|\nabla q\|^2_{L^2((0,S_0)\times\omega_0)}+\tau\left(\|\partial_{x_1}h_1\|^2_{L^2(Z)}+\|\partial_{x_1}h_2\|^2_{L^2(Z)}+|\partial_{x_1}h_3|^2_{L^2(J_1)}\right)\right).\end{aligned} \tag{4.10}$$

We recall that we use $\|\cdot\|$ for norms in the interior and $|\cdot|$ for norms at the boundary (see Notation 1.9). We prove this theorem in the remainder of this section.

4.2. A first estimate on the pressure

In order to prove Theorem 4.1, we exploit that q satisfies the third equation of (4.8), where D_φ is defined by (4.6). Since we do not have any boundary condition, we need to split the boundary value of q into high and low frequencies. More precisely, for $\mathcal{Q} \in H^2(\Omega)$, we introduce the Fourier coefficients of the trace of $\mathcal{Q}$:

$$a_k(\mathcal{Q}) := \begin{bmatrix} \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \mathcal{Q}(x_1, 0) e^{-ikx_1} \, \mathrm{d}x_1 \\ \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \mathcal{Q}(x_1, 1) e^{-ikx_1} \, \mathrm{d}x_1 \end{bmatrix} \quad (k \in \mathbb{Z}). \tag{4.11}$$

We then define the sets of low tangential frequencies and high tangential frequencies:

$$\mathrm{LF}_\tau := \{k \in \mathbb{Z},\ k^2 \leqslant \frac{\tau^2}{2} \inf |\partial_{x_2}\varphi|^2\}, \quad \mathrm{HF}_\tau := \{k \in \mathbb{Z},\ k^2 > \frac{\tau^2}{2} \inf |\partial_{x_2}\varphi|^2\}.$$

In the above definition, the infimum of $\partial_{x_2}\varphi$ is taken for $x \in \partial\Omega$ and $s \in [0, S_0]$.

Due to (4.1) and (4.3), we have

$$\inf |\partial_{x_2}\varphi| = \lambda e^{-s_0^2}.$$

Proposition 4.2. *There exist $\lambda_0 = \lambda_0(\widetilde{\psi}, S_0) > 0$ and $\tau_0 = \tau_0(\widetilde{\psi}, S_0) > 0$ such that for any $\lambda \geqslant \lambda_0$ and $\tau \geqslant \tau_0$, there exists $C = C(\lambda, \widetilde{\psi}, S_0) > 0$ such that for any $s \in [0, S_0]$, and any $\mathcal{Q} \in H^2(\Omega)$,*

$$\begin{aligned} \tau^3 \|\mathcal{Q}\|^2_{L^2(\Omega)} + \tau \|\nabla \mathcal{Q}\|^2_{L^2(\Omega)} + \tau |\partial_{x_2}\mathcal{Q}|^2_{L^2(\partial\Omega)} + \sum_{k \in \mathrm{LF}_\tau} \tau(\tau^2 + k^2) |a_k(\mathcal{Q})|^2 \\ \leqslant C \left(\|D_\varphi \mathcal{Q}\|^2_{L^2(\Omega)} + \tau^3 \|\mathcal{Q}\|^2_{L^2(\omega_0)} + \tau \|\nabla \mathcal{Q}\|^2_{L^2(\omega_0)} + \sum_{k \in \mathrm{HF}_\tau} \tau(\tau^2 + k^2) |a_k(\mathcal{Q})|^2 \right). \end{aligned} \tag{4.12}$$

Proof. We can decompose the operator D_φ (see (4.6)) as follows $D_\varphi = \mathcal{S} + \mathcal{A} + \mathcal{R}$, where

$$\mathcal{S} = -\Delta - \tau^2 |\nabla\varphi|^2, \quad \mathcal{A} = 2\tau\nabla\varphi \cdot \nabla + 2\tau(\Delta\varphi), \quad \mathcal{R} = -\tau(\Delta\varphi).$$

Then, after some standard computation, we can obtain that

$$\begin{aligned} \int_\Omega (\mathcal{S}\mathcal{Q})(\mathcal{A}\mathcal{Q}) \mathrm{d}x = \tau \int_\Omega \left(2\nabla^2\varphi(\nabla\mathcal{Q}, \nabla\mathcal{Q}) + \Delta\varphi |\nabla\mathcal{Q}|^2 \right) \mathrm{d}x + \tau^3 \int_\Omega \left(2\nabla^2\varphi(\nabla\varphi, \nabla\varphi) - |\nabla\varphi|^2 \Delta\varphi \right) |\mathcal{Q}|^2 \mathrm{d}x \\ - \tau \int_\Omega (\Delta^2\varphi) |\mathcal{Q}|^2 \mathrm{d}x + \mathcal{B}, \end{aligned} \tag{4.13}$$

where

$$\nabla^2\varphi = \left(\frac{\partial^2\varphi}{\partial x_i \partial x_j} \right)_{i,j}$$

and where $\mathcal{B}$ corresponds to the boundary terms:

$$\begin{aligned} \mathcal{B} = -2\tau \int_{\partial\Omega} \partial_n \mathcal{Q} (\nabla\varphi \cdot \nabla\mathcal{Q}) \mathrm{d}\Gamma + \tau \int_{\partial\Omega} (\partial_n\varphi) |\nabla\mathcal{Q}|^2 \mathrm{d}\Gamma - 2\tau \int_{\partial\Omega} (\Delta\varphi)(\partial_n\mathcal{Q}) \mathcal{Q} \mathrm{d}\Gamma \\ + \tau \int_{\partial\Omega} (\partial_n \Delta\varphi) \mathcal{Q}^2 \mathrm{d}\Gamma - \tau^3 \int_{\partial\Omega} |\nabla\varphi|^2 (\partial_n\varphi) \mathcal{Q}^2 \mathrm{d}\Gamma. \end{aligned}$$

Using (4.1), we have that $\partial_n \varphi < 0$ and we can simplify the above quantity:

$$\mathcal{B} = \tau \int_{\partial\Omega} |\partial_{x_2}\varphi| \left(\partial_{x_2}\mathcal{Q}\right)^2 \mathrm{d}\Gamma - \tau \int_{\partial\Omega} |\partial_{x_2}\varphi| \left(\partial_{x_1}\mathcal{Q}\right)^2 \mathrm{d}\Gamma - 2\tau \int_{\partial\Omega} (\partial_{x_2}^2\varphi)(n\cdot e_2)(\partial_{x_2}\mathcal{Q})\mathcal{Q}\mathrm{d}\Gamma \\ + \tau \int_{\partial\Omega} (\partial_{x_2}^3\varphi)(n\cdot e_2)\mathcal{Q}^2\mathrm{d}\Gamma + \tau^3 \int_{\partial\Omega} |\partial_{x_2}\varphi|^3 \mathcal{Q}^2\mathrm{d}\Gamma.$$

Combining the above relation with (4.3), there exists $\tau_1 = \tau_1(S_0) > 0$ such that for any $\tau \geqslant \tau_1$, we have

$$\mathcal{B} \geqslant \frac{1}{2}\tau\lambda\varphi_0 \int_{\partial\Omega} \left(\partial_{x_2}\mathcal{Q}\right)^2 \mathrm{d}\Gamma - \tau\lambda\varphi_0 \int_{\partial\Omega} \left(\partial_{x_1}\mathcal{Q}\right)^2 \mathrm{d}\Gamma + \frac{3}{4}\tau^3\lambda^3\varphi_0^3 \int_{\partial\Omega} \mathcal{Q}^2\mathrm{d}\Gamma \\ \geqslant \frac{1}{2}\tau\lambda\varphi_0 \int_{\partial\Omega} \left(\partial_{x_2}\mathcal{Q}\right)^2 \mathrm{d}\Gamma + 2\pi\tau\lambda\varphi_0 \sum_{k\in\mathbb{Z}} \left(\frac{3}{4}\tau^2\lambda^2\varphi_0^2 - k^2\right) |a_k(\mathcal{Q})|^2. \quad (4.14)$$

Using (4.3) and (4.2), there exist $C_1 = C_1(\widetilde{\psi})$, $C_2 = C_2(\widetilde{\psi})$, $\tau_2 = \tau_2(S_0, \widetilde{\psi})$ and $\lambda_1 = \lambda_1(\widetilde{\psi})$ such that for $\lambda \geqslant \lambda_1$ and $\tau \geqslant \tau_2$,

$$\tau \int_\Omega \left(2\nabla^2\varphi(\nabla\mathcal{Q},\nabla\mathcal{Q}) + \Delta\varphi|\nabla\mathcal{Q}|^2\right)\mathrm{d}x + \tau^3 \int_\Omega \left(2\nabla^2\varphi(\nabla\varphi,\nabla\varphi) - |\nabla\varphi|^2\Delta\varphi\right)|\mathcal{Q}|^2\mathrm{d}x - \tau\int_\Omega (\Delta^2\varphi)|\mathcal{Q}|^2\mathrm{d}x \\ \geqslant C_1 \int_\Omega \left(\tau\lambda^2\varphi\left|\nabla\mathcal{Q}\right|^2 + \tau^3\lambda^4\varphi^3\left|\mathcal{Q}\right|^2\right)\mathrm{d}x - C_2 \int_{\omega_0} \left(\tau\lambda^2\varphi\left|\nabla\mathcal{Q}\right|^2 + \tau^3\lambda^4\varphi^3\left|\mathcal{Q}\right|^2\right)\mathrm{d}x. \quad (4.15)$$

