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TURNPIKE PROPERTIES OF OPTIMAL BOUNDARY CONTROL PROBLEMS WITH RANDOM LINEAR HYPERBOLIC SYSTEMS

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Abstract. In many applications, in systems that are governed by linear hyperbolic partial differential equations some of the problem parameters are uncertain. If information about the probability distribution of the parametric uncertainty, distribution is available, the uncertain state of the system can be described using an intrinsic formulation through a polynomial chaos expansion. This allows to obtain solutions for optimal boundary control problems with random parameters. We show that similar to the deterministic case, a turnpike result holds in the sense that for large time horizons the optimal states for dynamic optimal control problems on a substantial part of the time interval approaches the optimal states for the corresponding uncertain static optimal control problem. We show turnpike results both for the full uncertain system as well as for a generalized polynomial chaos approximation.

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1. Introduction

Turnpike theory has its roots in mathematical economics (see [12]) but has been extended to optimization problems in various contexts, see for example the monograph [58]. Turnpike results for discrete time dynamics are presented in [24]. The relation between the turnpike property and dissipativity inequalities is studied in [25]. For optimal control problems with partial differential equations, the turnpike phenomenon has been investigated for example in [52] where distributed control is considered. A study of linear quadratic optimal control of general evolution equations is presented in [26]. In [13] manifold turnpikes are studied. Problems of optimal boundary control have been studied for example in [31], where the Neumann-control of a system that is governed by the wave equation is investigated and in [28], [27] for a system that is governed by a 2×2 linear hyperbolic partial differential equation. A survey of turnpike properties in optimal control is given in [23].

In this paper, we consider a system that is governed by a 2×2 linear hyperbolic equation. We prove that also for the system with uncertain parameters turnpike results hold. To be precise, we show an integral turnpike property in the sense that the normalized norm of the difference between the optimal dynamic control and the optimal static control converges to zero with the order $1/T$ as the time-horizon T tends to infinity. A number

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of applications where systems with uncertainty parameters appear are presented in [2], where applications in the control of water flow in channels and traffic flow is described. As a particular example of turnpike control applications in gas networks have been discussed recently, see *e.g.* [2] for the case of complete information. However, systems for gas transportation are typically subject to several sources of uncertainty with the friction coefficient as one particular example.

Recent work discusses the problem of the identification of those uncertain parameters in [35] as well as in [33] in particular for uncertain friction parameters in models for pipeline gas flow. Here, we are interested in concepts that allow to obtain a deterministic control in the situation where such uncertain parameters occur. In the applications, the full random state that is generated in such a situation is usually not observable. Possibly moments of the random solution like the expectation of the state and/or variance are available for feedback control. Hence, deterministic control is the relevant setting from an application point of view.

Besides this motivation, there is also a mathematically interesting aspect. If we consider random hyperbolic systems and stochastic controls, then, the results could be easily obtained from prior work [27], since the resulting equations are actually the same except for the random variable that acts in the equations as a parameter. When dealing with stochastic models, one may either follow a non-intrusive approach, *e.g.*, based on sampling (Monte–Carlo) [42, 50, 51] or based on collocation [1], or an intrusive [41, 55] approach. Since our control should take into account the full distribution of the stochastic state we follow here the intrusive approach by representing the stochastic input as a series of orthogonal functions, known as generalized polynomial chaos (gPC) expansions [5, 53, 56]. Equations for the coefficients of the series are obtained by Galerkin projection of the series. For hyperbolic partial differential equations the gPC approach has been recently explored *e.g.* in [8, 11, 16, 17, 19, 22, 40, 45, 48] mostly focusing on questions of hyperbolicity. In this paper the turnpike property will be explored for this approach. This is related to stabilization results under uncertainty as discussed *e.g.* in [14, 21]. Also results on (linear) kinetic equations as presented recently *e.g.* in [6, 7, 37, 38, 49, 57, 59] are closely related to the gPC of the underlying hyperbolic equations. A simulation study on turnpikes in stochastic LQ optimal control is presented in [44], where time-discrete systems with independent probabilistic constraints at each point in time are considered.

From a mathematical perspective, we are interested here in optimal control problems, with the aim of obtaining a deterministic optimal control that yields in a well-defined sense, an optimal performance of the uncertain system.

In order to evaluate the performance of a given control, the objective functions is defined in terms of the expected value of the system state. We consider objective functions of integral type that is given by integrals over the time horizon of the optimal control problem. This structure is essential to derive turnpike results also for uncertain systems.

2. Definition of the problem

For the convenience of the reader, we first state the initial boundary problem in the deterministic form. Let $L > 0$ and a natural number N be given. We consider a linear hyperbolic system in diagonal form that is defined by the space interval $x \in [0, L]$. For given functions $d_-,\ d_+ \in (C^1([0,\, L]))^N$ we assume that the inequality $d_-(x) < 0 < d_+(x)$ holds for all $x \in [0,\, L]$ and all components of d_+, d_-. Define the block diagonal matrices

$$D = \begin{pmatrix} d_+ & 0 \\ 0 & d_- \end{pmatrix},\ D' = \begin{pmatrix} (d_+)' & 0 \\ 0 & (d_-)' \end{pmatrix},\ |D| = \begin{pmatrix} |d_+| & 0 \\ 0 & |d_-| \end{pmatrix}. \tag{2.1}$$

Let $M(x)$ denote symmetric $2N \times 2N$ matrices that depends continuously on $x \in [0,\, L]$. Assume that for all $x \in [0,\, L]$ the matrices (2.2) are positive semi–definite, *i.e.*,

$$|D(x)|\, M(x) + M(x)\, |D(x)| \geq 0\ \forall x \in [0, L]. \tag{2.2}$$

For a given initial state $r^0 = (r_+^0, r_-^0) \in (L^\infty(0, L))^{2N}$ we consider a system that is governed by the initial boundary value problem

$$\begin{cases} r(0, x) = r^0(x), \\ \partial_t r(t,x) + D(x)\,\partial_x r(t,x) = -M(x)\,r(t,x), \\ r_+(t, 0) = u_+(t), \\ r_-(t, L) = u_-(t) \end{cases} \tag{2.3}$$

for $t \in (0,T)$ and $x \in (0, L)$ almost everywhere. Here, $r = (r_+, r_-)$ and the controls are $u_+, u_- \in (L^2(0, T))^N$. Under the previous assumptions the system (2.3) has a solution $r \in C([0,T], L^2((0, L); \mathbb{R}^{2N}))$. Moreover, for the boundary traces of the solution, we have $r_+(\cdot, L), r_-(\cdot, 0) \in (L^2(0, T))^N$. This follows by a Picard iteration along the characteristic curves. The corresponding result on the existence of differentiable solutions is given in [36] using a Lyapunov function approach, see also [3]. We also refer to [15] for H^s-stability of linear hyperbolic systems.

The system (2.3) is extended by a parametric uncertainty ω in the source term, *i.e.*, M is replaced by $M(x, \omega)$. In the case of transportation models, this is motivated by *e.g.* uncertain friction parameter [20, 29, 30], uncertain bottom topography in shallow-water systems [9, 47] or unknown relaxation rates in traffic flow models [18].

The parametric uncertainty ω is modeled as follows. Given a random variable $\overline{\omega}$ with values $\omega \in \mathbb{R}^d$ and probability density $\rho_\omega \in L^1(\mathbb{R}^d)$ we are interested in the solution of the random variable

$$r(t, x, \omega) \in C\left([0,T], L^2_{1\times\rho_\omega}((0, L) \times \mathbb{R}^d; \mathbb{R}^{2N})\right). \tag{2.4}$$

Here, $r = (r_+(t, x, \omega), r_-(t, x, \omega))$ is defined as the random solution to the set of equations (2.5) with random source term, *i.e.*,

$$\begin{cases} r(0, x, \omega) = r^0(x), \\ \partial_t r(t, x, \omega) + D(x)\,\partial_x r(t, x, \omega) = -M(x, \omega)\,r(t, x, \omega), \\ r_+(t, 0, \omega) = u_+(t), \\ r_-(t, L, \omega) = u_-(t) \end{cases} \tag{2.5}$$

almost surely in $\omega \in \mathbb{R}^d$, for $t \in (0,T)$ and $x \in (0, L)$ almost everywhere. Provided that $M(x, \omega) \in L^\infty((0, L) \times \mathbb{R}^d)$ we have that for $r^0 \in \left(L^2(0, L)\right)^{2N}$, the solution $r(t, x, \omega)$ fulfills the regularity (2.4).

