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SUFFICIENT CONDITIONS FOR THE CONTROLLABILITY OF WAVE EQUATIONS WITH A TRANSMISSION CONDITION AT THE INTERFACE

Ludovick Gagnon*

Abstract. We consider waves travelling in two different mediums each endowed with a different constant speed of propagation. At the interface between the two mediums, the refraction of the rays of the optic geometry obeys the Snell's law. We provide sufficient conditions on the geometry of the mediums and on the speed of propagation for the exact boundary controllability.

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1. Introduction

1.1. Statement of the problem

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and strictly convex domain. Let $\Omega_2 \subset \Omega$ be an open, bounded, strictly convex domain strictly included in Ω, meaning there exists $\Omega' \subset \Omega$ open and simply connected such that $\overline{\Omega}_2 \subset \Omega'$ and $\overline{\Omega'} \subset \Omega$. We define $\Omega_1 := \Omega \setminus \overline{\Omega}_2$. In this setting, the boundary of Ω_2 is $\partial\Omega_2$ and the boundary of Ω_1 is $\partial\Omega_1 = \partial\Omega \cup \partial\Omega_2$. We assume the boundary $\partial\Omega_2$ and $\partial\Omega$ to be of class $C^k, k \geq 3$ and with no contact of order $k-1$ with its tangents. The outward unit normal of $x \in \partial\Omega_2$ is denoted $n_2(x)$ and the outward unit normal of $x \in \partial\Omega$ is denoted $n(x)$. With these notations, the outward unit normal of Ω_1 is $-n_2(x)$ if $x \in \partial\Omega_1 \cap \partial\Omega_2$ and $n(x)$ if $x \in \partial\Omega_1 \cap \partial\Omega$. To ease the notations, we identify $\partial\Omega_1 \cap \partial\Omega$ with $\partial\Omega$ and $\partial\Omega_1 \cap \partial\Omega_2$ with $\partial\Omega_2$.

For $T > 0$, we consider the set of wave equations,

$$\begin{cases} (\partial_t^2 - c_i^2\Delta)u^i(t,x) = 0, & (t,x) \in (0,T) \times \Omega_i, \\ u^i(0,x) = u_0^i(x), u_t^i(0,x) = u_1^i(x), & x \in \Omega_i, \end{cases} \tag{1.1}$$

where $c_i > 0, i = 1,2$ and $(\mathbb{1}_{\Omega_1}u_0^1 + \mathbb{1}_{\Omega_2}u_0^2, \mathbb{1}_{\Omega_1}u_1^1 + \mathbb{1}_{\Omega_2}u_1^2) \in H_0^1(\Omega) \times L^2(\Omega)$. At the interface $\partial\Omega_2$, a transmission condition is imposed,

$$\begin{cases} u^1(t,x) = u^2(t,x), & (t,x) \in (0,T) \times \partial\Omega_2, \\ c_1^2\partial_{n_2}u^1(t,x) = c_2^2\partial_{n_2}u^2(t,x), & (t,x) \in (0,T) \times \partial\Omega_2, \end{cases} \tag{1.2}$$

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Université de Lorraine, CNRS, Inria équipe SPHINX, F-54000 Nancy, France.

* Corresponding author: ludovick.gagnon@inria.fr

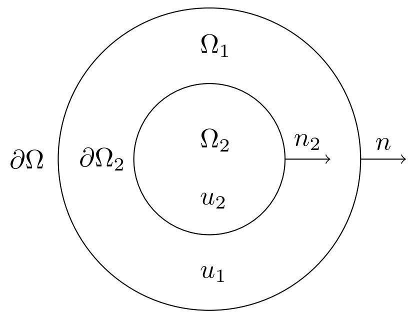

FIGURE 1. Representation of the spatial domain for (1.1)–(1.3) with $\Omega_2 = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$, $\Omega_1 = \{x \in \mathbb{R}^2 \mid 1 < \|x\| < 2\}$.

and a Dirichlet boundary condition is imposed on the exterior boundary $\partial\Omega$,

$$u^1(t,x) = 0, \qquad (t,x) \in (0,T) \times \partial\Omega. \tag{1.3}$$

Figure 1 is an example of the spatial configuration where the solutions to (1.1)–(1.3) are defined.

A straightforward computation shows that the energy of solutions to (1.1)–(1.3) is preserved along time,

$$E(u^1(t,.), u^2(t,.)) := \sum_{i=1}^{2} \int_{\Omega_i} \left|\partial_t u^i(t,x)\right|^2 + c_i^2 \left|\nabla u^i(t,x)\right|^2 \,\mathrm{dx} = E(u^1(0,.), u^2(0,.)).$$

We are interested, in this article, to establish sufficient geometric conditions for the observability of solutions to (1.1)–(1.3).

Definition 1.1. Let $\Gamma \subset \partial\Omega$. We say that (1.1)–(1.3) is observable in time $T > 0$ with respect to Γ if and only if there exists $c_T > 0$ such that for any initial data $(\mathbb{1}_{\Omega_1} u_0^1 + \mathbb{1}_{\Omega_2} u_0^2, \mathbb{1}_{\Omega_1} u_1^1 + \mathbb{1}_{\Omega_2} u_1^2) \in H_0^1(\Omega) \times L^2(\Omega)$, the solutions to (1.1)–(1.3) satisfy,

$$E(u^1(0,.), u^2(0,.)) \leq c_T \int_0^T \int_\Gamma \left|\partial_n u^1(t,x)\right|^2 \,\mathrm{d}\sigma\mathrm{dt}. \tag{1.4}$$

The observability of (1.1)–(1.3) finds applications in optical fibers and guided waves [14] and also in seismic prospection of Earth inner layers [15]. Moreover, it is well-known that the observability of (1.1)–(1.3) is equivalent ([11]) to the exact controllability of,

$$\begin{cases} (\partial_t^2 - c_i^2 \Delta) y^i(t,x) = 0, & (t,x) \in (0,T) \times \Omega_i, \\ y^1(t,x) = y^2(t,x), & (t,x) \in (0,T) \times \partial\Omega_2, \\ c_1^2 \partial_{n_2} y^1(t,x) = c_2^2 \partial_{n_2} y^2(t,x), & (t,x) \in (0,T) \times \partial\Omega_2, \\ y^1(t,x) = 0, & (t,x) \in (0,T) \times \partial\Omega \setminus \Gamma, \\ y^1(t,x) = v(t,x), & (t,x) \in (0,T) \times \Gamma, \\ y^i(0,x) = y_0^i(x), y_t^i(0,x) = y_1^i(x), & x \in \Omega_i, \end{cases} \tag{1.5}$$

with $(\mathbb{1}_{\Omega_1} y_0^1 + \mathbb{1}_{\Omega_2} y_0^2, \mathbb{1}_{\Omega_1} y_1^1 + \mathbb{1}_{\Omega_2} y_1^2) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $v \in L^2((0,T) \times \Gamma)$.

Definition 1.2. Let $\Gamma \subset \partial\Omega$. We say that (1.5) is exactly controllable in time $T > 0$ if and only if for any initial data $(\mathbb{1}_{\Omega_1} y_0^1 + \mathbb{1}_{\Omega_2} y_0^2, \mathbb{1}_{\Omega_1} y_1^1 + \mathbb{1}_{\Omega_2} y_1^2) \in L^2(\Omega) \times H^{-1}(\Omega)$ and final data $(\mathbb{1}_{\Omega_1} y_{0,T}^1 + \mathbb{1}_{\Omega_2} y_{0,T}^2, \mathbb{1}_{\Omega_1} y_{1,T}^1 + \mathbb{1}_{\Omega_2} y_{1,T}^2) \in$

$L^2(\Omega) \times H^{-1}(\Omega)$, there exists a control $v \in L^2((0,T)\times\Gamma)$ that drives the solution to (1.1)–(1.3) to,

$$(y^i(T,.), y^i_t(T,.)) = (y^i_{0,T}, y^i_{1,T}), \quad \text{in } L^2(\Omega_i) \times H^{-1}(\Omega_i),$$

for $i = 1, 2$.

1.2. The problematic

Let us present, formally, the main difficulty in the analysis of the observability of (1.1)–(1.3). In order to do so, we begin by recalling the main ideas behind the proof of the observability for the classical wave equation with the microlocal analysis approach. Consider the wave equation defined over Ω and $T > 0$,

$$\begin{cases} (\partial_{tt} - c^2\Delta)u(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.6}$$

where $c > 0$ and $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$. We recall that the energy of (1.6),

$$E(u(t,.)) := \int_\Omega |\partial_t u(t,x)|^2 + c^2 |\nabla u(t,x)|^2 \, \mathrm{dx},$$

is preserved along time. The observability of (1.6) amounts to prove, as for (1.1)–(1.3), the observability inequality,

$$E(u)(0) \le c_T \int_0^T \int_\Gamma |\partial_n u(t,x)|^2 \, \mathrm{d}\sigma \mathrm{d}t, \tag{1.7}$$

for $c_T > 0$ for any $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$. The proof of the observability inequality (1.7), using the microlocal analysis approach, consists to contradict the observability inequality (1.7) and to extract a subsequence of initial data $\{(u^n_0, u^n_1)\}$ weakly converging in the state space $H^1_0(\Omega) \times L^2(\Omega)$ to a non-null initial data. The contradiction is obtained if one can show that the subsequence converges strongly to zero in the state space. The standard procedure is to show the following on the sequence of solutions u_n associated to the initial data $\{(u^n_0, u^n_1)\}$:

1. The contradiction of the observability inequality (1.7) implies strong convergence to 0 of the solutions u_n over $\Gamma \times (0,T)$;
2. This strong convergence is propagated forward and backward in time along the generalized bicharacteristics of (1.6), thanks to the time-reversibility of the equation.

The generalized bicharacteristics of (1.6) correspond to the ray of the optic geometry or, equivalently, to the trajectory of billards. They propagate at constant speed c in straight line inside Ω and are reflected at the boundary according to the law "angle of incidence equal to the angle of refraction". Here and below, the incoming ray (before being reflected at the boundary) is denoted γ^-, and the outgoing ray γ^+. Using the two previous assertions, the contradiction is obtained if the *Geometric Control Condition* [1] (GCC) is satisfied, that is, if every generalized bicharacteristics of (1.6) intersect[1] Γ in time $T > 0$. Indeed, under GCC, the strong convergence is propagated everywhere in Ω, for any initial data, which implies that the subsequence must converge strongly to zero. Otherwise, if Γ does not satisfy GCC in time $T > 0$, then there exists at least one ray that never intersect Γ in time $T > 0$ on which the subsequence can concentrate. Such subsequence therefore violates the observability inequality (1.7). For simplicity, for the rest of the article, we shall refer to an observed

[1] Since Ω is convex, one does not need to assume non-diffractive intersection with Γ, see Section 2.1 for the definition.

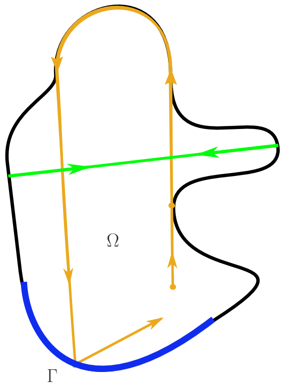

FIGURE 2. Representation of the propagation of rays for (1.6). The blue region is the observability region $\Gamma \subset \partial\Omega$. The ray in yellow is a ray observed by the observability region. The ray in green is a trapped ray for any time $T > 0$ for (Ω, Γ).

ray, a ray that intersects Γ non-diffractively. Figure 2 illustrates the propagation of rays in a domain Ω for which there exists a trapped ray for the chosen observability region Γ.

The strategy to prove the observability of (1.1)–(1.3) relies on a similar approach than for the classical wave equation: one argues by contradiction on the observability inequality (1.4) and extracts a subsequence of initial data. The observability region still provide strong convergence over $\Gamma \times (0, T)$ and one would want to propagate this strong convergence along the bicharacteristics to obtain a contradiction. The propagation of the bicharacteristics inside Ω_1 and Ω_2, as well as at the boundary $\partial\Omega$, is the same as for the classical wave equation (1.6). At the interface $\partial\Omega_2$, however, the propagation is more complex since the bicharacteristics obey the Snell's law. Indeed, assume an incoming ray γ_1^- from Ω_1 intersects $\partial\Omega_2$ at an angle θ_1 (the angle of the ray and the outward normal n_2). As for the classical wave equation, the ray is reflected at the same angle to an outgoing ray γ_1^+ in Ω_1. Moreover, if the Snell's law,

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}, \tag{1.8}$$

is not vacuous for the values of c_1, c_2 and θ_1, then a ray γ_2^+ is transmitted to Ω_2 with an angle θ_2 (with respect to the inward normal $-n_2$). If (1.8) is vacuous, then Ω_2 acts as an obstacle for the ray γ_1^- and only a reflection occurs. The same description holds for rays incoming from Ω_2. We stress here on one important aspect of the geometrical assumption made in the introduction: the strict convexity of Ω_2 and the hypothesis $c_2 > c_1$ result in (1.8) never being vacuous for the incoming rays from Ω_2, meaning that every rays from Ω_2 are reflected and transmitted on $\partial\Omega_2$. Figure 3 gives an example of the propagation of a bicharacteristic of (1.1)–(1.3).

Unlike the classical wave equation (1.6), the propagation of the strong convergence along the bicharacteristics is not straightforward due to the possible interference between bicharacteristics at the interface. Indeed, denote γ_1^+ a bicharacteristic intersecting Γ at time t and suppose it had intersected $\partial\Omega_2$ at a previous time $t' < t$. Denote θ_1 the angle of reflection of γ_1^+ on $\partial\Omega_2$. The bicharacteristic γ_1^+ may be the outgoing bicharacteristic from a reflection of an incoming bicharacteristic γ_1^- intersecting $\partial\Omega$ at time t' and angle θ_1. It may also be the transmitted bicharacteristic from γ_2^-, a bicharacteristic from Ω_2 intersecting $\partial\Omega_2$ at time t' with an angle θ_2, assuming the Snell's law (1.8) is not vacuous for θ_1 (if it is vacuous, then no interference is possible and γ_1^- is observed). Therefore, by the superposition principle, γ_1^+ may be obtained simultaneously as a reflection from Ω_1 and a transmission from Ω_2 at time t' (see Fig. 4). Interference between γ_1^- and γ_2^- may then prevent one to conclude on the observability of γ_1^- or γ_2^- from the observability of γ_1^+ at Γ alone.

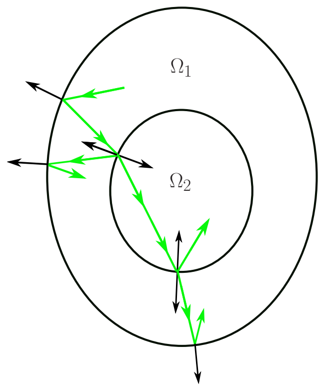

FIGURE 3. Example of the propagation of rays for (1.1)–(1.3).

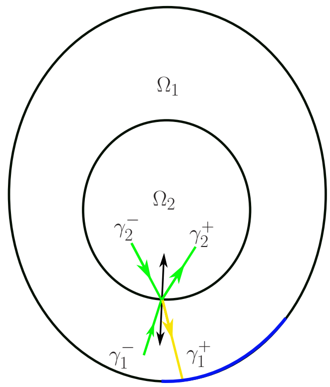

FIGURE 4. Example of a ray obtained as a superposition of two rays at $\partial\Omega_2$ for (1.1)–(1.3).

When four hyperbolic[2] rays $\gamma_1^\pm, \gamma_2^\pm$ on $\partial\Omega_2$ exist, we shall prove the following:

1. If two of the four rays $\gamma_1^\pm, \gamma_2^\pm$ are observed, then the other two are also observed (Cor. 2.3);
2. If only one of the four rays is observed, then no interference is possible. In other words, one may follow one of the three other rays to see if they are observed (Prop. 2.2).

One can then combine these two results recursively to conclude on the observability of a bicharacteristic. Such recursive use of Corollary 2.3 and Proposition 2.2 is stated as Corollary 2.6 (see Fig. 5 for an example of a recursive argument that will be used in this article for the hyperbolic rays). The observation of the gliding rays at the interface is more difficult to analyse, and a different approach is needed. We refer to Corollary 2.5 for the sufficient conditions for the observability of such rays.

We underline that Proposition 2.2 and Corollary 2.3 were already known in the literature. Indeed, Miller described the propagation of the semi-classical defect measure across a sharp interface for the wave equation, as well as the propagation of the semi-classical defect measure for the linear Schrödinger with a potential jump [12]. The general description of the propagation of the defect measure for a system of wave equations was given by Burq and Lebeau in [4]. Our present contribution is to introduce a geometrical construction that yield sufficient conditions on the observability of (1.1)–(1.3). Before stating our main result, we review the state of the art

[2]See Section 2.1 for a precise definition of the hyperbolic, gliding and grazing rays.

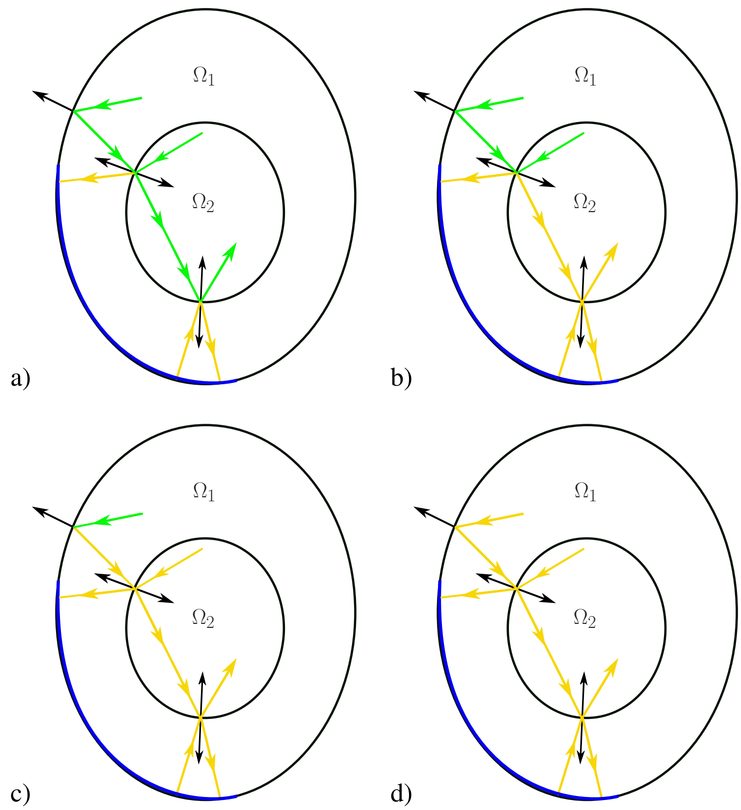

FIGURE 5. Recursive use of Corollary 2.3 and the propagation for the classical wave equation at the exterior boundary. Rays in yellow are observed and rays in green are not observed (yet).

concerning the observability of (1.1)–(1.3), introduce some notations and present the geometrical construction used in the proof of the main result.

