

NEW SECOND-ORDER OPTIMALITY CONDITIONS IN SUB-RIEMANNIAN GEOMETRY

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Abstract. We study the geometry of the second-order expansion of the extended end-point map for the sub-Riemannian geodesic problem. Translating the geometric reality into equations we derive new second-order necessary optimality conditions in sub-Riemannian Geometry. In particular, we find an ODE for velocity of an abnormal sub-Riemannian geodesics. It allows to divide abnormal minimizers into two classes, which we propose to call 2-normal and 2-abnormal extremals. In the 2-normal case the above ODE completely determines the velocity of a curve, while in the 2-abnormal case the velocity is undetermined at some, or at all points. With some enhancement of the presented results it should be possible to prove the regularity of all 2-normal extremals (the 2-abnormal case seems to require study of higher-order conditions) thus making a step towards solving the problem of smoothness of sub-Riemannian abnormal geodesics. As a by-product we present a new derivation of Goh conditions. We also prove that the assumptions weaker than these used in [Boarotto, Monti, Palmurella, 2020] to derive third-order Goh conditions, imply piece-wise- C^2 regularity of an abnormal extremal.

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1. INTRODUCTION

The problem of smoothness of abnormal geodesics. The question whether all minimizing sub-Riemannian geodesics are smooth dates back to the famous example of Montgomery [16] of a minimizing abnormal geodesics. It is commonly agreed [1, 17] to be one of the main open problems in sub-Riemannian geometry. Despite 30 years of efforts, there are not many general results. Among the latter we must mention the derivation of the second-order necessary conditions for optimality [2] (known as *Goh conditions*) and a technical masterpiece of Hakavouri and Le Donne [8] (see also [9]) which ruled out corner-like singularities (some previous results in this direction are [14, 19]). We are also aware of a very recent attempts to derive third-order (and higher) necessary conditions of optimality [4, 5]. We refer to [18] for a discussion of the problem and its history.

Keywords and phrases: Sub-Riemannian geometry, geodesic, abnormal minimizer, optimality conditions, 2-jet, Goh conditions.

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Basic ideas. In this paper we propose the following approach to the problem. Consider a sub-Riemannian manifold $(M, \mathcal{D}, \langle \cdot, \cdot \rangle)$ and the problem of minimizing the energy by a \mathcal{D} -horizontal curve $q(t)$, with $t \in [t_0, t_1]$, joining given two points $q_0 = q(t_0)$ and $q_1 = q(t_1)$ on M . We may (locally) reformulate this problem as an optimal control problem in a standard way – see page 6. Now let

$$\mathbf{End}^{[t_0, t]} : L^\infty([t_0, t], \mathbb{R}^k) \longrightarrow M \times \mathbb{R} ; \quad \mathbf{End}^{[t_0, t]}[u] = (q(t), E^{[t_0, t]}[u])$$

be the extended end-point map at time $t \in [t_0, t_1]$ related with this optimal control problem. Here $q(t)$ is the sub-Riemannian trajectory corresponding to the control $u \in L^\infty([t_0, t_1], \mathbb{R}^k)$ and $E^{[t_0, t]}[u]$ is the energy of this trajectory at time t .

In the next step, for a distinguished trajectory of the system $q(t) \in M$ corresponding to a given control $u \in L^\infty([t_0, t], \mathbb{R}^k)$, we study curves $s \mapsto \mathbf{End}^{[t_0, t]}[u + s \cdot \Delta u]$ built for all possible controls $\Delta u \in L^\infty([t_0, t], \mathbb{R}^k)$. More precisely, we are interested in 2-jets of such curves at $s = 0$, *i.e.*

$$\mathbf{End}^{[t_0, t]}[u + s \cdot \Delta u] \stackrel{\text{loc.}}{=} : \mathbf{End}^{[t_0, t]}[u] + s \cdot \mathbf{b}^{(1)}(t, \Delta u) + s^2 \cdot \mathbf{b}^{(2)}(t, \Delta u) + o(s^2).$$

One may think of the collection of pairs $(\mathbf{b}^{(1)}(t, \Delta u), \mathbf{b}^{(2)}(t, \Delta u))$ (for all possible Δu 's) as of quadratic approximations of the image of the end-point map around $\mathbf{End}^{[t_0, t]}[u]$.

It turns out that, for a given $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$, curves $\mathbf{b}^{(1)}(t, \Delta u)$ and $\mathbf{b}^{(2)}(t, \Delta u)$ satisfy quite a complicated system of ODEs (2.5)–(2.6). It is convenient to treat the latter as a control system (built on top of a fixed trajectory $q(t)$ of the initial sub-Riemannian control system) with Δu playing the role of the control.

The key point now is to relate the properties of this new control system with optimality of a sub-Riemannian trajectory $q(t)$. To do this we rely on two results:

- Firstly, the classical theory of Agrachev and Sarychev [2] gives information about the space of curves $\mathbf{b}^{(2)}(t, \Delta u)$ for these controls $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$ which satisfy $\mathbf{b}^{(1)}(t, \Delta u) = \mathbf{0}$.
- Secondly, thanks to the results of [11], we may change coordinates transforming the pairs $(\mathbf{b}^{(1)}(t, \Delta u), \mathbf{b}^{(2)}(t, \Delta u))$ into pairs $(\mathbf{q}^{(1)}(t, \Delta u), \mathbf{q}^{(2)}(t, \Delta u)) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$, for which the new control system we want to analyze takes a much simpler form (4.3)–(4.4).

Using the above we are able to relate the geometry of the sets

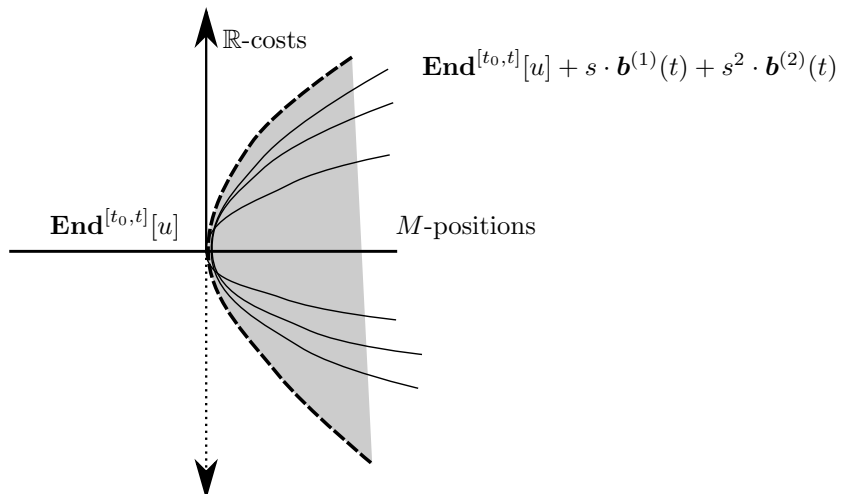
$$\{(\mathbf{b}^{(1)}(t, \Delta u), \mathbf{b}^{(2)}(t, \Delta u)) \mid \Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)\} \quad \text{where } t \in [t_0, t_1]$$

with the optimality of the sub-Riemannian trajectory $q(t)$.

The main result. Leaving aside some technical complications (we need to divide the curve $q(t)$ into pieces by excluding a finite number of points in the time interval $[t_0, t_1]$) it turns out that, if the curve $q(t)$ is an abnormal minimizing geodesics, then there exists a *Pontryagin covector* $\varphi(t) = (\varphi(t), 0) \in \mathbb{T}_{(q(t), E^{[t_0, t]}[u])}^*(M \times \mathbb{R})$ such that for each $t \in [t_0, t_1]$, the set

$$S_t := \{(\mathbf{b}^{(1)}(t, \Delta u), \langle \mathbf{b}^{(2)}(t, \Delta u), \varphi(t) \rangle) \mid \Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)\} \subset \mathbb{T}_{(q(t), E^{[t_0, t]}[u])}(M \times \mathbb{R}) \times \mathbb{R}$$

is bounded by a graph of a quadratic map $\Phi_2(t) : \mathbb{T}_{(q(t), E^{[t_0, t]}[u])}(M \times \mathbb{R}) \times \mathbb{T}_{(q(t), E^{[t_0, t]}[u])}(M \times \mathbb{R}) \rightarrow \mathbb{R}$ (see the figure below).



For a minimizing curve the second-order approximations of the image of the extended end-point map (grey area) must be bounded by a graph of a quadratic function (dashed line). Solid lines represent different quadratic approximations of the image of the extended end-point map $\mathbf{End}^{[t_0, t]}$.

Moreover, by taking this bounding map biggest possible, we may guarantee that $\Phi_2(t)$ has properties of the value function in the sense of Bellman. In particular, it turns out that for every control $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$ the function

$$t \mapsto \langle \mathbf{b}^{(2)}(t, \Delta u), \varphi(t) \rangle - \Phi_2(t)[\mathbf{b}^{(1)}(t, \Delta u), \mathbf{b}^{(1)}(t, \Delta u)]$$

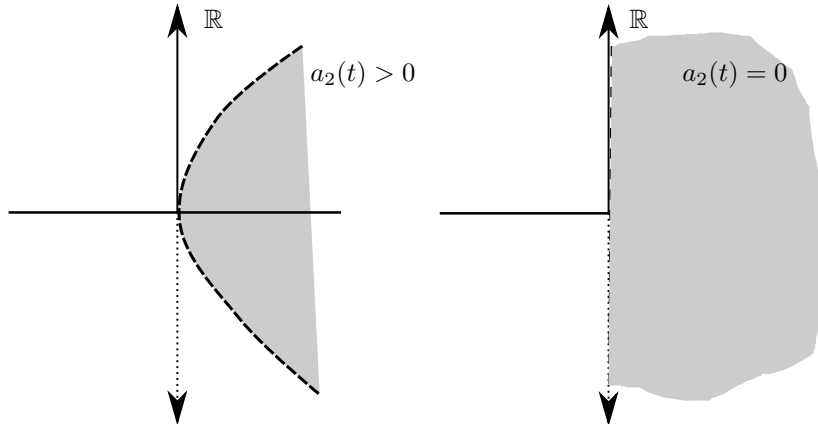
is non-decreasing. The above properties are stated as Theorem 3.3, which is the main result of this paper.

Discussion of the main result. By differentiation of the latter function with respect to t (we were able to show that $\Phi_2(t)$ is differentiable almost everywhere) we prove Theorem 3.4, obtaining a set of ODEs (3.6)–(3.7) and an inequality (3.5) that an abnormal minimizing SR geodesic should satisfy.

The derivation of these equations is, in our opinion, the most significant consequence of Theorem 3.3. (The well-known *Goh condition* is a simple consequence of these equations – see page 15.) The first of these equations is of the form

$$\langle X_i|_{q(t)}, \xi(t) \rangle + 2 a_2(t) \cdot u_i(t) = 0. \quad (1.1)$$

It describes the controls $u_i(t)$ of the SR geodesic $q(t)$ in terms of vector fields X_i spanning the distribution \mathcal{D} , and two functions $\xi(t)$, $a_2(t)$ related with the quadratic map $\Phi_2(t)$. Note that if $a_2(t) \neq 0$ we may explicitly derive $u_i(t)$ as a function of other quantities. This observation is a basis of division of abnormal extremals into sub-classes of *2-normal* and *2-abnormal extremals* proposed by us in Definition 3.5. The former class is defined by the property that $a_2(t) \neq 0$ for every $t \in [t_0, t_1]$, hence the controls u_i are solutions of algebraic equations. Geometrically the 2-normal case corresponds to a situation when, for every t , the graph of $\Phi_2(t)$ strictly separates the cost direction from the set S_t . By contrast, in the 2-abnormal case the controls $u_i(t)$ are undetermined at some times $t \in [t_0, t_1]$ and the above separation is not strict. Schematically the difference is depicted on the figure below.



In the 2-normal case (left), the image of the extended end-point map is separated from the cost direction \mathbb{R} , while in the 2-abnormal case (right) there is no separation.

The presence of equation (1.1) is very important from the point of view of the *regularity problem of SR geodesics*, at least in the 2-normal case. Since, the controls $u_i(t)$, and the maps $\xi(t)$ and $a_2(t)$ are algebraically related, it is clear that the regularity problem is linked with the t -regularity of maps $\xi(t)$ and $a_2(t)$ or, more generally, the whole quadratic map $\Phi_2(t)$. Unfortunately, in this paper we were only able to prove that these objects are of bounded variation, which is not enough for our purposes. However, I hope that other researchers, perhaps more experienced in optimal control theory, could help improving the regularity results.

Let us note that the regularity of 2-abnormal extremals seems to be a much more difficult task requiring, in my opinion, study of third- or higher-order expansions of the extended end-point map.

Supplementary results. The question whether the set S_t is bounded is closely related with a special case of a linear-quadratic optimal control problem (we speak about a *characteristic optimal control problem* of the SR trajectory $q(t)$). In our main result – Theorem 3.3 – we used only the existence of a bound (and the fact that it is actually a graph of a quadratic map). An interesting problem on its own is the question whether the boundary points on S_t are realizable by some controls from $L^\infty([t_0, t], \mathbb{R}^k)$, or to put it differently, does the characteristic optimal control problem has solutions. In general, this may not be the case as examples studied by us in Section 3.4 show. However, when the solutions exists they can provide a lot of information about the SR trajectory $q(t)$, in particular prove its regularity. We discuss this topic in detail in Section 5, providing both algebraic and differential (by means of the Pontryagin Maximum Principle) criteria for the existence of solutions of the characteristic optimal control problem.

In particular, we prove Lemma 5.9 in which we show that assumptions weaker than these of Theorem 1.2 of [4] imply that if the abnormal minimizing SR geodesic $q(t)$ is a 2-normal extremal in the sense of Definition 3.5, it must be piece-wise C^2 with a finite number of pieces.

Content of this paper. In the preliminary Section 2 we state the sub-Riemannian geodesic problem and reformulate it in a control-theoretic setting. We also give a brief introduction to jets, define the extended end-point map, and recall first order optimality conditions (for normal and abnormal extremals).

In Section 3 we state our main results – Theorems 3.3 and 3.4 (linking them with the Agrachev-Sarychev theory). Later we discuss these results, in particular, addressing the regularity problem, Goh conditions, and introducing the notions of 2-normal and 2-abnormal curves. We also study two examples – the Martinet system and a corner curve.

Section 4 contains the proof of Theorem 3.3. We begin it by some technical preparations in Section 4.1, including construction of, so called, *adapted coordinates*, and defining the *characteristic control system* and the

characteristic optimal control problem related with a SR trajectory. The actual proof, contained in Section 4.2 is rather long, technical, and requires a few separate steps.

In Section 5 we study additional information input on a SR geodesic if we can assure the existence of solutions of the characteristic optimal control problem. In particular we get some regularity results under this assumption.

We end by stating a few hypothesis in Section 6.

2. PRELIMINARIES

Regularity of curves. Throughout this work we will talk about curves on smooth manifolds of various classes of regularity. (Some emphasis will be put on improving the regularity of a curve by means of a bootstrap technique.) At the bottom level we have L^∞ -curves, *i.e. measurable and essentially bounded*. A step higher are ACB curves, *i.e. absolutely continuous with L^∞ derivatives*. The relation within these two is that if $\gamma(t)$ satisfies an ODE $\dot{\gamma}(t) \stackrel{\text{a.e.}}{=} f(t)$, where $f(t)$ is of class L^∞ then $\gamma(t)$ is ACB. This resembles the standard relation between C^r and C^{r+1} curves. Therefore it should not be very abusive to the standard mathematical jargon to denote the class L^∞ as C^{0-} , and the class ACB as C^{1-} (so that $C^{0-+1} = C^{1-}$ and also $C^{1-} \subset C^0$).

2.1. The sub-Riemannian geodesic problem

The sub-Riemannian geodesic problem. We shall be working within the following geometric setting. Let M be a smooth n -dimensional manifold and let $\mathcal{D} \subset TM$ be a smooth distribution of rank k on M . By $g(\cdot, \cdot) : \mathcal{D} \times_M \mathcal{D} \rightarrow \mathbb{R}$ we shall denote a positively-defined symmetric bilinear product on \mathcal{D} (called a *sub-Riemannian metric*). The triple (M, \mathcal{D}, g) constitutes a *sub-Riemannian structure* (*SR structure*, in short). Given a pair of points $q_0, q_1 \in M$ and an interval $[t_0, t_1] \subset \mathbb{R}$ we consider the following *SR geodesic problem*:

Find an absolutely continuous curve $q : [t_0, t_1] \rightarrow M$ satisfying the following conditions: (P)

- (P1) q is almost everywhere tangent to \mathcal{D} , *i.e.*, $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for a.e. $t \in [t_0, t_1]$.
- (P2) q joins q_0 and q_1 , that is, $q(t_0) = q_0$ and $q(t_1) = q_1$.
- (P3) in the set of all curves satisfying conditions (P1) and (P2), curve q realizes the minimum of the *energy functional*

$$q \mapsto \mathcal{E}[q] := \frac{1}{2} \int_{t_0}^{t_1} g(\dot{q}(\tau), \dot{q}(\tau)) \, d\tau.$$

It is well-known [17] that under the assumption that \mathcal{D} is bracket-generating, and that M is complete, problem (P) has a solution for every $q_0, q_1 \in M$ and every $[t_0, t_1] \subset \mathbb{R}$. A solution of the problem (P) is called a *minimizing SR geodesic*.

It follows easily from the Cauchy-Schwarz inequality that the minimizing SR geodesic has to be normalized, *i.e.*, $g(\dot{q}(t), \dot{q}(t))$ is constant on $[t_0, t_1]$. Therefore we may actually look for the solutions of (P) in the class of ACB curves. In the forthcoming control-theoretic formulation of the SR geodesic problem, this simple observation will allow us to consider the smaller class of L^∞ -controls, instead of all L^1 -controls.

Control-theoretic formulation. In a local version, problem (P) can be formulated as an optimal control problem. Indeed, choose k smooth, linearly independent vector fields X_1, \dots, X_k locally spanning the distribution \mathcal{D} . Note that, by applying the Gram-Schmidt orthogonalization, we may assume that the fields $\{X_i\}_{i=1, \dots, k}$ form an orthonormal basis of \mathcal{D} , *i.e.*, $g(X_i, X_j) = \delta_{ij}$. Now every ACB curve $q(t)$ almost everywhere tangent to \mathcal{D}

may be regarded as a solution of

$$\dot{q}(t) = \sum_{i=1}^k u^i(t) \cdot X_i \Big|_{q(t)}, \quad (\Sigma)$$

where $t \mapsto u(t) = (u^1(t), \dots, u^k(t)) \in \mathbb{R}^k$ is an L^∞ -map, and equation (Σ) is regarded in the sense of Caratheodory, *i.e.* the equality holds for almost every $t \in [t_0, t]$. We shall call such a curve q a *SR trajectory* corresponding to the *control* u .

Note that under the assumption that the fields X_i are normalized as $g(X_i, X_j) = \delta_{ij}$, the energy of such a trajectory is simply $E^{[t_0, t_1]}[u]$, where we define

$$E^{[t_0, t_1]}[u] := \frac{1}{2} \int_{t_0}^{t_1} \sum_{i=1}^k (u^i(\tau))^2 d\tau = \frac{1}{2} \|u\|_{L^2([t_0, t_1], \mathbb{R}^k)}^2. \quad (\text{E})$$

From now on we shall **study the SR geodesic problem (P) within such a local framework**, *i.e.* we take $M \stackrel{\text{loc.}}{=} \mathbb{R}^n$, and choose a global g -orthonormal basis of smooth vector fields $\{X_i\}_{i=1,2,\dots,k}$ spanning \mathcal{D} .

Remark 2.1 (Notation in the extended configuration space). It is convenient to think of both, the trajectory $q(t)$, and the energy $E^{[t_0, t_1]}$ at the same time, by introducing *the extended configuration space* $\mathbf{M} := M \times \mathbb{R}$. For a control $u \in L^\infty([t_0, t_1], \mathbb{R}^k)$ the *extended trajectory* is defined as $\mathbf{q}(t) := (q(t), E^{[t_0, t]}[u])$, where $q(t)$ satisfies (Σ) . Similar objects, consequently denoted by bold letters, will be considered thorough the paper.

