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ASYMPTOTIC ANALYSIS OF A FAMILY OF NON-LOCAL FUNCTIONALS ON SETS

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Abstract. We study the asymptotic behavior of a family of functionals which penalize a short-range interaction of convolution type between a finite perimeter set and its complement. We first compute the pointwise limit and we obtain a lower estimate on more regulars sets. Finally, some examples are discussed.

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1. Introduction

In this paper we study the asymptotic behavior, as $\varepsilon \to 0$, of the family of functionals

$$\mathscr{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{E^c \cap \Omega} f(G_\varepsilon * \chi_{E\cap\Omega})\,\mathrm{d}x.$$

where $\Omega \subset \mathbb{R}^N$ is open and bounded, $N > 1$, E is a set of finite perimeter in Ω, f is given and $G_\varepsilon(z) = \frac{1}{\varepsilon^N} G(\frac{z}{\varepsilon})$ where G is a suitable kernel. Our analysis has been inspired by a paper by Miranda *et al.* in [8] where the case $f(t) = t$ is considered and G is the Gauss-Weierstrass kernel, namely $G(z) = \frac{1}{(4\pi)^{N/2}} e^{-|z|^2/4}$ (see also [6, 7] for smoother sets and [1] for similar convolution approximation). More precisely, in [8] it is proven that the pointwise limit is, up to a constant, the perimeter of E in Ω. A more general kernel G has been investigated, in the context of optimal partition problems, by Esedoḡlu and Otto [5] where G is smooth and non-negative, radially symmetric and satisfying the following conditions:

$$\int_{\mathbb{R}^N} G(z)\,\mathrm{d}z = 1, \quad \int_{\mathbb{R}^N} |z| G(z)\,\mathrm{d}z < +\infty, \quad |\nabla G(z)| \lesssim G\left(\frac{z}{2}\right), \quad \nabla G(z)\cdot z \le 0.$$

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On the other hand, as in [5], Esedoḡlu and Otto consider only the case $f(t) = t$, but they prove a complete Γ-convergence result for the family $\{\mathscr{F}_\varepsilon\}_{\varepsilon>0}$ on finite perimeter sets with respect to the strong L^1-convergence. We point out that for the simpler class of functionals with $f(t) = t$ there is a much simpler proof of gamma convergence in L^1 subsequently given by Esedoḡlu and Jacobs in [4]. In particular, they deal with more general convolution kernels, which in particular can be non-radially symmetric, thus including anisotropy, and even change sign to a certain extent, which turns out to be necessary for the approximation of certain anisotropies. A very similar result has been obtained more recently by Berendsen and Pagliari [3]. As far as we know, the last result is due to Pagliari [9] where he essentially remove the radial symmetry of G and he obtain, as limit, an anisotropic perimeter.

In this paper we try to investigate the general situation. We assume that G is even, non-negative, supported on the unit closed ball and with $\int_{\mathbb{R}^N} G(z)\,\mathrm{d}z = 1$. First of all we are able to compute the pointwise limit, as $\varepsilon \to 0$, of $\mathscr{F}_\varepsilon(E)$ whenever f is C^1, non-decreasing and $f(0) = 0$. It turns out (see Thm. 3.1) that for any $E \subset \mathbb{R}^N$ with finite perimeter in Ω

$$\lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon(E) = \int_{\partial^* E\cap\Omega} \int_0^1 f\left(\int_{\{z\cdot\nu_E(x)\geq t\}} G(z)\,\mathrm{d}z\right) \mathrm{d}t\,\mathrm{d}\mathcal{H}^{N-1}(x)$$

where $\partial^* E$ is the reduced boundary of E and $\nu_E(x)$ is the outer unit normal at E. In view to have a Γ-convergence result we investigate also the lower estimate. Unfortunately, the technique of Esedoḡlu and Otto [5] does not work in our situation: it is crucial for them to switch the order of integration, that is impossible for us since we have f between the exterior integral and the convolution one. It seems that this difficulty cannot be easily overcome in the general situation. We are able to show (see Thm. 3.2) a Γ-liminf inequality only on graphs of C^1 functions with respect to the C^1-uniform convergence. Actually, it is easy to generalize such a inequality in the case of sets which are locally graphs of C^1 functions with respect to a suitable convergence (see Rem. 3.3). Finally, we also prove (see Thm. 3.6) that if f is also convex, then the pointwise limit is lower semicontinuous with respect to the strong L^1-convergence, which suggests that for f convex the Γ-limit in the strong L^1-convergence should be the pointwise limit. At the end of the paper we will also discuss some examples.

2. Notation and preliminaries

2.1. Notation

In what follows $N \in \mathbb{N}$ with $N \geq 1$. For any $r > 0$ and $x \in \mathbb{R}^N$ the notation $B^d_r(x)$ stands for the open ball in $\mathbb{R}^d$ centered at x with radius r, while $\mathbb{S}^{N-1} = \partial B^N_1(0)$. If $A \subseteq \mathbb{R}^N$ we also denote by $\mathcal{H}^k(A)$ the Hausdorff measure of A of dimension $k \in \{0, 1, \dots, N\}$ ($\mathcal{H}^0$ is the counting measure). If A_h, A are measurable subsets of $\mathbb{R}^N$, then $A_h \to A$ in $L^1(\mathbb{R}^N)$ (or $L^1_{\text{loc}}(\mathbb{R}^N)$) means that $\chi_{A_h} \to \chi_A$ in $L^1(\mathbb{R}^N)$ (respectively $L^1_{\text{loc}}(\mathbb{R}^N)$). Finally, for any $A \subseteq \mathbb{R}^N$ we let $A^c = \mathbb{R}^N \setminus A$.