1.3. State of the art

Few results are available in the literature regarding the observability problem of (1.1)–(1.3). To the best of our knowledge, it was first raised by Lions in Chapter VI of [11]. It was proven that if $c_2 > c_1$ and Ω_2 is star-shaped with respect to $x_0 \in \Omega_2$ (and without restriction on the convexity of Ω_2 or Ω), then the observability holds for $\Gamma = \Gamma(x_0)$ Chapter VI, Theorem 5.1 of [11] (see (1.9) for the definition of $\Gamma(x_0)$). The general case was left as an open problem. It was then noticed by Lebeau, Le Rousseau, Terpolilli and Trelat when $c_2 > c_1$ and $\Gamma = \partial\Omega$ that the observability holds using a microlocal argument [9]. This result is equivalent to Chapter VI, Theorem 5.1 of [11] under the hypothesis that Ω_2 and Ω are strictly convex. We also mention the work of Baudouin, Mercado and Osses [2] on global Carleman estimates for (1.1)–(1.3). The description of the propagation of the defect measure across the interface for the wave and Schrödinger equation was done in the work of Miller [12]. The exponential decay of the defect measure along time, by mean of gliding radiation in Ω_1, along a gliding ray for a strictly convex domain Ω_2 is in particular proved in [12].

Burq and Lebeau proved the propagation of defect measures for systems of wave equations and its application to exponential decay of the energy of solutions to the Lamé system [4] (see also [10]). The exact controllability of the Lamé system was obtained by Dehman and Raymond [6].

We finally cite the work of Miller on escape functions [13], and the hierarchy of different observability regions, from which the uniformly escaping geometry definition is very much inspired.

1.4. Notations

Let us fix some notations used throughout the article. We begin by recalling the definition of the $\Gamma(x_0)$ observability region for the wave equation ([11]). Let $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$. Then,

$$\Gamma(x_0) := \{x \in \partial\Omega \,|\, \langle (x - x_0), n(x)\rangle > 0\}, \tag{1.9}$$

where $\langle ., . \rangle$ denotes the $\mathbb{R}^2$ inner product. We recall that if Ω is strictly convex, then $\Gamma(x_0)$ is a connected part of $\partial\Omega$.

Consider the parametrisation $\delta(s), s \in [0, 1]$ of $\partial\Omega$ in the counter-clockwise direction. We fix $\delta(0) = \delta(1)$ to be such that $\delta(0) \in \Gamma(x_0)$. We also parametrise $\partial\Omega_2$ with $\delta_2(s), s \in [0, 1]$, in the counter-clockwise direction. The choice of the specific value of $\delta_2(0) = \delta_2(1)$ will be determined later on (see for instance step 2 in Sect. 1.5). In the geometrical construction, lines between two points $x, y \in \mathbb{R}^2$ will be denoted $l(x, y)$.

We recall now what is referred to the collision map $\mathcal{F}$ in the billiard literature ([5]) for Ω_1,

$$\begin{aligned} \mathcal{F} : (\partial\Omega \cup \partial\Omega_2) \times (\mathbb{R}^2 \setminus \{(0, 0)\}) &\longrightarrow (\partial\Omega \cup \partial\Omega_2) \times (\mathbb{R}^2 \setminus \{(0, 0)\}), \\ (x, \xi) &\longmapsto (x^1, \xi^1), \end{aligned}$$

where x^1 is the point of intersection with $\partial\Omega \cup \partial\Omega_2$ of the ray of Ω_1 starting from $x \in \partial\Omega \cup \partial\Omega_2$ and travelling in the $\xi \in \mathbb{R}^2 \setminus \{(0, 0)\}$ direction in straight line and where $\xi^1 \in \mathbb{R}^2 \setminus \{(0, 0)\}$ is the direction of the outgoing ray reflected according to the law of reflection of (1.6). Iterations of the collision map are denoted $\mathcal{F}^n, n \in \mathbb{Z}$, where $\mathcal{F}^{-n}$ for $n \in \mathbb{N}$ is nth pre-image of $\mathcal{F}$. For $(x, \xi) \in (\partial\Omega \cup \partial\Omega_2) \times (\mathbb{R}^2 \setminus \{(0, 0)\})$ we denote the projections $\Pi_x(x, \xi) = x$ and $\Pi_\xi(x, \xi) = \xi$. Notice that not all directions $\xi \in \mathbb{R}^2 \setminus \{(0, 0)\}$ are admissible directions for the domain of definition of $\mathcal{F}$ since half of those directions correspond to incoming rays, but it is customary to identify these directions to their unique outgoing direction [5]. We define in the same way $\mathcal{F}_2$, the collision map of Ω_2,

$$\mathcal{F}_2 : \partial\Omega_2 \times (\mathbb{R}^2 \setminus \{(0, 0)\}) \longrightarrow \partial\Omega_2 \times (\mathbb{R}^2 \setminus \{(0, 0)\}), \tag{1.10}$$

$$(x, \xi) \longmapsto (x^1, \xi^1), \tag{1.11}$$

At the boundary $\partial\Omega_2$, we identify $\mathcal{F}(x, \xi)$ to $\mathcal{F}_2(x, \xi)$ according to (1.8) in the following sense. Let $x \in \partial\Omega_2$ and consider the ray in Ω_2 starting from this point in the direction $\xi \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Then, from the strict convexity, it intersects $\partial\Omega_2$ non-diffractively at $\Pi_x(\mathcal{F}_2(x, \xi))$ with an angle,

$$\theta_2 = \arccos\left(\frac{\langle -n_2(\Pi_x(\mathcal{F}_2(x, \xi))), \Pi_\xi(\mathcal{F}_2(x, \xi))\rangle}{\|\Pi_\xi(\mathcal{F}_2(x, \xi))\|}\right).$$

Assuming $c_2 > c_1$, the Snell's law (1.8) is not vacuous and a ray is transmitted with an angle θ_1. The corresponding outgoing direction $\xi \in \mathbb{R}^2 \setminus \{(0, 0)\}$ is given by the formula,

$$\cos\theta_1 = \left\langle n_2(\Pi_x(\mathcal{F}_2(x, \xi))), \frac{\xi}{\|\xi\|} \right\rangle,$$

with the sign condition $\langle \xi, \Pi_\xi(\mathcal{F}_2(x, \xi))\rangle \geq 0$ so it corresponds to the outgoing ray.

1.5. Strategy of the proof and statement of the main results

We are now able to introduce formally the geometrical construction used in this paper to provide sufficient conditions for the observability (1.1)–(1.3). The main idea of the geometrical construction is to focus at each steps on specific parts of the domain Ω. More precisely, our attention will revolve around the boundary $\partial\Omega$ and $\partial\Omega_2$, as the support of the defect measure is transported along the bicharacteristics inside Ω_1 and Ω_2. Hence, different parts of the boundary of $\partial\Omega$ and $\partial\Omega_2$ will be considered at the time. Using an iterative argument, we will be able to conclude on the observation of the region under consideration with the help of the observability of the regions previously considered. At the end of the iterative procedure, there will be a remaining region, which cannot be treated with the present geometrical construction. The uniformly escaping geometry condition will be introduced so that all the rays from that region escape uniformly in order to ensure the observability of this remaining region.

Below is the description of the geometrical construction, with the main ideas of the construction as well as the intuition behind the propagation of the defect measure at the interface.

Step 1: Localisation of Ω_2 with respect to $\Gamma(x_0)$ and definition of Γ_1^0
First, we localise Ω_2 with respect to $\Gamma(x_0)$. To this end, define,

$$L(y) := \max_{x\in\partial\Omega_2} \langle y - x_0, x - y\rangle , \quad y \in \partial\Omega.$$

Now let $0 < s_1 < s_2 < 1$ be such that $\{\delta(s_1), \delta(s_2)\} = \partial\Gamma(x_0)$, where $\partial\Gamma(x_0)$ is the boundary of $\Gamma(x_0)$ understood as a set of $\partial\Omega$. The sign of $L(\delta(s_i)), i = 1, 2$ reveals the dynamic of the rays propagating toward $\Gamma(x_0)$ with respect to Ω_2 (see Fig 6). Indeed, notice that if $L(\delta(s_i)) = 0$ for $i = 1$ or $i = 2$, then the ray starting from $\delta(s_i)$ in the $-n(\delta(s_i))$ direction is grazing the domain Ω_2. Therefore, if $L(\delta(s_i)) > 0$ (resp. $L(\delta(s_i)) < 0$) for $i = 1$ or $i = 2$, then the ray starting from $\delta(s_i)$ in the $-n(\delta(s_i))$ direction encounters Ω_2 non-diffractively (resp. avoids Ω_2). In the case $L(\delta(s_i)) \geq 0$, the dynamic from $\delta(s)$ for $|s - s_i|, i = 1, 2$ small is delicate as both-half rays may encounter the interface. This region is therefore studied in the next steps (and we define $\Gamma_1^0 = \Gamma(x_0)$). Otherwise, we can extend the boundary of $\Gamma(x_0)$ by defining Γ_1^0 such that $\partial\Gamma_1^0 = \{\delta(s_1^{1,0}), \delta(s_2^{1,0})\}$ with $s_1 \leq s_1^{1,0} < s_2^{1,0} \leq s_2$ and such that $L(\delta(s)) \leq 0$ for $s \in [s_1, s_1^{1,0}]$ and $s \in [s_2^{1,0}, s_2]$ (see Fig. 7). We shall prove that such an extension satisfy, by definition of $\Gamma(x_0)$, the following property: every ray from $x \in \Gamma_1^0 \setminus \Gamma(x_0)$ and travelling in the $-n(x)$ direction never intersect Ω_2 non-diffractively and has a finite number of reflection over $\Gamma_1^0 \setminus \Gamma(x_0)$ before hitting $\Gamma(x_0)$. This latter assertion comes from the definition of $\Gamma(x_0)$, where the dynamic in the strictly convex region $\partial\Omega \setminus \Gamma(x_0)$ may be approximated by the dynamic of the convex region defined by the cone made by the line $l(x_0, \delta(s_i)), i = 1, 2$ (recall $\{\delta(s_1), \delta(s_2)\} = \partial\Gamma(x_0)$). We are able in fact to prove more: every ray from $\Gamma_1^0 \setminus \Gamma(x_0)$ is observed (see Lem. 3.1). Indeed, these rays are made of two-half ray, one of which being initially located above $-n$ (in the sense that the distance between the ray and x_0 increases locally in time), and forced to propagate toward $\Gamma(x_0)$. Hence the observability of these rays come from the classical propagation of the defect measure for the wave equation, as one half-ray does not intersect non-diffractively Ω_2 before intersecting $\Gamma(x_0)$.

Step 2: Definition of Γ_2^0
We create the boundary $\{\delta_2(s_1^{2,0}), \delta_2(s_2^{2,0})\} = \partial\Gamma_2^0$, with $s_1^{2,0}, s_2^{2,0} \in [0, 1]$, of $\Gamma_2^0 \subset \partial\Omega_2$ according to the previous step. Recall that $s_i, i = 1, 2$ are such that $\{\delta(s_1), \delta(s_2)\} = \partial\Gamma(x_0)$. If $L(\delta(s_i)) \geq 0$, $i = 1$ or $i = 2$, then the corresponding boundary of Γ_2^0 is defined by $s_i^{2,0}$ such that,

$$\langle \delta(s_i) - x_0, \delta_2(s_i^{2,0}) - \delta(s_i)\rangle = \max_{x\in\partial\Omega_2} \langle \delta(s_i) - x_0, x - \delta(s_i)\rangle = L(\delta(s_i)).$$

From the strict convexity of Ω_2, the maximum $\tilde{x} \in \partial\Omega_2$ of $\langle \delta(s_i) - x_0, x - \delta(s_i)\rangle, x \in \partial\Omega_2$ is unique and therefore there is only one corresponding $s_i^{2,0}$ such that $\delta_2(s_i^{2,0}) = \tilde{x}_i$ (up to reparametrisation if $s_i^{2,0} = 0$ or $s_i^{2,0} = 1$). Geometrically, the line between $\delta(s_i)$ and $\delta_2(s_i^{2,0})$ corresponds to a tangential contact with Ω_2 (see Fig. 8, on the left). If $L(\delta(s_i)) < 0$, $i = 1$ or $i = 2$, then $\Gamma(x_0)$ was extended to Γ_1^0 and the corresponding extended boundary

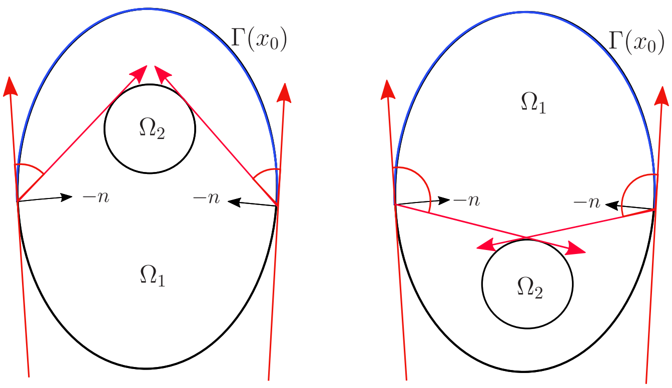

FIGURE 6. *Left*: case where Ω_2 and $\Gamma(x_0)$ are such that $L(\delta(s_i)) \geq 0$ for $i = 1, 2$. *Right*: case where Ω_2 and $\Gamma(x_0)$ are such that $L(\delta(s_i)) < 0$ for $i = 1, 2$.

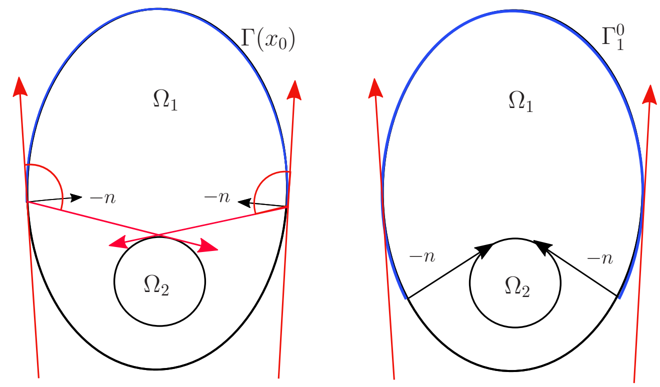

FIGURE 7. *Left*: Ω_2 and $\Gamma(x_0)$ are such that $L(\delta(s_i)) < 0$ for $i = 1, 2$. *Right*: extension of $\Gamma(x_0)$ to Γ_1^0.

$\delta(s_i^{1,0})$ satisfies $L(\delta(s_i^{1,0})) = 0$. Hence, define $s_i^{2,0}$ such that,

$$\langle \delta(s_i^{1,0}) - x_0, \delta_2(s_i^{2,0}) - \delta(s_i^{1,0}) \rangle = \max_{x \in \partial\Omega_2} \langle \delta(s_i^{1,0}) - x_0, x - \delta(s_i^{1,0}) \rangle = L(\delta(s_i^{1,0})) = 0.$$

Again, the uniqueness of $s_i^{2,0}$, up to reparametrisation, comes from the strict convexity of Ω_2. Geometrically, the line between $\delta_2(s_i^{2,0})$ and $\delta(s_i^{1,0})$ is tangential to Ω_2 and is in the $-n(\delta(s_i^{1,0}))$ direction from $\delta(s_i^{1,0})$ (see Fig. 8, on the right).

With the definition of $s_i^{2,0}$, Γ_2^0 is defined as follow. First, up to a reparametrisation, we consider $\delta_2(s)$ such that $0 < s_1^{2,0} < s_2^{2,0} < 1$. Then, $\Gamma_2^0 = \{\delta_2(s) \,|\, 0 \leq s < s_1^{2,0} \text{ or } s_2^{2,0} < s \leq 1\}$.

The definition of Γ_2^0 allows us to track the dynamic of the rays near the interface. First, consider a point x of the interface where there exists four half-rays according to Snell's law (1.8). Recall that observing only two is sufficient to conclude on the observability of all four of them. Notice that, at the boundary of $\partial\Gamma_2^0$, the tangential contact and the strict convexity of Ω_2 force every two half-rays of Ω_1 starting from $\delta_2(s_i^{2,0})$ to intersect $\Gamma_1^0 \cup \partial\Gamma_1^0$. Now, it suffices to remark that every two-half rays of Ω_1 starting from $x \in \Gamma_2^0$ are also forced to propagate toward Γ_1^0 (see Lem. 3.3).

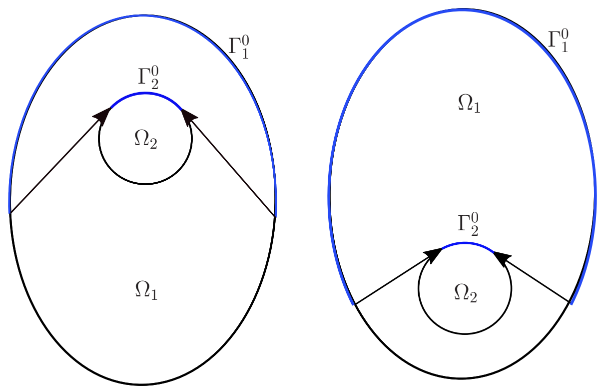

FIGURE 8. Creation of Γ_2. *Left*: case where Ω_2 and $\Gamma(x_0)$ are such that $L(\delta(s_i)) \geq 0$ for $i = 1, 2$. *Right*: case where Ω_2 and $\Gamma(x_0)$ are such that $L(\delta(s_i)) < 0$ for $i = 1, 2$.

Step 3: Extension of Γ_2^0 to Γ_2^1

We extend Γ_2^0 to Γ_2^1 in the following way. Draw the line between one point of $\partial\Gamma_2^0$, say $s_1^{2,0}$, and the opposite point of the boundary $\partial\Gamma_1^0$, $s_2^{1,0}$ in this case. If this line has no intersection with $\partial\Omega_2$, then there is no extension of Γ_2^0 from this boundary. Otherwise, define $\widetilde{\Gamma}_2^1 \supset \Gamma_2^0$ bounded by these intersections. We define $\Gamma_2^1 \supset \Gamma_2^0$ every point $x \in \widetilde{\Gamma}_2^1$ such that the ray starting from x and travelling in the direction $n_2(x)$ intersects Γ_1^0 (see Fig. 9).