2.1. Well-posedness of the initial-boundary value problem

In this section we study the well-posedness of the initial-boundary value problem (2.5). The proof relies on the method of characteristics, that are in this case in fact deterministic. Therefore, we obtain that the eigenvalues in the diagonal system matrix define two families of characteristics with (deterministic) slopes d_- and d_+, respectively. Define the sets

$$\Gamma_+ = \{0\} \times [0, L] \cup [0, T] \times \{0\},$$
$$\Gamma_- = \{0\} \times [0, L] \cup [0, T] \times \{L\}.$$

For $j \in \{1, ..., N\}$ we use the notation $d_\pm = (d_\pm^{(j)})_{j=1}^N$ with $d_\pm^{(j)} \in \mathbb{R}$. For $t \geq 0$ and the space variable $x \in [0, L]$ and $j \in \{1, ..., N\}$ we define the $\mathbb{R}^2$-valued function $\xi_\pm^{(j)}(s, x, t)$ as the solution of the initial value problem

$$\xi_\pm^{(j)}(t, x, t) = (t, x), \tag{2.6}$$
$$\partial_s \xi_\pm^{(j)}(s, x, t) = (1, d_\pm^{(j)}(\Pi_2 \xi_\pm^{(j)}(s, x, t)). \tag{2.7}$$

where Π_2 denoted the projection on the second component, that is for $(x_1, x_2) \in \mathbb{R}^2$ we have $\Pi_2(x_1, x_2) = x_2$. This implies that for all $j \in \{1, ..., N\}$ we have

$$\xi_+^{(j)}(s, x, t) = (s, x + \int_t^s d_+^{(j)}(\Pi_2 \xi_+^{(j)}(\tau, x, t)) \, \mathrm{d}\tau), \tag{2.8}$$

$$\xi_-^{(j)}(s, x, t) = (s, x - \int_t^s -d_-^{(j)}(\Pi_2 \xi_-^{(j)}(\tau, x, t)) \, \mathrm{d}\tau)). \tag{2.9}$$

Define the points

$$P_0^{(j)\pm}(t, x) = \Gamma_\pm \cap \{\xi_\pm^{(j)}(s, x, t), \, s \in \mathbb{R}\} \in \mathbb{R}^2.$$

For the t-component of $P_0^{(j)\pm}(t, x)$ we use the notation $t_0^{(j)\pm}(t, x) \geq 0$. Note again, that the solutions $\xi_\pm^{(j)}$ and $P_0^{(j)\pm}$ are independent of ω. For each $\omega \in \mathbb{R}^d$, the solution of (2.5) can be defined by rewriting the partial differential equation in the form of integral equations along the characteristic curves given by the functions $\xi_\pm^{(j)}(\cdot, x, t)$: For all $j \in \{1, ..., N\}$ for the j-th component of $r_\pm$ we define

$$r_\pm^{(j)}(t, x, \omega) = r_\pm^{(j)}(P_0^{(j)\pm}(t, x), \omega) - \int_{t_0^{(j)\pm}(t, x)}^t \Pi_{(j)\pm} M(x, \omega) \, r(\xi_\pm^{(j)}(s, x, t), \omega) \, \mathrm{d}s \tag{2.10}$$

where $\Pi_{(j)\pm}$ denotes the projection on the j-th $\pm$-components, that is for $(x_+, x_-) \in \mathbb{R}^{2N}$, it holds $\Pi_{(j)\pm}(x_+, x_-) = x_\pm^{(j)}$.

Note that almost everywhere the values of $r_\pm^{(j)}(P_0^{(j)\pm}(t, x), \omega)$ are given on $\Gamma_\pm$ either by the initial data, that is $\Pi_{(j)\pm} r^0$ (if the t-component of $P_0^{(j)\pm}(t, x)$ that is $t_0^{(j)\pm}(t, x)$ is zero), or the boundary data, that is u_+ at $x = 0$ or u_- at $x = L$. Here, the case that the x-component of $P_0^{(j)\pm}(t, x)$ is zero occurs only for the components of r_+ and the case that the x-component of $P_0^{(j)\pm}(t, x)$ is L only for components of r_-.

For sufficiently large finite time T, the characteristic curves starting at $t = 0$ reach a point at the terminal time T after a finite number of reflections at the boundaries $x = L$ or $x = 0$. For the deterministic case, the definition of the solutions of hyperbolic boundary value problems based on the integral equations (2.10) is presented for example in [4]. This can be adapted to the present case and we have the following result.

Theorem 2.1. *Let $T > 0$ be given. Assume that the entries of $M(x,\omega)$ are in $L^\infty((0, L) \times \mathbb{R}^d)$. Then for initial data $r_+^0, \, r_-^0 \in (L^2(0, L))^N$ and control functions $u_+, \, u_- \in (L^2(0, T))^N$ there exists a unique solution of (2.5) that satisfies the integral equations (2.10) along the characteristic curves with $r_+, \, r_- \in C\left([0, T], L^2_{1\times\rho_\omega}((0, L) \times \mathbb{R}^d; \mathbb{R}^N)\right)$ and the boundary conditions at $x = 0$ and $x = L$ almost everywhere in $[0, T]$.*

Proof. The proof is based upon Banach's fixed point theorem with the canonical fixed point iteration. It has to be shown that this map is a contraction in the Banach space

$$X = C\left([0, T], L^2_{1\times\rho_\omega}((0, L) \times \mathbb{R}^d; \mathbb{R}^{2N})\right).$$

For $(r_+, r_-) \in X$ and $j \in \{1, ..., N\}$ define the iteration map as

$$\Phi_\pm^{(j)}(r_+, r_-)(t, x, \omega) = r_\pm^{(j)}(P_0^{(j)\pm}(t, x), \omega) - \int_{t_0^{(j)\pm}(t, x)}^t \Pi_\pm^{(j)} M(x, \omega) \, r(\xi_\pm^{(j)}(s, x, t), \omega) \, \mathrm{d}s$$

where the values of $r_\pm^{(j)}(P_0^{(j)\pm}(t,\,x),\,\omega)$ are prescribed by the initial conditions and boundary conditions in (2.5). Then for $t_1,\,t_2\in[0,T]$, $t_1<t_2$ we have

$$\left(\int_{\mathbb{R}^d}\int_0^L \left|\Phi_\pm^{(j)}(r_+,\,r_-)(t_2,\,x,\,\omega)-\Phi_\pm^{(j)}(r_+,\,r_-)(t_1,\,x,\,\omega)\right|^2\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega\right)^{1/2} \tag{2.11}$$

$$\leq\left(\int_{\mathbb{R}^d}\int_0^L \left|r_\pm(P_0^{(j)\pm}(t_2,\,x),\,\omega)-r_\pm(P_0^{(j)\pm}(t_1,\,x),\,\omega)\right|^2\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega\right)^{1/2} \tag{2.12}$$

$$+\left(\int_{\mathbb{R}^d}\int_0^L \left|\int_{t_1}^{t_2}\Pi_\pm^{(j)}M(x,\,\omega)\,\left(r(\xi_\pm^{(j)}(s,\,x,\,t_2),\,\omega)-r(\xi_\pm^{(j)}(s,\,x,\,t_1),\,\omega)\right)\mathrm{d}s\right|^2\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega\right)^{1/2} \tag{2.13}$$

$$=:\Xi_1(t_1,\,t_2)+\Xi_2(t_1,\,t_2). \tag{2.14}$$

Since $(r_+,r_-)\in X$ and $P_0^{(j)\pm}(\cdot,\,x)$ depends Lipschitz-continuously on t we have $\lim_{t_2\to t_1}\Xi_1(t_1,\,t_2)=0$. Moreover,

$$\Xi_2(t_1,\,t_2)\leq\|M\|_\infty\left(\int_{\mathbb{R}^d}\int_0^L\left(\int_{t_1}^{t_2}\left|r(\xi_\pm^{(j)}(s,\,x,\,t_2),\,\omega)-r(\xi_\pm^{(j)}(s,\,x,\,t_1),\,\omega)\right|\,\mathrm{d}s\right)^2\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega\right)^{1/2}$$

$$\leq\sqrt{|t_2-t_1|}\,\|M\|_\infty^2\left(\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\int_0^L\left|r(\xi_\pm^{(j)}(s,\,x,\,t_2),\,\omega)-r(\xi_\pm^{(j)}(s,\,x,\,t_1),\,\omega)\right|^2\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega\,\mathrm{d}s\right)^{1/2}.$$

