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STABILITY IN DISTRIBUTION AND STABILIZATION OF SWITCHING JUMP DIFFUSIONS*

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Abstract. This paper aims to study stability in distribution of Markovian switching jump diffusions. The main motivation stems from stability and stabilizing hybrid systems in which there is no trivial solution. An explicit criterion for stability in distribution is derived. The stabilizing effects of Markov chains, Brownian motions, and Poisson jumps are revealed. Based on these criteria, stabilization problems of stochastic differential equations with Markovian switching and Poisson jumps are developed.

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1. Introduction

This work focuses on stability in distribution of a class of jump diffusions with Markovian switching. The underlying process is a two-component process $(X(\cdot), \alpha(\cdot))$, where $X(\cdot)$ describes the jump diffusion behavior and $\alpha(\cdot)$ is a continuous-time Markov chain having a finite state space. Recently, such a class of stochastic processes has received much attention in various settings for different domain of applications; see [13, 20] and references therein for comprehensive treatments and coverage of switching diffusions and [7, 9, 14–16, 18] for more recent progress in the fields.

Why is the consideration of stability in distribution important; why is it necessary? It is well known that in deterministic systems of differential equations, an important starting point is examination of equilibria. When one considers stochastic systems, in lieu of the equilibria, one often has to begin with stationary distributions. Thus to some extent, stationary distributions are frequently the primary concerns, especially when the systems have no equilibria. In addition, the stability in distribution is closely related to the concept of weak stability, which is a term used by Wonham [17]. Such weak stability concept implies the so-called recurrence under suitable conditions. That is, for a stochastic system given as the solutions of a differential equation, starting from a point

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outside an open set with compact closure, one wish to see if the trajectories will return to the open set in finite time infinitely often.

Most of the work in stability of switching diffusions and switching jump diffusions to date are concerned with stability in probability, moment stability, or almost sure stability, in which $x = 0$ is a trivial solution (an equilibrium point) to the corresponding equations and any other solution will converge to trivial solution in probability, in the pth moment for some $p > 0$, or in the almost sure sense. In contrast, we are interested in the cases that there is no equilibrium point of the differential equation, but there is still stability in the sense that all solutions converge in distribution to some probability measure. In [3], the authors considered stability in distribution of a semi-linear stochastic differential equation with Markovian switching of the form

$$\mathrm{d}X(t) = A(\alpha(t))X(t)\mathrm{d}t + \sigma(X(t), \alpha(t))\mathrm{d}w(t),$$

where $w(\cdot)$ is a standard Brownian motion. In an important development [24], Yuan and Mao provided sufficient conditions guaranteeing stability in distribution for nonlinear Markovian switching diffusions of the form

$$\mathrm{d}X(t) = b(X(t), \alpha(t))\mathrm{d}t + \sigma(X(t), \alpha(t))\mathrm{d}w(t), \tag{1.1}$$

where $\alpha(\cdot)$ is a finite-state Markov chain. Subsequently, in the work of [8], Nguyen provided much weaker conditions of by using localization arguments to further improve the criteria for stability in distribution. In [19], the authors have considered stability in distribution of a switching jump diffusion

$$\begin{aligned} &\mathrm{d}X(t) = b(X(t), \alpha(t))\mathrm{d}t + \sigma(X(t), \alpha(t))\mathrm{d}w(t) + \mathrm{d}J(t),\\ &J(t) = \int_0^t \int_\Gamma g(X(s^-), \alpha(s^-), \gamma)N(\mathrm{d}s, \mathrm{d}\gamma), \end{aligned} \tag{1.2}$$

where $b(\cdot)$, $\sigma(\cdot)$, and $g(\cdot)$ are suitable functions, and $N(t, \cdot)$ is a Poisson measure. For existence and uniqueness of solutions as well as the related maximum principles and Harnack inequalities, we refer to [6]. Related works on stability in distribution of the aforementioned systems can be found in [2, 4, 5]. Some criteria for invariant measures and stability in distribution of equation (1.1) and its generalizations with path-dependent and path-independent switching can be found in [1, 11, 14]. We refer to [15, 16, 22, 25, 26] for related works on stability in probability and exponential stability of equation (1.2). Recent efforts on stabilization in distribution of hybrid systems by certain feedback controls can be found in [12, 23]. Regarding equation (1.2), the criteria in [19] are given in terms of the existence of a set of Lyapunov functions $V(x, i)$ for $i \in \mathcal{M}$, where $\mathcal{M}$ is the state space of $\alpha(\cdot)$. Consequently, it is nontrivial to apply these criteria. We are not aware any work on explicit criteria for stability in distribution of equation (1.2). In this work, our first aim is to construct a general criterion for stability in distribution of equation (1.2). The novelty of our work lies in that in order to apply our criterion, one need only construct at most two Lyapunov functions $U(x)$ and $V(x)$ and in most common cases, it is sufficient to construct $U(x)$ only. Moreover, we reveal the contribution of the Markov chain $\alpha(\cdot)$ in the sense that equation (1.2) is stable in distribution if $\sum_{i\in\mathcal{M}} \nu_i\eta_i < 0$ and $\sum_{i\in\mathcal{M}} \nu_i\zeta_i < 0$, where $\nu = (\nu_1, \dots, \nu_m)$ is the stationary distribution of $\alpha(\cdot)$ and $\eta = (\eta_1, \dots, \eta_m)^\top$ and $\zeta = (\zeta_1, \dots, \zeta_m)^\top$ are certain vectors. Another distinct feature of our work is the construction of an explicit and easily verifiable criterion for stability for switching jump diffusions.

Treating stability of hybrid systems, motivated by [3, 12, 21, 23, 26], the following question arises. Can we apply feedback controls (or perturbations using Brownian motions and/or Poisson jumps) to stabilize a given system? Moreover, if a given system is not regular (not having global solutions), can we design certain feedback rules to regularize and stabilize it? To the best of our knowledge, these topics have not been well understood for stability in distribution. Using the criteria for stability in distribution developed in this work, we address these questions. We show that given any scalar switching differential equations, one can design feedback strategies so that the resulting switching jump diffusions are stable in distribution. Nevertheless, the multi-dimensional counterpart is rather challenging. By designing a novel treatment, we are able to treat a wide class of such stochastic dynamic systems so that we can regularize and stabilize the systems in distribution.

The contributions of our work in this paper can be summarized as follows.

(1) We focus on nonlinear stochastic differential equations with jumps and Markov switching, and provide sufficient conditions that are substantially weaker than the existing results and extend and further improve the results in [8].
(2) We give insight on how each of the components, namely, Brownian motion, switching, and jump process can contribute in a positive way to stability in distribution.
(3) We further obtain strategies to stabilize randomly switching ordinary differential equations.
(4) When the jump disappears in the dynamic systems, our results cover that of switching diffusions; when the Brownian motion also disappears, our results cover that of switching differential equations.

The rest of the work is organized as follows. Section 2 presents the problem formulation. Section 3 proceeds with criteria for stability in distribution. Section 4 develops strategies for stabilization in the sense of stability in distribution for the stochastic dynamic systems that we are interested in. Section 5 provides some examples for illustration. Finally, Section 6 concludes the paper with a few more remarks.

2. Formulation

We begin this section with the following notation.

Notation. Let $\mathbb{R}_+ = [0,\infty)$ and $\mathbb{N}$ be the set of positive integers. Let $C^2\big(\mathbb{R}^d,\mathbb{R}_+\big)$ be the set of all functions $V:\mathbb{R}^d\to\mathbb{R}_+$, which are twice continuously differentiable on $\mathbb{R}^d$ with $C^2\big(\mathbb{R}^d_0,\mathbb{R}_+\big)$ being the set of functions $V:\mathbb{R}^d\to\mathbb{R}_+$ that are twice continuously differentiable on $\mathbb{R}^d_0 := \mathbb{R}^d\setminus\{0\}$. For two real numbers c_1, c_2, $c_1\vee c_2$ denotes $\max\{c_1,c_2\}$. For a matrix $A\in\mathbb{R}^{d_1\times d_2}$, $A^\top$ denotes its transpose. For a matrix $A\in\mathbb{R}^{d\times d}$, its trace norm is given by $|A|=\sqrt{\operatorname{tr}(AA^\top)}$, while I_d denotes the $d\times d$ identity matrix. For $x=(x_1,\dots,x_d)^\top\in\mathbb{R}^d$, its Euclidean norm is denoted by $|x|=\big(\sum_{i=1}^d x_i^2\big)^{1/2}$. For a nonempty set $\Gamma\subset\mathbb{R}^d\setminus\{0\}$ and a probability measure π defined on Γ, denote by Γ_b the family of all bounded positive functions $h(y)$ on Γ with $\int_\Gamma \ln\big[h(y)\big]\pi(\mathrm{d}\gamma)<\infty$.

We work with a complete filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\})$ with the filtration $\{\mathcal{F}_t\}$ satisfying the usual condition (*i.e.*, it is right-continuous and $\mathcal{F}_0$ contains all the null sets). Assume that the Markov chain $\alpha(\cdot)$ and the d-dimensional standard Brownian motion $w(\cdot)$ are defined on $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\})$. Moreover, $\alpha(\cdot)$ and $w(\cdot)$ are $\{\mathcal{F}_t\}$-adapted and independent.

Suppose $\alpha(\cdot)$ takes values in $\mathcal{M}=\{1,\dots,m\}$ with the generator $Q=(q_{ij})\in\mathbb{R}^{m\times m}$, where $m\in\mathbb{N}$. Hence, $\alpha(\cdot)$ is described by a transition probability specification of the form

$$\mathbb{P}\{\alpha(t+\Delta t)=j|\alpha(t)=i\}=\begin{cases} q_{ij}\Delta t+o(\Delta t) & \text{if } i\neq j,\\ 1+q_{ii}\Delta t+o(\Delta t) & \text{if } i=j.\end{cases} \tag{2.1}$$

Note that $q_{ij}\ge 0$ if $i\neq j$ and $\sum_{j\in\mathcal{M}} q_{ij}=0$ for any $i\in\mathcal{M}$.

Let Γ be a subset of $\mathbb{R}^d\setminus\{0\}$ that is the range space of the impulsive jumps. For any subset B in Γ, $N(t,B)$ counts the number of impulses on $[0,t]$ with values in B, $b(\cdot,\cdot):\mathbb{R}^d\times\mathcal{M}\mapsto\mathbb{R}^d$, $\sigma(\cdot,\cdot):\mathbb{R}^d\times\mathcal{M}\mapsto\mathbb{R}^d\times\mathbb{R}^d$, and $g(\cdot,\cdot,\cdot):\mathbb{R}^d\times\mathcal{M}\times\Gamma\mapsto\mathbb{R}^d$ are suitable Borel functions under some precise conditions to be specified later.

Consider the dynamic system given by

$$\begin{aligned} \mathrm{d}X(t)&=b(X(t),\alpha(t))\mathrm{d}t+\sigma(X(t),\alpha(t))\mathrm{d}w(t)+\mathrm{d}J(t),\\ J(t)&=\int_0^t\int_\Gamma g(X(s^-),\alpha(s^-),\gamma)N(\mathrm{d}s,\mathrm{d}\gamma),\end{aligned} \tag{2.2}$$

with initial condition $X(0)=x_0, \alpha(0)=i_0$, where $N(t,B)$ is a Poisson measure such that the jump process $N(\cdot)$ is independent of the Brownian motion $w(\cdot)$ and the switching process $\alpha(\cdot)$. The compensated Poisson measure is defined by

$$\widetilde{N}(t,B)=N(t,B)-\lambda t\pi(B)\quad\text{for}\quad B\subset\Gamma,$$

where $\lambda\in(0,\infty)$ is known as the jump rate and $\pi(\cdot)$ is the jump distribution with $\pi(\Gamma)=1$. We have used the set up as in [22].

