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# STABILITY OF INVERSE SOURCE PROBLEM FOR A TRANSMISSION WAVE EQUATION WITH MULTIPLE INTERFACES OF DISCONTINUITY*

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**Abstract.** In this paper, we consider a transmission wave equation in $N$ embedded domains with multiple interfaces of discontinuous coefficients in $\mathbb{R}^2$. We study the global stability in determining the source term from a one-measurement data of wavefield velocity in a subboundary over a time interval. We prove the stability estimate for this inverse source problem by a combination of the local hyperbolic/elliptic Carleman estimates and the Fourier-Bros-Iagolniter transformation. Our method could be generalized to general dimensions since the weight functions and Carleman estimates are independent of the dimensions.

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## 1. Introduction

Let $\Omega_k$ $(k = 1, 2, \cdots, N)$ be a series of bounded connected open sets in $\mathbb{R}^2$. For $1 \le k \le N-1$, we assume that (1) $\overline{\Omega}_k \subset \Omega_{k+1}$; (2) The boundary of $\Omega_k$ (denoted by $S_k$) is smooth and strictly convex, which means the curvature is positive. We denote the inner (outer) side along $S_k$ by $S_k^-(S_k^+)$ and denote the jump of function $u$ across $S_k$ by $[u]_{S_k}$, *i.e.*, $[u]_{S_k} = u\big|_{S_k^+} - u\big|_{S_k^-}$.

In this paper, we consider a wave equation with discontinuous principal coefficient

$$a(x) = a_k, \quad x \in \Omega_k \setminus \overline{\Omega}_{k-1}, \quad 1 \le k \le N,$$

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where $a_k$ are constants satisfying $a_1 > a_2 > \cdots > a_N > 0$ and $\Omega_0 = \emptyset$. More precisely, we consider the following equation

$$\begin{cases} \partial_t^2 u(x,t) - a(x)\Delta u(x,t) = F(x,t), & \text{in } Q := \Omega_N \times (0,T), \\ u = 0, & \text{in } \Sigma := \partial\Omega_N \times (0,T), \\ u(x,0) = \partial_t u(x,0) = 0, & \text{in } \Omega_N, \end{cases} \tag{1.1}$$

with the transmission conditions on the interfaces

$$[u]_{S_k} = \left[ a \frac{\partial u}{\partial n_k} \right]_{S_k} = 0, \quad 1 \le k \le N-1, \tag{1.2}$$

where $\boldsymbol{n}_k$ denotes the outward unit normal vector of $S_k$. If $F \in L^2(Q)$, then equation (1.1) has a unique weak solution $u \in C([0,T]; H_0^1(\Omega_N)) \cap C^1([0,T]; L^2(\Omega_N))$, see [23]. Here we note that for each $F \in L^2(Q)$, $u$ solves the equation

$$\partial_t^2 u(x,t) - a(x)\Delta u(x,t) = F(x,t) \ \text{ in } Q$$

if and only if, for each $1 \le k \le N$, $u$ solves

$$\partial_t^2 u(x,t) - a_k \Delta u(x,t) = F(x,t) \ \text{ in } (\Omega_k \setminus \overline{\Omega}_{k-1}) \times (0,T).$$

System (1.1)–(1.2) arises naturally in geophysics, more precisely, in seismic prospection of the Earth's inner layers. There is much interest in recovering coefficients or the source term from boundary measurements. Condition $a_k > a_{k+1}$ and the convexity of the interfaces $S_k$ make the recovery be possible. In the case when $a_k < a_{k+1}$ or $S_k$ is not convex, the rays coming from the inner domain may be trapped and fail to arrive at the exterior boundary due to Snell's law.

In [4], Baudouin *et al.* prove the Lipschitz stability for a one-measurement inverse problem of identifying the coefficient in the zero-order term for $N = 2$, by using a global Carleman estimate and the method of Bukhgeim-Klibanov (B-K) [13, 14, 21]. Later in [25], Riahi proves the Hölder stability for the inverse problem of determining the spatial varied coefficient $a(x)$ in the principle term by two measurements, under an assumption that $a(x)$ is a piecewise constant near the interface. For the one-dimension case, we refer to Bellassoued and Yamamoto [8], in which the authors establish the global Carleman estimate for multiple interfaces. However, the weight function they used cannot work in higher dimension.

The B-K method is powerful for the formulation with a finite number of observations, there are many works on the uniqueness and the conditional stability, and the complete list is too long to be given here. To cite some of them, for example, see Baudouin *et al.* [3], Bellassoued [5], Bellassoued and Yamamoto [7], Imanuvilov and Yamamoto [18], where the key is to estabilish a global Carleman estimate for each problem. For further details on the Carleman estimate for hyperbolic equations, see for example, see Bellassoued and Yamamoto [8] and Imanuvilov [17]. On the other hand, for Carleman estimates for elliptic or parabolic equations with discontinuous coefficients , we refer to Benabdallah, Gaitan and Le Rousseau [11, 12], and Le Rousseau and Lerner [28].

In this article, we are interested in the case when there are more than one interface. Different from [4, 25], it is difficult to find a weight function satisfying conditions (a)–(f) listed in Section 2 in the whole domain when $N > 2$, hence we cannot expect to establish a global Carleman estimate. We have to apply local Carleman estimates which will inevitably create some remainder terms by using cut-off functions. In order to eliminate this remainder terms, we then combine an elliptic Carleman estimate and Fourier-Bros-Iagolniter (FBI) transformation [26, 27] which can turn the problem from hyperbolic case into elliptic one. Thus, the main achievements in this paper is that we combine the local Carleman estimates and the FBI transformation rather than the global Carleman estimate to prove the conditional stability of our inverse source problem. We remark that the method in this paper can work for general dimensions since the weight functions and Carleman estimates are independent of the dimensions.

Bellassoued and Yamamoto [6] also use FBI transformation to establish a weak observation estimate which shows the stability in the continuation of solutions from lateral boundary data on an arbitrarily small part of boundary, and prove logarithmic stability in the inverse coefficient problem with single measurement of arbitrarily small part of boundary data, provided that values of coefficients in neighborhood of the boundary are known, see also [9]. By following their strategy, we can determine the source term in our problem by one measurement of the solution on arbitrarily small part of boundary if the source term is known near the boundary.

### 1.1. Notations and the main theorem

Throughout this paper, for a function $v = v(x,t)$ with $x = (x_1, x_2) \in \mathbb{R}^2$, $t \in (0,T)$, we use the following notations:

$$\nabla v = (\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}), \quad \nabla_{x,t} v = (\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial t}), \quad D^2(v) = \Big(\frac{\partial^2 v}{\partial x_i \partial x_j}\Big)_{1\le i,j\le 2}.$$

For a multi-index $\alpha = (\alpha_1, \alpha_2)$, $\alpha_1, \alpha_2 \in \mathbb{N}$, and $X \in \mathbb{R}^2$, we define

$$|\alpha| = \alpha_1 + \alpha_2, \quad \frac{\partial^\alpha v}{\partial x^\alpha} = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}, \quad D^2(v)(X,X) = X^T D^2(v) X,$$

where $\cdot^T$ denotes the transpose of matrices. For a function $g : \mathbb{R} \to \mathbb{R}$, we denote $g \circ g(x) = g(g(x))$ and $g \circ g \circ g(x) = g(g(g(x)))$ and so on.

We arbitrarily choose two points $x_0, x_1 \in \Omega_1$ and fix $\delta > 0$ such that

$$\overline{B(x_0,\delta) \cup B(x_1,\delta)} \subset \Omega_1, \quad B(x_0,\delta) \cap B(x_1,\delta) = \emptyset.$$

Since $S_{N-1}$ is convex, let $y_*(x, x_j)$ be the unique point of $S_{N-1} \cap \{x_j + \lambda(x - x_j) : \lambda > 0\}$, $(j = 0,1)$ and $\boldsymbol{n}$ be the outward unit normal vector of $\Gamma := \partial\Omega_N$. We define the observation region by

$$\widetilde{\Gamma} = \bigcup_{j=0}^{1} \Big\{ x \in \Gamma : \nabla(\frac{|x - x_j|^2}{|y_*(x,x_j) - x_j|^2}) \cdot \boldsymbol{n} > 0 \Big\},$$

and set $\widetilde{\Sigma} = \widetilde{\Gamma} \times (0,T)$. In what follows, $C > 0$ denotes a generic constant which value may change from line to line.

**Theorem 1.1.** *Let $u$ be the solution of (1.1)–(1.2). Suppose that the source term is given by $F(x,t) = f(x)R(x,t)$ satisfying*

$$(1)\ |R(x,0)| \ge c_0 > 0, \quad (2)\ R \in H^1_{loc}(\mathbb{R}; L^\infty(\Omega_N)).$$

*Then if the observation time $T$ is large enough, there exist constants $C > 0$ and $\nu_1, \nu_2 > 0$ such that the following estimate holds for all $\gamma > 1$*

$$\|f\|^2_{L^2(\Omega_N)} \le C\gamma^{-\nu_1} \|v\|^2_{H^2(0,T;H^1(\Omega_N))} + C \underbrace{g \circ g \circ \cdots g}_{N-1}(\gamma) \Big\| \frac{\partial v}{\partial \boldsymbol{n}} \Big\|^2_{L^2(\widetilde{\Sigma})}. \tag{1.3}$$

*where $g(\gamma) = \mathrm{e}^{\nu_2 \gamma}$, $v = \partial_t u$ is the velocity of wavefield $u$.*

By Theorem 1.1 we can readily derive the stability and uniqueness of the inverse problem.

**Corollary 1.2.** *(Stability) Under the assumptions in Theorem 1.1, if $v$ satisfy $\|v\|_{H^2(0,T;H^1(\Omega_N))} < +\infty$ and the observation data satisfy $0 \le \big\|\frac{\partial v}{\partial \boldsymbol{n}}\big\|_{L^2(\widetilde{\Sigma})} \le 1$, then we have*

$$\|f\|^2_{L^2(\Omega_N)} \le C\Big(\underbrace{(\frac{1}{\nu_2}\ln)\circ(\frac{1}{\nu_2}\ln)\circ\cdots(\frac{1}{\nu_2}\ln)}_{N-1}\big(C+\frac{1}{\big\|\frac{\partial v}{\partial \boldsymbol{n}}\big\|_{L^2(\widetilde{\Sigma})}}\big)\Big)^{-\nu_1} \times\|v\|^2_{H^2(0,T;H^1(\Omega_N))} + C\Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|_{L^2(\widetilde{\Sigma})}. \tag{1.4}$$

*Proof.* We can assume $C$ is large enough and take

$$\gamma = (\frac{1}{\nu_2}\ln)\circ(\frac{1}{\nu_2}\ln)\circ\cdots(\frac{1}{\nu_2}\ln)\Big(C+\frac{1}{\big\|\frac{\partial v}{\partial \boldsymbol{n}}\big\|_{L^2(\widetilde{\Sigma})}}\Big) > 1.$$

□

**Corollary 1.3.** *(Uniqueness) Under the assumptions in Theorem 1.1, if $v$ satisfy $\|v\|_{H^2(0,T;H^1(\Omega_N))} < +\infty$ and the observation data $\frac{\partial v}{\partial \boldsymbol{n}} = 0$ on $\widetilde{\Sigma}$, then the source term $f(x) = 0$ in $\Omega_N$.*

The remainder of the paper is organized as follows. In Section 2, we give the local Carleman estimate for both hyperbolic and elliptic cases. In Section 3, we prove the main result, *i.e.*, Theorem 1.1. More particularly, we show the controllability of the source term by the solution and the boundary data in Section 3.1. And in Section 3.2, we show the controllability of the solution by the source term and the boundary data. Then in Section 3.3, we mix them together and prove Theorem 1.1. Section 4 is devoted to proving the Carleman estimate, *i.e.*, Lemma 2.1.

## 2. Carleman estimate in an annulus

In this section we present the Carleman estimate for the hyperbolic/elliptic operators in annuluses $\Omega_k \setminus \Omega_j$ $(1 \le j < k-1 \le N-1)$ and $\Omega_k \setminus \hat{\Omega}_0$ $(2 \le k \le N)$. Here $\hat{\Omega}_0$ is an arbitrarily fixed neighbourhood of $x_0$ contained in $\Omega_1$. For consistency, we use the notation $\Omega_k \setminus \Omega_j$ for $0 \le j < k-1 \le N-1$ and when we consider the case $j = 0$, we are actually concerning about $\Omega_k \setminus \hat{\Omega}_0$. For the following understanding, the $(k, j)$ can be regarded as being fixed.

In order to state the Carleman estimate, we need to choose a suitable weight function $\psi(x,t) = d(x) - \beta t^2$, $\beta > 0$ which satisfies some of the following conditions:

$(a)$ $\exists \delta_0 > 0,\ s.t.\ |\nabla\psi| \ge \delta_0, \quad \forall x \in \Omega_k \setminus (\Omega_j \cup S_{j+1} \cup \cdots \cup S_{k-1})$.

$(b)$ $[\psi]_{S_i} = [a\dfrac{\partial^\alpha \psi}{\partial x^\alpha}]_{S_i} = 0, \quad j < i < k,\ \forall|\alpha| \le 3$.

