ASYMPTOTIC ANALYSIS OF A FAMILY OF NON-LOCAL FUNCTIONALS ON SETS

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Abstract. We study the asymptotic behavior of a family of functionals which penalize a short-range interaction of convolution type between a finite perimeter set and its complement. We first compute the pointwise limit and we obtain a lower estimate on more regular sets. Finally, some examples are discussed.

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1. Introduction

In this paper we study the asymptotic behavior, as \( \varepsilon \to 0 \), of the family of functionals

\[
\mathcal{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{E \cap \Omega} f(G_\varepsilon * \chi_{E \cap \Omega}) \, dx.
\]

where \( \Omega \subset \mathbb{R}^N \) is open and bounded, \( N > 1 \), \( E \) is a set of finite perimeter in \( \Omega \), \( f \) is given and \( G_\varepsilon(z) = \frac{1}{\varepsilon^N} G(z/\varepsilon) \) where \( G \) is a suitable kernel. Our analysis has been inspired by a paper by Miranda et al. in [8] where the case \( f(t) = t \) is considered and \( G \) is the Gauss-Weierstrass kernel, namely \( G(z) = \frac{1}{(4\pi)^{N/2}} e^{-|z|^2/4} \) (see also [6, 7] for smoother sets and [1] for similar convolution approximation). More precisely, in [8] it is proven that the pointwise limit is, up to a constant, the perimeter of \( E \) in \( \Omega \). A more general kernel \( G \) has been investigated, in the context of optimal partition problems, by Esedo\'glu and Otto [5] where \( G \) is smooth and non-negative, radially symmetric and satisfying the following conditions:

\[
\int_{\mathbb{R}^N} G(z) \, dz = 1, \quad \int_{\mathbb{R}^N} |z|G(z) \, dz < +\infty, \quad |\nabla G(z)| \lesssim G \left( \frac{z}{2} \right), \quad \nabla G(z) \cdot z \leq 0.
\]

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On the other hand, as in [5], Esedoḡlu and Otto consider only the case \( f(t) = t \), but they prove a complete \( \Gamma \)-convergence result for the family \( \{ \mathcal{F}_\varepsilon \}_{\varepsilon > 0} \) on finite perimeter sets with respect to the strong \( L^1 \)-convergence. We point out that for the simpler class of functionals with \( f(t) = t \) there is a much simpler proof of gamma convergence in \( L^1 \) subsequently given by Esedoḡlu and Jacobs in [4]. In particular, they deal with more general convolution kernels, which in particular can be non-radially symmetric, thus including anisotropy, and even change sign to a certain extent, which turns out to be necessary for the approximation of certain anisotropies. A very similar result has been obtained more recently by Berendsen and Pagliari [3]. As far as we know, the last result is due to Pagliari [9] where he essentially remove the radial symmetry of \( G \). The simplest case \( f \) then is the reduced boundary of \( E \) and \( \nu_E(x) \) is the outer unit normal at \( E \) in view to have a \( \Gamma \)-convergence result we investigate also the lower estimate. Unfortunately, the technique of Esedoḡlu and Otto is the reduced boundary of \( E \) and \( \nu_E(x) \) is the outer unit normal at \( E \). In view to have a \( \Gamma \)-convergence result we investigate also the lower estimate. Unfortunately, the technique of Esedoḡlu and Otto [5] does not work in our situation: it is crucial for them to switch the order of integration, that is impossible for us since we have \( f \) between the exterior integral and the convolution one. It seems that this difficulty cannot be easily overcome in the general situation. We are able to show (see Thm. 3.2) a \( \Gamma \)-liminf inequality only on graphs of \( C^1 \) functions with respect to the \( C^1 \)-uniform convergence. Actually, it is easy to generalize such a inequality in the case of sets which are locally graphs of \( C^1 \) functions with respect to a suitable convergence (see Rem. 3.3). Finally, we also prove (see Thm. 3.6) that if \( f \) is also convex, then the pointwise limit is lower semicontinuous with respect to the strong \( L^1 \)-convergence, which suggests that for \( f \) convex the \( \Gamma \)-limit in the strong \( L^1 \)-convergence should be the pointwise limit. At the end of the paper we will also discuss some examples.

2. Notation and preliminaries

2.1. Notation

In what follows \( N \in \mathbb{N} \) with \( N \geq 1 \). For any \( r > 0 \) and \( x \in \mathbb{R}^N \) the notation \( B^d_r(x) \) stands for the open ball in \( \mathbb{R}^d \) centered at \( x \) with radius \( r \), while \( \mathbb{S}^{N-1} = \partial B^1_r(0) \). If \( A \subseteq \mathbb{R}^N \) we also denote by \( \mathcal{H}^k(A) \) the Hausdorff measure of \( A \) of dimension \( k \in \{0,1, \ldots, N\} \) (\( \mathcal{H}^0 \) is the counting measure). If \( A_h, A \) are measurable subsets of \( \mathbb{R}^N \), then \( A_h \rightarrow A \) in \( L^1(\mathbb{R}^N) \) (or \( L^1_{\text{loc}}(\mathbb{R}^N) \)) means that \( \chi_{A_h} \rightarrow \chi_A \) in \( L^1(\mathbb{R}^N) \) (respectively \( L^1_{\text{loc}}(\mathbb{R}^N) \)). Finally, for any \( A \subseteq \mathbb{R}^N \) we let \( A^c = \mathbb{R}^N \setminus A \).