We define an operator $\mathcal{G}$ as follows. If $V:\mathbb{R}^d\times\mathcal{M}\mapsto\mathbb{R}$ satisfying $V(\cdot,i)\in C^2\big(\mathbb{R}^d,\mathbb{R}_+\big)$ for each $i\in\mathcal{M}$, then

$$\begin{aligned}(\mathcal{G}V)(x,i)\ &=\frac{1}{2}\mathrm{tr}\big(\sigma^\top(x,i)V_{xx}(x,i)\sigma(x,i)\big)+V_x(x,i)b(x,i)+QV(x,\cdot)(i)\\&\quad+\lambda\int_\Gamma[V(x+g(x,i,\gamma),i)-V(x,i)]\pi(\mathrm{d}\gamma),\end{aligned}$$

where

$$V_x(x,i)=\Big(\frac{\partial V(x,i)}{\partial x_1},\dots,\frac{\partial V(x,i)}{\partial x_d}\Big),\quad V_{xx}(x,i)=\Big(\frac{\partial^2V(x,i)}{\partial x_k\partial x_l}\Big)_{d\times d},$$

and $QV(x,\cdot)(i)=\sum_{j\in\mathcal{M}}q_{ij}V(x,j)$. For notational simplicity, we also write $(QV)(x,i)=\sum_{j\in\mathcal{M}}q_{ij}V(x,j)$. Thus,

$$(QV)(x,i)=\sum_{j\in\mathcal{M}}q_{ij}V(x,j)=\sum_{j\in\mathcal{M},j\neq i}q_{ij}\big(V(x,j)-V(x,i)\big)\quad\text{for}\quad(x,i)\in\mathbb{R}^d\times\mathcal{M}.$$

Next, we introduce the functions $\overline{b}:\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{M}\to\mathbb{R}$, $\overline{\sigma}:\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{M}\to\mathbb{R}^d$, and $\overline{g}:\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{M}\times\Gamma\to\mathbb{R}^d$ by

$$\begin{aligned}&\overline{b}(x,y,i)=b(x,i)-b(y,i),\quad\overline{\sigma}(x,y,i)=\sigma(x,i)-\sigma(y,i),\\&\overline{g}(x,y,i,\gamma)=g(x,i,\gamma)-g(y,i,\gamma),\quad(x,y,i,\gamma)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathcal{M}\times\Gamma.\end{aligned}$$

We also define an operator $\overline{\mathcal{G}}$ as follows. If $U:\mathbb{R}^d\times\mathcal{M}\mapsto\mathbb{R}$ satisfying $U(\cdot,i)\in C^2\big(\mathbb{R}^d,\mathbb{R}_+\big)$ for each $i\in\mathcal{M}$, then

$$\begin{aligned}(\overline{\mathcal{G}}U)(x,y,i)\ &=\frac{1}{2}\mathrm{tr}\big(\overline{\sigma}^\top(x,y,i)U_{xx}(x-y,i)\overline{\sigma}(x,y,i)\big)+U_x(x-y,i)\overline{b}(x,y,i)\\&\quad+(QU)(x-y,i)+\lambda\int_\Gamma[U(x-y+\overline{g}(x,y,i,\gamma),i)-U(x-y,i)]\pi(\mathrm{d}\gamma).\end{aligned}$$

To proceed, we pose the following conditions.

(A1) For each $n\in\mathbb{N}$, there exists a constant $\widetilde{K}_n>0$ such that

$$|b(x,i)-b(y,i)|+|\sigma(x,i)-\sigma(y,i)|+\int_\Gamma|g(x,i,\gamma)-g(y,i,\gamma)|\pi(\mathrm{d}\gamma)\le\widetilde{K}_n|x-y|$$

whenever $|x|\vee|y|\le n$ and $i\in\mathcal{M}$. Moreover, $\sup_{i\in\mathcal{M}}\int_\Gamma|g(0,i,\gamma)|\pi(\mathrm{d}\gamma)<\infty$.

(A2) The Markov chain $\alpha(\cdot)$ is irreducible. That is, the system of equations

$$\nu Q=0,\quad\sum_{i\in\mathcal{M}}\nu_i=1$$

has a unique solution $\nu=(\nu_1,\dots,\nu_m)$ satisfying $\nu_i>0$ for each $i\in\mathcal{M}$.

(A3) There exist a function $V(\cdot) \in C^2(\mathbb{R}_0^d, \mathbb{R}_+)$, constants $\eta_{i,\delta}$ and $r_\delta > 0$ for $(i,\delta) \in \mathcal{M} \times (0,1)$ such that

$$\begin{aligned}&\lim_{|x|\to\infty} V(x) = \infty,\\ &(\mathcal{G}V^\delta)(x,i) \le \eta_{i,\delta} V^\delta(x) + r_\delta \ \text{ for } \ (x,i) \in \mathbb{R}_0^d \times \mathcal{M}, \quad \delta \in (0,1),\\ &\lim_{\delta\to 0} \eta_{i,\delta}/\delta = \eta_i \ \text{ and } \ \sum_{i\in\mathcal{M}} \nu_i \eta_i < 0,\end{aligned}$$

where $\nu = (\nu_1, \dots, \nu_m)$ is given in condition (A2) and $V^\delta(\cdot) = \big(V(\cdot)\big)^\delta$.

(A4) There exist a function $U(\cdot) \in C^2(\mathbb{R}_0^d, \mathbb{R}_+)$ and constants $\zeta_{i,\delta}$ for $(i,\delta) \in \mathcal{M} \times (0,1)$ such that

$$\begin{aligned}&U(0) = 0, \quad \inf_{|x|\ge r} U(x) > 0 \ \text{ for any } \ r > 0,\\ &(\overline{\mathcal{G}}U^\delta)(x,y,i) \le \zeta_{i,\delta} U^\delta(x-y) \ \text{ for } \ (x,y,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}, \quad x \ne y,\\ &\lim_{\delta\to 0} \zeta_{i,\delta}/\delta = \zeta_i \ \text{ and } \ \sum_{i\in\mathcal{M}} \nu_i \zeta_i < 0,\end{aligned} \tag{2.3}$$

where $U^\delta(\cdot) = \big(U(\cdot)\big)^\delta$.

Remark 2.1. Condition (A1) essentially is a local Lipschitz condition. It guarantees that for each $(x_0, i_0) \in \mathbb{R}^d \times \mathcal{M}$, equation (2.2) has a unique solution $\big(X^{x_0,i_0}(\cdot), \alpha^{i_0}(\cdot)\big)$. Using condition (A3), we show that this solution is global. That is, it is regular (see Def. 2.4). In order to verify each of conditions (A3) and (A4), one need only find one Lyapunov function. In condition (A2), $\nu = (\nu_1, \dots, \nu_m)$ is the stationary distribution of $\alpha(\cdot)$. The contribution of the switching process $\alpha(\cdot)$ is revealed explicitly *via* its contribution to the sums $\sum_{i\in\mathcal{M}} \nu_i \eta_i$ and $\sum_{i\in\mathcal{M}} \nu_i \zeta_i$.

Remark 2.2. Suppose $g(\cdot,\cdot,\cdot) \equiv 0$. Then equation (2.2) is simply a switching diffusion. Let $V(\cdot) \in C^2(\mathbb{R}_0^d, \mathbb{R}_+)$, a constant $\widetilde{c} > 0$, and $(\eta_1, \dots, \eta_m)^\top \in \mathbb{R}^m$ be such that $V(x) \ge 1$ for $x \in \mathbb{R}^d$,

$$\lim_{|x|\to\infty} V(x) = \infty, \quad (\mathcal{G}V)(x,i) \le \eta_i V(x) + \widetilde{c} \ \text{ for } \ (x,i) \in \mathbb{R}_0^d \times \mathcal{M}, \quad \sum_{i\in\mathcal{M}} \nu_i \eta_i < 0.$$

We claim that condition (A3) holds. Indeed, for $\delta \in (0,1)$,

$$\begin{aligned}(\mathcal{G}V^\delta)(x,i) &= \delta V^{\delta-1}(x)(\mathcal{G}V)(x,i) + \frac{\delta(\delta-1)}{2} V^{\delta-2}(x)\big|V_x(x)\sigma(x,i)\big|^2\\ &\le \delta\eta_i V^\delta(x) + \delta\widetilde{c}, \quad (x,i) \in \mathbb{R}_0^d \times \mathcal{M}.\end{aligned}$$

Hence, it is clear that condition (A3) is satisfied. Similarly, condition (A4) holds if there exist $U(\cdot) \in C^2(\mathbb{R}_0^d, \mathbb{R}_+)$ and $(\zeta_1, \dots, \zeta_m)^\top \in \mathbb{R}^m$ be such that $U(0) = 0$, $\inf_{|x|\ge r} U(x) > 0$ for any $r > 0$ and

$$\sum_{i\in\mathcal{M}} \nu_i \zeta_i < 0, \quad (\overline{\mathcal{G}}U)(x,y,i) \le \zeta_i U(x-y,i) \ \text{ for } \ (x,y,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}, x \ne y.$$

Remark 2.3. Assume (A1) and (A2). Suppose condition (A4) holds with $U(x) = |x|$ and there exist $K > 0$ and $r \in (0,2)$ such that

$$|\sigma(x,i)| \le K(|x|^r + 1) \ \text{ for } \ (x,i) \in \mathbb{R}^d \times \mathcal{M}. \tag{2.4}$$

Then condition (A3) is also satisfied. Indeed, letting $y = 0$ in (2.3) yields

$$(\overline{\mathcal{G}}U^\delta)(x, 0, i) \le \zeta_{i,\delta} U^\delta(x) \;\; \text{for} \;\; (x, i) \in \mathbb{R}^d_0 \times \mathcal{M}, \tag{2.5}$$

where $U^\delta(x) = |x|^\delta$ for $x \in \mathbb{R}^d$ and $\delta \in (0, 1)$. Direct computation yields that

$$U^\delta_x(x) = \delta |x|^{\delta-2} x, \quad U^\delta_{xx}(x) = \delta\big[|x|^{\delta-2} I_d + (\delta - 2)|x|^{\delta-4} x x^\top\big], \quad x \in \mathbb{R}^d_0. \tag{2.6}$$

In view of (2.4) and (2.6), for $\delta < 2 - r$, there exists a constant $r_\delta > 0$ such that

$$\begin{aligned} &\frac{1}{2}\mathrm{tr}\big(\overline{\sigma}^\top(x, 0, i) U^\delta_{xx}(x) \overline{\sigma}(x, 0, i)\big) \ge \frac{1}{2}\mathrm{tr}\big(\sigma^\top(x, i) U^\delta_{xx}(x) \sigma(x, i)\big) - \frac{r_\delta}{3}, \\ &U^\delta_x(x)\overline{b}(x, 0, i) \ge U^\delta_x(x) b(x, i) - \frac{r_\delta}{3}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &\lambda \int_\Gamma [U^\delta(x + \overline{g}(x, 0, i, \gamma)) - U^\delta(x)]\pi(\mathrm{d}\gamma) \\ &\quad \ge \lambda \int_\Gamma [\big|x + g(x, i, \gamma)\big|^\delta - |x|^\delta]\pi(\mathrm{d}\gamma) - \lambda \int_\Gamma |g(0, i, \gamma)|^\delta \pi(d\gamma) \\ &\quad = \lambda \int_\Gamma [U^\delta(x + g(x, i, \gamma)) - U^\delta(x)]\pi(\mathrm{d}\gamma) - \frac{r_\delta}{3} \end{aligned} \tag{2.8}$$

for any $(x, i) \in \mathbb{R}^d_0 \times \mathcal{M}$. It follows from (2.5), (2.7), and (2.8) that

$$(\mathcal{G}U^\delta)(x, i) \le \zeta_{i,\delta} U^\delta(x) + r_\delta \;\; \text{for} \;\; (x, i) \in \mathbb{R}^d_0 \times \mathcal{M}.$$

Thus, condition (A3) is satisfied with $V(x) = U(x) = |x|$ and $\eta_{i,\delta} = \zeta_{i,\delta}$ for $(i, \delta) \in \mathcal{M} \times (0, 1)$.

The regularity and stability in distribution of the process $(X(\cdot), \alpha(\cdot))$ are defined as follows. A system being regular essentially means that it has no finite explosion time, whereas stability in distribution is a weaker sense notion of stability for a stochastic dynamic system.

Definition 2.4.
(a) The process $\big(X(\cdot), \alpha(\cdot)\big)$ with initial data $(X(0), \alpha(0)) = (x_0, i_0)$ is said to be regular if for any $0 < T < \infty$,

$$\mathbb{P}\Big(\sup_{0 \le t \le T} |X^{x_0,i_0}(t)| = \infty\Big) = 0.$$

(b) The process $\big(X(\cdot), \alpha(\cdot)\big)$ is said to be stable in distribution if it is regular and there exists a probability measure $\pi(dx \times \{j\})$ on $\mathbb{R}^d \times \mathcal{M}$ such that its transition probability $p(t, x_0, i_0, dx \times \{j\})$ converges weakly to $\pi(dx \times \{j\})$ as $t \to \infty$ for any $(x_0, i_0) \in \mathbb{R}^d \times \mathcal{M}$. equation (2.2) is said to be stable in distribution if $\big(X(\cdot), \alpha(\cdot)\big)$ is stable in distribution.

To study the stability in distribution of equation (2.2), motivated by [24] and [19], we introduce properties (P1), (P2), and (P3) as follows.

Definition 2.5.
(a) Equation (2.2) is said to have property (P1) if for any $(x_0, i_0) \in \mathbb{R}^d \times \mathcal{M}$ and any $\varepsilon > 0$, there exists a constant $R > 0$ such that

$$\mathbb{P}\Big(\big|X^{x_0,i_0}(t)\big| \ge R\Big) < \varepsilon \;\; \text{for} \;\; t \ge 0.$$

(b) Equation (2.2) is said to have property (P2) if for any $\varepsilon > 0$ and any compact subset D of $\mathbb{R}^d$, there exists $T = T(\varepsilon, D) > 0$ such that

$$\mathbb{P}\Big(\big|X^{x_0,i_0}(t) - X^{y_0,i_0}(t)\big| < \varepsilon\Big) \geq 1 - \varepsilon \text{ for } t \geq T,$$

whenever $(x_0, y_0, i_0) \in D \times D \times \mathcal{M}$.