2.2. Finite perimeter sets

We recall some notion on finite perimeter sets in euclidean space; for details we refer to [2]. Let Ω be an open subset of $\mathbb{R}^N$. A measurable set $E \subseteq \mathbb{R}^N$ is said to be a *set of finite perimeter in* Ω if

$$\mathcal{P}(E,\Omega) = \sup\left\{\int_E \operatorname{div} X(x)\,\mathrm{d}x : X \in C^1_c(\Omega;\mathbb{R}^N),\ \|X\|_\infty \leq 1\right\} < +\infty.$$

The quantity $\mathcal{P}(E,\Omega)$ is called *perimeter of E in Ω*. Finite perimeter sets have nice boundary in a measure theoretical sense. Precisely, one can define a subset of E as the set of points x where there exists a unit vector $\nu_E(x)$ such that:

$$\frac{x-E}{r} \to \{y \in \mathbb{R}^N : y \cdot \nu_E(x) \geq 0\}, \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^N) \text{ as } r \to 0, \tag{2.1}$$

and which is referred to as the *outer normal to E at x*. The set where $\nu_E(x)$ exists is called the *reduced boundary of E* and is denoted by $\partial^* E$. It turns out that, for any E set of finite perimeter in Ω, we have $\mathcal{P}(E,\Omega) = \mathcal{H}^{N-1}(\partial^* E \cap \Omega)$. The reduced boundary of E plays the role of the topological boundary also in the sense of the integration by parts. Indeed, one can show that, if E is a set of finite perimeter in Ω, then the following Gauss-Green formula holds true:

$$\int_E \operatorname{div} X(x)\,\mathrm{d}x = \int_{\partial^* E} X(x)\cdot \nu_E(x)\,\mathrm{d}\mathcal{H}^{N-1}(x), \quad \forall X \in C^1_c(\Omega;\mathbb{R}^N). \tag{2.2}$$

Finite perimeter sets satisfy good properties for the Calculus of Variations: for instance, if E_h, E have finite perimeter in Ω and $E_h \overset{L^1}{\to} E$, then

$$\mathcal{P}(E,\Omega) \leq \liminf_{h\to+\infty} \mathcal{P}(E_h,\Omega).$$

3. Setting of the problem and main results

Let $N > 1$, let $G\colon \mathbb{R}^N \to [0,+\infty)$ be of class C^∞ such that

$$\operatorname{supp} G = \overline{B_1^N(0)}, \quad G(-x) = G(x), \quad \int_{\mathbb{R}^N} G(x)\,\mathrm{d}x = 1.$$

For any $\varepsilon > 0$ and for any $x \in \mathbb{R}^N$, let

$$G_\varepsilon(x) = \frac{1}{\varepsilon^N} G\left(\frac{x}{\varepsilon}\right).$$

We consider a continuous and non-decreasing function $f\colon [0,+\infty) \to \mathbb{R}$ with $f(0) = 0$. Let $\Omega \subset \mathbb{R}^N$ be open bounded. We denote by $\mathcal{P}_N(\Omega)$ the set of all sets of finite perimeter in Ω. For any $\varepsilon > 0$, we introduce the functional $\mathscr{F}_\varepsilon\colon \mathcal{P}_N(\Omega) \to [0,+\infty)$ defined by

$$\mathscr{F}_\varepsilon(E) = \frac{1}{\varepsilon}\int_{E^c\cap\Omega} f(G_\varepsilon * \chi_{E\cap\Omega})\,\mathrm{d}x. \tag{3.1}$$

In order to state our main results, we introduce the function $\theta\colon \mathbb{S}^{N-1} \to [0,+\infty)$ given by

$$\theta(\nu) = \int_0^1 f\left(\int_{\{x\cdot\nu \geq t\}} G(x)\,\mathrm{d}x\right)\mathrm{d}t. \tag{3.2}$$

Let $\mathscr{F}\colon \mathcal{P}_N(\Omega) \to [0,+\infty)$ be the functional given by

$$\mathscr{F}(E) = \int_{\partial^* E\cap\Omega} \theta(\nu_E(x))\,\mathrm{d}\mathcal{H}^{N-1}(x).$$

Our first main result concerns the pointwise limit of $\mathscr{F}_\varepsilon$ on $\mathcal{P}_N(\Omega)$.

Theorem 3.1. (Pointwise limit) *Assume f of class C^1. Let $E \in \mathcal{P}_N(\Omega)$. Then*

$$\lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon(E) = \mathscr{F}(E).$$

On the other hand, we are also able to prove a lower estimate on graphs.

Theorem 3.2. (Lower estimate) *Let $D \subset \mathbb{R}^{N-1}$ be open and bounded with Lipschitz boundary, let $u_h, u \in C^{1,1}(D)$, with $u_h, u > 0$ on D such that $u_h \to u$ uniformly in $C^1(D)$. Let E_h, E be given by*

$$E_h = \{(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R} : x \in D,\, 0 \le y \le u_h(x)\},$$

$$E = \{(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R} : x \in D,\, 0 \le y \le u(x)\}.$$

Then, for any positive infinitesimal sequence (ε_h) it holds

$$\liminf_{h\to+\infty} \mathscr{F}_{\varepsilon_h}(E_h) \ge \mathscr{F}(E).$$

Remark 3.3. It is not difficult to see that Theorem 3.2 can be generalized to uniformly $C^{1,1}$-regular sets in Ω with respect to a suitable notion of uniform convergence. Precisely, a set $E \subset \mathbb{R}^N$ is said to be uniformly $C^{1,1}$-regular set in Ω if there exist $L, \delta > 0$ such that for every $x \in \partial E \cap \Omega$ there exist $D^x \subseteq \mathbb{R}^{N-1}$ open and a function $u^x \in C^{1,1}(D^x)$ such that:

- $\partial E \cap \Omega \cap B^N_\delta(x)$ is the graph of u^x;
- $\|\nabla u^x\|_\infty \le L$.

On the set of all uniformly $C^{1,1}$-regular sets in Ω we put a convergence of sequences. Precisely, we say that E_h converges to E if there exist $\delta, L > 0$ such that for every $x \in \partial E \cap \Omega$ there exist $D^x \subseteq \mathbb{R}^{N-1}$ open and functions $u^x_h, u^x \in C^{1,1}(D^x)$ such that:

- $\partial E_h \cap \Omega \cap B^N_\delta(x), \partial E \cap \Omega \cap B^N_\delta(x)$ are the graphs of u^x_h, u^x respectively;
- $\|\nabla u^x_h\|_\infty \le L$ and $\|\nabla u^x\|_\infty \le L$;
- $u^x_h \to u^x$ uniformly in $C^1(D^x)$.

It is easy to see that with respect to this type of convergence the lower estimate

$$\liminf_{h\to+\infty} \mathscr{F}_{\varepsilon_h}(E_h) \ge \mathscr{F}(E)$$

follows as a simple consequence of Theorem 3.2.