2.2. Finite perimeter sets

We recall some notion on finite perimeter sets in euclidean space; for details we refer to [2]. Let \( \Omega \) be an open subset of \( \mathbb{R}^N \). A measurable set \( E \subseteq \mathbb{R}^N \) is said to be a set of finite perimeter in \( \Omega \) if

\[
\mathcal{P}(E, \Omega) = \sup \left\{ \int_E \operatorname{div} X(x) \, dx : X \in C^1_c(\Omega; \mathbb{R}^N), \|X\|_{\infty} \leq 1 \right\} < +\infty.
\]
The quantity $P(E, \Omega)$ is called the \textit{perimeter of $E$ in $\Omega$}. Finite perimeter sets have nice boundary in a measure theoretical sense. Precisely, one can define a subset of $E$ as the set of points $x$ where there exists a unit vector $\nu_E(x)$ such that:
\[
\frac{x - E}{r} \to \{ y \in \mathbb{R}^N : y \cdot \nu_E(x) \geq 0 \}, \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N) \text{ as } r \to 0,
\]
and which is referred to as the \textit{outer normal to $E$ at $x$}. The set where $\nu_E(x)$ exists is called the \textit{reduced boundary of $E$} and is denoted by $\partial^* E$. It turns out that, for any $E$ set of finite perimeter in $\Omega$, we have $P(E, \Omega) = \mathcal{H}^{N-1}(\partial^* E \cap \Omega)$. The reduced boundary of $E$ plays the role of the topological boundary also in the sense of the integration by parts. Indeed, one can show that, if $E$ is a set of finite perimeter in $\Omega$, then the following Gauss-Green formula holds true:
\[
\int_{E} \text{div} \, X(x) \, dx = \int_{\partial^* E} X(x) \cdot \nu_E(x) \, d\mathcal{H}^{N-1}(x), \quad \forall X \in C^1_c(\Omega; \mathbb{R}^N). \tag{2.2}
\]

Finite perimeter sets satisfy good properties for the Calculus of Variations: for instance, if $E_h, E$ have finite perimeter in $\Omega$ and $E_h \overset{L^1_1}{\to} E$, then
\[
P(E, \Omega) \leq \liminf_{h \to +\infty} P(E_h, \Omega).
\]

\section*{3. Setting of the problem and main results}

Let $N > 1$, let $G: \mathbb{R}^N \to [0, +\infty)$ be of class $C^\infty$ such that
\[
supp G = \overline{B^N(0)}, \quad G(-x) = G(x), \quad \int_{\mathbb{R}^N} G(x) \, dx = 1.
\]

For any $\varepsilon > 0$ and for any $x \in \mathbb{R}^N$, let
\[
G_\varepsilon(x) = \frac{1}{\varepsilon^N} G \left( \frac{x}{\varepsilon} \right).
\]

We consider a continuous and non-decreasing function $f: [0, +\infty) \to \mathbb{R}$ with $f(0) = 0$. Let $\Omega \subset \mathbb{R}^N$ be open bounded. We denote by $\mathcal{P}_N(\Omega)$ the set of all sets of finite perimeter in $\Omega$. For any $\varepsilon > 0$, we introduce the functional $\mathcal{F}_\varepsilon: \mathcal{P}_N(\Omega) \to [0, +\infty)$ defined by
\[
\mathcal{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{E \cap \Omega} f(G_\varepsilon * \chi_{E \cap \Omega}) \, dx. \tag{3.1}
\]

In order to state our main results, we introduce the function $\theta: S^{N-1} \to [0, +\infty)$ given by
\[
\theta(\nu) = \int_0^1 f \left( \int_{\{x \cdot \nu \geq t\}} G(x) \, dx \right) \, dt. \tag{3.2}
\]

Let $\mathcal{F}: \mathcal{P}_N(\Omega) \to [0, +\infty)$ be the functional given by
\[
\mathcal{F}(E) = \int_{\partial^* E \cap \Omega} \theta(\nu_E(x)) \, d\mathcal{H}^{N-1}(x).
\]
Our first main result concerns the pointwise limit of $F_{\varepsilon}$ on $\mathcal{P}_N(\Omega)$.

**Theorem 3.1. (Pointwise limit)** Assume $f$ of class $C^1$. Let $E \in \mathcal{P}_N(\Omega)$. Then

$$\lim_{\varepsilon \to 0} F_{\varepsilon}(E) = F(E).$$

On the other hand, we are also able to prove a lower estimate on graphs.

**Theorem 3.2. (Lower estimate)** Let $D \subset \mathbb{R}^{N-1}$ be open and bounded with Lipschitz boundary, let $u_h, u \in C^{1,1}(D)$, with $u_h, u > 0$ on $D$ such that $u_h \to u$ uniformly in $C^1(D)$. Let $E_h, E$ be given by

$$E_h = \{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} : x \in D, \ 0 \leq y \leq u_h(x)\},$$

$$E = \{(x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} : x \in D, \ 0 \leq y \leq u(x)\}.$$

Then, for any positive infinitesimal sequence $(\varepsilon_h)$ it holds

$$\liminf_{h \to +\infty} F_{\varepsilon_h}(E_h) \geq F(E).$$

**Remark 3.3.** It is not difficult to see that Theorem 3.2 can be generalized to uniformly $C^{1,1}$-regular sets in $\Omega$ with respect to a suitable notion of uniform convergence. Precisely, a set $E \subset \mathbb{R}^N$ is said to be uniformly $C^{1,1}$-regular set in $\Omega$ if there exist $L, \delta > 0$ such that for every $x \in \partial E \cap \Omega$ there exist $D_x \subset \mathbb{R}^{N-1}$ open and a function $u_x \in C^{1,1}(D_x)$ such that:

- $\partial E \cap \Omega \cap B_\delta(x)$ is the graph of $u^x$;
- $\|\nabla u^x\|_{\infty} \leq L$.