Finally, combining (4.13), (4.14) and (4.15), we deduce the existence of $\lambda_0 > 0$ and $\tau_0 > 0$ such that for any $\lambda \geqslant \lambda_0$ and $\tau \geqslant \tau_0$, there exist $C_3 = C_3(\lambda, \widetilde{\psi}, S_0) > 0$ and $C_4 = C_4(\lambda, \widetilde{\psi}, S_0) > 0$ such that

$$\|D_\varphi\mathcal{Q}\|^2_{L^2(\Omega)} \geqslant \frac{1}{2}\|(\mathcal{S}+\mathcal{A})\mathcal{Q}\|^2_{L^2(\Omega)} - \|\mathcal{R}\mathcal{Q}\|^2_{L^2(\Omega)} \geqslant \mathrm{Re}\left(\mathcal{S}\mathcal{Q},\mathcal{A}\mathcal{Q}\right)_{L^2(\Omega)} - \|\mathcal{R}\mathcal{Q}\|^2_{L^2(0,1)} \\ \geqslant C_3\left(\int_\Omega \left(\tau|\nabla\mathcal{Q}|^2 + \tau^3|\mathcal{Q}|^2\right)\ \mathrm{d}x + \tau\int_{\partial\Omega}\left(\partial_{x_2}\mathcal{Q}\right)^2\mathrm{d}\Gamma + \tau\sum_{k\in\mathrm{LF}_\tau}(\tau^2+k^2)|a_k(\mathcal{Q})|^2\right) \\ - C_4\left(\int_{\omega_0}\left(\tau|\nabla\mathcal{Q}|^2 + \tau^3|\mathcal{Q}|^2\right)\ \mathrm{d}x + \tau\sum_{k\in\mathrm{HF}_\tau} k^2|a_k(\mathcal{Q})|^2\right).$$

□

Applying the above result to $\mathcal{Q} = q$ solution of (4.8) and integrating into $(0, S_0)$, we deduce that

$$\tau^3\|q\|^2_{L^2(Z)} + \tau\|\nabla q\|^2_{L^2(Z)} + \tau|\partial_{x_2}q|^2_{L^2((0,S_0)\times\partial\Omega)} + \sum_{k\in\mathrm{LF}_\tau}\tau(\tau^2+k^2)|a_k(q)|^2_{L^2(0,S_0)} \\ \leqslant C\left(\tau^3\|q\|^2_{L^2((0,S_0)\times\omega_0)} + \tau\|\nabla q\|^2_{L^2((0,S_0)\times\omega_0)} + \sum_{k\in\mathrm{HF}_\tau}\tau(\tau^2+k^2)|a_k(q)|^2_{L^2(0,S_0)}\right). \quad (4.16)$$

We recall that $q(s,\cdot) \equiv 0$ if $|s - s_0| \geqslant S_0/6$ due to the support of χ (see (3.17)). Next, we will estimate the high tangential frequencies of the pressure.

4.3. Estimates in the high frequency regime

We define

$$Y := (0, S_0) \times (0,1), \quad I_i := (0, S_0) \times \{i\}, \ i = 0, 1.$$

We recall that (u, p) is defined by (3.18) and (f_1, f_2, f_3) is defined by (3.20). We define $(u^k, p^k, f_1^k, f_2^k, f_3^k)$ the Fourier coefficients of (u, p, f_1, f_2, f_3) in the x_1 direction. For instance

$$u^k(s, x_2) := \frac{1}{2\pi} \int_0^{2\pi} u(s, x_1, x_2) e^{-ikx_1} \, \mathrm{d}x_1 \quad ((s, x_2) \in Y).$$

Finally, with $\tau > 0$ and φ_0 defined by (4.3), we set

$$w^k = e^{\tau\varphi_0} u^k, \quad \pi^k = e^{\tau\varphi_0} p^k, \quad h_i^k = e^{\tau\varphi_0} f_i^k \quad (i = 1, 2, 3). \tag{4.17}$$

Note that h_i^k are the Fourier coefficients of the functions h_i defined by (4.9). Since φ_0 only depends on s, and using (4.3), (4.4), we have

$$a_k(q) = \begin{bmatrix} (\pi^k)_{|I_0} \\ (\pi^k)_{|I_1} \end{bmatrix}. \tag{4.18}$$

Let us define the following conjugated operators:

$$Q_{k,\varphi_0} := e^{\tau\varphi_0}(-\partial_s^2 - \partial_{x_2}^2 + k^2)e^{-\tau\varphi_0} = -\partial_s^2 - \partial_{x_2}^2 + k^2 + 2\tau\varphi_0'\partial_s - \tau^2(\varphi_0')^2 + \tau\varphi_0'',$$
$$S_{k,\varphi_0} := e^{\tau\varphi_0}(-\partial_s^2 + k^2)e^{-\tau\varphi_0} = -\partial_s^2 + k^2 + 2\tau\varphi_0'\partial_s - \tau^2(\varphi_0')^2 + \tau\varphi_0''.$$

Then, for $k \in \mathbb{Z}^*$, (3.19) transforms into

$$\begin{cases} Q_{k,\varphi_0} w_1^k + ik\pi^k = h_1^k & \text{in } Y, \\ Q_{k,\varphi_0} w_2^k + \partial_{x_2}\pi^k = h_2^k & \text{in } Y, \\ \operatorname{div}_k w^k = 0 & \text{in } Y, \\ w_1^k = 0 & \text{on } I_0 \cup I_1, \\ w_2^k = 0 & \text{on } I_0, \\ S_{k,\varphi} w_2^k = h_3^k + \pi^k & \text{on } I_1, \end{cases} \tag{4.19}$$

where

$$\operatorname{div}_k \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = ikf_1 + \partial_{x_2} f_2.$$

The relation (3.21) yields

$$w^k = 0 \quad \text{and} \quad \pi^k = 0 \quad \text{if } s \notin \left[\frac{1}{3}S_0, \frac{2}{3}S_0\right]. \tag{4.20}$$

Proposition 4.3. *If the solution of (4.19) satisfies (4.20), then there exist $\lambda_3 = \lambda_3(S_0) > 0$ and $C(S_0) > 0$ such that for any $\lambda \geqslant \lambda_3$ and $k \in \mathrm{HF}_\tau$,*

$$\|\pi^k\|_{L^2(I_0 \cup I_1)} \leqslant \frac{C}{(k^2 + \tau^2)^{1/4}} \left(\|h_1^k\|_{L^2(Y)} + \|h_2^k\|_{L^2(Y)} + |h_3^k|_{L^2(I_1)} \right).$$

Proof. The proof is based on energy estimates on (4.19). We first obtain H^2 estimates on w^k by using the high-frequency hypothesis to control the lower order terms. The estimate on π^k follows then from (4.19). We multiply the first line (4.19) by w_1^k, the second line (4.19) by w_2^k, and the last line (4.19) by w_2^k. Integrating by parts and summing up yield

$$\begin{aligned}\int_Y |\partial_s w^k|^2 \, \mathrm{d}y + \int_Y |\partial_{x_2} w^k|^2 \, \mathrm{d}y + k^2 \int_Y |w^k|^2 \, \mathrm{d}y - \tau^2 \int_Y (\varphi_0')^2 |w^k|^2 \, \mathrm{d}y \\ + \int_{I_1} |\partial_s w_2^k|^2 \, \mathrm{d}s + k^2 \int_{I_1} |w_2^k|^2 \, \mathrm{d}s - \tau^2 \int_{I_1} (\varphi_0')^2 |w_2^k|^2 \, \mathrm{d}s \\ = \mathrm{Re} \int_Y h_1^k \overline{w_1^k} \, \mathrm{d}y + \mathrm{Re} \int_Y h_2^k \overline{w_2^k} \, \mathrm{d}y + \mathrm{Re} \int_{I_1} h_3^k \overline{w_2^k} \, \mathrm{d}s. \end{aligned} \tag{4.21}$$

Now, since $k \in \mathrm{HF}_\tau$, we have

$$k^2 > \frac{\tau^2}{2} \inf |\partial_{x_2} \varphi|^2 = \frac{\tau^2}{2} \lambda^2 e^{-2s_0^2}.$$