This geometrical construction is at the heart of the proof of the main result, as it allows to track precisely the rays propagating from the interface inside Ω_2. Indeed, notice first that by construction, the ray starting from $x \in \Gamma_2^1$ in the $n_2(x)$ direction intersect Γ_1^0. Then, it follows that one of the two-half rays propagating from $x \in \Gamma_2^1$ inside Ω_1 is forced to propagate, roughly speaking, between Γ_2^1 and the ray in the $n_2(x)$ direction. This half-ray is therefore observed by Γ_1^0. It remains to prove that one of the three other half-ray is also observed. We follow the half-ray of Ω_2 propagating in the same tangential direction as the observed half ray of Ω_1 and we follow its propagation in the region defined by the line $l(\delta_2(s_1^{2,0}), \delta(s_2^{1,0}))$ and Γ_2^1. From the Snell's law and from the geometrical construction, we are able to prove that if a ray propagating in this region exits without intersecting once again Γ_2^1, then the other half-ray in Ω_1 intersects Γ_1^0. These types of rays are therefore also observed. We are left with the propagation of the half-ray in this region that encounters Γ_2^1 again. But either the ray intersects directly Γ_2^0, for which we conclude its observability, or it intersects $\Gamma_2^1 \setminus \Gamma_2^0$, and the same argument is used once again. The key here is that there is only a finite number of such intersections, and therefore either the ray intersects Γ_2^0 after a finite intersection with $\Gamma_2^1 \setminus \Gamma_2^0$, or it escapes through $l(\delta_2(s_1^{2,0}), \delta(s_2^{1,0}))$, and therefore two-half rays of Ω_1 are observed. The proof of the propagation of the defect measure over the region Γ_2^1 is found in Lemma 3.4.

Step 4: Extension of Γ_1^0 to Γ_1^1

We extend the boundary $\{\delta(s_1^{1,0}), \delta(s_2^{1,0})\}$ of Γ_1^0 to $\{\delta(s_1^{1,1}), \delta(s_2^{1,1})\}$ with $0 < s_1^{1,0} \leq s_1^{1,1} < s_2^{1,1} \leq s_2^{1,0} < 1$ such that the region $\Gamma_1^1 := \{\delta(s) \,|\, s \in [0, s_1^{1,1}) \cup (s_2^{1,1}, 1]\}$ has the property that every ray starting from $x \in \Gamma_1^1 \setminus \Gamma_1^0$ in the $-n$ direction intersect the boundary of Γ_2^1 (see Fig. 10).

The construction here is made so that one half-ray coming from Γ_1^1 either intersects Γ_1^0 or Γ_2^1, which are observable regions (see Lem. 3.5), or $\Gamma_1^1 \setminus \Gamma_0^1$ once again. We conclude in the latter case that the ray intersects Γ_1^0 or Γ_2^1 after a finite number of intersection with $\Gamma_1^1 \setminus \Gamma_0^1$ due to the fact that the propagation of rays in $\partial\Omega \setminus \Gamma(x_0)$ is approximated by the cone defined by the lines $l(\delta(s_i), x_0)), i = 1, 2$ with $\delta(s_i) \in \partial\Gamma(x_0)$.

Step 5: Iteration of step 3 and 4

We iterate step 3 and 4 to extend Γ_i^{n-1} to Γ_i^n for $i = 1, 2$ and $n \in \mathbb{N}$ until $\Gamma_2^{n-1} = \Gamma_2^n$. We prove in Lemma 3.6 that it happens in a finite number of iterations of step 3 and 4. More precisely, it happens either because there are no longer intersection of the lines defined in step 3 with the boundary of Ω_2, or because the rays starting

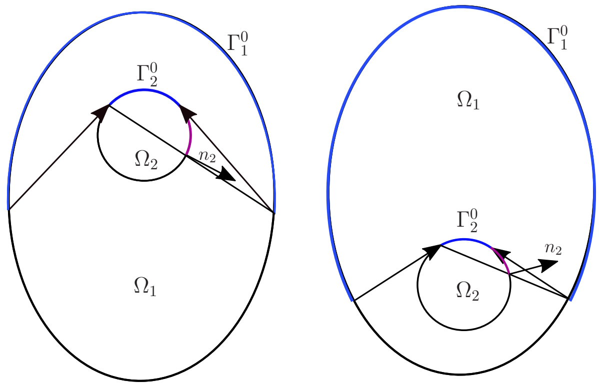

FIGURE 9. Extension of one of the boundary of Γ_2^0 in two different cases.

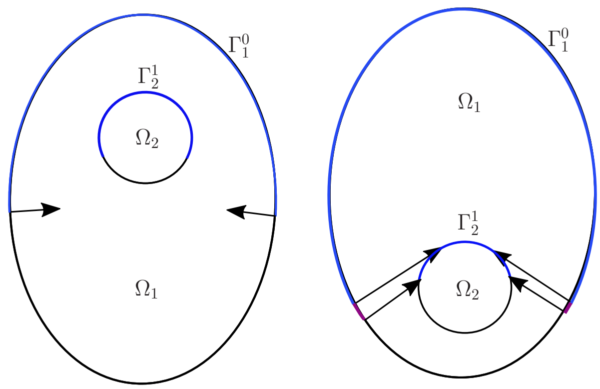

FIGURE 10. Extension of one of the boundary of Γ_1^0. *Left*: Γ_1^0 is not extended. *Right*: Γ_1^0 is extended.

from the extended region of Γ_2^n and propagating in the n_2 direction do not intersect Γ_1^n. Once the iteration process is over, we denote $\Gamma_i := \Gamma_i^n$ and its boundary $\{\delta(s_1^i), \delta(s_2^i)\}$ with $0 < s_1^i \le s_2^i < 1$.

In each step of the iteration process, we proved that the extended region is observable, meaning that every rays starting from this region are observed, or equivalently the observation of a ray intersecting this region. We are able to prove that the iteration process allows to obtain an observable region on Ω_2 that satisfies GCC.

Lemma 1.3. *Let Γ_2 be defined by the iteration process. Then there exists $x_0^2 \in \mathbb{R}^2 \setminus \overline{\Omega}_2$ such that $\Gamma_2 = \Gamma(x_0^2)$.*

Let us define $\Omega_1^f \subset \Omega_1$ the remaining part of Ω bounded by $\partial\Omega \setminus \Gamma_1$, $\partial\Omega_2 \setminus \Gamma_2$ and the lines $l(\delta(s_i^1), \delta_2(s_i^2))$ for $i = 1, 2$ (see Fig. 11). At this point, the iteration process allows us to conclude that every ray propagating in $\Omega_1 \setminus \Omega_1^f$ are observable. However, one cannot conclude on the observability of all the rays propagating inside Ω_1^f and Ω_2 with the previous arguments. To provide sufficient conditions for the observability of all the rays in Ω_1^f and Ω_2, we introduce the notion of uniformly escaping geometry. For convenience, we identify $n(\delta(s))^\perp$ with $\delta'(s)/\|\delta'(s)\|$, that is the unit tangential vector in the direction of the parametrisation.

Definition 1.4 (Uniformly escaping geometry)**.** We say that Ω_1^f is a uniformly escaping geometry if the application

$$\mathcal{M} : (\partial\Omega_2 \setminus \Gamma_2) \times (\mathbb{R}^2 \setminus \{(0,0)\}) \longrightarrow \qquad \mathbb{R}$$

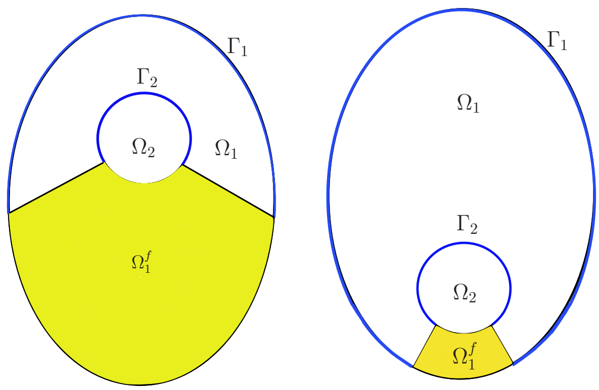

FIGURE 11. Definition of Ω_1^f.

$$(x,\xi) \quad \longmapsto \langle \xi, n\left(\Pi_x\left(\mathcal{F}\left(x,\xi\right)\right)\right)^{\perp}\rangle \tag{1.12}$$

is nondecreasing for $s \mapsto \mathcal{M}(\delta_2(s), n_2(\delta_2(s))), \delta_2(s) \in \partial\Omega_2 \setminus \Gamma_2$.

The name escaping geometry refers to the work of Miller in [13] on escape functions where GCC is reinterpreted in terms of escaping rays. In [13], a ray is said to have escaped for (1.6) through the observability region Γ if the ray intersects Γ in a non-diffractive way. Such a ray is then not reflected on Γ and is assumed to have escaped Ω to $\mathbb{R}^n \setminus \Omega$ as if there is no boundary Γ. The geometric control condition is then equivalent to every ray having exited Ω through Γ in time $T > 0$. The notion of uniformly escaping geometry is an adaptation of this definition in the context of rays propagating through an interface. To highlight how the notion of uniformly escaping geometry is related to the observability of (1.1)–(1.3), let us first specify how the rays escape Ω_1^f.

The boundary $l(\delta(s_i^1), \delta_2(s_i^2)), i = 1,2$ of Ω_1^f are assumed to be escaping for Ω_1^f, that is, every ray that intersects $l(\delta(s_i^1), \delta_2(s_i^2)), i = 1,2$ non-diffractively escape Ω_1^f (to $\Omega_1 \supset \Omega_1^f$). Rays are reflected in the usual sense on $\partial\Omega \setminus \Gamma_1$ and only rays intersecting $\partial\Omega_2 \setminus \Gamma_2$ in the $-n_2$ direction can escape through this boundary. This strong requirement is made so that the notion of uniformly escaping geometry is valid for any value of c_1 and c_2, since rays in the $-n_2$ direction are always transmitted through the interface $\partial\Omega_2$.

Behaviour of rays in Ω_1^f under UEG

The uniformly escaping geometry ensures that every rays propagating in Ω_1^f will be observed. Indeed, by definition, a ray starting in the $\mathcal{M}(x, n_2(x)) = 0, x \in \partial\Omega_2 \setminus \Gamma_2$ region and in the direction $n_2(x)$ satisfy $\mathcal{F}^2(x, n_2(x)) = (x, n_2(x))$ by definition. Since this is an escaping direction for $\partial\Omega_2 \setminus \Gamma_2$, this ray is assumed to have escaped Ω_1^f (see the light green ray in Fig. 12 on the left). A ray starting in the $\mathcal{M}(x, n_2(x)) < 0$ region in the $n_2(x)$ direction will eventually escape through $l(\delta(s_2^1), \delta_2(s_2^2)), \delta(s_2^1) \in \partial\Gamma_1, \delta_2(s_2^2) \in \partial\Gamma_2$ thanks to the nondecreasing assumption on $\mathcal{M}$ (see the light green ray in Fig. 12 on the right). The same description holds for rays in the $\mathcal{M}(x, n_2(x)) > 0$ region in the $n_2(x)$ direction. The complete picture can be deduced by this analysis. Indeed, consider a ray in the $M(x, n_2(x)) < 0$ region propagating in the opposite direction of the parametrisation of δ. Since the ray propagating in the $n_2(x)$ direction have escaped in the opposite direction of propagation, then the half-ray propagating in the same direction also escapes through the same boundary (see the dark green ray in Fig. 12 on the right). The case $M(x, n_2(x)) > 0$ is symmetric and one can follow either half-ray in the region $M(x, n_2(x)) = 0$ as both half-ray escapes through one of the two boundary $l(\delta(s_i^1), \delta_2(s_i^2)), \delta(s_i^1) \in \partial\Gamma_1, \delta_2(s_i^2) \in \partial\Gamma_2, i = 1,2$ (see the dark green ray in Fig. 12 on the left for the propagation of one of the half-ray).

We are finally now able to state the main results of this paper.

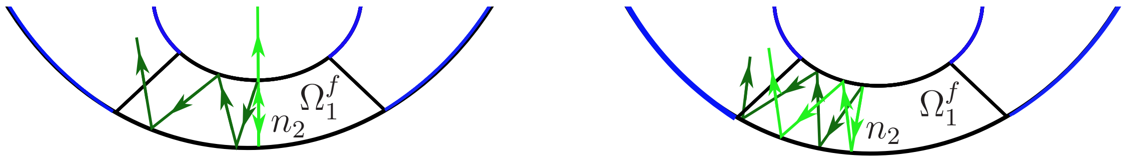

FIGURE 12. Example of a uniformly escaping geometry. *Left*: (light green) ray from x in the n_2 direction such that $\mathcal{M}(x, n_2(x)) = 0$ – (dark green) half-ray from the same point propagating in the negative tangential direction (with respect to δ'). *Right*: (light green) ray from x in the n_2 direction such that $\mathcal{M}(x, n_2(x)) < 0$ – (dark green) half-ray from the same point propagating in the negative tangential direction (with respect to δ').

Theorem 1.5. *Let $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$ and $\Gamma(x_0)$ be defined as in (1.9). If Ω_1^f is a uniformly escaping geometry, then for every $c_2 > c_1$, there exists a time $T > 0$ such that (1.1)–(1.3) is observable.*

The proof of Theorem 1.5 follows the geometrical construction as well as the arguments described above. In order to finish the proof, it is necessary to prove the observability of the gliding ray of $\partial\Omega_2$ that radiates into Ω_1. To do so, it suffices to remark that in the region Γ_2^0, the two corresponding half rays of Ω_1 are always observed, which is sufficient to prove the observability of the gliding ray in over Γ_2^0. The observability is then propagated along $\partial\Omega_2$ by considering an incoming gliding ray from Γ_2^0, which is observed, and to prove that a second incoming ray from Ω_1 is also observed, either by geometrical construction or by the UEG condition.

Remark 1.6. The minimal time $T > 0$ such that Theorem 1.5 holds is difficult to estimate, due to the complex path of the bicharacteristic flow for (1.1)–(1.3). Indeed, the bicharacteristic flow for (1.1)–(1.3) corresponds to a graph, and a stronger requirement than GCC is required to prevent the concentration along certain rays at the interface. From the geometrical construction above, the time of the minimal time $T > 0$ for Theorem 1.5 to hold is roughly estimated by adding the minimal time for all the rays of Ω_1^f to escape (this time allowing the ray to escape through $\partial\Omega_2$ according to Snell's law and not only in the $-n_2$ direction), the minimal time for rays to escape Ω_2 through Γ_2 and finally the time for the bicharacteristics of $\Omega_1 \setminus \Omega_1^f$ to reach $\Gamma(x_0)$.

Examples of geometrical configurations under the hypothesis of Theorem 1.5

Assume Ω and Ω_2 are concentric disks at the origin (such that $\Omega \setminus \Omega_2 \neq \emptyset$) and assume $x_0 = (0, y_0)$ with $y_0 < 0$ and $x_0 \notin \Omega$. If one chooses $\Gamma = \Gamma(x_0)$, then the iteration process yields a Ω_1^f region that satisfy the UEG condition. Indeed, the iteration process gives $\Gamma_1 = \Gamma(x_0)$, the lines $l(\delta(s_i^1), \delta_2(s_i^2)), i = 1, 2$ are those that start from $\delta(s_i^1)$ in the inward normal $-n$ direction and the boundary of Γ_2 is the intersection of these line with $\partial\Omega_2$ (it is an example of convergence of the iteration process such that (3.6) is satisfied). The domain Ω_1^f defined this way satisfy UEG. Moreover, if one moves the center of the disk Ω_2 on the negative y-axis, one still preserve the UEG condition. However, if the center of the disk Ω_2 is moved along the positive y-axis, then the UEG condition no longer holds.

Pathological behaviour of rays if Ω_1^f does not satisfy UEG

Rays propagating in Ω_1^f if it is not assumed to be a uniformly escaping geometry are difficult to analyse. One can exhibit examples where Ω_2 is an obstacle for certain directions for rays incoming from Ω_1. Figure 13 illustrate one example of trapped ray in Ω_1^f, despite the hypothesis $c_2 > c_1$. But even the hypothesis that there are no such trapped rays in Ω_1^f is not sufficient to conclude on the observability of (1.1)–(1.3). Notice that the weaker condition of assuming $\mathcal{M}$ is negative, null and positive on respective parts of $\partial\Omega \cap \Omega_1^f$ and $\partial\Omega_2 \cap \Omega_1^f$ is not sufficient since it does not prevent focusing dynamics between Ω_1^f and the interface that wouldn't leave Ω_1^f. Hence, only assuming that there exists an escape function [13] for Ω_1^f, or that Ω_1^f satisfies GCC (roughly

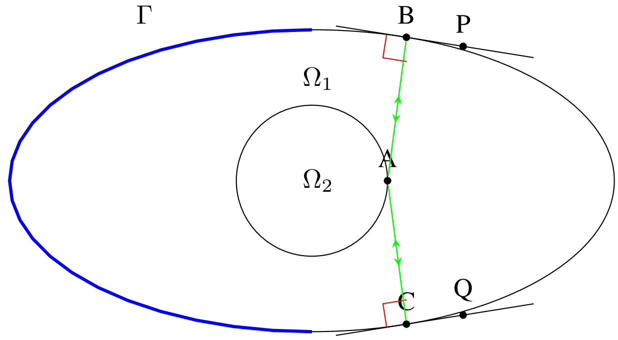

FIGURE 13. Representation of the spatial domain for (1.1)–(1.3) with trapped rays not encountering $\Gamma \subset \partial\Omega_1$ satisfying GCC. Here $\Omega_2 = \{x \in \mathbb{R}^2 \mid |x| < 1\}$, $\Omega_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1/4|^2 + |x_2/2|^2 < 1 \text{ and } x \notin \Omega_2\}$ and $c_1 = 1$, $c_2 = \sqrt{2}$.

speaking since the rays intersecting $\partial\Omega_2$ in the $-n_2$ direction escapes Ω_1^f) is not sufficient to conclude on the observability of (1.1)–(1.3). If Ω_1^f does not satisfy the uniformly escaping condition, one has to impose much more restrictive conditions to ensure observability.

Theorem 1.7. *Let $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$ and $\Gamma(x_0)$ be defined as in (1.9). If Ω_1^f is not a uniformly escaping geometry but if Ω_1^f and $c_2 > c_1$ are such that for every $x \in \partial\Omega \setminus \Gamma_1$ and $\xi \in \mathbb{R}^2$ such that $\Pi_x(\mathcal{F}(x,\xi)) \in \partial\Omega_2 \setminus \Gamma_2$ and $\Pi_x(\mathcal{F}^2(x,\xi)) \in \partial\Omega \setminus \Gamma_1$, the transmitted rays in Ω_2 are such that $\Pi_x(\mathcal{F}_2^2(x,\xi^{\pm})) \in \Gamma_2$, then (1.1)–(1.3) is observable in some time $T > 0$.*

Comments on the sufficient $\Gamma(x_0)$ condition

It seems unexpected to express the observability region Γ as a $\Gamma(x_0)$ region to state Theorem 1.5 and Theorem 1.7, as the proof relies on microlocal arguments from which one is able to prove the sharper GCC condition for the classical wave equation. This sharpness was already underlined in the paper of Bardos, Lebeau and Rauch ([1]), since they proved that there exists Ω strictly convex and Γ non-connected that satisfies GCC. Since $\Gamma(x_0)$ is connected for strictly convex Ω, then $\nexists x_0 \in \mathbb{R}^2$ such that $\Gamma = \Gamma(x_0)$ for such Γ. The result that $\Gamma(x_0)$ observability regions always satisfies GCC was proved later on by Miller ([13]) using escape functions. We prove here that GCC is more general than the $\Gamma(x_0)$ condition, even when Ω is strictly convex and Γ is connected.