On the set of all uniformly $C^{1,1}$-regular sets in $\Omega$ we put a convergence of sequences. Precisely, we say that $E_h$ converges to $E$ if there exist $\delta, L > 0$ such that for every $x \in \partial E \cap \Omega$ there exist $D^x \subset \mathbb{R}^{N-1}$ open and functions $u^x_h, u^x \in C^{1,1}(D^x)$ such that:

- $\partial E_h \cap \Omega \cap B_\delta(x), \partial E \cap \Omega \cap B_\delta(x)$ are the graphs of $u^x_h, u^x$ respectively;
- $\|\nabla u^x_h\|_{\infty} \leq L$ and $\|\nabla u^x\|_{\infty} \leq L$;
- $u^x_h \to u^x$ uniformly in $C^1(D^x)$.

It is easy to see that with respect to this type of convergence the lower estimate

$$\liminf_{h \to +\infty} F_{\varepsilon_h}(E_h) \geq F(E)$$

follows as a simple consequence of Theorem 3.2.

Combining Theorem 3.1 with Theorem 3.2 and Remark 3.3 we eventually obtain a $\Gamma$-convergence result.

**Corollary 3.4.** The family $\{F_{\varepsilon}\}_{\varepsilon > 0}$ $\Gamma$-converges to $F$ as $\varepsilon \to 0$ on uniformly $C^{1,1}$-regular sets with respect to the convergence introduced in Remark 3.3.

**Remark 3.5.** We do not expect compactness of equibounded sequences of uniformly $C^{1,1}$-regular sets. Nevertheless, at least if $f(t) \geq mt$ for some $m > 0$, equibounded sequences are compact in $L^1$. Indeed, if $(\varepsilon_h)$ is a
positive and infinitesimal sequence and \((E_h)\) be a sequence in \(\mathcal{P}_N(\Omega)\) with \(F_{\varepsilon}(E_h) \leq c\) for some \(c \geq 0\), we get
\[
c \geq F_{\varepsilon}(E_h) \geq \frac{m}{\varepsilon} \int_{E_h \cap \Omega} G_{\varepsilon} * \chi_{E_h \cap \Omega} \, dx
\]
and the compactness follows by Lemma A.4 of [5] (see also [1], Thm. 3.1).

The next and last result suggests that the \(\Gamma\)-limit on \(\mathcal{P}_N(\Omega)\) of the family \((F_{\varepsilon})_{\varepsilon > 0}\) with respect to the \(L^1\)-convergence could be really \(F\), at least if \(f\) is convex.

**Theorem 3.6.** If \(f\) is convex then the functional \(F : \mathcal{P}_N(\Omega) \to \mathbb{R}\) is lower semicontinuous with respect to the \(L^1\)-topology.

### 4. The Pointwise Limit

In this section we prove Theorem 3.1. The main idea comes from the technique used in [8]. We divide the proof in some steps.

**Step 1.** We claim that for any \(E \in \mathcal{P}_N\) we have
\[
F_{\varepsilon}(E) = \frac{1}{\varepsilon} \int_{\partial^* E} \int_0^\varepsilon X(\eta, x) \cdot \nu_E(x) \, d\eta \, d\mathcal{H}^{N-1}(x),
\] (4.1)

where for any \(\eta > 0\) and for any \(x \in \partial^* E\)
\[
X(\eta, x) = \frac{1}{\eta^N} \int_{E_{\varepsilon}} f'(G_{\eta} * \chi_{E}(y)) G \left( \frac{y - x}{\eta} \right) \frac{y - x}{\eta} \, dy.
\]

For any \(\eta > 0\) and any \(y \in \mathbb{R}^N\) we have, using the Gauss-Green formula (2.2),
\[
\frac{d}{d\eta} f(G_{\eta} * \chi_{E}(y)) = -f'(G_{\eta} * \chi_{E}(y)) \frac{1}{\eta^{N+1}} \int_{\mathbb{R}^N} \left( NG \left( \frac{y - x}{\eta} \right) + \nabla G \left( \frac{y - x}{\eta} \right) \cdot \frac{y - x}{\eta} \right) \chi_{E}(x) \, dx
\]
\[
= f'(G_{\eta} * \chi_{E}(y)) \frac{1}{\eta^N} \int_{\mathbb{R}^N} \text{div}_x \left( G \left( \frac{y - x}{\eta} \right) \frac{y - x}{\eta} \right) \chi_{E}(x) \, dx
\]
\[
= f'(G_{\eta} * \chi_{E}(y)) \frac{1}{\eta^N} \int_{\partial^* E} G \left( \frac{y - x}{\eta} \right) \frac{y - x}{\eta} \cdot \nu_E(x) \, d\mathcal{H}^{N-1}(x).
\]