Combining Theorem 3.1 with Theorem 3.2 and Remark 3.3 we eventually obtain a Γ-convergence result.

Corollary 3.4. *The family $\{\mathcal{F}_\varepsilon\}_{\varepsilon>0}$ Γ-converges to $\mathcal{F}$ as $\varepsilon \to 0$ on uniformly $C^{1,1}$-regular sets with respect to the convergence introduced in Remark 3.3.*

Remark 3.5. We do not expect compactness of equibounded sequences of uniformly $C^{1,1}$-regular sets. Nevertheless, at least if $f(t) \ge mt$ for some $m > 0$, equibounded sequences are compact in L^1. Indeed, if (ε_h) is a positive and infinitesimal sequence and (E_h) be a sequence in $\mathcal{P}_N(\Omega)$ with $\mathscr{F}_{\varepsilon_h}(E_h) \leq c$ for some $c \geq 0$, we get

$$c \geq \mathscr{F}_{\varepsilon_h}(E_h) \geq \frac{m}{\varepsilon} \int_{E^c \cap \Omega} G_\varepsilon * \chi_{E\cap\Omega} \, \mathrm{d}x$$

and the compactness follows by Lemma A.4 of [5] (see also [1], Thm. 3.1).

The next and last result suggests that the Γ-limit on $\mathcal{P}_N(\Omega)$ of the family $(\mathscr{F}_\varepsilon)_{\varepsilon>0}$ with respect to the L^1-convergence could be really $\mathscr{F}$, at least if f is convex.

Theorem 3.6. *If f is convex then the functional $\mathscr{F} : \mathcal{P}_N(\Omega) \to \mathbb{R}$ is lower semicontinuous with respect to the L^1-topology.*

4. The pointwise limit

In this section we prove Theorem 3.1. The main idea comes from the technique used in [8]. We divide the proof in some steps.

Step 1. We claim that for any $E \in \mathcal{P}_N$ we have

$$\mathscr{F}_\varepsilon(E) = \frac{1}{\varepsilon} \int_{\partial^* E} \int_0^\varepsilon X(\eta, x) \cdot \nu_E(x) \, \mathrm{d}\eta \, \mathrm{d}\mathcal{H}^{N-1}(x), \tag{4.1}$$

where for any $\eta > 0$ and for any $x \in \partial^* E$

$$X(\eta, x) = \frac{1}{\eta^N} \int_{E^c} f'(G_\eta * \chi_E(y)) G\left(\frac{y-x}{\eta}\right) \frac{y-x}{\eta} \, \mathrm{d}y.$$

For any $\eta > 0$ and any $y \in \mathbb{R}^N$ we have, using the Gauss-Green formula (2.2),

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\eta} f(G_\eta * \chi_E(y)) \\
&\quad = -f'(G_\eta * \chi_E(y)) \frac{1}{\eta^{N+1}} \int_{\mathbb{R}^N} \left(NG\left(\frac{y-x}{\eta}\right) + \nabla G\left(\frac{y-x}{\eta}\right) \cdot \frac{y-x}{\eta} \right) \chi_E(x) \, \mathrm{d}x \\
&\quad = f'(G_\eta * \chi_E(y)) \frac{1}{\eta^N} \int_{\mathbb{R}^N} \mathrm{div}_x \left(G\left(\frac{y-x}{\eta}\right) \frac{y-x}{\eta} \right) \chi_E(x) \, \mathrm{d}x \\
&\quad = f'(G_\eta * \chi_E(y)) \frac{1}{\eta^N} \int_{\partial^* E} G\left(\frac{y-x}{\eta}\right) \frac{y-x}{\eta} \cdot \nu_E(x) \, \mathrm{d}\mathcal{H}^{N-1}(x).
\end{aligned}$$

Now notice that, since $G_\varepsilon * \chi_E \to \chi_E$ in $L^1(\mathbb{R}^N)$ as $\varepsilon \to 0$, we can say that for any $y \in \mathbb{R}^N$

$$f(G_\varepsilon * \chi_E(y)) - f(\chi_E(y)) = \int_0^\varepsilon \frac{\mathrm{d}}{\mathrm{d}\eta} f(G_\eta * \chi_E(y)) \, \mathrm{d}\eta,$$

from which we get, using the fact that $f(0)=0$,

$$\begin{aligned}
\mathscr{F}_\varepsilon(E) &= \frac{1}{\varepsilon}\int_{E^c} f(G_\varepsilon * \chi_E(y)) - f(\chi_E(y))\,\mathrm{d}y \\
&= \frac{1}{\varepsilon}\int_{E^c}\int_0^\varepsilon \frac{\mathrm{d}}{\mathrm{d}\eta} f(G_\eta * \chi_E(y))\,\mathrm{d}\eta\,\mathrm{d}y \\
&= \frac{1}{\varepsilon}\int_{\partial^* E}\int_0^\varepsilon \frac{1}{\eta^N}\int_{E^c} f'(G_\eta * \chi_E(y)) G\left(\frac{y-x}{\eta}\right)\frac{y-x}{\eta}\,\mathrm{d}y\,\mathrm{d}\eta\cdot\nu_E(x)\,\mathrm{d}\mathcal{H}^{N-1}(x) \\
&= \frac{1}{\varepsilon}\int_{\partial^* E}\int_0^\varepsilon X(\eta,x)\cdot\nu_E(x)\,\mathrm{d}\eta\,\mathrm{d}\mathcal{H}^{N-1}(x)
\end{aligned}$$

hence (4.1).