On the other hand,

$$\sup_{[0,S_0]} |\varphi_0'| \leqslant S_0.$$

From the two previous relations, we deduce the existence of $\lambda_3 = \lambda_3(S_0) > 0$ such that for $\lambda \geqslant \lambda_3$ and for $k \in \mathrm{HF}_\tau$

$$k^2 - \tau^2 |\varphi_0'|^2 \geqslant \frac{1}{2}(\tau^2 + k^2).$$

Combining the above relation and (4.21) yields

$$\begin{aligned}(\tau^2 + k^2) \left\| \left[\partial_s w^k, \partial_{x_2} w^k\right] \right\|_{L^2(Y)}^2 + (\tau^2 + k^2)^2 \left\| w^k \right\|_{L^2(Y)}^2 + (\tau^2 + k^2) \left| \partial_s w_2^k \right|_{L^2(I_1)}^2 + (\tau^2 + k^2)^2 \left| w_2^k \right|_{L^2(I_1)}^2 \\ \lesssim \left\| h_1^k \right\|_{L^2(Y)}^2 + \left\| h_2^k \right\|_{L^2(Y)}^2 + \left| h_3^k \right|_{L^2(I_1)}^2 . \end{aligned} \tag{4.22}$$

Now, we write (4.19) under the form

$$\left\{ \begin{array}{l} \left(-\partial_s^2 - \partial_{x_2}^2 + k^2\right) w_1^k + ik\pi^k = H_1^k \quad \text{in } Y, \\ \left(-\partial_s^2 - \partial_{x_2}^2 + k^2\right) w_2^k + \partial_{x_2} \pi^k = H_2^k \quad \text{in } Y, \\ \mathrm{div}_k \, w^k = 0 \quad \text{in } Y, \\ w_1^k = 0 \quad \text{on } I_0 \cup I_1, \\ w_2^k = 0 \quad \text{on } I_0, \\ \left(-\partial_s^2 + k^2\right) w_2^k = H_3^k + \pi^k \quad \text{on } I_1, \end{array} \right. \tag{4.23}$$

where

$$\begin{aligned} H_1^k &:= -2\tau \varphi_0' \partial_s w_1^k + \tau^2 (\varphi_0')^2 w_1^k - \tau \varphi_0'' w_1^k + h_1^k, \\ H_2^k &:= -2\tau \varphi_0' \partial_s w_2^k + \tau^2 (\varphi_0')^2 w_2^k - \tau \varphi_0'' w_2^k + h_2^k, \\ H_3^k &:= -2\tau \varphi_0' \partial_s w_2^k + \tau^2 (\varphi_0')^2 w_2^k - \tau \varphi_0'' w_2^k + + h_3^k. \end{aligned}$$

From (4.22), we deduce

$$\left\|H_1^k\right\|_{L^2(Y)} + \left\|H_2^k\right\|_{L^2(Y)} + \left|H_3^k\right|_{L^2(I_1)} \lesssim \left\|h_1^k\right\|_{L^2(Y)} + \left\|h_2^k\right\|_{L^2(Y)} + \left|h_3^k\right|_{L^2(I_1)}. \tag{4.24}$$

We multiply the first line (4.23) by $-\partial_s^2 w_1^k$, the second line (4.23) by $-\partial_s^2 w_2^k$, and the last line (4.23) by $-\partial_s^2 w_2^k$. Integrating by parts, summing up and using (4.24) yield

$$\begin{aligned}\left\|\partial_s^2 w^k\right\|_{L^2(Y)}^2 + \left\|\partial_s \partial_{x_2} w^k\right\|_{L^2(Y)}^2 + k^2\left\|\partial_s w^k\right\|_{L^2(Y)}^2 + \left|\partial_s^2 w_2^k\right|_{L^2(I_1)}^2 + k^2\left|\partial_s w_2^k\right|_{L^2(I_1)}^2 \\ \lesssim \left\|h_1^k\right\|_{L^2(Y)}^2 + \left\|h_2^k\right\|_{L^2(Y)}^2 + \left|h_3^k\right|_{L^2(I_1)}^2.\end{aligned} \tag{4.25}$$

Next, we write (4.23) under the form

$$\begin{cases} \left(-\partial_{x_2}^2 + k^2\right) w_1^k + ik\pi^k = \widetilde{H}_1^k & \text{in } Y, \\ \left(-\partial_{x_2}^2 + k^2\right) w_2^k + \partial_{x_2}\pi^k = \widetilde{H}_2^k & \text{in } Y, \\ \operatorname{div}_k w^k = 0 & \text{in } Y, \\ w_1^k = 0 & \text{on } I_0 \cup I_1, \\ w_2^k = 0 & \text{on } I_0, \\ k^2 w_2^k = \widetilde{H}_3^k + \pi^k & \text{on } I_1, \end{cases} \tag{4.26}$$

where

$$\widetilde{H}_1^k := H_1^k + \partial_s^2 w_1^k, \quad \widetilde{H}_2^k := H_2^k + \partial_s^2 w_2^k, \quad \widetilde{H}_3^k := H_3^k + \partial_s^2 w_1^k.$$

From (4.24) and (4.25), we deduce

$$\left\|\widetilde{H}_1^k\right\|_{L^2(Y)} + \left\|\widetilde{H}_2^k\right\|_{L^2(Y)} + \left|\widetilde{H}_3^k\right|_{L^2(I_1)} \lesssim \left\|h_1^k\right\|_{L^2(Y)} + \left\|h_2^k\right\|_{L^2(Y)} + \left|h_3^k\right|_{L^2(I_1)}. \tag{4.27}$$

The first two lines of (4.26) can be written as

$$\nabla\left(e^{ikx_1}\pi^k\right) = e^{ikx_1}\begin{bmatrix}\widetilde{H}_1^k \\ \widetilde{H}_2^k\end{bmatrix} + \Delta\left(e^{ikx_1} w^k\right) \tag{4.28}$$

and thus

$$\left\|\nabla\left(e^{ikx_1}\pi^k\right)\right\|_{H^{-1}(\Omega)} \leqslant \left\|e^{ikx_1}\begin{bmatrix}\widetilde{H}_1^k \\ \widetilde{H}_2^k\end{bmatrix}\right\|_{H^{-1}(\Omega)} + \left\|\nabla\left(e^{ikx_1} w^k\right)\right\|_{L^2(\Omega)} \tag{4.29}$$

and using that $k \neq 0$, we deduce from the above estimate that

$$\left\|e^{ikx_1}\pi^k\right\|_{H^{-1}(\Omega)} \leqslant \left\|e^{ikx_1}\begin{bmatrix}\widetilde{H}_1^k \\ \widetilde{H}_2^k\end{bmatrix}\right\|_{H^{-1}(\Omega)} + \left\|\nabla\left(e^{ikx_1} w^k\right)\right\|_{L^2(\Omega)}. \tag{4.30}$$

Combining (4.29) and (4.30) with the Nečas inequality (see, for instance, [12], p. 231, Thm. IV.1.1), we deduce that

$$k\left\|e^{ikx_1}\pi^k\right\|_{L^2(\Omega)} \lesssim \left\|\widetilde{H}_1^k\right\|_{L^2(0,1)} + \left\|\widetilde{H}_2^k\right\|_{L^2(0,1)} + k\left\|\nabla\left(e^{ikx_1} w^k\right)\right\|_{L^2(\Omega)} \tag{4.31}$$

and thus, with (4.27) and (4.22),

$$k\left\|\pi^k\right\|_{L^2(Y)} \lesssim \left\|h_1^k\right\|_{L^2(Y)} + \left\|h_2^k\right\|_{L^2(Y)} + \left|h_3^k\right|_{L^2(I_1)}. \tag{4.32}$$

On the other hand, differentiating the divergence equation of system (4.26) with respect to x_2 and using (4.22), (4.27) yield

$$\|\partial_{x_2}^2 w_2^k\|_{L^2(Y)} + \|\partial_{x_2}\pi^k\|_{L^2(Y)} \lesssim \left\|h_1^k\right\|_{L^2(Y)} + \left\|h_2^k\right\|_{L^2(Y)} + \left|h_3^k\right|_{L^2(I_1)}.$$

Then, combining the above relation with (4.32) and with a trace inequality, we deduce

$$\|\pi^k\|_{L^2(I_0\cup I_1)} \lesssim \frac{1}{k^{1/2}}\left(\left\|h_1^k\right\|_{L^2(Y)} + \left\|h_2^k\right\|_{L^2(Y)} + \left|h_3^k\right|_{L^2(I_1)}\right).$$

Using that $k\in \mathrm{HF}_\tau$, we deduce the result. □

4.4. Proof of Theorem 4.1

From Theorem 4.3 and from (4.18), for $k\in\mathrm{HF}_\tau$

$$(\tau^2+k^2)\left|a^k(q)\right|^2_{L^2(0,S_0)} \lesssim k^2\left(\left\|h_1^k\right\|^2_{L^2(Y)} + \left\|h_2^k\right\|^2_{L^2(Y)} + \left|h_3^k\right|^2_{L^2(I_1)}\right)$$

and thus, with the Parseval formula,

$$\sum_{k\in\mathrm{HF}_\tau}(\tau^2+k^2)|a_k(q)|^2_{L^2(0,S_0)} \lesssim \|\partial_{x_1}h_1\|^2_{L^2(Z)} + \|\partial_{x_1}h_2\|^2_{L^2(Z)} + |\partial_{x_1}h_3|^2_{L^2(J_1)},$$

where we have used (4.17) and (4.9).