Now notice that, since \(G_{\varepsilon} * \chi_{E} \to \chi_{E}\) in \(L^1(\mathbb{R}^N)\) as \(\varepsilon \to 0\), we can say that for any \(y \in \mathbb{R}^N\)
\[
f(G_{\varepsilon} * \chi_{E}(y)) - f(\chi_{E}(y)) = \int_0^\varepsilon \frac{d}{d\eta} f(G_{\eta} * \chi_{E}(y)) \, d\eta,
\]
from which we get, using the fact that \( f(0) = 0 \),

\[
\mathcal{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{E^c} f(G_\varepsilon * \chi_E(y)) - f(\chi_E(y)) \, dy \\
= \frac{1}{\varepsilon} \int_{E^c} \int_0^\varepsilon \frac{d}{d\eta} f(G_\eta * \chi_E(y)) \, d\eta \, dy \\
= \frac{1}{\varepsilon} \int_{\partial E} \int_0^\varepsilon \frac{1}{\varepsilon^N} \int_{E^c} f'(G_\eta * \chi_E(y)) G\left(\frac{y - x}{\eta}\right) \frac{y - x}{\eta} \, dy \, d\eta \cdot \nu_E(x) \, d\mathcal{H}^{N-1}(x) \\
= \frac{1}{\varepsilon} \int_{\partial E} \int_0^\varepsilon X(\eta, x) \cdot \nu_E(x) \, d\eta \, d\mathcal{H}^{N-1}(x)
\]

hence (4.1).

**Step 2.** We claim that for any \( x \in \partial^* E \) we have

\[
\lim_{\varepsilon \to 0} X(\varepsilon, x) = \int_{\{z : \nu_E(x) \geq 0\}} f' \left( \int_{\{v - z : \nu_E(x) \geq 0\}} G(v) \, dv \right) G(z) \, dz. \tag{4.2}
\]

First of all we have

\[
X(\varepsilon, x) = \frac{1}{\varepsilon^N} \int_{E^c} f'(G_\varepsilon * \chi_E(y)) G\left(\frac{y - x}{\varepsilon}\right) \frac{y - x}{\varepsilon} \, dy \\
= \frac{1}{\varepsilon^N} \int_{E^c} f' \left( \frac{1}{\varepsilon^N} \int_{E} G\left(\frac{y - w}{\varepsilon}\right) \, dw \right) G\left(\frac{y - x}{\varepsilon}\right) \frac{y - x}{\varepsilon} \, dy.
\]

Performing first the change of variable \( y = x + \varepsilon z \) and then \( w = x + \varepsilon z - \varepsilon v \), we obtain

\[
X(\varepsilon, x) = \int_{\frac{x + \varepsilon z - w}{\varepsilon}} f' \left( \frac{1}{\varepsilon^N} \int_{E} G\left(\frac{x + \varepsilon z - w}{\varepsilon}\right) \, dw \right) G(z) \, dz \\
= \int_{\frac{x + \varepsilon z}{\varepsilon}} f' \left( \int_{\frac{x - w}{\varepsilon} + z} G(v) \, dv \right) G(z) \, dz.
\]

Passing to the limit as \( \varepsilon \to 0 \) using (2.1) and applying the Dominated convergence Theorem we easily get (4.2).

**Step 3.** We claim that for any \( x \in \partial^* E \) it holds

\[
\int_{\{z : \nu_E(x) \geq 0\}} f' \left( \int_{\{v - z : \nu_E(x) \geq 0\}} G(v) \, dv \right) G(z) \, dz \cdot \nu_E(x) = \theta(\nu_E(x)). \tag{4.3}
\]

First of all observe any \( z \in \mathbb{R}^N \) with \( z \cdot \nu_E(x) \geq 0 \) can be written in a unique way as \( z = \bar{z} + t \nu_E(x) \) with \( \bar{z} \cdot \nu_E(x) = 0 \) and \( t \geq 0 \). In particular, \( z \cdot \nu_E(x) = (\bar{z} + t \nu_E(x)) \cdot \nu_E(x) = t \). Moreover, since \( G \) is supported on
$B^1_N(0)$, we can consider $t \in [0, 1]$ obtaining

$$
\int_{\{z \cdot \nu_E(x) \geq 0\}} f'(\int_{\{(v-z) \cdot \nu_E(x) \geq 0\}} G(v) \, dv) G(z) \, dz \cdot \nu_E(x)
= \int_{\{z \cdot \nu_E(x) \geq 0\}} f'(\int_{\{(v-z) \cdot \nu_E(x) \geq 0\}} G(v) \, dv) G(z) \cdot \nu_E(x) \, dz
= \int_0^1 \int_{\{z \cdot \nu_E(x) = 0\}} f'(\int_{\{v \cdot \nu_E(x) \geq 0\}} G(v) \, dv) G(z + t \nu_E(x)) \, t \, dt \, dz
= \int_0^1 f'(\int_{\{v \cdot \nu_E(x) \geq 0\}} G(v) \, dv) \int_{\{z \cdot \nu_E(x) = 0\}} G(z + t \nu_E(x)) \, t \, dt \, dz
= \int_0^1 f'(\int_{\{v \cdot \nu_E(x) \geq 0\}} G(v) \, dv) \int_{\{z \cdot \nu_E(x) = 0\}} G(z) \, d\mathcal{H}^{N-1}(z) \, t \, dt.
$$

Finally, we remark that

$$
\frac{d}{dt} \int_{\{v \cdot \nu_E(x) \geq t\}} G(v) \, dv
= \lim_{h \to 0} \frac{1}{h} \left( \int_{\{v \cdot \nu_E(x) \geq t + h\}} G(v) \, dv - \int_{\{v \cdot \nu_E(x) \geq t\}} G(v) \, dv \right)
= -\lim_{h \to 0} \frac{1}{h} \int_{\{t \leq v \cdot \nu_E(x) \leq t + h\}} G(v) \, dv
= -\int_{\{v \cdot \nu_E(x) = t\}} G(v) \, dv.
$$

