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EQUIVALENCE OF THREE KINDS OF OPTIMAL CONTROL PROBLEMS FOR LINEAR HEAT EQUATIONS WITH MEMORY

Lijuan Wang[1,*] and Xiuxiang Zhou[2,**,***]

Abstract. This paper studies an equivalence theorem for three different kinds of optimal control problems, which are optimal time control problems, optimal norm control problems, and optimal target control problems. The controlled systems in this paper are internally controlled linear heat equations with memory.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \in \mathbb{N}^+ := \{1, 2, 3, \ldots\})$ be a bounded domain with a C^2 smooth boundary $\partial\Omega$. Let ω be an open and nonempty subset of Ω. χ_ω denotes the characteristic function of the set ω. For fixed $T > 0$, we write Q_T and Σ_T for the product sets $\Omega \times (0, T)$ and $\partial\Omega \times (0, T)$, respectively. Consider the following controlled linear heat equations with memory:

$$\begin{cases} y_t - \Delta y + \displaystyle\int_0^t a(t-s)y(s)\,\mathrm{d}s = \chi_\omega \chi_{(\tau,T)} u & \text{in} \quad Q_T, \\ y = 0 & \text{on} \quad \Sigma_T, \\ y(0) = y_0 & \text{in} \quad \Omega. \end{cases} \tag{1.1}$$

Here, $a \in L^1(0, T)$ is the memory kernel; $u \in L^\infty(0, T; L^2(\Omega))$ is a control; $y_0 \in L^2(\Omega)$ is an initial datum; $\tau \in [0, T]$ and $\chi_{(\tau,T)}$ stands for the characteristic function of the set (τ, T) (When $\tau = T$, we denote $\chi_{(\tau,T)} = 0$). It is well known that for each $u \in L^\infty(0, T; L^2(\Omega))$ and each $y_0 \in L^2(\Omega)$, (1.1) has a unique solution, denoted by $y(\cdot; y_0, \chi_{(\tau,T)}u)$, in $W^{1,2}(0, T; H^{-1}(\Omega)) \bigcap L^2(0, T; H_0^1(\Omega)) \subset C([0, T]; L^2(\Omega))$ (see, for instance, [7]). Throughout this paper, we denote the usual inner product and norm in $L^2(\Omega)$ by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively.

Now, we will formulate three kinds of optimal control problems considered in this paper. To this end, we set

$$\mathcal{Y}(y_0, \tau) := \{y(T; y_0, \chi_{(\tau,T)}u) : u \in L^\infty(0, T; L^2(\Omega))\},$$

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[1] School of Mathematics and Statistics, Computational Science Hubei Key Laboratory, Wuhan University, Wuhan 430072, China.

[2] School of Mathematics and Statistics, Lingnan Normal University, Zhanjiang 524048, China.

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[***] Corresponding author: zhouxiuxiang@163.com

which is the reachable set of (1.1) from y_0 over (τ, T). For three given constants $M \geq 0, r > 0$ and $\tau \in [0, T)$, we define the following sets of controls:

$$\mathcal{U}_{\tau,M} := \{v \in L^\infty(0,T;L^2(\Omega)) : \|v(t)\| \leq M \text{ for a.e. } t \in (\tau,T)\};$$
$$\mathcal{V}_{M,r} := \{v \in L^\infty(0,T;L^2(\Omega)) : \text{there exists } \tau \in [0,T) \text{ so that } v \in \mathcal{U}_{\tau,M}$$
$$\text{and } y(T;y_0,\chi_{(\tau,T)}v) \in B(z_d,r)\};$$
$$\mathcal{W}_{r,\tau} := \{v \in L^\infty(0,T;L^2(\Omega)) : y(T;y_0,\chi_{(\tau,T)}v) \in B(z_d,r)\},$$

where

$$z_d \in L^2(\Omega) \setminus \mathcal{Y}(y_0, 0), \tag{1.2}$$

$B(z_d, r)$ denotes the closed ball in $L^2(\Omega)$, centered at z_d and of radius r. It is clear that $\mathcal{U}_{\tau,M} \neq \emptyset$. We will show that $\mathcal{V}_{M,r} \neq \emptyset$ for some cases (see Lem. 3.3) and $\mathcal{W}_{r,\tau} \neq \emptyset$ (see Lem. 3.1).

Next, the optimal target, optimal time and optimal norm control problems are defined as follows:

$$(OP)^{\tau,M}: \quad r(\tau,M) := \inf\{\|y(T;y_0,\chi_{(\tau,T)}u) - z_d\| : u \in \mathcal{U}_{\tau,M}\};$$
$$(TP)^{M,r}: \quad \tau(M,r) := \sup\{\tau \in [0,T) : u \in \mathcal{U}_{\tau,M} \text{ and } y(T;y_0,\chi_{(\tau,T)}u) \in B(z_d,r)\};$$
$$(NP)^{r,\tau}: \quad M(r,\tau) := \inf\{\|u\|_{L^\infty(\tau,T;L^2(\Omega))} : u \in \mathcal{W}_{r,\tau}\}.$$

About problems $(OP)^{\tau,M}, (TP)^{M,r}$ and $(NP)^{r,\tau}$, three notes are given in order.

- *(i)* $r(\tau, M)$ is called the optimal distance for $(OP)^{\tau,M}$; u^* is called an optimal control to $(OP)^{\tau,M}$ if $u^* \in \mathcal{U}_{\tau,M}, u^*(t) = 0$ for a.e. $t \in (0,\tau)$ and $\|y(T;y_0,\chi_{(\tau,T)}u^*) - z_d\| = r(\tau,M)$.
- *(ii)* $\tau(M,r)$ is called the optimal time for $(TP)^{M,r}$; u^* is called an optimal control to $(TP)^{M,r}$ if $u^* \in \mathcal{U}_{\tau(M,r),M}, u^*(t) = 0$ for a.e. $t \in (0,\tau(M,r))$ and $y(T;y_0,\chi_{(\tau(M,r),T)}u^*) \in B(z_d,r)$.
- *(iii)* $M(r,\tau)$ is called the optimal norm for $(NP)^{r,\tau}$; u^* is called an optimal control to $(NP)^{r,\tau}$ if $u^* \in \mathcal{W}_{r,\tau}, u^*(t) = 0$ for a.e. $t \in (0,\tau)$ and $\|u^*\|_{L^\infty(\tau,T;L^2(\Omega))} = M(r,\tau)$.

Let

$$r_T(y_0) := \|y(T;y_0,0) - z_d\|. \tag{1.3}$$

Clearly, when $r \geq r_T(y_0)$, the null control is the optimal control to $(NP)^{r,\tau}$ for any $\tau \in [0,T)$, while the null control and T are the optimal control and the optimal time to $(TP)^{M,r}$ for any $M > 0$. Hence, we only study $(TP)^{M,r}$ and $(NP)^{r,\tau}$ for the case where $r < r_T(y_0)$.

Our main result (see Thm. 1.2) is based on the following hypothesis.

(H) If $z \in L^2(\Omega) \setminus \{0\}$, then $\chi_\omega p(t;z) \neq 0$ for a.e. $t \in (0,T)$, where $p(\cdot;z)$ is the solution to the equation:

$$\begin{cases} p_t + \Delta p - \displaystyle\int_t^T a(s-t)p(s)\,\mathrm{d}s = 0 & \text{in} \quad Q_T, \\ p = 0 & \text{on} \quad \Sigma_T, \\ p(T) = z & \text{in} \quad \Omega. \end{cases} \tag{1.4}$$

Remark 1.1. Let $a(t) = \alpha e^{\beta t}, t \in [0,T]$, where α and β are two constants. By similar arguments as those in [16] (see the proof of (3.17) in [16]), we can directly check that **(H)** holds.

The main result of this paper is the following equivalence theorem.