Combining this estimate with (4.16) we finally obtain the sought result.

□

5. Proof of the spectral inequality

The proof of the spectral inequality, that is (3.13) is based on interpolation estimates. More precisely, it will be a consequence of Theorem 5.3 stated below. In order to show such a result, we first recall some interpolation inequalities available in the literature and then we combine them with the global pressure estimates, that is Theorem 4.1 to show Theorem 5.3. The last part of this section is devoted to the proof of the spectral inequality from the interpolation inequality.

First, we need the following notation for this section:

$$\mathscr{O}_0 := \left(s_0-\frac{S_0}{6}, s_0+\frac{S_0}{6}\right)\times\omega_0, \quad \mathscr{O} := \left(s_0-\frac{S_0}{5}, s_0+\frac{S_0}{5}\right)\times\omega, \tag{5.1}$$

$$\widetilde{Z} := \left(s_0-\frac{S_0}{10}, s_0+\frac{S_0}{10}\right)\times\Omega, \quad \widetilde{J}_1 := \left(s_0-\frac{S_0}{10}, s_0+\frac{S_0}{10}\right)\times\Gamma_1, \tag{5.2}$$

$$\widehat{Z} := \left(s_0-\frac{S_0}{9}, s_0+\frac{S_0}{9}\right)\times\Omega, \quad \widehat{J}_1 := \left(s_0-\frac{S_0}{9}, s_0+\frac{S_0}{9}\right)\times\Gamma_1. \tag{5.3}$$

Note that $\widetilde{Z} \subset \widehat{Z}$ and $\widetilde{J_1} \subset \widehat{J_1}$.

5.1. Estimates on the velocity

The two components of the velocity satisfy different boundary conditions (see (3.16)). We start by an estimate on the first component U_1 that satisfies homogeneous Dirichlet boundary conditions. For the proof of this result, we refer to relation (1) in Section 3 of [36].

Theorem 5.1. *There exist $C_1 > 0$ and $\mu_1 \in (0,1)$ such that for all $w \in H^2(Z)$ such that $w_{|J_0 \cup J_1} = 0$,*

$$\|w\|_{H^1(\widetilde{Z})} \leqslant C_1 \|w\|_{H^1(\widehat{Z})}^{1-\mu_1} \left(\|\Delta_z w\|_{L^2(\widehat{Z})} + \|w\|_{L^2(\mathcal{O}_0)} \right)^{\mu_1}. \tag{5.4}$$

Note that Theorem 5.1 is stated with an H^1 observation in [36], but we can transform it into an L^2 observation as in (5.4) by using a cut-off function and integrations by parts.

For the estimate of U_2, we note that it satisfies a Ventcel boundary condition on J_1 and the Dirichlet boundary condition on J_0. Hence, we use the following result, which is basically a consequence of a Carleman estimate obtained in [13]. However for the sake of completeness, we prove the next result in Appendix A.

Theorem 5.2. *There exist $C_2 > 0$ and $\mu_2 \in (0,1)$ such that*

$$\|w\|_{H^1(\widetilde{Z})} + |w|_{H^1(\widetilde{J_1})} \leqslant C_2 \left(\|w\|_{H^1(\widehat{Z})} + |w|_{H^1(\widehat{J_1})} \right)^{1-\mu_2} \times \left(\|\Delta_z w\|_{L^2(\widehat{Z})} + \left| (\partial_{x_2} w)_{|J_1} - \Delta_{x_1,s} w_{|J_1} \right|_{L^2(\widehat{J_1})} + \|w\|_{L^2(\mathcal{O}_0)} \right)^{\mu_2}. \tag{5.5}$$

for all $w \in H^2(Z)$ such that $w_{|J_0} = 0$ and $w_{|J_1} \in H^2(J_1)$.

Note that both Theorem 5.1 and Theorem 5.2 hold for $\mu_3 \in (0,1)$ such that $\mu_3 \leqslant \mu_1$ and $\mu_3 \leqslant \mu_2$ (with a modification of the constants C_1 and C_2). Thus, with $\mu_3 = \min(\mu_1, \mu_2)$ and an adequate constant C_3, we can apply Theorem 5.1 with $w = U_1$ and Theorem 5.2 with $w = U_2$, where U satisfies (3.16) and we deduce

$$\|U\|_{H^1(\widetilde{Z})} + |U_2|_{H^1(\widetilde{J_1})} \leqslant C_3 \left(\|U\|_{H^1(\widehat{Z})} + |U_2|_{H^1(\widehat{J_1})} \right)^{1-\mu_3} \times \left(\|\nabla P\|_{L^2(\widehat{Z})} + \left| P_{|J_1} - m_{\mathcal{I}}(P) \right|_{L^2(\widehat{J_1})} + \|U\|_{L^2(\mathcal{O}_0)} \right)^{\mu_3}. \tag{5.6}$$

We have used here the fact that on J_1,

$$\partial_{x_2} U_2 = -\partial_{x_1} U_1 = 0.$$

We are going now to combine the above estimate with the estimates of the pressure terms obtained in Section 4.

5.2. Patching the estimates together

Combining the previous estimates, we can now prove the following result.

Theorem 5.3. *There exist $C > 0$ and $\mu \in (0,1)$ such that for any $\Lambda > 0$ and for any $(a_j)_j \in \mathbb{C}^{\mathbb{N}}$, the function U defined by (3.14) satisfies*

$$\|U\|_{H^1(\widetilde{Z})} + |U_2|_{H^1(\widetilde{J_1})} \leqslant C \left(\|U\|_{H^2(Z)} + |U_2|_{H^2(J_1)} \right)^{1-\mu} \|U\|_{H^2(\mathcal{O}_0)}^{\mu}.$$

Proof. We start with the estimate (4.10), where we recall that q is given by (3.18), (4.4) and h_i, $i = 1, 2, 3$ by (4.9):

$$\tau^3 \|e^{\tau\varphi}\chi P\|^2_{L^2(Z)} + \tau\|e^{\tau\varphi}\chi\nabla P\|^2_{L^2(Z)} + \tau^3|e^{\tau\varphi_0}\chi(P - m_{\mathcal{I}}(P))|^2_{L^2(J_1)} + \tau|e^{\tau\varphi_0}\chi\partial_{x_1}P|^2_{L^2(J_1)} + \tau|e^{\tau\varphi_0}\chi\partial_{x_2}P|^2_{L^2(J_1)} \leqslant C\Big(\tau^3\|e^{\tau\varphi}P\|^2_{L^2(\mathcal{O}_0)} + \tau\|e^{\tau\varphi}\nabla P\|^2_{L^2(\mathcal{O}_0)} + \tau\,\|e^{\tau\varphi_0}\partial_{x_1}f_1\|^2_{L^2(Z)} + \tau\,\|e^{\tau\varphi_0}\partial_{x_1}f_2\|^2_{L^2(Z)} + \tau\,|e^{\tau\varphi_0}\partial_{x_1}f_3|^2_{L^2(J_1)}\Big). \quad (5.7)$$

Note that f_1, f_2 (respectively f_3) are supported in

$$Z_8 := (\operatorname{supp}\chi') \times \mathcal{I} \times (0,1) \quad (\text{respectively in } J_8 := (\operatorname{supp}\chi') \times \mathcal{I} \times \{1\}),$$

and since $\operatorname{supp}\chi' \subset [s_0 - S_0/6, s_0 + S_0/6]$

$$\sup_{J_8}\varphi_0 = \sup_{Z_8}\varphi_0 \leqslant e^{-\frac{S_0^2}{36}}. \quad (5.8)$$

Hence, from (3.20)

$$\|e^{\tau\varphi_0}\partial_{x_1}f_1\|_{L^2(Z)} + \|e^{\tau\varphi_0}\partial_{x_1}f_2\|_{L^2(Z)} + |e^{\tau\varphi_0}\partial_{x_1}f_3|_{L^2(J_1)} \leqslant Ce^{\tau e^{-\frac{S_0^2}{36}}}\left(\|U\|_{H^2(Z)} + |U_2|_{H^2(J_1)}\right). \quad (5.9)$$

In $\widehat{Z}$ (respectively in $\widehat{J}_1$), $\chi(s) \equiv 1$, and

$$\inf_{\widehat{Z}}\varphi = \inf_{\widehat{Z}}\varphi_0 = \inf_{\widehat{J}_1}\varphi_0 = e^{-\frac{S_0^2}{81}}. \quad (5.10)$$