$(c)$ $a\dfrac{\partial \psi}{\partial n_i} \ge \delta_0, \quad \forall x \in S_i,\ j < i < k$.

$(d)$ $\psi(x, t_0)$ is a constant along $S_i$ at any moment $t = t_0$, *i.e.*, $\exists c_i$, *s.t.*

$$\psi(x,t) = c_i - \beta t^2, \quad \forall x \in S_i,\ j < i < k.$$

$(e)$ $\exists \delta_1 > 0,\ s.t.\ D^2(\psi)(X,X) \ge \delta_1|X|^2, \quad \forall x \in \Omega_k \setminus (\Omega_j \cup S_{j+1} \cup \cdots \cup S_{k-1}); X \in \mathbb{R}^2$.

$(f)$ $\beta < a_k\delta_1/2$.

Note that the condition (e) is valid for the two-dimensional case, but it could be generalized to general dimensions with suitable convexity assumptions [4]. We further set $\varphi(x,t) = \mathrm{e}^{\lambda\psi(x,t)}$ with $\lambda > 0$ and denote $\big(\Omega_k \setminus (\Omega_j \cup S_{j+1} \cup \cdots \cup S_{k-1})\big) \times (-T,T)$ by $Q_j^k$. We denote $C^3(Q_j^k)$ the set of functions for which the third derivative is continuous in $Q_j^k$. Notice that the weight function $\psi(x,t)$ depends on $(k,j)$. We have the following Carleman estimate with this weight function.

**Lemma 2.1.**
*(1) (hyperbolic case) If $\psi(x,t) \in C\big((\Omega_k \setminus \Omega_j) \times (-T,T)\big) \cap C^3(Q_j^k)$ satisfies conditions (a)-(f), then there exist constants $\lambda > 0$, $s_0 > 0$ and $C > 0$ such that for all $s > s_0$ and $u \in H^1\big(Q_j^k\big)$ satisfying $\partial_t^2 u - a\Delta u \in L^2(Q_j^k)$ and*

$$\begin{cases} [u]_{S_i} = \left[a\dfrac{\partial u}{\partial \boldsymbol{n}_i}\right]_{S_i} = 0, & j < i < k, \\ u = 0, & \text{on } \partial\Omega_k \times (-T,T), \\ u = 0, & \text{in a neighbourhood of } \partial\Omega_j, \\ u(x,\pm T) = \partial_t u(x,\pm T) = 0, & \text{in } \Omega_k \setminus \Omega_j, \end{cases} \tag{2.1}$$

*the following estimate holds*

$$\begin{aligned} &\int_{-T}^{T}\int_{\Omega_k\setminus\Omega_j} (s|\nabla_{x,t} w|^2 + s^3|w|^2 + |P_1 w|^2)\mathrm{d}x\mathrm{d}t \\ &\le C\Big(\int_{-T}^{T}\int_{\Omega_k\setminus\Omega_j} \mathrm{e}^{2s\varphi}|\partial_t^2 u - a\Delta u|^2\mathrm{d}x\mathrm{d}t + \int_{-T}^{T}\int_{\widetilde{\partial\Omega_k}} \mathrm{e}^{2s\varphi}\Big|\frac{\partial u}{\partial \boldsymbol{n}_k}\Big|^2\mathrm{d}\sigma\mathrm{d}t\Big), \end{aligned}$$

*where $w = \mathrm{e}^{s\varphi}u$, and*

$$P_1 = \partial_t^2 - a\Delta + s^2\lambda^2\varphi^2(|\partial_t\psi|^2 - a|\nabla\psi|^2)$$

*is a second-order linear operator, and $\widetilde{\partial\Omega_k} = \big\{x \in \partial\Omega_k : \frac{\partial\psi}{\partial n_k} > 0\big\}$.*

*(2) (elliptic case) If $\psi(x,t) \in C\big((\Omega_k \setminus \Omega_j) \times (-T,T)\big) \cap C^3(Q_j^k)$ satisfies conditions $(a)-(d)$, then there exist constants $\lambda > 0$, $s_0 > 0$ and $C > 0$ such that for all $s > s_0$ and $u \in H^1\big(Q_j^k\big)$ satisfying $\partial_t^2 u + a\Delta u \in L^2(Q_j^k)$ and (2.1), the following estimate holds*

$$\begin{aligned} &\int_{-T}^{T}\int_{\Omega_k\setminus\Omega_j} \mathrm{e}^{2s\varphi}(s|\nabla_{x,t} u|^2 + s^3|u|^2)\mathrm{d}x\mathrm{d}t \\ &\le C\Big(\int_{-T}^{T}\int_{\Omega_k\setminus\Omega_j} \mathrm{e}^{2s\varphi}|\partial_t^2 u + a\Delta u|^2\mathrm{d}x\mathrm{d}t + \int_{-T}^{T}\int_{\widetilde{\partial\Omega_k}} \mathrm{e}^{2s\varphi}\Big|\frac{\partial u}{\partial \boldsymbol{n}_k}\Big|^2\mathrm{d}\sigma\mathrm{d}t\Big). \end{aligned}$$

**Remark 2.2.** We give the following four remarks on Lemma 2.1.

(1) $P_1 w$ in the hyperbolic case of Lemma 2.1 plays a role in resulting in Corollary 2.3. We will give the proof of Lemma 2.1 in Section 4.

(2) In the case of multiple interfaces, the existence of the weight function satisfying conditions (a)-(f) is still unknown. Or we can say it is hard to construct. However, we can construct a weight function satisfying conditions (a)-(d). In the next Section 3, we will first apply the hyperbolic case of Lemma 2.1 to annulus $\Omega_{k+1} \setminus \Omega_{k-1}$ $(1 \le k \le N-1)$ in Section 3.1, namely an annulus with only one interface. Then in Section 3.2, we will apply the elliptic case of Lemma 2.1 to annulus $\Omega_N \setminus \Omega_{k-1}$ $(2 \le k \le N-1)$, namely an annulus with multiple interfaces.

(3) In [4], the authors construct a weight function satisfying conditions (a)-(f) when there is only one interface in the annulus. Let us consider annulus $\Omega_{k+1} \setminus \Omega_{k-1}$ $(1 \le k \le N-1)$, since $S_k$ is convex there is exactly one point $y_k(x)$ such that

$$y_k(x) \in S_k \cap \big\{x_0 + \lambda(x - x_0) : \lambda > 0\big\}.$$

We define the weight function

$$\psi_k(x,t) = \begin{cases} a_k \dfrac{|x-x_0|^2}{|y_k(x)-x_0|^2} - \beta t^2 + M - a_k, & x \in \Omega_{k+1} \setminus \Omega_k, \\ a_{k+1} \dfrac{|x-x_0|^2}{|y_k(x)-x_0|^2} - \beta t^2 + M - a_{k+1}, & x \in \Omega_k \setminus \Omega_{k-1}, \end{cases} \tag{2.2}$$

where $M$ is chosen to let $\psi_k$ be positive. By Proposition 1 in [4] (p. 263), $\psi_k$ satisfies conditions (a)–(e), and we choose $\beta = a_{k+1}\delta_1/3$ so that the condition (f) is satisfied.

(4) If we can construct a global weight function satisfying conditions (a)–(f) in the annulus with multiple interfaces, then we can just follow the strategy in [4] to simplify the proof without using the FBI transformation later applied in this paper and improve the stability rate in Theorem 1.1 further.

**Corollary 2.3.** *If $\psi(x,t) \in C\big((\Omega_k \setminus \Omega_j) \times (-T,T)\big) \cap C^3(Q_j^k)$ satisfies conditions (a)-(f), then there exist constants $\lambda > 0$, $s_0 > 0$ and $C > 0$ such that for all $s > s_0$ and $u \in H^1\big(Q_j^k\big)$ satisfying $\partial_t^2 u - a\Delta u \in L^2(Q_j^k)$ and (2.1), the following estimate holds*

$$\begin{aligned} &\sqrt{s} \int_{\Omega_k \setminus \Omega_j} |\partial_t w(x,0)|^2 \mathrm{d}x \\ \le{}& C\Big( \int_{-T}^{T} \int_{\Omega_k \setminus \Omega_j} \mathrm{e}^{2s\varphi} |\partial_t^2 u - a\Delta u|^2 \mathrm{d}x\mathrm{d}t + \int_{-T}^{T} \int_{\widetilde{\partial\Omega_k}} \mathrm{e}^{2s\varphi} \Big|\frac{\partial u}{\partial \boldsymbol{n}_k}\Big|^2 \mathrm{d}\sigma\mathrm{d}t \Big). \end{aligned}$$

*Proof.* By integration by parts we can check that

$$\begin{aligned} &\int_{-T}^{0} \int_{\Omega_k \setminus \Omega_j} P_1 w \partial_t w \ \mathrm{d}x\mathrm{d}t \\ =&\frac{1}{2} \int_{\Omega_k \setminus \Omega_j} |\partial_t w(x,0)|^2 \mathrm{d}x - \frac{1}{2} s^2\lambda^2 \int_{-T}^{0} \int_{\Omega_k \setminus \Omega_j} |w|^2 \partial_t \big(\varphi^2 (|\partial_t \psi|^2 - a|\nabla\psi|^2)\big) \ \mathrm{d}x\mathrm{d}t. \end{aligned}$$

So

$$\begin{aligned} \int_{\Omega_k \setminus \Omega_j} |\partial_t w(x,0)|^2 \mathrm{d}x \le& \int_{-T}^{T} \int_{\Omega_k \setminus \Omega_j} \frac{1}{\sqrt{s}} |P_1 w|^2 + \sqrt{s} |\partial_t w|^2 \ \mathrm{d}x\mathrm{d}t \\ &+ Cs^2 \int_{-T}^{T} \int_{\Omega_k \setminus \Omega_j} |w|^2 \ \mathrm{d}x\mathrm{d}t, \end{aligned}$$

then by the hyperbolic case of Lemma 2.1 we complete the proof. □

## 3. Stability of the inverse source problem

In this section, we apply Lemma 2.1 to obtain Lipschitz stability of inverse source problem for (1.1). We take the even extension of $R$ and $u$ to the interval $(-T,T)$. We call this function in the same way and let $v=\partial_t u$, then $v(x,t)$ satisfies the following system

$$\begin{cases}\partial_t^2 v(x,t)-a(x)\Delta v(x,t)=f(x)\partial_t R(x,t), & \text{in } \Omega_N\times(-T,T),\\ v=0, & \text{on } \Gamma\times(-T,T),\\ v(x,0)=0,\quad \partial_t v(x,0)=f(x)R(x,0), & \text{in } \Omega_N,\\ [v]_{S_k}=\left[a\dfrac{\partial v}{\partial n_k}\right]_{S_k}=0, & 1\le k\le N-1.\end{cases}$$

In order to fulfil the condition of Lemma 2.1, we take a cut-off function $\eta(t)\in C^\infty(\mathbb{R})$ which vanishes at $t=\pm T$. We let $\eta(t)$ satisfy $0\le\eta(t)\le 1$ and

$$\eta(t)=\begin{cases}1, & |t|\le 2T/3,\\ 0, & |t|\ge 3T/4.\end{cases}$$

We then set $y(x,t)=\eta(t)v(x,t)$ and it satisfies

$$\begin{cases}\partial_t^2 y(x,t)-a(x)\Delta y(x,t)=G(x,t)+H(x,t), & \text{in } \Omega_N\times(-T,T),\\ y=0, & \text{on } \Gamma\times(-T,T),\\ y(x,0)=0,\quad \partial_t y(x,0)=f(x)R(x,0), & \text{in } \Omega_N,\\ [y]_{S_k}=\left[a\dfrac{\partial y}{\partial n_k}\right]_{S_k}=0, & 1\le k\le N-1\end{cases}\tag{3.1}$$

where

$$G(x,t)=\eta(t)f(x)\partial_t R(x,t),\quad H(x,t)=2\eta'(t)\partial_t v+\eta''(t)v.$$

Recall that $x_0\in\Omega_1$ and $\overline{B(x_0,\delta)}\subset\Omega_1$. For $x\ne x_0$ and $1\le k\le N-1$, note that $y_k(x)$ is the unique point of $\{x_0+\lambda(x-x_0):\lambda>0\}\cap S_k$. So $\Omega_k$ can be represented by

$$\Omega_k=\left\{x\ne x_0:\frac{|x-x_0|^2}{|y_k(x)-x_0|^2}<1\right\}\cup\{x_0\}.$$

Since $\overline{\Omega}_{k-1}\subset\Omega_k$ $(1\le k\le N-1)$, we can choose constant $r_k>0$ such that

$$\overline{\Omega}_{k-1}\subset\left\{x\ne x_0:\frac{|x-x_0|^2}{|y_k(x)-x_0|^2}<1-6r_k\right\}\cup\{x_0\}\subset\Omega_k.$$

We can further let $r_1$ satisfy (since $\Omega_0$ is an empty set)

$$\left\{x\ne x_0:\frac{|x-x_0|^2}{|y_1(x)-x_0|^2}<1-4r_1\right\}\cup\{x_0\}\subset B(x_0,\delta).\tag{3.2}$$

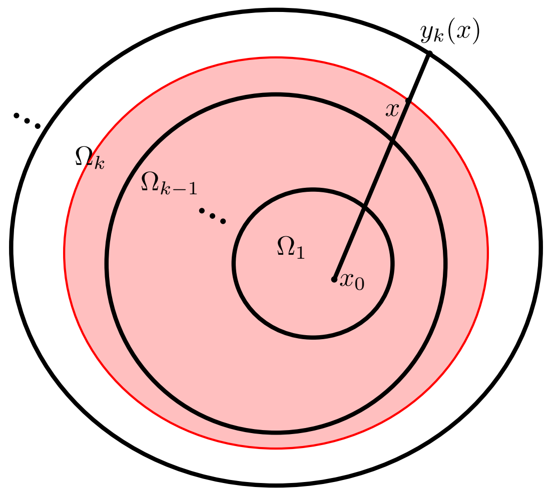



FIGURE 1. The sketch map of the domain $D_k(r)$ indicated with pink shaded area.