Since $(r_+,r_-)\in X$ this yields $\lim_{t_2\to t_1}\Xi_2(t_1,\,t_2)=0$. Hence we have $(\Phi_+(r_+,\,r_-),\,\Phi_-(r_+,\,r_-))\in X$. For (r_+,r_-), $(q_+,q_-)\in X$ and $j\in\{1,...,N\}$ and $t\in[0,\,T]$ we have

$$\left(\int_{\mathbb{R}^d}\int_0^L\left|\Phi_\pm^{(j)}(r_+,\,r_-)(t,\,x,\,\omega)-\Phi_\pm^{(j)}(q_+,\,q_-)(t,\,x,\,\omega)\right|^2\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega\right)^{1/2}$$

$$=\left(\int_{\mathbb{R}^d}\int_0^L\left|\int_{t_0^{(j)\pm}(t,\,x)}^{t}\Pi_\pm^{(j)}M(x,\,\omega)\,(r-q)(\xi_+^{(j)}(s,\,x,\,t),\,\omega)\,\mathrm{d}s\right|^2\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega\right)^{1/2}$$

$$\leq\|M\|_\infty\sqrt{T}\max_{s\in[0,\,T]}\left(\int_{\mathbb{R}^d}\int_0^L\left|(r-q)(\xi_\pm^{(j)}(s,\,x,\,t),\,\omega)\right|_{\mathbb{R}^N}^2\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega\right)^{1/2}.$$

Since D is continuously differentiable, the change of variables formula holds true under minimal assumptions (see [34]). This implies that the integral term on the right-hand side can be bounded by the norm of $(r-q)(s,\cdot,\cdot)$ in $L^2_{1\times\rho_\omega}((0,\,L)\times\mathbb{R}^d;\mathbb{R}^{2N})$. Hence if $\|M\|_\infty\sqrt{T}$ is sufficiently small, the map Φ is a contraction. This yields the existence of the solution on a time interval $[0,\,t_0]$ if t_0 is sufficiently small. By concatenation of a finite number of solutions on time intervals of the length t_0 this yields the solution on an arbitrarily large finite time interval $[0,\,T]$. $\square$

2.2. Dynamic optimal control problem

In this section, we define the problem of dynamic optimal Dirichlet boundary control for the hyperbolic system (2.5) with random coefficients.

Let a continuously differentiable convex function $f : \mathbb{R}^{4N} \to [0, \infty)$ be given. Note that f contains the state r and the control u, *i.e.*, $f = f(u_+, u_-, r_+, r_-)$ and is therefore $4N-$dimensional function. For the partial derivatives of f with respect to the two controls component we use the notation $f_u = (f_{u_+},\, f_{u_-})$ and for the partial derivatives with respect to the two components of the state we use the notation $f_r = (f_{r_+},\, f_{r_-})$. We assume that f satisfies the following condition: There exists a number $\kappa > 0$ such that for all $u_1,\, u_2,\, r_1,\, r_2 \in \mathbb{R}^{2N}$ we have

$$[f_u(u_1,\, r_1) - f_u(u_2,\, r_2)]^\top (u_1 - u_2) + [f_r(u_1,\, r_1) - f_r(u_2,\, r_2)]^\top (r_1 - r_2) \geq \kappa \, \|u_1 - u_2\|^2_{\mathbb{R}^{2N}}. \tag{2.15}$$

This condition implies that f is a convex function. Moreover, it implies that for all $r \in \mathbb{R}^{2N}$ the function $f(\cdot, r)$ is strongly convex (see *e.g.* [39], Cor. A1.32).

Assume that for all $u \in \left(L^2(0,\, T)\right)^{2N}$, $R \in L^2_{1\times\rho_\omega}((0,T) \times \mathbb{R}^d; \mathbb{R}^{2N})$ we have $f(u(\cdot),\, R(\cdot)) \in L^1(0,\, T) \times L^1_{\rho_\omega}(\mathbb{R}^d)$. Here, R is defined as the trace of r at the boundary $x \in \{0, L\}$, where $r = (r_+,\, r_-)$ is given by equation (2.5) with regularity (2.4).

We define the objective function

$$J_T(u) = \int_{\mathbb{R}^d} \int_0^T f(u_+(\tau),\, u_-(\tau),\, r_+(\tau,\, L, \omega),\, r_-(\tau,\, 0, \omega)) \,\mathrm{d}\tau \, \rho(\omega) \,\mathrm{d}\omega. \tag{2.16}$$

The objective function is the expected value of f. Inequality (2.15) holds for example if $f(u,\, r) = g_1(u) + g_2(r)$ with a strongly convex function g_1 and a convex function g_2. An example is the squared Euclidean norm

$$f(u, r) = \|u\|^2_{\mathbb{R}^{2N}} + \|r\|^2_{\mathbb{R}^{2N}}$$

or a general linear quadratic cost

$$f(u,\, r) = (u_+,\, u_-,\, r_+,\, r_-)^\top \, A \, (u_+,\, u_-,\, r_+,\, r_-) + v^\top \, (u_+,\, u_-,\, r_+,\, r_-) \tag{2.17}$$

with a symmetric positive definite matrix A and a vector $v \in \mathbb{R}^{4N}$.

In the sequel we use the notation

$$H(T) = \left(L^2(0,\, T)\right)^{2N}, \; H_r(T) = L^2_{1\times\rho_\omega}\left((0,\, T) \times \mathbb{R}^d; \mathbb{R}^{2N}\right) \tag{2.18}$$

with the scalar product $\langle u,\, v\rangle_{H_r(T)} = \int_{\mathbb{R}^d} \int_0^T u^\top(\tau,\, \omega)\, v(\tau,\, \omega) \,\mathrm{d}\tau \rho_\omega(\omega) \,\mathrm{d}\omega$ and the corresponding norm

$$\|r\|_{H_r(T)} = \left(\int_{\mathbb{R}^d} \int_0^T \|r(\tau,\, \omega)\|^2_{\mathbb{R}^2} \,\mathrm{d}\tau \rho_\omega(\omega) \,\mathrm{d}\omega \right)^{1/2}.$$

For $r = (r_+, r_-) \in \mathbb{R}^{2N}$, we use the notation $\|r\|_{\mathbb{R}^{2N}} = \sqrt{\sum_{k=1}^N (r_{+,k})^2 + (r_{-,k})^2}$.

Under assumption (2.15) and since the control to state mapping is linear, the objective J_T is strongly convex in the sense that there exists a constant $\kappa > 0$ such that for all $T > 0$ and all $u,\, v \in H(T)$ we have

$$\langle J_T'(u) - J_T'(v),\, u - v\rangle_{H(T)} \geq \kappa \|u - v\|^2_{H(T)}. \tag{2.19}$$

We consider the dynamic optimal control problem

$$\min_{u \in H(T)} J_T(u). \tag{2.20}$$

Due to the growth condition (2.15) the existence of an optimal control follows with the Direct Method of the Calculus of Variations. Before stating the optimality conditions we proof properties of the solution operators.

2.2.1. Adjoint operator for the system (2.5)

For the analysis of the boundary control problem (2.5) and (2.20), the study of adjoint operators is essential. We start with the definition of the operator that maps the control and the initial state to the boundary traces of the generated states.

For a given time $T > 0$, we define the operator $F_T\,(u,\, r^0)$ that maps the initial state $r^0 \in (L^2(0,\, L))^N$ and the boundary control $u = (u_+(\cdot),\, u_-(\cdot)) \in H(T)$ to the boundary trace

$$(r_+(\cdot,\, L,\, \omega),\, r_-(\cdot,\, 0,\, \omega)) \in H_r(T)$$

of the solution of the linear initial boundary value problem (2.5). Note that for simplicity we embed the space $H(T)$ in the space $H_r(T)$ by the identity. We therefore consider F_T as an operator

$$F_T : H_r(T) \times L^2((0, L); \mathbb{R}^{2N}) \to H_r(T).$$

Thus we have

$$F_T \begin{pmatrix} u_+(\cdot) \\ u_-(\cdot) \\ r^0_+(\cdot) \\ r^0_-(\cdot) \end{pmatrix} = \begin{pmatrix} r_+(\cdot, L, \omega) \\ r_-(\cdot,\, 0, \omega) \end{pmatrix}. \tag{2.21}$$

Lemma 2.2 states that the operator norm of F_T is uniformly bounded with respect to T. We use the following notation: For $v \in \mathbb{R}^N$, let

$$[v]_{\min} = \min_{j \in \{1,\dots,N\}} |v_j| \text{ and } \quad \|v\|_\infty = \max_{j \in \{1,\dots,N\}} |v_j|.$$

Similar to [27] in the sequel we assume that

$$|D(x)|\, M(x,\omega) + M(x,\omega)\, |D(x)| \geq 0 \tag{2.22}$$

holds true a.s. in ω and for all $x \in [0, L]$.