Integrating by parts we finally get

$$
\int_0^1 f'(\int_{\{v \cdot \nu_E(x) \geq 0\}} G(v) \, dv) \int_{\{z \cdot \nu_E(x) = 0\}} G(z) \, d\mathcal{H}^{N-1}(z) \, t \, dt
= -\int_0^1 \frac{d}{dt} f(\int_{\{v \cdot \nu_E(x) \geq 0\}} G(v) \, dv) \, t \, dt
= -f(\int_{\{v \cdot \nu_E(x) \geq 0\}} G(v) \, dv) \bigg|_0^1 + \int_0^1 f(\int_{\{v \cdot \nu_E(x) \geq 0\}} G(v) \, dv) \, dt
= \theta(\nu_E(x))
$$

where $\theta$ has been introduced in (3.2). This concludes the proof of (4.3).

**Step 4.** We easily conclude. Using (4.1), (4.2), (4.3), De l'Hôpital rule and the Dominated convergence Theorem
we deduce that
\[ \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(E) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\partial^* E} \int_0^\varepsilon X(\eta, x) \cdot \nu_E(x) \, d\eta \, dH^{N-1}(x) \]
\[ = \int_{\partial^* E} \lim_{\varepsilon \to 0} X(\eta, x) \cdot \nu_E(x) \, d\eta \, dH^{N-1}(x) \]
\[ = \int_{\partial^* E} \theta(\nu_E(x)) \, dH^{N-1}(x) \]
and this ends the proof of Theorem 3.1.

**Remark 4.1.** We remark that if \( E \) is a \( C^{1,1} \)-regular set in \( \Omega \) then the computation of the pointwise limit is easier. Indeed, for such sets the following geometric property holds true (for details see [10], Sect. I.2): there exists \( r > 0 \) such that the map
\[ \Psi_r : \partial E \times [0, r] \to \{ y \in E^c : d(y, \partial E) \leq r \}, \quad \Psi_r(x) = x + t \nu_E(x) \]
is a \( C^{1,1} \)-diffeomorphism. Thus, performing change of variable \( x = y - \varepsilon z \) we have
\[ \mathcal{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{\{ y \in E^c : d(y, \partial E) \leq \varepsilon \}} f \left( \frac{1}{\varepsilon^N} \int_E G\left( \frac{y - x}{\varepsilon} \right) \, dx \right) \, dy \]
\[ = \frac{1}{\varepsilon} \int_{\{ y \in E^c : d(y, \partial E) \leq \varepsilon \}} f \left( \int_{\frac{y - x}{\varepsilon}} G(z) \, dz \right) \, dy \]
\[ = \frac{1}{\varepsilon} \int_{\Psi_r(\partial E \times [0, \varepsilon])} f \left( \int_{\frac{y - x}{\varepsilon}} G(z) \, dz \right) \, dy. \]

For any \((x, t) \in \partial E \times [0, \varepsilon] \) let \( J_\varepsilon(x, t) = |\det D\Psi_\varepsilon(x)| \). Then, using also \( t = \varepsilon s \),
\[ \mathcal{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{\partial E} \int_0^\varepsilon f \left( \int_{\frac{y - E}{\varepsilon} + t \nu_E(x)} G(z) \, dz \right) \, J_\varepsilon(x, t) \, dH^{N-1}(x) \, dt \]
\[ = \int_{\partial E} \int_0^1 f \left( \int_{\frac{y - E}{\varepsilon} + s \nu_E(x)} G(z) \, dz \right) \, J_\varepsilon(x, \varepsilon s) \, dH^{N-1}(x) \, ds. \]

Since the regularity of \( E \) we have
\[ \lim_{\varepsilon \to 0} J_\varepsilon(x, \varepsilon s) = 1 \]
from which, applying again (2.1),
\[ \lim_{\varepsilon \to 0} \mathcal{F}_\varepsilon(E) = \int_{\partial E} \int_0^1 f \left( \int_{\{ v \nu_E(x) \geq t \}} G(z) \, dz \right) \, dt \, dH^{N-1}(x) = \mathcal{F}(E). \]
5. The lower estimate

In this section we will prove our second main result, that is Theorem 3.2. First of all at any \( x \in D \) we let

\[
\nu_h(x) = \frac{(-\nabla u_h(x), 1)}{\sqrt{1 + |\nabla u_h(x)|^2}}.
\]

It turns out that \( \nu_h(x) \) is the exterior unit normal to \( \partial^* E_h \) at \( (x, u_h(x)) \). For any \( \eta > 0 \) small enough

\[
D^\eta = \{ x \in D : d(x, \partial D) > \eta \}.
\]

It turns out that \( D^\eta \nsubseteq D \) in \( L^1 \) as \( \eta \to 0^+ \). If \( z \in \mathbb{R}^N \) we will use the notation \( z = (\bar{z}, z^N) \). We now divide the proof into several steps.

**Step 1:** We claim that for any \( \sigma > 0 \), for any \( x \in D^{3\sigma} \) and for any \( h \in \mathbb{N} \) with \( \varepsilon_h < \sigma \) we have

\[
\overline{B_2^{N-1}(0)} \subset \frac{x - D^{\sigma}}{\varepsilon_h}.
\]

Indeed, \( x \in D^{3\sigma} \) means that \( \overline{B_{2\sigma}^{N-1}(x)} \subset D^{\sigma} \). If now \( z \in \mathbb{R}^{N-1} \) and \( |z| \leq 2 \) then \( |x - \varepsilon_h z - x| \leq 2\varepsilon_h < 2\sigma \) which implies that \( x - \varepsilon_h z \in D^{\sigma} \) and then (5.1).