2.2. On jets

In the paper we speak about second-order optimality conditions in SR geometry. The term *second-order* means that we will consider the second-order Taylor expansions of various objects. Geometrically, we rather speak about *2-jets*. Below we recall some basic definitions and constructions from jet theory [12, 22].

Jets of curves. We say that two C^2 -curves $\gamma, \tilde{\gamma} : (-\varepsilon, \varepsilon) \rightarrow M$ are *tangent of order two* at $p = \gamma(0) = \tilde{\gamma}(0)$ if in some (and thus any) local coordinate system around p on M we have

$$\gamma'(0) = \tilde{\gamma}'(0) \quad \text{and} \quad \gamma''(0) = \tilde{\gamma}''(0).$$

In other words, maps $s \mapsto \gamma(s)$ and $s \mapsto \tilde{\gamma}(s)$ have the same order-two Taylor expansions at $s = 0$. The relation \sim_2 of being tangent of order two is an equivalence relation and its equivalence classes $[\gamma]_{\sim_2} = [\tilde{\gamma}]_{\sim_2}$ are called *2-jets* at p . We denote the set of all 2-jets at p by $\mathbb{T}_p^2 M$, and call it the *second tangent space of M at p* . In fact, the collection $\mathbb{T}^2 M := \bigcup_{p \in M} \mathbb{T}_p^2 M$ is a locally trivial bundle over M . We call it the *second tangent bundle of M* . It is an example of a *graded bundle* in the sense of [7], with the canonical multiplicative \mathbb{R} -action defined on representatives by $\lambda \mapsto (s \mapsto \gamma(\lambda \cdot s))$.

Given a curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ and its (local) second-order Taylor expansion

$$\gamma(s) \stackrel{\text{loc.}}{=} p + s \cdot \beta^{(1)} + s^2 \cdot \beta^{(2)} + o(s^2),$$

we may thus identify its 2-jet with a pair $(\beta^{(1)}, \beta^{(2)}) \simeq [\gamma]_{\sim_2}$. Although one should remember, that only the first term $\beta^{(1)}$ is a geometric object, representing a tangent vector at p , while the term $\beta^{(2)}$ on its own (without the presence of a particular $\beta^{(1)}$) has no intrinsic geometric meaning. In a particular situation when $M = V$ is a vector space we may, however, canonically identify $\mathbb{T}_p^2 V \simeq V \times V$, by considering the above Taylor expansion in any linear coordinates on V . Thus for a vector space, the bare term $\beta^{(2)}$ has a geometric interpretation of an element of V .

Jets of maps. Let $F : M \rightarrow N$ be a smooth map between manifolds. For a C^2 -curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ passing through $\gamma(0) = p \in M$ consider a (local) second-order Taylor expansion

$$F(\gamma(s)) \stackrel{loc}{=} F(p) + s \cdot b^{(1)} + s^2 \cdot b^{(2)} + o(s^2).$$

Then the pair $(b^{(1)}, b^{(2)})$ depends only on the 2-jet $[\gamma]_{\sim_2}$. Thus we may define the 2-jet of map F

$$\mathbb{T}^2 F : \mathbb{T}^2 M \longrightarrow \mathbb{T}^2 N \quad \text{via} \quad \mathbb{T}_p^2 M \ni [\gamma]_{\sim_2} \longmapsto [F(\gamma)]_{\sim_2} \in \mathbb{T}_{F(p)}^2 N.$$

$\mathbb{T}^2 F$ is a bundle map over $F : M \rightarrow N$.

Pairings on jets. As a particular example of the above, consider a composition of a smooth function $f : M \rightarrow \mathbb{R}$ and a C^2 -curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ passing through $\gamma(0) = p \in M$. The order-two Taylor expansion of $f(\gamma) : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ reads as

$$f(\gamma(s)) = f(p) + s \cdot D_p f[\gamma'(0)] + s^2 \cdot \{D_p f[\gamma''(0)] + D_p^2 f[\gamma'(0), \gamma'(0)]\} + o(s^2),$$

and it depends only on the 2-jet $[\gamma]_{\sim_2}$. Further, the resulting 2-jet $[f(\gamma)]_{\sim_2}$ is valued in the second tangent space of a vector space $\mathbb{T}_{f(p)}^2 \mathbb{R} \simeq \mathbb{R} \times \mathbb{R}$, and so the term of degree two

$$D_p f[\gamma''(0)] + D_p^2 f[\gamma'(0), \gamma'(0)] \tag{2.1}$$

has an intrinsic geometric meaning.

Note that for a fixed 2-jet $[f(\gamma)]_{\sim_2}$, the value of (2.1) depends only on the first and second derivatives $D_p f$ and $D_p^2 f$, *i.e.* the second-order Taylor expansion of f at p . Since we may construct smooth functions with arbitrary derivatives at a point, we may define via

$$\langle (\phi, \Phi), (b^{(1)}, b^{(2)}) \rangle := \langle \phi, b^{(2)} \rangle + \Phi[b^{(1)}, b^{(1)}] \tag{2.2}$$

a natural (and geometrically meaningful) pairing between a 2-jet of a curve $(b^{(1)}, b^{(2)}) \in \mathbb{T}_p^2 M$ and a pair (ϕ, Φ) consisting of a linear map $\phi : \mathbb{T}_p M \rightarrow \mathbb{R}$ and a symmetric bi-linear map $\Phi : \mathbb{T}_p M \times \mathbb{T}_p M \rightarrow \mathbb{R}$.

Remark 2.2. It is worth to mention that the pairs (ϕ, Φ) as above can be identified with the elements of $\mathbb{T}_p^{2*} M$, *i.e.* 2-covectors at $p \in M$ in the sense of Tulczyjew [23]. We define the latter as equivalence classes of smooth functions on M , saying that $f, \tilde{f} : M \rightarrow \mathbb{R}$ are equivalent if and only if $f(\gamma(s))$ and $\tilde{f}(\gamma(s))$ have the same second-order Taylor expansions at $s = 0$ for any smooth curve $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ passing through $p = \gamma(0)$. The pairing (2.2) can be thus interpreted as a map $\mathbb{T}_p^{2*} M \times \mathbb{T}_p^2 M \rightarrow \mathbb{R}$. We remark that it has an elegant description in terms of the Weyl algebra of dual numbers [12].

2.3. The end-point map and its second-order expansion

The end-point map and its second expansion. Denote by

$$\text{End}_{q_0}^{[t_0, t]} : L^\infty([t_0, t], \mathbb{R}^k) \longrightarrow M$$

the *end-point map* of the control system (Σ) at time t and with the initial condition q_0 at $t = t_0$. That is, for a control $u \in L^\infty([t_0, t], \mathbb{R}^k)$, $\text{End}_{q_0}^{[t_0, t]}[u]$ is a solution $q(t)$ of (Σ) satisfying $q(t_0) = q_0$.

We would like to study the following second-order expansion of this map (written in some local coordinate system)

$$\text{End}_{q_0}^{[t_0, t]}[u + s \cdot \Delta u] \stackrel{\text{loc}}{=} \text{End}_{q_0}^{[t_0, t]}[u] + s \cdot b^{(1)}(t, \Delta u) + s^2 \cdot b^{(2)}(t, \Delta u) + o(s^2), \quad (2.3)$$

where $\Delta u \in T_u L^\infty([t_0, t], \mathbb{R}^k) \simeq L^\infty([t_0, t], \mathbb{R}^k)$ is a control, and s is a real parameter. Of course $b^{(1)}(t, \Delta u) = D_u \text{End}_{q_0}^{[t_0, t]}[\Delta u]$, and one is tempted to write $b^{(2)}(t, \Delta u) = \frac{1}{2} D_u^2 \text{End}_{q_0}^{[t_0, t]}[\Delta u, \Delta u]$. The last equality makes, however, no sense as in general the second derivative of a manifold-valued map is not a well-defined geometric object – cf. Chapter 20 of [3]. Expansion (2.3) has a geometric meaning though – by our considerations in Ssec. 2.2 it describes the second tangent map $T^2 \text{End}_{q_0}^{[t_0, t]}$ evaluated on a 2-jet $[s \mapsto u + s \cdot \Delta u + s^2 \cdot 0]_{\sim 2} \in T_u^2 L^\infty([t_0, t], \mathbb{R}^k) \simeq L^\infty([t_0, t], \mathbb{R}^k) \times L^\infty([t_0, t], \mathbb{R}^k)$. Thus the pair $(b^{(1)}(t, \Delta u), b^{(2)}(t, \Delta u))$ is a local representative of some element in $T_{q(t)}^2 M$.

To calculate $b^{(1)}(t, \Delta u)$ and $b^{(2)}(t, \Delta u)$, denote by $q_s(t)$ a solution of the following ODE

$$\dot{q}_s(t) = \sum_{i=1}^k (u^i(t) + s \cdot \Delta u_i(t)) \cdot X_i \Big|_{q_s(t)} \quad \text{with} \quad q_s(t_0) = q_0. \quad (2.4)$$

Obviously $q_s(t) = \text{End}_{q_0}^{[t_0, t]}[u + s \cdot \Delta u]$. What is more, (2.4) is an ODE in the sense of Caratheodory, depending smoothly on the parameter s . As such it has a solution which also smoothly depends on s – see [6] – and, moreover, equations for $b^{(1)}(t, \Delta u) \stackrel{\text{loc}}{=} \partial_s|_{s=0} q_s(t)$ and $b^{(2)}(t, \Delta u) \stackrel{\text{loc}}{=} \frac{1}{2} \partial_s^2|_{s=0} q_s(t)$ are obtained by differentiating (2.4) with respect to s , leading to (for notation simplicity below we omit the dependence of $b^{(1)}(t, \Delta u)$ and $b^{(2)}(t, \Delta u)$ on Δu)

$$b^{(1)}(t) = \sum_{i=1}^k u_i(t) \cdot D_{q(t)} X_i \left[b^{(1)}(t) \right] + \sum_{i=1}^k \Delta u_i(t) \cdot X_i \Big|_{q(t)} \quad (2.5)$$

with the initial condition $b^{(1)}(t_0) = 0$; and

$$b^{(2)}(t) = \sum_{i=1}^k u_i(t) \left\{ D_{q(t)} X_i \left[b^{(2)}(t) \right] + \frac{1}{2} D_{q(t)}^2 X_i \left[b^{(1)}(t), b^{(1)}(t) \right] \right\} + 2 \sum_{i=1}^k \Delta u_i(t) \cdot D_{q(t)} X_i \left[b^{(1)}(t) \right], \quad (2.6)$$

where $b^{(1)}(t)$ is as above, and the initial condition is $b^{(2)}(t_0) = 0$.

The extended end-point map. Now we would like to add the energy (E) to the picture. We define the *extended end-point map*

$$\mathbf{End}_{q_0}^{[t_0, t]} : L^\infty([t_0, t], \mathbb{R}^k) \longrightarrow M \times \mathbb{R},$$

simply as $\mathbf{End}_{q_0}^{[t_0, t]}[u] := (\text{End}_{q_0}^{[t_0, t]}[u], E^{[t_0, t]}[u])$. For any $\Delta u \in L^\infty([t_0, t], \mathbb{R}^k)$ the second expansion of the energy term is just

$$E^{[t_0, t]}[u + s \cdot \Delta u] = E^{[t_0, t]}(u) + s \cdot c^{(1)}(t, \Delta u) + s^2 \cdot c^{(2)}(t, \Delta u), \quad (2.7)$$

where $c^{(1)}(t, \Delta u) = \int_{t_0}^t \sum_i u_i(\tau) \cdot \Delta u_i(\tau) d\tau$ and $c^{(2)}(t, \Delta u) = \frac{1}{2} \int_{t_0}^t \sum_i \Delta u_i(\tau)^2 d\tau$. Or, in differential terms:

$$\begin{aligned} \dot{c}^{(1)}(t) &= \sum_{i=1}^k u_i(t) \cdot \Delta u_i(t) \\ \dot{c}^{(2)}(t, \Delta u) &= \frac{1}{2} \sum_{i=1}^k \Delta u_i(t)^2 \end{aligned} \tag{2.8}$$

with the initial conditions $c^{(1)}(t_0) = 0$ and $c^{(2)}(t_0) = 0$.

Normal and abnormal extremals. Note that if for some t the vector space

$$\text{Im } D_u \mathbf{End}^{[t_0, t]} = \{(b^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u)) \mid \Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)\}$$

equals $\mathbb{T}_{(q(t), E^{[t_0, t]}[u])}(M \times \mathbb{R})$ then, by the open-mapping theorem, the map $\mathbf{End}^{[t_0, t]}$ is open at u , and hence the related trajectory cannot be a minimizing sub-Riemannian geodesic. In this way we obtained a necessary condition that a SR geodesics should satisfy. Curves that satisfy this conditions are known as *extremals*. We can further classify these according to the geometric properties of $\text{Im } D_u \mathbf{End}^{[t_0, t]}$.

Definition 2.3. Consider a sub-Riemannian trajectory $q(t)$ with $t \in [t_0, t_1]$ corresponding to the control $u \in L^\infty([t_0, t_1], \mathbb{R}^k)$. Curve $q(t)$ is called an *extremal* if $\text{Im } D_u \mathbf{End}^{[t_0, t]}$ is a proper subspace of $\mathbb{T}_{(q(t_1), E^{[t_0, t_1]}[u])}(M \times \mathbb{R})$. An extremal is called *normal* if the space $\text{Im } D_u \mathbf{End}^{[t_0, t]}$ does not contain the subspace $\{0\} \times \mathbb{R}$. If the image $\text{Im } D_u \mathbf{End}^{[t_0, t]}$ is contained in a space $V \times \mathbb{R}$, where $V \subset \mathbb{T}_{q(t_1)}M$ is some proper subspace, $q(t)$ is called *abnormal*. We will say an abnormal extremal has *co-rank* r if $\text{Im } D_u \mathbf{End}^{[t_0, t]}$ is a co-rank r -subspace of $\mathbb{T}_{(q(t_1), E^{[t_0, t_1]}[u])}(M \times \mathbb{R})$.

Strictly abnormal extremals are abnormal extremals which are not normal, that is $\text{Im } D_u \mathbf{End}^{[t_0, t]} = V \times \mathbb{R}$ for some proper subspace $V \subset \mathbb{T}_{q(t_1)}M$.

Normal extremals are known to be C^∞ -smooth and energy-minimizing on small distances [15, 17].

First-order abnormal optimality conditions. Let us recall the following standard fact [17].

Lemma 2.4. *Let a SR trajectory $q(t)$ corresponding to the control $u \in L^\infty([t_0, t_1], \mathbb{R}^k)$ be an abnormal extremal. Then there exists a nowhere-vanishing covector curve $\varphi(t) \in \mathbb{T}_{q(t)}^*M$ satisfying the following conditions:*

$$\begin{aligned} \langle X_i|_{q(t)}, \varphi(t) \rangle &= 0 \quad \text{for every } i = 1, 2, \dots, k \\ \langle b, \dot{\varphi}(t) \rangle + \left\langle \sum_i u_i(t) \cdot D_{q(t)} X_i[b], \varphi(t) \right\rangle &= 0 \quad \text{for every } b \in \mathbb{T}_{q(t)}M. \end{aligned} \tag{2.9}$$

We shall call such a $\varphi(t)$ a Pontryagin covector.

It is instructive to see that conditions (2.9) can be easily derived from the following one

$$\text{Im } D_u \mathbf{End}^{[t_0, t]} \subset \ker(\varphi(t), 0) \quad (\text{equivalently } \langle b^{(1)}(t, \Delta u), \varphi(t) \rangle = 0 \text{ for every } t \text{ and every } \Delta u),$$

which expresses the fact that $\text{Im } D_u \mathbf{End}^{[t_0, t]}$ is contained in a proper subspace $V \times \mathbb{R} \subset \mathbb{T}_{q(t)}M \times \mathbb{T}_{E^{[t_0, t]}[u]}\mathbb{R}$ for every t .

To see this it is enough to differentiate the above condition with respect to t to get:

$$0 = \langle \dot{b}^{(1)}(t, \Delta u), \varphi(t, \Delta u) \rangle + \langle b^{(1)}(t, \Delta u), \dot{\varphi}(t) \rangle \stackrel{(2.5)}{=} \\ \langle \sum_i u_i(t) D_{q(t)} X_i[b^{(1)}(t, \Delta u)] \rangle + \sum_i \Delta u_i(t) \langle X_i|_{q(t)}, \varphi(t) \rangle + \langle b^{(1)}(t, \Delta u), \dot{\varphi}(t) \rangle.$$

This should hold for every possible choice of the control Δu . Thus the part linear in Δu should vanish leading to $\langle X_i|_{q(t)}, \varphi(t) \rangle = 0$ for every $i = 1, 2, \dots, k$. Consequently we are left with

$$0 = \langle \sum_i u_i(t) D_{q(t)} X_i[b^{(1)}(t, \Delta u)] \rangle + \langle b^{(1)}(t, \Delta u), \dot{\varphi}(t) \rangle.$$

3. THE MAIN RESULT AND ITS DISCUSSION

In this section we will formulate our main result – Theorem 3.3 – and discuss its consequences. We would like to present our result as an extension of the classical Agrachev-Sarychev theory. However, for the sake of generalization, we prefer to use the language of 2-jets rather than the one of first and second derivatives originally used in [2]. For this reason our notation will be somehow non-standard.

3.1. The Agrachev-Sarychev Index Lemma

Statement of the Index Lemma. Recall that by a *negative index* of a quadratic form we understand the maximal dimension of a subspace on which this form is negatively-defined. We have the following classical result.

Lemma 3.1 (Index Lem. [2]). *Let $q(t)$, with $t \in [t_0, t_1]$, be an abnormal minimizing SR geodesic of co-rank r , corresponding to the control $u \in L^\infty([t_0, t_1], \mathbb{R}^k)$. Then there exists a non-vanishing Pontryagin covector $\varphi(t) \in T_{q(t)}^* M$ such that for each $t \in [t_0, t_1]$ we have*

- (i) $\text{Im } D_u \mathbf{End}_{q_0}^{[t_0, t]} \subset \ker \varphi(t)$, and
- (ii) *the negative index of the quadratic form (the Agrachev-Sarychev index, AS index in short)*

$$\langle D_u^2 \mathbf{End}_{q_0}^{[t_0, t]}[\cdot, \cdot], \varphi(t) \rangle : \ker D_u \mathbf{End}_{q_0}^{[t_0, t]} \times \ker D_u \mathbf{End}_{q_0}^{[t_0, t]} \longrightarrow \mathbb{R}$$

is smaller than r .

Above we identify $\varphi(t) \in T^* M$ with $\varphi(t) + 0 \cdot dr \in T^* M \times T^* \mathbb{R}$, so that the pairing between $\varphi(t)$ and the element $D_u \mathbf{End}_{q_0}^{[t_0, t]}[\Delta u]$ makes sense.

For a concise proof we refer to the recent monograph [21], Appendix B. Let us remark, that although the second derivative $D_u^2 \mathbf{End}_{q_0}^{[t_0, t]}$ has a local sens only, the above map $\langle D_u^2 \mathbf{End}_{q_0}^{[t_0, t]}[\cdot, \cdot], \varphi(t) \rangle$ is well-defined, since our attention is restricted to the kernel $\ker D_u \mathbf{End}_{q_0}^{[t_0, t]}$ – cf. our discussion of pairing of 2-jets preceding formula (2.2).

A simple corollary. We are most interested in the situation when the Agrachev-Sarychev index is actually zero, *i.e.* the respective quadratic map is non-negatively defined. It turns out that this is not so difficult to achieve by the following argument.