Theorem 1.2. *Suppose that* **(H)** *holds. The following conclusions are true:*

- (i) *Given $M > 0$ and $\tau \in [0,T)$, problems $(OP)^{\tau,M}, (TP)^{M,r(\tau,M)}$ and $(NP)^{r(\tau,M),\tau}$ have the same optimal control;*
- (ii) *Given $r \in (0, r_T(y_0))$ and $\tau \in [0,T)$, problems $(NP)^{r,\tau}, (OP)^{\tau,M(r,\tau)}$ and $(TP)^{M(r,\tau),r}$ have the same optimal control;*
- (iii) *Given $M > 0$ and $r \in [r(0,M), r_T(y_0))$, problems $(TP)^{M,r}, (NP)^{r,\tau(M,r)}$ and $(OP)^{\tau(M,r),M}$ have the same optimal control.*

Remark 1.3. Under the assumption of (1.2), we have that $r(\tau, M) > 0$ (see Lem. 2.2), which plays an important role in establishing Theorem 1.2. If $r(\tau, M) = 0$, then the target sets in problems $(TP)^{M,r(\tau,M)}$ and $(NP)^{r(\tau,M),\tau}$ would be $\{z_d\}$. It is unreasonable because (1.1) is not exactly controllable.

The equivalence theorems of different optimal control problems have been studied in [12, 13] and the references therein. The equivalence theorems not only play important roles in the studies of these problems, but also have wide applications (see, for instance, [12] and the references therein). Our study is partially motivated by [13], where an equivalence theorem for these three kinds of optimal control problems was studied and the controlled systems are internally controlled heat equations. However, the controlled systems in this paper are internally controlled heat equations with memory. In some cases, such as heat transfer and nuclear reactor dynamics, the effect of the past on the present is needed to consider. Such phenomena can be described by partial integro-differential equations (see, for instance, [5, 11, 18]). There exist many topics on partial integro-differential equations, such as Pontryagin's maximum principle of optimal control problems, controllability, observability, stability, minimal time control and so on (see, for instance, [1–4, 8, 9, 14–16, 19]). In [16], connections between minimal time and optimal norm control problems for linear heat equations with memory were discussed. Minimal time control is to ask for a control (from a constraint set), which drives the solution of the system from a given initial state to a given target set as soon as possible. It is a kind of optimal time control problem. In this paper, $(TP)^{M,r}$ is another kind of optimal time control problem. The aim of controls in $(TP)^{M,r}$ is to delay initiation of active control as late as possible, so that the corresponding solution reaches the given target set $B(z_d, r)$ at the ending time. To the best of our knowledge, the equivalence theorem of the above-mentioned three kinds of optimal control problems for heat equations with memory has not been touched on.

The rest of the paper is organized as follows. Section 2 presents some properties of optimal target control problems. Section 3 shows the existence of optimal controls to optimal norm control problems and optimal time control problems. Sections 4 and 5 establish the equivalence between optimal target control problems and optimal norm control problems, and the equivalence between optimal norm control problems and optimal time control problems, respectively. Section 6 gives the proof of Theorem 1.2.

2. Some properties of optimal target control problems

We start this section by discussing the following convergence result.

Lemma 2.1. *Let $\tau \in [0,T], \{\tau_n\}_{n\geq 1} \subset [0,T], u \in L^\infty(0,T;L^2(\Omega))$ and $\{u_n\}_{n\geq 1} \subset L^\infty(0,T;L^2(\Omega))$. Assume that $\chi_{(\tau_n,T)}u_n \to \chi_{(\tau,T)}u$ weakly star in $L^\infty(0,T;L^2(\Omega))$. Then there exists a subsequence of $\{n\}_{n\geq 1}$, still denoted by itself, so that*

$$y(\cdot; y_0, \chi_{(\tau_n,T)}u_n) \to y(\cdot; y_0, \chi_{(\tau,T)}u) \;\; \textit{strongly in} \;\; C([0,T]; L^2(\Omega)).$$

Proof. Denote

$$w_n(\cdot) := y(\cdot; y_0, \chi_{(\tau_n,T)}u_n) - y(\cdot; y_0, \chi_{(\tau,T)}u).$$

It is clear that

$$\begin{cases} (w_n)_t - \Delta w_n = \chi_\omega(\chi_{(\tau_n,T)}u_n - \chi_{(\tau,T)}u) - \displaystyle\int_0^t a(t-s)w_n(s)\,\mathrm{d}s & \text{in} \quad Q_T, \\ w_n = 0 & \text{on} \quad \Sigma_T, \\ w_n(0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.1}$$

Multiplying the first equation of (2.1) by $2w_n$, we obtain that for a.e. $t \in (0,T)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|w_n(t)\|^2 + 2\|\nabla w_n(t)\|^2 \le C\|w_n(t)\| + 2\|w_n(t)\|\int_0^t |a(t-s)|\|w_n(s)\|\,\mathrm{d}s. \tag{2.2}$$

Here and throughout the proof of this lemma, C denotes a generic positive constant independent of n. Integrating (2.2) over $(0,t)$, we get that

$$\begin{aligned} &\|w_n(t)\|^2 + 2\int_0^t \|\nabla w_n(s)\|^2\,\mathrm{d}s \\ \le\ & C + C\int_0^t \|w_n(s)\|^2\,\mathrm{d}s + 2\int_0^t \left(\int_0^r |a(r-s)|\|w_n(s)\|\,\mathrm{d}s\right)\|w_n(r)\|\,\mathrm{d}r. \end{aligned} \tag{2.3}$$

For the last term on the right hand side of (2.3), it follows from Hölder's inequality that

$$\begin{aligned} &2\int_0^t \left(\int_0^r |a(r-s)|\|w_n(s)\|\,\mathrm{d}s\right)\|w_n(r)\|\,\mathrm{d}r \\ &\le 2\left[\int_0^t \left(\int_0^r |a(r-s)|\|w_n(s)\|\,\mathrm{d}s\right)^2\mathrm{d}r\right]^{\frac{1}{2}}\left(\int_0^t \|w_n(r)\|^2\,\mathrm{d}r\right)^{\frac{1}{2}} \\ &\le 2\left[\int_0^t \left(\int_0^r |a(r-s)|\,\mathrm{d}s\right)\left(\int_0^r |a(r-s)|\|w_n(s)\|^2\,\mathrm{d}s\right)\mathrm{d}r\right]^{\frac{1}{2}}\left(\int_0^t \|w_n(r)\|^2\,\mathrm{d}r\right)^{\frac{1}{2}}. \end{aligned}$$

Since $a \in L^1(0,T)$, we obtain from the above estimate that

$$\begin{aligned} &2\int_0^t \left(\int_0^r |a(r-s)|\|w_n(s)\|\,\mathrm{d}s\right)\|w_n(r)\|\,\mathrm{d}r \\ &\le C\left(\int_0^t\int_0^r |a(r-s)|\|w_n(s)\|^2\,\mathrm{d}s\mathrm{d}r\right)^{\frac{1}{2}}\left(\int_0^t \|w_n(r)\|^2\,\mathrm{d}r\right)^{\frac{1}{2}} \\ &= C\left(\int_0^t\int_s^t |a(r-s)|\|w_n(s)\|^2\,\mathrm{d}r\mathrm{d}s\right)^{\frac{1}{2}}\left(\int_0^t \|w_n(r)\|^2\,\mathrm{d}r\right)^{\frac{1}{2}} \\ &\le C\int_0^t \|w_n(s)\|^2\,\mathrm{d}s, \end{aligned}$$

which, combined with (2.3), indicates that

$$\|w_n(t)\|^2 + 2\int_0^t \|\nabla w_n(s)\|^2 \, \mathrm{d}s \le C + C\int_0^t \|w_n(s)\|^2 \, \mathrm{d}s. \tag{2.4}$$

By (2.4) and Gronwall's inequality, we get that

$$\|w_n\|^2_{C([0,T];L^2(\Omega))} + \|w_n\|^2_{L^2(0,T;H_0^1(\Omega))} \le C. \tag{2.5}$$