Combining (5.7), (5.9) and (5.10), there exist $\tau_4, c_1, c_2 > 0$ such that for all $\tau \geqslant \tau_4$, we have

$$\|\nabla P\|^2_{L^2(\widehat{Z})} + |(P - m_{\mathcal{I}}(P))|^2_{L^2(\widehat{J}_1)} \leqslant e^{c_1\tau}\left(\|P\|^2_{L^2(\mathcal{O}_0)} + \|\nabla P\|^2_{L^2(\mathcal{O}_0)}\right) + e^{-c_2\tau}\left(\|U\|_{H^2(Z)} + |U_2|_{H^2(J_1)}\right). \quad (5.11)$$

On the other hand, we deduce from (5.6) and a Young inequality that

$$\|U\|_{H^1(\widetilde{Z})} + |U_2|_{H^1(\widetilde{J}_1)} \leqslant C_4 e^{-\frac{\mu_3 c_2}{2(1-\mu_3)}\tau}\left(\|U\|_{H^1(\widehat{Z})} + |U_2|_{H^1(\widehat{J}_1)}\right) + e^{\frac{c_2}{2}\tau}\left(\|\nabla P\|_{L^2(\widehat{Z})} + \left|P_{|J_1} - m_{\mathcal{I}}(P)\right|_{L^2(\widehat{J}_1)} + \|U\|_{L^2(\mathcal{O}_0)}\right)$$

and combining this relation with (5.11), we deduce the existence of $c_3, c_4 > 0$ such that for all $\tau \geqslant \tau_4$

$$\|U\|_{H^1(\widetilde{Z})} + |U_2|_{H^1(\widetilde{J}_1)} \lesssim e^{-c_3\tau}\left(\|U\|_{H^2(Z)} + |U_2|_{H^2(J_1)}\right) + e^{c_4\tau}\left(\|P\|^2_{L^2(\mathcal{O}_0)} + \|\nabla P\|^2_{L^2(\mathcal{O}_0)} + \|U\|_{L^2(\mathcal{O}_0)}\right)$$

Now, we can use c_P in (3.14) so that for all $s \in [0, S_0]$,

$$\int_{\omega_0} P(s,x)\,\mathrm{d}x = 0$$

and using the Poincaré-Wirtinger inequality, we deduce that

$$\|P\|^2_{L^2(\mathcal{O}_0)} \lesssim \|\nabla P\|^2_{L^2(\mathcal{O}_0)} \lesssim \|\Delta_z U\|_{L^2(\mathcal{O}_0)}.$$

We deduce that for some constants $c_5, c_6 > 0$, for all $\tau \geqslant \tau_4$,

$$\|U\|_{H^1(\widetilde{Z})} + |U_2|_{H^1(\widetilde{J}_1)} \lesssim e^{-c_5\tau}\left(\|U\|_{H^2(Z)} + |U_2|_{H^2(J_1)}\right) + e^{c_6\tau}\|U\|_{H^2(\mathcal{O}_0)}.$$

Optimizing this inequality with respect to $\tau \geqslant \tau_4$ (see, for instance, [53], [13], Lem. 8.4) allows us to conclude the proof of Theorem 5.3. □

5.3. From the interpolation inequality to the spectral inequality

Using Theorem 5.3, we are now in a position to prove Theorem 3.2. This inequality combined with Theorem 1.8 yields the main result of the article (Thm. 1.2).

Proof of Theorem 3.2. From (3.9) and (3.1), we deduce that

$$\|U\|^2_{H^1(\widetilde{Z})} + |U_2|^2_{H^1(\widetilde{J}_1)} \geqslant \|U\|^2_{L^2(\widetilde{Z})} + |U_2|^2_{L^2(\widetilde{J}_1)} \geqslant \int_{s_0-\frac{S_0}{10}}^{s_0+\frac{S_0}{10}} \sum_{\lambda_j\leqslant\Lambda} |a_j|^2 \cosh(\sqrt{\lambda_j}s)^2 \, \mathrm{d}s \gtrsim \sum_{\lambda_j\leqslant\Lambda} |a_j|^2$$

and

$$\|U\|^2_{H^2(Z)} + |U_2|^2_{H^2(J_1)} \lesssim e^{C\sqrt{\Lambda}} \sum_{\lambda_j\leqslant\Lambda} |a_j|^2$$

Combining Theorem 5.3 with the previous relations, we deduce that

$$\sum_{\lambda_j\leqslant\Lambda} |a_j|^2 \lesssim e^{C\sqrt{\Lambda}} \|U\|^2_{H^2(\mathcal{O}_0)}. \tag{5.12}$$

Let us introduce a cut-off function $\vartheta \in C_0^\infty(\mathcal{O})$ such that $0 \leqslant \vartheta \leqslant 1$ and $\vartheta = 1$ in $\mathcal{O}_0$. By integration by parts, we obtain

$$\|U\|^2_{H^2(\mathcal{O}_0)} \leqslant \sum_{\alpha_0+\alpha_1+\alpha_2\leqslant 2} \int_{\mathcal{O}} \vartheta \left|\partial_s^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} U\right|^2 \mathrm{d}s\mathrm{d}x \lesssim \sum_{\alpha_0+\alpha_1+\alpha_2\leqslant 4} \left\|\partial_s^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} U\right\|_{L^2(Z)} \|U\|_{L^2(\mathcal{O})}. \tag{5.13}$$

On the other hand, using the elliptic regularity of (3.3), the definition (3.14) of U and the orthogonality of the eigenfunctions $\left(\begin{bmatrix} u^{(j)} \\ \eta^{(j)} \end{bmatrix}\right)_j$ in $\mathcal{H}$, we have

$$\sum_{\alpha_0+\alpha_1+\alpha_2\leqslant 4} \left\|\partial_s^{\alpha_0}\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2} U\right\|_{L^2(Z)} \lesssim \Lambda^4 e^{C\sqrt{\Lambda}} \left(\sum_{\lambda_j\leqslant\Lambda} |a_j|^2\right)^{1/2}. \tag{5.14}$$

Combining (5.12), (5.13) and (5.14), we deduce

$$\sum_{\lambda_j\leqslant\Lambda} |a_j|^2 \lesssim e^{C\sqrt{\Lambda}} \|U\|^2_{L^2(\mathcal{O})}.$$

Thus Theorem 3.2 follows. □

Appendix A. Proof of Theorem 5.2

A.1 A Carleman estimate

The proof of Theorem 5.2 is mainly based on a Carleman estimate obtained in [13] that we recall here. We recall that Z, J_1, J_0 are defined by (3.15) whereas $\widetilde{Z}$ and $\widetilde{J}_1$ are defined by (5.2). In what follows, we consider $z^0 \in \widetilde{J}_1$, an open neighborhood V of z^0 in $\overline{Z}$ and a weight function $\varphi \in C^\infty(\overline{V})$. For any $\sigma \in \mathbb{R}$, we define

$$p_{\varphi,\sigma}(z,\xi,\tau) = |\xi|^2 - \tau^2|\nabla_z\varphi(z)|^2 - \sigma^2 + 2i\tau\xi\cdot\nabla_z\varphi(z), \quad (z \in V,\ \xi \in \mathbb{R}^3,\ \tau \in \mathbb{R}).$$

It is the principal symbol of the conjugated operator associated with $-\Delta_z - \sigma^2$, that is, of the operator

$$P_{\varphi,\sigma} = -e^{\tau\varphi}\left(\Delta_z + \sigma^2\right)e^{-\tau\varphi} = -\Delta_z + 2\tau\nabla_z\varphi\cdot\nabla_z - \tau^2|\nabla_z\varphi|^2 + \tau(\Delta_z\varphi) - \sigma^2.$$

We assume the following hypotheses on φ: sub-ellipticity on $\overline{V}$, that is the existence of $\tau_0 > 0$ such that for any $z \in V$, $\xi \in \mathbb{R}^3$, $|\sigma| \geqslant 1$ and $\tau \geqslant \tau_0|\sigma|$,

$$p_{\varphi,\sigma}(z,\xi,\tau) = 0 \implies \frac{1}{2i}\left\{\overline{p_{\varphi,\sigma}}, p_{\varphi,\sigma}\right\}(z,\xi,\tau) > 0. \tag{A.1}$$

and the two following conditions to handle Ventcel boundary conditions (see [13], conditions (23) and (24)):

$$\nabla_z\varphi \neq 0 \quad \text{in } \overline{V}, \quad \text{and} \quad \sup_{\overline{V}\cap\widetilde{J}_1} |\nabla_{s,x_1}\varphi| \leqslant \nu_0 \inf_{\overline{V}} |\partial_{x_2}\varphi|, \tag{A.2}$$

for $\nu_0 > 0$ small enough. We recall that the Poisson bracket is defined by

$$\left\{p^{(1)}, p^{(2)}\right\} = \sum_{j=1}^{3} \frac{\partial p^{(1)}}{\partial \xi_j}\frac{\partial p^{(2)}}{\partial z_j} - \frac{\partial p^{(2)}}{\partial \xi_j}\frac{\partial p^{(1)}}{\partial z_j}$$

where we set here $z_1 = s$, $z_2 = x_1$ and $z_3 = x_2$ to simplify.