STEP 2. We claim that for any $x\in\partial^* E$ we have

$$\lim_{\varepsilon\to 0} X(\varepsilon,x) = \int_{\{z\cdot\nu_E(x)\geq 0\}} f'\left(\int_{\{(v-z)\cdot\nu_E(x)\geq 0\}} G(v)\,\mathrm{d}v\right) G(z) z\,\mathrm{d}z. \tag{4.2}$$

First of all we have

$$\begin{aligned}
X(\varepsilon,x) &= \frac{1}{\varepsilon^N}\int_{E^c} f'(G_\varepsilon * \chi_E(y)) G\left(\frac{y-x}{\varepsilon}\right)\frac{y-x}{\varepsilon}\,\mathrm{d}y \\
&= \frac{1}{\varepsilon^N}\int_{E^c} f'\left(\frac{1}{\varepsilon^N}\int_E G\left(\frac{y-w}{\varepsilon}\right)\mathrm{d}w\right) G\left(\frac{y-x}{\varepsilon}\right)\frac{y-x}{\varepsilon}\,\mathrm{d}y.
\end{aligned}$$

Performing first the change of variable $y=x+\varepsilon z$ and then $w=x+\varepsilon z-\varepsilon v$, we obtain

$$\begin{aligned}
X(\varepsilon,x) &= \int_{\frac{E^c-x}{\varepsilon}} f'\left(\frac{1}{\varepsilon^N}\int_E G\left(\frac{x+\varepsilon z-w}{\varepsilon}\right)\mathrm{d}w\right) G(z) z\,\mathrm{d}z \\
&= \int_{\frac{E^c-x}{\varepsilon}} f'\left(\int_{\frac{x-E}{\varepsilon}+z} G(v)\,\mathrm{d}v\right) G(z) z\,\mathrm{d}z.
\end{aligned}$$

Passing to the limit as $\varepsilon\to 0$ using (2.1) and applying the Dominated convergence Theorem we easily get (4.2).

STEP 3. We claim that for any $x\in\partial^* E$ it holds

$$\int_{\{z\cdot\nu_E(x)\geq 0\}} f'\left(\int_{\{(v-z)\cdot\nu_E(x)\geq 0\}} G(v)\,\mathrm{d}v\right) G(z) z\,\mathrm{d}z\cdot\nu_E(x) = \theta(\nu_E(x)). \tag{4.3}$$

First of all observe any $z\in\mathbb{R}^N$ with $z\cdot\nu_E(x)\geq 0$ can be written in a unique way as $z=\bar{z}+t\nu_E(x)$ with $\bar{z}\cdot\nu_E(x)=0$ and $t\geq 0$. In particular, $z\cdot\nu_E(x)=(\bar{z}+t\nu_E(x))\cdot\nu_E(x)=t$. Moreover, since G is supported on

$\overline{B_1^N(0)}$, we can consider $t \in [0,1]$ obtaining

$$\begin{aligned}
&\int_{\{z\cdot\nu_E(x)\geq 0\}} f'\left(\int_{\{(v-z)\cdot\nu_E(x)\geq 0\}} G(v)\,\mathrm{d}v\right) G(z)z\,\mathrm{d}z\cdot\nu_E(x)\\
&\qquad= \int_{\{z\cdot\nu_E(x)\geq 0\}} f'\left(\int_{\{(v-z)\cdot\nu_E(x)\geq 0\}} G(v)\,\mathrm{d}v\right) G(z)z\cdot\nu_E(x)\,\mathrm{d}z\\
&\qquad= \int_0^1\int_{\{\bar z\cdot\nu_E(x)=0\}} f'\left(\int_{\{v\cdot\nu_E(x)\geq t\}} G(v)\,\mathrm{d}v\right) G(\bar z+t\nu_E(x))\,t\,\mathrm{d}t\,\mathrm{d}\bar z\\
&\qquad= \int_0^1 f'\left(\int_{\{v\cdot\nu_E(x)\geq t\}} G(v)\,\mathrm{d}v\right)\int_{\{\bar z\cdot\nu_E(x)=0\}} G(\bar z+t\nu_E(x))\,t\,\mathrm{d}\bar z\,\mathrm{d}t\\
&\qquad= \int_0^1 f'\left(\int_{\{v\cdot\nu_E(x)\geq t\}} G(v)\,\mathrm{d}v\right)\int_{\{z\cdot\nu_E(x)=t\}} G(z)\,\mathrm{d}\mathcal{H}^{N-1}(z)\,t\,\mathrm{d}t.
\end{aligned}$$

Finally, we remark that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\{v\cdot\nu_E(x)\geq t\}} G(v)\,\mathrm{d}v\\
&\qquad= \lim_{h\to 0}\frac{1}{h}\left(\int_{\{v\cdot\nu_E(x)\geq t+h\}} G(v)\,\mathrm{d}v - \int_{\{v\cdot\nu_E(x)\geq t\}} G(v)\,\mathrm{d}v\right)\\
&\qquad= -\lim_{h\to 0}\frac{1}{h}\int_{\{t\leq v\cdot\nu_E(x)\leq t+h\}} G(v)\,\mathrm{d}v\\
&\qquad= -\int_{\{v\cdot\nu_E(x)=t\}} G(v)\,\mathrm{d}v.
\end{aligned}$$

Integrating by parts we finally get

$$\begin{aligned}
&\int_0^1 f'\left(\int_{\{v\cdot\nu_E(x)\geq t\}} G(v)\,\mathrm{d}v\right)\int_{\{z\cdot\nu_E(x)=t\}} G(z)\,\mathrm{d}\mathcal{H}^{N-1}(z)\,t\,\mathrm{d}t\\
&\quad= -\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f\left(\int_{\{v\cdot\nu_E(x)\geq t\}} G(v)\,\mathrm{d}v\right) t\,\mathrm{d}t\\
&\quad= -f\left(\int_{\{v\cdot\nu_E(x)\geq t\}} G(v)\,\mathrm{d}v\right) t\Big|_0^1 + \int_0^1 f\left(\int_{\{v\cdot\nu_E(x)\geq t\}} G(v)\,\mathrm{d}v\right)\,\mathrm{d}t\\
&\quad= \theta(\nu_E(x))
\end{aligned}$$

where θ has been introduced in (3.2). This concludes the proof of (4.3).