(c) Equation (2.2) is said to have property (P3) if for any $T > 0$, any $\varepsilon > 0$ and any compact subset D of $\mathbb{R}^d$, there exists a constant $R > 0$ such that

$$\mathbb{P}\Big(\big|X^{x_0,i_0}(t)\big| \leq R \text{ for all } t \in [0, T]\Big) \geq 1 - \varepsilon \text{ for } (x_0, i_0) \in D \times \mathcal{M}.$$

Remark 2.6. Properties (P1) and (P2) are essentially those used in [24] and [19]. Nevertheless, in [24] and [19], the authors assume that the drift and diffusion coefficients satisfy the linear growth condition, which guarantees that property (P3) holds (see [24], Eq. (3.10), p. 282). Therefore, property (P3) is not stated explicitly in the aforementioned references. In this paper, we drop the linear growth condition, hence, we need property (P3).

3. Criteria for stability in distribution

As a preparation of the subsequent study, we first state a lemma, which is more or less a restatement of the Fredholm alternative; see Lemma A.12 of [20] for a proof.

Lemma 3.1. *Under condition* (A2)*, for any $\xi \in \mathbb{R}^m$, $Qc = \xi$ has a solution $c \in \mathbb{R}^m$ if and only if $\nu\xi = 0$, where ν is the stationary distribution associated with Q.*

Lemma 3.2. *Assume* (A1)–(A3)*. Then $(X(\cdot), \alpha(\cdot))$ with initial data $(X(0), \alpha(0)) = (x_0, i_0)$ is regular for any $(x_0, i_0) \in \mathbb{R}^d \times \mathcal{M}$. Moreover, equation* (2.2) *has properties* (P1) *and* (P3)*.*

Proof. The proof is divided into three steps.

Step 1: Consider the function

$$W(x, i) = (1 - \delta c_i)V^\delta(x), \quad (x, i) \in \mathbb{R}^d \times \mathcal{M},$$

where $c_1, c_2, \ldots, c_m$ are constants to be determined, $\delta \in (0, 1)$ is sufficiently small so that $1 - \delta c_i > 0$ for each $i \in \mathcal{M}$. We claim that we can choose $c_1, c_2, \ldots, c_m$, $\delta > 0$, $r > 0$, and $\beta > 0$ such that $(\mathcal{G}W)(x, i) \leq -\beta W(x, i) + r$ for $(x, i) \in \mathbb{R}^d_0 \times \mathcal{M}$.

Indeed, we have

$$\begin{aligned}(\mathcal{G}W)(x, i) &= (1 - \delta c_i)(\mathcal{G}V^\delta)(x, i) - \sum_{j\neq i, j\in\mathcal{M}} q_{ij}V^\delta(x)(c_j - c_i)\delta \\ &= \delta(1 - \delta c_i)\bigg(\frac{1}{\delta}(\mathcal{G}V^\delta)(x, i) - V^\delta(x) \sum_{j\neq i, j\in\mathcal{M}} q_{ij}\frac{c_j - c_i}{1 - \delta c_i}\bigg).\end{aligned} \tag{3.1}$$

Recall that $\sum\limits_{j\in\mathcal{M}} q_{ij} = 0$ for each $i \in \mathcal{M}$. We obtain

$$\begin{aligned}\sum_{j\neq i, j\in\mathcal{M}} q_{ij}\frac{c_j - c_i}{1 - \delta c_i} &= \sum_{j\in\mathcal{M}} q_{ij}c_j + \sum_{j\neq i, j\in\mathcal{M}} q_{ij}\frac{c_i(c_j - c_i)\delta}{1 - \delta c_i} \\ &= \sum_{j\in\mathcal{M}} q_{ij}c_j + O(\delta).\end{aligned} \tag{3.2}$$

It follows from (3.1), (3.2), and condition (A3) that

$$(\mathcal{G}W)(x,i) \ \le \delta(1-\delta c_i)V^\delta(x)\Big(\eta_i - \sum_{j\in\mathcal{M}} q_{ij}c_j + O(\delta)\Big) + (1-\delta c_i)r_\delta, \quad (x,i)\in\mathbb{R}_0^d\times\mathcal{M}. \tag{3.3}$$

By Lemma 3.1, the equation

$$Qc = \big(\eta_1,\eta_2,\dots,\eta_d\big)^\top - \big(\sum_{j\in\mathcal{M}} \nu_j\eta_j\big)\mathbb{1}$$

has a solution $c=(c_1,c_2,\dots,c_m)^\top\in\mathbb{R}^m$, where $\mathbb{1}=(1,1,\dots,1)^\top\in\mathbb{R}^m$. The numbers $c_1,c_2,\dots,c_m$ we just found are used in the definition of $W(\cdot,\cdot)$. Thus, we have

$$\eta_i - \sum_{j\in\mathcal{M}} q_{ij}c_j = \sum_{j\in\mathcal{M}} \nu_j\eta_j \ \text{ for } \ i\in\mathcal{M}.$$

Using this representation in (3.3), we obtain

$$(\mathcal{G}W)(x,i)\le \delta W(x,i)\Big(\sum_{j\in\mathcal{M}}\nu_j\eta_j + O(\delta)\Big) + (1-\delta c_i)r_\delta, \quad (x,i)\in\mathbb{R}_0^d\times\mathcal{M}.$$

Since $\sum_{j\in\mathcal{M}}\nu_j\eta_j<0$, we can choose $\delta>0$, $\beta>0$, and $r>0$ so that

$$(1/2)V^\delta(x)\le W(x,i)\le 2V^\delta(x), \quad (\mathcal{G}W)(x,i)\le -\beta W(x,i)+r \ \text{ for } \ (x,i)\in\mathbb{R}_0^d\times\mathcal{M}. \tag{3.4}$$

In view of (3.4), we can find a large $R>0$ such that

$$(\mathcal{G}W)(x,i)\le W(x,i) \ \text{ for } \ |x|\ge R, i\in\mathcal{M}, \inf_{|x|>R, i\in\mathcal{M}} W(x,i)\to\infty \quad \text{as } \ R\to\infty.$$

By using a standard argument, we can show that the process $\big(X(\cdot),\alpha(\cdot)\big)$ with initial data $(X(0),\alpha(0))=(x_0,i_0)$ is regular for any $(x_0,i_0)\in\mathbb{R}^d\times\mathcal{M}$; see Theorem 3.4.1 of [10].

Step 2: Let $(x_0,i_0)\in\mathbb{R}^d\times\mathcal{M}$ and $\varepsilon>0$. To establish property (P1), we use the same steps as in the proof of Lemma 4.1 of [24]. A sketch is given as follows. By using (3.4) and the Dynkin formula, we can show that $\sup_{t\ge0}\mathbb{E}|V^\delta\big(X^{x_0,i_0}(t)\big)|<\infty$. Let $C=\sup_{t\ge0}\mathbb{E}|V^\delta\big(X^{x_0,i_0}(t)\big)|$ and $R_1>C/\varepsilon$. Since $\lim_{|x|\to\infty}V(x)=\infty$, there exists a constant $R>0$ such that

$$V^\delta(x)\ge R_1 \ \text{ whenever } \ x\ge R.$$

This together with the Chebyshev inequality implies

$$\mathbb{P}\big(|X^{x_0,i_0}(t)|\ge R\big)\le\mathbb{P}\big(V^\delta\big(X^{x_0,i_0}(t)\big)\ge R_1\big)\le\frac{\mathbb{E}|V^\delta\big(X^{x_0,i_0}(t)\big)|}{R_1}\le\frac{C}{R_1}<\varepsilon, \quad t\ge0.$$

Thus, equation (2.2) has property (P1).

Step 3: Noting Remark 2.6, because no linear growth is assumed, we need to establish property (P3). To this end, let $T>0$, $\varepsilon>0$, a compact subset D of $\mathbb{R}^d$, and $(x_0,i_0)\in D\times\mathcal{M}$. Without loss of generality, we can suppose the function $V(\cdot)$ in condition (A3) satisfies $V(\cdot) \in C^2(\mathbb{R}^d, \mathbb{R}_+)$. Otherwise, we can work with a function $\widehat{V} \in C^2(\mathbb{R}^d, \mathbb{R}_+)$ with $\widehat{V}(x) = V(x)$ for $|x| \geq 1$. Define

$$\tau_n = \inf\{t \geq 0 : \left|X^{x_0,i_0}(t)\right| \geq n\} \quad \text{for} \quad n \in \mathbb{N}.$$

Since $\big(X^{x_0,i_0}(\cdot), \alpha^{i_0}(\cdot)\big)$ is regular, $\tau_n \to \infty$ almost surely as $n \to \infty$. In view of (3.4), there exists a constant $K > 0$ such that $(\mathcal{G}W)(x,i) \leq K/(2T)$ for any $(x,i) \in \mathbb{R}^d \times \mathcal{M}$ and $W(x,i) \leq K/2$ for any $(x,i) \in D \times \mathcal{M}$. Then the Dynkin formula yields

$$\begin{aligned} \mathbb{E}\big[W\big(X^{x_0,i_0}(\tau_n \wedge T), \alpha^{i_0}(\tau_n \wedge T)\big)\big] &= W(x_0,i_0) + \mathbb{E}\int_0^{\tau_n \wedge T} \mathcal{G}W(X^{x_0,i_0}(s), \alpha^{i_0}(s))\mathrm{d}s \\ &\leq K/2 + TK/(2T) = K. \end{aligned} \tag{3.5}$$

Let $\rho_n = \min\limits_{|x|=n} V^\delta(x)$. By condition (A3), $\rho_n \to \infty$ as $n \to \infty$. The first estimate in (3.4) and (3.5) imply

$$(1/2)\rho_n \mathbb{P}(\tau_n \leq T) \leq \mathbb{E}\big[W\big(X^{x_0,i_0}(\tau_n \wedge T), \alpha^{i_0}(\tau_n \wedge T)\big)\big] \leq K.$$

That is, $\mathbb{P}(\tau_n \leq T) \leq 2K/\rho_n$. Let $n = R \in \mathbb{N}$ be such that $2K/\rho_R \leq \varepsilon$. It follows that $\mathbb{P}(\tau_R \leq T) \leq \varepsilon$. Equivalently,

$$\mathbb{P}\Big(\left|X^{x_0,i_0}(t)\right| \leq R \ \text{ for all } \ t \in [0,T]\Big) \geq 1 - \varepsilon.$$

Note also that this estimate holds for all $(x_0, i_0) \in D \times \mathcal{M}$. Thus, equation (2.2) has property (P3). This completes the proof. □

The following lemma indicates that if $x_0, y_0 \in \mathbb{R}^d$, $i_0 \in \mathcal{M}$ and $x_0 \neq y_0$, then almost all sample paths of $X^{x_0,i_0}(t)$ and $X^{y_0,i_0}(t)$ will never intersect.

Lemma 3.3. *Assume* (A1)–(A3). *For any* $x_0, y_0 \in \mathbb{R}^d$, $i_0 \in \mathcal{M}$ *and* $x_0 \neq y_0$, *we have*

$$\mathbb{P}\big(\left|X^{x_0,i_0}(t) - X^{y_0,i_0}(t)\right| \neq 0 \ \textit{ for any } \ t \geq 0\big) = 1.$$

Proof. The proof is standard. It is a modification of Lemma 2.10 in [22]. We omit it for brevity. □

Lemma 3.4. *Assume* (A1)–(A4). *Then equation* (2.2) *has property* (P2).

Proof. Consider the function

$$W(x,i) = (1 - \delta c_i)U^\delta(x), \quad (x,i) \in \mathbb{R}^d \times \mathcal{M},$$

where $c_1, c_2, \ldots, c_m$ are constants to be determined, $\delta \in (0,1)$ is sufficiently small so that $1 - \delta c_i > 0$ for each $i \in \mathcal{M}$. We claim that we can choose $c_1, c_2, \ldots, c_m$, $\delta > 0$, and $\beta > 0$ such that $(\overline{\mathcal{G}}W)(x,y,i) \leq -\beta W(x-y,i)$ for $(x,y,i) \in (\mathbb{R}^d)^2 \times \mathcal{M}$ and $x \neq y$.