Then we have the following result proved in [13]:

Theorem A.1. *Assume $z^0 \in \widetilde{J}_1$ and V is an open neighborhood of z^0 in $\overline{Z}$. Assume also that $\varphi \in C^\infty(\overline{V})$ satisfies the conditions (A.1) and (A.2) for ν_0 small enough. Then, there exist $\tau_0 > 0$ and $C > 0$ such that for all $|\sigma| \geqslant 1$, for all $\tau \geqslant \tau_0|\sigma|$ and for all $w \in C_0^\infty(V)$,*

$$\tau^3\left\|e^{\tau\varphi}w\right\|_{L^2(V)}^2 + \tau\left\|e^{\tau\varphi}\nabla_z w\right\|_{L^2(V)}^2 + \tau^3\left|e^{\tau\varphi}w_{|J_1}\right|_{L^2\left(V\cap\widetilde{J}_1\right)}^2 + \tau\left|e^{\tau\varphi}\nabla_{s,x_1}w_{|J_1}\right|_{L^2\left(V\cap\widetilde{J}_1\right)}^2$$
$$+ \tau\left|e^{\tau\varphi}\partial_{x_2}w_{|J_1}\right|_{L^2\left(V\cap\widetilde{J}_1\right)}^2 \leqslant C\left(\left\|e^{\tau\varphi}(-\Delta_z - \sigma^2)w\right\|_{L^2(V)}^2 + \tau\left|e^{\tau\varphi}(\partial_{x_2}w_{|J_1} - \Delta_{s,x_1}w_{|J_1}\right|_{L^2\left(V\cap\widetilde{J}_1\right)}^2\right).$$

First to precise the above statement, by $w \in C_0^\infty(V)$ we mean that w is the restriction of a C^∞ function with compact support in V_0 where V_0 is an open set of $\mathbb{R}\times\mathcal{I}\times\mathbb{R}$ such that $V_0 \cap \overline{Z} = V$. Second, we use the above result in the case $\sigma = 1$, so that by taking τ_0 large enough, we obtain that for all $\tau \geqslant \tau_0$ and for all $w \in C_0^\infty(V)$,

$$\tau^3\left\|e^{\tau\varphi}w\right\|_{L^2(V)}^2 + \tau\left\|e^{\tau\varphi}\nabla_z w\right\|_{L^2(V)}^2 + \tau^3\left|e^{\tau\varphi}w_{|J_1}\right|_{L^2\left(V\cap\widetilde{J}_1\right)}^2 + \tau\left|e^{\tau\varphi}\nabla_{s,x_1}w_{|J_1}\right|_{L^2\left(V\cap\widetilde{J}_1\right)}^2$$
$$+ \tau\left|e^{\tau\varphi}\partial_{x_2}w_{|J_1}\right|_{L^2\left(V\cap\widetilde{J}_1\right)}^2 \leqslant C\left(\left\|e^{\tau\varphi}\Delta_z w\right\|_{L^2(V)}^2 + \tau\left|e^{\tau\varphi}\left(\partial_{x_2}w_{|J_1} - \Delta_{s,x_1}w_{|J_1}\right)\right|_{L^2\left(V\cap\widetilde{J}_1\right)}^2\right). \tag{A.3}$$

A.2 Interpolation estimates for the Ventcel boundary condition

Using the Carleman inequality of the previous section, one can deduce, in a classical way, an interpolation inequality. First let us define the weight function that we are going to use.

We consider the following norms on $\mathbb{R}\times\mathcal{I}\times\mathbb{R}$:

$$\left|(s,x_1,x_2)\right|_\lambda := \left(\frac{s^2}{\lambda^2}+\frac{x_1^2}{\lambda^2}+x_2^2\right)^{1/2}.$$

We consider $z^0=(s^*,x_1^*,1)\in\widetilde{J}_1$,

$$V := (s^*-\delta,s^*+\delta)\times(x_1^*-\delta,x_1^*+\delta)\times(1-\delta,1]\,,$$

with $\delta\in(0,1)$ small enough such that

$$(s^*-\delta,s^*+\delta)\subset\left(s_0-\frac{S_0}{9},s_0+\frac{S_0}{9}\right)$$

(see (5.3)). We also define $z^*=(s^*,x_1^*,0)$. Then we define

$$\psi(z):=|z-z^*|_\lambda\,,\quad \varphi=e^{-\lambda\psi}.$$

Lemma A.2. *There exists $\lambda_0>0$ such that for any $\lambda\geqslant\lambda_0$, the weight function φ satisfies* (A.1) *and* (A.2) *on V for some $\tau_0=\tau_0(\lambda)$.*

Proof. We assume that $\lambda\geqslant 1$ in all what follows. First, since $\nabla_z\psi\neq 0$ in $\overline{V}$, we deduce the first point of (A.2). For the second point of (A.2), we first notice that

$$\inf_{\overline{V}\cap\widetilde{J}_1}\psi=1,\quad \sup_{\overline{V}}\psi=\left(\frac{2\delta^2}{\lambda^2}+1\right)^{1/2}\leqslant 2. \tag{A.4}$$

We thus deduce

$$\sup_{\overline{V}\cap\widetilde{J}_1}|\nabla_{s,x_1}\varphi|\leqslant\sup_{\overline{V}\cap\widetilde{J}_1}\varphi=e^{-\lambda}\quad\text{and}\quad\inf_{\overline{V}}|\partial_{x_2}\varphi|\geqslant\lambda\frac{1-\delta}{2}e^{-\left(2\delta^2+\lambda^2\right)^{1/2}}.$$

Consequently there exists λ_1 such that the second point of (A.2) holds for $\lambda\geqslant\lambda_1$.

For (A.1), we compute the Poisson bracket:

$$\begin{aligned}\frac{1}{8\tau i}\{\overline{p_{\varphi,\sigma}},p_{\varphi,\sigma}\} &= \tau^2\left(\nabla_z^2\varphi\right)\nabla_z\varphi\cdot\nabla_z\varphi+\left(\nabla_z^2\varphi\right)\xi\cdot\xi\\ &= \tau^2\varphi^3\left(\lambda^4\left|\nabla_z\psi\right|^4-\lambda^3\left(\nabla_z^2\psi\right)\left(\nabla_z\psi\right)\cdot\left(\nabla_z\psi\right)\right)+\varphi\left(\lambda^2(\nabla_z\psi\cdot\xi)^2-\lambda\left(\nabla_z^2\psi\right)\xi\cdot\xi\right)\\ &\geqslant \tau^2\lambda^4\varphi^3\left|\nabla_z\psi\right|^4-\tau^2\lambda^3\varphi^3\left|\nabla_z^2\psi\right|\left|\nabla_z\psi\right|^2-\lambda\varphi\left|\nabla_z^2\psi\right||\xi|^2.\end{aligned}$$

Now, if $p_{\varphi,\sigma}(z,\xi,\tau)=0$, then $|\xi|^2=\tau^2\lambda^2\varphi^2|\nabla_z\psi(z)|^2+\sigma^2$ so that

$$\frac{1}{8\tau i}\{\overline{p_{\varphi,\sigma}},p_{\varphi,\sigma}\}\geqslant\tau^2\lambda^4\varphi^3\left|\nabla_z\psi\right|^4-2\tau^2\lambda^3\varphi^3\left|\nabla_z^2\psi\right|\left|\nabla_z\psi\right|^2-\lambda\varphi\left|\nabla_z^2\psi\right|\sigma^2.$$

From (A.4), there exist positive constants independent of λ such that

$$C_1 \leqslant |\nabla_z \psi| \leqslant C_2, \quad \left|\nabla_z^2 \psi\right| \leqslant C_3.$$

In particular there exist $C > 0$ and $\lambda_0 \geqslant \lambda_1$, such that for $\lambda \geqslant \lambda_0$,

$$\frac{1}{8\tau i}\{\overline{p_{\varphi,\sigma}}, p_{\varphi,\sigma}\} \geqslant C\tau^2\lambda^4\varphi^3 - \lambda\varphi\left|\nabla_z^2\psi\right|\sigma^2$$

and there exists $\tau_0 = \tau_0(\lambda)$ such that for $\tau \geqslant \tau_0|\sigma|$,

$$\frac{1}{8\tau i}\{\overline{p_{\varphi,\sigma}}, p_{\varphi,\sigma}\} \geqslant \frac{C}{2}\tau^2\lambda^4\varphi^3 > 0.$$

□

From now on, the value of λ shall be kept fixed. We define, for $\beta > 0$,

$$\widehat{Z}_\beta := \{z \in \widehat{Z} \ ; \ \operatorname{dist}(z, \widehat{J}_1) > \beta\}.$$