We define

$$D_k(r) := \Big\{ x \neq x_0 : \frac{|x-x_0|^2}{|y_k(x)-x_0|^2} < 1-r \Big\} \cup \{x_0\}.$$

It is easy to see that $\overline{\Omega}_{k-1} \subset D_k(\epsilon_2) \subset D_k(\epsilon_1) \subset \Omega_k$ for all $0 < \epsilon_1 < \epsilon_2 \leq 6r_k$. Figure 1 is the sketch map of the domain $D_k$ indicated with the pink shaded area. The red circle in Figure 1 is determined by

$$\frac{|x-x_0|^2}{|y_k(x)-x_0|^2} = 1-r.$$

### 3.1. Control $f(x)$ by $y(x,t)$ and the boundary

First we present the following lemma.

**Lemma 3.1.** *There exist constants $C > 0$, $T_{N-1} > 0$ and $\nu_{N-1} \in (0,1)$ such that*

$$\|f\|^2_{L^2(\Omega_N \setminus D_{N-1}(4r_{N-1}))} \leq C\Big(\|v\|^2_{H^1(Q)} + \Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big)^{1-\nu_{N-1}} \Big(\Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big)^{\nu_{N-1}}$$

*for all $T \geq T_{N-1}$.*

*Proof.* We take a cut-off function $\xi_{N-1}(x) \in C^\infty(\mathbb{R}^2)$ satisfying $0 \leq \xi_{N-1} \leq 1$ and

$$\xi_{N-1}(x) = \begin{cases} 1, & x \in \Omega_N \setminus D_{N-1}(5r_{N-1}), \\ 0, & x \in \overline{D_{N-1}(6r_{N-1})}. \end{cases}$$

Then we set $z_{N-1}(x,t) = \xi_{N-1}(x)y(x,t)$ and we have

$$\begin{cases} \partial_t^2 z_{N-1} - a\Delta z_{N-1} = \xi_{N-1}\big(\eta(t)f(x)\partial_t R(x,t) + H(x,t)\big) + S_{N-1}(x,t), \\ z_{N-1}(x,0) = 0, \quad \partial_t z_{N-1}(x,0) = \xi_{N-1}(x)f(x)R(x,0), \\ [z_{N-1}]_{S_{N-1}} = \left[a\dfrac{\partial z_{N-1}}{\partial \boldsymbol{n}_{N-1}}\right]_{S_{N-1}} = 0, \end{cases}$$

in annulus $(\Omega_N \setminus \Omega_{N-2}) \times (-T,T)$ with the Dirichlet boundary condition

$$z_{N-1}(x,t) = 0, \quad (x,t) \in \partial(\Omega_N \setminus \Omega_{N-2}) \times (-T,T),$$

and $S_{N-1}(x,t) = -2a\nabla y \cdot \nabla \xi_{N-1} - ay\Delta\xi_{N-1}$ is supported in $\overline{D_{N-1}(5r_{N-1}) \setminus D_{N-1}(6r_{N-1})} \times (-T,T)$. In this annulus, we take the weight function $\psi_{N-1}(x,t)$ by (2.2) for $k = N-1$. Here and henceforth we choose $\beta$ in the weight function satisfying condition (f) in Section 2. We further choose $T_{N-1}$ satisfying

$$\beta T_{N-1}^2 \geq 4\Big(\sup_{x\in\Omega_N\setminus\Omega_{N-1}} \frac{|x-x_0|^2}{|y_{N-1}(x)-x_0|^2} - 1\Big)a_{N-1} + 20r_{N-1}a_N,$$

then we have for all $T \geq T_{N-1}$,

$$\psi_{N-1}(x,t) \leq M - 5r_{N-1}a_N, \quad \forall\ T/2 < |t| < T, \quad x \in \Omega_N \setminus \Omega_{N-2}.$$

We set $\varphi_{N-1} = \mathrm{e}^{\lambda\psi_{N-1}}$ and $w_{N-1} = \mathrm{e}^{s\varphi_{N-1}}z_{N-1}$. By Corollary 2.3, we know that there exist $\lambda > 0$, $s_0 > 0$ and $C > 0$ such that for all $s > s_0$, we have

$$\begin{aligned} &\sqrt{s}\int_{\Omega_N\setminus\Omega_{N-2}} |\partial_t w_{N-1}(x,0)|^2 \mathrm{d}x \\ \leq & C\Big(\int_{-T}^{T}\int_{\Omega_N\setminus\Omega_{N-2}} \mathrm{e}^{2s\varphi_{N-1}}\Big(|\xi_{N-1}(x)\eta(t)f(x)\partial_t R(x,t)|^2 + |\xi_{N-1}(x)H(x,t)|^2 \\ &+ |S_{N-1}(x,t)|^2\Big)\mathrm{d}x\mathrm{d}t + \int_{-T}^{T}\int_{\widetilde{\Gamma}} \mathrm{e}^{2s\varphi_{N-1}}\Big|\frac{\partial z_{N-1}}{\partial \boldsymbol{n}}\Big|^2 \mathrm{d}\sigma\mathrm{d}t\Big). \end{aligned} \tag{3.3}$$

Since $|R(x,0)| \geq c_0 > 0$ and $R \in H^1\big(0,T;L^\infty(\Omega_N)\big)$, we have

$$\begin{aligned} \sqrt{s}\int_{\Omega_N\setminus\Omega_{N-2}} |\partial_t w_{N-1}(x,0)|^2 \mathrm{d}x &= \sqrt{s}\int_{\Omega_N\setminus\Omega_{N-2}} \mathrm{e}^{2s\varphi_{N-1}(x,0)}|\xi_{N-1}(x)f(x)R(x,0)|^2\mathrm{d}x \\ &\geq c_0^2\sqrt{s}\int_{\Omega_N\setminus\Omega_{N-2}} \mathrm{e}^{2s\varphi_{N-1}(x,0)}|\xi_{N-1}(x)f(x)|^2\mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} &\int_{-T}^{T}\int_{\Omega_N\setminus\Omega_{N-2}} \mathrm{e}^{2s\varphi_{N-1}(x,t)}|\xi_{N-1}(x)\eta(t)f(x)\partial_t R(x,t)|^2\mathrm{d}x\mathrm{d}t \\ \leq & 2\|R\|_{H^1(0,T;L^\infty(\Omega_N))}\int_{\Omega_N\setminus\Omega_{N-2}} \mathrm{e}^{2s\varphi_{N-1}(x,0)}|\xi_{N-1}(x)f(x)|^2\mathrm{d}x. \end{aligned}$$

Without loss of generality, we assume that $c_0^2\sqrt{s_0} \geq 2C\|R\|_{H^1(0,T;L^\infty(\Omega_N))} + 1$. Hence (3.3) implies

$$\begin{aligned}&\int_{\Omega_N\setminus\Omega_{N-2}} \mathrm{e}^{2s\varphi_{N-1}(x,0)}|\xi_{N-1}(x)f(x)|^2\mathrm{d}x\\ &\leq C\Big(\int_{-T}^{T}\int_{\Omega_N\setminus\Omega_{N-2}} \mathrm{e}^{2s\varphi_{N-1}(x,t)}\Big(|H(x,t)|^2+|S_{N-1}(x,t)|^2\Big)\mathrm{d}x\mathrm{d}t\\ &\quad+\int_{-T}^{T}\int_{\widetilde{\Gamma}}\mathrm{e}^{2s\varphi_{N-1}}\Big|\frac{\partial z_{N-1}}{\partial \boldsymbol{n}}\Big|^2\mathrm{d}\sigma\mathrm{d}t\Big),\end{aligned}\tag{3.4}$$

Since

$$\psi_{N-1}(x,t)\leq M-5r_{N-1}a_N,\quad \forall\ T/2<|t|<T,\quad x\in D_{N-1}(5r_{N-1})\setminus D_{N-1}(6r_{N-1}),$$

we have

$$\begin{aligned}&\int_{-T}^{T}\int_{\Omega_N\setminus\Omega_{N-2}} \mathrm{e}^{2s\varphi_{N-1}(x,t)}\Big(|H(x,t)|^2+|S_{N-1}(x,t)|^2\Big)\mathrm{d}x\mathrm{d}t\\ =&2\int_{2T/3}^{T}\int_{\Omega_N\setminus\Omega_{N-2}} \mathrm{e}^{2s\varphi_{N-1}(x,t)}|H(x,t)|^2\mathrm{d}x\mathrm{d}t\\ &+\int_{-T}^{T}\int_{D_{N-1}(5r_{N-1})\setminus D_{N-1}(6r_{N-1})} \mathrm{e}^{2s\varphi_{N-1}(x,t)}|S_{N-1}(x,t)|^2\mathrm{d}x\mathrm{d}t\\ \leq& C\mathrm{e}^{2s\mathrm{e}^{\lambda(M-5r_{N-1}a_N)}}\|v\|^2_{H^1(Q)}.\end{aligned}$$

Combining this inequality with (3.4), we obtain

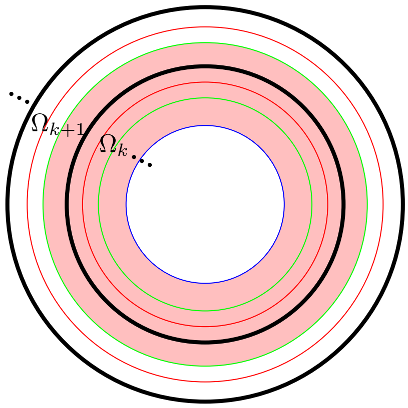
$$\begin{aligned}&\int_{\Omega_N\setminus D_{N-1}(4r_{N-1})} \mathrm{e}^{2s\mathrm{e}^{\lambda(M-4r_{N-1}a_N)}}|f(x)|^2\mathrm{d}x\\ \leq&\int_{\Omega_N\setminus\Omega_{N-2}} \mathrm{e}^{2s\varphi_{N-1}(x,0)}|\xi_{N-1}(x)f(x)|^2\mathrm{d}x\\ \leq& C\Big(\mathrm{e}^{2s\mathrm{e}^{\lambda(M-5r_{N-1}a_N)}}\|v\|^2_{H^1(Q)}+\mathrm{e}^{2s\mathrm{e}^{2\lambda M}}\Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Gamma})}\Big).\end{aligned}\tag{3.5}$$

This implies

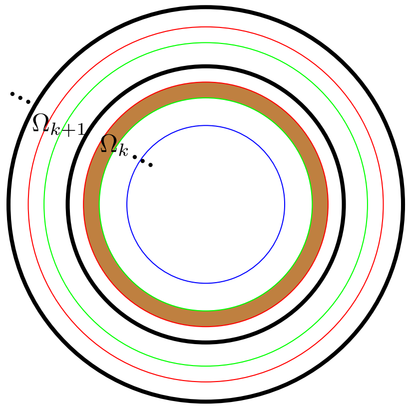
$$\begin{aligned}&\|f\|^2_{L^2(\Omega_N\setminus D_{N-1}(4r_{N-1}))}\\ \leq& C\Big(\mathrm{e}^{-2sk_1}\|v\|^2_{H^1(Q)}+\mathrm{e}^{2sk_2}\Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\Sigma^+)}\Big)\\ \leq& C\mathrm{e}^{2s_0k_2}\Big(\mathrm{e}^{-2(s-s_0)k_1}\big(\|v\|^2_{H^1(Q)}+\Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Gamma})}\big)+\mathrm{e}^{2(s-s_0)k_2}\Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Gamma})}\Big),\end{aligned}\tag{3.6}$$

for all $s>s_0$, where

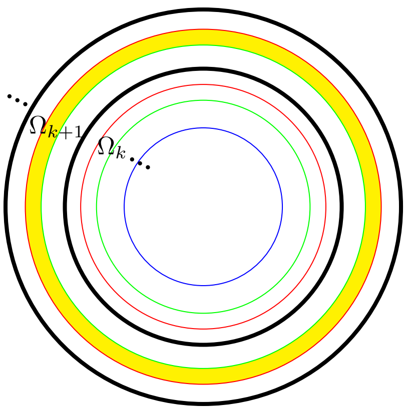
$$k_1=\mathrm{e}^{\lambda(M-4r_{N-1}a_N)}-\mathrm{e}^{\lambda(M-5r_{N-1}a_N)}>0,\quad k_2=\mathrm{e}^{2\lambda M}-\mathrm{e}^{\lambda(M-4r_{N-1}a_N)}>0.$$




(a) $A_k$




(b) $O_k$




(c) $O_{k+1}$

FIGURE 2. The sketch maps of domains $A_k$, $O_k$ and $O_{k+1}$ indicated with different shaded areas. (a) domain $A_k$ with pink shaded area, (b) domain $O_k$ with brown shaded area, (c) domain $O_{k+1}$ with yellow shaded area.