STEP 4. We easily conclude. Using (4.1), (4.2), (4.3), De l'Hôpital rule and the Dominated convergence Theorem we deduce that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\mathscr{F}_\varepsilon(E) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\partial^* E}\int_0^\varepsilon X(\eta,x)\cdot\nu_E(x)\,\mathrm{d}\eta\,\mathrm{d}\mathcal{H}^{N-1}(x)\\
&= \int_{\partial^* E}\lim_{\varepsilon\to 0} X(\eta,x)\cdot\nu_E(x)\,\mathrm{d}\eta\,\mathrm{d}\mathcal{H}^{N-1}(x)\\
&= \int_{\partial^* E}\theta(\nu_E(x))\,\mathrm{d}\mathcal{H}^{N-1}(x)
\end{aligned}$$

and this ends the proof of Theorem 3.1. □

Remark 4.1. We remark that if E is a $C^{1,1}$-regular set in Ω then the computation of the pointwise limit is easier. Indeed, for such sets the following geometric property holds true (for details see [10], Sect. I.2): there exists $r>0$ such that the map

$$\Psi_r\colon \partial E\times[0,r]\to\{y\in E^c : d(y,\partial E)\le r\},\quad \Psi_r(x)=x+t\nu_E(x)$$

is a $C^{1,1}$-diffeomorphism. Thus, performing change of variable $x=y-\varepsilon z$ we have

$$\begin{aligned}
\mathscr{F}_\varepsilon(E) &= \frac{1}{\varepsilon}\int_{\{y\in E^c: d(y,\partial E)\le\varepsilon\}} f\left(\frac{1}{\varepsilon^N}\int_E G\left(\frac{y-x}{\varepsilon}\right)\mathrm{d}x\right)\,\mathrm{d}y\\
&= \frac{1}{\varepsilon}\int_{\{y\in E^c: d(y,\partial E)\le\varepsilon\}} f\left(\int_{\frac{y-E}{\varepsilon}} G(z)\mathrm{d}z\right)\,\mathrm{d}y\\
&= \frac{1}{\varepsilon}\int_{\Psi_\varepsilon(\partial E\times[0,\varepsilon])} f\left(\int_{\frac{y-E}{\varepsilon}} G(z)\mathrm{d}z\right)\,\mathrm{d}y.
\end{aligned}$$

For any $(x,t)\in\partial E\times[0,\varepsilon]$ let $J_\varepsilon(x,t)=|\det D\Psi_\varepsilon(x)|$. Then, using also $t=\varepsilon s$,

$$\begin{aligned}
\mathscr{F}_\varepsilon(E) &= \frac{1}{\varepsilon}\int_{\partial E}\int_0^\varepsilon f\left(\int_{\frac{x-E}{\varepsilon}+\frac{t}{\varepsilon}\nu_E(x)} G(z)\mathrm{d}z\right)\,J_\varepsilon(x,t)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}t\\
&= \int_{\partial E}\int_0^1 f\left(\int_{\frac{x-E}{\varepsilon}+s\nu_E(x)} G(z)\mathrm{d}z\right)\,J_\varepsilon(x,\varepsilon s)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}s.
\end{aligned}$$

Since the regularity of E we have

$$\lim_{\varepsilon\to 0}J_\varepsilon(x,\varepsilon s)=1$$

from which, applying again (2.1),

$$\lim_{\varepsilon\to 0}\mathscr{F}_\varepsilon(E)=\int_{\partial E}\int_0^1 f\left(\int_{\{v\cdot\nu_E(x)\ge t\})} G(z)\,\mathrm{d}z\right)\,\mathrm{d}t\,\mathrm{d}\mathcal{H}^{N-1}(x)=\mathscr{F}(E).$$

5. The lower estimate

In this section we will prove our second main result, that is Theorem 3.2. First of all at any $x \in D$ we let

$$\nu_h(x) = \frac{(-\nabla u_h(x), 1)}{\sqrt{1 + |\nabla u_h(x)|^2}}.$$

It turns out that $\nu_h(x)$ is the exterior unit normal to $\partial^* E_h$ at $(x, u_h(x))$. For any $\eta > 0$ small enough

$$D^\eta = \{x \in D : d(x, \partial D) > \eta\}.$$

It turns out that $D^\eta \nearrow D$ in L^1 as $\eta \to 0^+$. If $z \in \mathbb{R}^N$ we will use the notation $z = (\bar{z}, z^N)$. We now divide the proof into several steps.

Step 1: We claim that for any $\sigma > 0$, for any $x \in D^{3\sigma}$ and for any $h \in \mathbb{N}$ with $\varepsilon_h < \sigma$ we have

$$\overline{B_2^{N-1}(0)} \subset \frac{x - D^\sigma}{\varepsilon_h}. \tag{5.1}$$

Indeed, $x \in D^{3\sigma}$ means that $\overline{B_{2\sigma}^{N-1}(x)} \subset D^\sigma$. If now $z \in \mathbb{R}^{N-1}$ and $|z| \leq 2$ then $|x - \varepsilon_h z - x| \leq 2\varepsilon_h < 2\sigma$ which implies that $x - \varepsilon_h z \in D^\sigma$ and then (5.1).

Step 2: For any $x \in D^{3\sigma}$, $s \in [0,1]$ and $\xi \in \mathbb{R}^{N-1}$ we let

$$a_h(x, s, \xi) = \frac{u_h(x) - u_h(x + \varepsilon_h s\overline{\nu_h(x)} - \varepsilon_h \xi)}{\varepsilon_h} + s\nu_h(x)^N.$$

We claim that

$$\lim_{h \to +\infty} a_h(x, s, \xi) = \nabla u(x) \cdot (\xi - s\overline{\nu(x)}) + s\nu(x)^N. \tag{5.2}$$

Indeed,

$$\begin{aligned}
&\frac{u_h(x + \varepsilon_h s\overline{\nu_h(x)} - \varepsilon_h \xi) - u_h(x)}{\varepsilon_h} \\
&= \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} \frac{\mathrm{d}}{\mathrm{d}t} u_h(x + t(s\overline{\nu_h(x)} - \xi))\,\mathrm{d}t \\
&= \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} \nabla u_h(x + t(s\overline{\nu_h(x)} - \xi)) \cdot (s\overline{\nu_h(x)} - \xi)\,\mathrm{d}t \\
&= \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} (\nabla u_h(x + t(s\overline{\nu_h(x)} - \xi)) - \nabla u(x + t(s\overline{\nu_h(x)} - \xi))) \cdot (s\overline{\nu_h(x)} - \xi)\,\mathrm{d}t \\
&\qquad + \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} (\nabla u(x + t(s\overline{\nu_h(x)} - \xi)) - \nabla u(x + t(s\overline{\nu(x)} - \xi))) \cdot (s\overline{\nu_h(x)} - \xi)\,\mathrm{d}t \\
&\qquad + \frac{1}{\varepsilon_h} \int_0^{\varepsilon_h} \nabla u(x + t(s\overline{\nu(x)} - \xi)) \cdot (s\overline{\nu_h(x)} - \xi)\,\mathrm{d}t =: I_1 + I_2 + I_3.
\end{aligned}$$

Concerning the first integral, we have

$$|I_1| \leq (s+|\xi|)\|\nabla u_h - \nabla u\|_\infty \to 0 \ \text{ as } h \to +\infty.$$

On the other hand, if L is the Lipschitz constant of ∇u, we get

$$I_2 \leq L(s+|\xi|)\|\overline{\nu_h} - \overline{\nu}\|_\infty \to 0 \ \text{ as } h \to +\infty.$$

Finally, for the third integral, let $g(t) = \nabla u(x + t(s\overline{\nu(x)} - \xi))$. Then g is continuous, hence

$$\lim_{h\to+\infty} \frac{1}{\varepsilon_h}\int_0^{\varepsilon_h} g(t)\,\mathrm{d}t = g(0)$$

from which

$$\lim_{h\to+\infty} I_3 = \nabla u(x)\cdot(s\overline{\nu(x)} - \xi)$$

as claimed.