Indeed, we have

$$\begin{aligned} (\overline{\mathcal{G}}W)(x,y,i) &= (1-\delta c_i)(\overline{\mathcal{G}}U^\delta)(x,y,i) - \sum_{j \neq i, j \in \mathcal{M}} q_{ij} U^\delta(x-y)(c_j - c_i)\delta \\ &= \delta(1-\delta c_i)\bigg(\frac{1}{\delta}(\overline{\mathcal{G}}U^\delta)(x,y,i) - U^\delta(x-y)\sum_{j \neq i, j \in \mathcal{M}} q_{ij}\frac{c_j - c_i}{1 - \delta c_i}\bigg). \end{aligned} \tag{3.6}$$

In view of (3.2), we have

$$\sum_{j\neq i,j\in\mathcal{M}} q_{ij}\frac{c_j-c_i}{1-\delta c_i}=\sum_{j\in\mathcal{M}} q_{ij}c_j+O(\delta). \tag{3.7}$$

It follows from (3.6), (3.7), and condition (A4) that

$$(\overline{\mathcal{G}}W)(x,y,i)\ \leq \delta(1-\delta c_i)U^\delta(x-y)\Big(\zeta_i-\sum_{j\in\mathcal{M}} q_{ij}c_j+O(\delta)\Big),\ \ (x,y,i)\in(\mathbb{R}^d)^2\times\mathcal{M}, x\neq y. \tag{3.8}$$

By Lemma 3.1, the equation

$$Qc=\big(\zeta_1,\zeta_2,\dots,\zeta_m\big)^\top-\big(\sum_{j\in\mathcal{M}}\nu_j\zeta_j\big)\mathbb{1}$$

has a solution $c=(c_1,c_2,\dots,c_m)^\top\in\mathbb{R}^m$, where $\mathbb{1}=(1,1,\dots,1)^\top\in\mathbb{R}^m$. The numbers $c_1,c_2,\dots,c_m$ we just found are used in the definition of $W(\cdot,\cdot)$. Thus, we have

$$\zeta_i-\sum_{j\in\mathcal{M}} q_{ij}c_j=\sum_{j\in\mathcal{M}}\nu_j\zeta_j \ \text{ for }\ i\in\mathcal{M}.$$

Using this representation in (3.8), we obtain

$$(\overline{\mathcal{G}}W)(x,y,i)\leq\delta W(x-y,i)\Big(\sum_{j\in\mathcal{M}}\nu_j\zeta_j+O(\delta)\Big),\ \ (x,y,i)\in(\mathbb{R}^d)^2\times\mathcal{M}, x\neq y.$$

Since $\sum_{j\in\mathcal{M}}\nu_j\zeta_j<0$, we can choose $\delta>0$ and $\beta>0$ so that

$$(1/2)U^\delta(x)\leq W(x,i)\leq 2U^\delta(x),\ (\overline{\mathcal{G}}W)(x,y,i)\leq-\beta W(x-y,i),\ (x,y,i)\in(\mathbb{R}^d)^2\times\mathcal{M}, x\neq y. \tag{3.9}$$

Let $\varepsilon>0$ and D be a compact subset of $\mathbb{R}^d$. Let $(x_0,y_0,i_0)\in D\times D\times\mathcal{M}$. For notational simplicity, we denote $X(t)=X^{x_0,i_0}(t)$, $Y(t)=X^{y_0,i_0}(t)$, and $\alpha(t)=\alpha^{i_0}(t)$. By (2.3), there exists a constant $r>0$ such that

$$\{x\in\mathbb{R}^d:|x|\geq\varepsilon\}\subset\{x\in\mathbb{R}^d:U^\delta(x)\geq r\}. \tag{3.10}$$

Let $\{\widehat{\tau}_n\}_n$ be the sequence of stopping times defined by

$$\widehat{\tau}_n=\inf\{t\geq 0:|X(t)-Y(t)|\geq n\}\ \text{ for }\ n\in\mathbb{N}.$$

Since the solutions of equation (2.2) are regular, $\widehat{\tau}_n\to\infty$ almost surely as $n\to\infty$. By Lemma 3.3 and the Dynkin formula, we obtain that for each $t>0$,

$$\begin{aligned}\mathbb{E}\big[e^{\beta(t\wedge\widehat{\tau}_n)}W\big(X(t\wedge\widehat{\tau}_n)-Y(t\wedge\widehat{\tau}_n),\alpha(t\wedge\widehat{\tau}_n)\big)\big]&=W(x_0-y_0,i_0)\\ +\mathbb{E}\int_0^{t\wedge\widehat{\tau}_n}e^{\beta s}\big[\beta W\big(X(s)-Y(s),\alpha(s)\big)&+(\overline{\mathcal{G}}W)(X(s),Y(s),\alpha(s))\big]\mathrm{d}s.\end{aligned}$$

This together with (3.9) implies

$$(1/2)\mathbb{E}\Big[e^{\beta(t\wedge\widehat{\tau}_n)}U^\delta\big(X(t\wedge\widehat{\tau}_n)-Y(t\wedge\widehat{\tau}_n)\big)\Big]\leq 2U^\delta(x_0-y_0).$$

Letting $n \to \infty$ gives

$$\mathbb{E}\big[U^\delta\big(X(t)-Y(t)\big)\big] \le 4e^{-\beta t}U^\delta(x_0-y_0) \ \text{ for } \ t \ge 0. \tag{3.11}$$

Let $T > 0$ be such that

$$4e^{-\beta T}U^\delta(x_0-y_0) \le r\varepsilon \ \text{ for } \ (x_0,y_0) \in D \times D. \tag{3.12}$$

Then for any $t \ge T$, we have from (3.10), (3.11), and (3.12) that

$$\begin{aligned}\mathbb{P}\big(|X(t)-Y(t)| \ge \varepsilon\big) &\le \mathbb{P}\Big(U^\delta\big(X(t)-Y(t)\big) \ge r\Big)\\ &\le \frac{\mathbb{E}\big[U^\delta\big(X(t)-Y(t)\big)\big]}{r}\\ &\le \varepsilon;\end{aligned}$$

that is, $\mathbb{P}\big(|X(t)-Y(t)| < \varepsilon\big) \ge 1-\varepsilon$ for $t \ge T$. Thus, equation (2.2) has property (P2). □

We are in a position to state our main results in this section.

Theorem 3.5. *Assume* (A1)–(A4). *Then the following assertions hold.*

(a) *Equation* (2.2) *is stable in distribution.*
(b) *Suppose further that there exist positive constants κ_1, κ_2, and p such that*

$$\kappa_1|x|^p \le U(x) \le \kappa_2|x|^p \ \text{ for } \ x \in \mathbb{R}^d. \tag{3.13}$$

Then there exists a constant $\rho > 0$ such that $\mathbb{E}|X^{x_0,i_0}(t) - X^{y_0,i_0}(t)|^\rho$ converges to zero exponentially fast as $t \to \infty$ for any $(x_0,y_0,i_0) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}$.

Proof.
(a) By virtue of Lemma 3.2 and Lemma 3.4, equation (2.2) has properties (P1), (P2), and (P3). For the rest of the proof, we use essentially the same steps as in the proof of Theorem 3.1 in [24], hence, we omit it for brevity.
(b) Recall from (3.11) that

$$\mathbb{E}\big[U^\delta\big(X(t)-Y(t)\big)\big] \le 4e^{-\beta t}U^\delta(x_0-y_0) \ \text{ for } \ t \ge 0, \tag{3.14}$$

where β and δ are two positive constants. Combining (3.13) and (3.14) yields

$$\mathbb{E}|X(t)-Y(t)|^{p\delta} \le (4\kappa_2/\kappa_1)e^{-\beta t}|x_0-y_0|^{p\delta} \ \text{ for } \ t \ge 0. \tag{3.15}$$

Let $\rho = p\delta$. Then (3.15) tells us that $\mathbb{E}|X(t)-Y(t)|^\rho$ converges to zero exponentially fast as $t \to \infty$. This completes the proof. □

Remark 3.6. Under certain conditions, Theorem 3.5 states that there exists a constant $\rho > 0$ so that $\mathbb{E}|X^{x_0,i_0}(t) - X^{y_0,i_0}(t)|^\rho$ converges to zero exponentially fast for any $(x_0,y_0,i_0) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}$. In this case, it is said that equation (2.2) is asymptotically flat in the ρth mean (see [3]).

Now we apply the general criterion established above to derive an explicit and verifiable criterion for stability in distribution. This criterion is obtained when we take $U(x) = V(x) = |x|$ in conditions (A3) and (A4). More criteria can be constructed if we choose $U(x) = V(x) = (x^\top Bx)^{1/2}$ for some positive definite matrix $B \in \mathbb{R}^{d\times d}$.

Theorem 3.7. *Assume* (A1)–(A2)*. Moreover, for each* $i \in \mathcal{M}$*, there are constants* $K_b(i)$*,* $K_\sigma(i)$*,* $K_d(i)$*, and a function* $K_g(i,\cdot) \in \Gamma_b$ *such that*

$$\begin{aligned}
&(x-y)^\top \big(b(x,i)-b(y,i)\big) \le K_b(i)|x-y|^2,\\
&|\sigma(x,i)-\sigma(y,i)|^2 \le K_\sigma(i)|x-y|^2,\\
&\big|(x-y)^\top \big(\sigma(x,i)-\sigma(y,i)\big)\big|^2 \ge K_d(i)|x-y|^4,\\
&|x+g(x,i,\gamma)-y-g(y,i,\gamma)| \le K_g(i,\gamma)|x-y|,
\end{aligned} \tag{3.16}$$

for all $x,y \in \mathbb{R}^d, \gamma \in \Gamma$ *and* $i \in \mathcal{M}$*. Define*

$$\zeta_i = K_b(i) + \frac{1}{2}K_\sigma(i) - K_d(i) + \lambda \int_\Gamma \ln\big[K_g(i,\gamma)\big]\pi(\mathrm{d}\gamma) \quad \textit{for} \quad i \in \mathcal{M}. \tag{3.17}$$

Suppose $\sum_{i\in\mathcal{M}} \nu_i\zeta_i < 0$*. Then the following assertions hold.*

(a) *equation* (2.2) *is stable in distribution.*
(b) *There exists a constant* $\rho > 0$ *such that* $\mathbb{E}|X^{x_0,i_0}(t) - X^{y_0,i_0}(t)|^\rho$ *converges to zero exponentially fast as* $t \to \infty$ *for any* $(x_0,y_0,i_0) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}$*.*

Proof. (a) Consider the functions $U(x) = |x|$ and $U^\delta(x) = |x|^\delta$ for $x \in \mathbb{R}^d$ and $\delta \in (0,1)$. Direct computation yields that

$$U^\delta_x(x) = \delta|x|^{\delta-2}x, \quad U^\delta_{xx}(x) = \delta\big[|x|^{\delta-2}I_d + (\delta-2)|x|^{\delta-4}xx^\top\big], \quad x \in \mathbb{R}^d_0.$$

Thus,

$$\begin{aligned}
(\overline{\mathcal{G}}U^\delta)(x,y,i) &= \delta\big|x-y\big|^{\delta-2}(x-y)^\top\overline{b}(x,y,i) + \frac{\delta}{2}\mathrm{tr}\Big[\big|x-y\big|^{\delta-2}\overline{\sigma}(x,y,i)\overline{\sigma}^\top(x,y,i)\Big]\\
&\quad + \frac{1}{2}\delta(\delta-2)\mathrm{tr}\Big[\big|x-y\big|^{\delta-4}(x-y)(x-y)^\top\overline{\sigma}(x,y,i)\overline{\sigma}^\top(x,y,i)\Big]\\
&\quad + \lambda\int_\Gamma \Big[\big|x-y+\overline{g}(x,y,i,\gamma)\big|^\delta - \big|x-y\big|^\delta\Big]\pi(\mathrm{d}\gamma)\\
&\le U^\delta(x-y)\bigg(\delta K_b(i) + \frac{\delta}{2}K_\sigma(i) + \frac{\delta}{2}(\delta-2)K_d(i) + \lambda\int_\Gamma \Big(\big|K_g(i,\gamma)\big|^\delta - 1\Big)\pi(\mathrm{d}\gamma)\bigg).
\end{aligned}$$

That is,

$$(\overline{\mathcal{G}}U^\delta)(x,y,i) \le \zeta_{i,\delta}U^\delta(x-y) \quad \text{for} \quad (x,y,i) \in (\mathbb{R}^d)^2 \times \mathcal{M}, x \ne y,$$

where

$$\zeta_{i,\delta} = \delta K_b(i) + \frac{\delta}{2}K_\sigma(i) + \frac{\delta}{2}(\delta-2)K_d(i) + \lambda\int_\Gamma \Big(\big|K_g(i,\gamma)\big|^\delta - 1\Big)\pi(\mathrm{d}\gamma) \quad \text{for} \quad (i,\delta) \in \mathcal{M} \times (0,1).$$

We have $\lim_{\delta\to 0} \zeta_{i,\delta}/\delta = \zeta_i$ given by (3.17) and $\sum_{i\in\mathcal{M}} \nu_i\zeta_i < 0$. Hence, condition (A4) holds. By virtue of Remark 2.3, condition (A3) is also satisfied. Thus, by Theorem 3.5, equation (2.2) is stable in distribution.