This, along with (2.1), and standard energy estimate for heat equation (see Rem. 3.4.1 of Chap. 3 in [17]), yields that

$$\begin{aligned} &\|w_n\|_{L^2(0,T;H^2(\Omega))} + \|w_n\|_{W^{1,2}(0,T;L^2(\Omega))} + \|w_n\|_{C([0,T];H_0^1(\Omega))} \\ \le\, & C\Big\|\chi_{(\tau_n,T)}u_n - \chi_{(\tau,T)}u - \int_0^t a(t-s)w_n(s)\,\mathrm{d}s\Big\|_{L^2(0,T;L^2(\Omega))} \\ \le\, & C. \end{aligned}$$

Hence, according to Ascoli's Theorem (see Thm. 3.1 of Chap. 3 in [10]), there is a subsequence of $\{n\}_{n\ge1}$, still denoted in the same manner, and a function $w \in W^{1,2}(0,T;L^2(\Omega)) \bigcap L^2(0,T;H^2(\Omega)\bigcap H_0^1(\Omega))$, so that

$$\begin{aligned} w_n \to w \quad & \text{weakly in } \ W^{1,2}(0,T;L^2(\Omega)) \textstyle\bigcap L^2(0,T;H^2(\Omega)\bigcap H_0^1(\Omega)), \\ & \text{and strongly in } \ C([0,T];L^2(\Omega)). \end{aligned} \tag{2.6}$$

Passing to the limit for $n \to +\infty$ in (2.1), by (2.6), we get that $w = 0$ and

$$y(\cdot;y_0,\chi_{(\tau_n,T)}u_n) \to y(\cdot;y_0,\chi_{(\tau,T)}u) \ \text{ strongly in } \ C([0,T];L^2(\Omega)).$$

In summary, we complete the proof of Lemma 2.1. □

The next result shows the existence of optimal controls to optimal target control problems.

Lemma 2.2. *Let $M \ge 0$ and $\tau \in [0,T)$. Then $(OP)^{\tau,M}$ has optimal controls and $r(\tau,M) > 0$.*

Proof. When $M = 0$, it is clear that the null control is the optimal control to $(OP)^{\tau,M}$. When $M > 0$, by the definition of $r(\tau,M)$, there exists a sequence $\{u_n\}_{n\ge1} \subset \mathcal{U}_{\tau,M}$ so that

$$\|y(T;y_0,\chi_{(\tau,T)}u_n) - z_d\| \to r(\tau,M) \ \text{ as } \ n \to +\infty. \tag{2.7}$$

Since $\{u_n\}_{n\ge1} \subset \mathcal{U}_{\tau,M}$, there is a subsequence of $\{n\}_{n\ge1}$, still denoted by itself, and a control $\widetilde{u} \in \mathcal{U}_{\tau,M}$, so that

$$u_n \to \widetilde{u} \ \text{ weakly star in } \ L^\infty(\tau,T;L^2(\Omega)).$$

This, together with Lemma 2.1 and (2.7), implies that $\|y(T;y_0,\chi_{(\tau,T)}\widetilde{u}) - z_d\| = r(\tau,M)$. Furthermore, since $\widetilde{u} \in \mathcal{U}_{\tau,M}$, we conclude that $\chi_{(\tau,T)}\widetilde{u}$ is an optimal control to $(OP)^{\tau,M}$.

To prove $r(\tau,M) > 0$, let $u^* \in \mathcal{U}_{\tau,M}$ be an optimal control to $(OP)^{\tau,M}$. We note that

$$r(\tau,M) = \|y(T;y_0,\chi_{(\tau,T)}u^*) - z_d\| \ \text{ and } \ y(T;y_0,\chi_{(\tau,T)}u^*) \in \mathcal{Y}(y_0,0).$$

This, along with (1.2), yields that $r(\tau,M) > 0$.

Hence, we end the proof of Lemma 2.2. □

Lemma 2.3. *Let $M \geq 0$ and $\tau \in [0,T)$. Then u^* is an optimal control to $(OP)^{\tau,M}$ if and only if $u^* \in \mathcal{U}_{\tau,M}$, $u^*(t) = 0$ for a.e. $t \in (0,\tau)$, and*

$$\int_\tau^T \langle \chi_\omega p^*(t), u^*(t) \rangle \,\mathrm{d}t = \max_{v \in \mathcal{U}_{\tau,M}} \int_\tau^T \langle \chi_\omega p^*(t), v(t) \rangle \,\mathrm{d}t, \tag{2.8}$$

where $p^*(\cdot) := p(\cdot; z_d - y(T; y_0, \chi_{(\tau,T)} u^*))$ is the solution of (1.4) (with z replaced by $z_d - y(T; y_0, \chi_{(\tau,T)} u^*)$).

Proof. When $M = 0$, it is trivial. It suffices to consider the case $M > 0$. For any $u_1, u_2 \in \mathcal{U}_{\tau,M}$ and $\varepsilon \in [0,1]$, we set

$$\begin{cases} \widetilde{u}_\varepsilon := (1-\varepsilon) u_1 + \varepsilon u_2, \\ y_i(t) := y(t; y_0, \chi_{(\tau,T)} u_i), \quad t \in [0,T], \quad i = 1,2, \\ \widetilde{y}_\varepsilon(t) := y(t; y_0, \chi_{(\tau,T)} \widetilde{u}_\varepsilon), \quad t \in [0,T]. \end{cases}$$

It is clear that

$$\begin{cases} (\widetilde{y}_\varepsilon - y_1)_t - \Delta(\widetilde{y}_\varepsilon - y_1) = \chi_\omega \chi_{(\tau,T)} (\widetilde{u}_\varepsilon - u_1) - \displaystyle\int_0^t a(t-s)(\widetilde{y}_\varepsilon - y_1)(s) \,\mathrm{d}s & \text{in} \quad Q_T, \\ \widetilde{y}_\varepsilon - y_1 = 0 & \text{on} \quad \Sigma_T, \\ (\widetilde{y}_\varepsilon - y_1)(0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.9}$$

Multiplying the first equation of (2.9) by $p(\cdot; z_d - y_1(T))$ and integrating it over Q_T, we obtain that

$$\langle \widetilde{y}_\varepsilon(T) - y_1(T), z_d - y_1(T) \rangle = \varepsilon \int_\tau^T \langle \chi_\omega (u_2 - u_1), p(t; z_d - y_1(T)) \rangle \,\mathrm{d}t. \tag{2.10}$$

Since

$$2\langle \widetilde{y}_\varepsilon(T) - y_1(T), y_1(T) - z_d \rangle = \|\widetilde{y}_\varepsilon(T) - z_d\|^2 - \|y_1(T) - z_d\|^2 - \|\widetilde{y}_\varepsilon(T) - y_1(T)\|^2,$$

it follows from (2.10) that

$$\begin{aligned} &\|\widetilde{y}_\varepsilon(T) - z_d\|^2 - \|y_1(T) - z_d\|^2 \\ =& \|\widetilde{y}_\varepsilon(T) - y_1(T)\|^2 - 2\varepsilon \int_\tau^T \langle \chi_\omega (u_2 - u_1), p(t; z_d - y_1(T)) \rangle \,\mathrm{d}t \\ =& \varepsilon^2 \|y(T; 0, \chi_{(\tau,T)}(u_2 - u_1))\|^2 - 2\varepsilon \int_\tau^T \langle \chi_\omega (u_2 - u_1), p(t; z_d - y_1(T)) \rangle \,\mathrm{d}t. \end{aligned} \tag{2.11}$$

We first prove the necessity. Let u^* be an optimal control to $(OP)^{\tau,M}$. Clearly, $u^* \in \mathcal{U}_{\tau,M}$ and $u^*(t) = 0$ for a.e. $t \in (0,\tau)$. Arbitrarily take $u \in \mathcal{U}_{\tau,M}$. Choosing $u_1 = u^*$ and $u_2 = u$ in (2.11), we get that

$$\begin{aligned} &\int_\tau^T \langle \chi_\omega (u - u^*), p^*(t) \rangle \,\mathrm{d}t \\ =& \frac{\varepsilon}{2} \|y(T; 0, \chi_{(\tau,T)}(u - u^*))\|^2 - \frac{1}{2\varepsilon} \|\widetilde{y}_\varepsilon(T) - z_d\|^2 + \frac{1}{2\varepsilon} \|y(T; y_0, \chi_{(\tau,T)} u^*) - z_d\|^2 \\ \leq& \frac{\varepsilon}{2} \|y(T; 0, \chi_{(\tau,T)}(u - u^*))\|^2. \end{aligned}$$

Passing to the limit for $\varepsilon \to 0^+$ in the above inequality, we observe that

$$\int_\tau^T \langle \chi_\omega(u-u^*), p^*(t)\rangle \,\mathrm{d}t \le 0 \ \text{ for all } \ u \in \mathcal{U}_{\tau,M}.$$

This implies (2.8).