Now minimizing the right-hand side of (3.6) with respect to $s - s_0 > 0$, we have

$$\|f\|^2_{L^2(\Omega_N \setminus D_{N-1}(4r_{N-1}))} \leq C\Big(\|v\|^2_{H^1(Q)} + \Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big)^{1-\nu_{N-1}} \Big(\Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Gamma})}\Big)^{\nu_{N-1}},$$

where $\nu_{N-1} = \frac{k_1}{k_1+k_2}$. The proof of Lemma 3.1 is complete. □

In the following we extend the estimation layer by layer and finally cover almost the whole domain $\Omega_N \setminus B(x_0, \delta)$. More precisely, we set

$$A_k := D_{k+1}(2r_{k+1}) \setminus D_k(4r_k), \quad O_k := D_k(r_k) \setminus D_k(2r_k).$$

We will control $f(x)$ in the annulus $A_k$ by $y(x,t)$ in $O_{k+1} \times (-\frac{T}{2}, \frac{T}{2})$. Figure 2 shows the sketch maps of domains $A_k$, $O_k$ and $O_{k+1}$ indicated with pink, brown and yellow shaded areas respectively.

**Lemma 3.2.** *For $1 \leq k \leq N-2$, there exist constants $C>0$, $T_k>0$ and $\nu_k \in (0,1)$ such that*

$$\|f\|^2_{L^2(A_k)} \leq C\Big(\|v\|^2_{H^1(Q)} + \|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(O_{k+1}))}\Big)^{1-\nu_k}\Big(\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(O_{k+1}))}\Big)^{\nu_k},$$

*for all $T \geq T_k$.*

*Proof.* Similar to the proof of Lemma 3.1. We take a cut-off function $\xi_k(x) \in C^\infty(\mathbb{R}^2)$ satisfying $0 \leq \xi_k \leq 1$ and

$$\xi_k(x) = \begin{cases} 1, & x \in D_{k+1}(2r_{k+1}) \setminus D_k(5r_k), \\ 0, & x \in D_k(6r_k) \cup \big(\Omega_N \setminus D_{k+1}(r_{k+1})\big). \end{cases}$$

Then we set $z_k(x,t) = \xi_k(x)y(x,t)$ and we have

$$\begin{cases} \partial_t^2 z_k - a\Delta z_k = \xi_k\big(\eta(t)f(x)\partial_t R(x,t) + H(x,t)\big) + S_k(x,t), \\ z_k(x,0) = 0, \quad \partial_t z_k(x,0) = \xi_k(x)f(x)R(x,0), \\ [z_k]_{S_k} = \left[a\dfrac{\partial z_k}{\partial \boldsymbol{n}_k}\right]_{S_k} = 0, \end{cases}$$

in annulus $(\Omega_{k+1} \setminus \Omega_{k-1}) \times (-T,T)$ with the Dirichlet boundary condition

$$z_k(x,t) = 0, \quad (x,t) \in \partial(\Omega_{k+1} \setminus \Omega_{k-1}) \times (-T,T),$$

and $S_k(x,t) = -2a\nabla y \cdot \nabla \xi_k - ay\Delta\xi_k$ vanishes outside $O_{k+1} \cup D_k(5r_k) \setminus D_k(6r_k)$. In order to adapt with the concepts in Section 2, here and henceforth we should replace $\Omega_0$ by $D_1(6r_1)$ when we mention $\Omega_2 \setminus \Omega_0$.

In this annulus, we take the weight function (2.2) and set $\varphi_k(x,t) = \mathrm{e}^{\lambda\psi_k(x,t)}$. Following the steps in Lemma 3.1, there exists $T_k > 0$, for all $T \geq T_k$, we deduce

$$\begin{aligned} &\int_{\Omega_{k+1}\setminus\Omega_{k-1}} \mathrm{e}^{2s\varphi_k(x,0)}|\xi_k(x)f(x)|^2 \mathrm{d}x \\ \leq\, & C\int_{-T}^{T}\int_{\Omega_{k+1}\setminus\Omega_{k-1}} \mathrm{e}^{2s\varphi_k(x,t)}\Big(|H(x,t)|^2 + |S_k(x,t)|^2\Big)\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} &\int_{-T}^{T}\int_{\Omega_{k+1}\setminus\Omega_{k-1}} \mathrm{e}^{2s\varphi_k(x,t)}|H(x,t)|^2 \mathrm{d}x\mathrm{d}t \\ =\; & 2\int_{2T/3}^{T}\int_{\Omega_{k+1}\setminus\Omega_{k-1}} \mathrm{e}^{2s\varphi_k(x,t)}|H(x,t)|^2 \mathrm{d}x\mathrm{d}t \\ \leq\; & C\mathrm{e}^{2s\mathrm{e}^{\lambda(M-5r_k a_{k+1})}}\|v\|^2_{H^1(Q)}, \end{aligned} \tag{3.8}$$

$$
\begin{aligned}
&\int_{-T}^{T}\int_{\Omega_{k+1}\setminus\Omega_{k-1}} \mathrm{e}^{2s\varphi_k(x,t)}|S_k(x,t)|^2\mathrm{d}x\mathrm{d}t \\
=\ &\int_{-T}^{T}\Big(\int_{D_k(5r_k)\setminus D_k(6r_k)}+\int_{O_{k+1}}\Big)\mathrm{e}^{2s\varphi_k(x,t)}|S_k(x,t)|^2\mathrm{d}x\mathrm{d}t \qquad (3.9)\\
\leq\ & C\Big(\mathrm{e}^{2s\mathrm{e}^{\lambda(M-5r_ka_{k+1})}}\|v\|^2_{H^1(Q)}+\mathrm{e}^{2s\mathrm{e}^{2\lambda M}}\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(O_{k+1}))}\Big),
\end{aligned}
$$

Now we combine (3.7)–(3.9) together to get

$$
\begin{aligned}
&\int_{A_k}\mathrm{e}^{2s\mathrm{e}^{\lambda(M-4r_ka_{k+1})}}|f(x)|^2\mathrm{d}x\\
\leq&\int_{\Omega_{k+1}\setminus\Omega_{k-1}}\mathrm{e}^{2s\varphi_k(x,0)}|\xi_k(x)f(x)|^2\mathrm{d}x\\
\leq& C\Big(\mathrm{e}^{2s\mathrm{e}^{\lambda(M-5r_ka_{k+1})}}\|v\|^2_{H^1(Q)}+\mathrm{e}^{2s\mathrm{e}^{2\lambda M}}\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(O_{k+1}))}\Big).
\end{aligned}
$$

Then by the argument like Lemma 3.1, we can find $\nu_k\in(0,1)$ such that

$$
\|f\|^2_{L^2(A_k)}\leq C\Big(\|v\|^2_{H^1(Q)}+\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(O_{k+1}))}\Big)^{1-\nu_k}\Big(\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(O_{k+1}))}\Big)^{\nu_k}.
$$

The proof of Lemma 3.2 is complete. □

**Corollary 3.3.** *We set*

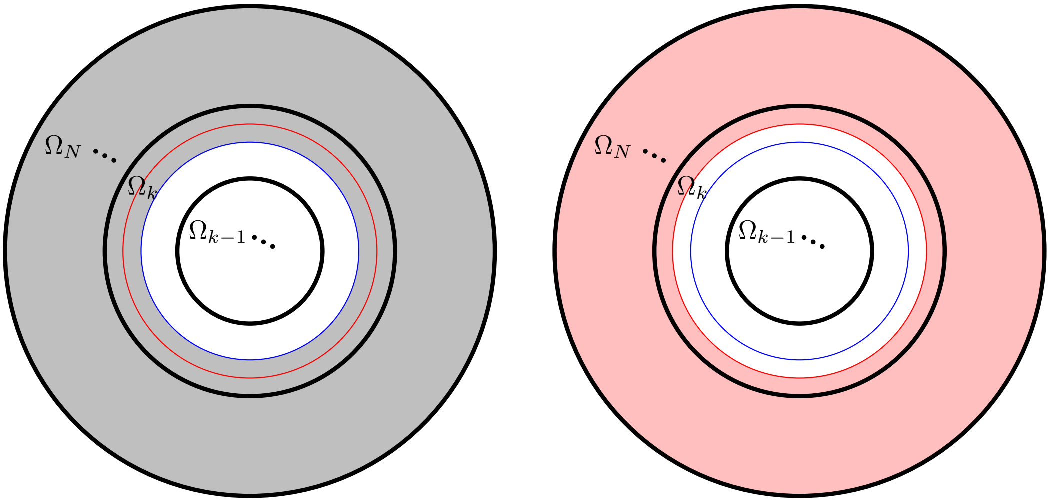
$$
\widetilde{A}_k:=\Omega_N\setminus D_k(4r_k),\quad \widetilde{O}_k:=\Omega_N\setminus D_k(2r_k)
$$

*for $k=1,\cdots,N-1$ and $\widetilde{O}_N=\emptyset$. There exist constants $C>0$ and $p,p'>1$ independent of $k$ such that, for all $\varepsilon\in(0,1)$ and $T\geq\max\{T_1,T_2,\dots,T_{N-1}\}$,*

$$
\|f\|^2_{L^2(\widetilde{A}_k)}\leq C\varepsilon^p\|v\|^2_{H^1(Q)}+C\varepsilon^{-p'}\Big(\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_{k+1}))}+\Big\|\frac{\partial v}{\partial\boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big).
$$

*Proof.* For the convenience, we present the sketch maps of domains $\widetilde{A}_k$ and $\widetilde{O}_k$, which are shown in Figures 3a and 3b indicated with gray and pink shaded area respectively. By Lemma 3.1 and Lemma 3.2, let $\sigma=\min(\nu_1,\nu_2,\dots,\nu_{N-1})$, we have

$$
\begin{aligned}
\|f\|^2_{L^2(\Omega_N\setminus D_{N-1}(4r_{N-1}))}\leq& C\Big(\|v\|^2_{H^1(Q)}+\Big\|\frac{\partial v}{\partial\boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big)^{1-\nu_{N-1}}\Big(\Big\|\frac{\partial v}{\partial\boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big)^{\nu_{N-1}}\\
\leq& C\Big(\|v\|^2_{H^1(Q)}+\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_{k+1}))}+\Big\|\frac{\partial v}{\partial\boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big)^{1-\sigma}\\
&\cdot\Big(\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_{k+1}))}+\Big\|\frac{\partial v}{\partial\boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big)^{\sigma},
\end{aligned}
$$




(a) $\widetilde{A}_k$ (b) $\widetilde{O}_k$

FIGURE 3. The sketch maps of domains $\widetilde{A}_k$ and $\widetilde{O}_k$ indicated with different shaded areas. (a) domain $\widetilde{A}_k$ with gray shaded area, (b) domain $\widetilde{O}_k$ with pink shaded area.

and the same upper bound for $\|f\|^2_{L^2(A_j)}$, $j \geq k$. Thus, we have

$$\begin{aligned}\|f\|^2_{L^2(\widetilde{A}_k)} &\leq \|f\|^2_{L^2(\Omega_N\setminus D_{N-1}(4r_{N-1}))} + \sum_{j=k}^{N-2}\|f\|^2_{L^2(A_j)}\\ &\leq C\Big(\|v\|^2_{H^1(Q)} + \|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_{k+1}))} + \Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big)^{1-\sigma}\\ &\qquad\cdot\Big(\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_{k+1}))} + \Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big)^{\sigma}.\end{aligned}$$

Then by the Young inequality with $p=\frac{1}{1-\sigma}$ and $p'=\frac{1}{\sigma}$, we have

$$\begin{aligned}\|f\|^2_{L^2(\widetilde{A}_k)} &\leq C\varepsilon^p\Big(\|v\|^2_{H^1(Q)} + \|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_{k+1}))} + \Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big)\\ &\qquad + C\varepsilon^{-p'}\Big(\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_{k+1}))} + \Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big)\\ &\leq C\varepsilon^p\|v\|^2_{H^1(Q)} + C\varepsilon^{-p'}\Big(\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_{k+1}))} + \Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big).\end{aligned}$$

□

### 3.2. Control $y(x,t)$ by $f(x)$ and the boundary

We cannot follow the strategy in Section 3.1 to control $y(x,t)$ due to the extra variable $t$ compared with $f(x)$. In order to make it feasible we need to use the FBI transformation $F_{\gamma,t_0}$ with respect to the variable $t$. It is defined by

$$F_{\gamma,t_0}y(x,\omega):=\sqrt{2\gamma}\int_{\mathbb{R}}\mathrm{e}^{-\gamma(t_0-t+\mathrm{i}\omega)^2}y(x,t)\mathrm{d}t,$$

where $\gamma>0$, $t_0\in[-\frac{T}{2}-1,\frac{T}{2}+1]$ and $\omega\in[-3,3]$. One fact we easily obtain is that, for any $x\in\Omega_N$,

$$\begin{aligned}|F_{\gamma,t_0}y(x,\omega)|^2&\le 2\gamma\Big(\int_{\mathbb{R}}\mathrm{e}^{-\gamma(t_0-t)^2+\gamma\omega^2}|y(x,t)|\mathrm{d}t\Big)^2\\&\le\sqrt{2\pi\gamma}\mathrm{e}^{2\gamma\omega^2}\int_{-T}^{T}|y(x,t)|^2\mathrm{d}t.\end{aligned}$$

Further noting that

$$\begin{aligned}\partial_\omega F_{\gamma,t_0}y(x,\omega)&=-\mathrm{i}\sqrt{2\gamma}\int_{\mathbb{R}}\partial_t\mathrm{e}^{-\gamma(t_0-t+\mathrm{i}\omega)^2}y(x,t)\mathrm{d}t\\&=\mathrm{i}\sqrt{2\gamma}\int_{\mathbb{R}}\mathrm{e}^{-\gamma(t_0-t+\mathrm{i}\omega)^2}\partial_t y(x,t)\mathrm{d}t,\end{aligned}$$

we have

$$\|F_{\gamma,t_0}y\|^2_{H^1(\Omega_N\times(-3,3))}\le\mathrm{e}^{C_1\gamma}\|y\|^2_{H^1(Q)},\tag{3.10}$$

for some $C_1>0$.