(b) It is shown in part (a) that condition (A4) holds with $U(x) = |x|$ for $x \in \mathbb{R}^d$. Hence, by virtue of Theorem 3.5, there exists a constant $\rho > 0$ such that $\mathbb{E}|X^{x_0,i_0}(t) - X^{y_0,i_0}(t)|^\rho$ converges to zero exponentially fast as $t \to \infty$ for any $(x_0,y_0,i_0) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}$. □

4. Stabilization of switching jump diffusions

Based on the criteria developed in the preceding sections, we proceed to investigate stabilizing effects owing to Markov chains, Brownian motions, and Poisson jumps, respectively. Because of our work in this paper focuses on stability in distribution, the stabilization, in fact, is in the sense of the so-called weak stabilization. Such a term was probably first coined in the earlier work of Wonham [17].

Example 4.1. We consider equation (2.2), where conditions (A1)–(A2) and (3.16) are satisfied. Recall that

$$\zeta_i = K_b(i) + \frac{1}{2}K_\sigma(i) - K_d(i) + \lambda \int_\Gamma \ln\big[K_g(i,\gamma)\big]\pi(\mathrm{d}\gamma) \quad \text{for} \quad i \in \mathcal{M}.$$

Note that the switching jump diffusion $X(\cdot)$ may viewed as m jump diffusions that interact and switch back and forth due to the switching mechanism. These jump diffusions are denoted by $X^{(1)}(\cdot), X^{(2)}(\cdot), \dots, X^{(m)}(\cdot)$ given by

$$\begin{aligned} &\mathrm{d}X^{(i)}(t) = b\big(X^{(i)}(t), i\big)\mathrm{d}t + \sigma\big(X^{(i)}(t), i\big)\mathrm{d}w(t) + \mathrm{d}J^{(i)}(t), \\ &J^{(i)}(t) = \int_0^t \int_\Gamma g\big(X^{(i)}(s^-), i), \gamma\big)N(\mathrm{d}s, \mathrm{d}\gamma), \quad i \in \mathcal{M}. \end{aligned}$$

By virtue of Theorem 3.7, the stability of overall system (2.2) does not require all $\zeta_i < 0$, but only their average $\sum_{i\in\mathcal{M}} \nu_i\zeta_i < 0$.

For each jump diffusion $X^{(i)}(\cdot)$ above, we can apply Theorem 3.7 to verify the stability in distribution. In particular, suppose there exists $i_0 \in \mathcal{M}$ such that $\zeta_{i_0} < 0$. Then $X^{(i_0)}(\cdot)$ is stable in distribution. In such a case, we can design a suitable Markov switching $\alpha(\cdot)$ so that the i_0th subsystem is dominant and $\sum_{i\in\mathcal{M}} \nu_i\zeta_i < 0$. Thus, the switching process $\alpha(\cdot)$ can work as a stabilizing force.

Example 4.2. Consider the switching differential equation

$$\mathrm{d}X(t) = b(X(t), \alpha(t))\mathrm{d}t. \tag{4.1}$$

Equation (2.2) can be seen as a perturbed version of equation (4.1). Suppose that there are constants $K_b(i)$ for $i \in \mathcal{M}$ such that

$$(x-y)^\top\big(b(x,i) - b(y,i)\big) \le K_b(i)|x-y|^2, \quad (x,y,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}.$$

Can we add a suitable noise to equation (4.1) so that the resulting system given by equation (2.2) is stable in distribution? The answer is positive since we can always choose $\sigma(\cdot,\cdot)$ and $g(\cdot,\cdot)$ so that (3.16) holds. For simplicity, we can take

$$\sigma(x,i) = \lambda(i)\mathrm{diag}(x) + C(i), \quad g(x,i,\gamma) = \big(\mu(i) - 1\big)x,$$

where $\lambda(i) \in \mathbb{R}, \mu(i) > 0$ and $C(i) \in \mathbb{R}^{d\times d}$ for each $i \in \mathcal{M}$. Then $\sigma(\cdot,\cdot)$ and $g(\cdot,\cdot)$ satisfies (3.16) with

$$K_\sigma(i) = \lambda^2(i), \quad K_d(i) = \lambda^2(i), \quad K_g(i,\gamma) = \mu(i) \quad \text{for} \quad i \in \mathcal{M}, \gamma \in \Gamma.$$

If we choose $\lambda(i)$ and $\mu(i)$ such that

$$\sum_{i\in\mathcal{M}} \nu_i\big[K_b(i) - \frac{1}{2}\lambda^2(i) + \lambda \ln \mu(i)\big] < 0,$$

by virtue of Theorem 3.7, equation (2.2) is stable in distribution. Thus, by adding a suitable noise to an unstable system, we can make it stable in distribution.

Consider a switching ordinary differential equation given by equation (4.1). In [21], the authors have shown that if the solutions of equation (4.1) are not regular, one can add a feedback control term, which is of the form $\sigma(X(t),\alpha(t))dw(t)$ to suppress the finite explosion time. Then to ensure stability (in the sense of almost sure exponential stability), one can add another feedback control $\widehat{\sigma}(X(t),\alpha(t))d\widehat{w}(t)$. Here, a question arises naturally: can we use the same strategy to regularize and stabilize a given system in the sense of stability in distribution? We proceed to provide an affirmative answer for any given scalar switching differential equations.

Theorem 4.3. *Consider a scalar switching jump diffusion*

$$\begin{aligned} &\mathrm{d}X(t)=b(X(t),\alpha(t))\mathrm{d}t+\sigma(X(t),\alpha(t))dw(t)+\int_\Gamma g(X(t^-),\alpha(t^-),\gamma)N(\mathrm{d}t,\mathrm{d}\gamma),\\ &X(0)=x_0\in\mathbb{R},\quad \alpha(0)=i_0\in\mathcal{M}, \end{aligned} \tag{4.2}$$

where conditions (A1) *and* (A2) *are satisfied. Moreover,*

(a) *there exist constants $\kappa_i>0$ and $\mu_i\ge 0$ for $i\in\mathcal{M}$ such that $\sigma(x,i)=\kappa_i b(x,i)+\mu_i x$ for $(x,i)\in\mathbb{R}\times\mathcal{M}$;*
(b) *there exists constants $\xi_i>0$ for $i\in\mathcal{M}$ such that*

$$\begin{aligned} &\big|x+g(x,i,\gamma)-y-g(y,i,\gamma)\big|\le \xi_i|x-y| \quad \text{for}\quad (x,y,i,\gamma)\in\mathbb{R}\times\mathbb{R}\times\mathcal{M}\times\Gamma,\\ &\sum_{i\in\mathcal{M}}\nu_i\bigg(\frac{1-2\kappa_i\mu_i}{2\kappa_i^2}+\lambda\ln\xi_i\bigg)<0. \end{aligned} \tag{4.3}$$

Then equation (4.2) *is stable in distribution.*

Proof. For $(x,y,i)\in\mathbb{R}\times\mathbb{R}\times\mathcal{M}$, we define

$$h(x,y,i)=\frac{\overline{b}(x,y,i)}{x-y}\quad\text{if}\quad x\ne y,\quad h(x,y,i)=0\quad\text{if}\quad x=y.$$

Let $U(x)=|x|$ and $U^\delta(x)=|x|^\delta$ for $x\in\mathbb{R}$ and $\delta\in(0,1)$. Then for $(x,y,i)\in\mathbb{R}\times\mathbb{R}\times\mathcal{M}$ and $x\ne y$,

$$\begin{aligned} (\overline{\mathcal{G}}U^\delta)(x,y,i) &=\delta\big|x-y\big|^{\delta-2}(x-y)\overline{b}(x,y,i)+\frac{\delta(\delta-1)}{2}\big|x-y\big|^{\delta-2}\big|\kappa_i\overline{b}(x,y,i)+\mu_i(x-y)\big|^2\\ &\quad+\lambda\int_\Gamma\Big[\big|x-y+\overline{g}(x,y,i,\gamma)\big|^\delta-\big|x-y\big|^\delta\Big]\pi(\mathrm{d}\gamma)\\ &\le\delta\big|x-y\big|^\delta\bigg(h(x,y,i)-\frac{(1-\delta)}{2}\kappa_i^2|h(x,y,i)|^2-(1-\delta)\kappa_i\mu_i h(x,y,i)\\ &\quad-\frac{1-\delta}{2}\mu_i^2+\frac{\lambda}{\delta}\big(\xi_i^\delta-1\big)\bigg). \end{aligned} \tag{4.4}$$

Observe that for any $a\in\mathbb{R}$,

$$\begin{aligned} a-\frac{(1-\delta)}{2}\kappa_i^2a^2-(1-\delta)\kappa_i\mu_i a &=-\frac{(1-\delta)}{2}\bigg(\kappa_i a-\frac{1-(1-\delta)\kappa_i\mu_i}{(1-\delta)\kappa_i}\bigg)^2+\frac{\big(1-(1-\delta)\kappa_i\mu_i\big)^2}{2(1-\delta)\kappa_i^2}\\ &\le\frac{\big(1-(1-\delta)\kappa_i\mu_i\big)^2}{2(1-\delta)\kappa_i^2}. \end{aligned} \tag{4.5}$$

This together with (4.4) implies that

$$(\overline{\mathcal{G}}U^\delta)(x,y,i) \ \leq \delta U^\delta(x-y)\Big[\frac{\big(1-(1-\delta)\kappa_i\mu_i\big)^2}{2(1-\delta)\kappa_i^2} - \frac{1-\delta}{2}\mu_i^2 + \frac{\lambda}{\delta}\Big(\xi_i^\delta - 1\Big)\Big].$$

That is, $(\overline{\mathcal{G}}U^\delta)(x,y,i) \leq \zeta_{i,\delta}U^\delta(x-y)$ where

$$\zeta_{i,\delta} = \delta\Big[\frac{\big(1-(1-\delta)\kappa_i\mu_i\big)^2}{2(1-\delta)\kappa_i^2} - \frac{1-\delta}{2}\mu_i^2 + \frac{\lambda}{\delta}\Big(\xi_i^\delta - 1\Big)\Big] \ \text{ for } \ (i,\delta)\in\mathcal{M}\times(0,1). \tag{4.6}$$

We have

$$\lim_{\delta\to 0}\frac{\zeta_{i,\delta}}{\delta} = \frac{1-2\kappa_i\mu_i}{2\kappa_i^2} + \lambda\ln\xi_i, \quad i\in\mathcal{M}. \tag{4.7}$$

By (4.3), condition (A4) holds.

We proceed to verify condition (A3). Denote $h_0(x,i) = \dfrac{b(x,i)}{x}$ for $x\neq 0, i\in\mathcal{M}$. For $x\neq 0$ we have

$$\begin{aligned}
(\mathcal{G}U^\delta)(x,i) \ &= \delta\big|x\big|^{\delta-2}xb(x,i) + \frac{\delta(\delta-1)}{2}|x|^{\delta-2}|\kappa_i b(x,i)+\mu_i x|^2 \\
&\quad +\lambda\int_\Gamma\Big[\big|x+g(x,i,\gamma)\big|^\delta - |x|^\delta\Big]\pi(\mathrm{d}\gamma) \\
&\leq \delta|x|^\delta\Big[h_0(x,i) - \frac{1-\delta}{2}\kappa_i^2|h_0(x,i)|^2 - (1-\delta)\kappa_i\mu_i h_0(x,i) - \frac{1-\delta}{2}\mu_i^2\Big] \\
&\quad +\lambda\int_\Gamma\Big[\big|x+\overline{g}(x,0,i,\gamma)\big|^\delta - |x|^\delta + |g(0,i,\gamma)|^\delta\Big]\pi(\mathrm{d}\gamma) \\
&\leq \delta U^\delta(x)\Big[h_0(x,i) - \frac{1-\delta}{2}\kappa_i^2|h_0(x,i)|^2 - (1-\delta)\kappa_i\mu_i h_0(x,i) \\
&\quad -\frac{1-\delta}{2}\mu_i^2 + \frac{\lambda}{\delta}\Big(\xi_i^\delta - 1\Big)\Big] + \lambda\int_\Gamma|g(0,i,\gamma)|^\delta\pi(\mathrm{d}\gamma) \\
&\leq \delta U^\delta(x)\zeta_{i,\delta} + r_\delta,
\end{aligned} \tag{4.8}$$

where $\zeta_{i,\delta}$ is given by (4.6) and $r_\delta = \lambda\int_\Gamma|g(0,i,\gamma)|^\delta\pi(\mathrm{d}\gamma)$. Note that we have employed the estimate (4.5) to derive the last line of (4.8). By condition (A1), $r_\delta<\infty$. By (4.7) and (4.3), condition (A3) is satisfied. Thus, by virtue of Theorem 3.5, equation (4.2) is stable in distribution. □

Remark 4.4. By virtue of Theorem 4.3, we can choose constants κ_i, μ_i, and ξ_i for $i\in\mathcal{M}$ to regularize and stabilize equation (4.1). That is, under suitable design, arbitrary ordinary differential equations of the form (4.1) can be regularized. Moreover, it is possible to regularize and stabilize equation (4.1) by using the feedback control $\sigma(X(t),\alpha(t))dw(t)$ only. That is, we can take $g(\cdot,\cdot,\cdot)\equiv 0$.