We next show the sufficiency. Suppose that $u^* \in \mathcal{U}_{\tau,M}$ satisfies (2.8) and $u^*(t) = 0$ for a.e. $t \in (0,\tau)$. For arbitrarily fixed $u \in \mathcal{U}_{\tau,M}$, let $\varepsilon = 1, u_1 = u^*$ and $u_2 = u$ in (2.11). We have that

$$\begin{aligned} & \|y(T; y_0, \chi_{(\tau,T)}u) - z_d\|^2 - \|y(T; y_0, \chi_{(\tau,T)}u^*) - z_d\|^2 \\ = \ & \|y(T; 0, \chi_{(\tau,T)}(u-u^*))\|^2 - 2\int_\tau^T \langle \chi_\omega(u-u^*), p^*(t)\rangle \,\mathrm{d}t \ge 0, \end{aligned}$$

which indicates that

$$\|y(T; y_0, \chi_{(\tau,T)}u) - z_d\| \ge \|y(T; y_0, \chi_{(\tau,T)}u^*) - z_d\| \ \text{ for all } \ u \in \mathcal{U}_{\tau,M}.$$

This yields that u^* is an optimal control to $(OP)^{\tau,M}$.

In summary, we finish the proof of Lemma 2.3. □

Corollary 2.4. *Suppose that* **(H)** *holds. For each $M \ge 0$ and $\tau \in [0,T)$, any optimal control u^* to the problem $(OP)^{\tau,M}$ satisfies that*

$$\|u^*(t)\| = M \ \text{ for a.e. } \ t \in (\tau, T). \tag{2.12}$$

Consequently, $(OP)^{\tau,M}$ has a unique optimal control.

Proof. Let u^* be an optimal control to $(OP)^{\tau,M}$. According to (2.8), it holds that

$$\langle \chi_\omega p^*(t), u^*(t)\rangle = \max_{v \in B(0,M)} \langle \chi_\omega p^*(t), v\rangle \ \text{ for a.e. } \ t \in (\tau, T), \tag{2.13}$$

where $B(0,M)$ denotes the closed ball in $L^2(\Omega)$, centered at the origin, and of radius M. Since $p^*(T) \neq 0$, it follows from **(H)** that $\|\chi_\omega p^*(t)\| \neq 0$ for a.e. $t \in (0,T)$. Thus, it follows from (2.13) that

$$u^*(t) = M\frac{\chi_\omega p^*(t)}{\|\chi_\omega p^*(t)\|} \ \text{ for a.e. } \ t \in (\tau, T).$$

This implies (2.12).

Suppose that $v^* \in \mathcal{U}_{\tau,M}$ is also an optimal control to $(OP)^{\tau,M}$. It is obvious that $(u^*+v^*)/2$ is an optimal control to $(OP)^{\tau,M}$. According to (2.12), it holds that

$$\|u^*(t)\| = \|v^*(t)\| = \|(u^*(t)+v^*(t))/2\| = M \ \text{ for a.e. } \ t \in (\tau, T).$$

Hence,

$$\|u^*(t) - v^*(t)\|^2 = 2(\|u^*(t)\|^2 + \|v^*(t)\|^2) - \|u^*(t)+v^*(t)\|^2 = 0 \ \text{ for a.e. } \ t \in (\tau, T),$$

which indicates that $u^*(\cdot) = v^*(\cdot)$.

This completes the proof. □

3. The existence of optimal controls to $(NP)^{r,\tau}$ and $(TP)^{M,r}$

Lemma 3.1. *Suppose that* **(H)** *holds. For each* $\tau \in [0,T)$*, the system* (1.1) *is approximately controllable at time* T*, i.e.,*

$$\overline{\mathcal{Y}(y_0,\tau)}^{\|\cdot\|} = L^2(\Omega).$$

Proof. Without loss of generality, we assume $y_0 = 0$. Then we define $G : L^\infty(0,T;L^2(\Omega)) \to L^2(\Omega)$ as follows:

$$Gu := y(T;0,\chi_{(\tau,T)}u) \text{ for all } u \in L^\infty(0,T;L^2(\Omega)).$$

Clearly, (1.1) is approximately controllable at time T if and only if the range of G is dense in $L^2(\Omega)$. Furthermore, by the duality argument (see [6]), it suffices to show that

$$\text{Ker}G^* = \{0\}. \tag{3.1}$$

Indeed, by (1.1) and (1.4), we have that for each $u \in L^\infty(0,T;L^2(\Omega))$ and each $z \in L^2(\Omega)$,

$$\int_\tau^T \langle \chi_\omega u(t), p(t;z)\rangle \,\mathrm{d}t = \langle Gu, z\rangle.$$

This implies that

$$G^*(z) = \chi_\omega \chi_{(\tau,T)} p(\cdot;z) \text{ for all } z \in L^2(\Omega).$$

Hence, by **(H)**, we see that (3.1) holds.

This completes the proof. □

Lemma 3.2. *Suppose that* **(H)** *holds. For each* $\tau \in [0,T)$ *and* $r \in (0,+\infty)$*, the problem* $(NP)^{r,\tau}$ *has optimal controls. Moreover,* $M(r,\tau) \in (0,+\infty)$ *when* $\tau \in [0,T)$ *and* $r \in (0, r_T(y_0))$*.*

Proof. Let $\tau \in [0,T)$ and $r \in (0,+\infty)$ be fixed. By Lemma 3.1, we have that $\mathcal{W}_{r,\tau} \neq \emptyset$. Then, according to the definition of $M(r,\tau)$, there exists a sequence $\{u_n\}_{n\geq 1} \subset L^\infty(0,T;L^2(\Omega))$ so that

$$y(T;y_0,\chi_{(\tau,T)}u_n) \in B(z_d,r) \text{ and } \lim_{n\to+\infty} \|u_n\|_{L^\infty(\tau,T;L^2(\Omega))} = M(r,\tau). \tag{3.2}$$

It follows from the second relation of (3.2) that there is a subsequence of $\{n\}_{n\geq 1}$, still denoted by itself, and a control $\widetilde{u} \in L^\infty(0,T;L^2(\Omega))$, so that

$$u_n \to \widetilde{u} \text{ weakly star in } L^\infty(\tau,T;L^2(\Omega)), \tag{3.3}$$

and

$$\|\widetilde{u}\|_{L^\infty(\tau,T;L^2(\Omega))} \leq \liminf_{n\to+\infty} \|u_n\|_{L^\infty(\tau,T;L^2(\Omega))} = M(r,\tau). \tag{3.4}$$

Moreover, by (3.3) and Lemma 2.1, there is a subsequence of $\{n\}_{n\geq 1}$, still denoted in the same manner, so that

$$y(T;y_0,\chi_{(\tau,T)}u_n) \to y(T;y_0,\chi_{(\tau,T)}\widetilde{u}) \text{ strongly in } L^2(\Omega).$$

This, along with the first relation in (3.2), yields that

$$\|y(T;y_0,\chi_{(\tau,T)}\widetilde{u})-z_d\|\leq r. \tag{3.5}$$

By (3.4) and (3.5), we see that $\chi_{(\tau,T)}\widetilde{u}$ is an optimal control to $(NP)^{r,\tau}$.

Arbitrarily fix $\tau\in[0,T)$ and $r\in(0,r_T(y_0))$. From the above fact, we have that $M(r,\tau)<+\infty$. Seeking for a contradiction, we suppose that $M(r_0,\tau)=0$ for some $r_0\in(0,r_T(y_0))$. Then the null control is an optimal control to $(NP)^{r_0,\tau}$ and

$$\|y(T;y_0,0)-z_d\|\leq r_0<r_T(y_0),$$

which, combined with the definition of $r_T(y_0)$ (see (1.3)), leads to a contradiction.