Lemma 1.8. *There exist $\Omega \subset \mathbb{R}^2$ open, strictly convex domains and $\Gamma \subset \partial\Omega$ connected that satisfy GCC and such that $\nexists x_0 \in \mathbb{R}^2$ such that $\Gamma = \Gamma(x_0)$.*

We prove Lemma 1.8 with the specific dynamic of the billiards on the ellipse. More precisely, we use the 1-dimensional folliation of its phase space [5]. The (strong) Birkhoff's conjecture states that this 1-dimensional folliation, or integrability, is particuliar to the ellipse.

Conjecture 1.9 (Birkhoff)**.** *The only integrable billiards in two dimension are the ellipses.*

Figure 14 illustrates an observable region $\Gamma \subset \partial\Omega$ given by Lemma 1.8. This constitutes one example of the rather surprising corollary.

Corollary 1.10. *There exists $\Omega \subset \mathbb{R}^2$ open, strictly convex domains and $\Gamma \subset \partial\Omega$ such that Γ and $\partial\Omega \setminus \Gamma$ satisfy GCC.*

Under the assumption that Γ is connected, the hypothesis $\Gamma = \Gamma(x_0)$ is therefore not necessary condition. The $\Gamma(x_0)$ condition however provides a better understanding on the propagation of rays starting from $x \in \partial\Omega \setminus \Gamma(x_0)$

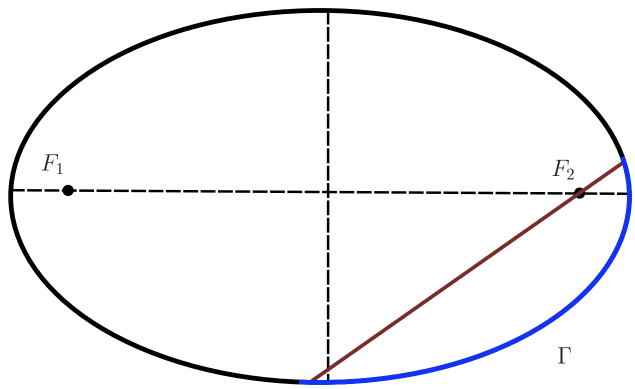

FIGURE 14. Example of an observable region $\Gamma \subset \partial\Omega$ for (1.6).

and moving in the $-n(x)$ direction. Indeed, the propagation of any such rays for any strictly convex Ω can be approximated by replacing $\partial\Omega \setminus \Gamma(x_0)$ by the cone defined by $l(x_0, x_i)$, $x_i \in \partial\Gamma(x_0)$, $i = 1, 2$. The rays propagating in the cone provide a lower bound on the displacement toward $\Gamma(x_0)$ for any rays propagating in the $-n$ direction starting from $\partial\Omega \setminus \Gamma(x_0)$. This property is fundamental in the present analysis and will be referred as the conical assumption.

1.6. Structure of the paper

The paper is organized as follow. In Section 2.1, we describe the generalized bicharacteristics for (1.1)–(1.3) following closely the presentation in Section 2 of [6]. The propagation of the defect measure is then presented in Section 2.2. It relies on the previous work of Burq and Lebeau on general system of wave equations with application to the Lamé system [4] and is presented following [6]. The microlocal analysis at the interface is very similar and we therefore only sketch the proofs. Section 3 is devoted to the proof of Theorem 1.5 and 1.7. Lemma 1.8 is proved in Section 4.

2. The generalized bicharacteristics and the propagation theorem

2.1. Definition of the generalized bicharacteristics

Let $M_i = \mathbb{R}_t \times \Omega_i$ for $i = 1, 2$. We consider $T^*(M_i) = \{(t, x, \tau_i, \xi_i) \in \mathbb{R}^6 \,|\, (t, x) \in \mathbb{R}_t \times \Omega_i\}$. The principal symbol of $P_i = \partial_t^2 - c_i^2 \Delta$ is given by

$$p_i(t, x; \tau_i, \xi_i) = \tau_i^2 - c_i^2 |\xi_i|^2.$$

Propagation of the bicharacteristics inside Ω_i

The bicharacteristics γ_i live in Char $P_i := \{(t, x, \tau_i, \xi_i) \in T^*(M_i) \,|\, p_i(t, x; \tau_i, \xi_i) = 0\}$ and are obtained as the solution of,

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}(t(\sigma), x(\sigma), \tau_i(\sigma), \xi_i(\sigma)) = H_{p_i}(t(\sigma), x(\sigma), \tau_i(\sigma), \xi_i(\sigma)), \quad \sigma \in \mathbb{R}, \tag{2.1}$$

where H_{p_i} is the Hamiltonian associated to p_i. The dual coordinates (τ_i, ξ_i) solution to (2.1) are constant with respect to σ and the rays – the projection over $\mathbb{R}_t \times \Omega_i$ of the bicharacteristics – are solution to,

$$\begin{cases} \dot{t}(\sigma) = -2\tau_i, & \sigma \in \mathbb{R}, \\ \dot{x}(\sigma) = 2c_i^2 \xi_i, & \sigma \in \mathbb{R}. \end{cases}$$

Description of the bicharacteristics near $\partial\Omega$

For $x \in \partial\Omega$, consider the local geodesic coordinates $(x_n, x') \in \mathbb{R}^2$ such that x is located at the origin, that is $x = (x_n, x') = (0,0)$ and, locally, $\Omega = \{x_n > 0\}, \partial\Omega = \{x^n = 0\}$. To simplify the notations, we further identify $x' > 0$ with the counter-clockwise of δ. Likewise, we write the dual coordinate $\xi_1 = (\xi_1', \xi_1^n)$ where $\xi_1' > 0$ is the tangential direction in the counter-clockwise direction and $\xi_1^n > 0$ is in the direction of the inward normal $-n(x)$. In these local coordinates, the Laplacian takes locally the form,

$$\Delta = \partial_{x_n}^2 + R(x_n, x', D_{x'}), \tag{2.2}$$

where $R(x_n, x', D_{x'})$ is a second order tangential elliptic operator of real principal symbol $r(x_n, x', \xi_1')$. Since Ω is assumed to be strictly convex, every bicharacteristics defined on Char P_1 intersect $\partial\Omega$ in a *non-diffractive* way. In other words, if we denote σ_0 the moment when the ray $x(\sigma) \in \Omega$, for $\sigma < \sigma_0$ small, intersects $\partial\Omega$, then the bicharacteristic defined on $T^*(\mathbb{R}_t \times \mathbb{R}^2)$ and in the direction ξ_1 escapes Ω for $\sigma > \sigma_0$ small.

We decompose $T^*(\partial\Omega \times (0,T)) = \mathcal{H} \cup \mathcal{G} \cup \mathcal{E}$ where,

$$\begin{aligned} \mathcal{H} &:= \left\{(t, x', \tau_1, \xi_1') \in T^*(\partial\Omega \times (0,T)) \,|\, \tau_1^2/c_1^2 - r(0, x', \xi_1') > 0\right\}, \\ \mathcal{G} &:= \left\{(t, x', \tau_1, \xi_1') \in T^*(\partial\Omega \times (0,T)) \,|\, \tau_1^2/c_1^2 - r(0, x', \xi_1') = 0\right\}, \\ \mathcal{E} &:= \left\{(t, x', \tau_1, \xi_1') \in T^*(\partial\Omega \times (0,T)) \,|\, \tau_1^2/c_1^2 - r(0, x', \xi_1') < 0\right\}. \end{aligned} \tag{2.3}$$

These sets are the hyperbolic, glancing and elliptic set, respectively, of $T^*(\partial\Omega \times (0,T))$. From the strict convexity of Ω, the glancing set $\mathcal{G}$ of $T^*(\partial\Omega \times (0,T))$ is reduced to the gliding set $\mathcal{G}^+$.

Reflection: $\rho_1 \in \mathcal{H}$.

If a bicharacteristic intersects $\partial\Omega$ transversally, that is $x(\sigma_0) \in \partial\Omega$ for some $\sigma_0 \in \mathbb{R}$, then,

$$p_1(t, x, \tau_1, \xi_1', \xi_1^n)(\sigma_0) = \tau_1^2 - c_1^2(|\xi_1^n|^2 + r(0, x'(\sigma_0), \xi_1')) = 0,$$

admits two real roots when seen as a polynomial of the variable ξ_1^n. Therefore at these points,

$$\begin{aligned} \lim_{\sigma \to \sigma_0^+} \xi_1^+(\sigma) &= \left(\xi_1'(\sigma_0), \sqrt{\frac{\tau_1^2}{c_1^2} - r(0, x'(\sigma_0), \xi_1')}\right), \\ \lim_{\sigma \to \sigma_0^-} \xi_1^-(\sigma) &= \left(\xi_1'(\sigma_0), -\sqrt{\frac{\tau_1^2}{c_1^2} - r(0, x'(\sigma_0), \xi_1')}\right). \end{aligned} \tag{2.4}$$

The bicharacteristic components $(x^\pm(\sigma), \xi_1^\pm)$ obey,

$$\begin{cases} x^+(\sigma) = x + 2c_1^2 \sigma \xi_1^+, & 0 < \sigma < \epsilon, \\ x^-(\sigma) = x + 2c_1^2 \sigma \xi_1^-, & -\epsilon < \sigma < 0, \end{cases} \tag{2.5}$$

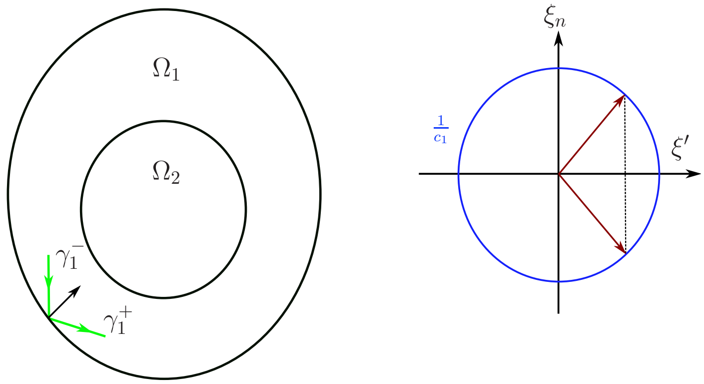

FIGURE 15. Rays $(t, x(t))$ of the bicharacteristics near $\rho_1 \in \mathcal{H}$.

for $\epsilon > 0$ sufficiently small (see Fig. 15). For $\sigma > 0$ (resp. $\sigma < 0$), let $\gamma_1^+(\sigma, \rho_1) := (t(\sigma), x^+(\sigma), \tau_1, \xi_1^+)$ (resp. $\gamma_1^-(\sigma, \rho_1) := (t(\sigma), x^-(\sigma), \tau_1, \xi_1^-)$) the outgoing (resp. incoming) bicharacteristic of ρ_1. The generalized bicharacteristic path satisfies $\gamma(0, \rho_1) = \rho_1$ and

$$\gamma(0, \rho_1) = \begin{cases} \gamma_1^+(\sigma, \rho_1), & 0 < \sigma < \epsilon, \\ \gamma_1^-(\sigma, \rho_1), & -\epsilon < \sigma < 0, \end{cases} \tag{2.6}$$

for $\epsilon > 0$ sufficiently small.

Gliding rays: $\rho_1 \in \mathcal{G}^+$.

The gliding ray on $\partial\Omega$ is described by the following equation on the bicharacteristic,

$$\begin{cases} \widetilde{\gamma_1}(\sigma) = (t(\sigma), x(\sigma), \tau_1(\sigma), \xi_1(\sigma)), \sigma \in [a, b], \text{such that } (t(\sigma), x(\sigma), \tau_1(\sigma), \xi_1(\sigma)) \in \mathcal{G}^+ \\ \dot{\tau_1}(\sigma) = 0, \dot{t}(\sigma) = -2\tau_1, \dot{x}(\sigma) = 2c_1\xi_1'(\sigma) \text{ and } D\xi_1(\sigma) = 0, \\ \text{where } D \text{ denotes the covariant derivative over } \partial\Omega_1. \end{cases} \tag{2.7}$$

In other words, $\sigma \in [a, b] \to x(\sigma) \in \partial\Omega$ is a geodesic curve such that $\dot{x}(\sigma) = 2c_i\xi_1'(\sigma)$ (see Fig. 16).

Description of the bicharacteristics near $\partial\Omega_2$.

We use again the local geodesic coordinates near $\partial\Omega_2$. In this case, locally, $\Omega_2 = \{x_n > 0\}, \partial\Omega_2 = \{x_n = 0\}$, $\Omega_1 = \{x_n < 0\}$ and $x' > 0$ is in the counter-clockwise direction of the parametrisation of δ_2. For the dual component $\xi_i = (\xi_i', \xi_i^n), i = 1, 2$, $\xi_1^n > 0$ (resp. $\xi_2^n < 0$) is in the direction of the inward normal $n_2(x)$ (with respect to Ω_1) (resp. $-n_2(x)$) and $\xi_i' > 0$ is in the counter-clockwise direction of the parametrisation of δ_2. The abuse of notation introduced in the introduction shorten the notation $T^*((\partial\Omega_1 \cap \partial\Omega_2) \times (0, T)) \times T^*(\partial\Omega_2 \times (0, T))$ to $T^*(\partial\Omega_2 \times (0, T))$.

For $(\rho_1, \rho_2) \in T^*(\partial\Omega_2 \times (0, T))$, the Snell's law follows from the transmission boundary condition (1.2). Indeed, taking the tangential derivative of the relation $u_1 = u_2$ implies

$$\xi_1' = \xi_2', \tag{2.8}$$

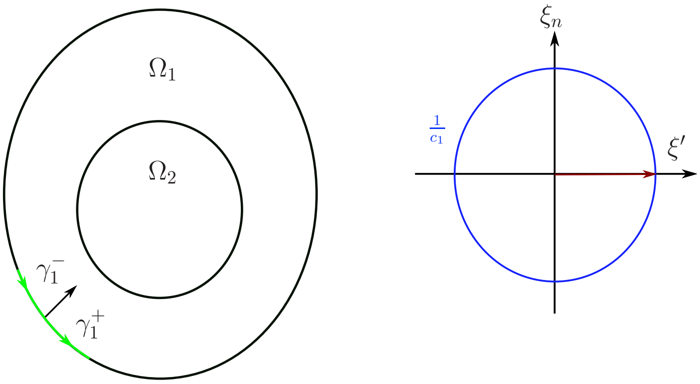

FIGURE 16. Rays $(t, x(t))$ of the bicharacteristics near $\rho_1 \in \mathcal{G}^+$.

which yields (1.8) or, in local geodesic coordinates, a correspondance between rays near the boundary through the relations (2.4) or (2.7). Therefore $T^*(\partial\Omega_2 \times (0,T))$ is decomposed, using the strict convexity of Ω_2, as $(\mathcal{H}^1 \times \mathcal{H}^2) \cup (\mathcal{H}^1 \times \mathcal{G}^{2,+}) \cup (\mathcal{H}^1 \times \mathcal{E}^2) \cup (\mathcal{G}^{1,-} \times \mathcal{E}^2) \cup (\mathcal{E}^1 \times \mathcal{E}^2)$, where,

$$\begin{aligned}
\mathcal{H}^1 \times \mathcal{H}^2 =& \{ (\rho_1, \rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid \tau_i^2/c_i^2 > r(0,x',\xi_i'), i = 1,2\}, \\
\mathcal{H}^1 \times \mathcal{G}^{2,+} =& \{ (\rho_1, \rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid \tau_1^2/c_1^2 > r(0,x',\xi_1'), \tau_2^2/c_2^2 = r(0,x',\xi_2')\}, \\
\mathcal{H}^1 \times \mathcal{E}^2 =& \{ (\rho_1, \rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid \tau_1^2/c_1^2 > r(0,x',\xi_1'), \tau_2^2/c_2^2 < r(0,x',\xi_2')\}, \\
\mathcal{G}^{1,-} \times \mathcal{E}^2 =& \{ (\rho_1, \rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid \tau_1^2/c_1^2 = r(0,x',\xi_1'), \tau_2^2/c_2^2 < r(0,x',\xi_2')\}, \\
\mathcal{E}^1 \times \mathcal{E}^2 =& \{ (\rho_1, \rho_2) \in T^*(\partial\Omega_2 \times (0,T)) \mid \tau_i^2/c_i^2 < r(0,x',\xi_i'), i = 1,2\}.
\end{aligned} \tag{2.9}$$

Reflection–Transmission: $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{H}^2$.

If a bicharacteristic, incoming from Ω_i, intersects $\partial\Omega_2$ transversally and such that the Snell's law (1.8) is not vacuous and $|\theta_2| \neq \pi/2$, then,

$$p_i(t,x,\tau_i,\xi_i',\xi_i^n)(\sigma_0) = \tau_i^2 - c_i^2(|\xi_i^n|^2 + r(0,x'(\sigma_0),\xi_i')) = 0, \tag{2.10}$$

admits two real roots for $i = 1,2$ as a polynomial of the variable ξ_i^n. Therefore at these points,

$$\begin{aligned}
\lim_{\sigma\to\sigma_0^+} \xi_i^+(\sigma) &= \left(\xi_i'(\sigma_0), (-1)^i \sqrt{\frac{\tau_i^2}{c_i^2} - r(0,x'(\sigma_0),\xi_i')} \right), \\
\lim_{\sigma\to\sigma_0^-} \xi_i^-(\sigma) &= \left(\xi_i'(\sigma_0), (-1)^{i+1} \sqrt{\frac{\tau_i^2}{c_i^2} - r(0,x'(\sigma_0),\xi_i')} \right).
\end{aligned} \tag{2.11}$$

For $\sigma > 0$, let $\gamma_i^+(\sigma,\rho_i) = (t(\sigma), x_i^+(\sigma), \tau_i(\sigma), \xi_i^+(\sigma))$ be the reflected bicharacteristic, let $\gamma_j^+(\sigma,\rho_j) = (t(\sigma), x_j^+(\sigma), \tau_j(\sigma), \xi_j^+(\sigma))$ be the transmitted bicharacteristic and let $\gamma_i^-(\sigma,\rho_i) = (t(\sigma), x_i^-(\sigma), \tau_i(\sigma), \xi_i^-(\sigma))$ be the incoming bicharacteristic of ρ_i ($x_{i,j}^{\pm}$ being defined by (2.5)). The generalized bicharacteristic path is such

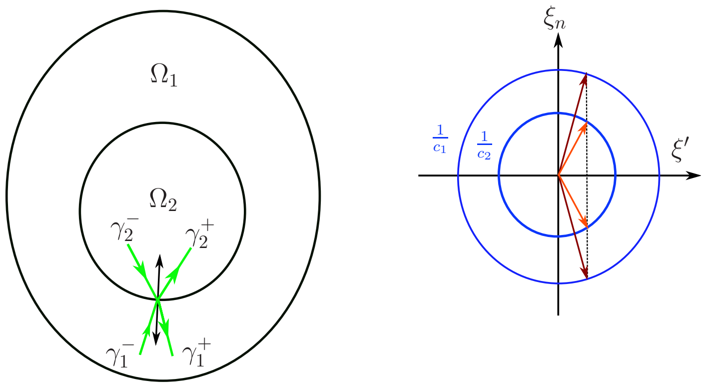

FIGURE 17. Rays $(t, x(t))$ of the bicharacteristics near a point $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{H}^2$.

that $\gamma(0, \rho_i) = \rho_i$ and,

$$\gamma(\sigma, \rho_i) = \begin{cases} \gamma_i^+(\sigma, \rho_i), & 0 < \sigma < \epsilon, \\ \gamma_j^+(\sigma, \rho_j), & 0 < \sigma < \epsilon, \\ \gamma_i^-(\sigma, \rho_i), & -\epsilon < \sigma < 0, \end{cases}$$

for $\epsilon > 0$ sufficiently small. Figure 17 illustrates the outgoing bicharacteristics from an incoming bicharacteristic γ_1^- or γ_2^- at a point $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{H}^2$.