We apply the FBI transformation to (3.1) and get

$$\begin{aligned}\partial_\omega^2F_{\gamma,t_0}y+a\Delta F_{\gamma,t_0}y=&\sqrt{2\gamma}\int_{\mathbb{R}}\mathrm{e}^{-\gamma(t_0-t+\mathrm{i}\omega)^2}\Big(-\partial_t^2y(x,t)+a\Delta y(x,t)\Big)\mathrm{d}t\\=&-F_{\gamma,t_0}G-F_{\gamma,t_0}H(x,t)\end{aligned}\tag{3.11}$$

with

$$[F_{\gamma,t_0}y]_{S_k}=\left[a\frac{\partial F_{\gamma,t_0}y}{\partial\boldsymbol{n}_k}\right]_{S_k}=0,\qquad 1\le k\le N-1,$$

and

$$F_{\gamma,t_0}y=0,\quad x\in\partial\Omega_N.$$

Since $H(x,t)$ is supported in $2T/3\le|t|\le 3T/4$ and $|t_0|\le\frac{T}{2}+1$, we have

$$\begin{aligned}|F_{\gamma,t_0}H|^2&\le 2\gamma\Big(\int_{\mathbb{R}}\mathrm{e}^{-\gamma(\frac{T}{6}-1)^2+9\gamma}|H(x,t)|\mathrm{d}t\Big)^2\\&\le\mathrm{e}^{-C_2T\gamma+C_1\gamma}\int_{-T}^{T}|H(x,t)|^2\mathrm{d}t\end{aligned}\tag{3.12}$$

for some positive constants $C_1, C_2$. Moreover, we also have

$$|F_{\gamma,t_0}G|^2 \le \mathrm{e}^{C_1\gamma}\int_{-T}^{T}|G(x,t)|^2\mathrm{d}t \le C\mathrm{e}^{C_1\gamma}|f(x)|^2. \tag{3.13}$$

**Lemma 3.4.** *For $2\le k\le N-1$, there exist constants $C>0$ and $l_k\in(0,1)$ such that*

$$\|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_k\times(-1,1))} \le C\Big(\|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_{k+1}\times(-1,1))}+\alpha_k\Big)^{1-l_k} \times\Big(\|F_{\gamma,t_0}y\|^2_{H^1(\Omega_N\times(-3,3))}\Big)^{l_k},$$

*where*

$$\alpha_k=\|F_{\gamma,t_0}G\|^2_{L^2(\widetilde{A}_k\times(-3,3))}+\|F_{\gamma,t_0}H\|^2_{L^2(\widetilde{A}_k\times(-3,3))}+\left\|\frac{\partial F_{\gamma,t_0}y}{\partial \boldsymbol{n}}\right\|^2_{L^2(\widetilde{\Gamma}\times(-3,3))}.$$

*Proof.* Firstly we take a function $\phi_k(x)\le 0$ satisfying the following three conditions:
(i)

$$\begin{aligned}\phi_k(x)\le -8,&\ \text{ in }\ D_k(3r_k),\\ -5\le\phi_k(x)\le -3,&\ \text{ in }\ D_{k+1}(2r_{k+1})\setminus D_k(2r_k).\end{aligned}$$

In the case when $k=N-1$, it becomes $-5\le\phi_{N-1}(x)\le-3$ in $\Omega_N\setminus D_{N-1}(2r_{N-1})$.
(ii)

$$\begin{aligned}&[\phi_k(x)]_{S_j}=\left[a\frac{\partial^\alpha\phi_k(x)}{\partial x^\alpha}\right]_{S_j}=0,\quad \frac{\partial\phi}{\partial\boldsymbol{n}_j}>0,\quad \text{for } j\ge k,\ 0<|\alpha|\le 3,\\ &|\nabla\phi_k(x)|\ne 0,\quad \text{in }\ \Omega_N\setminus\big(\Omega_{k-1}\cup S_k\cup\cdots\cup S_{N-1}\big).\end{aligned}$$

Note that $\phi_k(x)\in C^3\big(\Omega_N\setminus(\Omega_{k-1}\cup S_k\cup\cdots\cup S_{N-1})\big)$.
(iii)

$$\Big\{x\in\partial\Omega_N:\nabla\phi_k(x)\cdot(x-x_0)>0\Big\}\subset\widetilde{\Gamma}.$$

The existence of $\phi_k(x)$ will be presented in the Remark 3.5 below. Then we define

$$\tilde{\psi}_k(x,\omega)=\phi_k(x)-\omega^2,\quad \tilde{\varphi}_k(x,\omega)=\mathrm{e}^{\lambda\tilde{\psi}_k(x,\omega)},$$

and take a cut-off function $\chi\in C^\infty(\mathbb{R})$ satisfying $0\le\chi(t)\le1$ and

$$\chi(t)=\begin{cases}1, & -7\le t\le -1,\\ 0, & t\le -8 \textit{ or } t\ge 0.\end{cases}$$

We set

$$Y(x,\omega)=\chi(\tilde{\psi}_k)F_{\gamma,t_0}y(x,\omega),$$

then from (3.11) we have

$$\begin{aligned}&\partial_\omega^2 Y + a\Delta Y\\ =& -\chi(\tilde{\psi}_k)F_{\gamma,t_0}G - \chi(\tilde{\psi})F_{\gamma,t_0}H + [\partial_\omega^2 + a\Delta, \chi(\tilde{\psi}_k)]F_{\gamma,t_0}y,\end{aligned}$$

with

$$Y(x, \pm 3) = 0,$$

where $[A, B] = AB - BA$ denotes the commutator. We note that $[\partial_\omega^2 + a\Delta, \chi(\tilde{\psi}_k)]$ is a linear differential operator of the first order and is supported in

$$\Big\{(x,\omega) : -8 \le \tilde{\psi}_k \le -7 \ \text{ or } \ -1 \le \tilde{\psi}_k \le 0\Big\}.$$

By (ii) we can verify that

$$[Y]_{S_j} = \left[a\frac{\partial Y}{\partial \boldsymbol{n}_j}\right]_{S_j} = 0, \quad j \ge k$$

for all $\omega \in [-3, 3]$, then we apply the elliptic case of Lemma 2.1 to $Y(x, \omega)$ in $\Omega_N \setminus \Omega_{k-1} \times [-3, 3]$. There exist constants $\lambda, s_0 > 0$ such that for all $s > s_0$, we have

$$\begin{aligned}&\int_{-3}^3\int_{\Omega_N\setminus\Omega_{k-1}} (s^3|Y|^2 + s|\nabla_{x,\omega}Y|^2)\mathrm{e}^{2s\tilde{\varphi}_k}\mathrm{d}x\mathrm{d}\omega\\ \le& C\int_{-3}^3\int_{\Omega_N\setminus\Omega_{k-1}} (|\chi(\tilde{\psi}_k)F_{\gamma,t_0}G|^2 + |\chi(\tilde{\psi}_k)F_{\gamma,t_0}H|^2\\ &+ \big|[\partial_\omega^2 + a\Delta, \chi(\tilde{\psi}_k)]F_{\gamma,t_0}y\big|^2)\mathrm{e}^{2s\tilde{\varphi}_k}\mathrm{d}x\mathrm{d}\omega + C\int_{-3}^3\int_{\widetilde{\Gamma}}\Big|\frac{\partial Y}{\partial \boldsymbol{n}}\Big|^2\mathrm{e}^{2s\tilde{\varphi}_k}\mathrm{d}\sigma\mathrm{d}\omega\\ \le& C\int_{-3}^3\int_{\Omega_N\setminus D_k(4r_k)} (|F_{\gamma,t_0}G|^2 + |F_{\gamma,t_0}H|^2)\mathrm{e}^{2s}\mathrm{d}x\mathrm{d}\omega\\ &+ C\int_{-8\le\tilde{\psi}_k\le-7} (|F_{\gamma,t_0}y|^2 + |\nabla_{x,\omega}F_{\gamma,t_0}y|^2)\mathrm{e}^{2s\mathrm{e}^{-7\lambda}}\mathrm{d}x\mathrm{d}\omega\\ &+ C\int_{-1\le\tilde{\psi}_k\le 0} (|F_{\gamma,t_0}y|^2 + |\nabla_{x,\omega}F_{\gamma,t_0}y|^2)\mathrm{e}^{2s}\mathrm{d}x\mathrm{d}\omega\\ &+ C\int_{-3}^3\int_{\widetilde{\Gamma}}\Big|\frac{\partial Y}{\partial \boldsymbol{n}}\Big|^2\mathrm{e}^{2s}\mathrm{d}\sigma\mathrm{d}\omega.\end{aligned}$$

Since $-6 \le \tilde{\psi}_k \le -3$ in $\big(D_{k+1}(2r_{k+1}) \setminus D_k(2r_k)\big) \times (-1, 1)$ and $\{(x, \omega) : -1 \le \tilde{\psi}_k \le 0\} \subset \widetilde{O}_{k+1} \times (-1, 1)$, we obtain

$$\begin{aligned}&\mathrm{e}^{2s\mathrm{e}^{-6\lambda}}\|F_{\gamma,t_0}y\|^2_{H^1(D_{k+1}(2r_{k+1})\setminus D_k(2r_k)\times(-1,1))}\\ =& \mathrm{e}^{2s\mathrm{e}^{-6\lambda}}\Big(\|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_k\times(-1,1))} - \|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_{k+1}\times(-1,1))}\Big)\\ \le& C\mathrm{e}^{2s}\Big(\alpha_k + \|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_{k+1}\times(-1,1))}\Big) + C\mathrm{e}^{2s\mathrm{e}^{-7\lambda}}\|F_{\gamma,t_0}y\|^2_{H^1(\Omega_N\times(-3,3))}.\end{aligned}$$

Therefore, there exist positive constants $k_1, k_2$ such that

$$\|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_k\times(-1,1))} \leq C\mathrm{e}^{k_1 s}\Big(\alpha_k + \|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_{k+1}\times(-1,1))}\Big) + C\mathrm{e}^{-k_2 s}\|F_{\gamma,t_0}y\|^2_{H^1(\Omega_N\times(-3,3))},$$

then we minimize the right-hand side with respect to $s$ and complete the proof of Lemma 3.4 with $l_k = \frac{k_2}{k_1+k_2}$. $\square$

**Remark 3.5.** In this remark, we show the existence of $\phi(x)$. We can construct $\phi_k(x)$ in each annulus $\Omega_j \setminus \Omega_{j-1}$ $(k \leq j \leq N)$ according to the following four steps.

(1) When $x \in \Omega_k \setminus \Omega_{k-1}$, we set

$$h_1(x) = \epsilon\frac{|x-x_0|^2}{|y_k(x)-x_0|^2} - 5, \quad 0<\epsilon<1.$$

We take a cut-off function $0 \leq \theta(t) \leq 1$ satisfying $\theta'(t) \geq 0$,

$$\theta(t) = \begin{cases} 0, & t \leq 2, \\ 1, & t \geq 3, \end{cases}$$

then in this annulus we define

$$\phi_k(x) = -4\theta\Big((1-\frac{|x-x_0|^2}{|y_k(x)-x_0|^2})/r_k\Big) + h_1(x).$$

When $x \in \Omega_k \setminus D_k(2r_k)$,

$$\theta\Big((1-\frac{|x-x_0|^2}{|y_k(x)-x_0|^2})/r_k\Big) = 0 \text{ and } \phi_k(x) = h_1(x) \in [-5,-3].$$

When $x \in D_k(3r_k)$,

$$\theta\Big((1-\frac{|x-x_0|^2}{|y_k(x)-x_0|^2})/r_k\Big) = 1 \text{ and } \phi_k(x) = \epsilon\frac{|x-x_0|^2}{|y_k(x)-x_0|^2} - 9 < -8$$

as desired. Moreover, we can verify that

$$|\nabla\phi_k\cdot(x-x_0)| = (\frac{4\theta'}{r_k}+\epsilon)\frac{2|x-x_0|^2}{|y_k(x)-x_0|^2} > 0,$$

so conditions (i) and (ii) are satisfied in this area.