Hence, we finish the proof of Lemma 3.2. □

Lemma 3.3. *Suppose that* **(H)** *holds. For each $M\geq 0$ and $r\in(0,r_T(y_0))$, the problem $(TP)^{M,r}$ has optimal controls if and only if $M\geq M(r,0)$.*

Proof. Firstly, we prove the necessity. Let u^* be an optimal control to $(TP)^{M,r}$, where $M\geq 0$ and $r\in(0,r_T(y_0))$. Then we have that

$$\tau(M,r)\in[0,T),u^*\in\mathcal{U}_{\tau(M,r),M}\ \text{ and }\ y(T;y_0,\chi_{(\tau(M,r),T)}u^*)\in B(z_d,r),$$

which indicate that $\chi_{(\tau(M,r),T)}u^*\in\mathcal{W}_{r,0}$. By the definition of $M(r,0)$, we find that

$$M(r,0)\leq\|\chi_{(\tau(M,r),T)}u^*\|_{L^\infty(0,T;L^2(\Omega))}\leq M.$$

Next, we show the sufficiency. Arbitrarily take $M\geq M(r,0)$ and $r\in(0,r_T(y_0))$. According to Lemma 3.2, $(NP)^{r,0}$ has an optimal control, denoted by v^*. Clearly,

$$y(T;y_0,\chi_{(0,T)}v^*)\in B(z_d,r)\ \text{ and }\ \|v^*\|_{L^\infty(0,T;L^2(\Omega))}=M(r,0)\leq M.$$

These, along with the definition of $\tau(M,r)$, imply that there exist two sequences $\{\tau_n\}_{n\geq1}\subset[0,T)$ and $\{u_n\}_{n\geq1}$ with $u_n\in\mathcal{U}_{\tau_n,M}$ (for each $n\geq1$) satisfying

$$\lim_{n\to+\infty}\tau_n=\tau(M,r)\ \text{ and }\ y(T;y_0,\chi_{(\tau_n,T)}u_n)\in B(z_d,r). \tag{3.6}$$

Therefore, there is a subsequence of $\{n\}_{n\geq1}$, still denoted by itself, and a control $\widetilde{u}\in L^\infty(0,T;L^2(\Omega))$ with $\|\widetilde{u}\|_{L^\infty(0,T;L^2(\Omega))}\leq M$, so that

$$\chi_{(\tau_n,T)}u_n\to\chi_{(\tau(M,r),T)}\widetilde{u}\ \text{ weakly star in }\ L^\infty(0,T;L^2(\Omega)).$$

Moreover, by Lemma 2.1, there exists a subsequence of $\{n\}_{n\geq1}$, still denoted in the same manner, so that

$$y(T;y_0,\chi_{(\tau_n,T)}u_n)\to y(T;y_0,\chi_{(\tau(M,r),T)}\widetilde{u})\ \text{ strongly in }\ L^2(\Omega). \tag{3.7}$$

By the second relation of (3.6) and (3.7), we see that

$$y(T;y_0,\chi_{(\tau(M,r),T)}\widetilde{u})\in B(z_d,r).$$

Hence, $\chi_{(\tau(M,r),T)}\widetilde{u}$ is an optimal control to $(TP)^{M,r}$.

This completes the proof. □

4. Equivalence of optimal target and norm control problems

Lemma 4.1. *Suppose that* **(H)** *holds. Let* $\tau \in [0, T)$. *Then the function* $r(\tau, \cdot)$ *is strictly monotonically decreasing and continuous from* $[0, +\infty)$ *onto* $(0, r_T(y_0)]$. *Moreover,*

$$r = r(\tau, M(r, \tau)) \ \textit{for each} \ r \in (0, r_T(y_0)], \tag{4.1}$$

and

$$M = M(r(\tau, M), \tau) \ \textit{for each} \ M \geq 0. \tag{4.2}$$

Proof. The proof will be divided into four steps.

Step 1. The map $M \to r(\tau, M)$ is strictly decreasing.

Let $0 \leq M_1 < M_2$. We show that $r(\tau, M_1) > r(\tau, M_2)$. By contradiction, we suppose that $r(\tau, M_2) \geq r(\tau, M_1)$. Then the optimal control u_1 to $(OP)^{\tau, M_1}$ satisfies that

$$u_1 \in \mathcal{U}_{\tau, M_1} \subset \mathcal{U}_{\tau, M_2} \ \text{ and } \ \|y(T; y_0, \chi_{(\tau,T)} u_1) - z_d\| = r(\tau, M_1) \leq r(\tau, M_2).$$

These imply that u_1 is also an optimal control to $(OP)^{\tau, M_2}$. By Corollary 2.4, we find that

$$\|u_1(t)\| = M_2 \ \text{ for a.e. } \ t \in (\tau, T),$$

which leads to a contradiction since $u_1 \in \mathcal{U}_{\tau, M_1}$ and $M_1 < M_2$.

Step 2. The map $M \to r(\tau, M)$ is continuous.

Let $0 \leq M_1 < M_2$ and let u_2 be the optimal control to $(OP)^{\tau, M_2}$. By Step 1, we obtain that

$$\begin{aligned} r(\tau, M_1) &> r(\tau, M_2) = \|y(T; y_0, \chi_{(\tau,T)} u_2) - z_d\| \\ &\geq \left\| y\Big(T; y_0, \chi_{(\tau,T)} \tfrac{M_1}{M_2} u_2\Big) - z_d \right\| - \left\| y\Big(T; 0, \chi_{(\tau,T)} \tfrac{M_2 - M_1}{M_2} u_2\Big) \right\| \\ &= \left\| y\Big(T; y_0, \chi_{(\tau,T)} \tfrac{M_1}{M_2} u_2\Big) - z_d \right\| - \tfrac{M_2 - M_1}{M_2} \|y(T; 0, \chi_{(\tau,T)} u_2)\|. \end{aligned} \tag{4.3}$$

On one hand, by the definition of $r(\tau, M_1)$ and the fact that $\frac{M_1}{M_2} u_2 \in \mathcal{U}_{\tau, M_1}$, we have that

$$\left\| y\Big(T; y_0, \chi_{(\tau,T)} \frac{M_1}{M_2} u_2\Big) - z_d \right\| \geq r(\tau, M_1). \tag{4.4}$$

On the other hand, by similar arguments as those led to (2.5), we get that

$$\|y(T; 0, \chi_{(\tau,T)} u_2)\| \leq C M_2, \tag{4.5}$$

where $C > 0$ is a constant independent of M_1 and M_2. It follows from (4.3) to (4.5) that

$$r(\tau, M_1) > r(\tau, M_2) \geq r(\tau, M_1) - C(M_2 - M_1),$$

which indicates that

$$|r(\tau, M_1) - r(\tau, M_2)| \le C(M_2 - M_1).$$

Step 3. It holds that $r(\tau, 0) = r_T(y_0)$ and $\lim_{M\to+\infty} r(\tau, M) = 0$.

The first equality follows from the definitions of $r_T(y_0)$ and $r(\tau, 0)$. We now prove the second equality. Let $\varepsilon > 0$ and fix it. According to Lemma 3.1, there exists a control $u_\varepsilon \in L^\infty(0, T; L^2(\Omega))$ so that

$$y(T; y_0, \chi_{(\tau,T)} u_\varepsilon) \in B(z_d, \varepsilon).$$

It is clear that

$$u_\varepsilon \in \mathcal{U}_{\tau,M} \;\text{ for all }\; M \ge \|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}.$$

By the definition of $r(\tau, M)$, we get that

$$r(\tau, M) \le \|y(T; y_0, \chi_{(\tau,T)} u_\varepsilon) - z_d\| \le \varepsilon \;\text{ for all }\; M \ge \|u_\varepsilon\|_{L^\infty(\tau,T;L^2(\Omega))},$$

which indicates that $\lim_{M\to+\infty} r(\tau, M) = 0$.