Lemma 2.2. *Assume that* (2.22) *holds. The operator F_T is uniformly bounded with respect to T as an operator from the Hilbert space $H_r(T) \times (L^2(0, L))^{2N}$ to $H_r(T)$. For the corresponding operator norm of F_T for all $T > 0$ we have*

$$\|F_T\| \leq \max\left\{1,\ \sup_{x \in [0,\, L]} \left\{ \frac{1}{\sqrt{[d_+(x)]_{\min}}},\ \frac{1}{\sqrt{[d_-(x)]_{\min}}} \right\}\right\}. \tag{2.23}$$

Moreover, we have

$$\|F_T(u,\, r^0)\|^2_{H_r(T)} \leq \max_{x \in [0,\, L]} \left\{ \frac{1}{[d_+(x)]_{\min}},\ \frac{1}{[d_-(x)]_{\min}} \right\} \|r^0\|^2_{(L^2(0,\, L))^{2N}} + \|u\|^2_{H_r(T)}. \tag{2.24}$$

We define the operator $\tilde{F}_T\,(u) = F_T\,(u,\,0)$ as operator $H_r(T)$ to $H_r(T)$ and for the corresponding operator norm we obtain $\|\tilde{F}_T\| \leq 1$.

Proof. Let $(u_+, u_-) \in (L^2(0,T))^{2N} \subset H_r(T)$ be given. Let r_+, r_- with regularity (2.4) denote the generated solution of (2.5). We have

$$\begin{aligned}
&\|u\|^2_{H_r(T)} - \|F_T(u,\, r^0)\|^2_{H_r(T)}\\
&= \int_{\mathbb{R}^d} \int_0^T \|r_+(t,0,\,\omega)\|^2 + \|r_-(t,L,\,\omega)\|^2 - \|r_+(t,L,\,\omega)\|^2 - \|r_-(t,0,\,\omega)\|^2 \,\mathrm{d}t\, \rho_\omega(\omega)\,\mathrm{d}\omega\\
&= -\int_{\mathbb{R}^d} \int_0^T \int_0^L \left(\|r_+(t,x,\,\omega)\|^2\right)_x - \left(\|r_-(t,x,\,\omega)\|^2\right)_x \,\mathrm{d}x\,\mathrm{d}t\, \rho_\omega(\omega)\,\mathrm{d}\omega\\
&= -\int_{\mathbb{R}^d} \int_0^T \int_0^L 2\, r^\top\, |D|^{-1}\, D\, r_x \,\mathrm{d}x\,\mathrm{d}t\, \rho_\omega(\omega)\,\mathrm{d}\omega.
\end{aligned}$$

Due to the PDE in (2.5) we have $r^\top\,|D|^{-1}\,D\,r_x = -r^\top|D|^{-1}\,r_t - r^\top\,|D|^{-1}\,M\,r$, hence

$$\|u\|^2_{H_r(T)} - \|F_T(u,\, r^0)\|^2_{H_r(T)} = \int_{\mathbb{R}^d} \int_0^T \int_0^L 2\, r^\top |D|^{-1}\, r_t + 2\, r^\top\, |D|^{-1}\, M\, r \,\mathrm{d}x\,\mathrm{d}t\, \rho_\omega(\omega)\,\mathrm{d}\omega.$$

Therefore we have

$$\begin{aligned}
&\|u\|^2_{H_r(T)} - \|F_T(u,\, r^0)\|^2_{H_r(T)}\\
&= \int_{\mathbb{R}^d} \int_0^L \int_0^T \left(r^\top |D|^{-1}\, r\right)_t + 2\, r^\top\, |D|^{-1} M\, r \,\mathrm{d}t\,\mathrm{d}x\, \rho_\omega(\omega)\,\mathrm{d}\omega\\
&= \int_{\mathbb{R}^d} \int_0^L \left(r^\top |D|^{-1}\, r\right)\Big|_{t=0}^T + \int_0^T 2\, r^\top\, |D|^{-1} M\, r \,\mathrm{d}t\,\mathrm{d}x\, \rho_\omega(\omega)\,\mathrm{d}\omega\\
&= \int_{\mathbb{R}^d} \int_0^L r^\top(T,x,\omega) |D(x)|^{-1}\, r(T,x,\omega) - (r^0(x))^\top |D(x)|^{-1}\, r^0(x) \,\mathrm{d}x\, \rho_\omega(\omega)\,\mathrm{d}\omega\\
&\quad + \int_{\mathbb{R}^d} \int_0^L \int_0^T 2\, r^\top(t,x,\omega)\, |D(x)|^{-1} M(x,\omega)\, r(t,x,\omega) \,\mathrm{d}t\,\mathrm{d}x\, \rho_\omega(\omega)\,\mathrm{d}\omega.
\end{aligned}$$

Since $r^\top(T,x,\omega)|D(x)|^{-1}\,r(T,x,\omega) \geq 0$ and $2\,r^\top|D|^{-1}Mr = r^\top(|D|^{-1}M + M^\top|D|^{-1})r$ due to inequality (2.22) this yields

$$\begin{aligned}
&\|u\|^2_{H_r(T)} - \|F_T(u,\, r^0)\|^2_{H_r(T)}\\
&\geq -\int_{\mathbb{R}^d} \int_0^L (r^0)^\top(x) |D(x)|^{-1}\, r^0(x) \,\mathrm{d}x\, \rho_\omega(\omega)\,\mathrm{d}\omega\\
&\quad + \int_{\mathbb{R}^d} \int_0^L \int_0^T r^\top(t,x,\omega)\, \left[|D(x)|^{-1}\, M(x,\omega) + M(x,\,\omega)|D(x)|^{-1}\right] r(t,x,\omega)\, \,\mathrm{d}t\,\mathrm{d}x\, \rho_\omega(\omega)\,\mathrm{d}\omega\\
&\geq -\sup_{x\in[0,\,L]} \left\{\tfrac{1}{[d_+(x)]_{\min}},\, \tfrac{1}{[d_-(x)]_{\min}}\right\} \int_0^L (r^0(x))^\top\, r^0(x)\,\mathrm{d}x.
\end{aligned}$$

Hence (2.23) and (2.24) follow. $\square$

We are interested in the adjoint operator $F_T^* : H_r(T) \to H_r(T) \times L^2((0,L);\mathbb{R}^{2N})$ that satisfies for all $z_T \in H_r(T) = \mathcal{D}(F_T^*)$ the equation

$$\langle F_T\,(u,\, r^0),\, z_T\rangle_{H_r(T)} \tag{2.25}$$

$$= \int_{\mathbb{R}^d} \left[\int_0^T \langle u(t), \begin{pmatrix} F_T^*(z_T)(t,\omega)_1 \\ F_T^*(z_T)(t,\omega)_2 \end{pmatrix} \rangle_{\mathbb{R}^{2N}}\, \mathrm{d}t + \int_0^L \langle r^0(x), \begin{pmatrix} F_T^*(z_T)(x,\omega)_3 \\ F_T^*(z_T)(x,\omega)_4 \end{pmatrix} \rangle_{\mathbb{R}^{2N}}\, \mathrm{d}x \right] \rho_\omega(\omega)\, \mathrm{d}\omega$$

where $\langle \cdot\,,\,\cdot \rangle_{\mathbb{R}^{2N}}$ denotes the usual scalar product in $\mathbb{R}^{2N}$. Due to (2.23) we have the inequality

$$\|F_T^*\| \le \max\left\{1,\ \sup_{x\in[0,\,L]} \left\{ \frac{1}{\sqrt{[d_+(x)]_{\min}}},\ \frac{1}{\sqrt{[d_-(x)]_{\min}}} \right\}\right\}. \tag{2.26}$$

Similar as in [3], we determine an explicit representation of F_T^* in the following lemma.