(2) When $x \in \Omega_{k+1} \setminus \Omega_k$, we set

$$h_2(x) = \epsilon\frac{a_k}{a_{k+1}}\frac{|x-x_0|^2}{|y_k(x)-x_0|^2} - 5 + \epsilon(1-\frac{a_k}{a_{k+1}}),$$

$$h_3(x) = \epsilon\frac{|x-x_0|^2}{|y_{k+1}(x)-x_0|^2} - 4 - \frac{1}{k+1},$$

we take $\epsilon$ small enough such that $-5 \le h_2(x) \le h_3(x) \le -3$. Then in this annulus we define

$$\phi_k(x) = \theta\Big((1 - \frac{|x-x_0|^2}{|y_{k+1}(x)-x_0|^2})/r_{k+1}\Big)\big(h_2(x) - h_3(x)\big) + h_3(x).$$

We can verify that when $x \in D_{k+1}(3r_{k+1}) \setminus \Omega_k$, $\phi_k(x) = h_2(x)$. And when $x \in \Omega_{k+1} \setminus D_{k+1}(2r_{k+1})$, $\phi_k(x) = h_3(x)$, we have

$$\begin{aligned}|\nabla\phi_k \cdot (x-x_0)| =& \Big(\frac{\theta'}{r_{k+1}}(\phi_3 - \phi_2) + \epsilon(1-\theta)\Big)\frac{2|x-x_0|^2}{|y_{k+1}(x)-x_0|^2} \\ &+ \epsilon\theta\frac{a_k}{a_{k+1}}\frac{2|x-x_0|^2}{|y_k(x)-x_0|^2} > 0.\end{aligned}$$

(3) When $x \in \Omega_{j+1} \setminus \Omega_j$, $k < j < N-1$, we define $\phi_k(x)$ in the same way as (2). We set

$$\phi_{4j}(x) = \epsilon\frac{a_j}{a_{j+1}}\frac{|x-x_0|^2}{|y_j(x)-x_0|^2} - 4 - \frac{1}{j} + \epsilon(1 - \frac{a_j}{a_{j+1}}),$$

$$\phi_{5j}(x) = \epsilon\frac{|x-x_0|^2}{|y_{j+1}(x)-x_0|^2} - 4 - \frac{1}{j+1},$$

and define

$$\phi_k(x) = \theta\Big((1 - \frac{|x-x_0|^2}{|y_{j+1}(x)-x_0|^2})/r_{j+1}\Big)\big(h_{4j}(x) - h_{5j}(x)\big) + h_{5j}(x).$$

(4) When $x \in \Omega_N \setminus \Omega_{N-1}$, we define

$$\phi_k(x) = \epsilon\frac{a_{N-1}}{a_N}\frac{|x-x_0|^2}{|y_{N-1}(x)-x_0|^2} - 4 - \frac{1}{N-1} + \epsilon(1 - \frac{a_{N-1}}{a_N}),$$

then we can see that the condition (iii) is also true by the definition of $\widetilde{\Gamma}$.

**Corollary 3.6.** *For $2 \le k \le N-1$, there exist constants $C > 0$ and $d_k \in (0,1)$ such that*

$$\|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_k\times(-1,1))} \le C\alpha_k^{1-d_k}\Big(\|F_{\gamma,t_0}y\|^2_{H^1(\Omega_N\times(-3,3))}\Big)^{d_k}.$$

*Proof.* Let us fix $k$, for $j \ge k$, by Lemma 3.4, we get

$$\begin{aligned}\|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_j\times(-1,1))} &\le C\Big(\alpha_j + \|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_{j+1}\times(-1,1))}\Big)^{1-l_j} \\ &\quad\times\Big(\|F_{\gamma,t_0}y\|^2_{H^1(\Omega_N\times(-3,3))}\Big)^{l_j} \\ &\le C\Big(\alpha_k + \|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_{j+1}\times(-1,1))}\Big)^{1-l_j} \\ &\quad\times\Big(\|F_{\gamma,t_0}y\|^2_{H^1(\Omega_N\times(-3,3))}\Big)^{l_j},\end{aligned}$$

together with

$$\|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_{N-1}\times(-1,1))}\leq C\alpha_k^{1-l_{N-1}}\Big(\|F_{\gamma,t_0}y\|^2_{H^1(\Omega_N\times(-3,3))}\Big)^{l_{N-1}}.$$

Hence we can reach the conclusion by using mathematical induction. □

**Lemma 3.7.** *For $2\leq k\leq N-1$, there exists $T_k'=\frac{(1+p_k)C_1}{C_2}$, where $p_k=\frac{1+d_k}{1-d_k}$ and $C_1,C_2$ are as in (3.12), such that the following estimate holds true for $T\geq T_k'$:*

$$\|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_k\times(-1,1))}\leq C\Big(\mathrm{e}^{p_k\gamma}\big(\|f\|^2_{L^2(\widetilde{A}_k)}+\Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\big)+\mathrm{e}^{-C_1\gamma}\|v\|^2_{H^1(Q)}\Big).$$

*Proof.* By (3.10), (3.12) and (3.13) we easily have

$$\alpha_k\leq C\Big(\mathrm{e}^{C_1\gamma}\|f\|^2_{L^2(\widetilde{A}_k)}+\mathrm{e}^{-C_2T\gamma+C_1\gamma}\|v\|^2_{H^1(Q)}+\mathrm{e}^{C_1\gamma}\Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big). \tag{3.14}$$

By Corollary 3.6 and the Young inequality we have for all $\epsilon>0$

$$\|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_k\times(-1,1))}\leq C\epsilon^{\frac{1}{d_k}}\|F_{\gamma,t_0}y\|^2_{H^1(\Omega_N\times(-3,3))}+C\epsilon^{-\frac{1}{1-d_k}}\alpha_k.$$

We take $\epsilon=\mathrm{e}^{-2C_1d_k\gamma}$, combining with (3.10) and (3.14), then we have

$$\begin{aligned}\|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_k\times(-1,1))}\leq C\Big(&\mathrm{e}^{-C_1\gamma}\|v\|^2_{H^1(Q)}+\mathrm{e}^{(p_kC_1-C_2T)\gamma}\|v\|^2_{H^1(Q)}\\&+\mathrm{e}^{p_k\gamma}\|f\|^2_{L^2(\widetilde{A}_k)}+\mathrm{e}^{p_k\gamma}\Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\Big).\end{aligned}$$

This completes the proof of Lemma 3.7. □

Now in order to finish this subsection we need to show that the $H^1$-norm of $y$ in $\widetilde{O}_k\times(-\frac{T}{2},\frac{T}{2})$ can be controlled by $\|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_k\times(-1,1))}$. We set $y_\gamma(x,t)=F_{\gamma,t}y(x,0)$, then we have

$$y_\gamma(x,t)=\sqrt{2\gamma}\int_{\mathbb{R}}\mathrm{e}^{-\gamma(t-\xi)^2}y(x,\xi)\mathrm{d}\xi=K_\gamma*y(x,t),$$

where

$$K_\gamma(t)=\sqrt{2\gamma}\mathrm{e}^{-\gamma t^2}.$$

**Lemma 3.8.** *For $2\leq k\leq N-1$, we have*

$$\|y_\gamma\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_k))}\leq C\Big(\mathrm{e}^{p_k\gamma}\big(\|f\|^2_{L^2(\widetilde{A}_k)}+\Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}\big)+\mathrm{e}^{-C_1\gamma}\|v\|^2_{H^1(Q)}\Big).$$

*Proof.* By Cauchy Integral Formula, we have for $0<\rho<1$ and $t\in[-\frac{T}{2},\frac{T}{2}]$,

$$\begin{aligned}|y_\gamma(x,t)|^2 &= \left|\frac{1}{2\pi \mathrm{i}}\oint_{|z-t|=\rho}\frac{y_\gamma(x,z)}{z-t}\mathrm{d}z\right|^2\\ &= \left|\frac{1}{2\pi}\int_0^{2\pi} y_\gamma(x,t+\rho\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}\theta\right|^2\\ &\le \frac{1}{2\pi}\int_0^{2\pi}|y_\gamma(x,t+\rho\mathrm{e}^{\mathrm{i}\theta})|^2\mathrm{d}\theta.\end{aligned}$$

Then integrating this inequality for $\rho\in(\frac{1}{2},1)$, we get

$$\begin{aligned}\frac{1}{2}|y_\gamma(x,t)|^2 &\le \frac{1}{\pi}\int_{\frac{1}{2}}^1\int_0^{2\pi}|y_\gamma(x,t+\rho\mathrm{e}^{\mathrm{i}\theta})|^2\rho\mathrm{d}\theta\mathrm{d}\rho\\ &= \frac{1}{\pi}\iint_{\frac{1}{2}\le|t_0-t|^2+\omega^2\le 1}|y_\gamma(x,t_0+\mathrm{i}\omega)|^2\mathrm{d}t_0\mathrm{d}\omega\\ &\le \frac{1}{\pi}\int_{-1}^1\int_{-\frac{T}{2}-1}^{\frac{T}{2}+1}|F_{\gamma,t_0}y(x,\omega)|^2\mathrm{d}t_0\mathrm{d}\omega.\end{aligned}$$

This estimate holds for all $x\in\widetilde{O}_k$ and $t\in[-\frac{T}{2},\frac{T}{2}]$. Therefore, we have

$$\begin{aligned}\|y_\gamma\|^2_{L^2(\widetilde{O}_k\times(-\frac{T}{2},\frac{T}{2}))} &\le \frac{2T}{\pi}\int_{\widetilde{O}_k}\int_{-1}^1\int_{-\frac{T}{2}-1}^{\frac{T}{2}+1}|F_{\gamma,t_0}y(x,\omega)|^2\mathrm{d}t_0\mathrm{d}\omega\mathrm{d}x\\ &= \frac{2T}{\pi}\int_{-\frac{T}{2}-1}^{\frac{T}{2}+1}\|F_{\gamma,t_0}y\|^2_{L^2(\widetilde{O}_k\times(-1,1))}\mathrm{d}t_0.\end{aligned}$$

By using the same argument to the first-order derivatives of $y_\gamma$, we conclude that

$$\|y_\gamma\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_k))} \le \frac{2T}{\pi}\int_{-\frac{T}{2}-1}^{\frac{T}{2}+1}\|F_{\gamma,t_0}y\|^2_{H^1(\widetilde{O}_k\times(-1,1))}\mathrm{d}t_0.$$

Then by Lemma 3.7 we complete the proof of Lemma 3.8. □

**Lemma 3.9.** *For $2\le k\le N-1$ and $\forall\gamma>0$, the following estimate holds*

$$\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_k))}\le C\Big(\mathrm{e}^{p_k\gamma}(\|f\|^2_{L^2(\widetilde{A}_k)}+\Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})})+\frac{1}{\gamma^2}\|v\|^2_{H^2(-T,T;H^1(\Omega_N))}\Big).$$

*Proof.* We denote the Fourier transform of $y(x,t)$ for $t$ by $\widehat{y}(x,\tau)$, then

$$\widehat{y}-\widehat{y_\gamma}=\widehat{y}-\widehat{K_\gamma * y}=(1-\widehat{K_\gamma})\widehat{y}.$$

Since

$$|1-\widehat{K_\gamma}|=|1-\mathrm{e}^{-\frac{\tau^2}{4\gamma}}|\le\frac{\tau^2}{4\gamma},$$

by the Parseval identity we have

$$\begin{aligned}\|y-y_\gamma\|^2_{L^2(\widetilde{O}_k\times(-\frac{T}{2},\frac{T}{2}))} &\le \|\widehat{y}-\widehat{y_\gamma}\|^2_{L^2(\widetilde{O}_k\times\mathbb{R})}\\ &\le \|\frac{\tau^2}{4\gamma}\widehat{y}\|^2_{L^2(\widetilde{O}_k\times\mathbb{R})}\\ &= (\frac{1}{4\gamma})^2\|\partial_t^2 y\|^2_{L^2(\widetilde{O}_k\times(-T,T))}.\end{aligned}$$

Similarly, we have

$$\|y-y_\gamma\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_k))} \le (\frac{1}{4\gamma})^2\|\partial_t^2 y\|^2_{L^2(-T,T;H^1(\widetilde{O}_k))}.$$

Therefore, combining with Lemma 3.8 we obtain

$$\begin{aligned}\|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_k))} &\le 2\|y-y_\gamma\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_k))} + 2\|y_\gamma\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_k))}\\ &\le C\Big(\mathrm{e}^{p_k\gamma}(\|f\|^2_{L^2(\widetilde{A}_k)} + \Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}) + \frac{1}{\gamma^2}\|v\|^2_{H^2(-T,T;H^1(\Omega_N))}\Big).\end{aligned}$$

The proof of Lemma 3.9 is complete. □

### 3.3. End of the proof of Theorem 1.1

For simplicity, we set

$$\begin{aligned}F_k &= \|f\|^2_{L^2(\widetilde{A}_k)}, & Y_k &= \|y\|^2_{L^2(-\frac{T}{2},\frac{T}{2};H^1(\widetilde{O}_k))},\\ V &= \|v\|^2_{H^2(-T,T;H^1(\Omega_N))}, & B &= \Big\|\frac{\partial v}{\partial \boldsymbol{n}}\Big\|^2_{L^2(\widetilde{\Sigma})}.\end{aligned}$$