Step 4. Proofs of (4.1) and (4.2).

We first claim (4.1). By the definition of $r_T(y_0)$, one can easily check that

$$M(r_T(y_0), \tau) = 0 \;\text{ and }\; r_T(y_0) = r(\tau, 0) = r(\tau, M(r_T(y_0), \tau)). \tag{4.6}$$

For each $r \in (0, r_T(y_0))$, let $\widetilde{u}$ be an optimal control to $(NP)^{r,\tau}$ (see Lem. 3.2). Then $\widetilde{u} \in \mathcal{U}_{\tau,M(r,\tau)}$ and $\|y(T; y_0, \chi_{(\tau,T)} \widetilde{u}) - z_d\| \le r$. By the definition of $r(\tau, M(r, \tau))$, we have that

$$r(\tau, M(r, \tau)) \le r.$$

Suppose that $r(\tau, M(r, \tau)) < r$. Since the function $r(\tau, \cdot)$ is continuous and strictly monotonically decreasing, there is a $M_0 \in (0, M(r, \tau))$ so that $r(\tau, M_0) = r$. Hence, according to Corollary 2.4, the optimal control u_0 to $(OP)^{\tau,M_0}$ satisfies that

$$\|u_0\|_{L^\infty(\tau,T;L^2(\Omega))} = M_0 < M(r, \tau) \;\text{ and }\; \|y(T; y_0, \chi_{(\tau,T)} u_0) - z_d\| = r. \tag{4.7}$$

It follows from the second equality in (4.7) that $u_0 \in \mathcal{W}_{r,\tau}$. By the definition of $M(r, \tau)$, we see that $M(r, \tau) \le \|u_0\|_{L^\infty(\tau,T;L^2(\Omega))}$, which contradicts the first inequality in (4.7). Hence,

$$r(\tau, M(r, \tau)) = r \;\text{ for all }\; r \in (0, r_T(y_0)).$$

This, together with the second equality in (4.6), yields (4.1).

Next, we show (4.2). According to Steps 1–3, it is clear that

$$r(\tau, M) \in (0, r_T(y_0)] \;\text{ for all }\; M \ge 0 \;\text{ and }\; \tau \in [0, T).$$

Then by (4.1), we obtain that

$$r(\tau, M) = r(\tau, M(r(\tau, M), \tau)) \text{ for all } M \geq 0 \text{ and } \tau \in [0, T),$$

which, combined with Step 1, indicates (4.2).

In summary, we finish the proof of Lemma 4.1. □

The main result of this section reads as follows.

Theorem 4.2. *Suppose that* **(H)** *holds.*

(i) *Given $M \geq 0$ and $\tau \in [0, T)$, the optimal control to $(OP)^{\tau,M}$ is an optimal control to $(NP)^{r(\tau,M),\tau}$.*
(ii) *Given $\tau \in [0, T)$ and $r \in (0, r_T(y_0)]$, any optimal control to $(NP)^{r,\tau}$ is the optimal control to $(OP)^{\tau,M(r,\tau)}$.*
(iii) *Given $\tau \in [0, T)$ and $r \in (0, r_T(y_0)]$, $(NP)^{r,\tau}$ holds the bang-bang property (i.e., any optimal control u^* satisfies that $\|u^*(t)\| = M(r, \tau)$ for a.e. $t \in (\tau, T)$) and has a unique optimal control.*

Proof. (i) Let u_1 be the optimal control to $(OP)^{\tau,M}$. This, along with Corollary 2.4, yields that

$$u_1 \in L^\infty(0, T; L^2(\Omega)), \|u_1\|_{L^\infty(\tau,T;L^2(\Omega))} = M, u_1(t) = 0 \text{ for a.e. } t \in (0, \tau),$$

and

$$y(T; y_0, \chi_{(\tau,T)}u_1) \in B(z_d, r(\tau, M)).$$

It follows from the above and (4.2) that u_1 is an optimal control to $(NP)^{r(\tau,M),\tau}$.

(ii) Let u_2 be an arbitrary optimal control to $(NP)^{r,\tau}$. Then

$$u_2 \in L^\infty(0, T; L^2(\Omega)), \|u_2\|_{L^\infty(\tau,T;L^2(\Omega))} = M(r, \tau), u_2(t) = 0 \text{ for a.e. } t \in (0, \tau),$$

and

$$y(T; y_0, \chi_{(\tau,T)}u_2) \in B(z_d, r),$$

which, combined with (4.1) and Corollary 2.4, indicate that u_2 is the optimal control to $(OP)^{\tau,M(r,\tau)}$.

(iii) The bang-bang property and the uniqueness of optimal control follow from (ii) and Corollary 2.4. □

5. Equivalence of optimal norm and time control problems

Lemma 5.1. *Suppose that* **(H)** *holds. Let $r \in (0, r_T(y_0))$. Then the function $M(r, \cdot)$ is strictly monotonically increasing and continuous from $[0, T)$ onto $[M(r, 0), +\infty)$. Moreover,*

$$M = M(r, \tau(M, r)) \text{ for each } M \in [M(r, 0), +\infty) \tag{5.1}$$

and

$$\tau = \tau(M(r, \tau), r) \text{ for each } \tau \in [0, T). \tag{5.2}$$

Proof. The proof will be carried out by the following five steps.

Step 1. The map $\tau \to M(r, \tau)$ is strictly increasing over $[0, T)$.

Let $0 \leq \tau_1 < \tau_2 < T$. We claim that $M(r,\tau_1) < M(r,\tau_2)$. By contradiction, suppose that $M(r,\tau_2) \leq M(r,\tau_1)$. According to Theorem 4.2, the optimal control u to $(NP)^{r,\tau_2}$ satisfies

$$u \in L^\infty(0,T;L^2(\Omega)),\ \|u\|_{L^\infty(\tau_2,T;L^2(\Omega))} = M(r,\tau_2) \leq M(r,\tau_1)$$

and

$$y(T;y_0,\chi_{(\tau_2,T)}u) \in B(z_d,r).$$

These imply that $\chi_{(\tau_2,T)}u$ is the optimal control to $(NP)^{r,\tau_1}$. It follows from Theorem 4.2 and Lemma 3.2 that

$$\|\chi_{(\tau_2,T)}u(t)\| = M(r,\tau_1) > 0 \ \text{ for a.e. } \ t \in (\tau_1,\tau_2).$$

This leads to a contradiction since $\chi_{(\tau_2,T)}u(t) = 0$ for a.e. $t \in (\tau_1,\tau_2)$.

Step 2. The function $M(r,\cdot)$ is left-continuous.

Let $0 \leq \tau_1 < \tau_2 < \cdots < \tau_n < \cdots < \tau < T$ and $\lim\limits_{n\to+\infty} \tau_n = \tau$. We are going to show that

$$\lim_{n\to+\infty} M(r,\tau_n) = M(r,\tau). \tag{5.3}$$

If (5.3) does not hold, then by Step 1, we get that

$$\lim_{n\to+\infty} M(r,\tau_n) = M(r,\tau) - \delta \ \text{ for some } \ \delta \in (0, M(r,\tau)). \tag{5.4}$$

Let $u_n \in \mathcal{U}_{\tau_n,M(r,\tau_n)}$ be the optimal control to $(OP)^{\tau_n,M(r,\tau_n)}$. According to Lemma 2.3, it holds that

$$\int_{\tau_n}^T \langle \chi_\omega p_n(t), u_n(t)\rangle \,\mathrm{d}t \geq \int_{\tau_n}^T \langle \chi_\omega p_n(t), v_n(t)\rangle \,\mathrm{d}t \ \text{ for all } \ v_n \in \mathcal{U}_{\tau_n,M(r,\tau_n)}, \tag{5.5}$$

where $p_n(\cdot) := p(\cdot; z_d - y(T;y_0,\chi_{(\tau_n,T)}u_n))$. Moreover, it follows from (4.1), Corollary 2.4, Step 1 and (5.4) that

$$r = r(\tau_n, M(r,\tau_n)) = \|y(T;y_0,\chi_{(\tau_n,T)}u_n) - z_d\| \ \text{ for all } \ n \geq 1 \tag{5.6}$$

and

$$\|u_n\|_{L^\infty(\tau_n,T;L^2(\Omega))} = M(r,\tau_n) \leq M(r,\tau) - \delta. \tag{5.7}$$