For two vectors $d_\pm,\, z_\pm \in \mathbb{R}^N$ we use the notation $(d_\pm z_\pm)$ for the vector where the corresponding components of $d_\pm$ and $z_\pm$ are multiplied, that is

$$\begin{pmatrix} d_\pm^{(1)} \\ d_\pm^{(2)} \\ \vdots \\ d_\pm^{(N)} \end{pmatrix} \begin{pmatrix} z_\pm^{(1)} \\ z_\pm^{(2)} \\ \vdots \\ z_\pm^{(N)} \end{pmatrix} = \begin{pmatrix} d_\pm^{(1)} z_\pm^{(1)} \\ d_\pm^{(2)} z_\pm^{(2)} \\ \vdots \\ d_\pm^{(N)} z_\pm^{(N)} \end{pmatrix} \in \mathbb{R}^N. \tag{2.27}$$

Lemma 2.3. *For $z_T = (z_+^T,\, z_-^T) \in \mathcal{D}(F_T^*) = H_r(T)$, define $z = (z_+, z_-)$ with regularity (2.4) as the solution of the adjoint system*

$$\begin{cases} z(T,\, x,\, \omega) = 0,\ x \in (0, L), \\ z_t(t,\, x,\, \omega) + D(x)\, z_x(t,\, x,\, \omega) = [M(x,\, \omega) - D'(x)]\ z(t,\, x,\, \omega), \\ (d_+(L) z_+(t,\, L,\, \omega)) = z_+^T(t,\, \omega), \\ (|d_-(0)|\, z_-(t,\, 0,\, \omega)) = z_-^T(t,\, \omega). \end{cases} \tag{2.28}$$

Then we have

$$F_T^* \begin{pmatrix} z_+^T \\ z_-^T \end{pmatrix} = \begin{pmatrix} (d_+(0)\, z_+(\cdot,\, 0,\, \omega)) \\ (|d_-(L)|\, z_-(\cdot,\, L,\, \omega)) \\ \int_{\mathbb{R}^d} z_+(0,\, \cdot,\, \omega) \rho_\omega(\omega) \mathrm{d}\omega \\ \int_{\mathbb{R}^d} z_-(0,\, \cdot,\, \omega) \rho_\omega(\omega) \mathrm{d}\omega \end{pmatrix}. \tag{2.29}$$

Proof. Let r denote the solution of (2.5). Using integration by parts we obtain the equation

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} \int_0^T \int_0^L \langle z,\, r_t + D\, r_x + M\, r\rangle_{\mathbb{R}^{2N}}\, \mathrm{d}x\, \mathrm{d}t\, \rho_\omega(\omega)\, \mathrm{d}\omega \\ &= \int_{\mathbb{R}^d} \left[\int_0^T \int_0^L -r^\top z_t\, \mathrm{d}x\, \mathrm{d}t + \int_0^L \left[r(t,\, x)^\top z(t,\, x) \right]\Big|_{t=0}^T \mathrm{d}x \right] \rho_\omega(\omega)\, \mathrm{d}\omega \\ &+ \int_{\mathbb{R}^d} \left[\int_0^T \int_0^L -r^\top D\, z_x\, \mathrm{d}x\, \mathrm{d}t - \int_0^T \int_0^L r^\top D'\, z\, \mathrm{d}x\, \mathrm{d}t \right] \rho_\omega(\omega)\, \mathrm{d}\omega \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^d}\int_0^T \left[r(t,\,x)^\top D(x)\,z(t,\,x)\right]\Big|_{x=0}^{L}\,\mathrm{d}t\,\rho_\omega(\omega)\,\mathrm{d}\omega\\
&+\int_{\mathbb{R}^d}\int_0^T\int_0^L r^\top M\,z\,\mathrm{d}x\,\mathrm{d}t\,\rho_\omega(\omega)\,\mathrm{d}\omega\\
=&\int_{\mathbb{R}^d}\int_0^T\int_0^L -r^\top\left[z_t+D\,z_x+D'\,z-M\,z\right]\mathrm{d}x\,\mathrm{d}t\,\rho_\omega(\omega)\,\mathrm{d}\omega\\
&+\int_{\mathbb{R}^d}\left[\int_0^T \left[r(t,\,x,\,\omega)^\top D(x)\,z(t,\,x,\,\omega)\right]\Big|_{x=0}^{L}\,\mathrm{d}t-\int_0^L r^0(x)^\top z(0,\,x,\,\omega)\,\mathrm{d}x\right]\rho_\omega(\omega)\,\mathrm{d}\omega.
\end{aligned}
$$

Due to (2.28) we obtain

$$
\begin{aligned}
0=&\int_{\mathbb{R}^d}\left[\int_0^T \left[r(t,\,x,\,\omega)^\top D(x)\,z(t,\,x,\,\omega)\right]\Big|_{x=0}^{L}\,\mathrm{d}t-\int_0^L r^0(x)^\top z(0,\,x,\,\omega)\,\mathrm{d}x\right]\rho_\omega(\omega)\,\mathrm{d}\omega\\
=&\int_{\mathbb{R}^d}\left[\int_0^T \left(d_+\,z_+(t,\,x,\,\omega)\right)^\top r_+(t,\,x,\,\omega)\Big|_{x=0}^{L}\,\mathrm{d}t+\int_0^T \left(d_-\,z_-(t,\,x,\,\omega)\right)^\top r_-(t,\,x,\,\omega)\Big|_{x=0}^{L}\,\mathrm{d}t\right]\rho_\omega(\omega)\,\mathrm{d}\omega\\
&-\int_{\mathbb{R}^d}\int_0^L r^0(x)^\top z(0,\,x,\,\omega)\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega.
\end{aligned}
$$

Thus we have

$$
\int_{\mathbb{R}^d}\int_0^T \left(d_+(0)\,z_+(t,\,0,\,\omega)\right)^\top r_+(t,\,0,\,\omega)+\left(d_-(0)\,z_-(t,\,0,\,\omega)\right)^\top r_-(t,\,0,\,\omega)\,\mathrm{d}t\,\rho_\omega(\omega)\,\mathrm{d}\omega \tag{2.30}
$$

$$
\begin{aligned}
=&\int_{\mathbb{R}^d}\int_0^T \left(d_+(L)\,z_+(t,\,L,\,\omega)\right)^\top r_+(t,\,L,\,\omega)+\left(d_-(L)\,z_-(t,\,L,\,\omega)\right)^\top r_-(t,\,L,\,\omega)\,\mathrm{d}t\,\rho_\omega(\omega)\,\mathrm{d}\omega\\
&-\int_{\mathbb{R}^d}\int_0^L r^0(x)^\top z(0,\,x,\,\omega)\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega.
\end{aligned}
$$

Due to the definition of F_T and (2.28) this implies the equation

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_0^T \langle F_T\,(u,\,r^0)(t),\,z_T(t,\,\omega)\rangle_{\mathbb{R}^{2N}}\,\mathrm{d}t\,\rho_\omega(\omega)\,\mathrm{d}\omega\\
=&\int_{\mathbb{R}^d}\int_0^T r_+(t,\,L,\,\omega)^\top z_+^T(t,\,\omega)+r_-(t,\,0,\,\omega)^\top z_-^T(t,\,\omega)\,\mathrm{d}t\,\rho_\omega(\omega)\,\mathrm{d}\omega\\
=&\int_{\mathbb{R}^d}\int_0^T \left(d_+(L)\,r_+(t,\,L,\,\omega)\right)^\top z_+(t,\,L,\,\omega)+\left(|d_-(0)|\,r_-(t,\,0,\,\omega)\right)^\top z_-(t,\,0,\,\omega)\,\mathrm{d}t\,\rho_\omega(\omega)\,\mathrm{d}\omega\\
=&\int_{\mathbb{R}^d}\int_0^T \left(d_+(0)\,r_+(t,\,0,\,\omega)\right)^\top z_+(t,\,0,\,\omega)+\left(|d_-(L)|\,r_-(t,\,L,\,\omega)\right)^\top z_-(t,\,L,\,\omega)\,\mathrm{d}t\,\rho_\omega(\omega)\,\mathrm{d}\omega\\
&+\int_{\mathbb{R}^d}\int_0^L r^0(x)^\top z(0,\,x,\,\omega)\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega
\end{aligned}
$$

where the last equality follows from (2.30). With (2.5) this yields the equation

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \int_0^T \langle F_T\,(u,\, r^0)(t),\, z_T(t,\,\omega)\rangle_{\mathbb{R}^{2N}}\,\mathrm{d}t\,\rho_\omega(\omega)\,\mathrm{d}\omega \\
&= \int_{\mathbb{R}^d} \int_0^T u_+(t)^\top\,(d_+(0)\,z_+(t,\,0,\,\omega)) + u_-(t)^\top\,(|d_-(L)|\,z_-(t,\,L,\,\omega))\,\mathrm{d}t\,\rho_\omega(\omega)\,\mathrm{d}\omega \\
&\quad + \int_{\mathbb{R}^d} \int_0^L r^0(x)^\top\,z(0,\,x,\,\omega)\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega \\
&= \left\langle \begin{pmatrix} u \\ r^0 \end{pmatrix},\, F_T^*(z_T) \right\rangle_{H_r(T)\times L^2((0,L);\mathbb{R}^{2N})}.
\end{aligned}
$$

Therefore, (2.43) holds. □

The forward and the adjoint operator are both bounded uniformly in the time horizon T. This is important to show the turnpike property later.