As we shall prove later (see Lem. 4.5), function $t \mapsto \text{ind}_- \langle D_u^2 \mathbf{End}_{q_0}^{[t_0, t]}[\cdot, \cdot], \varphi(t) \rangle$, assigning to each $t \in [t_0, t_1]$ the respective AS index, is non-decreasing. On the other hand, by the result of Lemma 3.1 this function is bounded. It follows that there are at most $r - 1$ points of growth on the interval $[t_0, t_1]$. Inbetween these points the index must be zero. Therefore for the price of excluding a finite number of points from the trajectory (or,

to see it differently, for the price of dividing the trajectory into a finite number of pieces), we may actually set the AS index to zero. Now we have:

Lemma 3.2. *Under the assumptions of Lemma 3.1, there exists a non-vanishing Pontryagin covector $\varphi(t) \in \mathbb{T}_{q(t)}^* M$ such that*

$$\langle b^{(1)}(t, \Delta u), \varphi(t) \rangle = 0$$

for every $t \in [t_0, t_1]$ and every $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$.

Further, there exist at most $r + 1$ points $\tau_0 = t_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_r = t_1$ such that for each subinterval $[t'_0, t'_1] \subset (\tau_j, \tau_{j+1})$ the following inequality

$$\langle b^{(2)}(t, \Delta u), \varphi(t) \rangle \geq 0 \tag{3.1}$$

holds for every $\Delta u \in \ker D_u \mathbf{End}^{[t'_0, t'_1]} \subset L^\infty([t'_0, t'_1], \mathbb{R}^k)$ and every $t \in [t'_0, t'_1]$.

3.2. Optimality conditions of degree 2

The main result. Our main result – Theorem 3.3 – may be seen as a twofold generalization of Lemma 3.2. First of all, we generalize inequality (3.1) by adding a quadratic part to it. Thanks to that controls Δu can be arbitrary, not just from the kernel of $D_u \mathbf{End}^{[t_0, t_1]}$ as was previously the case. Since we are no longer restricted in the choice of controls Δu , we will be able to add the monotonicity condition (3.4) related with the discussed inequality. This is the part that was completely absent in the original Agrachev-Sarychev theory. It has, however, important consequences, allowing us to derive new equations for abnormal geodesic in Theorem 3.4.

Theorem 3.3. *Let $q(t)$, with $t \in [t_0, t_1]$, be a strictly abnormal minimizing SR geodesic of co-rank r , corresponding to the control $u \in L^\infty([t_0, t_1], \mathbb{R}^k)$. For $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$ consider the second-order expansion of the extended end-point map in the form*

$$\begin{aligned} \mathbf{End}^{[t_0, t_1]}[u + s \cdot \Delta u] &= \left(\mathbf{End}^{[t_0, t_1]}[u + s \cdot \Delta u], E^{[t_0, t_1]}[u + s \cdot \Delta u] \right) \stackrel{loc.}{=} \\ &\left(q(t), E^{[t_0, t_1]}[u] \right) + s \left(b^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u) \right) + s^2 \left(b^{(2)}(t, \Delta u), c^{(2)}(t, \Delta u) \right) + o(s^2). \end{aligned}$$

Then there exists a non-vanishing Pontryagin covector $\varphi(t) \in \mathbb{T}_{q(t)}^* M$ such that

$$\langle b^{(1)}(t, \Delta u), \varphi(t) \rangle = 0 \tag{3.2}$$

for every $t \in [t_0, t_1]$ and every $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$.

Further there exists a division of the interval $[t_0, t_1]$ by at most $r(n - r - k + 1) + 1$ points $\tau_0 = t_0 < \tau_1 < \dots < \tau_N = t_1$ such that on each sub-interval $[t'_0, t'_1] \subset (\tau_i, \tau_{i+1})$ there exists a family of symmetric bi-linear maps $\Phi_2(t) : \mathbb{T}_{\mathbf{End}^{[t'_0, t'_1]}[u]}(M \times \mathbb{R}) \times \mathbb{T}_{\mathbf{End}^{[t'_0, t'_1]}[u]}(M \times \mathbb{R}) \rightarrow \mathbb{R}$ parameterized by $t \in [t'_0, t'_1]$ satisfying the following conditions:

- $-\Phi_2(t)[(b^{(1)}, c^{(1)}), (b^{(1)}, c^{(1)})]$ equals to the infimum

$$\inf \{ \langle b^{(2)}(t, \Delta u), \varphi(t) \rangle \mid \Delta u \in L^\infty([t'_0, t'_1], \mathbb{R}^k), b^{(1)}(t, \Delta u) = b^{(1)}, c^{(1)}(t, \Delta u) = c^{(1)} \}.$$

We conclude that for every $\Delta u \in L^\infty([t'_0, t'_1], \mathbb{R}^k)$ and $t \in [t'_0, t'_1]$ the following inequality holds

$$\langle b^{(2)}(t, \Delta u), \varphi(t) \rangle + \Phi_2(t)[(b^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u)), (b^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u))] \geq 0; \tag{3.3}$$

- the assignment $t \mapsto \Phi_2(t)$ is a map of bounded variation, hence differentiable almost everywhere,
- moreover, for every $\Delta u \in L^\infty([t'_0, t'_1], \mathbb{R}^k)$ the assignment

$$t \longmapsto \langle b^{(2)}(t, \Delta u), \varphi(t) \rangle + \Phi_2(t)[(b^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u)), (b^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u))] \quad (3.4)$$

is non-decreasing.

The proof is rather long and technical, and will be given in Section 4.

Explanation of the main result. Let us briefly discuss the above result. At the fundamental level it says that there is a relation between optimality of an abnormal curve and the geometry of the second-order expansion of the extended end-point map $\mathbf{End}^{[t_0, t]}$ for each time t . This relation is expressed by (3.2) – an equality involving the first derivatives of this map (the standard first-order optimality condition) – and (3.3) – an inequality involving 2-jets of $s \mapsto \mathbf{End}^{[t_0, t]}[u + s \cdot \Delta u]$. The latter inequality is stated by means of a pairing of a 2-jet and 2-covector $((\phi(t), 0), \Phi_2(t))$ on $M \times \mathbb{R}$ – cf. Remark 2.2. To understand the meaning of these conditions let $\mathbf{f} : M \times \mathbb{R} \rightarrow \mathbb{R}$ be any function realizing this 2-covector, i.e. such that $D\mathbf{f}[\cdot] = \langle \cdot, (\varphi(t), 0) \rangle$ and $D^2\mathbf{f}[\cdot, \cdot] = \Phi_2(t)[\cdot, \cdot]$. Then conditions (3.2)–(3.3) imply that, up to the terms of order higher than 2, the curves $s \mapsto \mathbf{End}^{[t_0, t]}[u + s \cdot \Delta u]$ lie on a one side of a level set of \mathbf{f} – cf. our considerations on page 7. To put it differently: there is a quadratic hypersurface which bounds the 2-jets of curves $s \mapsto \mathbf{End}^{[t_0, t]}[u + s \cdot \Delta u]$ for every possible choice of control Δu . Since $D\mathbf{f} = (\varphi(t), 0)$ is trivial on \mathbb{R} -component, this hypersurface is 1-tangent to the energy direction \mathbb{R} in $M \times \mathbb{R}$. The question whether we have also 2-tangency of these objects will be a basis of a distinction between 2-normal and 2-abnormal curves, which we shall introduce later in this section.

At this point it is worth to explain a presumable contradiction, pointed out by one of the reviewers of this paper. Namely, in the Agrachew-Sarychew Lemma 3.1, since our attention is restricted to controls Δu in the kernel of $D_u \mathbf{End}_{q_0}^{[t_0, t_1]}$, we can state conditions on the second derivative of the end-point map $D_u^2 \mathbf{End}_{q_0}^{[t_0, t_1]}$ – which now becomes the dominant term of the expansion of $\mathbf{End}_{q_0}^{[t_0, t_1]}[u + s \cdot \Delta u]$. On the contrary, assumptions of Theorem 3.3 do not require $D_u \mathbf{End}_{q_0}^{[t_0, t_1]}[\Delta u]$ to vanish, hence the “access” to the second derivative $D_u^2 \mathbf{End}_{q_0}^{[t_0, t_1]}[\Delta u, \Delta u]$ does not seem so obvious. Note, however, that we do not study the end-point map, $\mathbf{End}_{q_0}^{[t_0, t_1]}[u + s \cdot \Delta u]$ directly, but rather through the composition $\mathbf{f} \circ \mathbf{End}_{q_0}^{[t_0, t_1]}[u + s \cdot \Delta u]$. The latter expands as (cf. the last paragraph of Section 2.2)

$$\begin{aligned} \mathbf{f} \circ \mathbf{End}_{q_0}^{[t_0, t_1]}[u + s \cdot \Delta u] &= \mathbf{f} \circ \mathbf{End}_{q_0}^{[t_0, t_1]}[u] + s \cdot D\mathbf{f} \left[D_u \mathbf{End}_{q_0}^{[t_0, t_1]}[\Delta u] \right] \\ &+ s^2 \cdot \left(D\mathbf{f} \left[D_u^2 \mathbf{End}_{q_0}^{[t_0, t_1]}[\Delta u, \Delta u] \right] + D^2\mathbf{f} \left[D_u \mathbf{End}_{q_0}^{[t_0, t_1]}[\Delta u], D_u \mathbf{End}_{q_0}^{[t_0, t_1]}[\Delta u] \right] \right) + o(s^2) \end{aligned}$$

Now $D\mathbf{f}$ is the Pontryagin covector which annihilates $D_u \mathbf{End}_{q_0}^{[t_0, t_1]}[\Delta u]$, making the quadratic part the dominant term in the expansion. Theorem 3.3 gives us precisely information about this quadratic term, for $D\mathbf{f} = (\varphi(t), 0)$ and $D^2\mathbf{f}[\cdot, \cdot] = \Phi_2(t)[\cdot, \cdot]$.

The important part of the assertion is that we have a direct description of what this quadratic map $\Phi_2(t)$ is. Namely, $-\Phi_2(t)[(b^{(1)}, c^{(1)}), (b^{(1)}, c^{(1)})]$ is defined as a solution of the (optimal control) problem of minimizing the value $\langle b^{(2)}(t, \Delta u), \varphi(t) \rangle$ in the class of controls Δu that give fixed values $b^{(1)}(t, \Delta u) = b^{(1)}$ and $c^{(1)}(t, \Delta u) = c^{(1)}$. In other words, we cannot further improve inequality (3.3) – it describes the envelope of the space of all 2-jets of $s \mapsto \mathbf{End}^{[t_0, t]}[u + s \cdot \Delta u]$. This description allows to calculate $\Phi_2(t)$ in concrete examples. Let us stress, that neither the finiteness of the solution of the above optimal control problem, nor the property that it is actually quadratic in $(b^{(1)}, c^{(1)})$ are trivial facts.

There is a caveat to what we said above. Namely the described properties of the expansion of the extended end-point map may not be true on the whole geodesic, but only on its sub-intervals obtained after a removal of a finite number of points. These points play essentially the same role as the points at which the AS index grows (cf.

Lem. 3.2) – their removal guarantees the positive-definiteness of the quadratic maps considered in the discussed results. In principle, it should be possible to reformulate the conditions (3.3) of Theorem 3.3 as a finite-index condition for some global map quadratic in Δu (as it is done in the original Agrachev-Sarychev Lem. 3.1). However, in such a formulation one would not be able to state condition (3.4). As we shall see shortly, the latter condition enables the derivation of the evolution equations, hence is important for the study of regularity of the trajectory. Therefore, we see no satisfactory way of getting rid of the point-removal procedure. On the other hand, isolated point singularities may be treated by the methods of [8].

Equations of an abnormal geodesic. Conditions (3.2)–(3.3) are formulated for every $t \in [t_0, t_1]$. On the other hand, the elements $b^{(i)}(t, \Delta u)$ and $c^{(i)}(t, \Delta u)$, for $i = 1, 2$ encoding the 2-jet of $s \mapsto \mathbf{End}^{[t_0, t]}[u + s \cdot \Delta u]$ satisfy (2.5)–(2.6) and (2.8), and thus can be derived by an integration along $[t_0, t]$. Therefore, we should expect that optimality conditions at time t are somehow *compatible* with the optimality conditions at all preceding times.

For condition (3.2) this “compatibility” is expressed by the fact that $\varphi(t)$ is a Pontryagin covector – we have seen that at the very end of Section 2. By the same token, condition (3.4) is responsible for “compatibility” of conditions (3.3) at different times t . This will become clear in the proof of Theorem 3.3. At this moment we shall see that an argument very similar to the one above allows to derive from (3.4) concrete relations between the objects appearing in the assertion of Theorem 3.3. In particular, we get equations that every abnormal minimizing SR geodesic should satisfy. These were unknown to date.

Theorem 3.4. *Within the assumptions and notation of Theorem 3.3, consider the Pontryagin covector $\varphi(t)$ and a family of symmetric bi-linear maps $\Phi_2(t)$ defined on a segment $[t'_0, t'_1] \ni t$.*

Then we may decompose $\Phi_2(t)$ as

$$\Phi_2(t)[(b^{(1)}, c^{(1)}), (b^{(1)}, c^{(1)})] = \Phi_2(t)[b^{(1)}, b^{(1)}] + \langle b^{(1)}, \xi(t) \rangle c^{(1)} + a_2(t)(c^{(1)})^2,$$

where $\Phi_2(t) : T_{q(t)}M \times T_{q(t)}M \rightarrow \mathbb{R}$ is a symmetric bi-linear map, $\xi(t) : T_{q(t)}M \rightarrow \mathbb{R}$ a linear map, and $a_2(t) \in \mathbb{R}$ a number.

These objects satisfy the following conditions:

- maps $\Phi_2(t)$, $\xi(t)$ and $a_2(t)$ are of bounded variation, hence differentiable almost everywhere.
- for almost every $t \in [t'_0, t'_1]$ the quadratic form

$$\begin{aligned} & \left\langle \sum_{i=1}^k u_i(t) D_{q(t)}^2 X_i[b^{(1)}, b^{(1)}], \varphi(t) \right\rangle - \dot{\Phi}_2(t)[b^{(1)}, b^{(1)}] - 2\Phi_2(t) \left[\sum_{i=1}^k u_i(t) D_{q(t)} X_i[b^{(1)}, b^{(1)}] \right] + \\ & - \left(\langle b^{(1)}, \dot{\xi}(t) \rangle + \left\langle \sum_{i=1}^k u_i(t) D_{q(t)} X_i[b^{(1)}, \xi(t) \right\rangle \right) \cdot c^{(1)} - a_2(t) \cdot (c^{(1)})^2 \end{aligned} \quad (3.5)$$

is non-negative for every $(b^{(1)}, c^{(1)}) = (b^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u))$.

- for almost every $t \in [t'_0, t'_1]$ the following equations are satisfied for every $b^{(1)} = b^{(1)}(t, \Delta u)$ and each $i = 1, 2, \dots, k$

$$\langle X_i|_{q(t)}, \xi(t) \rangle + 2 a_2(t) \cdot u_i(t) = 0 \quad (3.6)$$

$$2 \langle D_{q(t)} X_i[b^{(1)}], \varphi(t) \rangle + 2\Phi_2(t) [X_i|_{q(t)}, b^{(1)}] + \langle b^{(1)}, \xi(t) \rangle \cdot u_i(t) = 0. \quad (3.7)$$

Proof. Since $\Phi_2(t)$ is symmetric bilinear, it is obvious that it can be decomposed into parts $\Phi_2(t)$, $\xi(t)$ and $a_2(t)$ as stated in the assertion. Further these new objects must also be maps of bounded variation, since $\Phi_2(t)$ was such.

Now fix the control Δu and consider the map

$$t \longmapsto F(t) := \langle b^{(2)}(t, \Delta u), \varphi(t) \rangle + \Phi_2(t)[(b^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u)), (b^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u)).$$

It is non-decreasing by the assertion of Theorem 3.3, and differentiable almost-everywhere, as $\Phi_2(t)$ is of bounded variation and all the other objects appearing in the formula are at least absolutely continuous. Thus whenever defined $F(t)$ should have a non-negative derivative. We can calculate this derivative using evolution equations (2.5), (2.6), (2.8) and (2.9) to arrive at

$$F'(t) = (\text{a part linear in } \Delta u(t)) + (\text{a part quadratic in } b^{(1)}(t, \Delta u) \text{ and } c^{(1)}(t, \Delta u)).$$

This should be non-negative for every possible Δu , so the linear part must vanish. But this linear part is just (we skip simple but lengthy calculations)

$$\sum_i \Delta u_i(t) \left\{ \left(2 \langle D_{q(t)} X_i[b^{(1)}(t, \Delta u)], \varphi(t) \rangle + 2\Phi_2(t) \left[X_i|_{q(t)}, b^{(1)}(t, \Delta u) \right] + \langle b^{(1)}(t, \Delta u), \xi(t) \rangle \cdot u_i(t) \right) \right. \\ \left. + c^{(1)}(t, \Delta u) \left(\langle X_i|_{q(t)}, \xi(t) \rangle + 2 a_2(t) \cdot u_i(t) \right) \right\}.$$

It is easy to observe that the above expression is zero for every possible $\Delta u(t)$ if and only if (3.6)–(3.7) hold.

Now we are left with $F'(t) =$ a part quadratic in $b^{(1)}(t, \Delta u)$ and $c^{(1)}(t, \Delta u)$, which should be non-negative. This is precisely (again we skip simple calculations) condition (3.5) \square

3.3. Discussion of the main result

Let us now discuss some particular consequences of Theorems 3.3 and 3.4.

Problem of regularity. Observe that if $\xi(t)$ and/or $a_2(t)$ are non-zero (we shall call $q(t)$ a *2-normal extremal* in such a situation – see the next paragraph), then at least one of equations (3.6)–(3.7) gives us an algebraic equation for the controls $u_i(t)$ (and thus an ODE for the trajectory) expressed by means of the data of Theorem 3.3.

Since all $a_2(t)$, $\xi(t)$, and $\Phi_2(t)$ are maps of bounded variation, and $X_i|_{q(t)}$ shares the same regularity as $q(t)$, we can only conclude that in such a situation $u_i(t)$'s are functions of bounded variation. **If, however, the time-regularity of quadratic map $\Phi_2(t)$ could be improved, we could also improve the regularity of the controls, and thus the SR geodesic.** Recall that maps $\Phi_2(t)$ are defined as solutions of a certain optimization problem (to be more precise, Bellmann value function – we shall discuss that in detail in Sect. 4.1). Therefore the question whether 2-normal extremals are smooth boils down to another regularity question in optimal control theory. I was personally unable to get any general results in this direction (yet some more specific ones are collected in Sect. 5), but perhaps someone more experienced in optimal control theory could solve this problem.

2-normality and 2-abnormality. Observe that if $a_2(t) \neq 0$, equation (3.6) provides information about the controls $u_i(t)$. Basing on this fact we propose the following definition:

Definition 3.5. Let $q(t)$, with $t \in [t_0, t_1]$ be a strictly abnormal sub-Riemannian trajectory such that the pair $(\varphi(t), \Phi_2(t))$ described by Theorem 3.3 exists for every $t \in (t_0, t_1]$. If for every $t \in (t_0, t_1]$ we have $\Phi_2(t)[(0, 1), (0, 1)] = a_2(t) \neq 0$ we call $q(t)$ a *2-normal extremal*. Otherwise we say that $q(t)$ is a *2-abnormal extremal*.

By the results of Theorem 3.3 every strictly abnormal SR trajectory is either non-minimizing, or can be divided into a finite number of pieces, each of them being either 2-normal or 2-abnormal extremal.

The division of abnormal trajectories into the classes of 2-normal and 2-abnormal extremals proposed above shares some natural similarities with an analogous distinction between normal and abnormal trajectories in degree one. Namely, 2-normal extremals are characterized by a property that the second-order approximation of their extended end-point map can be strictly separated from the direction of the decreasing cost (see figure on page 4). Further they satisfy an ODE (although we were not able to prove their regularity from this property). On the other hand, for 2-abnormal extremals this separation may not be strict, and equation (3.6) gives us no information about the control $u(t)$ at the points where $a_2(t) = 0$.

Note that if $a_2(t) \equiv 0$, equations (3.8) and (3.6) are the same as equations for an abnormal extremal given at page ???. Thus we have two abnormal Pontryagin covectors $\varphi(t)$ and $\xi(t)$ in that case (although the latter may be trivial).