We choose

$$\varepsilon = \gamma^{-\frac{2}{p+p'}}, \quad \gamma > 1$$

in Corollary 3.3 so we have

$$F_k \le C\gamma^{-\frac{2p}{p+p'}}V + C\gamma^{\frac{2p'}{p+p'}}(Y_{k+1}+B) \tag{3.15}$$

for $1 \le k \le N-1$. By Lemma 3.9 we get

$$Y_{k+1} \le C\mathrm{e}^{p_k\gamma}(F_{k+1}+B) + C\frac{1}{\gamma^2}V. \tag{3.16}$$

Inserting (3.16) into (3.15), we obtain for $1 \le k \le N-2$,

$$F_k \le C\gamma^{-\frac{2p}{p+p'}}V + C\mathrm{e}^{(1+p_k)\gamma}(F_{k+1}+B), \tag{3.17}$$

and

$$F_{N-1} \le C\gamma^{-\frac{2p}{p+p'}}V + C\gamma^{\frac{2p'}{p+p'}}B, \tag{3.18}$$

since $Y_N = 0$. We claim that for $1 \le k \le N-1$, there exist $c_k > 0$ such that

$$F_k \le C\gamma^{-\beta_0} V + C \underbrace{f_k \circ f_k \circ \cdots f_k}_{N-k}(\gamma) B, \quad \gamma > 1, \tag{3.19}$$

where

$$\beta_0 = \frac{2p}{p+p'}, \quad f_k(\gamma) = \mathrm{e}^{c_k \gamma}, \quad f_k \circ f_k(\gamma) = f_k(f_k(\gamma)).$$

We prove (3.19) by mathematical induction. When $k = N-1$, (3.19) holds for $c_{N-1} = 1$. We assume that (3.19) holds for $k+1$. Then by (3.17) we know

$$\begin{aligned} F_k &\le C\gamma^{-\beta_0} V + C\mathrm{e}^{(p_k+1)\gamma}(F_{k+1} + B) \\ &\le C\gamma^{-\beta_0} V + C\mathrm{e}^{(p_k+1)\gamma}\Big(\tau^{-\beta_0} V + \underbrace{f_{k+1} \circ f_{k+1} \circ \cdots f_{k+1}}_{N-k-1}(\tau) B + B\Big) \end{aligned}$$

holds for $\tau > 1$ and $\gamma > 1$. We take

$$\tau = \mathrm{e}^{\frac{2(1+p_k)}{\beta_0}\gamma},$$

then we can see that (3.19) is true for $k$ with

$$c_k = \max\Big(\frac{2(1+p_k)}{\beta_0}, c_{k+1}\Big).$$

We set $\tau_0 = \max(c_1, c_2, \ldots, c_{N-1})$ and $g_0(\gamma) = \mathrm{e}^{\tau_0 \gamma}$, then (3.19) yields that for $1 \le k \le N-1$,

$$F_k \le C\gamma^{-\beta_0} V + C \underbrace{g_0 \circ g_0 \circ \cdots g_0}_{N-1}(\gamma) B.$$

Finally by (3.2) we conclude that

$$\begin{aligned} \|f\|^2_{L^2(\Omega_N \setminus B(x_0,\delta))} &\le \sum_{k=1}^{N-1} F_k \\ &\le C\gamma^{-\beta_0} V + C \underbrace{g_0 \circ g_0 \circ \cdots g_0}_{N-1}(\gamma) B. \end{aligned}$$

We use the whole same argument to $x_1$ and we can claim that there exist constants $\beta_1, \tau_1, C > 0$ such that

$$\|f\|^2_{L^2(\Omega_N \setminus B(x_1,\delta))} \le C\gamma^{-\beta_1} V + C \underbrace{g_1 \circ g_1 \circ \cdots g_1}_{N-1}(\gamma) B,$$

with $g_1(\gamma) = \mathrm{e}^{\tau_1 \gamma}$. Thus we complete the proof of Theorem 1.1.

## 4. Proof of Lemma 2.1

For universality, we consider

$$L := \rho\partial_t^2 - a\Delta,$$

where $\rho \in \{-1, 1\}$. We will aim to prove Lemma 2.1 in two cases:

**Case 1**: $\rho = 1$, the weight function $\psi(x,t)$ satisfies conditions (a)-(f).

**Case 2**: $\rho = -1$, the weight function $\psi(x,t)$ satisfies conditions (a)-(d).

As presented in Lemma 2.1, we set

$$\varphi = \mathrm{e}^{\lambda\psi}, \quad w = \mathrm{e}^{s\varphi}u.$$

We further set

$$E(\psi) = \rho|\partial_t\psi|^2 - a|\nabla\psi|^2,$$

$$Pw = \mathrm{e}^{s\varphi}L(\mathrm{e}^{-s\varphi}w) = P_1w + P_2w + Rw,$$

where

$$\begin{aligned}
&P_1w = \rho\partial_t^2 w - a\Delta w + s^2\lambda^2\varphi^2 E(\psi)w,\\
&P_2w = -s\lambda\varphi(L(\psi) + \gamma)w - s\lambda^2\varphi E(\psi)w - 2s\lambda\varphi(\rho\partial_t\psi\partial_t w - a\nabla\psi\cdot\nabla w),\\
&Rw = \gamma s\lambda\varphi w.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|Pw\|^2_{L^2(Q_j^k)} &\geq \frac{1}{2}\|P_1w + P_2w\|^2_{L^2(Q_j^k)} - \|Rw\|^2_{L^2(Q_j^k)}\\
&\geq \frac{1}{2}\|P_1w\|^2_{L^2(Q_j^k)} + (P_1w, P_2w)_{L^2(Q_j^k)} - \|Rw\|^2_{L^2(Q_j^k)}.
\end{aligned}$$

Here $(\cdot\ ,\ \cdot)_{L^2(Q_j^k)}$ is the inner product in $L^2(Q_j^k)$. And we write

$$(P_1w, P_2w)_{L^2(Q_j^k)} = \sum_{i=j+1}^{k}\int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} P_1wP_2w \,\mathrm{d}x\mathrm{d}t.$$

Using the integration by parts as [4] does, we have

$$\int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} P_1wP_2w \,\mathrm{d}x\mathrm{d}t = I_i + X_i + B_i^- - B_{i-1}^+, \tag{4.1}$$

where $I_i$ is the sum of the interior terms

$$\begin{aligned} I_i = {} & 2s\lambda \int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} \varphi\partial_t^2\psi|\partial_t w|^2 \,\mathrm{d}x\mathrm{d}t + \gamma\rho s\lambda \int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} \varphi|\partial_t w|^2 \,\mathrm{d}x\mathrm{d}t \\ & + 2s\lambda^2 \int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} \varphi\big(\rho\partial_t\psi\partial_t w - a\nabla w\cdot\nabla\psi\big)^2 \,\mathrm{d}x\mathrm{d}t \\ & + 2s\lambda \int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} a^2\varphi D^2(\psi)(\nabla w,\nabla w) \,\mathrm{d}x\mathrm{d}t \\ & - \gamma s\lambda \int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} a\varphi|\nabla w|^2 \,\mathrm{d}x\mathrm{d}t + 2s^3\lambda^4 \int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} \varphi^3 E(\psi)^2|w|^2 \,\mathrm{d}x\mathrm{d}t \\ & + 2s^3\lambda^3 \int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} \varphi^3\big(|\partial_t\psi|^2\partial_t^2\psi + a^2D^2(\psi)(\nabla\psi,\nabla\psi)\big)|w|^2 \,\mathrm{d}x\mathrm{d}t \\ & - \gamma s^3\lambda^3 \int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} \varphi^3 E(\psi)|w|^2 \,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

And $X_i$ in (4.1) is the sum of the remaining interior terms, in such a way that

$$|X_i| \le Cs \int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} |w|^2 \,\mathrm{d}x\mathrm{d}t,$$

for some constant $C$ independent of $s$.

And $B_i^-$ in (4.1) is the boundary term on $S_i^-$ (*i.e.* $\partial\Omega_i$):

$$\begin{aligned} B_i^- = {} & s\lambda \int_{-T}^{T}\int_{S_i^-} \left(a^2\varphi|\nabla w|^2\frac{\partial\psi}{\partial\boldsymbol{n}_i} - 2a^2\varphi(\nabla\psi\cdot\nabla w)\frac{\partial w}{\partial\boldsymbol{n}_i}\right) \mathrm{d}\sigma\mathrm{d}t \\ & - \frac{1}{2}s\lambda \int_{-T}^{T}\int_{S_i^-} a\varphi|w|^2\frac{\partial L(\psi)}{\partial\boldsymbol{n}_i} \,\mathrm{d}\sigma\mathrm{d}t \\ & - \frac{1}{2}s\lambda^2 \int_{-T}^{T}\int_{S_i^-} a\varphi(L(\psi)+\gamma)|w|^2\frac{\partial\psi}{\partial\boldsymbol{n}_i} \,\mathrm{d}\sigma\mathrm{d}t \\ & + s\lambda \int_{-T}^{T}\int_{S_i^-} a\varphi(L(\psi)+\gamma)w\frac{\partial w}{\partial\boldsymbol{n}_i} \,\mathrm{d}\sigma\mathrm{d}t + s\lambda^2 \int_{-T}^{T}\int_{S_i^-} a\varphi E(\psi)w\frac{\partial w}{\partial\boldsymbol{n}_i} \,\mathrm{d}\sigma\mathrm{d}t \\ & - \frac{1}{2}s\lambda^3 \int_{-T}^{T}\int_{S_i^-} a\varphi E(\psi)|w|^2\frac{\partial\psi}{\partial\boldsymbol{n}_i} \,\mathrm{d}\sigma\mathrm{d}t - \frac{1}{2}s\lambda^2 \int_{-T}^{T}\int_{S_i^-} a\varphi|w|^2\frac{\partial E(\psi)}{\partial\boldsymbol{n}_i} \,\mathrm{d}\sigma\mathrm{d}t \\ & + 2s\lambda \int_{-T}^{T}\int_{S_i^-} a\rho\varphi\partial_t\psi\partial_t w\frac{\partial w}{\partial\boldsymbol{n}_i} \,\mathrm{d}\sigma\mathrm{d}t - s\lambda \int_{-T}^{T}\int_{S_i^-} a\rho\varphi|\partial_t w|^2\frac{\partial\psi}{\partial\boldsymbol{n}_i} \,\mathrm{d}\sigma\mathrm{d}t \\ & + s^3\lambda^3 \int_{-T}^{T}\int_{S_i^-} a\varphi^3E(\psi)|w|^2\frac{\partial\psi}{\partial\boldsymbol{n}_i} \,\mathrm{d}\sigma\mathrm{d}t, \end{aligned}$$

and $B_{i-1}^+$ is obtained by replacing $S_i^-$ and $\boldsymbol{n}_i$ in $B_i^-$ by $S_{i-1}^+$ and $\boldsymbol{n}_{i-1}$ correspondingly.

We note that there is a minus sign before $B_{i-1}^{+}$ in (4.1) because $\boldsymbol{n}_{i-1}$ is the normal vector inward $\Omega_i \setminus \Omega_{i-1}$ while $\boldsymbol{n}_i$ is the outward normal vector. Hence we have

$$(P_1 w, P_2 w)_{L^2(Q_j^k)} = \sum_{i=j+1}^{k} I_i + \sum_{i=j+1}^{k} X_i + \sum_{i=j+1}^{k-1} (B_i^- - B_i^+) + B_k^- - B_j^+.$$

Since $u$ vanishes near $\partial\Omega_j$, the last term $B_j^+$ is zero. The aim is to show that $B_i^- - B_i^+ \geq 0$ and

$$\sum_{i=j+1}^{k} I_i \geq c \int_{-T}^{T} \int_{\Omega_k \setminus \Omega_j} s^3|w|^2 + s|\nabla w|^2 + s|\partial_t w|^2 \, \mathrm{d}x\mathrm{d}t \tag{4.2}$$

for some positive constant $c$ independent of $s$.