Step 3: Let $M = \sup_h \|u_h\|_\infty$ and let $\sigma \in (0, M/2)$. We claim that for any $h \in \mathbb{N}$ with $\varepsilon_h < \sigma$ it holds

$$\mathscr{F}_{\varepsilon_h}(E_h) \geq \int_{D^{3\sigma}}\int_0^1 f\left(\int_{B_1^{N-1}(0)}\int_{a_h(x,s,\xi)}^1 G(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\right)\mathrm{d}s\,\sqrt{1+|\nabla u_h(x)|^2}\,\mathrm{d}x. \tag{5.3}$$

Indeed, first of all notice that

$$\{z \in E_h^c : B_{\varepsilon_h}(z)\cap E_h \neq \emptyset\} \supset \{(x - r\overline{\nu_h(x)}, u_h(x) + r\nu_h(x)^N) : x \in D^\sigma,\, r\in(0,\varepsilon_h)\},$$

As a consequence,

$$\mathscr{F}_{\varepsilon_h}(E_h) \geq \frac{1}{\varepsilon_h}\int_{D^{3\sigma}}\int_0^{\varepsilon_h} f\left(G_{\varepsilon_h} * \chi_{E_h}(x - r\overline{\nu_h(x)}, u_h(x) + r\nu_h(x)^N)\right)\,dr\,\sqrt{1+|\nabla u_h(x)|^2}\,\mathrm{d}x.$$

We concentrate now on the term $G_{\varepsilon_h} * \chi_{E_h}(x - r\overline{\nu_h(x)}, u_h(x) + r\nu_h(x)^N)$ and we rewrite it in a suitable way by performing some changes of variables. First of all, by noticing that $E_h = \{(z,w)\in D\times\mathbb{R} : 0 \leq w \leq u_h(z)\}$ we have

$$\begin{aligned}
&G_{\varepsilon_h} * \chi_{E_h}(x - r\overline{\nu_h(x)}, u_h(x) + r\nu_h(x)^N)\\
&\geq \int_{D^\sigma}\frac{1}{\varepsilon_h^N}\int_0^{u_h(z)} G\left(\frac{x - r\overline{\nu_h(x)} - z}{\varepsilon_h}, \frac{u_h(x) + r\nu_h(x)^N - w}{\varepsilon_h}\right)\mathrm{d}w\,\mathrm{d}z.
\end{aligned}$$

We now perform the change of variables in the following order:

$$\eta = \frac{u_h(x) + r\nu_h(x)^N - w}{\varepsilon_h}, \quad \xi = \frac{x + r\overline{\nu_h(x)} - z}{\varepsilon_h}.$$

We obtain

$$G_{\varepsilon_h} * \chi_{E_h}(x - r\overline{\nu_h(x)}, u_h(x) + r\nu_h(x)^N) \geq \int_{\frac{x+r\overline{\nu_h(x)}-D^\sigma}{\varepsilon_h}} \int_{a_h(x,r/\varepsilon_h,\xi)}^{\frac{u_h(x)+r\nu_h(x)^N}{\varepsilon_h}} G(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

Recalling that f is non-decreasing and operating the change of variable $r = \varepsilon_h s$ we arrive to

$$\mathscr{F}_{\varepsilon_h}(E_h) \geq \int_{D^{3\sigma}} \int_0^1 f\left(\int_{\frac{x-D^\sigma}{\varepsilon_h}+s\overline{\nu_h(x)}} \int_{a_h(x,s,\xi)}^{\frac{u_h(x)}{\varepsilon_h}+s\nu_h(x)^N} G(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\right) \mathrm{d}s\,\sqrt{1+|\nabla u_h(x)|^2}\,\mathrm{d}x.$$

Now, since (5.1) we deduce that for any $x \in D^{3\sigma}$ and for any $s \in [0,1]$

$$\frac{x-D^\sigma}{\varepsilon_h} + s\overline{\nu_h(x)} \supset \overline{B_2^{N-1}(0)} + s\overline{\nu_h(x)} \supset \overline{B_1^{N-1}(0)}.$$

Moreover, using $\sigma < M/2$ we get also

$$\frac{u_h(x)}{\varepsilon_h} + s\nu_h(x)^N > 1.$$

As a consequence, recalling that G is supported on $\overline{B_1^N(0)}$ we obtain (5.3).

Step 4: Passing to the limit as $h \to +\infty$ in (5.3), using Fatou's Lemma (5.2) and the Dominated convergence Theorem we obtain

$$\begin{aligned}
&\liminf_{h\to+\infty} \mathscr{F}_{\varepsilon_h}(E_h) \\
&\qquad \geq \int_{D^{3\sigma}} \int_0^1 f\left(\liminf_{h\to+\infty} \int_{\overline{B_1^{N-1}(0)}} \int_{a_h(x,s,\xi)}^1 G(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\right) \mathrm{d}s \sqrt{1+|\nabla u(x)|^2}\,\mathrm{d}x \\
&\qquad = \int_{D^{3\sigma}} \int_0^1 f\left(\int_{\overline{B_1^{N-1}(0)}} \int_{\nabla u(x)\cdot(\xi - s\overline{\nu(x)})+s\nu(x)^N}^1 G(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\right) \mathrm{d}s\,\sqrt{1+|\nabla u(x)|^2}\,\mathrm{d}x.
\end{aligned}$$