Finally, note that the classes of 2-normal and 2-abnormal extremals are, in general, not disjoint as an abnormal trajectory of co-rank greater than one may admit several distinct Pontryagin covectors.

Derivation of Goh conditions. Equations (3.6)–(3.7) allow for an easy derivation of *Goh conditions*. Put $b^{(1)} = X_j$ in (3.7) to get

$$2\langle D X_i[X_j], \varphi \rangle \stackrel{(3.7)}{=} -2\Phi_2[X_i, X_j] - \langle X_j, \xi \rangle u_i(t) \stackrel{(3.6)}{=} -2\Phi_2[X_i, X_j] + a_2 \cdot u_j(t) u_i(t).$$

The right-hand side of the above expression is symmetric in i and j , hence

$$\langle [X_i, X_j], \varphi \rangle = \langle D X_i[X_j], \varphi \rangle - \langle D X_j[X_i], \varphi \rangle = 0.$$

The Goh condition may be seen as a vanishing of a non-symmetric part of the derivatives (3.6)–(3.7) due to the symmetric nature of the quadratic form $\Phi_2(t)$.

Other observations. If $a_2(t) \equiv 0$ (or more generally if $\dot{a}_2(t) = 0$), then the bi-linear form (3.5) contains a bare term linear in $c^{(1)}$. This term should vanish for every possible choice of $b^{(1)}$, since otherwise the form would not be positively-defined. This reasoning gives the evolution equation for $\xi(t)$:

$$\langle b^{(1)}, \dot{\xi}(t) \rangle + \left\langle \sum_{i=1}^k u_i(t) \cdot D_{q(t)} X_i[b^{(1)}], \xi(t) \right\rangle = 0 \quad \text{for every } b^{(1)} \in \text{Im} \left(D_u \text{End}_{q_0}^{[t_0, t]} \right), \quad (3.8)$$

and so, $\xi(t)$ is a (possibly trivial) *Pontryagin covector*.

The question whether the bi-linear form (3.5) is strictly positive or has some null-directions is closely related to the question whether the infimum value of the quadratic map $\Phi_2(t)$ is actually achieved by some control. We will discuss it in more detail in Section 5. At this point note that if the bi-linear form (3.5) is constantly zero, and if the objects $\Phi_2(t)$, $\xi(t)$ and $a_2(t)$ are absolutely continuous w.r.t. t then, in addition to the above equation (3.8), we get $\dot{a}_2(t) \equiv 0$ and, at the end of the day, also evolution equation for $\Phi_2(2)$:

$$\left\langle \sum_{i=1}^k u_i(t) D_{q(t)}^2 X_i[b^{(1)}, b^{(1)}], \varphi \right\rangle - \dot{\Phi}_2[b^{(1)}, b^{(1)}] - 2\Phi_2 \left[\sum_{i=1}^k u_i(t) D_{q(t)} X_i[b^{(1)}], b^{(1)} \right] = 0 \quad (3.9)$$

which should hold for every $b^{(1)} \in \text{Im} \left(D_u \text{End}_{q_0}^{[t_0, t]} \right)$. Informally speaking, these equations mean that the pair $(\varphi(t), \Phi_2(t))$ has a natural evolution compatible with the flow of the control vector field $q \mapsto \sum_i u_i(t) X_i(q)$. For more details the Reader is directed to [11].

3.4. Examples

Example – Martinet system. Consider $M = \mathbb{R}^3 \ni (x, y, z)$ and a distribution \mathcal{D} spanned by orthonormal vector fields $X_1 = \partial_x$, and $X_2 = (1-x)\partial_y + \frac{x^2}{2}\partial_z$. Let $q(t) = (0, t, 0)$ be a trajectory corresponding to the control $(u_1(t) \equiv 0, u_2(t) \equiv 1)$. It is well-known that $q(t)$ is an abnormal minimizing geodesic, and that $\phi(t) = dz$ is the related adjoint curve [17]. We want to see how Theorem 3.3 works in this case.

First we would like to derive equations (2.5) and (2.6) for the considered situation. Let us decompose

$$b^{(1)}(t) = b_x^{(1)}(t) \cdot \partial_x + b_y^{(1)}(t) \cdot \partial_y + b_z^{(1)}(t) \cdot \partial_z \quad \text{and} \quad b^{(2)}(t) = b_x^{(2)}(t) \cdot \partial_x + b_y^{(2)}(t) \cdot \partial_y + b_z^{(2)}(t) \cdot \partial_z.$$

Now note that

$$\sum_{i=1}^2 u_i(t) D_{q(t)} X_i [b^{(1)}] = -dx[b^{(1)}] \cdot \partial_y \quad \text{and} \quad \sum_{i=1}^2 u_i(t) D_{q(t)}^2 X_i [b^{(1)}, b^{(1)}] = dx \otimes dx [b^{(1)}, b^{(1)}] \cdot \partial_z.$$

From the above observation equation (2.5) reads as

$$b_x^{(1)}(t) = \Delta u_1(t), \quad b_y^{(1)}(t) = -b_x^{(1)}(t) + \Delta u_2(t), \quad b_z^{(1)}(t) = 0, \quad \text{and} \quad c^{(1)}(t) = \Delta u_2(t).$$

In particular,

$$b_y^{(1)}(t) = - \int_0^t b_x^{(1)}(\tau) d\tau + \int_0^t \Delta u_2(\tau) d\tau = - \int_0^t b_x^{(1)}(\tau) d\tau + c^{(1)}(t). \quad (3.10)$$

Similarly, (2.6) gives us

$$b_x^{(2)}(t) = 0, \quad b_y^{(2)}(t) = -b_x^{(2)}(t) - \Delta u_2(t) b_x^{(1)}(t), \quad b_z^{(2)}(t) = b_x^{(1)}(t)^2, \quad \text{and} \quad c^{(2)}(t) = \frac{1}{2} (\Delta u_1(t)^2 + \Delta u_2(t)^2).$$

In consequence, by the Cauchy-Schwarz inequality we can estimate

$$\langle b^{(2)}(t), \phi(t) \rangle = b_z^{(2)}(t) = \int_0^t b_x^{(1)}(\tau)^2 d\tau \stackrel{\text{C-S}}{\geq} \frac{1}{t} \cdot \left(\int_0^t b_x^{(1)}(\tau) d\tau \right)^2 = \frac{1}{t} \cdot [c^{(1)}(t) - b_y^{(1)}(t)]^2.$$

It is easy to see that the above inequality is always strict, yet on the other hand, it can be satisfied with an arbitrary accuracy. It follows that every minimal 2-jet is finite, but the minimum is never attained. We conclude that $-\Phi_2(t)[(b_1, c_1), (b_1, c_1)] = \frac{1}{t} \cdot [c^{(1)}(t) - b_y^{(1)}(t)]^2$ and, consequently,

$$\Phi_2(t) = \frac{1}{t} dy \otimes dy, \quad \xi(t) = -\frac{2}{t} dy \quad \text{and} \quad a_2(t) = \frac{1}{t}.$$

Since $a_2(t) \neq 0$ the considered trajectory is a *2-normal extremal* in the sense of Definition 3.5. We leave to the Reader checking that conditions (3.6)–(3.7) hold for the above data. Further, condition (3.5) reads as

$$\left(b_x^{(1)}(t) + \frac{1}{t} b_y^{(1)}(t) \right)^2 - \left(\frac{2}{t^2} b_y^{(1)}(t) + \frac{2}{t} b_x^{(1)}(t) \right) c^{(1)}(t) + \frac{1}{t^2} c^{(1)}(t)^2 = \left(b_x^{(1)}(t) + \frac{1}{t} b_y^{(1)}(t) - \frac{1}{t} c^{(1)}(t) \right)^2 \geq 0$$

and of course holds.

Example – a corner on a free Carnot group. The example below was studied by Hakavouri and Le Donne as communicated to us by [13]. It turns out to be a 2-abnormal extremal.

Consider a free Carnot group G of rank 2 and step 4. That is, the Lie algebra of left-invariant vector fields on G is generated by two fields X_1, X_2 , whose Lie brackets of order higher than 4 vanish, and Lie brackets of order less or equal 4 have no relations apart from the obvious ones arising either from the skew-symmetry, or from the Jacobi identity. It follows that all left-invariant vector fields on G are spanned (over \mathbb{R}) by fields: $X_1, X_2, Y := [X_1, X_2], Z_1 := [X_1, Y] = [X_1, [X_1, X_2]], Z_2 := [X_2, -Y] = [X_2, [X_2, X_1]], W_1 := [X_1, Z_1] = [X_1, [X_1, [X_1, X_2]]], W_2 := [X_2, Z_2] = [X_2, [X_2, [X_2, X_1]]],$ and $W_3 := [X_2, Z_1] = [X_1, -Z_2] = [X_1, [X_2, [X_1, X_2]]].$ Let us consider a left-invariant SR geodesic problem on G , constituted by setting $\mathcal{D}_q = \text{span}_{\mathbb{R}}\{X_1, X_2\}|_q$ and the assumption that the fields X_1 and X_2 are orthonormal.

In such a setting let $q : [-1, 1] \rightarrow G$ be the *corner* controlled by $u(t) = (\chi_{[-1,0]}(t), \chi_{[0,1]}(t))$, *i.e.*

$$q(t) = \begin{cases} \exp(t \cdot X_1) & \text{for } t \in [-1, 0] \\ \exp(t \cdot X_2) & \text{for } t \in [0, 1] \end{cases}.$$

It turns out that $q(t)$ is a strictly abnormal SR extremal, with

$$\text{Im } D_u \mathbf{End}^{[-1,1]} = \text{span}_{\mathbb{R}}\{X_1, X_2, Y, Z_1, Z_2, W_1, W_2\}|_{q(1)} \oplus \mathbb{R}.$$

In consequence, there is only one (up to rescaling) candidate for the Pontryagin covector $\varphi(t) \in \mathbb{T}_{q(t)}^* G$ uniquely defined by conditions

$$\langle W_3|_{q(t)}, \varphi(t) \rangle = 1, \text{ and } \langle \cdot, \varphi(t) \rangle = 0 \text{ on the space } \text{span}_{\mathbb{R}}\{X_1, X_2, Y, Z_1, Z_2, W_1, W_2\}|_{q(t)} \subset \mathbb{T}_{q(t)} G.$$

In this particular situation we were able to calculate $\langle \mathbf{b}^{(2)}(1, \Delta u), \varphi(1) \rangle$ for every control $\Delta u = (\Delta u_1, \Delta u_2) \in L^\infty([-1, 1], \mathbb{R}^2)$. We skip the details of this computation, as it is rather long and would require introduction of some additional theory regarding invariant SR structures on Lie groups. To state our result we need to introduce the function $f : G \rightarrow \mathbb{R}$ defined by

$$f(\exp(w_3 W_3) \exp(w_2 W_2) \exp(w_1 W_1) \exp(z_2 Z_2) \exp(z_1 Z_1) \exp(y Y) \exp(x_2 X_2) \exp(x_1 X_1) q(1)) := w_3$$

(obviously $D_{q(1)} f = \varphi(1)$). Let also $\rho \in \mathbb{T}_{q(1)}^* G$ be a covector defined by conditions

$$\langle X_1|_{q(1)}, \rho \rangle = 1, \text{ and } \langle \cdot, \rho \rangle = 0 \text{ on the space } \text{span}_{\mathbb{R}}\{X_2, Y, Z_1, Z_2, W_1, W_2, W_3\}|_{q(1)},$$

i.e. $\langle b^{(1)}(1, \Delta u), \rho \rangle$ is the X_1 -component of the vector $b^{(1)}(1, \Delta u) \in \mathbb{T}_{q(1)} G$.

It turns out that for every $\Delta u \in L^\infty([-1, 1], \mathbb{R}^2)$ the pair $(b^{(1)}(1, \Delta u), b^{(2)}(1, \Delta u))$ is subject to the following equality

$$\begin{aligned} & \langle \mathbf{b}^{(2)}(1, \Delta u), \varphi(1) \rangle + D_{q(1)}^2 f[b^{(1)}(1, \Delta u), b^{(1)}(1, \Delta u)] + \frac{1}{2} \langle b^{(1)}(1, \Delta u), \rho \rangle^2 \\ & = \int_0^1 t \left(\int_{-1}^t \Delta u_1(s) ds \right)^2 dt - \int_{-1}^0 t \left(\int_{-1}^t \Delta u_2(s) ds \right)^2 dt. \end{aligned}$$

Clearly the right-hand side is positively defined. We conclude that in the considered case inequality (3.3) holds with $\Phi_2(1)[(b^{(1)}, c^{(1)}), (b^{(1)}, c^{(1)})] = -D_{q(1)}^2 f[b^{(1)}, b^{(1)}] - \frac{1}{2} \langle b^{(1)}, \rho \rangle^2$. As the dependence on $c^{(1)}$ is trivial, $a_2(1) = 0$, and hence we deal with a *2-abnormal extremal* in the sense of Definition 3.5. It is worth to mention that, by the results of [8], a corner curve cannot be a SR geodesic.

4. PROOF OF THEOREM 3.3

4.1. The characteristic optimal control problem

Let us briefly sketch the content of this section. We would like to see Theorem 3.3 as an improvement on Lemma 3.2. If so, then the new part is the presence of the quadratic map $\Phi_2(t)$ related with minimizing the value of $\langle b^{(2)}(t, \Delta u), \varphi(t) \rangle$ while fixing the values of $b^{(1)}(t, \Delta u)$ and $c^{(1)}(t, \Delta u)$.

We shall formalize this minimization procedure by considering the value function $\widehat{Q}_\tau(\cdot)$ of a certain optimal control problem (the *characteristic control problem* of the SR trajectory). Giving a precise definition of this problem, as well as of an underlying *characteristic control system*, will conclude Section 4.1. It is worth to mention, that for technical reasons we will prefer to study curves $(b^{(1)}(t, \Delta u), b^{(2)}(t, \Delta u))$ in a special coordinate system – *adapted coordinates* – introduced in [11]. After introducing the above notions we will sketch the strategy of the proof at the beginning of the next Section 4.2.

Technical preparations (adapted coordinates). In the proof of Theorem 3.3 we are, in particular, interested in the geometry of the set of pairs $(b^{(1)}(t, \Delta u), b^{(2)}(t, \Delta u))$, which represent the 2-jets of $s \mapsto \text{End}^{[t_0, t_1]}[u + s \cdot \Delta u]$ at $s = 0$. These objects are described by the system of ODEs (2.5)–(2.6) which, for us, would be convenient to treat as a control system, steered by $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$. There are, however, two issues with such an approach. First of all, equations (2.5)–(2.6) are quite complicated, depending not only linearly on Δu , but also on the basic control u by formulas that involve derivatives of vector fields X_i . Secondly, for each t the pair $(b^{(1)}(t, \Delta u), b^{(2)}(t, \Delta u))$ belongs to a different space – $\mathbb{T}_{q(t)}^2 M$. It turns out that both complications can be solved simultaneously by a suitable change of coordinates. It will be given by a family of maps $\Psi^{(2)}(t) : \mathbb{T}_{q(t)}^2 M \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ constructed as follows [11].

Let $u \in L^\infty([t_0, t], \mathbb{R}^k)$ be a control, $q(t)$ the related trajectory of (Σ) , and choose a basis $(\psi_a)_{a=1,2,\dots,n}$ of the cotangent space $\mathbb{T}_{q_0}^* M$. First define a 1-parameter family of linear maps $\Phi^{(1)}(t) : \mathbb{T}_{q(t)} M \rightarrow \mathbb{R}^n$ by setting

$$\begin{aligned} \dot{\Phi}^{(1)}(t)[b] + \sum_i u_i(t) \cdot \Phi^{(1)}(t) [D_{q(t)} X_i[b]] &= 0 \quad \text{for every } b \in \mathbb{T}_{q(t)} M \\ \Phi^{(1)}(t_0) &= (\psi_1, \dots, \psi_n), \end{aligned} \tag{4.1}$$

and a 1-parameter family of symmetric bi-linear maps $\Phi^{(2)}(t) : \mathbb{T}_{q(t)} M \times \mathbb{T}_{q(t)} M \rightarrow \mathbb{R}^n$ by setting

$$\begin{aligned} \dot{\Phi}^{(2)}(t)[b, b] + \sum_i u_i(t) \cdot \left\{ \Phi^{(1)}(t) [D_{q(t)}^2 X_i[b, b]] + 2\Phi^{(2)}(t) [D_{q(t)} X_i[b], b] \right\} &= 0 \quad \text{for every } b \in \mathbb{T}_{q(t)} M \\ \Phi^{(2)}(t_0) &= (0, \dots, 0). \end{aligned}$$

Now the map $\Psi^{(2)}(t)$ is defined as follows:

$$\Psi^{(2)}(t)(b^{(1)}, b^{(2)}) = \left(\Phi^{(1)}(t) [b^{(1)}], \Phi^{(1)}(t) [b^{(2)}] + \frac{1}{2!} \Phi^{(2)}(t) [b^{(1)}, b^{(1)}] \right),$$

where $(b^{(1)}, b^{(2)}) \in \mathbb{T}_{q(t)}^2 M$. After [11] we call $\Psi^{(2)}(t)$ the *transformation of adapted coordinates*. The name “adapted” means “adapted to the geometry of the evolution equation (Σ) ” in the sense that $\Psi^{(2)}(t)$ acts on 2-jets in the same way as the flow of the control vector field $X_u(t, q) = \sum_i u_i(t) X_i(q)$. In particular, this manifests itself by the fact that passing to adapted coordinates simplifies the form of equations (2.5)–(2.6).

Lemma 4.1 ([11]). *For every $t \in [t_0, t_1]$ the map $\Phi^{(1)}(t)$ is an isomorphism between the fibre $\mathbb{T}_{q(t)} M$ and \mathbb{R}^n . By the same token, for every $t \in [t_0, t_1]$ the map $\Psi^{(2)}(t)$ is an isomorphism between the fibre $\mathbb{T}_{q(t)}^2 M$ and $\mathbb{R}^n \times \mathbb{R}^n$.*

Given $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$ set

$$\left(q^{(1)}(t, \Delta u), q^{(2)}(t, \Delta u) \right) := \Psi^{(2)}(t) \left(b^{(1)}(t, \Delta u), b^{(2)}(t, \Delta u) \right). \quad (4.2)$$

Then the curve $(q^{(1)}(t, \Delta u), q^{(2)}(t, \Delta u))$ is subject to the following evolution equations

$$\dot{q}^{(1)}(t, \Delta u) = \sum_i \Delta u_i(t) \cdot Y_i^{(1)}(t) \quad (4.3)$$

$$\dot{q}^{(2)}(t, \Delta u) = \sum_i \Delta u_i(t) \cdot Y_i^{(2)}(t, q^{(1)}(t, \Delta u)), \quad (4.4)$$

where, after denoting the inverse of $\Phi^{(1)}(t)$ by $A(t)$, we have

$$Y_i^{(1)}(t) := \Phi^{(1)}(t) \left[X_i|_{q(t)} \right] \quad \text{and} \quad Y_i^{(2)}(t, q^{(1)}) := \Phi^{(1)}(t) \left[D_{q(t)} X_i[A(t)q^{(1)}] \right] + \Phi^{(2)}(t) \left[X_i, A(t)q^{(1)} \right].$$

Finally, if the control u is of class C^r , with $r = 0_-, 1_-, 1, 2, 3, \dots$, then $\Phi^{(1)}(t)$, the inverse of $\Phi^{(1)}(t)$, and $\Phi^{(2)}(t)$ are of class C^{r+1} with respect to t .

The above result can be checked by a direct calculation. The C^{r+1} -regularity follows easily from the evolution equations for $\Phi^{(1)}(t)$ and $\Phi^{(2)}(t)$. For details and discussion, we refer to the original publication [11].