Notice that when $(\rho_1, \rho_2) \in \mathcal{H}^1 \times (\mathcal{H}^2 \cup \mathcal{G}^{2,+})$, the assumption $c_1 < c_2$ and the Snell's condition (1.8) implies,

$$\theta_1 < \theta_2, \quad \forall (\rho_1, \rho_2) \in \mathcal{H}^1 \times (\mathcal{H}^2 \cup \mathcal{G}^{2,+}). \tag{2.12}$$

Reflection–Gliding transmission: $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{G}^{2,+}$.

If an incoming bicharacteristic from Ω_1 intersects $\partial\Omega_2$ traversally at $x(\sigma_0) \in \partial\Omega_2$ for some $\sigma_0 \in \mathbb{R}$ at the critical angle $\sin\theta_1 = c_1/c_2$, then $p_1 = 0$ defined by (2.10) admits two real roots. The dual variable $\xi_2 = (\xi_2^n, \xi_2')$ is such that $\xi_2^n = 0$ and $\xi_1' = \xi_2'$. Using the same notations than for the reflected-transmitted case for the half-bicharacteristics $\gamma_1^\pm$, the generalized bicharacteristic path is such that

$$\gamma(\sigma, \rho_1) = \begin{cases} \gamma_1^+(\sigma, \rho_1), & 0 < \sigma < \epsilon, \\ \widetilde{\gamma}_2(\sigma, \rho_2), & 0 < \sigma < \epsilon, \\ \gamma_1^-(\sigma, \rho_1), & -\epsilon < \sigma < 0, \end{cases}$$

where $\widetilde{\gamma}_2$ is defined as the gliding ray for Ω_2 following the definition (2.7) of gliding rays on $\partial\Omega$. Figure 18 illustrates the outgoing bicharacteristics from an incoming bicharacteristic γ_1^- or $\widetilde{\gamma}_2^-$ at a point $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{G}^{2,+}$.

Reflection in Ω_1*:* $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{E}^2$.

If the angle of incidence of the incoming bicharacteristic from Ω_1 is strictly greater than the critical angle

$$\sin\theta_1 = c_1/c_2, \tag{2.13}$$

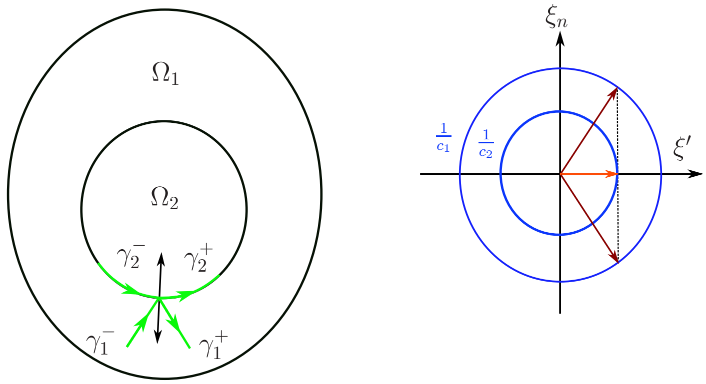

FIGURE 18. Rays $(t, x(t))$ of the bicharacteristics near $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{G}^{2,+}$.

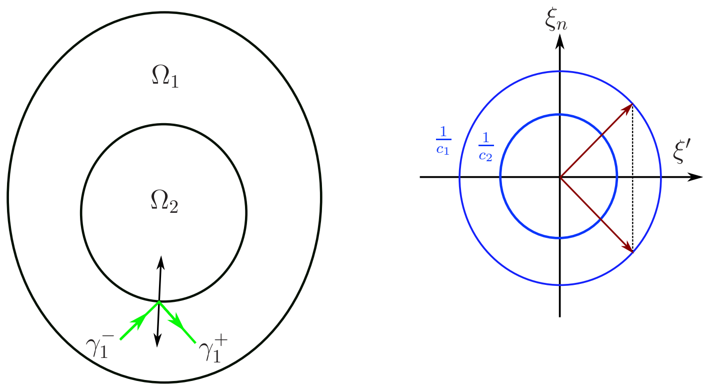

FIGURE 19. Rays $(t, x(t))$ of the bicharacteristics near $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{E}^2$.

it is reflected in Ω_1 according to (2.4) and (2.6) without transmission to Ω_2. In this case the domain Ω_2 acts as an obstacle. Figure 19 illustrates an example of reflection in Ω_1 without transmission in Ω_2.

Strictly diffractive rays: $(\rho_1, \rho_2) \in \mathcal{G}^{1,-} \times \mathcal{E}^2$.

In this case, the contact between the incoming ray and $\partial\Omega_2$ is *strictly diffractive*, meaning that the incoming bicharacteristic from Ω_1 defined over $T^*(\mathbb{R}_t \times \mathbb{R}^2)$ stays in Ω_1 for $|\sigma - \sigma_0| > \epsilon$ for $\epsilon > 0$ small and where σ_0 is the moment when the bicharacteristic interesected $\partial\Omega_1$. In this case, there are no transmission in Ω_2 under the hypothesis $c_2 > c_1$. Figure 20 illustrates an example of a grazing ray.

Full description of the generalized bicharacteristic flow of (1.1)–(1.3).

We gather the description above to describe the generalized bicharacteristic flow of (1.1)–(1.3). The bicharacteristics propagate in straight line and constant speed c_i in Ω_i. They are connected to the broken bicharacteristics near $\partial\Omega$ and $\partial\Omega_2$ according to their angle of incidence. As there are no tangential angle of incidence for $\partial\Omega$ under the strict convexity assumption of Ω, the gliding ray of $\partial\Omega$ is not connected to the bicharacteristic flow in Ω_1. Therefore a gliding ray on $\partial\Omega$ exists if and only if it started as a gliding ray on $\partial\Omega$. The gliding ray of $\partial\Omega_2$ is connected to the bicharacteristic flow in Ω by the critical angle of incidence (2.13). A gliding ray on

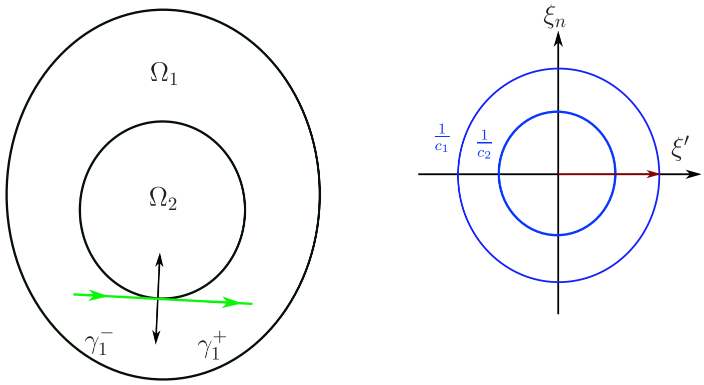

FIGURE 20. Rays $(t, x(t))$ of the bicharacteristics near $(\rho_1, \rho_2) \in \mathcal{G}^{1,-} \times \mathcal{E}^2$.

$\partial\Omega_2$ radiates in Ω_1 as it emits transmitted rays γ_1^+ (according to the description $(\rho_1, \rho_2) \in H^1 \times \mathcal{G}^{2,+}$) along its trajectory on $\partial\Omega_2$.

2.2. Microlocal defect measures

If $(u_i^k)_{k\in\mathbb{N}}$ are bounded sequences of solutions of (1.1)–(1.3) weakly converging to 0 in $H^1_{loc}((0,T) \times \Omega_i)$, then one can associate to each sequences a microlocal defect measure μ_i. Moreover, the measures μ_1 and μ_2 are each supported respectively in Char $P_1 \cup \mathcal{G}^+ \cup \mathcal{H}^1 \cup \mathcal{G}^{1,-}$ and Char $P_2 \cup \mathcal{H}^2 \cup \mathcal{G}^{2,+}$, where the hyperbolic and gliding sets are defined in (2.3) and (2.9). The property the support of the microlocal defect measure does not include the elliptic set is known as the elliptic regularity theorem for the microlocal defect measures [4] and the fact that $\mu_1 \mathbb{1}_{\mathcal{H}} = 0$ comes from the fact that the microlocal defect measure introduced in [8] and used in [4] does not charge the hyperbolic set (see for instance [4], Lem. 2.6).

We now analyse the property of these measures. In the interior $T^*(\mathbb{R}_t \times \Omega_i)$, since the solutions u^1 and u^2 of (1.1)–(1.3) evolve independently in $\Omega_i, i = 1, 2$, we make use of the classical measure propagation theorem of Gerard [7], stating that the support of the defect measure is transported along the bicharacteristic flow in Ω_i. Near the boundary $\partial\Omega$, the analysis of [3, 8] yields the propagation theorem for the wave equation with homogeneous Dirichlet boundary condition.

Corollary 2.1. *For $\rho_1 \in \mathcal{H} \times \mathcal{G}^+$, we have the following equivalence*

$$\big((\gamma_1^-) \cap supp(\mu_1)\big) = \emptyset \Leftrightarrow \big((\gamma_1^+) \cap supp(\mu_1)\big) = \emptyset.$$

An adaptation of the results of [4] (see also [6]) allows the characterisation of the propagation of the support of the measure along the generalized bicharacteristics near $\partial\Omega_2$.

Proposition 2.2. *With the above notations, we have*

1. *If $(\rho_1, \rho_2) \in (\mathcal{H}^1 \cup \mathcal{G}^{1,-} \cup \mathcal{E}^1) \times \mathcal{E}^2$, then $\mu_2 = 0$ near ρ_2. Therefore,*
   (a) *if $\rho_1 \in \mathcal{E}^1$, then $\mu_1 = 0$ near ρ_1,*
   (b) *otherwise, $\rho_1 \in \mathcal{H}^1 \cup \mathcal{G}^{1,-}$ and the support of μ_1 propagates from γ_1^- to γ_1^+.*
2. *otherwise assume $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{H}^2$. Then if $\gamma_i^- \cap supp(\mu_i) = \emptyset$, for $i = 1$ or $i = 2$, then for $j = 3 - i$, the support of μ_j propagates from γ_j^- to γ_j^+ and from γ_j^- to γ_i^+.*

As a consequence of the conversion of the total mass ([4], Proof Thm. 4), we obtain,

Corollary 2.3. *For $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{H}^2$, we have the following equivalence*

$$\big((\gamma_1^-) \cap supp(\mu_1)\big) \cup \big((\gamma_2^-) \cap supp(\mu_2)\big) = \emptyset$$
$$\Updownarrow$$
$$\big((\gamma_1^+) \cap supp(\mu_1)\big) \cup \big((\gamma_2^+) \cap supp(\mu_2)\big) = \emptyset.$$

Remark 2.4. 1. If one of the two statements in Corollary 2.3 holds true, then $\mu_i = 0, i = 1, 2$ near (ρ_1, ρ_2).
2. By time-reversibility, one also obtain the characterisation of two half-rays from Ω_1 or Ω_2. If these two half-rays of, say Ω_1, do not intersect the support of the measure μ_1, then the two half-rays in Ω_2 do not intersect the support of μ_2.

We finally state the propagation of the defect measure for the gliding region, based on the work of Miller [12].

Corollary 2.5. *Let $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{G}^{2,+}$. Then,*

$$\big((\gamma_1^-) \cap supp(\mu_1)\big) \cup \big((\gamma_1^+) \cap supp(\mu_1)\big) = \emptyset$$
$$\Downarrow \tag{2.14}$$
$$\big((\gamma_2^-) \cap supp(\mu_2)\big) \cup \big((\gamma_2^+) \cap supp(\mu_2)\big) = \emptyset, \tag{2.15}$$

and

$$\big((\gamma_1^-) \cap supp(\mu_1)\big) \cup \big((\gamma_2^-) \cap supp(\mu_2)\big) = \emptyset$$
$$\Updownarrow \tag{2.16}$$
$$\big((\gamma_1^+) \cap supp(\mu_1)\big) \cup \big((\gamma_2^+) \cap supp(\mu_2)\big) = \emptyset. \tag{2.17}$$

We begin by proving Proposition 2.2.
Proof:
1. Let $(\rho_1, \rho_2) \in (\mathcal{H}^1 \cup \mathcal{G}^{1,-} \cup \mathcal{E}^1) \times \mathcal{E}^2$. It is a classical result from the elliptic theory at the boundary that $\mu_2 = 0$ near $\rho_2 \in \mathcal{E}^2$ (see for instance [4], Appendix A.1). Therefore 1.a) and 1.b) follow from the classical propagation results for the wave equation at the boundary.
2. Consider now $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{H}^2$. We explain how to adapt the results of Appendix A.2 in [4]. We recall the notations of Section 2.1. The local geodesic coordinates (x_n, x') are such that, locally, we have $\Omega_2 = \{x_n > 0\}, \partial\Omega_2 = \{x_n = 0\}$ and $\Omega_1 = \{x_n < 0\}$. Recall that near this point, the local geodesic coordinates implies that the Laplacian takes the form (2.2), and therefore the wave equations operator write,

$$P_i = c_i^2(-D_{x_n}^2 + \tilde{R}_i(x_n, x', D_{x'}, D_t)),$$

where, in the hyperbolic region, $\tilde{R}$ is a second order tangential elliptic differential operator ([4, 6]),

$$\tilde{R}_i(x_n, x', D_{x'}, D_t) = \frac{D_t^2}{c_i^2} - R(x_n, x', D_{x'}),$$

with real principal symbol,

$$\tilde{r}(x_n, x', \xi', \tau) = \frac{\tau^2}{c_i^2} - r(x_n, x', \xi_i').$$

Near a hyperbolic point, one can then use Lemma A.1 of [4] to factorise the pseudodifferential operators in two different ways,

$$P_i = -c_i^2(D_{x_n} - \Lambda^+(x_n, x', D_{x'}, D_t))(D_{x_n} - \Lambda^-(x_n, x', D_{x'}, D_t)) + T(x_n, x', D_{x'}, D_t),$$
$$P_i = -c_i^2(D_{x_n} - \widetilde{\Lambda}^-(x_n, x', D_{x'}, D_t))(D_{x_n} - \widetilde{\Lambda}^+(x_n, x', D_{x'}, D_t)) + \widetilde{T}(x_n, x', D_{x'}, D_t),$$

where $\Lambda^\pm$ and $\widetilde{\Lambda}^\pm$ are tangential pseudodifferential operators of order 1 and such that $\sigma_1(\Lambda^\pm) = \pm\sqrt{\tilde{r}}$ and $\sigma_1(\widetilde{\Lambda}^\pm) = \pm\sqrt{\tilde{r}}$ near (ρ_1, ρ_2) and where T and $\widetilde{T}$ are tangential pseudodifferential operators of order $-\infty$. A microlocalisation near of $\gamma_i^\pm$ is done using $q_0(x', \xi_i')$ a symbol of order 0 and equal to 1 in a conical neighborhood of ρ_i and of compact support. Let us remark that at a point $(\rho_1, \rho_2) \in \mathcal{H}^1 \times (\mathcal{H}^2 \cup \mathcal{G}^{2,+})$, the tangential components of $\rho_1 = (t, x_n, x', \tau, \xi_1^n, \xi_1')$ and $\rho_2 = (t, x_n, x', \tau, \xi_2^n, \xi_2')$ are equal: $(t, x', \tau, \xi_1') = (t, x', \tau, \xi_2')$. Therefore, the same symbol q_0 may be used for the microlocalisation near ρ_1 and ρ_2. The symbol is then propagated by the Hamiltonian,

$$(c_i\partial_{x_n} \mp H_{\sqrt{\tilde{r}}})q_i^\pm = 0, \quad q_i^\pm|_{x_n=0} = q_0.$$

Consider $\varphi \in C_0^\infty(\mathbb{R}_x)$ to be equal to 1 near 0 and of compact support near 0. If we denote $Q_i^\pm = \mathrm{Op}(\varphi q_i^\pm)$ and $Q_0 = \mathrm{Op}(\varphi q_0)$, then we obtain the same results as in Appendix A.2 of [4], that is, if we consider bounded sequences $(u_1^k, u_2^k) \subset H^1((-1,0) \times Y) \times H^1((0,1) \times Y)$, where $Y = \{(t, x') \,|\, |(t, x')| < 1\}$, such that,

$$P_1 u_1 \to 0 \text{ in } L^2((-1,0) \times Y),$$
$$P_2 u_2 \to 0 \text{ in } L^2((0,1) \times Y).$$

and if we suppose, without loss of generality, that $\gamma_1^- \cap \mathrm{supp}(\mu_1) = \emptyset$, then (recall that we can deduce that $u_i^k|_{x=0}$ and $D_x u_i^k|_{x=0}$ are bounded in $H^1_{\rho_i}$ and $L^2_{\rho_i}$ respectively),

$$c_1^2 Q_0(D_{x_n} u_1^k|_{x_n=0} - \Lambda^- u_1^k|_{x_n=0}) \to 0 \text{ in } L^2(Y).$$

Applying the tangential pseudodifferential operator Λ^- to the Dirichlet boundary condition yields,

$$\Lambda^- u_1^k = \Lambda^- u_2^k, \text{ in } L^2(Y),$$

and a Lopatinski condition on u_2^k is obtained,

$$c_1^2 Q_0 \left(\frac{c_2^2 D_{x_n} u_2^k}{c_1^2}\bigg|_{x_n=0} - \Lambda^- u_2^k|_{x_n=0} \right) \to 0 \text{ in } L^2(Y). \tag{2.18}$$

In particular, the traces of solution u_k^2 are decoupled from the traces of u_k^1. The wave equation in Ω_2 can therefore be solved locally, independently of the solution u_k^1 over Ω_1, and the support of the defect measure propagates from γ_2^- to γ_2^+ thanks to the Lopatinski condition [4]. The boundary conditions of u_k^2 implies boundary conditions for u_k^1 which implies that the support of γ_2^- propagates to γ_1^+. This conclude the proof of Proposition 2.2.