### 4.1. The interior

**Case 1**: $\rho = 1$, $\psi$ satisfies conditions (a)-(f).

In this case, we have

$$\begin{aligned} I_i \geq\ & s\lambda \int_{-T}^{T} \int_{\Omega_i \setminus \Omega_{i-1}} \varphi(-4\beta + \gamma)|\partial_t w|^2 \, \mathrm{d}x\mathrm{d}t \\ & + s\lambda \int_{-T}^{T} \int_{\Omega_i \setminus \Omega_{i-1}} a\varphi(2a\delta_1 - \gamma)|\nabla w|^2 \, \mathrm{d}x\mathrm{d}t \\ & + s^3\lambda^3 \int_{-T}^{T} \int_{\Omega_i \setminus \Omega_{i-1}} \varphi^3\big(2\lambda E(\psi)^2 - (4\beta + \gamma)E(\psi)\big)|w|^2 \, \mathrm{d}x\mathrm{d}t \\ & + 2s^3\lambda^3 \int_{-T}^{T} \int_{\Omega_i \setminus \Omega_{i-1}} \varphi^3(-2a\beta + a^2\delta_1)|\nabla\psi|^2|w|^2 \, \mathrm{d}x\mathrm{d}t. \end{aligned}$$

Since $2\beta < a_k\delta_1 \leq a\delta_1$ in $\Omega_k \setminus \Omega_j$ by condition (f), we take

$$\gamma = a_k\delta_1 + 2\beta,$$

then

$$2a\delta_1 - \gamma \geq -4\beta + \gamma > 0.$$

On the other hand, since

$$2\lambda E(\psi)^2 - (4\beta + \gamma)E(\psi) \geq -\frac{(4\beta + \gamma)^2}{8\lambda},$$

we take $\lambda$ large enough such that

$$\frac{(4\beta + \gamma)^2}{8\lambda} < 2(-2a\beta + a^2\delta_1)\delta_0^2 \leq 2(-2a\beta + a^2\delta_1)|\nabla\psi|^2.$$

Hence we have

$$I_i \geq c\int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} s^3|w|^2 + s|\nabla w|^2 + s|\partial_t w|^2 \ \mathrm{d}x\mathrm{d}t,$$

with

$$c = \min\Big\{-4\beta+\gamma, a(2a\delta_1-\gamma), 2(-2a\beta+a^2\delta_1)\delta_0^2 - \frac{(4\beta+\gamma)^2}{8\lambda}\Big\} > 0.$$

**Case 2**: $\rho = -1$, $\psi$ satisfies conditions (a)-(d).

In this case, we have

$$|E(\psi)| = |\partial_t\psi|^2 + a|\nabla\psi|^2 \geq a_k\delta_0^2,$$

and

$$\begin{aligned}
I_i \geq\ & s\lambda\int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} \varphi(-4\beta-\gamma)|\partial_t w|^2 \ \mathrm{d}x\mathrm{d}t \\
&+ 2s\lambda\int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} a^2\varphi D^2(\psi)(\nabla w,\nabla w)\ \mathrm{d}x\mathrm{d}t \\
&- \gamma s\lambda\int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} a\varphi|\nabla w|^2\ \mathrm{d}x\mathrm{d}t + 2s^3\lambda^4\int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} \varphi^3 E(\psi)^2|w|^2\ \mathrm{d}x\mathrm{d}t \\
&+ 2s^3\lambda^3\int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} \varphi^3\big(|\partial_t\psi|^2\partial_t^2\psi + a^2D^2(\psi)(\nabla\psi,\nabla\psi)\big)|w|^2\ \mathrm{d}x\mathrm{d}t \\
&- \gamma s^3\lambda^3\int_{-T}^{T}\int_{\Omega_i\setminus\Omega_{i-1}} \varphi^3 E(\psi)|w|^2\ \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Since $\psi\in C^2(Q_j^k)$, there exists $M>0$ such that $|D^2(\psi)(X,X)|\leq M|X|^2$ for $X\in\mathbb{R}^2$. We take $\gamma$ which satisfies

$$-4\beta-\gamma>0,\quad -a\gamma-2a^2M>0.$$

Then we choose $\lambda$ large enough such that the term of $s^3\lambda^4$ can absorb the last two terms, and we get (4.2).

## 4.2. The boundary

In both cases $\psi$ satisfies conditions (a)-(d) and

$$[u]_{S_i} = \left[a\frac{\partial u}{\partial \boldsymbol{n}_i}\right]_{S_i} = 0.$$

It is not difficult to verify that the $r$th term of $B_i^-$ and $B_i^+$ are the same for each $r \in \{3,4,7,8,9\}$. So

$$\begin{aligned}
B_i^+ - B_i^- = & s\lambda \int_{-T}^{T}\int_{S_i} \Big( \left[a^2\varphi|\nabla w|^2 \frac{\partial \psi}{\partial \boldsymbol{n}_i}\right]_{S_i} - 2\left[a^2\varphi(\nabla\psi\cdot\nabla w)\frac{\partial w}{\partial \boldsymbol{n}_i}\right]_{S_i} \Big) \mathrm{d}\sigma \mathrm{d}t \\
& - \frac{1}{2}s\lambda \int_{-T}^{T}\int_{S_i} \left[a\varphi|w|^2 \frac{\partial L(\psi)}{\partial \boldsymbol{n}_i}\right]_{S_i} \mathrm{d}\sigma \mathrm{d}t \\
& + s\lambda^2 \int_{-T}^{T}\int_{S_i} \left[a\varphi E(\psi) w \frac{\partial w}{\partial \boldsymbol{n}_i}\right]_{S_i} \mathrm{d}\sigma \mathrm{d}t \\
& - \frac{1}{2}s\lambda^3 \int_{-T}^{T}\int_{S_i} \left[a\varphi E(\psi)|w|^2 \frac{\partial \psi}{\partial \boldsymbol{n}_i}\right]_{S_i} \mathrm{d}\sigma \mathrm{d}t \\
& + s^3\lambda^3 \int_{-T}^{T}\int_{S_i} \left[a\varphi^3 E(\psi)|w|^2 \frac{\partial \psi}{\partial \boldsymbol{n}_i}\right]_{S_i} \mathrm{d}\sigma \mathrm{d}t.
\end{aligned}$$

We denote $\boldsymbol{\tau}_i$ the tangent vector of $S_i$, then $\left[\frac{\partial w}{\partial \boldsymbol{\tau}_i}\right]_{S_i} = 0$ since $[w]_{S_i} = 0$. Moreover $\psi$ is constant on $S_i$ at any fixed time $t$, thus

$$\nabla\psi\cdot\nabla w = \frac{\partial \psi}{\partial \boldsymbol{n}_i}\frac{\partial w}{\partial \boldsymbol{n}_i}, \quad \text{on } S_i.$$

We have

$$\begin{aligned}
& \left[a^2\varphi|\nabla w|^2 \frac{\partial \psi}{\partial \boldsymbol{n}_i}\right]_{S_i} - 2\left[a^2\varphi(\nabla\psi\cdot\nabla w)\frac{\partial w}{\partial \boldsymbol{n}_i}\right]_{S_i} \\
= & -\left[a^2\varphi\left|\frac{\partial w}{\partial \boldsymbol{n}_i}\right|^2 \frac{\partial \psi}{\partial \boldsymbol{n}_i}\right]_{S_i} + \left[a^2\varphi\left|\frac{\partial w}{\partial \boldsymbol{\tau}_i}\right|^2 \frac{\partial \psi}{\partial \boldsymbol{n}_i}\right]_{S_i} \\
= & -(\frac{1}{a_{i+1}} - \frac{1}{a_i})\varphi(a\frac{\partial w}{\partial \boldsymbol{n}_i})^2(a\frac{\partial \psi}{\partial \boldsymbol{n}_i}) + (a_{i+1}-a_i)\varphi(\frac{\partial w}{\partial \boldsymbol{\tau}_i})^2(a\frac{\partial \psi}{\partial \boldsymbol{n}_i}) \\
\leq & -(\frac{1}{a_{i+1}} - \frac{1}{a_i})\varphi(a\frac{\partial w}{\partial \boldsymbol{n}_i})^2(a\frac{\partial \psi}{\partial \boldsymbol{n}_i}),
\end{aligned} \tag{4.3}$$

since $a_{i+1} < a_i$ and $a\frac{\partial \psi}{\partial \boldsymbol{n}_i} \geq \delta_0 > 0$. And

$$\begin{aligned}
\left[a\varphi|w|^2 \frac{\partial L(\psi)}{\partial \boldsymbol{n}_i}\right]_{S_i} & = -\left[a^2\varphi|w|^2 \frac{\partial(\Delta\psi)}{\partial \boldsymbol{n}_i}\right]_{S_i} \\
& = (a_i - a_{i+1})\varphi|w|^2(a\frac{\partial(\Delta\psi)}{\partial \boldsymbol{n}_i}),
\end{aligned} \tag{4.4}$$

$$\begin{aligned}\left[a\varphi E(\psi)w\frac{\partial w}{\partial \boldsymbol{n}_i}\right]_{S_i} &= -\left[a^2\varphi|\nabla\psi|^2 w\frac{\partial w}{\partial \boldsymbol{n}_i}\right]_{S_i}\\ &= -(\frac{1}{a_{i+1}}-\frac{1}{a_i})\varphi|a\nabla\psi|^2 w(a\frac{\partial w}{\partial \boldsymbol{n}_i})\\ &\leq (\frac{1}{a_{i+1}}-\frac{1}{a_i})\varphi\Big(\frac{\delta_0}{\lambda}(a\frac{\partial w}{\partial \boldsymbol{n}_i})^2+\frac{2\lambda|a\nabla\psi|^2}{\delta_0}|w|^2\Big),\end{aligned} \tag{4.5}$$

$$\begin{aligned}\left[a\varphi E(\psi)|w|^2\frac{\partial \psi}{\partial \boldsymbol{n}_i}\right]_{S_i} &= -\left[a^2\varphi|\nabla\psi|^2 |w|^2\frac{\partial \psi}{\partial \boldsymbol{n}_i}\right]_{S_i}\\ &= -(\frac{1}{a_{i+1}}-\frac{1}{a_i})\varphi|a\nabla\psi|^2 |w|^2(a\frac{\partial \psi}{\partial \boldsymbol{n}_i}).\end{aligned} \tag{4.6}$$

Gathering (4.3)-(4.6), we obtain

$$\begin{aligned}B_i^+ - B_i^- \leq &- s\lambda\int_{-T}^{T}\int_{S_i}(\frac{1}{a_{i+1}}-\frac{1}{a_i})\varphi(a\frac{\partial w}{\partial \boldsymbol{n}_i})^2(a\frac{\partial \psi}{\partial \boldsymbol{n}_i}-\delta_0)\mathrm{d}\sigma\mathrm{d}t\\ &- s\lambda\int_{-T}^{T}\int_{S_i}(\frac{1}{a_{i+1}}-\frac{1}{a_i})\varphi|w|^2F(s)\mathrm{d}\sigma\mathrm{d}t,\end{aligned}$$

with

$$F(s) = s^2\lambda^2\varphi^2(a\frac{\partial \psi}{\partial \boldsymbol{n}_i})^3 - \frac{1}{2}\lambda^2\big(a\frac{\partial \psi}{\partial \boldsymbol{n}_i}\big)^3 - \frac{2\lambda^2|a\nabla\psi|^2}{\delta_0} - \frac{a_ia_{i-1}}{2}\big(a\frac{\partial(\Delta\psi)}{\partial \boldsymbol{n}_i}\big).$$

Since $a\frac{\partial \psi}{\partial \boldsymbol{n}_i} \geq \delta_0$, there exists $s_0 > 0$ such that $F(\lambda) > 0$ for all $s > s_0$. So we get $B_i^+ - B_i^- \leq 0$.

The remaining term

$$\begin{aligned}B_k^- =& s\lambda\int_{-T}^{T}\int_{S_k^-}\Big(a^2\varphi|\nabla w|^2\frac{\partial \psi}{\partial \boldsymbol{n}_k} - 2a^2\varphi(\nabla\psi\cdot\nabla w)\frac{\partial w}{\partial \boldsymbol{n}_k}\Big)\ \mathrm{d}\sigma\mathrm{d}t\\ =& - s\lambda\int_{-T}^{T}\int_{\partial\Omega_k} a_k^2\varphi\left|\frac{\partial w}{\partial \boldsymbol{n}_k}\right|^2\frac{\partial \psi}{\partial \boldsymbol{n}_k}\ \mathrm{d}\sigma\mathrm{d}t\\ \geq& - s\lambda\int_{-T}^{T}\int_{\widetilde{\partial\Omega_k}} a_k^2\varphi\left|\frac{\partial w}{\partial \boldsymbol{n}_k}\right|^2\frac{\partial \psi}{\partial \boldsymbol{n}_k}\ \mathrm{d}\sigma\mathrm{d}t,\end{aligned}$$

since $u = 0$ on $\partial\Omega_k$. Now we carry all together

$$\begin{aligned}\|Pw\|^2_{L^2(Q_j^k)} - \frac{1}{2}\|P_1w\|^2_{L^2(Q_j^k)} \geq& c\int_{-T}^{T}\int_{\Omega_k\setminus\Omega_j} s^3|w|^2 + s|\nabla w|^2 + s|\partial_t w|^2\ \mathrm{d}x\mathrm{d}t\\ &- Cs\int_{-T}^{T}\int_{\Omega_k\setminus\Omega_j}|w|^2\ \mathrm{d}x\mathrm{d}t\\ &- Cs\int_{-T}^{T}\int_{\widetilde{\partial\Omega_k}}\left|\frac{\partial w}{\partial \boldsymbol{n}_k}\right|^2\ \mathrm{d}\sigma\mathrm{d}t.\end{aligned}$$

Thus we complete the proof for the hyperbolic case of Lemma 2.1. For the elliptic case of Lemma 2.1, it is obvious by noticing that

$$\mathrm{e}^{2s\varphi}|\nabla u|^2 \leq 2|\nabla w|^2 + 2s^2|\nabla\varphi|^2|w|^2,$$

$$\mathrm{e}^{2s\varphi}|\partial_t u|^2 \leq 2|\partial_t w|^2 + 2s^2|\partial_t\varphi|^2|w|^2.$$

The proof of Lemma 2.1 is complete.

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