The characteristic control system and its reachable sets. Since maps $\Phi^{(1)}(t)$ constructed in Lemma 4.1 are reversible, a pair $(q^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u))$ contains the same information as a pair $(b^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u))$, describing the first term of the 2-jet expansion of the extended end-point map $\mathbf{End}_{q_0}^{[t_0, t]}$. Thanks to the simple form of equation (4.3) the evolution of the former pair is, however, much easier to analyse. Therefore, to understand the geometry of $\mathbf{End}_{q_0}^{[t_0, t]}$, we shall now study the pairs $(q^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u)) \in \mathbb{R}^n \times \mathbb{R}$. In accordance with Remark 2.1 denote $\mathbf{q}^{(1)}(t, \Delta u) := (q^{(1)}(t, \Delta u), c^{(1)}(t, \Delta u))$ and $\mathbf{0} := (0_n, 0) \in \mathbb{R}^n \times \mathbb{R}$.

Definition 4.2. By a *characteristic control system* of the trajectory $q(t)$, with $t \in [t_0, t_1]$, we will understand the following control system in $\mathbb{R}^{n+1} \ni \mathbf{q}^{(1)} = (q^{(1)}, c^{(1)})$:

$$\begin{aligned} \dot{\mathbf{q}}^{(1)}(t) &= \sum_i \Delta u_i(t) \cdot Y_i^{(1)}(t) \\ c^{(1)}(t) &= \sum_{i=1}^k \Delta u_i(t) \cdot u_i(t) \end{aligned} \quad (\Lambda_1)$$

Here $Y_i^{(1)}(t) = \Phi^{(1)}(t) \left[X_i|_{q(t)} \right]$, and we treat $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$ as a control.

By $\mathcal{R}_{(\Lambda_1)}(t_0, \tau)$ we shall denote *time- τ reachable set of the system (Λ_1)* for the initial conditions $\mathbf{q}^{(1)}(t_0) = \mathbf{0}$. That is

$$\mathcal{R}_{(\Lambda_1)}(t_0, \tau) := \{ \mathbf{q}^{(1)}(\tau) \mid \mathbf{q}^{(1)}(t) \text{ satisfy } (\Lambda_1) \text{ for some } \Delta u \in L^\infty([t_0, \tau], \mathbb{R}^k) \text{ with } \mathbf{q}^{(1)}(t_0) = \mathbf{0} \}$$

We will also need another notion related with the characteristic control system. Choose $\tau \in [t_0, t_1]$. We will say that a vector $\mathbf{v} \in \mathbb{R}^{n+1}$ is *controllable to zero from time τ* if there exists a trajectory $\mathbf{q}^{(1)}(t)$ of the control system (Λ_1) with the initial condition $\mathbf{q}^{(1)}(\tau) = \mathbf{v}$ such that $\mathbf{q}^{(1)}(t_1) = \mathbf{0}$.

Pontryagin covector in adapted coordinates. Observe that equation (4.1) defining the map $\Phi^{(1)}(t)$ looks exactly the same, as the evolution equation of the Pontryagin covector (2.9). Therefore it is easy to translate conditions of being a Pontryagin covector to the setting of adapted coordinates.

Proposition 4.3. *A curve $\varphi(t) \in \mathbb{T}_{q(t)}M$ is a Pontryagin covector (as defined by conditions (2.9)) if and only if there exists a covector $\psi_0 \in (\mathbb{R}^n)^*$ such that*

$$\begin{aligned}\varphi(t) &= \Phi^{(1)}(t)^* \psi_0 \quad \text{and} \\ \langle Y_i^{(1)}(t), \psi_0 \rangle &= 0 \quad \text{for every } i = 1, 2, \dots, k \text{ and } t \in [t_0, t_1].\end{aligned}$$

Further if $\varphi(t)$ is as above, and if the pairs $(b^{(1)}(t, \Delta u), b^{(2)}(t, \Delta u))$ and $(q^{(1)}(t, \Delta u), q^{(2)}(t, \Delta u))$ are related by (4.2), we have

$$\begin{aligned}\langle b^{(1)}(t, \Delta u), \varphi(t) \rangle &= \langle q^{(1)}(t, \Delta u), \psi_0 \rangle \quad \text{and} \\ \langle b^{(2)}(t, \Delta u), \varphi(t) \rangle &= \langle q^{(2)}(t, \Delta u), \psi_0 \rangle + \frac{1}{2} \langle \Phi^{(2)}(t)[b^{(1)}(t, \Delta u), b^{(1)}(t, \Delta u)], \Phi^{(1)}(t)^* \psi_0 \rangle \\ &= \langle q^{(2)}(t, \Delta u), \psi_0 \rangle + \frac{1}{2} \langle A(t)\Phi^{(2)}(t)[A(t)q^{(1)}(t, \Delta u), A(t)q^{(1)}(t, \Delta u)], \psi_0 \rangle,\end{aligned}$$

where $A(t)$ denotes the inverse of $\Phi^{(1)}(t)$.

Recall that in Lemmas 3.1 and 3.2 we study a pairing $\langle b^{(2)}(t, \Delta u), \varphi(t) \rangle$ for a Pontryagin covector $\varphi(t)$.

As we see, when restricting our attention to $\Delta u \in \ker \mathbf{End}^{[t_0, t]}$ (as is the case in the discussed Lemmas) the pairing $\langle b^{(2)}(t, \Delta u), \varphi(t) \rangle$ equals to $\langle q^{(2)}(t, \Delta u), \psi_0 \rangle$. Without such a restriction both pairings are equal up to a term quadratic in $q^{(1)}(t, \Delta u)$.

The characteristic optimal control problem. Motivated by Lemmas 3.1 and 3.2 and our observations from the previous paragraph we define the following optimization problem for trajectories of the characteristic control system (Λ_1) .

Definition 4.4. Let $q(t)$, with $t \in [t_0, t_1]$, be a SR trajectory corresponding to the control $u \in L^\infty([t_0, t_1], \mathbb{R}^k)$. Choose $\psi_0 \in (\mathbb{R}^n)^*$, $\tau \in [t_0, t_1]$ and $\mathbf{v} \in \mathcal{R}_{(\Lambda_1)}(t_0, \tau)$. By a *characteristic optimal control problem* (characteristic OCP, in short) of the trajectory $q(t)$ we understand a problem of minimizing the functional

$$Q_{t_0, \tau}[\Delta u] := \langle q^{(2)}(\tau, \Delta u), \psi_0 \rangle$$

in the class of controls $\Delta u \in L^\infty([t_0, \tau], \mathbb{R}^k)$ such that $\mathbf{q}^{(1)}(t_0, \Delta u) = \mathbf{0}$ and $\mathbf{q}^{(1)}(\tau, \Delta u) = \mathbf{v}$. (In other words, Δu steers (Λ_1) from $\mathbf{0}$ to \mathbf{v} on $[t_0, \tau]$.) Note that by (4.4)

$$Q_{t_0, \tau}[\Delta u] = \int_{t_0}^{\tau} \sum_i \Delta u_i(t) \cdot \langle Y_i^{(2)}(t, q^{(1)}(t, \Delta u)), \psi_0 \rangle dt,$$

hence we are actually solving a Lorange-type optimal control problem.

Putting for the moment aside the question of the existence of solutions aside, define *the minimal 2-jet* to be the Bellman value function related with the characteristic OCP.

$$\widehat{Q}_{t_0, \tau}(\mathbf{v}) := \inf\{Q_{t_0, \tau}[\Delta u] \mid \Delta u \in L^\infty([t_0, \tau], \mathbb{R}^k) \text{ is a control steering } (\Lambda_1) \text{ from } \mathbf{0} \text{ to } \mathbf{v} \text{ on } [t_0, \tau]\}.$$

We admit a possibility that $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ equals to $-\infty$.

Finally, note that the cost functional $Q_{t_0, \tau}[\cdot]$ is quadratic, in the sense that $Q_{t_0, \tau}[\Delta u] = B_{t_0, \tau}[\Delta u, \Delta u]$, where $B_{t_0, \tau}[\cdot, \cdot] : L^\infty([t_0, t_1], \mathbb{R}^k) \times L^\infty([t_0, t_1], \mathbb{R}^k) \rightarrow \mathbb{R}$ is a symmetric bilinear map defined by

$$2 B_{t_0, \tau}[\Delta u, \Delta v] := \int_{t_0}^{\tau} \sum_i \langle \Delta u_i(t) \cdot Y_i^{(2)}(t, q^{(1)}(t, \Delta v)) + \Delta v_i(t) \cdot Y_i^{(2)}(t, q^{(1)}(t, \Delta u)), \psi_0 \rangle dt.$$

4.2. Properties of the minimal 2-jet

Strategy of the proof. In the remaining part of this section, we are going to investigate properties of the minimal 2-jet $\widehat{Q}_{t_0, \tau}(\cdot)$ under the assumption that $q(t)$ is an abnormal minimizing SR geodesic. A crucial part in the proof will be showing Lemma 4.12 stating that $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ is finite for all possible τ 's and \mathbf{v} 's, provided that the underlying SR trajectory is suitably divided into pieces. This will be done in several steps as listed below:

- First, we will prove Lemma 4.5 and the related Corollary 4.6 stating that for an abnormal minimizing SR geodesic the value function $\widehat{Q}_{t_0, \tau}(\mathbf{0})$ is finite after a suitable division of the trajectory. This is basically a reformulation of Lemma 3.2.
- Secondly, in Lemma 4.8 and Corollary 4.9, we show that if an element \mathbf{v} is controllable to zero from time τ then $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ is finite
- Lastly, in Lemma 4.11 we show that, after a suitable division of the trajectory, every element \mathbf{v} of the reachable set of (Λ_1) is controllable to zero.

With all the above points to conclude the proof it is enough to prove Theorem 4.13 which states that if $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ is finite for every possible \mathbf{v} , then it is quadratic in \mathbf{v} .

Step 1 – setting the AS index to zero. To simplify the notation let us denote the kernel of the first derivative of the extended end-point map $\mathbf{End}^{[t_0, t]}$ at u by

$$\mathcal{U}_0(t_0, t) := \ker D_u \mathbf{End}^{[t_0, t]} = \{\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k) \mid b^{(1)}(t, \Delta u) = 0, c^{(1)}(t, \Delta u) = 0\}.$$

Recall that in Lemma 3.1 we study quadratic maps

$$\mathcal{Q}_{[t_0, t]} : \mathcal{U}_0(t_0, t) \longrightarrow \mathbb{R} \quad \mathcal{Q}_{[t_0, t]} : \Delta u \longmapsto \langle b^{(2)}(t, \Delta u), \varphi(t) \rangle \stackrel{\text{Proposition 4.3}}{=} \langle q^{(2)}(t, \Delta u), \psi_0 \rangle,$$

where $\varphi(t)$ is a certain Pontryagin covector. In the last equality we use the notation from Proposition 4.3. Given a covector ψ_0 as above we may repeat the above construction on subintervals $[t'_0, t'_1] \subset [t_0, t_1]$ to produce analogous objects: $\mathcal{U}_0(t'_0, t'_1)$ and $\mathcal{Q}_{[t'_0, t'_1]}$, respectively. Actually, the extension of an L^∞ -map $\Delta u : [t'_0, t'_1] \rightarrow \mathbb{R}^k$ by zero on $[t_0, t'_0] \cup (t'_1, t_1]$ gives us a canonical inclusion $L^\infty([t'_0, t'_1], \mathbb{R}^k) \subset L^\infty([t_0, t_1], \mathbb{R}^k)$, which allows to interpret each $\mathcal{U}_0(t'_0, t'_1)$ as a subset of $\mathcal{U}_0(t_0, t_1)$ and, consequently, $\mathcal{Q}_{[t'_0, t'_1]}$ as a restriction of $\mathcal{Q}_{[t_0, t_1]}$ to $\mathcal{U}_0(t'_0, t'_1)$. With this simple observation the below result is now easy to prove. It is an obvious reformulation of Lemma 3.2.

Lemma 4.5. *Under the assumptions of Lemma 3.1, there exist a non-zero covector $\psi_0 \in (\mathbb{R}^n)^*$, and at most $r + 1$ points $\tau_0 = t_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_{r'} = t_1$ such that for each subinterval $[t'_0, t'_1] \subset (\tau_j, \tau_{j+1}]$, the quadratic map*

$$\mathcal{Q}_{[t'_0, t'_1]} : \mathcal{U}_0(t'_0, t'_1) \longrightarrow \mathbb{R}$$

is non-negatively defined.

Proof. We take a Pontryagin covector $\varphi(t)$ as in the assertion of Lemma 3.1 and let ψ_0 be the corresponding element of $(\mathbb{R}^n)^*$ given by Proposition 4.3. Using this element we define the map $\mathcal{Q}_{[t_0, t]}$ as above.

Since for $t < t'$ we have a canonical inclusion $\mathcal{U}_0(t_0, t) \subset \mathcal{U}_0(t_0, t')$, the integer-valued function

$$I : [t_0, t_1] \ni t \longmapsto \text{the negative index of } \mathcal{Q}_{[t_0, t_1]} \text{ on } \mathcal{U}_0(t_0, t)$$

is non-decreasing. Further, as being negatively defined on a subspace is an open property, I is left semi-continuous. By Lemma 3.1 the map I has at most $r - 1$ points of growth. Together with t_0 and t_1 this gives us at most $r + 1$ points $\tau_0 = t_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_{r'} = t_1$, with the property that I is constant on $(\tau_j, \tau_{j+1}]$. Take now any $[t'_0, t'_1] \subset (\tau_j, \tau_{j+1})$ and assume that $\mathcal{Q}_{[t_0, t_1]}$ is not non-negatively defined on $\mathcal{U}_0(t'_0, t'_1)$. For $\tau_j + \varepsilon < t'_0$ the sets $\mathcal{U}_0(t_0, \tau_j + \varepsilon)$ and $\mathcal{U}_0(t'_0, t'_1)$ are $\mathcal{Q}_{[t_0, t_1]}$ -orthogonal (consisting of controls with disjoint supports, for which $q^{(1)}$ vanishes at the end-points), it follows that the negative index of $\mathcal{Q}_{[t_0, t_1]}$ on $\mathcal{U}_0(t_0, \tau_j + \varepsilon) \cup \mathcal{U}_0(t'_0, t'_1) \subset \mathcal{U}_0(t_0, t'_1)$ is greater than the negative index of $\mathcal{Q}_{[t_0, t_1]}$ on $\mathcal{U}_0(t_0, \tau_j + \varepsilon)$. This is, however, impossible as the index function I had no growth-points between $\tau_j + \varepsilon$ and $\tau_{j+1} \geq t'_1$. \square

From the above we easily get

Corollary 4.6. *Under the assumptions and notation of Lemma 4.5, consider the restriction of an abnormal minimizing SR geodesic $q(t)$ to a subinterval $[t'_0, t'_1] \subset (\tau_j, \tau_{j+1})$. Then the related value function $\widehat{Q}_{t'_0, t}(\mathbf{0}) = 0$ for every $t \in [t'_0, t'_1]$.*

In this context it is convenient to state the following simple fact.

Proposition 4.7. *If $\widehat{Q}_{t_0, t}(\mathbf{0})$ is finite then it equals to 0.*

Proof. Of course taking a trivial control $\Delta u \equiv 0$ gives us $Q_{t_0, t}[\Delta u] = 0$, and hence $\widehat{Q}_{t_0, t}(\mathbf{0}) \leq 0$. Assume that there exists some $\Delta v \in \mathcal{U}_0(t_0, t)$ such that $Q_{t_0, t}[\Delta v] = -c < 0$. Note that, since $\mathcal{U}_0(t_0, t)$ is a linear space, and since the functional $Q_{t_0, t}[\cdot]$ is quadratic, for every $\lambda \in \mathbb{R}$ we have $\lambda \cdot \Delta v \in \mathcal{U}_0(t_0, t)$ and $Q_{t_0, t}[\lambda \cdot \Delta v] = -\lambda^2 c$. Taking $\lambda \rightarrow \infty$ gives us $\widehat{Q}_{t_0, t}(\mathbf{0}) = -\infty$. We get a contradiction with the assumption that $\widehat{Q}_{t_0, t}(\mathbf{0})$ was finite. \square

Step 2 – Bellman-like inequalities. The minimal 2-jet $\widehat{Q}_{t_0, t}(\cdot)$ is defined as the Bellman's value function for an optimal control problem. Thus it should not be surprising that it shares its basic properties. Namely, it turns out, that for a fixed control $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$, the distance between $Q_{t_0, t}[\Delta u]$ and the infimum value $\widehat{Q}_{t_0, t}(\mathbf{v} = \mathbf{q}^{(1)}(t, \Delta u))$ is a non-decreasing function of time t . This property will be of crucial importance later.

Lemma 4.8 (Bellman's inequality). *For every $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$, function*

$$t \longmapsto Q_{t_0, t}[\Delta u] - \widehat{Q}_{t_0, t}(\mathbf{q}^{(1)}(t, \Delta u))$$

is non-decreasing.

Proof. Fix a control $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$, and assume for simplicity that $Q_{t_0, t}[\Delta u] - \widehat{Q}_{t_0, t}(\mathbf{q}^{(1)}(t, \Delta u)) = 5$, while $Q_{t_0, t'}[\Delta u] - \widehat{Q}_{t_0, t'}(\mathbf{q}^{(1)}(t', \Delta u)) = 1$ for some $t_0 \leq t < t' \leq t_1$. Since $\widehat{Q}_{t_0, t}(\cdot)$ is defined as the infimum, there is some control $\Delta u' \in L^\infty([t_0, t], \mathbb{R}^k)$ such that $\mathbf{q}^{(1)}(t, \Delta u) = \mathbf{q}^{(1)}(t, \Delta u')$, while $Q_{t_0, t}[\Delta u'] - \widehat{Q}_{t_0, t}(\mathbf{q}^{(1)}(t, \Delta u)) < 5 - 1 = 4$. Thus

$$Q_{t_0, t}[\Delta u'] - Q_{t_0, t}[\Delta u] < -1. \tag{4.5}$$

Now consider a concatenated control $\widetilde{\Delta u} := \Delta u'|_{[t_0, t]} \circ \Delta u|_{[t, t']} \in L^\infty([t_0, t'], \mathbb{R}^k)$. Obviously, $\mathbf{q}^{(1)}(t, \widetilde{\Delta u}) = \mathbf{q}^{(1)}(t, \Delta u')$ and $\mathbf{q}^{(1)}(t', \widetilde{\Delta u}) = \mathbf{q}^{(1)}(t', \Delta u)$ as both controls $\widetilde{\Delta u}$ and Δu equal on $[t, t']$ and share the same initial point $\mathbf{q}^{(1)}(t, \Delta u)$. On the other hand,

$$Q_{t_0, t'}[\widetilde{\Delta u}] = Q_{t_0, t}[\Delta u'] + \int_t^{t'} \sum \Delta u_i(\tau) \cdot \langle Y_i^{(2)}(\tau, \mathbf{q}^{(1)}(\tau, \Delta u)), \psi_0 \rangle d\tau$$

$$= Q_{t_0,t}[\Delta u'] + Q_{t_0,t'}[\Delta u] - Q_{t_0,t}[\Delta u] \stackrel{(4.5)}{<} Q_{t_0,t'}[\Delta u] - 1.$$

It follows that

$$\begin{aligned} Q_{t_0,t'}[\widetilde{\Delta u}] - \widehat{Q}_{t_0,t'}(\mathbf{q}^{(1)}(t', \widetilde{\Delta u})) &= Q_{t_0,t'}[\widetilde{\Delta u}] - \widehat{Q}_{t_0,t'}(\mathbf{q}^{(1)}(t', \Delta u)) \\ &< Q_{t_0,t'}[\Delta u] - \widehat{Q}_{t_0,t'}(\mathbf{q}^{(1)}(t', \Delta u)) - 1 = 0. \end{aligned}$$

This is a contradiction, with the definition of $\widehat{Q}_{t_0,t'}(\cdot)$ as the infimum. \square

Combining the above with the results from the previous step of the proof we get a result that states that if we can steer an element $\mathbf{v} \in \mathbb{R}^{n+1}$ to zero, then the minimal 2-jet is finite at this point.