Next, we address the question of stabilization of multi-dimensional systems. Using the idea as in [21], we add two feedback terms, one of them regularizes the system and the other ensures the stability. Nevertheless, when such stochastic feedback strategies are used, we cannot find a function $U(\cdot)$ to verify condition (A4). Hence, a new approach is needed to establish property (P2).

Consider equation (4.1) on $\mathbb{R}^d$. We pose conditions to ensure the stabilization. Assume that the function $b(\cdot,i)$ is locally Lipschitz continuous for each $i\in\mathcal{M}$ with the left-side Lipschitz coefficient $b_L(\cdot)$ defined by

$$b_L(r) := \sup_{|x|\vee|y|\leq r, x\neq y, i\in\mathcal{M}}\frac{(x-y)^\top(b(x,i)-b(y,i))}{|x-y|^2}, \quad r>0.$$

Suppose there exists a differential function $\widetilde{b}:\mathbb{R}_+\mapsto[1,\infty)$ satisfying

(B0) $\lim_{r\to\infty}\widetilde{b}(r)=\infty$;
(B1) there exists a constant $c_0>0$ such that

$$0\le r\frac{d}{dr}\widetilde{b}(r)\le c_0\widetilde{b}(r),\quad r>0;$$

(B2) $b_L(r)\le\dfrac{1}{6}(\widetilde{b}(r))^2$ for $r>0$.

Remark 4.5.
(a) If $b(\cdot,\cdot)$ has a polynomial growth together with its left-side Lipschitz coefficient $b_L(\cdot)$, we can choose $\widetilde{b}(r)=c(r^n+1)$ for some $c>0$ and $n>0$. Particularly, if $b_j(\cdot,i)$ is differentiable and its gradient has a polynomial growth for each (i,j), then we can choose $\widetilde{b}(r)=c(r^n+1)$ for some $c>0$ and $n>0$.
(b) We slightly modify the proofs below, to show that we can in fact, improve (B1) by using (B1') below.

(B1') there exists a constant $\delta_0\in(0,1/4)$ such that

$$0\le r^{1-\delta_0}\frac{d}{dr}\widetilde{b}(r)\le c_0\widetilde{b}(r),\quad r>0.$$

With this improvement, we can find $\widetilde{b}(\cdot)$ if $b(\cdot,\cdot)$ and its left-side Lipschitz coefficient are bounded by $c(e^{(|x|+1)^{\delta_0}/2})$ for some constant $c>0$. However, to ease the exposition, we impose (B1) instead of (B1'). Whether or not we can stochastic stabilize equation (4.1) if $b(\cdot,\cdot)$ or $b_L(\cdot)$ has an exponential growth rate is an open question.

Now we assume that we can find a function $\widetilde{b}(\cdot)$ satisfying assumptions (B0)–(B2). We will show that it is possible to construct two stochastic controls through two independent scalar Brownian motions to regularize and stabilize equation (4.1).

Consider the following system where the set of real numbers $\{\mu_i\}_{i\in\mathcal{M}}$ is to be specified later:

$$\begin{aligned}&\mathrm{d}X(t)=b(X(t),\alpha(t))\mathrm{d}t+B(X(t))X(t)\mathrm{d}w(t)+\mu_{\alpha(t)}X(t)d\widehat{w}(t),\\&X(0)=x_0\in\mathbb{R}^d,\quad\alpha(0)=i_0\in\mathcal{M},\quad B(x)=\widetilde{b}(|x|).\end{aligned}\tag{4.9}$$

where $w(\cdot)$ and $\widehat{w}(\cdot)$ are two independent (one dimensional) Brownian motions. In the rest of this section, we suppose that assumptions (B0)–(B2) and condition (A2) hold. Moreover, we work with a fixed value of $\delta\in(0,1/4]$.

Lemma 4.6. *Equation* (4.9) *has properties* (P1) *and* (P3).

Proof. Let $V(x)=|x|^2+1$ and $V^\delta(x)=|V(x)|^\delta$ for $x\in\mathbb{R}^d$. Then

$$(\mathcal{G}V)(x,i)=2x^\top b(x,i)+\big(|B(x)|^2+\mu_i^2\big)|x|^2,\quad(x,i)\in\mathbb{R}^d\times\mathcal{M},$$

and

$$\begin{aligned}(\mathcal{G}V^\delta)(x,i)&=\delta V^{\delta-1}(x)(\mathcal{G}V)(x,i)+\frac{\delta(\delta-1)}{2}V^{\delta-2}(x)\big(|V_x(x)B(x)x|^2+|V_x(x)\mu_ix|^2\big)\\&=\delta(|x|^2+1)^\delta\Big[\frac{2x^\top b(x,i)+\big(|B(x)|^2+\mu_i^2\big)|x|^2}{|x|^2+1}-2(1-\delta)\frac{|x|^4\big(|B(x)|^2+\mu_i^2\big)}{(|x|^2+1)^2}\Big].\end{aligned}\tag{4.10}$$

In view of (B2),

$$\frac{2x^\top b(x,i)}{|x|^2+1} \le \frac{1}{3}\frac{|B(x)|^2|x|^2}{|x|^2+1} + |b(0,i)|.$$

Note that

$$C_0 := \sup_{(x,i)\in\mathbb{R}^d\times\mathcal{M}} \left(\frac{4}{3}\frac{|B(x)|^2|x|^2}{|x|^2+1} - \frac{3}{2}\frac{|x|^4|B(x)|^2}{(|x|^2+1)^2} + |b(0,i)|\right) < \infty \tag{4.11}$$

and

$$\frac{\mu_i^2|x|^2}{|x|^2+1} - 2(1-\delta)\frac{\mu_i^2|x|^4}{(|x|^2+1)^2} \le \frac{\mu_i^2|x|^2}{|x|^2+1}\left(1-\frac{3}{2}\frac{|x|^2}{|x|^2+1}\right) \le \mu_i^2\mathbf{1}_{\{|x|\le 2\}}.$$

Note also that $\delta(|x|^2+1)^\delta\mu_i^2\mathbf{1}_{\{|x|\le 2\}} \le \mu_i^2\mathbf{1}_{\{|x|\le 2\}}$ since $\delta\in(0,1/4]$. As a result,

$$(\mathcal{G}V^\delta)(x,i) \le \mu_i^2\mathbf{1}_{\{|x|\le 2\}} + \delta V^\delta(x)\Big(C_0 - (\frac{1}{2}-2\delta)\frac{|x|^4|B(x)|^2}{(x^2+1)^2}\Big),\ \ (x,i)\in\mathbb{R}^d\times\mathcal{M}, \tag{4.12}$$

which together with assumption (B0) implies the existence of positive constants $\overline{C}_1$ and $\overline{C}_2$ independent of $\{\mu_i\}_{i\in\mathcal{M}}$ such that

$$(\mathcal{G}V^\delta)(x,i) \le \overline{C}_1 + \mu_i^2 - \overline{C}_2 V^\delta(x),\ \ (x,i)\in\mathbb{R}^d\times\mathcal{M}. \tag{4.13}$$

Hence, condition (A3) holds. By Lemma 3.2, equation (4.9) has properties (P1) and (P3). □

The rest of the section is devoted to proving that equation (4.9) has property (P2) when $\mu^M := \max\{\mu_i : i\in\mathcal{M}\}$ is sufficiently large. Because the drift coefficient $b(\cdot,\cdot)$ and its left-side Lipschitz coefficient $b_L(\cdot)$ can have a highly nonlinear growth rate, it seems practically impossible to construct a function $U(\cdot)$ satisfying condition (A4) for a general nonlinear function $b(\cdot,\cdot)$. As a result, Lemma 3.4 is not applicable here. We overcome this technical difficulty by proving a number of lemmas below.

Lemma 4.7. *Let $\mu^M = \max\{\mu_i : i\in\mathcal{M}\}$. Then there exist positive constants C_1, C_2, C_3, and λ independent of μ^M and initial data $(x_0,i_0)\in\mathbb{R}^d\times\mathcal{M}$ such that*

$$\mathbb{E}(|X(t)|^2+1)^\delta \le C_1\big(1+(\mu^M)^2\big) + e^{-\lambda t}(|x_0|^2+1)^\delta,\ \ t\ge 0, \tag{4.14}$$

and

$$\mathbb{E}\int_0^t \frac{|X(s)|^4|B(X(s)))|^2}{(|X(s)|^2+1)^{2-\delta}}\mathrm{d}s \le C_2\big(1+(\mu^M)^2\big)t + C_3(|x_0|^2+1)^\delta \text{ for any } t\ge 0. \tag{4.15}$$

Proof. In view of (4.13), there exist positive constants $\overline{C}_1$ and $\overline{C}_2$ independent of $\{\mu_i\}_{i\in\mathcal{M}}$ and initial data $(x_0,i_0)\in\mathbb{R}^d\times\mathcal{M}$ such that

$$\big(\mathcal{G}(|x|^2+1)^\delta\big)(x,i) \le \overline{C}_1 + (\mu_M)^2 - \overline{C}_2(|x|^2+1)^\delta,\ \ (x,i)\in\mathbb{R}^d\times\mathcal{M}.$$

By Itô's formula, we obtain

$$\mathbb{E}e^{\overline{C}_2 t}\big(|X(t)|^2+1\big)^{\delta} \le (|x_0|^2+1)^{\delta} + \big(\overline{C}_1 + (\mu^M)^2\big)\int_0^t e^{\overline{C}_2 s}\mathrm{d}s,$$

which leads to

$$e^{\overline{C}_2 t}\mathbb{E}\big(|X(t)|^2+1\big)^{\delta} \le (|x_0|^2+1)^{\delta} + \frac{\overline{C}_1 + (\mu^M)^2}{\overline{C}_2}e^{\overline{C}_2 t}.$$

Then there exist positive constants C_1 and λ so that (4.14) holds.

We proceed to prove (4.15). In view of (4.12), we have

$$\begin{aligned}\mathbb{E}\big(|X(t)|^2+1\big)^{\delta} &\le (|x_0|^2+1)^{\delta} + (\mu^M)^2 t + \delta C_0 \mathbb{E}\int_0^t (|X(s)|^2+1)^{\delta}\mathrm{d}s \\ &\quad -\delta(\frac{1}{2}-2\delta)\mathbb{E}\int_0^t \frac{|X(s)|^4|B(X(s)|^2}{(|X(s)|^2+1)^{2-\delta}}\mathrm{d}s,\end{aligned}$$

which implies

$$\delta(\frac{1}{2}-2\delta)\mathbb{E}\int_0^t \frac{|X(s)|^4|B(X(s)|^2}{(|X(s)|^2+1)^{2-\delta}}\mathrm{d}s \le (|x_0|^2+1)^{\delta} + (\mu^M)^2 t + \delta C_0 \int_0^t \mathbb{E}(|X(s)|^2+1)^{\delta}\mathrm{d}s. \tag{4.16}$$

Then, substituting (4.14) into (4.16), we obtain (4.15) for some positive constants C_2 and C_3 independent of $\{\mu_i\}_{i\in\mathcal{M}}$ and initial data (x_0, i_0). This completes the proof. □

Lemma 4.8. *Suppose $M(t)$ is a continuous martingale with $M(0)=0$ and its quadratic variation $[M]_t$ satisfying $\mathbb{E}[M]_t \le L_M(t+1)$, where L_M is a positive constant. Then for any $\varepsilon_1>0$ and $\varepsilon_2>0$, there exists an $n_0 = n_0(L_M,\varepsilon_1,\varepsilon_2)>0$ such that*

$$\mathbb{P}\left\{\frac{|M(t)|}{t} \le \varepsilon_2 \text{ for all } t \ge n_0^2\right\} \ge 1-\varepsilon_1.$$

Proof. Define events

$$\Omega_n := \left\{\omega : \sup_{0\le t\le n^2}|M(t)| \ge n^2\varepsilon_2/2\right\}, \quad n=1,2,\dots$$

By the Markov inequality and the Burkholder-Davis-Gundy inequality,

$$\mathbb{P}(\Omega_n) \le \frac{\mathbb{E}\big(\sup_{0\le t\le n^2}|M(t)|^2\big)}{(n^2\varepsilon_2/2)^2} \le \frac{4\mathbb{E}([M(t)]_{n^2})}{n^4\varepsilon_2^2/4} \le \frac{16L_M(n^2+1)}{n^4}.$$

Now, for any $\varepsilon_1>0$, there exists $n_0>3$ such that $\sum_{n=n_0}^{\infty}\frac{16L_M(n^2+1)}{n^4} < \varepsilon_1$. Then

$$\mathbb{P}\big\{\Omega\setminus\cup_{n=n_0}^{\infty}\Omega_n\big\} \ge 1-\sum_{n=n_0}^{\infty}\frac{16L_M(n^2+1)}{n^4} \ge 1-\varepsilon_1.$$