**Step 2:** For any \( x \in D^{3\sigma}, s \in [0, 1] \) and \( \xi \in \mathbb{R}^{N-1} \) we let

\[
a_h(x, s, \xi) = \frac{u_h(x) - u_h(x + \varepsilon_h s \nu_h(x) - \varepsilon_h \xi)}{\varepsilon_h} + s \nu_h(x)^N.
\]

We claim that

\[
\lim_{h \to +\infty} a_h(x, s, \xi) = \nabla u(x) \cdot (\xi - s \nu(x)) + s \nu(x)^N.
\]

Indeed,

\[
u_h(x + \varepsilon_h s \nu_h(x) - \varepsilon_h \xi) - u_h(x)\\= \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} \frac{d}{dt} u_h(x + t(s \nu_h(x) - \xi)) dt\\= \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} \nabla u_h(x + t(s \nu_h(x) - \xi)) \cdot (s \nu_h(x) - \xi) dt\\= \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} (\nabla u_h(x + t(s \nu_h(x) - \xi)) - \nabla u(x + t(s \nu_h(x) - \xi))) \cdot (s \nu_h(x) - \xi) dt\\+ \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} (\nabla u(x + t(s \nu(x) - \xi)) - \nabla u(x + t(s \nu(x) - \xi))) \cdot (s \nu_h(x) - \xi) dt\\+ \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} \nabla u(x + t(s \nu(x) - \xi)) \cdot (s \nu_h(x) - \xi) dt =: I_1 + I_2 + I_3.
\]
Concerning the first integral, we have
\[ |I_1| \leq (s + |\xi|) \| \nabla u_h - \nabla u \|_\infty \to 0 \text{ as } h \to +\infty. \]

On the other hand, if \( L \) is the Lipschitz constant of \( \nabla u \), we get
\[ I_2 \leq L(s + |\xi|) \| \nu_h - \nu \|_\infty \to 0 \text{ as } h \to +\infty. \]

Finally, for the third integral, let \( g(t) = \nabla u(x + t(s\nu(x) - \xi)) \). Then \( g \) is continuous, hence
\[ \lim_{h \to +\infty} \frac{1}{\varepsilon_h} \int_{\varepsilon_h}^1 g(t) \, dt = g(0) \]
from which
\[ \lim_{h \to +\infty} I_3 = \nabla u(x) \cdot (s\nu(x) - \xi) \]
as claimed.

**Step 3:** Let \( M = \sup_h \| u_h \|_\infty \) and let \( \sigma \in (0, M/2) \). We claim that for any \( h \in \mathbb{N} \) with \( \varepsilon_h < \sigma \) it holds
\[ \mathcal{F}_{\varepsilon_h}(E_h) \geq \int_{D^\sigma} \int_0^1 f \left( \int_{B_{1}^{N-1}(0)} \int_0^1 G(\xi, \eta) \, d\eta \, d\xi \right) \, ds \sqrt{1 + |\nabla u_h(x)|^2} \, dx. \]

Indeed, first of all notice that
\[ \{ z \in E_h^c : B_{\varepsilon_h}(z) \cap E_h \neq \emptyset \} \supset \{(x - r\nu_h(x), u_h(x) + r\nu_h(x)^N) : x \in D^\sigma, r \in (0, \varepsilon_h)\}, \]
As a consequence,
\[ \mathcal{F}_{\varepsilon_h}(E_h) \geq \frac{1}{\varepsilon_h} \int_{D^\sigma} \int_0^{\varepsilon_h} f \left( G_{\varepsilon_h} \chi_{E_h}(x - r\nu_h(x), u_h(x) + r\nu_h(x)^N) \right) \, dr \sqrt{1 + |\nabla u_h(x)|^2} \, dx. \]

We concentrate now on the term \( G_{\varepsilon_h} \chi_{E_h}(x - r\nu_h(x), u_h(x) + r\nu_h(x)^N) \) and we rewrite it in a suitable way by performing some changes of variables. First of all, by noticing that \( E_h = \{(z, w) \in D \times \mathbb{R} : 0 \leq w \leq u_h(z)\} \), we have
\[ G_{\varepsilon_h} \chi_{E_h}(x - r\nu_h(x), u_h(x) + r\nu_h(x)^N) \geq \int_{D^\sigma} \frac{1}{\varepsilon_h^N} \int_0^{u_h(z)} G \left( \frac{x - r\nu_h(x) - z}{\varepsilon_h}, \frac{u_h(x) + r\nu_h(x)^N - w}{\varepsilon_h} \right) \, dw \, dz. \]

We now perform the change of variables in the following order:
\[ \eta = \frac{u_h(x) + r\nu_h(x)^N - w}{\varepsilon_h}, \quad \xi = \frac{x + r\nu_h(x) - z}{\varepsilon_h}. \]
We obtain
\[
G_{\varepsilon h} * \chi_{E_h}(x - \varepsilon_h r, u_h(x) + \varepsilon_h r \nu(x)) \geq \int_{\varepsilon_h (x, r/\varepsilon_h, \xi)} \int_{x + \varepsilon_h r}^{x + \varepsilon_h r + r \nu(x)} G(\xi, \eta) \, d\eta \, d\xi.
\]