Corollary 4.9. *Under the assumptions and notation of Lemma 4.5, consider the restriction of an abnormal minimizing SR geodesic $q(t)$ to a subinterval $[t'_0, t'_1] \subset (\tau_j, \tau_{j+1}]$. Let $\mathbf{v} \in \mathcal{R}_{(\Lambda_1)}(t'_0, \tau) \subset \mathbb{R}^{n+1}$ be a vector controllable to zero from time $\tau \in [t'_0, t'_1]$ (in the sense of Def. 4.2). Then the minimal 2-jet $\widehat{Q}_{t'_0, \tau}(\mathbf{v})$ is finite.*

Proof. Let $\Delta u_1 \in L^\infty([t'_0, \tau], \mathbb{R}^k)$ be a control steering (Λ_1) from $\mathbf{0}$ to \mathbf{v} (it exists, since \mathbf{v} belongs to the reachable set $\mathcal{R}_{(\Lambda_1)}(t'_0, \tau)$), and let $\Delta u_2 \in L^\infty([\tau, t'_1], \mathbb{R}^k)$ be a control steering (Λ_1) from \mathbf{v} to $\mathbf{0}$. Consider the concatenated control $\Delta u = \Delta u_1|_{[t'_0, \tau]} \circ \Delta u_2|_{[\tau, t'_1]}$. The related trajectory $\mathbf{q}^{(1)}(t, \Delta u)$ satisfies $\mathbf{q}^{(1)}(\tau, \Delta u) = \mathbf{v}$ and $\mathbf{q}^{(1)}(t'_1, \Delta u) = \mathbf{0}$. By the results of Lemma 4.8 we have

$$\begin{aligned} Q_{t'_0, \tau}[\Delta u] - \widehat{Q}_{t'_0, \tau}(\mathbf{v} = \mathbf{q}^{(1)}(\tau, \Delta u)) &\leq Q_{t'_0, t'_1}[\Delta u] - \widehat{Q}_{t'_0, t'_1}(\mathbf{0} = \mathbf{q}^{(1)}(t'_1, \Delta u)) \stackrel{\text{Proposition 4.7}}{=} \\ &Q_{t'_0, t'_1}[\Delta u] - 0, \end{aligned}$$

hence $\widehat{Q}_{t'_0, \tau}(\mathbf{v})$ is bounded from below. \square

Step 3 – controllability to zero. Now we prove that, after a suitable division of a SR trajectory (not necessarily an abnormal geodesic), the characteristic control system (Λ_1) is controllable to zero from every point of its reachable set. We start with a few simple properties of the reachable sets.

Proposition 4.10. *Consider a SR trajectory $q(t)$, with $t \in [t_0, t_1]$, corresponding to a control $u \in L^\infty([t_0, t_1], \mathbb{R}^k)$. The reachable sets $\mathcal{R}_{(\Lambda_1)}(t_0, \tau)$ of the related characteristic control system (Λ_1) have the following properties:*

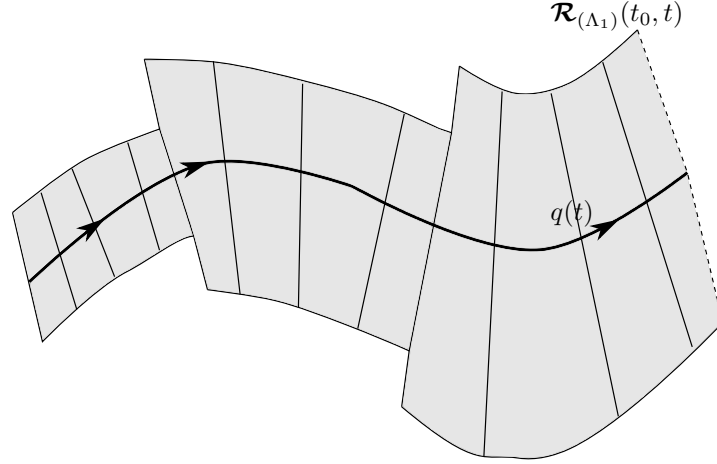
- (i) for each $\tau \in (t_0, t_1]$, the set $\mathcal{R}_{(\Lambda_1)}(t_0, \tau)$ is a linear subspace of \mathbb{R}^{n+1} ;
- (ii) if $\tau < \tau'$ then $\mathcal{R}_{(\Lambda_1)}(t_0, \tau) \subseteq \mathcal{R}_{(\Lambda_1)}(t_0, \tau')$;
- (iii) there exists $t'_1 \in (t_0, t_1)$ such that $\mathcal{R}_{(\Lambda_1)}(t_0, \tau) = \mathcal{R}_{(\Lambda_1)}(t_0, t_1)$ for every $\tau \in (t'_1, t_1]$.

Proof. The first point is obvious as the system is linear. So is the second, since each control Δu defined on $[t_0, \tau]$ can be extended to a control on $[t_0, \tau']$ by setting $\Delta u(t) = 0$ on $(\tau, \tau']$.

The third point follows from (ii) and the fact that the monotonous integer-valued function $\tau \mapsto \dim \mathcal{R}_{(\Lambda_1)}(t_0, \tau)$ is left semi-continuous. Indeed, assume that for some $\tau_0 \in [t_0, t_1]$ we have $\dim \mathcal{R}_{(\Lambda_1)}(t_0, \tau_0) > \dim \mathcal{R}_{(\Lambda_1)}(t_0, \tau)$ for all $\tau < \tau_0$. If this is the case then there exist a covector $\psi \in (\mathbb{R}^{n+1})^*$ annihilating $\mathcal{R}_{(\Lambda_1)}(t_0, \tau)$ for every $\tau < \tau_0$, but not $\mathcal{R}_{(\Lambda_1)}(t_0, \tau_0)$. If so, choose a control Δu_0 such that

$$\langle \psi, \mathbf{q}^{(1)}(\tau_0, \Delta u_0) \rangle \neq 0.$$

The map $\tau \mapsto \langle \psi, \mathbf{q}^{(1)}(\tau, \Delta u_0) \rangle$ is, however, defined by an integral of a MB function over $[t_0, \tau]$ and thus is continuous with respect to τ , so necessarily $\langle \psi, \mathbf{q}^{(1)}(\tau, \Delta u_0) \rangle \neq 0$ for some $\tau < \tau_0$. On the other hand, $\mathbf{q}^{(1)}(\tau, \Delta u_0) \in \mathcal{R}_{(\Lambda_1)}(t_0, \tau) \subset \ker \psi$. The contradiction ends the proof. \square



The dimension of the reachable set $\mathcal{R}_{(\Lambda_1)}(t_0, t)$ grows along the trajectory $q(t)$.

It turns out that after cutting the trajectory to a finite number of pieces all elements of the reachable sets are controllable to zero.

Lemma 4.11. *Consider a SR trajectory $q(t)$, with $t \in [t_0, t_1]$, corresponding to a control $u \in L^\infty([t_0, t_1], \mathbb{R}^k)$. There exist at most $n + 1$ -points $t_0 = \tau_0 < \tau_1 < \dots < \tau_s = t_1$ such that for each $\tau \in [\tau_i, \tau_{i+1})$ all points of the reachable set $\mathcal{R}_{(\Lambda_1)}(\tau_i, \tau)$ of the related characteristic control system (Λ_1) are controllable to zero from time τ .*

Proof. Denote by $(\overline{\Lambda_1})$ the control system (Λ_1) with reversed time (i.e we control from t_1 to t_0), and let $\mathcal{R}_{(\overline{\Lambda_1})}(t_1, \tau)$ denote its time τ -reachable set. Obviously, a vector $\mathbf{v} \in \mathbb{R}^{n+1}$ is controllable to zero from time $\tau \in [t_0, t_1]$ if and only if $\mathbf{v} \in \mathcal{R}_{(\overline{\Lambda_1})}(t_1, \tau)$. Observe that $\mathcal{R}_{(\overline{\Lambda_1})}(t_1, t_0) = \mathcal{R}_{(\Lambda_1)}(t_0, t_1)$ as both linear spaces are defined by the same integrals differing only by the direction of integration. System $(\overline{\Lambda_1})$ has properties analogous to those of (Λ_1) stated in Proposition 4.10. Therefore there exists a maximal $\tau_1 > t_0$ such that for every $\tau \in [t_0, \tau_1)$

$$\mathcal{R}_{(\overline{\Lambda_1})}(t_1, \tau) = \mathcal{R}_{(\overline{\Lambda_1})}(t_1, t_0) = \mathcal{R}_{(\Lambda_1)}(t_0, t_1) \stackrel{\text{Proposition 4.10 (ii)}}{\supset} \mathcal{R}_{(\Lambda_1)}(t_0, \tau),$$

Proving the assertion for $i = 0$. Now repeat the argument to the system on the new interval $[\tau_1, t_1]$. Note that by choosing τ_1 maximal possible we have $\dim \mathcal{R}_{(\Lambda_1)}(\tau_1, t_1) = \dim \mathcal{R}_{(\overline{\Lambda_1})}(t_1, \tau_1) < \dim \mathcal{R}_{(\overline{\Lambda_1})}(t_1, t_0) = \dim \mathcal{R}_{(\Lambda_1)}(t_0, t_1)$, so the procedure will end in at most n steps when the dimension of the reachable set falls to zero. \square

Conclusion – a suitable division of the SR geodesic. Now we can combine the two divisions: the one described in Corollary 4.6 which guarantees finiteness of the minimal 2-jet at zero, and the one from Lemma 4.11 which guarantees controllability to zero of any point of the reachable set, to get:

Lemma 4.12. *Assume that a SR trajectory $q(t)$, with $t \in [t_0, t_1]$, corresponding to a control $u \in L^\infty([t_0, t_1], \mathbb{R}^k)$, is a minimizing abnormal SR geodesic of co-rank r . Then there exists a non-zero covector $\psi_0 \in (\mathbb{R}^n)^*$, and at most $r(n - r - k + 1) + 1$ points $\tau_0 = t_0 < \tau_1 < \dots < \tau_N = t_1$ such that on each sub-interval $[t'_0, t'_1] \subset (\tau_i, \tau_{i+1})$ the related value function $\widehat{Q}_{t'_0, \tau}(\mathbf{v})$ is finite for every $\tau \in [t'_0, t'_1]$ and every $\mathbf{v} \in \mathcal{R}_{(\Lambda_1)}(t'_0, \tau)$.*

Proof. We first choose ψ_0 as in Lemma 4.5 and, according to Corollary 4.6, divide $[t_0, t_1]$ into at most r pieces of the form $(\tau_j, \tau_{j+1}]$ such that the $\widehat{Q}_{t'_0, \tau_{j+1}}(\mathbf{0}) = 0$ for any $[t'_0, \tau_{j+1}] \subset (\tau_j, \tau_{j+1}]$. Now by Lemma 4.11 we divide $[t'_0, \tau_{j+1}]$ into at most n pieces to guarantee that on each of these all elements of the reachable set are controllable

to zero. By Corollary 4.9 the minimal 2-jet $\widehat{Q}_{t'_0, \tau}(\mathbf{v})$ is finite for all elements \mathbf{v} of the reachable set. Actually, it follows from the proof of Lemma 4.11 that those new division points are the points at which the dimension of the reachable set $\mathcal{R}_{(\overline{\Lambda}_1)}(\tau_{j+1}, \tau)$ (for system (Λ_1) with reversed time) grows. Thus, first of all, these new division points do not depend on the choice of t'_0 , hence in fact we have a division of the whole $[\tau_j, \tau_{j+1}]$. Secondly, the number of pieces of the new division is actually not greater than $n - r - k + 1$, as the dimension of $\mathcal{R}_{(\overline{\Lambda}_1)}(\tau_{j+1}, \tau)$ is at most $n + 1 - r$ (by the assumption that $q(t)$ is of co-rank r) and at least k (as (Λ_1) is controlled by k independent vector fields). This ends the proof. \square

Step 4 – the shape of the minimal 2-jet. Now let us address the situation when the minimal 2-jet $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ is finite for all possible choices of $\tau \in [t_0, t_1]$ and $\mathbf{v} \in \mathcal{R}_{(\Lambda_1)}(t_0, \tau)$. By the results of Lemma 4.12 such a situation can be achieved by dividing a minimizing abnormal geodesics into a finite number of pieces. Our goal is to derive the properties of $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ in such a situation.

Theorem 4.13. *Consider the characteristic OCP of the trajectory $q(t)$ corresponding to the control $u \in L^\infty([t_0, t_1], \mathbb{R}^k)$. Assume that for some covector $\psi_0 \in (\mathbb{R}^n)^*$ the map $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ is finite for all possible choices of $\tau \in [t_0, t_1]$ and $\mathbf{v} \in \mathcal{R}_{(\Lambda_1)}(t_0, \tau)$. Then*

- $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ is quadratic with respect to \mathbf{v} .
- for a fixed \mathbf{v} , function $\tau \mapsto \widehat{Q}_{t_0, \tau}(\mathbf{v})$ is non-increasing.

Proof. To prove the first part of the assertion, fix $\tau \in [t_0, t_1]$ choose $\varepsilon > 0$, and choose any basis $\{\mathbf{v}_1, \dots, \mathbf{v}_s\}$ of $\mathcal{R}_{(\Lambda_1)}(t_0, \tau)$. For $i = 1, 2, \dots, s$ denote $\mathbf{v}_{s+i} := -\mathbf{v}_i$ and let P be the convex polygon with vertices $\pm \mathbf{v}_i$. Since $\widehat{Q}_{t_0, \tau}(\cdot)$ is defined as an infimum, for every $j = 1, 2, \dots, 2s$ we may find a control $\Delta u_j \in L^\infty([t_0, \tau], \mathbb{R}^k)$ such that $\mathbf{b}^{(1)}(\tau, \Delta u_j) = \mathbf{v}_j$ and $Q_{t_0, \tau}[\Delta u_j] \leq \widehat{Q}_{t_0, \tau}(\mathbf{v}_j) + \varepsilon$. Since for every $\Delta v \in \mathcal{U}_0(t_0, \tau)$ we have

$$Q_{t_0, \tau}[\Delta u_j + \Delta v] = Q_{t_0, \tau}[\Delta u_j] + 2B_{t_0, \tau}[\Delta u_j, \Delta v] + Q_{t_0, \tau}[\Delta v] \geq \widehat{Q}_{t_0, \tau}(\mathbf{v}_j = \mathbf{b}^{(1)}(\tau, \Delta u_j + \Delta v)),$$

the condition $Q_{t_0, \tau}[\Delta u_j] \leq \widehat{Q}_{t_0, \tau}(\mathbf{v}_j) + \varepsilon$ is equivalent to the following one:

$$2B_{t_0, \tau}[\Delta u_j, \Delta v] + Q_{t_0, \tau}[\Delta v] \geq -\varepsilon \quad \text{for every } \Delta v \in \mathcal{U}_0(t_0, \tau).$$

Now for every convex combination $\sum_{j=1}^{2s} \lambda_j \cdot \Delta u_j$ consider $F(\sum_j \lambda_j \cdot \mathbf{v}_j) := Q_{t_0, \tau}[\sum_j \lambda_j \cdot \Delta u_j]$. Obviously, as $\mathbf{b}^{(1)}(\tau, \sum_j \lambda_j \cdot \Delta u_j) = \sum_j \lambda_j \cdot \mathbf{b}^{(1)}(\tau, \Delta u_j) = \sum_j \lambda_j \cdot \mathbf{v}_j$, and $Q_{t_0, \tau}[\cdot]$ is a quadratic function, then F is a quadratic function on the polygon P . We further claim that $\widehat{Q}_{t_0, \tau}(\mathbf{v}) \leq F(\mathbf{v}) \leq \widehat{Q}_{t_0, \tau}(\mathbf{v}) + \varepsilon$. The first inequality is obvious as $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ is the infimum value of the functional $Q_{t_0, \tau}[\cdot]$ and $F(\mathbf{v}) = Q_{t_0, \tau}[\sum_j \lambda_j \cdot \Delta u_j]$ a particular one. The second inequality follows easily by an addition of analogous inequalities for particular Δu_j 's. Namely for every $\Delta v \in \mathcal{U}_0(t_0, \tau)$ we have

$$\begin{aligned} 2B_{t_0, \tau}[\sum_j \lambda_j \cdot \Delta u_j, \Delta v] + Q_{t_0, \tau}[\Delta v] &= 2B_{t_0, \tau}[\sum_j \lambda_j \cdot \Delta u_j, \Delta v] + (\sum_j \lambda_j) \cdot Q_{t_0, \tau}[\Delta v] \\ &= \sum_{j=1}^{2s} \lambda_j \cdot (2B_{t_0, \tau}[\Delta u_j, \Delta v] + Q_{t_0, \tau}[\Delta v]) \geq \sum_{j=1}^{2s} \lambda_j \cdot (-\varepsilon) = -\varepsilon. \end{aligned}$$

At the end of the day we showed that the function $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ can be ε -uniformly approximated by a quadratic function $F(\mathbf{v})$ on the polygon $P \ni \mathbf{v}$. Note that a uniform limit of a sequence of quadratic maps is also a quadratic map, thus $\widehat{Q}_{t_0, \tau}(\cdot)$ is quadratic on P . Since P was arbitrary the assertion follows.

The second part of the assertion is straightforward. As extending a given control $\Delta u \in L^\infty([t_0, \tau], \mathbb{R}^k)$ to a control $\Delta u' \in L^\infty([t_0, \tau'], \mathbb{R}^k)$ by a concatenation with the zero control, *i.e.* $\Delta u' = \Delta u|_{[t_0, \tau]} \circ 0|_{[\tau, \tau']}$, gives $Q_{t_0, \tau}[\Delta u] = Q_{t_0, \tau'}[\Delta u']$, we have $\widehat{Q}_{t_0, \tau}(\mathbf{v}) \geq \widehat{Q}_{t_0, \tau'}(\mathbf{v})$ for every $\tau' > \tau$ and $\mathbf{v} \in \mathcal{R}_{(\Lambda_1)}(t_0, \tau)$. \square

End of the proof. Now we have all the pieces ready to conclude the proof of Theorem 3.3. Namely, Lemma 4.12 gives us a division of an abnormal SR geodesic $q(t)$ into a finite number of pieces, such that $\widehat{Q}_{t'_0, \tau}(\mathbf{v})$ is finite for every possible $\mathbf{v} \in \mathcal{R}_{(\Lambda_1)}(t'_0, \tau)$, where $[t'_0, t'_1] \ni \tau$ lies inside a single piece. By the definition of the minimal 2-jet as an infimum, we know that

$$Q_{t'_0, \tau}[\Delta u] - \widehat{Q}_{t'_0, \tau}(\mathbf{q}^{(1)}(\tau, \Delta u)) \geq 0.$$

Further the map

$$\tau \longmapsto Q_{t'_0, \tau}[\Delta u] - \widehat{Q}_{t'_0, \tau}(\mathbf{q}^{(1)}(\tau, \Delta u)) = \langle \mathbf{q}^{(2)}(t, \Delta u), \psi_0 \rangle - \widehat{Q}_{t'_0, \tau}(\mathbf{q}^{(1)}(\tau, \Delta u))$$

is non-increasing by Lemma 4.8. What is more, $\widehat{Q}_{t'_0, \tau}(\mathbf{v})$ is quadratic in \mathbf{v} by the results of Theorem 4.13. Finally, as the map $\tau \mapsto \widehat{Q}_{t'_0, \tau}(\mathbf{v})$ is monotone, the coefficients of this quadratic form (say, in the standard basis of \mathbb{R}^{n+1}) are differences of a finite number of monotone function. Consequently, they are of bounded variation.