□

We now turn to the proof of Corollary 2.3.

Proof:

Let us consider the case where γ_1^- and γ_2^- intersects the boundary $\partial\Omega_2$ at a $\mathcal{H}^1 \times \mathcal{H}^2$ point. Consider $\big((\gamma_1^-) \cap \mathrm{supp}(\mu_1)\big) \cup \big((\gamma_2^-) \cap \mathrm{supp}(\mu_2)\big) = \emptyset$, we obtain from the proof of Proposition 2.2

$$c_i^2 Q_0 (D_x u_i^k|_{x=0} - \Lambda^- u_i^k|_{x=0}) \to 0 \text{ in } L^2(Y). \tag{2.19}$$

Using the Neumann condition yields,

$$(c_1^2 - c_2^2) Q_0 (\Lambda^- u_i^k|_{x=0}) \to 0 \text{ in } L^2(Y),$$

which is a tangential pseudodifferential operator of order 1 and elliptic at ρ_i. It follows that,

$$c_i^2 Q_0 D_x u_i^k|_{x=0} \to 0 \text{ in } L^2(Y).$$

The ellipticity implies that u_i^k converges to 0 in H^1 and the standard propagation results implies Corollary 2.3. □

We now prove Corollary 2.5. Its proof relies on [4] for the Lopatinski condition for the hyperbolic rays and [12] for the equation linking the defect measure of the gliding ray of Ω_2 and the Dirichlet trace of the hyperbolic ray of Ω_1. The notations of [12] are quite different from the present article, so we explain below how to deduce the proof of Corollary 2.5 from [12].

Proof:

Let us first prove (2.14). Assume $\gamma_1^\pm \cap \mathrm{supp}(\mu_1) = \emptyset$. Then, from the proof of Corollary 2.3, we know that both Dirichlet and Neumann traces of u_k^1 converges strongly to 0 in $H^1((0,1)\times Y)$ and $L^2((0,1)\times Y)$, respectively. From Proposition 1.14 in [12] (see also the remark below [12], Prop. 1.14), the Dirichlet trace of u_k^1 is the source term of the gliding ray, and since it converges strongly to zero, we deduce $\gamma_2^\pm \cap \mathrm{supp}(\mu_2) = \emptyset$. The proof of (2.16) may be deduced by the conservation of energy.

□

The following corollary follows from Proposition 2.2 and Corollary 2.3. We provide a sufficient condition for the propagation of the measure of the generalized bicharacteristic flow of (1.1)–(1.3) as illustrated in Figure 5. To this end, let us introduce some notations.

Let $(\rho_1, \rho_2) \in T^*(\partial\Omega_2 \times (0,T))$. We denote $(\rho_1^1, \rho_2^1) \in T^*(\partial\Omega_2 \times (0,T))$ the coordinates of the intersection of γ_1^+ with $\partial\Omega_2$ (after reflection on $\partial\Omega$). Denote $(\rho_1^2, \rho_2^2) \in T^*(\partial\Omega_2 \times (0,T))$ the coordinates of the intersection of γ_2^+ with $\partial\Omega_2$. Denote $(\rho_1^{-a}, \rho_2^{-a}) \in T^*(\partial\Omega_2 \times (0,T))$, for $a = 1,2$ the intersection of γ_a^- with $\partial\Omega_2$. Finally, for a finite sequence $a_i = \pm1, \pm2, 1 \leq i \leq n$, denote $(\rho_1^{a_1\ldots a_n}, \rho_2^{a_1\ldots a_n})$ the iteration of this procedure with the requirement that a sequence possesses consecutive 2 and -2 (or the converse) if and only if the ray is a trapped ray of Ω_2. Otherwise consecutives 1 and -1 (or the converse) or 2 and -2 (or the converse) consists to following a ray say γ_1^+ forward in time and then backward in time after the intersection with $\partial\Omega_2$ which is not described by the bicharacteristic flow of (1.1)–(1.3).

Corollary 2.6. *For $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{H}^2$, if there exists a finite sequence $\{a_i\}_{1\leq i\leq n}, a_i = \pm1, \pm2$ such that for every $1 \leq i \leq n-1$, $(\rho_1^{a_1\ldots a_i}, \rho_2^{a_1\ldots a_i}) \in \mathcal{H}^1 \times \mathcal{H}^2$ and such that there exists a half-ray $\gamma_{|b|}^{sign(b)}(\rho_1^{a_1\ldots a_i}, \rho_2^{a_1\ldots a_i}) \cap supp(\mu_{|b|}) = \emptyset$ where $b = \pm1$ or $b = \pm2$ is such that $b \neq -a_i$ or $b \neq a_{i+1}$ and such that there exist two half-rays $\gamma_{|b_j|}^{sign(b_j)}(\rho_1^{a_1\ldots a_n}, \rho_2^{a_1\ldots a_n}) \cap supp(\mu_{|b_j|}) = \emptyset$ where $j = 1,2$, $b_j = \pm1$ or $b_j = \pm2$ are such that $b_j \neq -a_n, j = 1,2$, then $\mu_1, \mu_2 = 0$ near (ρ_1, ρ_2).*

Proof:

Proposition 2.2 allows to use a recursive argument on the sequence of half-rays defined by the sequence $\{a_i\}_{1\leq i\leq n}$. We prove the result for $n = 1$. Since two-half rays do not intersect the support of their respective measure near $(\rho_1^{a_1}, \rho_2^{a_1})$, we conclude by Corollary 2.3 that $\mu_1, \mu_2 = 0$ near $(\rho_1^{a_1}, \rho_2^{a_1})$. Corollary 2.3 implies $\gamma_{|a_1|}^{\mathrm{sign}(-a_1)}(\rho_1^{a_1}, \rho_2^{a_1}) \cup \mathrm{supp}(\mu_{|a_1|}) = \emptyset$. Moreover $\gamma_{|a_1|}^{\mathrm{sign}(-a_1)}(\rho_1^{a_1}, \rho_2^{a_1})$ intersects $\partial\Omega_2$ at (ρ_1, ρ_2) by definition. By

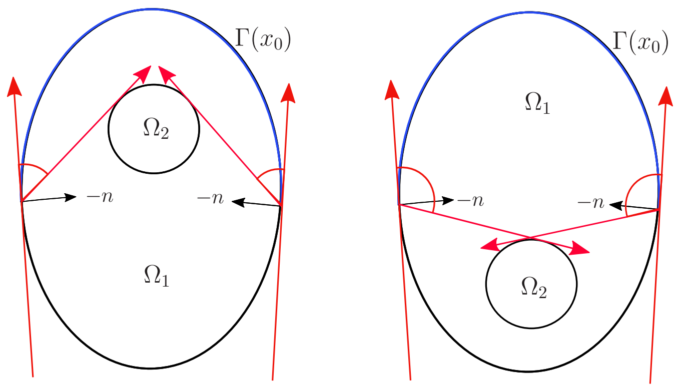

FIGURE 21. Localisation of Ω_2. Case $L(\delta(s_i)) < 0, i = 1, 2$ on the left and $L(\delta(s_i)) < 0, i = 1, 2$ on the right.

hypothesis, there exists another half-ray that does not intersect the support of its respective measure. Therefore we conclude this case by Corollary 2.3. The general case follows easily by iterating this process.

□

3. Geometric argument in the case of two wave equations with a transmission condition

We provide in this section the proof of Theorem 1.5 and Theorem 1.7 based on the geometrical construction introduced in Section 1.5. We begin by proving Theorem 1.5 which is divided in several steps. The proof of Theorem 1.7 is then presented, which only differs from the proof of Theorem 1.5 at the last step.

Proof:

Step 1: Definition of Γ_1^0

First, recall that the parametrisation of δ is made such that $\delta(0) = \delta(1) \in \Gamma(x_0)$ and that the parametrisation of δ is counter-clockwise. Hence, let $0 < s_1 < s_2 < 1$ be such that $\{\delta(s_1), \delta(s_2)\} \in \partial\Gamma(x_0)$. We denote,

$$L(y) := \max_{x\in\partial\Omega_2} \langle y - x_0, x - y\rangle , \quad y \in \partial\Omega. \tag{3.1}$$

The function L defined by (3.1) helps localising Ω_2 with respect to $\Gamma(x_0)$ (see Fig. 21). Indeed, if $L(\gamma(s_i)) \geq 0$ for $i = 1$ (resp. $i = 2$), then we define $s_1^{1,0} := s_1$ (resp. $s_2^{1,0} := s_2$). Otherwise, if $L(\delta(s_i)) < 0$ for $i = 1$ (resp. $i = 2$), denote $s_1^{1,0}$ (resp. $s_2^{1,0}$) the smallest $s > s_1$ (resp. largest $s < s_2$) such that $L(\delta(s)) = 0$. From the strict convexity of Ω, it is not difficult to see that $s_i^{1,0}$ satisfies $0 < s_1 \leq s_1^{1,0} < s_2^{1,0} \leq s_2 < 1$. With the definition of $s_i^{1,0}$ for $i = 1, 2$, we define,

$$\Gamma_1^0 := \{\delta(s) \mid s \in [0, s_1^{1,0}) \cup (s_2^{1,0}, 1]\}.$$

Figure 22 illustrates the extension of Γ_1^0.

We prove the following for Γ_1^0.

Lemma 3.1. *Neighborhoods of points $\rho_1 \in T^*(\Gamma_1^0 \times (0, T))$ do not belong to the support of the measure μ_1.*

Proof:

Since $\Gamma(x_0)$ is an observable region, we consider $\rho_1 = (t, x, \tau_1, \xi_1) \in T^*\left((\Gamma_1^0 \setminus \Gamma(x_0)) \times (0, T)\right)$. The standard elliptic theory implies that $\mu_1 = 0$ near $\rho_1 \in \mathcal{E}$ and the gliding ray of $\partial\Omega$ eventually reaches $\Gamma(x_0)$ so $\mu_1 = 0$

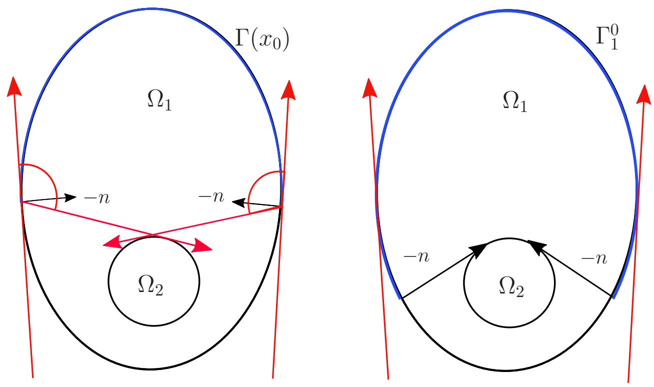

FIGURE 22. *Left*: case where $L(\delta(s_i)) > 0, i = 1, 2$. *Right*: extension of $\Gamma(x_0)$ to Γ_1^0.

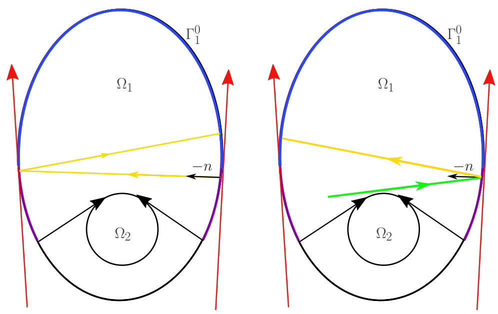

FIGURE 23. *Left*: ray from $x \in \Gamma_1^0 \setminus \Gamma(x_0)$ in the $-n(x)$ direction. *Right*: propagation of the half-ray toward $\Gamma(x_0)$.

near $\rho_1 \in \mathcal{G}^+$. Therefore, suppose $\rho_1 \in \mathcal{H}$. In this case, the conical assumption of $\Gamma(x_0)$ ensures that for every outgoing rays from $\rho_1 = (t, x, \tau_1, -n(x)/c_1) \in \mathcal{H}$, there exists $n \in \mathbb{N}^*$ such that $\Pi_x(\mathcal{F}^n(x, -n(x)/c_1)) \in \Gamma(x_0)$ and the bicharacteristic intersects $\Gamma(x_0)$ in a non-diffractive way. Notice that by construction, no such rays intersects $\partial\Omega_2$. Therefore, by Corollary 2.1, one obtains that $\mu_1 = 0$ near $\rho_1 = (t, x, \tau_1, -n(x)/c_1) \in \mathcal{H}$. In the general case $\rho_1 = (t, x, \tau_1, \xi_1) \in \mathcal{H}$, consider the local geodesic coordinates (x', x_n) such that $x' > 0$ is associated to the direction of parametrisation of $\delta(s)$. Then, for $x = \delta(s)$ such that $s \in [s_1, s_1^{1,0})$ (resp. $s \in (s_2^{1,0}, s_2]$), then either ρ_1 is the half-ray such that $\xi_1' < 0$ (resp. $\xi_1' > 0$) and propagates locally toward $\Gamma(x_0)$, or $\xi_1' > 0$ (resp. $\xi_1' < 0$) and then the second half-ray, according to (2.4), is such that its tangential component is negative (resp. positive) and this half-ray propagates locally toward $\Gamma(x_0)$. The half-ray propagating locally towards $\Gamma(x_0)$ propagates in fact globally towards $\Gamma(x_0)$ since it is located above (with respect to x_0) the ray in the $-n(x)$ direction, which encounters $\Gamma(x_0)$ (see Fig. 23). Therefore this half-ray intersects $\Gamma(x_0)$ non-diffractively and we conclude by Corollary 2.1.

□

Step 2: Definition of Γ_2^0

We define $(s_1^{2,0}, s_2^{2,0}) \in [0,1]$ to be such that,

$$\delta_2(s_i^{2,0}) = \text{argmax}(L(\delta(s_i^{1,0}))),\ i = 1, 2. \tag{3.2}$$

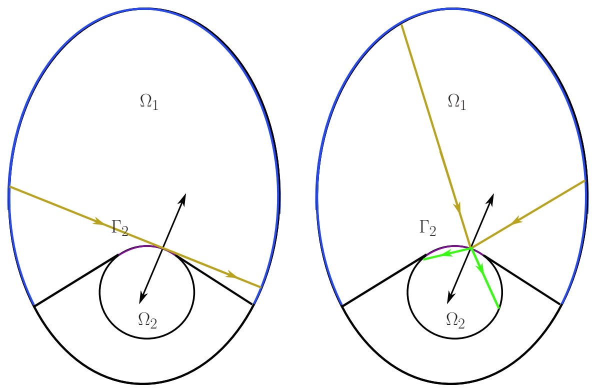

FIGURE 24. *Left*: the two-half rays of $(\rho_1, \rho_2) \in \mathcal{G}^{1,-} \times \mathcal{E}^2$ intersect Γ_1^0 non-diffractively. *Right*: the two half-rays $\gamma_1^{\pm}$ of $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{H}^2$ intersect Γ_1^0 non-diffractively.

Since Ω and Ω_2 are strictly convex, the existence and uniqueness of such points $\delta_2(s_i^{2,0})$ for $i = 1$ and $i = 2$ is ensured. We first prove,

Lemma 3.2. *Let $\delta_2(s_i^{2,0})$ for $i = 1, 2$ be defined by (3.2). Then, $\delta_2(s_1^{2,0}) \neq \delta_2(s_2^{2,0})$.*

Proof:
It follows easily from the strict convexity of Ω_2 in the case where $L(\delta(s_i^{1,0})) > 0$ for $i = 1$ or $i = 2$ so we focus on the the case where $L(\delta(s_i^{1,0})) = 0, i = 1, 2$. But in this case, by definition (3.1) of L, $\Pi_x(\mathcal{F}(\delta(s_i), -n(\delta(s_i))/c_i)) = \delta(s_j), i + j = 3$ and $\Pi_x(\mathcal{F}^2(\delta(s_i), -n(\delta(s_i))/c_i)) = \delta(s_i), i = 1, 2$. Therefore this ray is trapped and is not observed by $\Gamma(x_0)$, which is a contradiction with the definition of $\Gamma(x_0)$ since it satisfies GCC.

□

Since $\delta_2(s_1^{2,0}) \neq \delta_2(s_2^{2,0})$, we reparametrise $\delta_2(s)$ counter-clockwise so that (3.2) holds and such that $0 < s_1^{2,0} < s_2^{2,0} < 1$. We define,

$$\Gamma_2^0 := \{\delta_2(s) \,|\, s \in [0, s_1^{2,0}) \cup (s_2^{2,0}, 1]\}. \tag{3.3}$$

We have from Lemma 3.2 that $\Gamma_2^0 \neq \emptyset$. We prove the following.

Lemma 3.3. *Neighborhoods of points $(\rho_1, \rho_2) \in T^*(\Gamma_2^0 \times (0, T))$ do not belong to the support of the measures μ_1, μ_2.*

Proof:
By construction, the line $l(\delta(s_i^{1,0}), \delta_2(s_i^{2,0}))$ is tangent at $\delta_2(s_i^{2,0}) \in \partial\Omega_2$ for $i = 1, 2$. Moreover, the bicharacteristic starting from $\delta(s_i^{1,0})$ and passing through $\delta_2(s_i^{2,0})$ intersects Γ_1^0 non-diffractively. Therefore, the two half-bicharacteristics $\gamma_1^-(\rho_1)$ and $\gamma_1^+(\rho_1)$ of $(\rho_1, \rho_2) \in \mathcal{G}^{1,-} \times \mathcal{E}^2 \subset T^*(\Gamma_2^0 \times (0, T))$ intersect Γ_1^0 in a non-diffractive way (see Fig. 24). Proposition 2.2 implies that $\mu_1, \mu_2 = 0$ near $(\rho_1, \rho_2) \in \mathcal{G}^{1,-} \times \mathcal{E}^2 \subset T^*(\Gamma_2^0 \times (0, T))$. It also follows from Corollary 2.3 and Corollary 2.5 that $\mu_1, \mu_2 = 0$ near $(\rho_1, \rho_2) \in \mathcal{H}^1 \times (\mathcal{H}^2 \cup \mathcal{G}^{2,+} \cup \mathcal{E}^2) \subset T^*(\Gamma_2^0 \times (0, T))$ since both half-bicharacteristics $\gamma_1^+(\rho_1)$ and $\gamma_1^-(\rho_1)$ intersects Γ_1^0 non-diffractively. Finally, $\mu_1, \mu_2 = 0$ near $(\rho_1, \rho_2) \in \mathcal{E}^1 \times \mathcal{E}^2 \subset T^*(\Gamma_2^0 \times (0, T))$ follows from the classical elliptic theory.