Lemma A.3. *Assume $z^0 \in \widetilde{J}_1$. There exist an open neighborhood $\widetilde{V}$ of z^0 in $\overline{Z}$, $\mu, \beta \in (0,1)$ and $C > 0$ such that for any $v \in H^2(Z)$ with $v_{|_{J_1}} \in H^2(J_1)$,*

$$\|v\|_{H^1(\widetilde{V})} + \left|v_{|J_1}\right|_{H^1(\widetilde{V}\cap\widetilde{J}_1)} \leqslant C\left(\|v\|_{H^1(\widehat{Z})} + \left|v_{|J_1}\right|_{H^1(\widehat{J}_1)}\right)^{1-\mu} \times\left(\|\Delta_z v\|_{L^2(\widehat{Z})} + \left|\partial_{x_2} v_{|J_1} - \Delta_{s,x_1} v_{|J_1}\right|_{L^2(\widehat{J}_1)} + \|v\|_{H^1(\widehat{Z}_\beta)}\right)^{\mu}. \quad \text{(A.5)}$$

Proof. Standard computation shows the existence of $r_3 > 1$ such that

$$|z - z^*|_\lambda = r_3 \quad z = (s, x_1, 1) \in J_1 \implies (s, x_1) \in \left(s^* - \frac{\delta}{2}, s^* + \frac{\delta}{2}\right) \times \left(x_1^* - \frac{\delta}{2}, x_1^* + \frac{\delta}{2}\right).$$

We consider $r_2 \in (1, r_3)$. We can also check the existence of $r_1 \in (0, \delta)$ such that

$$\{z = (s, x_1, x_2) \in Z \ ; \ 1 - r_1 \leqslant x_2 \leqslant 1, \quad |z - z^*|_\lambda \leqslant r_3\} \subset V.$$

We consider two cut-off functions $\chi_0, \chi_1 \in C^\infty(\mathbb{R} \times \mathcal{I} \times \mathbb{R})$ such that

$$\chi_0(z) = \begin{cases} 0 & \text{if} \quad 0 < x_2 < 1 - r_1 \\ 1 & \text{if} \quad 1 - r_1/2 < x_2 < 1, \end{cases} \qquad \chi_1(z) = \begin{cases} 0 & \text{if} \quad |z - z^*|_\lambda > r_3 \\ 1 & \text{if} \quad |z - z^*|_\lambda \leqslant r_2. \end{cases}$$

Let us consider $v \in C^\infty(Z)$ and let us apply the Carleman estimate (A.3) to $w = \chi_0\chi_1 v \in C_0^\infty(V)$. In the right-hand side of this estimate, we have

$$\Delta_z(\chi_0\chi_1 v) = (\chi_0\chi_1)\Delta_z v + 2\nabla_z(\chi_0\chi_1)\cdot\nabla_z v + v\Delta_z(\chi_0\chi_1)$$

Note that in $\operatorname{supp}\chi_0\chi_1$, $\varphi \leqslant C_3 := e^{-\lambda(1-r_1)}$. The two last terms in the above relation are included in

$$V \cap (\operatorname{supp}\nabla\chi_0) \subset V \cap \{x_2 \in [1 - r_1, 1 - r_1/2]\}$$

and on this set, $\varphi \leqslant C_3$ or in

$$V \cap (\operatorname{supp} \nabla\chi_1) \subset V \cap \{r_2 \leqslant |z - z^*|_\lambda \leqslant r_3\}$$

and on this set, $\varphi \leqslant C_1 := e^{-\lambda r_2}$. Therefore,

$$\|e^{\tau\varphi}\Delta_z w\|_{L^2(V)} \lesssim e^{C_3\tau}\|\Delta_z v\|_{L^2(V)} + e^{C_3\tau}\|v\|_{H^1(V\cap\{x_2\in[1-r_1,1-r_1/2]\})} + e^{C_1\tau}\|v\|_{H^1(V)} \tag{A.6}$$

and similarly,

$$\left|e^{\tau\varphi}\left(\partial_{x_2} w_{|J_1} - \Delta_{s,x_1} w_{|J_1}\right)\right|_{L^2\left(V\cap\widetilde{J}_1\right)} \lesssim e^{C_3\tau}\left|\partial_{x_2} v_{|J_1} - \Delta_{s,x_1} v_{|J_1}\right|_{L^2\left(V\cap\widetilde{J}_1\right)} + e^{C_1\tau}\left|v_{|J_1}\right|_{H^1\left(V\cap\widetilde{J}_1\right)}. \tag{A.7}$$

There exists $r_4 > 0$ such that

$$\left\{z \in \mathbb{R}\times\mathcal{I}\times\mathbb{R} \; ; \; \left|z - z^0\right| \leqslant r_4\right\} \subset \left\{z = (s, x_1, x_2) \in \mathbb{R}\times\mathcal{I}\times\mathbb{R} \; ; \; 1 - \frac{r_1}{2} < x_2, \quad |z - z^*|_\lambda < r_2\right\}.$$

Then on the set

$$\widetilde{V} := \left\{z \in Z \; ; \; \left|z - z^0\right| < r_4\right\} \subset V$$

we have $\chi_0\chi_1 = 1$ and $\varphi \geqslant C_2 := e^{-\lambda \sup_{\widetilde{V}} \psi}$, with $C_2 \in (C_1, C_3)$. Combining this with (A.3), (A.6) and (A.7), we deduce that for all $\tau \geqslant \tau_0$,

$$\begin{aligned}\|v\|_{H^1(\widetilde{V})} + \left|v_{|J_1}\right|_{H^1\left(\widetilde{V}\cap\widetilde{J}_1\right)} &\\ \lesssim e^{(C_3-C_2)\tau}&\left(\|\Delta_z v\|_{L^2(V)} + \left|\partial_{x_2} v_{|J_1} - \Delta_{s,x_1} v_{|J_1}\right|_{L^2\left(V\cap\widetilde{J}_1\right)} + \|v\|_{H^1(V\cap\{x_2\in[1-r_1,1-r_1/2]\})}\right)\\ &+ e^{-(C_2-C_1)\tau}\left(\|v\|_{H^1(V)} + \left|v_{|J_1}\right|_{H^1\left(V\cap\widetilde{J}_1\right)}\right).\end{aligned} \tag{A.8}$$

Optimizing this inequality with respect to τ yields the interpolation inequality for v smooth. A density argument permits to conclude the proof of Theorem A.3. □

Proof of Theorem 5.2. By a compactness argument, one can deduce from Lemma A.3 an interpolation result on a neighborhood of $\widetilde{J}_1$. Then we combine this with classical interpolation estimates (see [36]) in the interior and at the boundary J_0 where Dirichlet boundary condition hold to conclude. A similar proof is done in [13] (see Lem. 8.3). □

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References

[1] M. Badra and T. Takahashi, Maximal regularity for the Stokes system coupled with a wave equation: application to the system of interaction between a viscous incompressible fluid and an elastic wall. *J. Evol. Equ.* **22** (2022) 71.
[2] M. Badra and T. Takahashi, Feedback stabilization of a fluid-rigid body interaction system. *Adv. Differ. Equ.* **19** (2014) 1137–1184.
[3] M. Badra and T. Takahashi, Feedback stabilization of a simplified 1d fluid-particle system. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **31** (2014) 369–389.
[4] M. Badra and T. Takahashi, Feedback boundary stabilization of 2D fluid–structure interaction systems. *Discrete Contin. Dyn. Syst.* **37** (2017) 2315–2373.
[5] M. Badra and T. Takahashi, Gevrey regularity for a system coupling the Navier-Stokes system with a beam equation. *SIAM J. Math. Anal.* **51** (2019) 4776–4814.
[6] M. Badra and T. Takahashi, Gevrey regularity for a system coupling the Navier-Stokes system with a beam: the non-flat case. *Funkcialaj Ekvacioj* **65** (2022) 63–109.

[7] H. Beirão da Veiga, On the existence of strong solutions to a coupled fluid–structure evolution problem. *J. Math. Fluid Mech.* **6** (2004) 21–52.

[8] M. Bellassoued and J. Le Rousseau, Carleman estimates for elliptic operators with complex coefficients. Part I: Boundary value problems. *J. Math. Pures Appl.* **104** (2015) 657–728.

[9] M. Bellassoued and J. Le Rousseau, Carleman estimates for elliptic operators with complex coefficients. Part II: Transmission problems. *J. Math. Pures Appl.* **115** (2018) 127–186.

[10] M. Boulakia and S. Guerrero, Local null controllability of a fluid-solid interaction problem in dimension 3. *J. Eur. Math. Soc. (JEMS)* **15** (2013) 825–856.

[11] M. Boulakia and A. Osses, Local null controllability of a two-dimensional Fluid–structure interaction problem. *ESAIM: COCV* **14** (2008) 1–42.

[12] F. Boyer and P. Fabrie, Mathematical tools for the study of the incompressible Navier-Stokes equations and related models. Vol. 183 of *Applied Mathematical Sciences*. Springer, New York (2013).

[13] R. Buffe, Stabilization of the wave equation with Ventcel boundary condition, *J. Math. Pures Appl.* **108** (2017) 207–259.