Summing up all the above observations, we have proven a version of Theorem 3.3 with the requested quadratic map equal to $-\widehat{Q}_{t'_0, \tau}(\mathbf{v})$. The only problem is that the data is expressed in terms of adapted coordinates $(\mathbf{q}^{(1)}(t, \Delta u), \mathbf{q}^{(2)}(t, \Delta u))$, instead of jets $(\mathbf{b}^{(1)}(t, \Delta u), \mathbf{b}^{(2)}(t, \Delta u))$ as in the original formulation. To end the proof we have just to invert the transformation of adapted coordinates (4.2). Note that, by the results of Proposition 4.3, $Q_{t'_0, \tau}[\mathbf{q}^{(1)}(\tau, \Delta u)] = \langle \mathbf{q}^{(2)}(t, \Delta u), \psi_0 \rangle$ equals to $\langle \mathbf{b}^{(2)}(t, \Delta u), \varphi(t) \rangle$ up to a term quadratic in $\mathbf{b}^{(1)}(t, \Delta u)$; and that the passage between $\mathbf{b}^{(1)}(t, \Delta u)$ and $\mathbf{q}^{(1)}(t, \Delta u)$ is linear. Therefore the quadratic character of the minimal 2-jet will be preserved under this transformation. So, still by the results of Proposition 4.3, will be the bounded variation regularity of this map. \square

5. SUPPLEMENTARY RESULTS – ON SOLUTIONS OF THE CHARACTERISTIC OCP

In Section 4.2 we were able to describe the shape of the minimal 2-jet $\widehat{Q}_{t_0, t}(\mathbf{v})$, *i.e.* the infimum of the set of values of $Q_{t_0, t}[\Delta u]$ under the condition that $\mathbf{q}^{(1)}(t, \Delta u) = \mathbf{v}$. This, in turn, allowed us to prove Theorem 3.3 describing sub-Riemannian optimality conditions of degree two. So far, however, we didn't address the question whether these infimum values are realized by some controls, *i.e.* if the characteristic OCP actually has solutions. In general this will not happen, as may be seen in the examples discussed in Section 3.4. The reason is that $\widehat{Q}_{t_0, t}(\mathbf{v})$ is defined as an infimum taken over elements Δu belonging to an affine subspace of finite codimension in $L^\infty([t_0, t_1], \mathbb{R}^k)$, and that we lack any sort of universal (weakly) compactness results in such situations. Below we will discuss some conditions that can guarantee the existence of controls realizing the infimum.

First of all, in Section 5.1, we study the characteristic OCP from the algebraic point of view, exploiting mostly the fact that the problem is quadratic in controls. It turns out that an important role is played by $\text{Sol}_{t_0, t}(\mathbf{0})$ – the space of solutions of the characteristic OCP with the end-point $\mathbf{0}$. We identify that one of the assumptions made in [4] implies that this space has finite codimension in the space of all controls $L^\infty([t_0, t], \mathbb{R}^k)$. We formalize this as *Monti's condition* and prove, in Lemma 5.5, that this finite codimension assumption implies that the characteristic OCP has solutions for an arbitrary end-point.

If the existence of solutions of the characteristic OCP can be somehow assured, we can apply to them the Pontryagin Maximum Principle (*PMP*, in short) to get additional information about the underlying SR trajectory $q(t)$. Precisely this situation is considered in Section 5.2. In particular, by combining our main result with the conclusions of the PMP, we prove Lemma 5.9 stating that if Monti's conditions are assumed, then the abnormal SR geodesics consist of a finite number of pieces which are either C^2 or 2-abnormal.

5.1. Algebraic approach to the characteristic OCP, Monti's condition

Algebraic criteria for the existence of solutions. We begin our discussion with simple algebraic criteria for a control to be a solution of the characteristic OCP.

Proposition 5.1. *Let $\Delta u \in L^\infty([t_0, t], \mathbb{R}^k)$ be such that $\mathbf{q}^{(1)}(t, \Delta u) = \mathbf{v}$. Then Δu is a solution of the characteristic OCP if and only if*

- (i) $B_{t_0, t}[\Delta u, \Delta v] = 0$ for every $\Delta v \in \mathcal{U}_0(t_0, t)$ and
- (ii) $\widehat{Q}_{t_0, t}(\mathbf{0})$ is finite (hence equal to zero by Proposition 4.7).

Proof. First let us prove that conditions (i)–(ii) imply optimality of the control Δu . Note that controls $\Delta u, \Delta u' \in L^\infty([t_0, t], \mathbb{R}^k)$ are such that $\mathbf{q}^{(1)}(t, \Delta u) = \mathbf{q}^{(1)}(t, \Delta u') = \mathbf{v}$ if and only if $\Delta v := \Delta u' - \Delta u$ belongs to $\mathcal{U}_0(t_0, t)$. Now by our assumptions $Q_{t_0, t}[\Delta v] \geq \widehat{Q}_{t_0, t}(\mathbf{0}) = 0$ and $B_{t_0, t}[\Delta u, \Delta v] = 0$. Therefore we have

$$\begin{aligned} Q_{t_0, t}[\Delta u'] &= Q_{t_0, t}[\Delta u + \Delta v] = Q_{t_0, t}[\Delta u] + 2B_{t_0, t}[\Delta u, \Delta v] + Q_{t_0, t}[\Delta v] \\ &= Q_{t_0, t}[\Delta u] + Q_{t_0, t}[\Delta v] \geq Q_{t_0, t}[\Delta u], \end{aligned}$$

Since $\Delta u'$ was arbitrary, Δu is indeed a solution of the characteristic OCP.

Assume now that Δu is a solution of the characteristic OCP. This implies, in particular, that $\widehat{Q}_{t_0, t}(\mathbf{v})$ is finite for $\mathbf{v} = \mathbf{q}^{(1)}(t, \Delta u)$. The latter condition suffices to prove (ii). Indeed, take any $\Delta v \in \mathcal{U}_0(t_0, t)$ and any $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} Q_{t_0, t}[\Delta u + \lambda \Delta v] &= Q_{t_0, t}[\Delta u] + 2B_{t_0, t}[\Delta u, \lambda \Delta v] + Q_{t_0, t}[\lambda \Delta v] \\ &= Q_{t_0, t}[\Delta u] + 2\lambda \cdot B_{t_0, t}[\Delta u, \Delta v] + \lambda^2 \cdot Q_{t_0, t}[\Delta v]. \end{aligned}$$

Thus if $Q_{t_0, t}[\Delta v]$ would be negative, we could make $Q_{t_0, t}[\Delta u + \lambda \Delta v]$ as low as possible by taking λ big enough. Therefore the assumption of finiteness of $\widehat{Q}_{t_0, t}(\mathbf{v})$ requires $Q_{t_0, t}[\Delta v] \geq 0$ for every $\Delta v \in \mathcal{U}_0(t_0, t)$. In consequence, $\widehat{Q}_{t_0, t}(\mathbf{0}) \geq 0$ and thus $\widehat{Q}_{t_0, t}(\mathbf{0}) = 0$ by Proposition 4.7.

To prove point (i) take $\Delta v \in \mathcal{U}_0(t_0, t)$. Then for each $\lambda \in \mathbb{R}$ the control $\Delta u + \lambda \Delta v$ belongs to $L^\infty([t_0, t], \mathbb{R}^k)$, and ends at $\mathbf{q}^{(1)}(t, \Delta u + \lambda \cdot \Delta v) = \mathbf{v}$. Thus the map

$$\mathbb{R} \ni \lambda \longmapsto Q_{t_0, t}[\Delta u + \lambda \Delta v] = Q_{t_0, t}[\Delta u] + 2\lambda \cdot B_{t_0, t}[\Delta u, \Delta v] + \lambda^2 \cdot Q_{t_0, t}[\Delta v]$$

has a minimum at $\lambda = 0$, which implies $B_{t_0, t}[\Delta u, \Delta v] = 0$. □

Spaces of solution of the characteristic OCP. We conclude that the space of solutions of the characteristic OCP has a simple structure.

Proposition 5.2. *The set of solutions of the characteristic OCP*

$$\text{Sol}_{t_0, t} := \{\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k) \mid Q_{t_0, t}[\Delta u] = \widehat{Q}_{t_0, t}(\mathbf{q}^{(1)}(t, \Delta u))\}$$

is a vector space. Further the space of solutions satisfying $\mathbf{q}^{(1)}(t, \Delta u) = \mathbf{v}$

$$\text{Sol}_{t_0, t}(\mathbf{v}) := \{\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k) \mid Q_{t_0, t}[\Delta u] = \widehat{Q}_{t_0, t}(\mathbf{v}), \mathbf{q}^{(1)}(t, \Delta u) = \mathbf{v}\}$$

is an affine space modeled on

$$\text{Sol}_{t_0, t}(\mathbf{0}) = \{\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k) \mid Q_{t_0, t}[\Delta u] = \widehat{Q}_{t_0, t}(\mathbf{0}), \mathbf{q}^{(1)}(t, \Delta u) = \mathbf{0}\}.$$

Proof. By Proposition 5.1, each solution $\Delta u \in L^\infty([t_0, t], \mathbb{R}^k)$ is characterized by the condition that the map

$$B_{t_0, t}[\Delta u, \cdot] : \mathcal{U}_0(t_0, t) \longrightarrow \mathbb{R}$$

is null. Note that $B_{t_0, t}[\Delta u, \cdot]$ is linear in Δu . This ends the first part of the proof.

Let now $\Delta u, \Delta u'$ be two elements of $\text{Sol}_{t_0, t}(\mathbf{v})$. Then, of course $\Delta v := \Delta u' - \Delta u$ satisfies $\mathbf{q}^{(1)}(t, \Delta v) = \mathbf{0}$. Further

$$\begin{aligned} 0 &= \widehat{Q}_{t_0, t}(\mathbf{v}) - \widehat{Q}_{t_0, t}(\mathbf{v}) = Q_{t_0, t}[\Delta u'] - Q_{t_0, t}[\Delta u] = Q_{t_0, t}[\Delta u + \Delta v] - Q_{t_0, t}[\Delta u] \\ &= 2B_{t_0, t}[\Delta u, \Delta v] + Q_{t_0, t}[\Delta v]. \end{aligned}$$

By the results of Proposition 5.1 the bilinear term vanishes, and so $Q_{t_0, t}[\Delta v] = 0 = \widehat{Q}_{t_0, t}(\mathbf{0})$, i.e. $\Delta u' - \Delta u \in \text{Sol}_{t_0, t}(\mathbf{0})$. Finally if put $\mathbf{v} = \mathbf{0}$ in the above calculation, we would get that a difference of two elements of $\text{Sol}_{t_0, t}(\mathbf{0})$ lies in $\text{Sol}_{t_0, t}(\mathbf{0})$. We conclude that the latter is a vector space. This ends the proof. \square

The space $\text{Sol}_{t_0, \tau}(\mathbf{0})$ has some interesting properties.

Proposition 5.3. *Assume that the minimal 2-jet $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ is finite for every $\mathbf{v} \in \mathcal{R}_{(\Lambda_1)}(t_0, \tau)$. Then for every $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$, and every $\Delta v \in \text{Sol}_{t_0, \tau}(\mathbf{0})$ we have*

$$B_{t_0, \tau}[\Delta u, \Delta v] = 0.$$

Proof. Take any such Δu and Δv . Since $Q_{t_0, \tau}[\Delta v] = \widehat{Q}_{t_0, \tau}(\mathbf{0}) = 0$, for every $\lambda \in \mathbb{R}$ we have

$$Q_{t_0, \tau}[\Delta u + \lambda \cdot \Delta v] = Q_{t_0, \tau}[\Delta u] + 2\lambda B_{t_0, \tau}[\Delta u, \Delta v] + \lambda^2 Q_{t_0, \tau}[\Delta v] = Q_{t_0, \tau}[\Delta u] + 2\lambda B_{t_0, \tau}[\Delta u, \Delta v]$$

Note that $\mathbf{q}^{(1)}(\tau, \Delta u + \lambda \Delta v) = \mathbf{q}^{(1)}(\tau, \Delta u)$, as $\mathbf{q}^{(1)}(\tau, \Delta v) = \mathbf{0}$. If $B_{t_0, \tau}[\Delta u, \Delta v] \neq 0$, then by taking λ big or small enough we would find a control steering (Λ_1) from $\mathbf{0}$ to $\mathbf{q}^{(1)}(\tau, \Delta u)$ with arbitrarily small cost. This contradicts the finiteness of $\widehat{Q}_{t_0, \tau}(\mathbf{q}^{(1)}(\tau, \Delta u))$. \square

Monti's condition. Proposition 5.1 shows that solutions of the characteristic OCP are related with the kernel of the linear maps $B_{t_0, \tau}[\Delta u, \cdot] : \mathcal{U}_0(t_0, \tau) \rightarrow \mathbb{R}$. On the other hand, the previous Proposition 5.3 shows that the space $\text{Sol}_{t_0, \tau}(\mathbf{0})$ is contained in that kernel. It turns out that if the space $\text{Sol}_{t_0, \tau}(\mathbf{0})$ is big enough it may guarantee the existence of solutions of the characteristic OCP. To formalize this property let us propose the following definition.

Definition 5.4. Let $q(t)$ with $t \in [t_0, t_1]$ be a sub-Riemannian trajectory and consider the related characteristic OCP. We say that the trajectory satisfies *Monti's condition* at time $\tau \in [t_0, t_1]$ if the space $\text{Sol}_{t_0, \tau}(\mathbf{0})$ is of finite codimension in $L^\infty([t_0, \tau], \mathbb{R}^k)$ (equivalently: of finite codimension in $\mathcal{U}_0(t_0, \tau)$, as the latter is a space of finite codimension in $L^\infty([t_0, \tau], \mathbb{R}^k)$ being the kernel of the linear map $\mathbf{q}^{(1)} : L^\infty([t_0, \tau], \mathbb{R}^k) \rightarrow \mathbb{R}^{n+1}$).

We propose the name Monti's condition as this is one of the assumptions used in [4] to derive the third-order optimality conditions.¹ This assumption is, however, a strong one. Further it may not be satisfied, as is the case in the two examples discussed in Section 3.4.

¹Actually, the assumption of Theorem 1.2 in [4] translated to the notation of this paper is that the space of elements

$$\text{dom}(\mathcal{D}^3 F) := \{\Delta v \in L^\infty([t_0, \tau], \mathbb{R}^k) \mid \mathbf{q}^{(1)}(\tau, \Delta v) = 0 \text{ and } \mathbf{q}^{(11)}(\tau, \Delta u, \Delta v) \in \text{Im } \mathbf{q}^{(1)}(\tau, \cdot) \text{ for every } \Delta u \in L^\infty([t_0, t], \mathbb{R}^k)\}$$

is of finite co-dimension in $\mathcal{U}_0(t_0, t)$. Here, $2\mathbf{q}^{(11)}(\tau, \Delta u, \Delta v) := \mathbf{q}^{(2)}(\tau, \Delta u + \Delta v) - \mathbf{q}^{(2)}(\tau, \Delta u) - \mathbf{q}^{(2)}(\tau, \Delta v)$ is the bi-linear form related with the quadratic map $\mathbf{q}^{(2)}(\tau, \cdot)$.

Note that, in particular, every element $\Delta v \in \text{dom}(\mathcal{D}^3 F)$ satisfies $\mathbf{q}^{(11)}(\tau, \Delta v, \Delta v) = \mathbf{q}^{(2)}(\tau, \Delta v) \in \text{Im } \mathbf{q}^{(1)}(\tau, \cdot) \subset \ker \psi_0$, where ψ_0 is as Proposition 4.3. Consequently, $Q_\tau[\Delta v] = \langle \mathbf{q}^{(2)}(\tau, \Delta v), \psi_0 \rangle = 0$ and thus $Q_\tau[\Delta v] = 0 = \widehat{Q}_\tau(\mathbf{0})$, i.e. $\Delta v \in \text{Sol}_\tau(\mathbf{0})$ whenever

Lemma 5.5. *Let $q(t)$ with $t \in [t_0, t_1]$ be a sub-Riemannian trajectory. Consider the related characteristic OCP and assume that the minimal 2-jet $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ is finite for every $\mathbf{v} \in \mathcal{R}_{(\Lambda_1)}(t_0, \tau)$. If the trajectory satisfies Monti's condition at time τ , then the set $\text{Sol}_{t_0, t}(\mathbf{v})$ is non-empty for every $\mathbf{v} \in \mathcal{R}_{(\Lambda_1)}(t_0, \tau)$.*

Proof. We will show that Monti's condition actually reduces the characteristic OCP to a finite-dimensional optimization problem.

Consider a control $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$ satisfying $\mathbf{q}^{(1)}(\tau, \Delta u) = \mathbf{v}$, and let us introduce any splitting $\mathcal{U}_0(t, \tau) = V \oplus \text{Sol}_{t_0, \tau}(\mathbf{0})$, where V is finite-dimensional. The quadratic map $\Delta v \mapsto Q_{t_0, \tau}[\Delta v] = B_{t_0, \tau}[\Delta v, \Delta v]$ is positively-defined on V , and so we may find a $B_{t_0, \tau}[\cdot, \cdot]$ -orthonormal basis $\{\Delta v_\alpha\}_{\alpha \in A}$ of V . Denote $\lambda_\alpha := B_{t_0, \tau}[\Delta u, \Delta v_\alpha]$ and consider $\Delta u' = \Delta u - \sum_{\alpha \in A} \lambda_\alpha \cdot \Delta v_\alpha$. By construction $\mathbf{q}^{(1)}(\tau, \Delta u') = \mathbf{q}^{(1)}(\tau, \Delta u) = \mathbf{v}$, and it is straightforward to check that $B_{t_0, \tau}[\Delta u', \Delta v_\alpha] = 0$ for every $\alpha \in A$. Since, by Proposition 5.3, also $B_{t_0, \tau}[\Delta u', \Delta v] = 0$ for every $\Delta v \in \text{Sol}_{t_0, \tau}(\mathbf{0})$, the linear map $B_{t_0, \tau}[\Delta u', \cdot]$ vanishes on the whole $\mathcal{U}_0(t, \tau) = V \oplus \text{Sol}_{t_0, \tau}(\mathbf{0})$, and thus $Q_{t_0, \tau}[\Delta u'] = \widehat{Q}_{t_0, \tau}(\mathbf{v})$ by Proposition 5.1. \square

5.2. Solutions of the characteristic OCP in light of the Pontryagin Maximum Principle

Once the existence of solutions of the characteristic OCP is established (for example by assuming Monti's condition), we may use the Pontryagin Maximum Principle (PMP, in short) [20] to derive necessary conditions for optimality. They will give us a lot of information about the underlying trajectory. When discussing solutions of the characteristic OCP we will always speak about normal or abnormal *extremals of the characteristic OCP*, not to be confused with the *sub-Riemannian extremal* that constitutes the characteristic OCP itself.

The PMP for the characteristic OCP. For the characteristic OCP the *Pontryagin Hamiltonian* $H : \mathbb{R} \times \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1})^* \times \mathbb{R}^k \rightarrow \mathbb{R}$ is defined as

$$\begin{aligned} H(t, (q^{(1)}, c^{(1)}), (\psi, a), \Delta v) &= H(t, q^{(1)}, (\psi, a), \Delta v) \\ &:= \left\langle \sum_i \Delta v_i \cdot Y_i^{(1)}(t), \psi \right\rangle + a \sum_i \Delta v_i \cdot u_i(t) + p_0 \left\langle \sum_i \Delta v_i \cdot Y_i^{(2)}(t, q^{(1)}), \psi_0 \right\rangle. \end{aligned}$$

Here, $(\psi, a) \in (\mathbb{R}^{n+1})^*$ is a covector dual to the pair $(q^{(1)}, c^{(1)}) \in \mathbb{R}^{n+1}$ and p_0 is either -1 (in the normal case) or 0 (in the abnormal case). Actually, we may assume that the pair (ψ, a) belongs to the dual of the reachable set $\mathcal{R}_{(\Lambda_1)}(t_0, t_1)$. Note that since H does not depend on $c^{(1)}$, the dual object a will always be constant. The following result is immediate.

Proposition 5.6. *Let the control $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$, together with a covector curve $(\psi(t), a(t))$, be an extremal of the the characteristic OCP related with the SR trajectory $q(t)$, corresponding to the control $u \in L^\infty([t_0, t_1], \mathbb{R}^k)$. Then in the abnormal case ($p_0 = 0$) both $\psi(t)$ and $a(t)$ are constant. In the normal case ($p_0 = -1$) $a(t)$ is constant, while $\psi(t)$ is subject to the evolution equation*

$$\langle b, \dot{\psi}(t) \rangle = \left\langle \sum_i \Delta u_i(t) \cdot Y_i^{(2)}(t, b), \psi_0 \right\rangle \quad (5.1)$$

where $b \in \mathbb{R}^n$ is arbitrary.