Recalling that \(f\) is non-decreasing and operating the change of variable \(r = \varepsilon_h s\) we arrive to
\[
\mathcal{F}_{\varepsilon h}(E_h) \geq \int_{D^{3\sigma}} \int_0^1 f \left( \int_{x - D^\sigma}^{x + D^\sigma} \int_{\varepsilon_h (x, s, \xi)}^{\varepsilon_h (x) + s \nu(x)} G(\xi, \eta) \, d\eta \, d\xi \right) \, ds \sqrt{1 + |\nabla u_h(x)|^2} \, dx.
\]

Now, since (5.1) we deduce that for any \(x \in D^{3\sigma}\) and for any \(s \in [0, 1]\)
\[
\frac{x - D^\sigma}{\varepsilon_h} + s \nu_h(x) \supset B_2^{N-1}(0) + s \nu_h(x) \supset B_2^{N-1}(0).
\]

Moreover, using \(\sigma < M/2\) we get also
\[
\frac{u_h(x)}{\varepsilon_h} + s \nu_h(x) > 1.
\]

As a consequence, recalling that \(G\) is supported on \(B_1^N(0)\) we obtain (5.3).

**Step 4:** Passing to the limit as \(h \to +\infty\) in (5.3), using Fatou’s Lemma (5.2) and the Dominated convergence Theorem we obtain
\[
\liminf_{h \to +\infty} \mathcal{F}_{\varepsilon h}(E_h)
\]

By the arbitrariness of \(\sigma\) small we get
\[
\liminf_{h \to +\infty} \mathcal{F}_{\varepsilon h}(E_h)
\]

By the arbitrariness of \(\sigma\) small we get
\[
\lim_{h \to +\infty} \mathcal{F}_{\varepsilon h}(E_h)
\]

**Step 5:** We conclude the proof showing that
\[
\int_D \int_0^1 f \left( \int_{B_2^{N-1}(0)} \int_{\nabla u(x) \cdot (\xi - \varepsilon_h \nu(x)) + \nu_h(x)} G(\xi, \eta) \, d\eta \, d\xi \right) \, dx \sqrt{1 + |\nabla u(x)|^2} = \mathcal{F}(E).
\]
First of all, we notice that
\[ \eta = \nabla u(x) \cdot (\xi - s\nu(x)) + s\nu(x) = \nabla u(x) \cdot \xi + s\sqrt{1 + |\nabla u(x)|^2} \]
is the equation of an affine hyperplane in \( \mathbb{R}^N \) orthogonal to \( \nu(x) \) whose distance from the origin is
\[ \frac{s\sqrt{1 + |\nabla u(x)|^2}}{\sqrt{1 + |\nabla u(x)|^2}} = s. \]
As a consequence,
\[ \int_{B_1^{N-1}(0)} \int_1^1 \nabla u(x) \cdot (\xi - s\nu(x)) + s\nu(x) G(\xi, \eta) d\eta d\xi = \int_{\{z \nu_E(x, u(x)) \geq s\}} G(z) dz \]
from which
\[ \int_D \int_0^1 f \left( \int_{B_1^{N-1}(0)} \int_1^1 \nabla u(x) \cdot (\xi - s\nu(x)) + s\nu(x) G(\xi, \eta) d\eta d\xi \right) ds \sqrt{1 + |\nabla u(x)|^2} dx \]
\[ = \int_D \int_0^1 f \left( \int_{\{z \nu_E(x, u(x)) \geq s\}} G(z) dz \right) ds \sqrt{1 + |\nabla u(x)|^2} dx \]
\[ = \int_{\partial E} \int_0^1 f \left( \int_{\{z \nu_E(y) \geq s\}} G(z) dz \right) ds dH^{N-1}(y) = \mathcal{F}(E) \]
and the proof is complete. \( \square \)

6. \( L^1 \)-LOWER SEMICONTINUITY OF \( \mathcal{F} \)

We are going to prove Theorem 3.6. It is well known (see for instance [2], Thm. 5.14) that is sufficient to check that the positively one-homogeneous extension of \( \theta \) given by
\[ \tilde{\theta}(v) = \begin{cases} |v| \theta \left( \frac{v}{|v|} \right) dt & \text{if } v \neq 0, \\ 0 & \text{if } v = 0, \end{cases} \]
is convex. First of all, by direct computation for each \( v \in \mathbb{R}^N \) with \( v \neq 0 \) we have
\[ \theta \left( \frac{v}{|v|} \right) = \int_0^1 f \left( \int_{\{z \geq |v|t\}} G(z) dz \right) dt \frac{|v|}{|v|} \int_0^{|v|} f \left( \int_{\{z \geq s\}} G(z) dz \right) ds \]
from which we obtain
\[ \tilde{\theta}(v) = \begin{cases} \int_0^{|v|} f \left( \int_{\{z \geq s\}} G(z) dz \right) ds & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases} \]
Now it is east to see that $\tilde{\theta}$ is convex. Indeed, since $f$ is convex there exist $(\alpha_h), (\beta_h)$ such that

$$f = \lim_{h \to +\infty} f_h$$

uniformly on compact sets, where $f_h(t) = \alpha_h t + \beta_h$.