2.2.2. Necessary optimality conditions for the dynamic problem

In this section we analyse at the necessary optimality conditions of the convex dynamic optimal control problem (2.20) to determine the structure of the optimal control. Since J_T is convex, it is subdifferentiable. The necessary optimality conditions of (2.20) are

$$
0 \in J_T'(u^{(\delta,\,T)}). \tag{2.31}
$$

for the optimal control $u^{(\delta,\,T)}$ and for the subdifferential J_T'.

We rewrite (2.16) using the operator F_T as follows: For all $u \in H(T)$ and r that satisfy (2.5), we have

$$
J_T(u) = \int_{\mathbb{R}^d} \int_0^T f(u(t),\, F_T\,(u,\, r^0))\,\mathrm{d}t\,\rho(\omega)\,\mathrm{d}\omega. \tag{2.32}
$$

Next, we characterize the subdifferential. Let $\tilde{u} = u + \delta^{(1)}$ with a control variation $\delta^{(1)} \in (L^2(0,\,T))^{2N}$ and let $\tilde{F}_T$ be as defined in Lemma 2.2. Then, for the corresponding adjoint operator, Lemma 2.3 implies also the existence of an operator

$$
\tilde{F}_T^* : H_r(T) \to H_r(T)
$$

such that

$$
\tilde{F}_T^* \begin{pmatrix} z_+^T(\cdot) \\ z_-^T(\cdot) \end{pmatrix} = \begin{pmatrix} (d_+(0)\,z_+(\cdot,\,0,\cdot)) \\ (|d_-(L)|\,z_-(\cdot,L,\cdot)) \end{pmatrix}.
$$

Note that since $\tilde{F}_T(u) = F_T(u, 0)$ we obtain as in equation (2.25)

$$
\langle \tilde{F}_T(u),\, z_T\rangle_{H_r(T)} = \int_{\mathbb{R}^d} \left[\int_0^T \langle u(t),\, \begin{pmatrix} \tilde{F}_T^*(z_T)(t,\omega)_1 \\ \tilde{F}_T^*(z_T)(t,\omega)_2 \end{pmatrix} \rangle_{\mathbb{R}^{2N}}\,\mathrm{d}t \right] \rho_\omega(\omega)\,\mathrm{d}\omega
$$

and hence $\tilde{F}_T^*$ is precisely given by the first two components of F_T^* given by equation (2.29).

Furthermore, since J_T is convex, we have

$$J_T(\tilde u) \geq J_T(u) + \int_{\mathbb{R}^d}\int_0^T f_u(u,\, F_T(u,\, r^0))\,\delta^{(1)} + f_R(u,\, F_T(u,\, r^0))\,\tilde F_T(\delta^{(1)})\,\mathrm{d}t\,\rho(\omega)\,\mathrm{d}\omega$$

$$= J_T(u) + \int_{\mathbb{R}^d}\int_0^T \left[f_u(u,\, F_T(u,\, r^0)) + \tilde F_T^* f_R(u,\, F_T(u,\, r^0))\right]\,\delta^{(1)}\,\mathrm{d}t\,\rho(\omega)\,\mathrm{d}\omega.$$

This implies that the subdifferential is given by

$$J_T'(u) = \int_{\mathbb{R}^d} f_u(u,\, F_T(u,\, r^0)) + \tilde F_T^* f_R(u,\, F_T(u,\, r^0))\,\rho(\omega)\,\mathrm{d}\omega.$$

Since $F_T, \tilde F_T^*$ are linear and bounded operators and f is continuously differentiable, we obtain that J_T is Frechet differentiable on $L^2(0, T; \mathbb{R}^{2N})$ and with a slight abuse of notation we denote the Frechet derivative also by J_T'.

Hence (2.31) yields the optimality conditions that are stated in the following lemma. Due to the convexity of the optimal control problem (2.20), they are necessary and sufficient.

Lemma 2.4. *The control $u^{(\delta,\,T)}$ is a solution of the dynamic optimal control problem (2.20) if and only if there exist a multiplier*

$$p^{(\delta,\,T)} = (p_+^{\delta,T}, p_-^{\delta,T}) \in H_r(T) \tag{2.33}$$

such that the optimality system

$$\int_{\mathbb{R}^d}\left(f_u(u^{(\delta,\,T)},\, F_T\,(u^{(\delta,\,T)},\, r^0)) + \begin{pmatrix} \left(d_+(0)\; p_+^{(\delta,T)}(\cdot,\, 0,\, \omega)\right) \\ \left(|d_-(L)|\; p_-^{(\delta,T)}(\cdot,\, L,\, \omega)\right) \end{pmatrix}\right)\rho(\omega)\,\mathrm{d}\omega = 0 \tag{2.34}$$

holds, that is for $(t,\, x) \in (0,\, T) \times (0,\, L)$ almost everywhere we have

$$\left\{\begin{array}{l} R^{(\delta,\,T)}(0,\, x) = r^0, \\ R_t^{(\delta,\,T)} + D\, R_x^{(\delta,\,T)} = -M\, R^{(\delta,\,T)}, \\ R_+^{(\delta,\,T)}(t,\, 0,\, \omega) = u_+^{(\delta,\,T)}(t), \\ R_-^{(\delta,\,T)}(t,\, L,\, \omega) = u_-^{(\delta,\,T)}(t), \\ p^{(\delta,\,T)}(T,\, x,\, \omega) = 0, \\ p_t^{(\delta,\,T)} + D\, p_x^{(\delta,\,T)} = [M - D']\; p^{(\delta,\,T)}, \\ \left(d_+(L)\, p_+^{(\delta,\,T)}(t,\, L,\, \omega)\right) = f_{R_+}(u^{(\delta,\,T)}(t),\, R_+^{(\delta,\,T)}(t, L, \omega), R_-^{(\delta,\,T)}(t, 0, \omega)), \\ \left(|d_-(0)| p_-^{(\delta,\,T)}(t,\, 0,\, \omega)\right) = f_{R_-}(u^{(\delta,\,T)}(t),\, R_+^{(\delta,\,T)}(t, L, \omega), R_-^{(\delta,\,T)}(t, 0, \omega)) \end{array}\right. \tag{2.35}$$

and for $R^{(\delta,\,T)}(t, \cdot, \omega) = (R_+^{(\delta,\,T)}(t, L, \omega), R_-^{(\delta,\,T)}(t, 0, \omega))$

$$\left\{\begin{array}{l} \int_{\mathbb{R}^d}\left(f_{u_+}(u^{(\delta,\,T)}(t),\, R^{(\delta,\,T)}(t, \cdot, \omega)) + \left(d_+(0)\, p_+^{(\delta,\,T)}(t,\, 0,\, \omega)\right)\right)\rho(\omega)\,\mathrm{d}\omega = 0, \\ \int_{\mathbb{R}^d}\left(f_{u_-}(u^{(\delta,\,T)}(t),\, R^{(\delta,\,T)}(t, \cdot, \omega)) + \left(|d_-(L)|\, p_-^{(\delta,\,T)}(t,\, L,\, \omega)\right)\right)\rho(\omega)\,\mathrm{d}\omega = 0. \end{array}\right. \tag{2.36}$$

We comment on the proof the previous Lemma. Since J_T' is differentiable, a necessary condition for $u^{(\delta,T)}$ to be optimal, is given $0 = J_T'(u^{(\delta,T)})$ which reads for all control variations $\delta^{(1)} \in (L^2(0,T))^{2N}$

$$0 = \int_0^T \int_{\mathbb{R}^d} f_u(u^{(\delta,T)},\, F_T(u^{(\delta,T)},\, r^0)) + \tilde{F}_T^* f_R(u^{(\delta,T)},\, F_T(u^{(\delta,T)},\, r^0))\, \delta^{(1)} \mathrm{d}t \rho(\omega)\, \mathrm{d}\omega.$$

Denote by $p^{(\delta,T)} = \tilde{F}_T^*(f_R(u^{(\delta,T)},\, F_T(u^{(\delta,T)},\, r^0))$ the adjoint state, where $\tilde{F}_T^* = ((F_T^*)_1, (F_T^*)_2)$ and F_T^* is given by Lemma 2.3. This implies that

$$\int_{\mathbb{R}^d} \Big(f_{u_+}(u^{(\delta,\,T)}(t),\, R^{(\delta,\,T)}(t,\cdot,\omega)) + \Big(d_+(0)\, p_+^{(\delta,\,T)}(t,\,0,\,\omega) \Big) \Big)\, \rho(\omega)\, \mathrm{d}\omega = 0,$$

and

$$\int_{\mathbb{R}^d} \Big(f_{u_-}(u^{(\delta,\,T)}(t),\, R^{(\delta,\,T)}(t,\cdot,\omega)) + \Big(|d_-(L)|\, p_-^{(\delta,\,T)}(t,\,L,\,\omega) \Big) \Big)\, \rho(\omega)\, \mathrm{d}\omega = 0$$

holds true. Since $R^{(\delta,T)} = F_T(u^{(\delta,T)}, r^0)$ the assertion follows.