□

Step 3. Extension of Γ_2^{n-1} to Γ_2^n

For $n \in \mathbb{N}$, consider the line $l(\delta(s_1^{1,n-1}), \delta_2(s_2^{2,n-1}))$ (resp. $l(\delta(s_2^{1,n-1}), \delta_2(s_1^{2,n-1}))$). Then there is at most one intersection of this line with $\partial\Omega_2$, aside from $\delta_2(s_2^{2,n-1})$ (resp. $\delta_2(s_2^{1,n-1})$). When it exists, we denote this

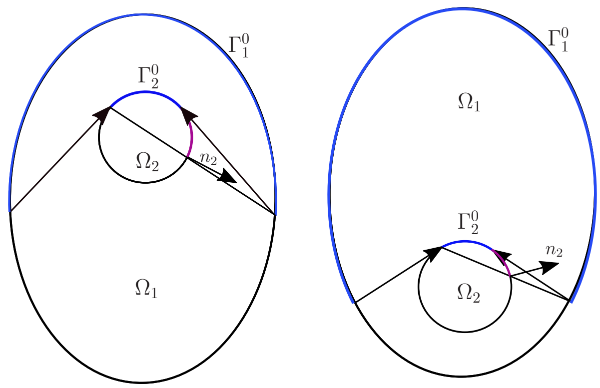

FIGURE 25. Extension of one of the boundary of Γ_2^{n-1} to Γ_2^n in two different cases.

intersection $\delta_2(\check{s}_1^{2,n})$ with $\check{s}_1^{2,n} \in (0,1)$ (resp. $\delta_2(\check{s}_2^{2,n})$ with $\check{s}_2^{2,n} \in (0,1)$). The uniqueness of the intersection is guaranteed by the strict convexity of Ω_2. When such an intersection exists, we extend Γ_2^{n-1} to Γ_2^n such that,

$$\Gamma_2^n := \{\delta_2(s) \,|\, s \in [0, \check{s}_1^{2,n}) \cup (\check{s}_2^{2,n}, 1] \text{ and } \Pi_x(\mathcal{F}(\delta_2(s), n_2(\delta_2(s))) \in \Gamma_1^{n-1}\}, \tag{3.4}$$

and such that $\Gamma_2^n \supseteq \Gamma_2^{n-1}$ is open and connected. The boundary of Γ_2^n is denoted $\delta_2(s_i^{2,n}), 0 < s_1^{2,n} < s_2^{2,n} < 1$. See Figure 25 for a representation of the extension on one side of Γ_2^{n-1}.

We proceed to prove.

Lemma 3.4. *Neighborhoods of points $(\rho_1, \rho_2) \in T^*(\Gamma_2^n \times (0,T)), n \in \mathbb{N}^*$ do not belong to the support of the measures μ_1, μ_2*

Proof:

Consider $\rho_1 = (t, \delta_2(s), \tau_1, \xi_1) \in T^*(\Gamma_2^n \times (0,T))$ with $s_2^{2,n} < s \le s_2^{2,n-1}$. The case $s_1^{2,n-1} \le s < s_1^{2,n}$ is covered in the same fashion. We recall that the case $0 \le s < s_1^{2,n-1}$ and $s_2^{2,n-1} < s \le 1$ were covered in the previous iteration.

By definition of Γ_2^n, $\Pi_x(\mathcal{F}(\delta_2(s), n_2(\delta_2(s))/c_1)) \in \Gamma_1^{n-1}$ and the intersection of the bicharacteristic is non-diffractive. Therefore, the half-bicharacteristic propagating (for small σ) in the positive tangential direction, say γ_1^-, intersects non-diffractively Γ_1^{n-1}. Notice that this is sufficient to conclude the cases $(\rho_1, \rho_2) \in (\mathcal{H}^1 \cup \mathcal{G}^{1,-}) \times \mathcal{E}^2$ by Corollary 2.3. We are also able to conclude the that neighborhood of $(\rho_1, \rho_2) \in \mathcal{H}^1 \times G^{2,+}$ does not belong to the support of μ_1, μ_2 using a continuity argument. Indeed, consider first $x = \delta_2(s_2^{2,n-1})$. At this point, the half ray γ_1^- is observed as proved above. Moreover, the incoming gliding ray from Γ_2^{n-1} is also observed. Therefore, we conclude from Corollary 2.5 that $(\rho_1, \rho_2) \in \mathcal{H}^1 \times G^{2,+}$ does not belong to the support of μ_1, μ_2 for $x = \delta_2(s_2^{2,n-1})$. Then, by continuity of δ_2 and of the gliding ray, one is able to repeat the same argument for $x = \delta_2(s)$ for decreasing values of s starting from $s_2^{2,n-1}$ up until $s_2^{2,n}$. The case $(\rho_1, \rho_2) \in \mathcal{E}^1 \times \mathcal{E}^2$ follows by the standard elliptic theory. The analysis of the remaining case $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{H}^2$ is divided in two cases: either γ_1^+ (recall (2.11)) intersects non-diffractively Γ_1^{n-1} or $\partial\Omega \setminus \Gamma_1^{n-1}$. In the first case, we are able to conclude by Corollary 2.3 that $\mu_1, \mu_2 = 0$ near these points $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{H}^2$ (see Fig. 26, on the left).

Otherwise, consider the line $l(\delta_2(s_1^{2,n-1}), \delta(s_2^{1,n-1}))$. Since γ_1^+ does not intersect Γ_1^{n-1}, the angle θ_1 between γ_1^+ and the normal $n_2(\delta_2(s))$ is strictly larger than the angle θ between the line $l(\delta_2(s_1^{2,n-1}), \delta(s_2^{1,n-1}))$ and $n_2(\delta_2(s))$. The Snell's law implies that $\theta_2 > \theta_1$. Therefore γ_2^-, the half-bicharacteristic propagating in the positive tangential direction for σ small, associated to $\rho_2 = (t, x, \tau_2, \xi_2)$, is confined to the region bounded between the line $l(\delta_2(s_1^{2,n-1}), \delta(s_2^{1,n-1}))$ and $\hat{\Gamma}_2^n := \{\delta_2(s) \,|\, s_1^{2,n-1} < s < s_2^{2,n}\} \subset \Gamma_2^n$ (see Fig. 26, on the right). The half-ray therefore intersects non-diffractively $\hat{\Gamma}_2^n$ at a point $\Pi_x(\mathcal{F}_2(x, \xi_2))$. If $\Pi_x(\mathcal{F}_2(x, \xi_2)) \in \Gamma_2^{n-1}$, then one concludes by Corollary 2.3 that $\mu_1, \mu_2 = 0$ near (ρ_1, ρ_2). Otherwise, $\Pi_x(\mathcal{F}_2(x, \xi_2)) \in \hat{\Gamma}_2^n \setminus \Gamma_2^{n-1}$ in which case

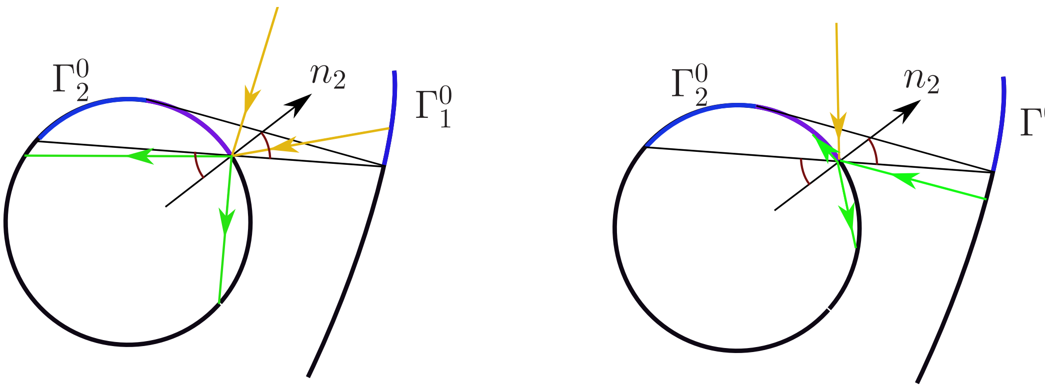

FIGURE 26. *Left*: both half-rays (in yellow) in Ω_1 intersect Γ_1. *Right*: an incoming half-ray from $\partial\Omega_1 \setminus \Gamma_1$. In this case, one of the transmitted half-ray is confined the region bounded by the line $l(\delta_2(s_2^{2,n-1}), \delta(s_1^{1,n-1}))$ and $\hat{\Gamma}_2^n$.

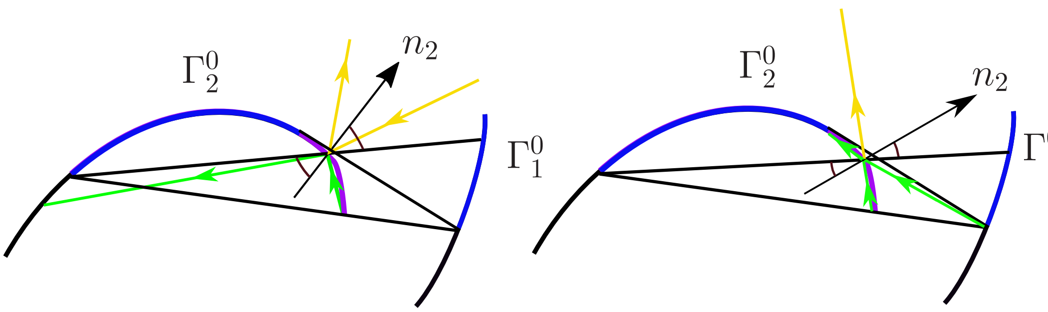

FIGURE 27. *Left*: the incoming half-ray $\gamma_2^+(\rho_{\mathcal{F}_2,2})$ as an angle of incidence smaller or equal than θ'. The two half-rays in Ω_1 therefore intersectΓ_1^{n-1}. *Right*: the incoming half-ray $\gamma_2^+(\rho_{\mathcal{F}_2,2})$ as an angle of incidence smaller or equal than θ'. In this case, only one half-ray of Ω_1 intersects Γ_1^{n-1} and the outgoing half-ray of Ω_2 is restricted to the region bounded by $\hat{\Gamma}_2^n$ and the line $l(\delta_2(s_2^{2,n-1}), \Pi_x(\mathcal{F}_2(x, \xi_2)))$.

we have to carry on with the analysis (which is possible thanks to Corollary 2.6 and γ_1^- intersecting Γ_1^{n-1}). Let us denote $(\rho_{\mathcal{F}_2,1}, \rho_{\mathcal{F}_2,2}) \in \mathcal{H}^1 \times \mathcal{H}^2$ the coordinates associated to the intersection of γ_2^- with $\Gamma_2^n \setminus \Gamma_2^{n-1}$. In this case, $\gamma_2^-(\rho_2) = \gamma_2^+(\rho_{\mathcal{F}_2,2})$.

Consider the line $l(\delta_2(s_1^{2,n-1}), \Pi_x(\mathcal{F}_2(x, \xi_2)))$. By construction, this line intersects Γ_1^{n-1}. Consider $\theta_{\mathcal{F}_2,2}$ the angle of incidence (and therefore the angle of reflection) at $\Pi_x(\mathcal{F}_2(x, \xi_2))$. Again, the analysis relies on a dichotomic argument. If $\theta_{\mathcal{F}_2,2}$ is less or equal than the angle θ' made between $-n_2(\Pi_x(\mathcal{F}_2(x, \xi_2)))$ and the line $l(\delta_1(s_2^{2,n-1}), \Pi_x(\mathcal{F}_2(x, \xi_2)))$, the half-ray $\gamma_2^-(\rho_{\mathcal{F}_2,2})$ intersects $\partial\Omega_2 \setminus \Gamma_2^n$. But in such case, since $\theta_{\mathcal{F}_2,1} < \theta_{\mathcal{F}_2,2}$ where $\theta_{\mathcal{F}_2,1}$ is the angle between $\gamma_1^\pm(\rho_{\mathcal{F}_2,1})$ and $n_2(\Pi_x(\mathcal{F}_2(x, \xi_2)))$, both half-rays $\gamma_1^\pm(\rho_{\mathcal{F}_2,1})$ intersects Γ_1^{n-1} in a non-diffractive way (see Fig. 27, on the left). Therefore we conclude by Corollary 2.6 that $\mu_1, \mu_2 = 0$ near (ρ_1, ρ_2). Otherwise, only $\gamma_1^-(\rho_{\mathcal{F}_2,1})$ intersects Γ_1^{n-1} and $\gamma_2^-(\rho_{\mathcal{F}_2,2})$ intersects $\hat{\Gamma}_2^n$ at a point $\Pi_x(\mathcal{F}_2^2(x, \xi_2))$ (see Fig. 27, on the right). If $\Pi_x(\mathcal{F}_2^2(x, \xi_2)) \in \Gamma_2^{n-1}$, we conclude by Corollary 2.6 that $\mu_1, \mu_2 = 0$ near (ρ_1, ρ_2). Otherwise, $\Pi_x(\mathcal{F}_2^2(x, \xi_2)) \in \hat{\Gamma}_2^n \setminus \Gamma_2^{n-1}$, in which case we iterate the present argument (thanks to Cor. 2.6 and to $\gamma_1^-(\rho_{\mathcal{F}_2,1})$ intersecting Γ_1^{n-1}).

We are able to conclude after a finite number of steps that $\mu_1, \mu_2 = 0$ near (ρ_1, ρ_2) since, by hypothesis, Ω_2 has no contact of infinite order with his boundary. Therefore, there exists $n \in \mathbb{N}^*$

such that $\Pi_x(\mathcal{F}_2^n(x,\xi_2)) \in \Gamma_2^{n-1}$ or $\Pi_x(\mathcal{F}_2^n(x,\xi_2)) \in \partial\Omega_2 \setminus \hat{\Gamma}_2^n$, in which cases $\gamma_1^{\pm}(\rho_{\mathcal{F}_2^n,1})$ intersect Γ_1^{n-1}. □

Step 4: Extension of Γ_1^{n-1} to Γ_1^n

We now extend Γ_1^{n-1} to Γ_1^n. Denote $s_1^{1,n}$ the largest $s \geq s_1^{1,n-1}$ such that,

$$\Pi_x(\mathcal{F}(\delta(s), -n(\delta(s)))) = \delta_2(s_1^{2,n-1}).$$

Likewise, denote $s_2^{1,n}$ the smallest $s \leq s_2^{1,n-1}$ such that,

$$\Pi_x(\mathcal{F}(\delta(s), -n(\delta(s)))) = \delta_2(s_2^{2,n-1}).$$

Then Γ_1^n is the open and connected part of $\partial\Omega$ such that $\Gamma_1^{n-1} \subset \Gamma_1^n$ and $\{\delta(s_1^{1,n}), \delta(s_2^{1,n})\} = \partial\Gamma_1^n$. The proof of Lemma 3.1, with the slight modification that an half-ray can intersect Γ_2^n, gives the proof of Lemma 3.5.

Lemma 3.5. *Neighborhoods of points $\rho_1 \in T^*(\Gamma_1^n \times (0,T))$ do not belong to the support of the measure μ_1.*

Step 5: Iteration until $\Gamma_2^{n-1} = \Gamma_2^n$

We iterate step 3 and 4 until $\Gamma_2^{n-1} = \Gamma_2^n$ for $n \in \mathbb{N}^*$. We begin by proving the following.

Lemma 3.6. *For any Ω and Ω_2 as defined in Section 1.1 and any $\Gamma(x_0)$ be defined as (1.9), there exists $n \in \mathbb{N}$ such that $\Gamma_2^{n-1} = \Gamma_2^n$.*

Proof:
To prove Lemma 3.6, it suffices to prove that there are no accumulation points for the sequence $\{(\delta_2(s_1^{2,n}), \delta_2(s_2^{2,n}))\}_{n\in\mathbb{N}}$, except when,

$$(\delta_2(s_1^{2,n-1}), \delta_2(s_2^{2,n-1})) = (\delta_2(s_1^{2,n}), \delta_2(s_2^{2,n})) \text{ for some } n \in \mathbb{N}^*. \tag{3.5}$$

Denote $\theta_{n,i}, i = 1,2$ the angle between $-n(\delta(s_i^{1,n}))$ and the line $l(\delta(s_i^{1,n}), \delta_2(s_j^{2,n})), j = 3-i$. Then it is easy to see that there exists $\theta_{min} > 0$ such that $\theta_{n,1} + \theta_{n,2} \geq \alpha_{min}, i = 1,2$ and for all $n \in \mathbb{N}$ until (3.5) thanks to the fact that $\Gamma(x_0)$ defines a conical region of $\partial\Omega \setminus \Gamma(x_0)$, that Ω and Ω_2 are strictly convex and that there are uniform lower and upper bound for the distance between $\partial\Omega_2$ and $\partial\Omega \setminus \Gamma(x_0)$.

□

Therefore the iteration process ends for a finite $n \in \mathbb{N}^*$ either because there are no longer intersections between the line $l(\delta(s_1^{1,n-1}), \delta_2(s_2^{2,n-1}))$ and $\partial\Omega_2$ and the line $l(\delta(s_2^{1,n-1}), \delta_2(s_1^{2,n-1}))$ and $\partial\Omega_2$ or because,

$$\Pi_\xi(\mathcal{F}^2(\delta(s_i), -n(\delta(s_i)))) = -n(\delta(s_i)). \tag{3.6}$$

At the end of the iteration process, one defines $\Gamma_1 := \Gamma_1^n$ and $\Gamma_2 := \Gamma_2^n$ and $\delta(s_i^1) \in \partial\Gamma_1, \delta_2(s_i^2) \in \partial\Gamma_2, i = 1,2$ and $s_1^i < s_2^i, i = 1,2$. We highlight at this point that $\Gamma_1 \subsetneq \partial\Omega$ and $\Gamma_2 \subsetneq \partial\Omega_2$. However, we are able to prove Lemma 3.7.

Lemma 3.7. *There exists $x_0^2 \in \mathbb{R}^2 \setminus \overline{\Omega}_2$ such that $\Gamma_2(x_0^2) = \Gamma_2$.*

Proof:
If $\delta(s_i^1) \in \partial\Gamma_1, \delta_2(s_i^2) \in \partial\Gamma_2, i = 1,2$ are such that (3.6), $\delta'(s_i^1) = \delta_2'(s_i^2)$. Since $\Gamma(x_0) \subset \Gamma_1$, then the tangents at $\delta_2(s_i^2), i = 1,2$ must intersect at a point $x_0^2 \in \mathbb{R}^2 \setminus \Omega_2$ such that $\Gamma_2(x_0^2) = \Gamma_2$. If, say $\delta_2(s_1^2)$, is such that there are no intersections between $\partial\Omega_2$ and $l(\delta(s_2^2), \delta_2(s_1^2))$, then this line has an intersection with the tangent line at $\delta_2(s_2^2)$ if $\delta_2(s_2^2)$ satisfies (3.6) or the line $l(\delta(s_1^2), \delta_2(s_2^2))$, defining in either cases $x_0^2 \in \mathbb{R}^2 \setminus \overline{\Omega}_2$.