Note that for $\omega \in \Omega \setminus \cup_{n=n_0}^{\infty}\Omega_n$, we have $\sup_{0\le t\le n^2}|M(t)| \le n^2\varepsilon_2/2$ for all $n \ge n_0$. As a result, if $\omega \in \Omega \setminus \cup_{n=n_0}^{\infty}\Omega_n$, we have for any $t > n_0^2$ that

$$\frac{|M(t)|}{t} \le \frac{(N_t+1)^2}{N_t^2}\frac{\varepsilon_2}{2} \le \varepsilon_2,$$

where N_t is the greatest integer smaller than t. Thus, we have

$$\mathbb{P}\left\{\frac{|M(t)|}{t} \le \varepsilon_2 \text{ for all } t \ge n_0^2\right\} \ge 1-\varepsilon_1.$$

This completes the proof. □

Lemma 4.9. *Let C_0 be given by (4.11) and*

$$C_4 = \sup_{x\in\mathbb{R}^d}\left\{\frac{\delta}{2}(x^2+1)^{\delta/2}\left[C_0 - \frac{1-2\delta}{4}\frac{|x|^4|B(x)|^2}{(x^2+1)^2}\right]\right\},\quad C_5 = \frac{32}{\delta(1-2\delta)}. \tag{4.17}$$

For any constants $H \ge 1$, $r > 0, \varepsilon_1 > 0, \varepsilon_2 > 0$ there exists $n_1 := n_1(r, \mu^M, \varepsilon_1, \varepsilon_2) > 0$ such that

$$\mathbb{P}\left\{\frac{1}{t}\int_0^t \mathbf{1}_{\{|X(s)|>H\}}|B(X(s)|^2 ds \le \frac{C_5(C_4+(\mu^M)^2)+\varepsilon_2}{H^\delta} \text{ for all } t \ge n_1\right\} \ge 1-\varepsilon_1 \tag{4.18}$$

for any initial data $(x_0, i_0) \in \mathbb{R}^d \times \mathcal{M}$ satisfying $|x_0| \le r$.

Proof. Similar to (4.12),

$$\begin{aligned}\left(\mathcal{G}(|x|^2+1)^{\delta/2}\right)(x,i) \le& \mu_i^2\mathbf{1}_{\{|x|\le 2\}} + \frac{\delta}{2}(x^2+1)^{\delta/2}\left[C_0 - \frac{1-2\delta}{2}\frac{|x|^4|B(x)|^2}{(x^2+1)^2}\right]\\ \le& (\mu^M)^2 + C_4 - \frac{\delta(1-2\delta)}{8}\frac{|x|^4|B(x)|^2}{(x^2+1)^{2-\delta/2}}\end{aligned} \tag{4.19}$$

(C_4 is given by (4.17)), which is finite because $\lim_{|x|\to\infty} B(x) = \infty$. By Itô's formula, we obtain from (4.19) that

$$0 \le (|X(t)|^2+1)^{\delta/2} \le (|x_0|^2+1)^{\delta/2} + \left((\mu^M)^2 + C_4\right)t - \frac{\delta(1-2\delta)}{8}\int_0^t \frac{|X(s)|^4|B(X(s)|^2}{(|X(s)|^2+1)^{2-\delta/2}}\mathrm{d}s + M_0(t) \tag{4.20}$$

where

$$M_0(t) := \delta\int_0^t \frac{B(X(t)))|X(s)|^2dw(s) + \mu_{\alpha(s)}|X(s)|^2d\widehat{w}(s)}{\left(|X(s)|^2+1\right)^{1-\delta/2}}.$$

It follows from (4.20) that

$$\frac{1}{t}\int_0^t \frac{|X(s)|^4|B(X(s)|^2}{(|X(s)|^2+1)^{2-\delta/2}}\mathrm{d}s \le \frac{8}{\delta(1-2\delta)}\left[\frac{(|x_0|^2+1)^{\delta/2}+M_0(t)}{t} + (\mu^M)^2 + C_4\right]. \tag{4.21}$$

By (4.15) and the fact that $B(x) \geq 1$ for any $x \in \mathbb{R}^d$,

$$\begin{aligned}\mathbb{E}\int_0^t \frac{|X(s)|^4}{(|X(s)|^2+1)^{2-\delta}}\mathrm{d}s &\leq \mathbb{E}\int_0^t \frac{|X(s)|^4|B(X(s)))|^2}{(|X(s)|^2+1)^{2-\delta}}\mathrm{d}s\\ &\leq C_2\big(1+(\mu^M)^2\big)t + C_3(|x_0|^2+1)^\delta,\end{aligned}$$

where C_2 and C_3 are constants independent of μ^M and initial data $(x_0, i_0) \in \mathbb{R}^d \times \mathcal{M}$. Consequently, there is a constant $L_{M_0} = L_{M_0}(r, \mu^M) > 0$ such that $\mathbb{E}[M_0]_t \leq L_{M_0}(1+t)$ for $t \geq 0$. An application of Lemma 4.8 to $M_0(t)$ implies that, for any $\varepsilon_1, \varepsilon_2$, there exists $n_1 := n_1(r, \mu^M, \varepsilon_1, \varepsilon_2)$ satisfying

$$\mathbb{P}\left\{\frac{8}{\delta(1-2\delta)}\left[\frac{(|x_0|^2+1)^{\delta/2}+M_0(t)}{t}\right] \leq \frac{\varepsilon_2}{4} \text{ for all } t \geq n_1\right\} \geq 1-\varepsilon_1 \quad \text{if} \quad |x_0| \leq r,$$

which together with (4.21) implies

$$\mathbb{P}\left\{\frac{1}{t}\int_0^t \frac{|X(s)|^4|B(X(s)|^2}{(|X(s)|^2+1)^{2-\delta/2}} \leq \frac{C_5}{4}(C_4+(\mu^M)^2)+\frac{\varepsilon_2}{4} \text{ for all } t \geq n_1\right\} \geq 1-\varepsilon_1, \tag{4.22}$$

where $C_5 = \frac{32}{\delta(1-2\delta)}$. Because

$$\mathbf{1}_{\{|x|>H\}}|B(x)|^2 \leq \frac{1}{H^\delta}\frac{4|x|^4|B(x)|^2}{(|x|^2+1)^{2-\delta/2}} \quad \text{for any} \quad x \in \mathbb{R}^d, H \geq 1, \tag{4.23}$$

we can easily derive (4.18) from (4.22). □

Lemma 4.10. *Let $U(x) = |x|^2$ for $x \in \mathbb{R}^d$. Then for any $(x, y, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}$ and $x \neq y$,*

$$(\overline{\mathcal{G}}\ln U)(x,y,i) \leq \frac{C_6}{2}\big(\mathbf{1}_{\{|x|>H\}}|B(x)|^2+\mathbf{1}_{\{|y|>H\}}|B(y)|^2\big)+C_6|\widetilde{b}(H)|^2-\mu_i^2,$$

where $C_6 = 6+4c_0^2$ and c_0 is given in assumption (B1).

Proof. We have

$$(\overline{\mathcal{G}}U)(x,y,i) = 2(x-y)^\top(b(x,i)-b(y,i)) + |B(x)x-B(y)y|^2+\mu_i^2|x-y|^2 \tag{4.24}$$

and

$$\begin{aligned}(\overline{\mathcal{G}}\ln U)(x,y,i) =& \frac{2(x-y)^\top(b(x,i)-b(y,i))+|B(x)x-B(y)y|^2}{|x-y|^2}-\mu_i^2\\ &-2\frac{\big|(x-y)^\top\big(B(x)x-B(y)y\big)\big|^2}{|x-y|^4}.\end{aligned} \tag{4.25}$$

Recall that $\widetilde{b}:\mathbb{R}^d\to(1,\infty)$ is a differentiable function and $B(x)=\widetilde{b}(|x|)$ for $x\in\mathbb{R}^d$. Suppose without loss of generality that $|x|\le|y|$. By the mean value theorem, we have

$$\begin{aligned}|B(x)x-B(y)y|^2=&|B(x)x-B(x)y+B(x)y-B(y)y|^2\\ \le&2|B(x)|^2|x-y|^2+2|B(x)-B(y)|^2|y|^2\\ \le&2|x-y|^2|B(x)|^2+2|y|^2\sup_{r\le|y|}\Big|\frac{d}{dr}\widetilde{b}(r)\Big|^2(|x|-|y|)^2\\ \le&2|x-y|^2|B(x)|^2+2|y|^2\sup_{r\le|y|}\Big|\frac{d}{dr}\widetilde{b}(r)\Big|^2|x-y|^2.\end{aligned}\tag{4.26}$$

By assumption (B1), $|y|\sup_{r\le|y|}\Big|\dfrac{d}{dr}\widetilde{b}(r)\Big|\le c_0|B(x)|$. This together with (4.26) implies

$$|B(x)x-B(y)y|^2\le 2(1+c_0^2)|x-y|^2\big(|B(x)|^2+|B(y)|^2\big),\tag{4.27}$$

which holds for any $x,y\in\mathbb{R}^d$. Moreover, by assumption (B2),

$$2(x-y)^\top\big(b(x,i)-b(y,i)\big)\le|x-y|^2(|B(x)|^2+|B(y)|^2).\tag{4.28}$$

Putting (4.27) and (4.28) into (4.25) yields

$$\begin{aligned}&(\overline{\mathcal{G}}\ln U)(x,y,i)\le(3+2c_0^2)(|B(x)|^2+|B(y)|^2)-\mu_i^2\\ &\le(3+2c_0^2)\Big(\mathbf{1}_{\{|x|>H\}}|B(x)|^2+\mathbf{1}_{\{|y|>H\}}|B(y)|^2+\mathbf{1}_{\{|x|\le H\}}|B(x)|^2+\mathbf{1}_{\{|y|\le H\}}|B(y)|^2\Big)-\mu_i^2\\ &\le\frac{C_6}{2}\big(\mathbf{1}_{\{|x|>H\}}|B(x)|^2+\mathbf{1}_{\{|y|>H\}}|B(y)|^2\big)+C_6|\widetilde{b}(H)|^2-\mu_i^2,\end{aligned}\tag{4.29}$$

where $C_6=6+4c_0^2$. Note that $\mathbf{1}_{\{|x|\le H\}}|B(x)|^2\le|\widetilde{b}(H)|^2$ and $\mathbf{1}_{\{|y|\le H\}}|B(y)|^2\le|\widetilde{b}(H)|^2$. This completes the proof. □

Remark 4.11. Define

$$\Lambda:=\frac{C_6C_5(C_4+(\mu^M)^2)}{H^\delta}+C_6|\widetilde{b}(H)|^2-\sum_{i\in\mathcal{M}}\mu_i^2\nu_i,\tag{4.30}$$

where C_0 is given by (4.11),

$$C_4=\sup_{x\in\mathbb{R}^d}\left\{\frac{\delta}{2}(x^2+1)^{\delta/2}\left[C_0-\frac{1-2\delta}{4}\frac{|x|^4|B(x)|^2}{(x^2+1)^2}\right]\right\},\quad C_5=\frac{32}{\delta(1-2\delta)},\quad C_6=6+4c_0^2.$$

It is easy to see that $\Lambda<0$ if we choose H and then μ^M to be sufficiently large. Indeed, we can first select sufficiently large H that $\frac{C_6C_5}{H^\delta}<\frac{\min_{i\in\mathcal{M}}\{\nu_i\}}{2}$ and then select one μ_{i^*} satisfying

$$\frac{\mu_{i^*}^2\nu_{i^*}}{2}\ge C_6|\widetilde{b}(H)|^2+\frac{C_6C_5C_4}{H^\delta}.$$

Then we can verify that $\Lambda<0$.

Lemma 4.12. *Suppose*

$$\Lambda = \frac{C_6 C_5 (C_4 + (\mu^M)^2)}{H^\delta} + C_6 |\widetilde{b}(H)|^2 - \sum_{i\in\mathcal{M}} \mu_i^2 \nu_i < 0.$$

For any $\varepsilon_1 > 0$ *and a compact set* D*, there exists* $T = T(\varepsilon_1, D, \Lambda) > 0$ *such that*

$$\mathbb{P}\left\{ \frac{\ln \left| X^{x_0,i_0}(t) - X^{y_0,i_0}(t) \right|^2}{t} < \frac{\Lambda}{5} \right\} \geq 1 - \varepsilon_1 \quad \textit{for} \quad t \geq T,$$

whenever $(x_0, y_0, i_0) \in D \times D \times \mathcal{M}$*. As a result, equation* (4.9) *has property* (P2).