2.3. Static optimal control problem

In this section we consider the static optimal control problem corresponding to problem (2.20). For this purpose define the space

$$\tilde{H}_r := L^2_{\rho_\omega}(\mathbb{R}^d; \mathbb{R}^{2N}). \tag{2.37}$$

The state $R^{(\sigma)}$ is a function of x and the parametric uncertainty ω and we have

$$R^{(\sigma)}(x,\omega) \in L^2_{1\times\rho_\omega}((0,\,L) \times \mathbb{R}^d; \mathbb{R}^{2N}). \tag{2.38}$$

The state $R^{(\sigma)}$ is the unique solution to the linear system of ordinary differential equations

$$\begin{cases} D(x)\, R_x^{(\sigma)}(x,\,\omega) = -M(x,\,\omega)\, R^{(\sigma)}(x,\,\omega), \\ R_+^{(\sigma)}(0,\,\omega) = u_+^{(\sigma)}, \\ R_-^{(\sigma)}(L,\,\omega) = u_-^{(\sigma)} \end{cases} \tag{2.39}$$

with $x \in (0,\,L)$ and for some $u^{(\sigma)} = \begin{pmatrix} u_+^{(\sigma)} \\ u_-^{(\sigma)} \end{pmatrix} \in \mathbb{R}^{2N}$. Note that we will later also embed $u^{(\sigma)} \in \tilde{H}_r$. To be more precise, note that $R^{(\sigma)}$ has higher regularity as function of x while the regularity with respect to ω cannot be improved. It is possible to write the explicit solution $R^{(\sigma)}$ but for the analysis of the adjoint operator the form (2.39) is more advantageous.

The static optimization problem is then given by

$$\begin{cases} \min\limits_{u^{(\sigma)} \in \mathbb{R}^{2N}} \int_{\mathbb{R}^d} f(u^{(\sigma)},\, R_+^{(\sigma)}(L,\,\omega),\, R_-^{(\sigma)}(0,\,\omega))\, \rho_\omega(\omega)\, \mathrm{d}\omega \\ \text{subject to (2.39).} \end{cases} \tag{2.40}$$

where we assume as above that f fulfills condition (2.15). Since $R^{(\sigma)}$ depends linearly on $u^{(\sigma)}$ the problem (2.40) is a strictly convex finite–dimensional optimization problem. Hence, it posses a unique solution $u^{(\sigma)}$.

2.3.1. Adjoint operator for the static problem

We define the static operator $F_{(\sigma)} : \tilde{H}_r \to \tilde{H}_r$ that maps the boundary control $u^{(\sigma)} = (u_+^{(\sigma)},\, u_-^{(\sigma)}) \in \mathbb{R}^{2N} \subset \tilde{H}_r$ to the point $(R_+^{(\sigma)}(L,\,\omega),\, R_-^{(\sigma)}(0,\,\omega))$, where $R^{(\sigma)}$ solves (2.39). Thus we have

$$F_{(\sigma)} \begin{pmatrix} u_+^{(\sigma)} \\ u_-^{(\sigma)} \end{pmatrix} = \begin{pmatrix} R_+^{(\sigma)}(L,\,\cdot) \\ R_-^{(\sigma)}(0,\,\cdot) \end{pmatrix}. \tag{2.41}$$

In the following Lemma 2.5 we obtain an explicit representation of the adjoint operator $F^*_{(\sigma)}$ that on $\tilde{H}_r$ is given by the relation

$$\int_{\mathbb{R}^d} \langle F_{(\sigma)}\,(u^{(\sigma)}),\, z\rangle_{\mathbb{R}^{2N}}\, \rho_\omega(\omega)\,\mathrm{d}\omega = \int_{\mathbb{R}^d} \langle u^{(\sigma)},\, F^*_{(\sigma)}(z)\rangle_{\mathbb{R}^{2N}}\, \rho_\omega(\omega)\,\mathrm{d}\omega.$$

Lemma 2.5. *For a random variable $z = (z_+,\, z_-)^\top \in \tilde{H}_r$ define $(z_+^{(\sigma)}(x,\omega),\, z_-^{(\sigma)}(x,\omega))$ with regularity (2.38) as the unique solution of the adjoint system*

$$\begin{cases} z_x^{(\sigma)}(x,\omega) = D^{-1}(x)\,[M(x,\omega) - D'(x)]\; z^{(\sigma)}(x,\omega), \\ \Big(d_+(L)\, z_+^{(\sigma)}(L,\,\omega)\Big) = z_+(\omega), \\ \Big(|d_-(0)|\, z_-^{(\sigma)}(0,\,\omega)\Big) = z_-(\omega). \end{cases} \tag{2.42}$$

Then we have

$$F^*_{(\sigma)} \begin{pmatrix} z_+(\cdot) \\ z_-(\cdot) \end{pmatrix} = \begin{pmatrix} \Big(d_+(0)\, z_+^{(\sigma)}(0,\,\cdot)\Big) \\ \Big(|d_-(L)|\, z_-^{(\sigma)}(L,\,\cdot)\Big) \end{pmatrix}. \tag{2.43}$$

Proof. Due to (2.39) we have the equation

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} \int_0^L \left(z^{(\sigma)}\right)^\top \left[D\, R_x^{(\sigma)} + M\, R^{(\sigma)}\right]\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega \\ &= \int_{\mathbb{R}^d} - \int_0^L \left(R^{(\sigma)}\right)^\top D\, z_x^{(\sigma)}\,\mathrm{d}x - \int_0^L \left(R^{(\sigma)}\right)^\top D'\, z^{(\sigma)}\,\mathrm{d}x \\ &\quad + \left[\left(z^{(\sigma)}(x)\right)^\top D\, R^{(\sigma)}(x)\right]\Big|_{x=0}^L + \int_0^L \left(R^{(\sigma)}\right)^\top M\, z^{(\sigma)}\,\mathrm{d}x\,\rho_\omega(\omega)\,\mathrm{d}\omega \\ &= \int_{\mathbb{R}^d} - \int_0^L \left(R^{(\sigma)}\right)^\top \left(D\, z_x^{(\sigma)} + [D' - M]\; z^{(\sigma)}\right)\,\mathrm{d}x \\ &\quad + \left[\left(z^{(\sigma)}(x,\omega)\right)^\top D\, R^{(\sigma)}(x,\omega)\right]\Big|_{x=0}^L \rho_\omega(\omega)\,\mathrm{d}\omega. \end{aligned}$$

Due to (2.42) this implies

$$0 = \int_{\mathbb{R}^d} \left[\left(z^{(\sigma)}(x,\omega)\right)^\top D\, R^{(\sigma)}(x,\omega)\right]\Big|_{x=0}^L \rho_\omega(\omega)\,\mathrm{d}\omega.$$

Thus we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left[\left(d_+(0)\, z_+^{(\sigma)}(0,\, \omega) \right)^\top \, R_+^{(\sigma)}(0,\, \omega) + \left(d_-(0)\, z_-^{(\sigma)}(0,\, \omega) \right)^\top \, R_-^{(\sigma)}(0,\, \omega) \right] \, \rho_\omega(\omega)\, \mathrm{d}\omega \\
= &\int_{\mathbb{R}^d} \left[\left(d_+(L)\, z_+^{(\sigma)}(L,\, \omega) \right)^\top \, R_+^{(\sigma)}(L,\, \omega) + \left(d_-(L)\, z_-^{(\sigma)}(L,\, \omega) \right)^\top \, R_-^{(\sigma)}(L,\, \omega) \right] \, \rho_\omega(\omega)\, \mathrm{d}\omega.
\end{aligned}$$