By the arbitrariness of σ small we get

$$\begin{aligned}
&\liminf_{h\to+\infty} \mathscr{F}_{\varepsilon_h}(E_h) \\
&\qquad \geq \int_D \int_0^1 f\left(\int_{\overline{B_1^{N-1}(0)}} \int_{\nabla u(x)\cdot(\xi - s\overline{\nu(x)})+s\nu(x)^N}^1 G(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\right) \mathrm{d}s\,\sqrt{1+|\nabla u(x)|^2}\,\mathrm{d}x.
\end{aligned}$$

step 5: We conclude the proof showing that

$$\int_D \int_0^1 f\left(\int_{\overline{B_1^{N-1}(0)}} \int_{\nabla u(x)\cdot(\xi - s\overline{\nu(x)})+s\nu(x)^N}^1 G(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\right) \mathrm{d}s\,\sqrt{1+|\nabla u(x)|^2}\,\mathrm{d}x = \mathscr{F}(E).$$

First of all, we notice that

$$\eta = \nabla u(x) \cdot (\xi - s\overline{\nu(x)}) + s\nu(x)^N = \nabla u(x) \cdot \xi + s\sqrt{1 + |\nabla u(x)|^2}$$

is the equation of an affine hyperplane in $\mathbb{R}^N$ orthogonal to $\nu(x)$ whose distance from the origin is

$$\frac{s\sqrt{1 + |\nabla u(x)|^2}}{\sqrt{1 + |\nabla u(x)|^2}} = s.$$

As a consequence,

$$\int_{\overline{B_1^{N-1}(0)}} \int_{\nabla u(x)\cdot(\xi - s\overline{\nu(x)}) + s\nu(x)^N}^{1} G(\xi, \eta)\, \mathrm{d}\eta\, \mathrm{d}\xi = \int_{\{z\cdot \nu_E(x,u(x)) \geq s\}} G(z)\, \mathrm{d}z$$

from which

$$\begin{aligned}
&\int_D \int_0^1 f\left(\int_{\overline{B_1^{N-1}(0)}} \int_{\nabla u(x)\cdot(\xi - s\overline{\nu(x)}) + s\nu(x)^N}^{1} G(\xi, \eta)\, \mathrm{d}\eta\, \mathrm{d}\xi \right) \mathrm{d}s\ \sqrt{1 + |\nabla u(x)|^2}\, \mathrm{d}x \\
&\quad = \int_D \int_0^1 f\left(\int_{\{z\cdot \nu_E(x,u(x)) \geq s\}} G(z)\, \mathrm{d}z \right) \mathrm{d}s\ \sqrt{1 + |\nabla u(x)|^2}\, \mathrm{d}x \\
&\quad = \int_{\partial E} \int_0^1 f\left(\int_{\{z\cdot \nu_E(y) \geq s\}} G(z)\, \mathrm{d}z \right) \mathrm{d}s\, \mathrm{d}\mathcal{H}^{N-1}(y) = \mathscr{F}(E)
\end{aligned}$$

and the proof is complete. □

6. L^1-lower semicontinuity of $\mathscr{F}$

We are going to prove Theorem 3.6. It is well known (see for instance [2], Thm. 5.14) that is sufficient to check that the positively one-homogeneous extension of θ given by

$$\tilde{\theta}(v) = \begin{cases} |v|\theta\left(\dfrac{v}{|v|}\right) \mathrm{d}t & \text{if } v \neq 0, \\ 0 & \text{if } v = 0, \end{cases}$$

is convex. First of all, by direct computation for each $v \in \mathbb{R}^N$ with $v \neq 0$ we have

$$\theta\left(\frac{v}{|v|}\right) = \int_0^1 f\left(\int_{\{z\cdot v \geq |v|t\}} G(z)\, \mathrm{d}z \right) \mathrm{d}t \overset{|v|t=s}{=} \frac{1}{|v|} \int_0^{|v|} f\left(\int_{\{z\cdot v \geq s\}} G(z)\, \mathrm{d}z \right) \mathrm{d}s$$

from which we obtain

$$\tilde{\theta}(v) = \begin{cases} \displaystyle\int_0^{|v|} f\left(\int_{\{z\cdot v \geq s\}} G(z)\, \mathrm{d}z \right) \mathrm{d}s & \text{if } v \neq 0, \\ 0 & \text{if } v = 0. \end{cases}$$

Now it is east to see that $\tilde{\theta}$ is convex. Indeed, since f is convex there exist $(\alpha_h),(\beta_h)$ such that

$$f=\lim_{h\to+\infty} f_h \text{ uniformly on compact sets, where } f_h(t)=\alpha_h t+\beta_h.$$

For any $h\in\mathbb{N}$ let

$$\tilde{\theta}_h(v)=\begin{cases} \displaystyle\int_0^{|v|} f_h\left(\int_{\{z\cdot v\geq s\}} G(z)\,\mathrm{d}z\right)\mathrm{d}s & \text{if } v\neq 0,\\ 0 & \text{if } v=0.\end{cases}$$

Since $f_h\to f$ uniformly on $[0,1]$ we can say that $\tilde{\theta}_h\to\tilde{\theta}$ pointwise. In order to conclude it is sufficient to show that $\tilde{\theta}_h$ is convex. For any $v\neq 0$ we let $\hat{v}=\frac{v}{|v|}$. Then

$$\begin{aligned}
\tilde{\theta}_h(v)&=\alpha_h\int_0^{|v|}\int_{\{z\cdot v\geq s\}} G(z)\,\mathrm{d}z\,\mathrm{d}s+\beta_h|v|\\
&=\alpha_h\int_0^{|v|}\int_{\{\bar{z}\cdot v=0\}}\int_{s/|v|}^{+\infty} G(\bar{z}+t\hat{v})\,\mathrm{d}\bar{z}\,\mathrm{d}t\,\mathrm{d}s+\beta_h|v|\\
&=\alpha_h\int_0^{+\infty}\int_{\{\bar{z}\cdot v=0\}}\int_0^{t|v|} G(\bar{z}+t\hat{v})\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\bar{z}+\beta_h|v|\\
&=\alpha_h|v|\int_0^{+\infty}\int_{\{\bar{z}\cdot v=0\}} tG(\bar{z}+t\hat{v})\,\mathrm{d}t\,\mathrm{d}\bar{z}+\beta_h|v|\\
&=\alpha_h\int_{\{\bar{z}\cdot v\geq 0\}} G(z)z\cdot v\,\mathrm{d}z+\beta_h|v|\\
&=\frac{\alpha_h}{2}\int_{\mathbb{R}^N} G(z)|z\cdot v|\,\mathrm{d}z+\beta_h|v|
\end{aligned}$$

where the last equality follows since G is even. Notice that the last expression is convex in v and this ends the proof.