[14] R. Buffe and L. Gagnon, Spectral inequality for an Oseen operator in a two dimensional channel (2021), working paper or preprint.

[15] S. Čanić, B. Muha and M. Bukač, Fluid–structure interaction in hemodynamics: modeling, analysis, and numerical simulation, in Fluid–structure interaction and biomedical applications, Adv. Math. Fluid Mech. Birkhäuser/Springer, Basel (2014), pp. 79–195.

[16] A. Chambolle, B. Desjardins, M.J. Esteban and C. Grandmont, Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. *J. Math. Fluid Mech.* **7** (2005) 368–404.

[17] F.W. Chaves-Silva and G. Lebeau, Spectral inequality and optimal cost of controllability for the Stokes system. *ESAIM: COCV* **22** (2016) 1137–1162.

[18] S.P. Chen and R. Triggiani, Proof of extensions of two conjectures on structural damping for elastic systems. *Pacific J. Math.* **136** (1989) 15–55.

[19] N. Cîndea, S. Micu, I. Rovenţa and M. Tucsnak, Particle supported control of a fluid-particle system. *J. Math. Pures Appl.* **104** (2015) 311–353.

[20] I.A. Djebour, Local null controllability of a fluid-rigid body interaction problem with Navier slip boundary conditions. *ESAIM: COCCV* **27** (2021) 46.

[21] A. Doubova and E. Fernández-Cara, Some control results for simplified one-dimensional models of fluid-solid interaction. *Math. Models Methods Appl. Sci.* **15** (2005) 783–824.

[22] E. Fernández-Cara, S. Guerrero, O.Y. Imanuvilov and J.-P. Puel, Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl.* **83** (2004) 1501–1542.

[23] C. Grandmont, Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate. *SIAM J. Math. Anal.* **40** (2008) 716–737.

[24] C. Grandmont and M. Hillairet, Existence of global strong solutions to a beam-fluid interaction system. *Arch. Ration. Mech. Anal.* **220** (2016) 1283–1333.

[25] C. Grandmont, M. Hillairet and J. Lequeurre, Existence of local strong solutions to fluid-beam and fluid-rod interaction systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **36** (2019) 1105–1149.

[26] O. Imanuvilov and T. Takahashi, Exact controllability of a fluid-rigid body system. *J. Math. Pures Appl.* **87** (2007) 408–437.

[27] O.Y. Imanuvilov, On exact controllability for the Navier-Stokes equations, *ESAIM: COCV* **3** (1998) 97–131.

[28] D. Jerison and G. Lebeau, Nodal sets of sums of eigenfunctions, in Harmonic analysis and partial differential equations (Chicago, IL, 1996), Chicago Lectures in Math., 223–239, Univ. Chicago Press, Chicago, IL (1999).

[29] J. Le Rousseau, M. Léautaud and L. Robbiano, Controllability of a parabolic system with a diffuse interface. *J. Eur. Math. Soc. (JEMS)* **15** (2013) 1485–1574.

[30] J. Le Rousseau, G. Lebeau and L. Robbiano, Elliptic Carleman estimates and applications to stabilization and controllability. Volume I. Dirichlet boundary conditions on Euclidean space. *Prog. Nonlinear Differ. Equ. Appl.*, vol. 97, Cham: Birkhäuser (2021).

[31] J. Le Rousseau, G. Lebeau and L. Robbiano, Elliptic Carleman estimates and applications to stabilization and controllability. Volume II. General Boundary Conditions on Riemannian Manifolds. *Prog. Nonlinear Differ. Equ. Appl.*, vol. 97, Birkhäuser, Cham (2022).

[32] J. Le Rousseau and N. Lerner, Carleman estimates for anisotropic elliptic operators with jumps at an interface. *Anal. PDE* **6** (2013) 1601–1648.

[33] J. Le Rousseau and L. Robbiano, Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations. *Arch. Ratl. Mech. Anal.* **195** (2010) 953–990.

[34] J. Le Rousseau and L. Robbiano, Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces. *Invent. Math.* **183** (2011) 245–336.

[35] M. Léautaud, Spectral inequalities for non-selfadjoint elliptic operators and application to the null-controllability of parabolic systems. *J. Funct. Anal.* **258** (2010) 2739–2778.

[36] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur. *Comm. Partial Differ. Equ.* **20** (1995) 335–356.

[37] D. Lengeler, Weak solutions for an incompressible, generalized Newtonian fluid interacting with a linearly elastic Koiter type shell. *SIAM J. Math. Anal.* **46** (2014) 2614–2649.

[38] D. Lengeler and M. Růžička, Weak solutions for an incompressible Newtonian fluid interacting with a Koiter type shell. *Arch. Ratl. Mech. Anal.* **211** (2014) 205–255.

[39] J. Lequeurre, Existence of strong solutions to a fluid–structure system. *SIAM J. Math. Anal.* **43** (2011) 389–410.

[40] Y. Liu, T. Takahashi and M. Tucsnak, Single input controllability of a simplified fluid–structure interaction model. *ESAIM: COCV* **19** (2013) 20–42.

[41] D. Maity, A. Roy and T. Takahashi, Existence of strong solutions for a system of interaction between a compressible viscous fluid and a wave equation. *Nonlinearity* **34** (2021) 2659–2687.

[42] D. Maity and T. Takahashi, L^p Theory for the Interaction Between the Incompressible Navier–Stokes System and a Damped Plate. *J. Math. Fluid Mech.* **23** (2021) 103.

[43] D. Maity and T. Takahashi, Existence and uniqueness of strong solutions for the system of interaction between a compressible Navier-Stokes-Fourier fluid and a damped plate equation. *Nonlinear Anal. Real World Appl.* **59** (2021) Paper No. 103267, 34.

[44] L. Miller, A direct Lebeau-Robbiano strategy for the observability of heat-like semigroups. *Discrete Contin. Dyn. Syst. Ser. B* **14** (2010) 1465–1485.

[45] S. Mitra, Observability and unique continuation of the adjoint of a linearized simplified compressible fluid-structure model in a 2D channel. *ESAIM: COCV* **27** (2021) Paper No. S18, 51.

[46] B. Muha and S. Čanić, Existence of a weak solution to a nonlinear fluid-structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls. *Arch. Ratl. Mech. Anal.* **207** (2013) 919–968.

[47] B. Muha and S. Čanić, Existence of a weak solution to a nonlinear Fluid–structure interaction problem modeling the flow of an incompressible, viscous fluid in a cylinder with deformable walls. *Arch. Ratl. Mech. Anal.* **207** (2013) 919–968.

[48] B. Muha and S. Čanić, A nonlinear, 3D fluid–structure interaction problem driven by the time-dependent dynamic pressure data: a constructive existence proof. *Commun. Inf. Syst.* **13** (2013) 357–397.

[49] B. Muha and S. Čanić, Fluid-structure interaction between an incompressible, viscous 3D fluid and an elastic shell with nonlinear Koiter membrane energy. *Interfaces Free Bound.* **17** (2015) 465–495.

[50] A. Quarteroni, M. Tuveri and A. Veneziani, Computational vascular fluid dynamics: problems, models and methods. *Comput. Visualizat. Sci.* **2** (2000) 163–197.

[51] M. Ramaswamy, A. Roy and T. Takahashi, Remark on the global null controllability for a viscous Burgers-particle system with particle supported control. *Appl. Math. Lett.* **107** (2020) 106483, 7.

[52] J.-P. Raymond, Feedback stabilization of a fluid–structure model. *SIAM J. Control Optim.* **48** (2010) 5398–5443.

[53] L. Robbiano, Cost function and control of solutions of hyperbolic equations. *Asymptotic Anal.* **10** (1995) 95–115.

[54] A. Roy and T. Takahashi, Local null controllability of a rigid body moving into a Boussinesq flow. *Math. Control Relat. Fields* **9** (2019) 793–836.

[55] A. Roy and T. Takahashi, Stabilization of a rigid body moving in a compressible viscous fluid. *J. Evol. Equ.* **21** (2021) 167–200.

[56] D.L. Russell, A unified boundary controllability theory for hyperbolic and parabolic partial differential equations. *Studies in Appl. Math.* **52** (1973) 189–211.

[57] T. Takahashi, M. Tucsnak and G. Weiss, Stabilization of a fluid-rigid body system. *J. Differ. Equ.* **259** (2015) 6459–6493.

[58] S. Trifunović and Y. Wang, Weak solution to the incompressible viscous fluid and a thermoelastic plate interaction problem in 3D. *Acta Math. Sci. Ser. B (Engl. Ed.)* **41** (2021) 19–38.

[59] S. Trifunović and Y.-G. Wang, Existence of a weak solution to the Fluid–structure interaction problem in 3D. *J. Differ. Equ.* **268** (2020) 1495–1531.

[60] M. Tucsnak and G. Weiss, Observation and control for operator semigroups, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel (2009).