Next, on one hand, by (5.7) and Lemma 2.1, there exists a subsequence of $\{u_n\}_{n\geq1}$, still denoted in the same way, and a control $\widetilde{u} \in L^\infty(0,T;L^2(\Omega))$ with $\|\widetilde{u}\|_{L^\infty(0,T;L^2(\Omega))} \leq M(r,\tau) - \delta$, so that

$$\chi_{(\tau_n,T)}u_n \to \chi_{(\tau,T)}\widetilde{u} \ \text{ weakly star in } \ L^\infty(0,T;L^2(\Omega)) \tag{5.8}$$

and

$$y(T;y_0,\chi_{(\tau_n,T)}u_n) \to y(T;y_0,\chi_{(\tau,T)}\widetilde{u}) \ \text{ strongly in } \ L^2(\Omega). \tag{5.9}$$

On the other hand, by similar arguments as those led to (2.5), we have that

$$\|p_n - \widetilde{p}\|_{C([0,T];L^2(\Omega))} \le C\|y(T;y_0,\chi_{(\tau_n,T)}u_n) - y(T;y_0,\chi_{(\tau,T)}\widetilde{u})\|_{L^2(\Omega)} \to 0, \tag{5.10}$$

where $\widetilde{p}(\cdot) := p(\cdot; z_d - y(T;y_0,\chi_{(\tau,T)}\widetilde{u}))$ and $C > 0$ is a constant independent of n. It follows from (5.6), (5.9) and (4.1) that

$$\|y(T;y_0,\chi_{(\tau,T)}\widetilde{u}) - z_d\| = r = r(\tau, M(r,\tau)). \tag{5.11}$$

Finally, by taking a control $v \in \mathcal{U}_{\tau,M(r,\tau)-\delta}$, we see that

$$\frac{M(r,\tau_n)}{M(r,\tau)-\delta}\chi_{(\tau,T)}v \in \mathcal{U}_{\tau_n,M(r,\tau_n)}.$$

This, along with (5.5), yields that

$$\int_{\tau_n}^T \langle \chi_\omega p_n(t), u_n(t)\rangle \,\mathrm{d}t \ge \int_\tau^T \left\langle \chi_\omega p_n(t), \frac{M(r,\tau_n)}{M(r,\tau)-\delta} v(t)\right\rangle \mathrm{d}t.$$

By (5.4), (5.8) and (5.10), we can pass to the limit in the above inequality to get that

$$\int_\tau^T \langle \chi_\omega \widetilde{p}(t), \widetilde{u}(t)\rangle \,\mathrm{d}t \ge \int_\tau^T \langle \chi_\omega \widetilde{p}(t), v(t)\rangle \,\mathrm{d}t \;\text{ for all }\; v \in \mathcal{U}_{\tau,M(r,\tau)-\delta},$$

which indicates that

$$\int_\tau^T \langle \chi_\omega \widetilde{p}(t), \widetilde{u}(t)\rangle \,\mathrm{d}t = \max_{v\in\mathcal{U}_{\tau,M(r,\tau)-\delta}} \int_\tau^T \langle \chi_\omega \widetilde{p}(t), v(t)\rangle \,\mathrm{d}t.$$

According to Lemma 2.3 again, the above equality shows that $\chi_{(\tau,T)}\widetilde{u}$ is an optimal control to $(OP)^{\tau,M(r,\tau)-\delta}$. Thus, we have that

$$\|y(T;y_0,\chi_{(\tau,T)}\widetilde{u}) - z_d\| = r(\tau, M(r,\tau)-\delta).$$

This, together with (5.11), implies that

$$r(\tau, M(r,\tau)) = r(\tau, M(r,\tau)-\delta).$$

It follows from the latter equality and Lemma 4.1 that

$$M(r,\tau) = M(r,\tau) - \delta,$$

which leads to a contradiction. Hence, (5.3) holds.

Step 3. The function $M(r,\cdot)$ is right-continuous.

Take a monotone decreasing sequence $\{\tau_n\}_{n\ge1} \subset (\tau,T)$ so that $\lim_{n\to+\infty}\tau_n = \tau$. It suffices to show that

$$\lim_{n\to+\infty} M(r,\tau_n) = M(r,\tau).$$

If this does not hold, then by the monotonicity of $M(r,\cdot)$, we get that

$$\lim_{n\to+\infty} M(r,\tau_n)=M(r,\tau)+\delta \ \text{ for some } \ \delta>0.$$

By similar arguments as those in Step 2, we can obtain that

$$r(\tau,M(r,\tau))=r(\tau,M(r,\tau)+\delta).$$

This contradicts the strict monotonicity of $r(\tau,\cdot)$.

Step 4. $\lim\limits_{\tau\to T} M(r,\tau)=+\infty$.

By contradiction, we suppose that $0<\tau_1<\tau_2<\cdots<\tau_n\to T$ and $\lim\limits_{n\to+\infty} M(r,\tau_n)=M_0<+\infty$. Let u_n be the optimal control to $(NP)^{r,\tau_n}$. Then

$$\chi_{(\tau_n,T)}u_n\to 0 \ \text{ weakly star in } \ L^\infty(0,T;L^2(\Omega)).$$

According to Lemma 2.1, there is a subsequence of $\{n\}_{n\geq1}$, still denoted in the same manner, so that

$$y(\cdot;y_0,\chi_{(\tau_n,T)}u_n)\to y(\cdot;y_0,0) \ \text{ in } \ C([0,T];L^2(\Omega)). \tag{5.12}$$

It follows from (5.12) that

$$r_T(y_0)=\|y(T;y_0,0)-z_d\|=\lim_{n\to\infty}\|y(T;y_0,\chi_{(\tau_n,T)}u_n)-z_d\|\leq r,$$

which contradicts the fact that $r<r_T(y_0)$.

Step 5. Proofs of (5.1) and (5.2).

Arbitrarily fix $M\geq M(r,0)$. According to Lemma 3.3, $(TP)^{M,r}$ has an optimal control u^*. Thus,

$$u^*\in L^\infty(0,T;L^2(\Omega)),\ \|u^*\|_{L^\infty(\tau(M,r),T;L^2(\Omega))}\leq M$$

and

$$y(T;y_0,\chi_{(\tau(M,r),T)}u^*)\in B(z_d,r).$$

These, along with the definition of $M(r,\tau(M,r))$, imply that

$$M\geq M(r,\tau(M,r)).$$

We now prove that $M=M(r,\tau(M,r))$. By contradiction, we suppose that $M>M(r,\tau(M,r))$. By Steps 1-4, there exists a $\tau_0\in(\tau(M,r),T)$ so that $M(r,\tau_0)=M$. Clearly, the optimal control u_0 to $(NP)^{r,\tau_0}$ satisfies that $u_0\in L^\infty(0,T;L^2(\Omega))$, $\|u_0\|_{L^\infty(\tau_0,T;L^2(\Omega))}=M(r,\tau_0)=M$ and $y(T;y_0,\chi_{(\tau_0,T)}u_0)\in B(z_d,r)$. From the latter and the definition of $\tau(M,r)$, it follows that $\tau(M,r)\geq\tau_0$, which contradicts $\tau_0\in(\tau(M,r),T)$. Hence, $M=M(r,\tau(M,r))$.

Furthermore, it follows from Step 1 and $\tau\in[0,T)$ that $M(r,\tau)\geq M(r,0)$. By (5.1), we deduce that

$$M(r,\tau)=M(r,\tau(M(r,\tau),r)).$$

Using Step 1 again, we obtain $\tau = \tau(M(r,\tau), r)$.