□

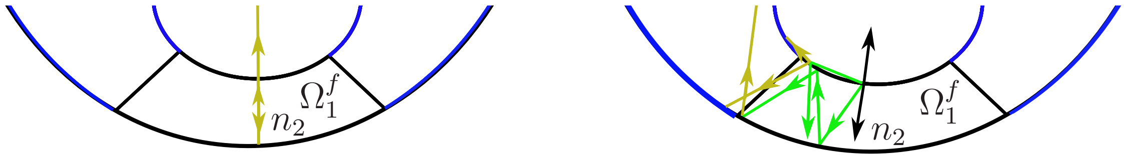

FIGURE 28. *Left*: the ray starting from x in the n_2 direction such that $\mathcal{M}(x, n_2(x)) = 0$ is directly observed when transmitted to Ω_2. *Right*: the propagation of the half-rays in the negative tangential direction from a point x such that $\mathcal{M}(x, n_2(x)) < 0$.

We then define $\Omega_1^f \subset \Omega_1$ the remaining part of the geometry bounded by $\partial\Omega \setminus \Gamma_1$, $\partial\Omega_2 \setminus \Gamma_2$ and the lines $l(\delta(s_i^1), \delta_2(s_i^2)), \delta(s_i^1) \in \partial\Gamma_1, \delta_2(s_i^2) \in \partial\Gamma_2, i = 1, 2$ (see Fig. 11).

Step 6: Uniformly escaping geometry Ω_1^f.

We finally prove,

Lemma 3.8. *Suppose Ω_1^f is a uniformly escaping geometry. Then $\mu_1, \mu_2 = 0$ near $(\rho_1, \rho_2) \in T^*(\Omega_1^f \times (0, T))$.*

When Ω_1^f is a uniformly escaping geometry, it suffices to let the bicharacteristics propagates until they exit Ω_1^f to obtain Lemma 3.8. We recall the definition (1.12) of $\mathcal{M}$.
Proof:

We first begin by describing the dynamic of $\rho_1 = (x, t, \tau_1, \xi_1) \in \mathcal{H}^1 \cap T^*((\partial\Omega_2 \setminus \Gamma_2) \times (0, T))$, without assumptions on $c_1 < c_2$. Notice that, under the uniformly escaping geometry assumption, in the region $\mathcal{M}(x, \xi) > 0$ (resp. $\mathcal{M}(x, \xi) < 0$), the half-ray propagating in the direction of parametrisation (resp. in the opposite direction) exits Ω_1^f through the line $l(\delta(s_2^1), \delta_2(s_2^2))$ (resp. $l(\delta(s_1^1), \delta_2(s_1^2))$). Indeed, it suffices to notice that in this region, $\mathcal{M}(x, n_2(x)/c_1) > 0$ exits Ω_1^f through through the line $l(\delta(s_2^1), \delta_2(s_2^2))$ (resp. $l(\delta(s_1^1), \delta_2(s_1^2))$), thanks to the monotonicity of $\mathcal{M}(x, n_2(x))$. We are therefore able to conclude on the observability of $(\rho_1, \rho_2) \in (\mathcal{H}^1 \cup \mathcal{G}^{1,-}) \times \mathcal{E}^2$ where $\mathcal{M}(x, n_2(x)) > 0$ or $\mathcal{M}(x, n_2(x)) < 0$. Now consider the region where $\mathcal{M}(x, n_2(x)) = 0$. By definition of $\mathcal{M}$, $\mathcal{F}^2(x, n_2(x)) = (x, n_2(x))$. Moreover, using Lemma 3.7, we obtain that $\Pi_x(\mathcal{F}_2(x, -n_2(x))) \in \Gamma_2$ which allows to conclude that $\mu_1, \mu_2 = 0$ near $\rho_1 = (t, x, \tau_1, n_2(x)/c_1), \rho_2 = (t, x, \tau_2, -n_2(x)/c_2)$ (see Fig. 28, on the left). In the case $(\rho_1, \rho_2) \in (\mathcal{H}^1 \cup \mathcal{G}^{1,-}) \times \mathcal{E}^2$ such that $\mathcal{M}(x, n_2(x)) = 0$ and $\xi_1 \neq n_2(x)/c_1$, the half-rays escape Ω_1^f through the lines $l(\delta(s_i^1), \delta_2(s_i^2))$.

Now let $(\rho_1, \rho_2) \in \mathcal{H}^1 \times \mathcal{H}^2$ and, without loss of generality, suppose $\mathcal{M}(x, \xi_1) > 0$. Consider $\rho_2 = (t, x, \tau_2, \xi_2) \in \mathcal{H}^2$ and denote the half-ray γ_2^+ outgoing from $\rho_2 = (t, x, \tau_2, \xi_2)$ and moving, locally in σ, in the direction of the parametrisation. If $\Pi_x(\mathcal{F}(x, \xi_1)) \in \Gamma_1$ and $\Pi_x(\mathcal{F}_2(x, \xi_2)) \in \Gamma_2$, then we conclude that $\mu_1, \mu_2 = 0$ near (ρ_1, ρ_2) by Corollary 2.3. Otherwise, $\Pi_x(\mathcal{F}(x, \xi_1)) \in \partial\Omega \setminus \Gamma_1$ or $\Pi_x(\mathcal{F}_2(x, \xi_2)) \in \partial\Omega_2 \setminus \Gamma_2$ and one needs to carry on with the analysis (see Fig. 28, on the right).

We describe first the situation for $\Pi_x(\mathcal{F}_2(x, \xi_2)) \in \partial\Omega_2 \setminus \Gamma_2$. The intersection is non-diffractive at this point and we denote $(\rho_{\mathcal{F}_2,1}, \rho_{\mathcal{F}_2,2}) \in \mathcal{H}^1 \times \mathcal{H}^2$. At this point, we follow two-half rays, the outgoing half-ray $\gamma_2^+(\rho_{\mathcal{F}_2,2})$ (recall that $\gamma_2^+(\rho_2) = \gamma_2^-(\rho_{\mathcal{F}_2,2})$) and $\gamma_1^+\rho_{\mathcal{F}_2,2})$, the half-ray, locally, in the same direction of propagation than $\gamma_2^+(\rho_{\mathcal{F}_2,2})$. We recall that the geometrical configuration may be so that $\mathcal{M}(\Pi_x(\mathcal{F}_2(x, \xi_2)), n_2(\Pi_x(\mathcal{F}_2(x, \xi_2)))) \leq 0$, but the direction of propagation locally of both rays ensure that they propagate toward Γ_1, as seen in the previous paragraph, and Γ_2. Hence, from Lemma 3.7, we conclude that there exists $n^* \in \mathbb{N}$ such that $\Pi_x(\mathcal{F}_2^{n^*}(x, \xi_2)) \in \Gamma_2$.

Let us now describe the situation for $\Pi_x(\mathcal{F}(x, \xi_1)) \in \partial\Omega \setminus \Gamma_1$. Since we are considering the case $\mathcal{M}(x, \xi_1) > 0$, γ_1^+ move, not only locally but globally, in the direction of the parametrisation. If $\Pi_x(\mathcal{F}^2(x, \xi_1)) \in \partial\Omega_2 \setminus \Gamma_2$, then the (possibly) transmitted ray of Ω_2 falls in the description of the previous paragraph and the reflected ray $\gamma_1^+(\rho_{\mathcal{F}^2(x,\xi_1),1})$ propagate in the direction of the parametrisation. Otherwise, if $\Pi_x(\mathcal{F}^2(x, \xi_1)) \in \partial\Omega \setminus \Gamma_1$, then the monotonicity of $\mathcal{M}$ ensures that the outgoing ray $\gamma_1^+(\rho_{\mathcal{F}^2(x,\xi_1),1})$ propagates in the direction of the parametrisation. We conclude that there exists $n' \in \mathbb{N}$ such that $\Pi_x(\mathcal{F}^{n'}(x,\xi_1)) \in \Gamma_1$.

We are then able to conclude thanks to Corollary 2.6. Indeed, for $(\rho_1,\rho_2) \in \mathcal{H}^1 \times \mathcal{H}^2$, then one has to follow the ray given by the previous description according to the sign of $\mathcal{M}$. When the ray intersects $\partial\Omega_2 \setminus \Gamma_2$, one has to follow the new rays γ_1^+ or γ_2^+ which propagates according to the sign of $\mathcal{M}$. There exist a uniform time of observability for these rays from Lemma 3.7, the monotonicity of $\mathcal{M}$ and the assumption that $\partial\Omega$ and $\partial\Omega_2$ has no contact of order $k-1$ with its tangents.

We finally turn to the case $(\rho_1,\rho_2) \in \mathcal{H}^1 \times \mathcal{G}^{2,+}$, the case $(\rho_1,\rho_2) \in \mathcal{E}^1 \times \mathcal{E}^2$ being deduced from the classical elliptic theory. In this case, one uses the continuity argument in the proof of Lemma 3.4 for the gliding ray along $\partial\Omega_2$ to propagate, point by point, the observability of $\gamma_2^- \in \Gamma_2$ to $\partial\Omega_2 \setminus \Gamma_2$. More precisely, define $s_1^2 \leq \tilde{s}_1^2 \leq s_2^2$ (resp. $s_1^2 \leq \tilde{s}_2^2 \leq s_2^2$) such that $\mathcal{M}(\delta_2(s), n_2(\delta_2(s))) \leq 0$ for $s \in [s_1^2, \tilde{s}_1^2]$ (resp. $\mathcal{M}(\delta_2(s), n_2(\delta_2(s))) \leq 0$ for $s \in [\tilde{s}_2^2, s_2^2]$). We concentrate on the case $s \in [\tilde{s}_2^2, s_2^2]$, the other case being treated similarly. We explain how to adapt the argument of Lemma 3.4. We first prove that for $s = s_2^2$ and neighborhood of $(\rho_1,\rho_2) \in \mathcal{H}^1 \times \mathcal{G}^{2,+}$ does not intersect the support of μ_1 and μ_2. Indeed, it suffices to remark that the incoming gliding ray γ_2^- comes from Γ_2, where it was proved that $\gamma_2^- \cap \operatorname{supp}(\mu_2) = \emptyset$. On the other hand, notice that the incoming ray γ_1^-, comes from Γ_1 since $c_2 > c_1$ and, by definition, the ray from $\delta_2(s_2^2)$ intersects $\gamma(s_2^1)$. Therefore $\gamma_1^- \cap \operatorname{supp}(\mu_1) = \emptyset$ and Corollary 2.5 yields the desired result in this particular case. Now, notice that this argument still holds by continuity for $s \in (\check{s}_2^2, s_2^2]$ where $\check{s}_2^2$ is such that the incoming ray γ_1^- at the critical angle $\theta_1 = \arcsin(c_1/c_2)$ intersects $\delta(s_2^1)$. Indeed, for such s, $\gamma_1^- \cap \operatorname{supp}(\mu_1) = \emptyset$ and by starting with $s = s_2^2$ and decreasing s up until $s = \check{s}_2^2$, one can use the previous points to prove that $\gamma_2^- \cap \operatorname{supp}(\mu_1) = \emptyset$ using the continuity of the defect measure [12]. For $s \in [\tilde{s}_2^2, \check{s}_2^2]$ we use a similar argument. Indeed, the incoming ray γ_1^- does not intersect directly Γ_1, but exits Ω_1^f through the line $l(\delta(s_2^1), \delta_2(s_2^2))$ under the UEG condition. If the ray γ_1^- only have reflections over $\partial\Omega$ before exiting Ω_1^f, then one obtains that $\gamma_1^- \cap \operatorname{supp}(\mu_1) = \emptyset$. Otherwise, the ray γ_1^- may intersect $\partial\Omega_2 \setminus \Gamma_2$ either at points $(\mathcal{H}^1 \times \mathcal{H}^2) \cup (\mathcal{G}^{1,-} \times \mathcal{E}^2)$, for which the observability was already deduced above (and therefore $\gamma_1^- \cap \operatorname{supp}(\mu_1) = \emptyset$) or at a point $\mathcal{H}^1 \times \mathcal{G}^{2,+}$. It suffices to use the same continuity argument by starting with $s = \check{s}_2^2$ until $s = \tilde{s}_2^2$. The key here is that if there exists $\hat{s} \in [\tilde{s}_2^2, \check{s}_2^2]$ such that γ_1^- has a $\mathcal{H}^1 \times \mathcal{G}^{2,+}$ contact, then it is for a $s > \hat{s}$, for which the observability was already proven. This recursion argument allows to prove, using Corollary 2.5, that neighborhoods of $(\rho_1,\rho_2) \in \mathcal{H}^1 \times \mathcal{G}^{2,+}$ does not intersect the support of μ_1 and μ_2.

□

We prove Theorem 1.7

Proof:

We follow the proof of Theorem 1.5 until Step 6.

Step 6: non-uniformly escaping geometry Ω_1^f

The remaining case is the event where Ω_1^f is a trapping region. In such case, c_1, c_2 and Ω_1^f are assumed to satisfy the following: for every $\rho = (t,x,\tau,\xi) \in T^*((\partial\Omega \setminus \partial\Omega_1) \times (0,T))$, $\Pi_x(\mathcal{F}(x,\xi)) \in \partial\Omega_2 \setminus \Gamma_2$ then $\mathcal{F}_2^{\pm}(\mathcal{F}(x,\xi)) \in \Gamma_2$. Therefore, $\mu_1 = 0$ near ρ and $\mu_1, \mu_2 = 0$ near $(\rho_1,\rho_2) \in T^*(\partial\Omega \setminus \Gamma_1) \times T^*(\partial\Omega_2 \setminus \Gamma_2)$ as $\mathcal{F}^{-1}(x,\xi_1) \in \Gamma_1$ for ρ_1 not described by the previous analysis.

□

4. Unexpected observability regions using the integrability of the ellipse

Consider Ω to be an ellipse of focii F_1 and F_2. We assume that the center of the ellipse is on $(0,0)$ and that the major axis lies on the x axis. We denote the focii with the natural notations $F_1 = (c,0)$ and $F_2 = (-c,0), c > 0$, the endpoints of the major axis $(-a,0)$ and $(a,0)$ and the endpoints of the minor axis $(0,-b)$ and $(0,b)$, with $a, b > 0$ and $c = \sqrt{a^2 - b^2}$. Let $x_1 \in \partial\Omega$ (without loss of generality we assume $x_1 = (x_1^1, x_1^2)$ such that $x_1^1, x_2^2 < 0$). Then, consider $x_2 \in \partial\Omega, x_1 \neq x_2$ to be the only point intersected by the line $l(x_1, F_1)$. Then,

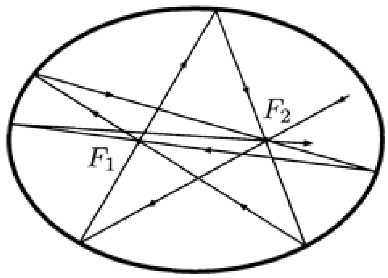

FIGURE 29. Example of a ray converging to the main axis ([5]).

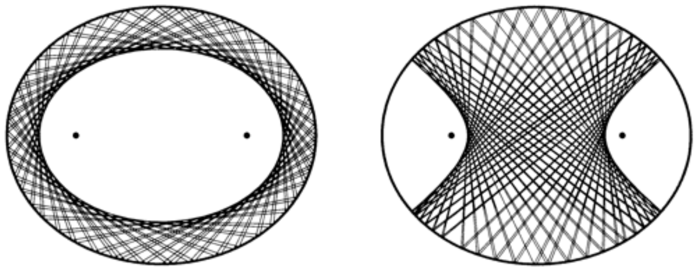

FIGURE 30. *Left*: examples of elliptic caustics. *Right*: examples of hyperbolic caustics ([5]).

Lemma 4.1. *Every $\Gamma \subset \partial\Omega$ open and connected including x_1 and x_2 satisfy GCC.*

It is easy to see that if Lenght(Γ) is stricly larger than half the perimeter of Ω, then Γ may be written of the form $\Gamma(x_0)$ for $x_0 \in \mathbb{R}^2 \setminus \overline{\Omega}$. The case where Lenght($\Gamma$) is less or equal than the perimeter of Ω is the new part of the proof. The proof relies on the dynamic of the billiards. Proof of these dynamics may be found in [5].

Proof:

Major and minor axis

The bicharacteristics travelling along the major and minor axis are periodic ones, as they go back and forth on these axis. Since one $(a, 0), (-b, 0) \in \Gamma$, these rays are observed.

Bicharacteristics going through one foci

A bicharacteristic going through one foci bounces off $\partial\Omega$ and go through the other foci. Every such bicharacteristics converge to the major axis and are therefore observed by Γ (see Fig. 29).

Hyperbolic caustics

Every bicharacteristics that cross the line between F_1 and F_2 will cross this line again after bouncing off $\partial\Omega$. Such a bicharacteristic has all his rays tangential to an hyperbol (see Fig. 30 on the right). Since $\{(x, y) \in \partial\Omega \,|\, x \geq 0, y \leq 0, \} \subset \Gamma$, it is easy to see that every such bicharacteristics have to intersect Γ in some time $T > 0$.

Elliptic caustics

Every bicharacteristics that do not cross the line between F_1 and F_2 and the focii won't cross the line again after bouncing off $\partial\Omega$. Moreover, each such rays is tangential to an ellipse of focii F_1 and F_2 with smaller minor and major axis (see Fig. 30 on the left). Notice first that every elliptic caustic starting of $\partial\Omega \cap \{y \leq 0\}$ and crossing the line $l(F_1,(a,0))$ either starts from Γ or intersects Γ. Then, it is easy to see that every bicharacteristics starting from $\partial\Omega \cap \{y \geq 0\}$ and crossing the line $l(F_2,(-a,0))$ has to intersect Γ.

Gliding ray

The gliding ray intersects Γ in some time $T > 0$.

□

Remark 4.2. Notice that when $F_1 \to F_2$, one recovers that GCC is equivalent to $\Gamma(x_0)$ for the circle if Γ is connected.

5. Conclusion

The use of domains in $\mathbb{R}^2$ was crucial in our analysis. In higher dimension, the proof does not follow immediately. Indeed, one uses in step 3 of the proof of Theorem 1.5 the fact that the transmitted rays in Ω_2 are confined in a certain region. This ensures that the rays propagating in this region either intersects Γ_2^1 or exit this region, in which case we obtain observability. This is no longer the case in dimension 3. One can only prove that the rays move toward Γ_2^1 but one can construct examples where the rays intersect regions of $\partial\Omega_2 \setminus \partial\Gamma_2^1$ before reaching Γ_2^1. One can circumvent this issue if one considers domains in $\mathbb{R}^3$ obtained by the revolution around an axis of strictly convex curves and by considering $\Gamma(x_0)$ such that x_0 lies along the axis of revolution. The symmetry allows one to use the proof of Theorem 1.5 straightforwardly. In the general case however one needs to analyse the rays transmitted back to Ω_1 through $\partial\Omega_2 \setminus \partial\Gamma_2^1$.

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