For any $h \in \mathbb{N}$ let

$$\tilde{\theta}_h(v) = \begin{cases} 
\int_0^{\|v\|} f_h \left( \int_{\{z \cdot v \geq s\}} G(z) \, dz \right) \, ds & \text{if } v \neq 0, \\
0 & \text{if } v = 0.
\end{cases}$$

Since $f_h \to f$ uniformly on $[0, 1]$ we can say that $\tilde{\theta}_h \to \tilde{\theta}$ pointwise. In order to conclude it is sufficient to show that $\tilde{\theta}_h$ is convex. For any $v \neq 0$ we let $\hat{v} = \frac{v}{\|v\|}$. Then

$$\tilde{\theta}_h(v) = \alpha_h \int_0^{\|v\|} \int_{\{z \cdot v \geq s\}} G(z) \, dz \, ds + \beta_h \|v\|$$

$$= \alpha_h \int_0^{\|v\|} \int_{\{z \cdot v = 0\}} \int_s^{+\infty} G(\bar{z} + t\hat{v}) \, d\bar{z} \, dt \, ds + \beta_h \|v\|$$

$$= \alpha_h \int_0^{+\infty} \int_{\{z \cdot v = 0\}} \int_0^{\|v\|} G(\bar{z} + t\hat{v}) \, ds \, d\bar{z} \, dt + \beta_h \|v\|$$

$$= \alpha_h \|v\| \int_0^{+\infty} \int_{\{z \cdot v = 0\}} tG(\bar{z} + t\hat{v}) \, dt \, d\bar{z} + \beta_h \|v\|$$

$$= \alpha_h \int_{\{z \cdot v \geq 0\}} G(z)z \cdot v \, dz + \beta_h \|v\|$$

$$= \alpha_h \frac{1}{2} \int_{\mathbb{R}^N} G(z) |z \cdot v| \, dz + \beta_h \|v\|$$

where the last equality follows since $G$ is even. Notice that the last expression is convex in $v$ and this ends the proof.

7. Some examples

In this section we characterize the limit functional $\mathcal{F}$ in some interesting cases.

7.1. $G$ radially symmetric

Assume $G(z) = g(|z|)$ for some $g: [0, +\infty) \to \mathbb{R}$. Take $\nu \in S^{N-1}$ and $t \geq 0$. Notice that the quantity

$$\int_{\{z \cdot v \geq t\}} G(z) \, dz$$

does not depend on $\nu$. Take now $E \in \mathcal{P}_N$ and $x \in \partial^* E$. We have

$$\int_0^1 f \left( \int_{\{z \cdot \nu_E(x) \geq t\}} G(z) \, dz \right) \, dt = c$$
where \( c \) is a constant that depends only on \( N, f \) and \( G \). Then

\[
\mathcal{F}(E) = c \mathcal{H}^{N-1}(\partial^* E).
\]

### 7.2. The case \( f(t) = t \)

When \( f \) is the identity function for any \( E \in \mathcal{P}_N \) and for any \( x \in \partial^* E \) we have

\[
\theta(\nu_E(x)) = \int_0^1 \int_{H_{\nu_E(x) + t\nu_E(x)}} G(z) \, dz \, dt
\]

\[
= \int_0^1 \int_{\{\bar{z} \cdot \nu_E(x) = 0\}} \int_t^1 G(\bar{z} + s\nu_E(x)) \, ds \, d\bar{z} \, dt
\]

\[
= \int_0^1 \int_{\{\bar{z} \cdot \nu_E(x) = 0\}} \int_0^s G(\bar{z} + s\nu_E(x)) \, dt \, d\bar{z} \, ds
\]

\[
= \int_{H_{\nu_E(x)}} s G(\bar{z} + s\nu_E(x)) \, d\bar{z} \, ds
\]

\[
= \int_{H_{\nu_E(x)}} G(z) \cdot \nu_E(x) \, dz
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^N} G(z)|z \cdot \nu_E(x)| \, dz.
\]

Then the limit \( \mathcal{F} \) is given by

\[
\mathcal{F}(E) = \frac{1}{2} \int_{\partial^* E} \int_{\mathbb{R}^N} G(z)|z \cdot \nu_E(x)| \, dz \, d\mathcal{H}^{N-1}(x).
\]

This is in accordance to [9].

**Remark 7.1.** If \( N > 1 \) and \( G \) is radially symmetric we have, if \( g: [0, +\infty) \to \mathbb{R} \) is such that \( G(z) = g(|z|) \),

\[
\frac{1}{2} \int_{\mathbb{R}^N} G(z)|z \cdot \nu_E(x)| \, dz = \frac{1}{2} \int_{\mathbb{R}^N} g(|z|)|z \cdot \nu_E(x)| \, dz
\]

\[
= \frac{1}{2} \int_0^{+\infty} \int_{\mathbb{S}^{N-1}} g(r)|\xi \cdot \nu_E(x)| \, dr \, d\mathcal{H}^{N-1}(\xi)
\]

\[
= |B_1^{N-1}(0)| \int_0^{+\infty} g(r) \, dr
\]

\[
= \frac{|B_1^{N-1}(0)|}{\mathcal{H}^{N-1}(\mathbb{S}^{N-1})} \int_{\mathbb{R}^N} G(z)|z| \, dz
\]

since it is well known that for any \( \nu \in \mathbb{S}^{N-1} \) it holds

\[
\frac{1}{2} \int_{\mathbb{S}^{N-1}} |\xi \cdot \nu| \, d\mathcal{H}^{N-1}(\xi) = |B_1^{N-1}(0)|.
\]
We thus deduce that
\[ \mathcal{F}(E) = c_{N,G} \mathcal{H}^{N-1}(E), \quad c_{N,G} = \frac{|B_1^{N-1}(0)|}{\mathcal{H}^{N-1}(S^{N-1})} \int_{\mathbb{R}^N} G(|z|) |z| \, dz. \]

This is in accordance to [5].

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**REFERENCES**