In summary, we finish the proof of Lemma 5.1. $\square$

Theorem 5.2. *Suppose that* **(H)** *holds.*

(i) *Given $r \in (0, r_T(y_0))$ and $M \geq M(r, 0)$, any optimal control to $(TP)^{M,r}$ is the optimal control to $(NP)^{r,\tau(M,r)}$.*
(ii) *Given $\tau \in [0, T)$ and $r \in (0, r_T(y_0))$, the optimal control to $(NP)^{r,\tau}$ is an optimal control to $(TP)^{M(r,\tau),r}$.*
(iii) *For each $r \in (0, r_T(y_0))$ and each $M \geq M(r, 0)$, $(TP)^{M,r}$ holds the bang-bang property (i.e., any optimal control u^* satisfies that $\|u^*(t)\| = M$ for a.e. $t \in (\tau(M, r), T)$) and has a unique optimal control.*

Proof. (i) Let u be an arbitrary optimal control to $(TP)^{M,r}$. Then

$$u(t) = 0 \text{ for a.e. } t \in (0, \tau(M, r)), \ \|u\|_{L^\infty(\tau(M,r),T;L^2(\Omega))} \leq M$$

and

$$y(T; y_0, \chi_{(\tau(M,r),T)}u) \in B(z_d, r).$$

These, along with (5.1) and Theorem 4.2, imply that u is the optimal control to $(NP)^{r,\tau(M,r)}$.

(ii) According to Theorem 4.2, the optimal control u to $(NP)^{r,\tau}$ satisfies that

$$u(t) = 0 \text{ for a.e. } t \in (0, \tau), \ \|u\|_{L^\infty(\tau,T;L^2(\Omega))} = M(r, \tau)$$

and

$$y(T; y_0, \chi_{(\tau,T)}u) \in B(z_d, r).$$

These, together with (5.2), yield that u is an optimal control to $(TP)^{M(r,\tau),r}$.

(iii) The bang-bang property and the uniqueness of optimal control follow from (i), Theorem 4.2 and (5.1). $\square$

6. Proof of Theorem 1.2

We start this section by establishing the relations of two functions $r(\cdot, M)$ and $\tau(M, \cdot)$ for each $M > 0$.

Lemma 6.1. *Suppose that* **(H)** *holds. Let $M > 0$. Then*

$$r = r(\tau(M, r), M) \ \text{for each} \ r \in [r(0, M), r_T(y_0)) \tag{6.1}$$

and

$$\tau = \tau(M, r(\tau, M)) \ \text{for each} \ \tau \in [0, T). \tag{6.2}$$

Proof. Let

$$\mathcal{B}_1 := \{(M, r) : r \in (0, r_T(y_0)), M \geq M(r, 0)\}$$

and

$$\mathcal{B}_2 := \{(M, r) : M > 0, r \in [r(0, M), r_T(y_0))\}.$$

Firstly, we claim that

$$\mathcal{B}_1 = \mathcal{B}_2. \tag{6.3}$$

To this end, on one hand, we arbitrarily fix $(M, r) \in \mathcal{B}_1$. Since $r \in (0, r_T(y_0))$, it follows from Lemma 3.2 that $M \geq M(r, 0) > 0$. Moreover, by Lemma 4.1, we get that

$$r(0, M) \leq r(0, M(r, 0)) = r,$$

which indicates that $(M, r) \in \mathcal{B}_2$. On the other hand, we arbitrarily take $(M, r) \in \mathcal{B}_2$. By Lemma 2.2, we have that $r(0, M) > 0$. Since $r \geq r(0, M)$, it follows from Lemma 4.1 that

$$r(0, M) \leq r = r(0, M(r, 0)) \text{ and } M(r, 0) \leq M.$$

These imply $(M, r) \in \mathcal{B}_1$. Hence, (6.3) holds.

Next, let $M > 0$ be fixed. It follows from (6.3), (5.1) and (4.1) that

$$r(\tau(M, r), M) = r(\tau(M, r), M(r, \tau(M, r))) = r \text{ for each } r \in [r(0, M), r_T(y_0)).$$

Meanwhile, by (4.2) and (5.2), we obtain that

$$\tau(M, r(\tau, M)) = \tau(M(r(\tau, M), \tau), r(\tau, M)) = \tau \text{ for each } \tau \in [0, T).$$

This completes the proof. □

Finally, we present the proof of Theorem 1.2.

Proof of Theorem 1.2. Let (P_1) and (P_2) be two optimal control problems. By $(P_1) \Rightarrow (P_2)$, we mean that the optimal control to (P_1) is the optimal control to (P_2). The proof will be carried out by three steps.

Step 1. $(OP)^{\tau,M} \Rightarrow (TP)^{M,r(\tau,M)} \Rightarrow (NP)^{r(\tau,M),\tau} \Rightarrow (OP)^{\tau,M}$ for each $M > 0$ and $\tau \in [0, T)$.

On one hand, by Theorem 4.2, Theorem 5.2 and (4.2), we see that

$$(OP)^{\tau,M} \Rightarrow (NP)^{r(\tau,M),\tau} \Rightarrow (TP)^{M(r(\tau,M),\tau),r(\tau,M)} = (TP)^{M,r(\tau,M)}. \tag{6.4}$$

On the other hand, by (6.3) and Theorem 5.2, we get that

$$(TP)^{M,r(\tau,M)} \Rightarrow (NP)^{r(\tau,M),\tau(M,r(\tau,M))}.$$

This, along with (6.2), yields that

$$(TP)^{M,r(\tau,M)} \Rightarrow (NP)^{r(\tau,M),\tau}. \tag{6.5}$$

Furthermore, it follows from Theorem 4.2 and (4.2) that

$$(NP)^{r(\tau,M),\tau} \Rightarrow (OP)^{\tau,M(r(\tau,M),\tau)} = (OP)^{\tau,M}. \tag{6.6}$$

Combining (6.4)–(6.6) leads to

$$(OP)^{\tau,M} \Rightarrow (TP)^{M,r(\tau,M)} \Rightarrow (NP)^{r(\tau,M),\tau} \Rightarrow (OP)^{\tau,M}.$$

Step 2. $(NP)^{r,\tau} \Rightarrow (OP)^{\tau,M(r,\tau)} \Rightarrow (TP)^{M(r,\tau),r} \Rightarrow (NP)^{r,\tau}$ for each $r \in (0, r_T(y_0))$ and $\tau \in [0,T)$.

By Theorem 4.2, Step 1 and (4.1), one can get that

$$(NP)^{r,\tau} \Rightarrow (OP)^{\tau,M(r,\tau)} \Rightarrow (TP)^{M(r,\tau),r(\tau,M(r,\tau))} = (TP)^{M(r,\tau),r}. \tag{6.7}$$

Moreover, it follows from Theorem 5.2 and (5.2) that

$$(TP)^{M(r,\tau),r} \Rightarrow (NP)^{r,\tau(M(r,\tau),r)} = (NP)^{r,\tau}.$$

This, together with (6.7), implies that

$$(NP)^{r,\tau} \Rightarrow (OP)^{\tau,M(r,\tau)} \Rightarrow (TP)^{M(r,\tau),r} \Rightarrow (NP)^{r,\tau}.$$

Step 3. $(TP)^{M,r} \Rightarrow (NP)^{r,\tau(M,r)} \Rightarrow (OP)^{\tau(M,r),M} \Rightarrow (TP)^{M,r}$ for each $M > 0$ and $r \in [r(0,M), r_T(y_0))$.

By (6.3), Theorem 5.2, Theorem 4.2 and (5.1), we obtain that

$$(TP)^{M,r} \Rightarrow (NP)^{r,\tau(M,r)} \Rightarrow (OP)^{\tau(M,r),M(r,\tau(M,r))} = (OP)^{\tau(M,r),M}. \tag{6.8}$$

It follows from Step 1 and (6.1) that

$$(OP)^{\tau(M,r),M} \Rightarrow (TP)^{M,r(\tau(M,r),M)} = (TP)^{M,r}.$$

This, along with (6.8), yields that

$$(TP)^{M,r} \Rightarrow (NP)^{r,\tau(M,r)} \Rightarrow (OP)^{\tau(M,r),M} \Rightarrow (TP)^{M,r}.$$

In summary, we finish the proof of Theorem 1.2. □

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