$\overline{Q_\tau(\mathbf{0})}$ is finite. Summing up, if $\widehat{Q}_\tau(\mathbf{0})$ is finite then

$$\text{dom}(\mathcal{D}^3 F) \subset \text{Sol}_\tau(\mathbf{0}) \subset L^\infty([t_0, \tau], \mathbb{R}^k)$$

and thus the condition of Definition 5.4 is weaker than the assumption made in [4].

Further, as the Hamiltonian $H(t, q^{(1)}, (\psi, a), \Delta v)$ is a linear function of Δv , and we have no bounds on the control, the Maximum Principle gives us the following conditions

$$\langle Y_i^{(1)}(t), \psi(t) \rangle + a \cdot u_i(t) + p_0 \langle Y_i^{(2)}(t, q^{(1)}), \psi_0 \rangle = 0 \quad (5.2)$$

for every $i = 1, 2, \dots, k$.

Existence of extremals implies regularity of the SR trajectory. The existence of certain extremals of the PMP for the characteristic OCP allows to deduce regularity of the underlying SR trajectory $q(t)$.

Lemma 5.7. *Under the assumptions of Proposition 5.6*

- if Δu is an abnormal extremal of the characteristic OCP with $a \neq 0$, the SR trajectory $q(t)$ has to be normal and thus C^∞ -smooth;
- if Δu is a normal extremal of the characteristic OCP with $a \neq 0$, then the SR trajectory $q(t)$ is C^1 ,
- if Δu is a normal extremal of the characteristic OCP with $a \neq 0$, and the SR trajectory $q(t)$ additionally satisfies Goh conditions, then $q(t)$ is C^2 .

Proof. To prove the first part, let $(\psi, a) \in (\mathbb{R}^{n+1})^*$ be the abnormal covector curve of the characteristic OCP (it is constant by Proposition 5.6). The Maximum Principle (5.2) gives us in this case

$$\langle Y_i^{(1)}(t), \psi \rangle + a \cdot u_i(t) = 0 \quad \text{for } i = 1, 2, \dots, k.$$

These are precisely the equations of the normal SR trajectory. In particular, we get $u_i(t) = -\frac{1}{a} \langle Y_i^{(1)}(t), \psi \rangle$. Now note that on the left-hand side we have something of class C^r (with $r = 0_-, 1_-, 1, 2, \dots$) while, by the results of Lemma 4.1, on the right-hand side something of class C^{r+1} . Hence $u_i(t)$ must be C^∞ -smooth by the bootstrap argument.

A similar argument works also for a normal extremal of the characteristic OCP. In that case, by (5.2) we have

$$u_i(t) = -\frac{1}{a} \left(\langle Y_i^{(1)}(t), \psi(t) \rangle - \langle Y_i^{(2)}(t, q^{(1)}(t)), \psi_0 \rangle \right).$$

Now note that, if $u(t)$ was of class L^∞ , then, by Lemma 4.1, $Y_i^{(1)}(t)$ and $Y_i^{(2)}(t, \cdot)$ are absolutely continuous with respect to t . Further the covector curve $\psi(t)$, as well as the trajectory $q^{(1)}(t, \Delta u)$, are Lipschitz. Since $Y_i^{(2)}(t, q^{(1)})$ depends linearly on $q^{(1)}$, we conclude that $u_i(t)$ is absolutely continuous and, consequently, $q(t)$ is C^1 . This ends the proof of the second part.

For the last part we differentiate (5.2) (which we now know is possible almost everywhere) to get

$$\begin{aligned} a \dot{u}_i(t) &= -\langle \partial_t Y_i^{(1)}(t), \psi(t) \rangle - \langle Y_i^{(1)}(t), \dot{\psi}(t) \rangle + \langle \partial_t Y_i^{(2)}(t, q^{(1)}(t)), \psi_0 \rangle + \langle Y_i^{(2)}(t, \dot{q}^{(1)}(t)), \psi_0 \rangle \stackrel{(5.1), (\Lambda_1)}{=} \\ &\quad - \langle \partial_t Y_i^{(1)}(t), \psi(t) \rangle + \langle \partial_t Y_i^{(2)}(t, q^{(1)}(t)), \psi_0 \rangle \\ &\quad - \left\langle \sum_j \Delta u_j(t) \cdot Y_j^{(2)}(t, Y_i^{(1)}(t)), \psi_0 \right\rangle + \left\langle Y_i^{(2)}(t, \sum_j \Delta u_j(t) \cdot Y_j^{(1)}(t)), \psi_0 \right\rangle \\ &= -\langle \partial_t Y_i^{(1)}(t), \psi(t) \rangle + \langle \partial_t Y_i^{(2)}(t, q^{(1)}(t)), \psi_0 \rangle \\ &\quad \sum_j \Delta u_j \langle Y_i^{(2)}(t, Y_j^{(1)}(t)) - Y_j^{(2)}(t, Y_i^{(1)}(t)), \psi_0 \rangle \stackrel{\text{Lem. 4.1}}{=} \\ &\quad - \langle \partial_t Y_i^{(1)}(t), \psi(t) \rangle + \langle \partial_t Y_i^{(2)}(t, q^{(1)}(t)), \psi_0 \rangle + \sum_j \Delta u_j \langle \Phi^{(1)}(t)[X_i, X_j], \psi_0 \rangle, \end{aligned}$$

By Lemma 4.1 the t -dependence of $Y_i^{(1)}(t)$ and $Y_i^{(2)}(t, \cdot)$ is C^1 (as $u_i(t)$'s are now absolutely continuous), and the term involving the Lie bracket vanishes because of the Goh condition. By repeating our reasoning from the previous point, $\dot{u}_i(t)$ is absolutely continuous and hence $q(t)$ is actually C^2 . \square

With a little more effort, combining the information provided by the PMP with our previous results about optimality conditions of degree two, we can prove that actually solutions of the characteristic OCP are precisely the normal extremals of the PMP. Further, the existence of particular solutions of the characteristic OCP guarantees the C^2 -regularity of the underlining SR trajectory.

Lemma 5.8. *Let $q(t)$ with $t \in [t_0, t_1]$ be a strictly abnormal SR extremal.*

- (i) *Assume that the minimal 2-jet $\widehat{Q}_{t_0, t}(\mathbf{0})$ is finite. Then the control $\Delta u \in L^\infty([t_0, t], \mathbb{R}^k)$ is a solution of the characteristic OCP if and only if it is a normal extremal of the characteristic OCP.*
- (ii) *Assume that the assertion of Theorem 4.13 holds. In particular, let $\widehat{Q}_{t_0, t}(\mathbf{v}) = \Psi_2(t)[\mathbf{v}, \mathbf{v}]$ for a quadratic map $\Psi_2(t) : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$. Then if the control $\Delta u \in L^\infty([t_0, t], \mathbb{R}^k)$ is such a solution of the characteristic OCP that $\Psi_2(t)[(0, 1), \mathbf{q}^{(1)}(t, \Delta u)] \neq 0$, the SR trajectory $q(t)$ is of class C^2 .*

Proof. If Δu is a solution of the characteristic OCP it must be either a normal, or an abnormal extremal of the PMP. The abnormal case can be easily excluded. Indeed, let Δu be an abnormal extremal corresponding to a covector $(\psi, a) \in (\mathbb{R}^{n+1})^*$. By the results of previous Lemma 5.7, if $a \neq 0$, then $q(t)$ must be a normal SR extremal, which is impossible since it is assumed to be strictly abnormal.

In the case $a = 0$, by (5.2) the covector ψ annihilates all control fields $Y_i^{(1)}(t)$ and, in consequence, the whole reachable set of (Λ_1) , so $\psi = 0$ and $a = 0$ which is impossible, as the covector defined by the PMP has to be nowhere-vanishing.

Now we can concentrate on the case where Δu is a normal extremal of the characteristic OCP. Let $(\psi(t), a) \in (\mathbb{R}^{n+1})^*$ be the related covector curve. For any other control $\Delta v \in L^\infty([t_0, t_1], \mathbb{R}^k)$ we would like to calculate the bi-linear form $B_{t_0, t}[\Delta u, \Delta v]$. Note that

$$\begin{aligned} \left\langle \sum_i \Delta v_i(t) \cdot Y_i^{(2)}(t, q^{(1)}(t, \Delta u)), \psi_0 \right\rangle &\stackrel{(5.2)}{=} \left\langle \sum_i \Delta v_i(t) \cdot Y_i^{(1)}(t), \psi(t) \right\rangle + a \cdot \sum_i \Delta v_i(t) \cdot u_i(t) \stackrel{(\Lambda_1)}{=} \\ &\langle \dot{q}^{(1)}(t, \Delta v), \psi(t) \rangle + a \cdot c^{(1)}(t, \Delta v). \end{aligned}$$

On the other hand,

$$\left\langle \sum_i \Delta u_i(t) \cdot Y_i^{(2)}(t, q^{(1)}(t, \Delta v)), \psi_0 \right\rangle \stackrel{(5.1)}{=} \langle q^{(1)}(t, \Delta v), \dot{\psi}(t) \rangle$$

By adding the two above equalities and integrating over $[t_0, t]$ we get

$$B_{t_0, t}[\Delta u, \Delta v] = \langle q^{(1)}(t, \Delta v), \psi(t) \rangle + a \cdot c^{(1)}(t, \Delta v). \quad (5.3)$$

Our first conclusion is that whenever $\Delta v \in \mathcal{U}_0(t_0, t)$ then $q^{(1)}(t, \Delta v) = 0$ and $c^{(1)}(t, \Delta v) = 0$, and thus $B_{t_0, t}[\Delta u, \Delta v] = 0$. By Proposition 5.1 this (together with the finiteness of $\widehat{Q}_{t_0, t}(\mathbf{0})$) implies that Δu is optimal, proving point (i).

To prove (ii), take any $\Delta v \in L^\infty([t_0, t_1], \mathbb{R}^k)$, $\varepsilon \in \mathbb{R}$, and denote $\mathbf{q}^{(1)}(t, \Delta v) = \mathbf{v} = (v, c) \in \mathbb{R}^n \times \mathbb{R}$. We have

$$\begin{aligned} Q_{t_0, t}[\Delta u + \varepsilon \cdot \Delta v] &= Q_{t_0, t}[\Delta u] + \varepsilon B_{t_0, t}[\Delta u, \Delta v] + \varepsilon^2 Q_{t_0, t}[\Delta v] \\ &= \widehat{Q}_{t_0, t}(\mathbf{q}^{(1)}(t, \Delta u)) + \varepsilon \langle \mathbf{v}, (\psi(t), a) \rangle + \varepsilon^2 Q_{t_0, t}[\Delta v] \\ &\geq \widehat{Q}_{t_0, t}(\mathbf{q}^{(1)}(t, \Delta u)) + \varepsilon \langle \mathbf{v}, (\psi(t), a) \rangle + \varepsilon^2 \widehat{Q}_{t_0, t}(\mathbf{v}), \end{aligned}$$

and thus

$$\widehat{Q}_{t_0,t}(\mathbf{q}^{(1)}(t, \Delta u) + \varepsilon \mathbf{v}) \geq \widehat{Q}_{t_0,t}(\mathbf{q}^{(1)}(t, \Delta u)) + \varepsilon \langle \mathbf{v}, (\psi(t), a) \rangle + \varepsilon^2 \widehat{Q}_{t_0,t}(\mathbf{v}). \quad (5.4)$$

On the other hand, by Theorem 4.13 we know that the minimal 2-jet $\widehat{Q}_{t_0,t}(\cdot)$ is a quadratic function and hence

$$\begin{aligned} \widehat{Q}_{t_0,t}(\mathbf{q}^{(1)}(t, \Delta u) + \varepsilon \mathbf{v}) &= \Psi_2[\mathbf{q}^{(1)}(t, \Delta u) + \varepsilon \mathbf{v}, \mathbf{q}^{(1)}(t, \Delta u) + \varepsilon \mathbf{v}] \\ &= \Psi_2[\mathbf{q}^{(1)}(t, \Delta u), \mathbf{q}^{(1)}(t, \Delta u)] + 2\varepsilon \Psi_2[\mathbf{v}, \mathbf{q}^{(1)}(t, \Delta u)] + \varepsilon^2 \Psi_2[\mathbf{v}, \mathbf{v}] \\ &= \widehat{Q}_{t_0,t}(\mathbf{q}^{(1)}(t, \Delta u)) + 2\varepsilon \Psi_2[\mathbf{v}, \mathbf{q}^{(1)}(t, \Delta u)] + \varepsilon^2 \widehat{Q}_{t_0,t}(\mathbf{v}) \\ &\stackrel{(5.4)}{\geq} \widehat{Q}_{t_0,t}(\mathbf{q}^{(1)}(t, \Delta u)) + \varepsilon \langle \mathbf{v}, (\psi(t), a) \rangle + \varepsilon^2 \widehat{Q}_{t_0,t}(\mathbf{v}), \end{aligned}$$

implying

$$2\varepsilon \Psi_2[\mathbf{v}, \mathbf{q}^{(1)}(t, \Delta u)] \geq \varepsilon \langle \mathbf{v}, (\psi(t), a) \rangle.$$

As ε and $\mathbf{v} = (v, c)$ were arbitrary, the above inequality may hold if and only if

$$\langle v, \psi(t) \rangle + c \cdot a = \langle \mathbf{v}, (\psi(t), a) \rangle = 2\Psi_2[\mathbf{v}, \mathbf{q}^{(1)}(t, \Delta u)].$$

In particular, taking $\mathbf{v} = (0, 1)$ we get $a = 2\Psi_2(t)[(0, 1), \mathbf{q}^{(1)}(t, \Delta u)]$. By Lemma 5.7 if the latter number is different than zero and the Goh conditions are satisfied, the trajectory $q(t)$ is of class C^2 . Note however, that if the assertion of Theorem 4.13 holds, then also the assertion of Theorem 3.3 is true (with a single piece). To show it suffices to repeat the reasoning from the last paragraph of Section 4.2. But then Goh conditions are true by the results of Section 3.3. This ends the proof. \square

Regularity under Monti's condition. We end our consideration by stating the following consequences of Monti's condition.

Lemma 5.9. *Let $q(t)$, with $t \in [t_0, t_1]$, be a strictly abnormal minimizing SR geodesic. Assume additionally that it satisfies Monti's condition at time t_1 (in the sense of Definition 5.4).*

Consider the division of $q(t)$ into a finite number of pieces provided by Theorem 3.4. Then on each of these pieces $q(t)$ is either a 2-abnormal extremal in the sense of Definition 3.5, or is of class C^2 .

Proof. Consider the division of $q(t)$ into pieces $[\tau_i, \tau_{i+1}]$ provided by Theorem 3.4. Take any $[t'_0, t'_1] \subset (\tau_i, \tau_{i+1})$. Let $\Phi_2(t)$ be the related quadratic map and $a_2(t) \in \mathbb{R}$ its $\mathbb{R} \times \mathbb{R}$ -component. Now we have two possibilities, either the piece of $q(t)$, with $t \in [t'_0, t'_1]$, is a 2-abnormal extremal, or $a_2(t) \neq 0$ for every t (the 2-normal case). Let us assume that we have the latter situation.

Observe that if the Monti's condition holds at t_1 then also it holds at t'_1 on the shorter trajectory as all we have to do is to take the finite-codimensional extension $\text{Sol}_{t_0, t_1}(\mathbf{0}) \subset L^\infty([t_0, t_1], \mathbb{R}^k)$ and intersect it with $L^\infty([t'_0, t'_1], \mathbb{R}^k)$. In particular, by Lemma 5.5, the characteristic OCP on $[t'_0, t'_1]$ has solutions for every possible end-point condition. Let $\Delta u \in L^\infty([t'_0, t'_1], \mathbb{R}^k)$ be such a solution with $\mathbf{q}^{(1)}(t'_1, \Delta u) = (0, 1)$. Recall that by the argument from the last paragraph of Section 4.2 the quadratic maps $\Psi_2(t)$ and $\Phi_2(t)$ have the same $\mathbb{R} \times \mathbb{R}$ -component, hence

$$\Psi_2(t'_1)[(0, 1), \mathbf{q}^{(1)}(t'_1, \Delta u)] = \Psi_2(t'_1)[(0, 1), (0, 1)] = \Phi_2[(0, 1), (0, 1)] = a_2(t) \neq 0.$$

Now Lemma 5.8 point (ii) guarantees that $q(t)$ is of class C^2 on $[t'_0, t'_1]$. This ends the proof. \square

6. FURTHER STUDY

The presented material provokes a few interesting questions:

- Is it possible to improve the time-regularity of the minimal 2-jet $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ in Theorem 4.13? Answering this question would be an important step in solving the regularity problem of SR geodesics. Namely, if the τ -regularity of $\widehat{Q}_{t_0, \tau}(\mathbf{v})$ could be improved to, say being of class C^k , then in the 2-normal case equation (3.6) would guarantee that the control $u(t)$ is also of class C^k , and thus the trajectory $q(t)$ be of class C^{k+1} on each piece from a finite collection.

We **hypothesize** that it is possible to prove that if the control $u(t)$ is of class C^r (where $r = 0_-, 1_-, 1, 2, \dots$) then the dependence $\tau \mapsto \widehat{Q}_{t_0, \tau}(\mathbf{v})$ is of class C^{r+1} . This, by the standard bootstrap argument, will imply that every 2-normal trajectory $q(t)$ is piece-wise smooth (with a finite number of pieces), thus smooth by the results of [8].

- For 2-abnormal trajectories, as equation (3.6) does not determines the control $u(t)$, the question of regularity remains open regardless of the regularity properties of the minimal 2-jet $\widehat{Q}_{t_0, \tau}(\mathbf{v})$. It seems that for such curves, one needs to understand the geometry of third (and higher) order expansion of the extended end-point map. Some research in this direction was initiated in [4], yet a fully satisfactory theory is yet to be developed.
- Sub-Riemannian normal extremals are known to be locally minimizing. It is natural to ask whether the same is true for 2-normal extremals. We **hypothesize** that the answer to this question is positive, and could be provided by extending the methods from [10].
- Our proof is based on Agrachev-Sarychev Index Lemma (Lem. 3.1). It seems to us that is is possible to (slightly) modify the proof of this result to guarantee the following result.

Hypothesis. Let $q(t)$, with $t \in [t_0, t_1]$ be a minimizing strictly abnormal SR geodesic of co-rank r . Then there exists a covector $\varphi_0 : T_{q(t_1)}M \rightarrow \mathbb{R}$, and a symmetric 2-form $\Phi_2 : T_{q(t_1)}(M \times \mathbb{R}) \times T_{q(t_1)}(M \times \mathbb{R}) \rightarrow \mathbb{R}$ such that:

- $\langle \mathbf{b}^{(1)}(t_1, \Delta u), \varphi_0 \rangle = 0$ for every $\Delta u \in L^\infty([t_0, t_1], \mathbb{R}^k)$
- the negative index of the 2-form

$$L^\infty([t_0, t_1], \mathbb{R}^k) \ni \Delta u \longmapsto \langle \mathbf{b}^{(2)}(t_1, \Delta u), \varphi_0 \rangle - \Phi_2[\mathbf{b}^{(1)}(t_1, \Delta u), \mathbf{b}^{(1)}(t_1, \Delta u)]$$

is at most r .

In the standard version the bi-linear term Φ_2 is not present, as the 2-form is restricted to a subspace $\ker D_u \mathbf{End}_{q_0}^{[t_0, t_1]} \subset L^\infty([t_0, t_1], \mathbb{R}^k)$.

It should be also possible to prove that necessarily $\Phi_2[(0, 1), (0, 1)] \geq 0$, and so $a_2(t)$ described by Theorem 3.3 is always non-negative.

Such an improvement on Agrachev-Sarychev Lemma will allow to simplify the cutting procedure that we used in order to guarantee the finiteness of all minimal 2-jets $\widehat{Q}_{t_0, \tau}(\mathbf{v})$, and thus reduce the estimate on the maximal number of divisions in Theorem 3.3.

We plan to address these questions in a future series of publications.

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