7. Some examples

In this section we characterize the limit functional $\mathscr{F}$ in some interesting cases.

7.1. G radially symmetric

Assume $G(z)=g(|z|)$ for some $g\colon[0,+\infty)\to\mathbb{R}$. Take $\nu\in\mathbb{S}^{N-1}$ and $t\geq 0$. Notice that the quantity

$$\int_{\{z\cdot\nu\geq t\}} G(z)\,\mathrm{d}z$$

does not depend on ν. Take now $E\in\mathcal{P}_N$ and $x\in\partial^*E$. We have

$$\int_0^1 f\left(\int_{\{z\cdot\nu_E(x)\geq t\}} G(z)\,\mathrm{d}z\right)\mathrm{d}t=c$$

where c is a constant that depends only on N, f and G. Then

$$\mathscr{F}(E) = c\,\mathcal{H}^{N-1}(\partial^* E).$$

7.2. The case $f(t) = t$

When f is the identity function for any $E \in \mathcal{P}_N$ and for any $x \in \partial^* E$ we have

$$\begin{aligned}
\theta(\nu_E(x)) &= \int_0^1 \int_{H_{\nu_E(x)}+t\nu_E(x)} G(z)\,\mathrm{d}z\,\mathrm{d}t \\
&= \int_0^1 \int_{\{\bar{z}\cdot\nu_E(x)=0\}} \int_t^1 G(\bar{z}+s\nu_E(x))\,\mathrm{d}s\,\mathrm{d}\bar{z}\,\mathrm{d}t \\
&= \int_0^1 \int_{\{\bar{z}\cdot\nu_E(x)=0\}} \int_0^s G(\bar{z}+s\nu_E(x))\,\mathrm{d}t\,\mathrm{d}\bar{z}\,\mathrm{d}s \\
&= \int_0^1 \int_{\{\bar{z}\cdot\nu_E(x)=0\}} s\,G(\bar{z}+s\nu_E(x))\,\mathrm{d}\bar{z}\,\mathrm{d}s \\
&= \int_{H_{\nu_E(x)}} G(z) z\cdot\nu_E(x)\,\mathrm{d}z \\
&= \frac{1}{2}\int_{\mathbb{R}^N} G(z)|z\cdot\nu_E(x)|\,\mathrm{d}z.
\end{aligned}$$

Then the limit $\mathscr{F}$ is given by

$$\mathscr{F}(E) = \frac{1}{2}\int_{\partial^* E}\int_{\mathbb{R}^N} G(z)|z\cdot\nu_E(x)|\,\mathrm{d}z\,\mathrm{d}\mathcal{H}^{N-1}(x).$$

This is in accordance to [9].

Remark 7.1. If $N > 1$ and G is radially symmetric we have, if $g\colon [0,+\infty) \to \mathbb{R}$ is such that $G(z) = g(|z|)$,

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^N} G(z)|z\cdot\nu_E(x)|\,\mathrm{d}z &= \frac{1}{2}\int_{\mathbb{R}^N} g(|z|)|z\cdot\nu_E(x)|\,\mathrm{d}z \\
&= \frac{1}{2}\int_0^{+\infty}\int_{\mathbb{S}^{N-1}} g(r)r|\xi\cdot\nu_E(x)|\,dr\,\mathrm{d}\mathcal{H}^{N-1}(\xi) \\
&= |B_1^{N-1}(0)|\int_0^{+\infty} g(r)r\,dr \\
&= \frac{|B_1^{N-1}(0)|}{\mathcal{H}^{N-1}(\mathbb{S}^{N-1})}\int_{\mathbb{R}^N} G(z)|z|\,\mathrm{d}z
\end{aligned}$$

since it is well known that for any $\nu \in \mathbb{S}^{N-1}$ it holds

$$\frac{1}{2}\int_{\mathbb{S}^{N-1}} |\xi\cdot\nu|\,\mathrm{d}\mathcal{H}^{N-1}(\xi) = |B^{N-1}(0)|.$$

We thus deduce that

$$\mathscr{F}(E) = c_{N,G}\,\mathcal{H}^{N-1}(E), \quad c_{N,G} = \frac{|B_1^{N-1}(0)|}{\mathcal{H}^{N-1}(\mathbb{S}^{N-1})}\int_{\mathbb{R}^N} G(z)|z|\,\mathrm{d}z.$$

This is in accordance to [5].

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References

[1] G. Alberti and G. Bellettini, A non-local anisotropic model for phase transitions: asymptotic behaviour of rescaled energies. *Eur. J. Appl. Math.* **9** (1998) 261–284.

[2] L. Ambrosio, N. Fusco and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems. Oxford University Press (2000).

[3] J. Berendsen and V. Pagliari, On the asymptotic behaviour of nonlocal perimeters. *ESAIM: COCV* **25** (2019), no. 48.

[4] S. Esedoglu and M. Jacobs, Convolution kernels and stability of threshold dynamics methods. *SIAM J. Numer. Anal.* **55** (2017) 2123–2150.

[5] S. Esedoḡlu and F. Otto, Threshold dynamics for networks with arbitrary surface tensions. *Comm. Pure Appl. Math.* **68** (2015) 808–864.

[6] P. Gilkey and M. van den Berg, Heat content asymptotics of a Riemannian mainfold with boundary. *J. Funct. Anal.* **120** (1994) 48–71.

[7] M. Ledoux, Semigroup proofs of the isoperimetric inequality in Euclidean and Gauss space. *Bull. Sci. Math.* **118** (1994) 485–510.

[8] M. Miranda Jr., D. Pallara, F. Paronetto and M. Preunkert, Short-time heat flow and functions of bounded variation in $\mathbb{R}$. *Ann. Fac. Sci. Toulouse Math.* **16** (2007) 125–145.

[9] V. Pagliari, Halfspaces minimise nonlocal perimeter: a proof via calibrations. *Ann. Mat. Pura Appl.* **199** (2020) 1685–1696.

[10] J. Wloka, Partial Differential Equations. Cambridge University Press (1987).