Proof. Let $(x_0, y_0, i_0) \in D \times D \times \mathcal{M}$. For notational simplicity, we denote $X(t) = X^{x_0,i_0}(t)$, $Y(t) = X^{y_0,i_0}(t)$, and $\alpha(t) = \alpha^{i_0}(t)$. By Itô's formula and (4.29), we have

$$\begin{aligned} \frac{\ln |X(t) - Y(t)|^2}{t} \leq & \frac{\ln |x_0 - y_0|^2}{t} + \frac{C_6}{2} \frac{1}{t} \int_0^t \Big(\mathbf{1}_{\{|X(s)|>H\}} |B(X(s))|^2 + \mathbf{1}_{\{|Y(s)|>H\}} |B(Y(s)|^2 \Big) \mathrm{d}s \\ & - \frac{1}{t} \int_0^t \mu^2_{\alpha(s)} \mathrm{d}s + C_6 |\widetilde{b}(H)|^2 + \frac{M_1(t) + M_2(t)}{t}, \end{aligned} \tag{4.31}$$

where

$$M_1(t) = 2 \int_0^t \frac{(X(t) - Y(t))^\top (B(X(s))X(s) - B(Y(s)Y(s))}{|X(s) - Y(s)|^2} \mathrm{d}w(t), \quad M_2(t) = 2 \int_0^t \mu_{\alpha(t)} d\widehat{w}(t).$$

In view of (4.27), we have the following estimate for the quadratic variation of M_1.

$$\begin{aligned} [M_1]_t = & 2 \int_0^t \left(\frac{(X(t) - Y(t))^\top \big(B(X(s))X(s) - B(Y(s))Y(s)\big)}{|X(s) - Y(s)|^2} \right)^2 \mathrm{d}t \\ \leq & 4(1 + c_0^2) \int_0^t (|B(X(s)|^2 + |B(Y(s)|^2) \mathrm{d}s, \end{aligned} \tag{4.32}$$

which together with (4.15) and (4.23) implies that $\mathbb{E}[M_1]_t \leq L_1(t+1)$ for some $L_1 = L_1(D) > 0$. Clearly, $\mathbb{E}[M_2]_t \leq 4(\mu^M)^2 t$. In view of Lemma 4.8, there exists $n_2 = n_2(\varepsilon_1, D, \Lambda) > 0$ such that

$$\mathbb{P}\left\{ \frac{M_1(t) + M_2(t)}{t} \leq \frac{|\Lambda|}{5} \quad \text{for all} \quad t \geq n_2 \right\} \geq 1 - \varepsilon_1/4. \tag{4.33}$$

In view of (4.18), there exists $n_3 = n_3(\varepsilon_1, D, \Lambda) > 0$ such that

$$\mathbb{P}\left\{ \frac{1}{t} \int_0^t \mathbf{1}_{\{|X(s)|>H\}} |B(X(s)|^2 ds \leq \frac{C_5(C_4 + (\mu^M)^2)}{H^\delta} + \frac{|\Lambda|}{5C_6} \quad \text{for all} \quad t \geq n_3 \right\} \geq 1 - \varepsilon_1/4 \tag{4.34}$$

and

$$\mathbb{P}\left\{ \frac{1}{t} \int_0^t \mathbf{1}_{\{|Y(s)|>H\}} |B(Y(s)|^2 ds \leq \frac{C_5(C_4 + (\mu^M)^2)}{H^\delta} + \frac{|\Lambda|}{5C_6} \quad \text{for all} \quad t \geq n_3 \right\} \geq 1 - \varepsilon_1/4. \tag{4.35}$$

Because of the ergodicity of $\alpha(t)$, there exists $n_4 = n_4(\varepsilon_1, D, \Lambda) > 0$ such that

$$\mathbb{P}\left\{\frac{\ln|x_0 - y_0|^2}{t} - \frac{1}{t}\int_0^t \mu^2_{\alpha(s)}\mathrm{d}s \le -\sum_{i\in\mathcal{M}} \mu_i^2\nu_i + \frac{|\Lambda|}{5} \text{ for all } t \ge n_4\right\} \ge 1 - \varepsilon_1/4. \tag{4.36}$$

Putting (4.33), (4.34), (4.35), and (4.36) into (4.31) yields

$$\mathbb{P}\left\{\frac{\ln|X(t) - Y(t)|^2}{t} \le \frac{\Lambda}{5} < 0 \text{ for all } t \ge \max\{n_2, n_3, n_4\}\right\} \ge 1 - \varepsilon_1.$$

The conclusion follows. □

The following theorem summarizes our results above.

Theorem 4.13. *Assume* (B0)–(B2) *and* (A2). *Let* Λ *be given by* (4.30). *If* $\Lambda < 0$, *then equation* (4.9) *is stable in distribution.*

5. Examples

Example 5.1. We consider the scalar regime-switching diffusion with Poisson jumps perturbations. Suppose that $\mathcal{M} = \{1, 2\}$ and for each $i \in \mathcal{M}$, there are constants λ_i, $\widehat{\lambda}_i$, μ_i, $\widehat{\mu}_i$ such that

$$\begin{aligned}\mathrm{d}X(t) &= \Big(\lambda_{\alpha(t)}X(t) + \widehat{\lambda}_{\alpha(t)}\Big)\mathrm{d}t + \Big(\mu_{\alpha(t)}X(t) + \widehat{\mu}_{\alpha(t)}\Big)\mathrm{d}w(t) \\ &\quad + \int_\Gamma g(X(t^-), \alpha(t^-), \gamma)N(\mathrm{d}t, \mathrm{d}\gamma).\end{aligned} \tag{5.1}$$

We consider equation (5.1) with $\lambda_1 = 1$, $\lambda_2 = -1/2$, $\widehat{\mu}_1 = 1$, $\widehat{\mu}_2 = 2$, $\widehat{\lambda}_1 = \widehat{\lambda}_2 = \mu_1 = \mu_2 = 0$, and

$$Q = \begin{pmatrix} -4 & 4 \\ 2.5 & -2.5 \end{pmatrix}, \quad \lambda = 1, \quad g(x, 1, \gamma) = \frac{\sin x}{2} + 1, \quad g(x, 2, \gamma) = -(3/4)x + 3, \quad x \in \mathbb{R}.$$

It can be seen that $|x + g(x, i, \gamma) - y - g(y, i, \gamma)| \le K_g(i)|x - y|$, where $K_g(1) = 3/2$ and $K_g(2) = 1/4$. Thus, the assumptions of Theorem 3.7 are satisfied. The values of ζ_1 and ζ_2 defined in (3.17) are $\zeta_1 = \lambda_1 + \lambda \ln|K_g(1)| = 1 + \ln|3/2|$ and $\zeta_2 = \lambda_2 + \lambda \ln|K_g(2)| = -(1/2) + \ln|1/4|$. Here $\nu = (\nu_1, \nu_2) = (5/13, 8/13)$. Hence, $\sum_{i=1}^2 \nu_i\zeta_i \approx -0.62 < 0$. Thus, equation (5.1) is stable in distribution. Moreover, by virtue of Theorem 3.7, there exists a constant $\rho > 0$ such that $\mathbb{E}|X^{x_0,i_0}(t) - X^{y_0,i_0}(t)|^\rho$ converges to zero exponentially fast for any $(x_0, y_0, i_0) \in \mathbb{R} \times \mathbb{R} \times \mathcal{M}$.

Example 5.2. Consider a switching dynamic system given by

$$\mathrm{d}X(t) = b(X(t), \alpha(t))\mathrm{d}t. \tag{5.2}$$

Let $X(0) = 1$, $\mathcal{M} = \{1, 2\}$, $b(x, 1) = x^2 + 5$ and $b(x, 2) = x^2 - 1$. Then the two subequations are given by

$$\mathrm{d}X(t) = \big(X^2(t) + 5\big)\mathrm{d}t, \quad \mathrm{d}X(t) = \big(X^2(t) - 1\big)\mathrm{d}t.$$

It can be seen that neither equation has global solutions. Figure 1(a) provides several trajectories of equation (5.2) with differential initial states $X(0)$ and initial state $\alpha(0) = 1$ under the same realizations of $\alpha(\cdot)$.

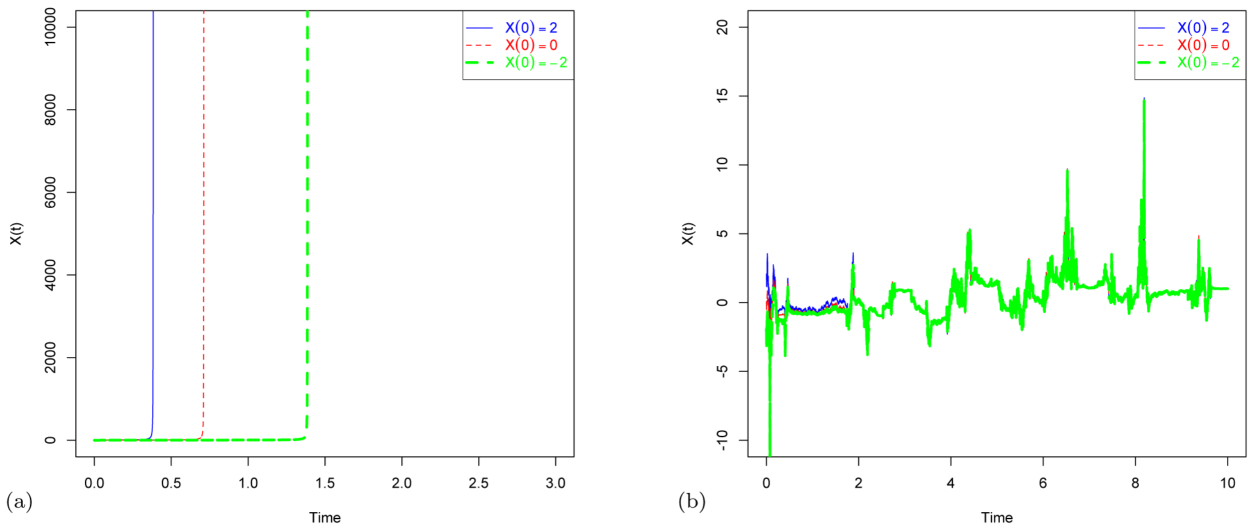

FIGURE 1. (a) Trajectories of $X(t)$ given by equation (5.2) with differential initial states $X(0)$ and initial state $\alpha(0) = 1$ under the same realizations of $\alpha(\cdot)$; (b) Trajectories of $X(t)$ given by equation (5.3) with differential initial states $X(0)$ and initial state $\alpha(0) = 1$ under the same realizations of $\alpha(\cdot)$, $w(\cdot)$, and $N(\cdot, \cdot)$.

To regularize the system, we add a feedback control of the form $\sigma(X(t), \alpha(t))\mathrm{d}w(t)$ where $\sigma(x, i) = b(x, i)$. In addition, to stabilize the system, we add another feedback control of the form $\int_\Gamma \rho_{\alpha(t^-)} X(t^-) N(\mathrm{d}t, \mathrm{d}\gamma)$. The resulting equation is

$$\mathrm{d}X(t) = b\big(X(t), \alpha(t)\big)\mathrm{d}t + b\big(X(t), \alpha(t)\big)\mathrm{d}w(t) + \int_\Gamma \rho_{\alpha(t^-)} X(t^-) N(\mathrm{d}t, \mathrm{d}\gamma). \tag{5.3}$$

For $Q = \begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix}$, we let $\lambda = 2$, $\rho_1 = 0.5$, and $\rho_2 = -3/4$. By using Theorem 4.3, we have

$$\kappa_1 = \kappa_2 = 1, \quad \mu_1 = \mu_2 = 0, \quad \xi_1 = 1.5, \quad \xi_2 = 0.25, \quad \nu_1 = 0.4, \quad \nu_2 = 0.6.$$

Consequently,

$$\sum_{i \in \mathcal{M}} \nu_i \left(\frac{1 - 2\kappa_i \mu_i}{2\kappa_i^2} + \lambda \ln \xi_i \right) \approx -0.84.$$

Thus, equation (5.3) is stable in distribution. To visualize the regularization and stabilization effects of the feedback controls as well as the exponentially contractive property of equation (5.3), we plot several trajectories of equation (5.3) in Figure 1(b).

Note that in Figure 1(b), there are three trajectories starting from the initial values $(x_0, i_0) = (2, 1)$, $(0, 1)$, and $(-2, 1)$ under the same realizations of $\alpha(\cdot)$, $w(\cdot)$ and $N(\cdot, \cdot)$. It can be seen that each trajectory converges to the others very fast as t increases and they are almost identical for $t \geq 3$.

6. Concluding remarks

This paper has been devoted to the study of Markovian switching jump diffusions. We have further explored the asymptotic behaviors of switching diffusions with Poisson jumps in which there might be no equilibrium point. The criteria for stability in distribution are established. The stabilization effects of Markov chains, Brownian motions, and Poisson jumps are investigated. Our results offer new insight and effective treatments to regularization and stabilization of switching jump diffusion systems. Although the paper is devoted to Markovian switching jump diffusions, when the jump part disappear, our results cover that of hybrid systems with a Markovian switching.

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