This implies the equation

$$\begin{aligned}
&\int_{\mathbb{R}^d} \langle F_{(\sigma)}\, (u^{(\sigma)}),\, z \rangle_{\mathbb{R}^{2N}}\, \rho_\omega(\omega)\, \mathrm{d}\omega \\
= &\int_{\mathbb{R}^d} \left[R_+^{(\sigma)}(L,\, \omega)\, z_+ + R_-^{(\sigma)}(0,\, \omega)\, z_- \right] \, \rho_\omega(\omega)\, \mathrm{d}\omega. \\
= &\int_{\mathbb{R}^d} \left[\left(d_+(L)\, R_+^{(\sigma)}(L,\, \omega) \right)^\top \, z_+^{(\sigma)}(L,\, \omega) + \left(|d_-(0)|\, R_-^{(\sigma)}(0,\, \omega) \right)^\top \, z_-^{(\sigma)}(0,\, \omega) \right] \, \rho_\omega(\omega)\, \mathrm{d}\omega \\
= &\int_{\mathbb{R}^d} \left[\left(d_+(0)\, R_+^{(\sigma)}(0,\, \omega) \right)^\top \, z_+^{(\sigma)}(0,\, \omega) + \left(|d_-(L)|\, R_-^{(\sigma)}(L,\, \omega) \right)^\top \, z_-^{(\sigma)}(L,\, \omega) \right] \, \rho_\omega(\omega)\, \mathrm{d}\omega \\
= &\int_{\mathbb{R}^d} \left(u_+^{(\sigma)}\, d_+(0) \right)^\top \, z_+^{(\sigma)}(0,\, \omega)\, \rho_\omega(\omega)\, \mathrm{d}\omega + \int_{\mathbb{R}^d} \left(u_-^{(\sigma)}\, |d_-(L)| \right)^\top \, z_-^{(\sigma)}(L,\, \omega)\, \rho_\omega(\omega)\, \mathrm{d}\omega \\
= &\int_{\mathbb{R}^d} \left\langle u^{(\sigma)},\, F_{(\sigma)}^*(z) \right\rangle_{\mathbb{R}^{2N}} \, \rho_\omega(\omega)\, \mathrm{d}\omega
\end{aligned}$$

with $F_{(\sigma)}^*$ as defined in (2.43). □

2.3.2. Necessary optimality conditions for the static problem

Let $u^{(\sigma)}$ denote the optimal control solving problem (2.40) and $R^{(\sigma)}$ the state generated by $u^{(\sigma)}$ as a solution of (2.39).

As in the dynamic case, we obtain Frechet differentiability of the cost functional in the static case. Using the same derivation as in the previous section we obtain a characterization of the Frechet differential in terms of the static operators $F_{(\sigma)}$ and $F_{(\sigma)}^*$, respectively. A necessary condition for optimality is that the Frechet differential vanishes and hence we obtain as necessary optimality conditions of (2.40):

$$\int_{\mathbb{R}^d} \left(f_u(u^{(\sigma)},\, F_{(\sigma)}(u^{(\sigma)})) + F_{(\sigma)}^* f_R(u^{(\sigma)},\, F_{(\sigma)}(u^{(\sigma)})) \right) \, \rho_\omega(\omega)\, \mathrm{d}\omega = 0. \tag{2.44}$$

This is equivalent to the optimality system for $R^{(\sigma)} = R^{(\sigma)}(x, \omega)$ and $z^{(\sigma)} = z^{(\sigma)}(x, \omega)$

$$\begin{cases}
D\, R_x^{(\sigma)} = -M\, R^{(\sigma)}, \\
R_+^{(\sigma)}(0) = u_+^{(\sigma)}, \\
R_-^{(\sigma)}(L) = u_-^{(\sigma)}, \\
D\, z_x^{(\sigma)} = [M - D']\ z^{(\sigma)}, \\
\left(d_+(L)\, z_+^{(\sigma)}(L, \omega) \right) = f_{R_+}(u^{(\sigma)},\, F_{(\sigma)}\, (u^{(\sigma)})), \\
\left(|d_-(0)| z_-^{(\sigma)}(0, \omega) \right) = f_{R_-}(u^{(\sigma)},\, F_{(\sigma)}\, (u^{(\sigma)}))
\end{cases} \tag{2.45}$$

and

$$\begin{cases} \int_{\mathbb{R}^d} \left(f_{u_+}(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) + \left(d_+(0)\, z_+^{(\sigma)}(0,\omega) \right) \right) \rho_\omega(\omega)\, \mathrm{d}\omega = 0, \\ \int_{\mathbb{R}^d} \left(f_{u_-}(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) + \left(|d_-(L)|\, z_-^{(\sigma)}(L,\omega) \right) \right) \rho_\omega(\omega)\, \mathrm{d}\omega = 0. \end{cases} \tag{2.46}$$

3. Turnpike result

In this section we show that the optimal controls that solve the dynamic optimal control problem with random dynamics (2.20) and the corresponding static optimal control problem with a random ordinary differential equation (2.40) satisfy an integral turnpike property. The integral turnpike property asserts that there is a constant $\zeta_0 > 0$ such that for all $T > 0$ we have

$$\int_0^T \left\| u^{(\delta,T)}(\tau) - u^{(\sigma)} \right\|_{\mathbb{R}^{2N}}^2 \, \mathrm{d}\tau \leq \zeta_0. \tag{3.1}$$

The turnpike inequality (3.1) implies that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T u^{(\delta,T)}(\tau)\, \mathrm{d}\tau = u^{(\sigma)}. \tag{3.2}$$

Thus, asymptotically for $T \to \infty$ the average value of the dynamic optimal control converges to the optimal static control. In fact, the convergence is of the order $O(\frac{1}{\sqrt{T}})$.

We also obtain an integral turnpike property for the state. This inequality concerns the uncontrolled boundary traces of the state that can be used for the observation of the state (similar as in [31], where the uncontrolled boundary traces have been used in the tracking term in the objective functional of an optimal control problem). Due to the uncertainty in the system, this observation is a random variable. Our inequality asserts that it is uniformly bounded in the $H_r(T)$-norm. Due to (2.24) we have

$$\|F_T(u^{(\delta,T)}, r^0) - F_{(\sigma)}(u^{(\sigma)})\|_{H^r(T)} \leq \|F_T(u^{(\delta,T)} - u^{(\sigma)},\, 0)\|_{H^r(T)} + \|F_T(0,\, r^0 - R^{(\sigma)})\|_{H^r(T)}$$

$$\leq \left(\int_0^T \left\| u^{(\delta,T)}(\tau) - u^{(\sigma)} \right\|_{\mathbb{R}^{2N}}^2 \, \mathrm{d}\tau \right)^{1/2} + \max_{x\in[0,\, L]} \left\{ \frac{1}{\sqrt{[d_+(x)]_{\min}}},\, \frac{1}{\sqrt{[d_-(x)]_{\min}}} \right\} \|r^0 - R^{(\sigma)}\|_{(L^2(0,\, L))^{2N}}.$$

With ζ_0 from (3.1), this yields an upper bound that is independent of T, see (3.4) below. Similarly, using (3.3) below we obtain a turnpike property for the adjoint state in the following sense: There exists $\zeta_1 > 0$ such that for all $T > 0$ the adjoint states $p^{(\delta,T)}$ as in (2.35) and $z^{(\sigma)}$ as in (2.45) fulfill

$$\|(p_+^{(\delta,T)}(\cdot, 0, \omega) - z_+^{(\sigma)}(0,\omega), p_-^{(\delta,T)}(\cdot, L, \omega) - z_-^{(\sigma)}(L,\omega))\|_{H^r(T)} \leq \zeta_1.$$

Our main result is stated in the following theorem.

Theorem 3.1. *Assume that the gradients f_R are Lipschitz continuous, i.e., there exists a Lipschitz constant $\hat{L}$ such that for all $T > 0$ for all $u_1,\, u_2 \in H_r(T)$, $R_1,\, R_1 \in H_r(T)$ we have*

$$\|f_R(u_1,\, R_1) - f_R(u_2,\, R_2)\|_{H_r(T)}^2 \leq \hat{L}^2 \left(\|u_1 - u_2\|_{H_r(T)}^2 + \|R_1 - R_2\|_{H_r(T)}^2 \right). \tag{3.3}$$

Then there exists a constant $\zeta_0 > 0$ such that (3.1) holds for all $T > 0$. Moreover, we have

$$\|F_T(u^{(\delta,T)}, r^0) - F_{(\sigma)}(u^{(\sigma)})\|_{H^r(T)} \leq \zeta_0{}^{\frac{1}{2}} + \max_{x\in[0,\, L]} \left\{ \tfrac{1}{[d_+(x)]_{\min}}, \tfrac{1}{[d_-(x)]_{\min}} \right\}^{\frac{1}{2}} \|r^0 - R^{(\sigma)}\|_{(L^2(0,\, L))^{2N}}. \tag{3.4}$$

Remark 3.2. An explicit estimate of ζ_0 is possible. To this end, define

$$c_0 = \max_{x\in[0,\, L]} \left\{ \tfrac{1}{\sqrt{[d_+(x)]_{\min}}}, \tfrac{1}{\sqrt{[d_-(x)]_{\min}}} \right\} \|r^0\|_{(L^2(0,\, L))^{2N}}, \tag{3.5}$$

$$c_1 = \max_{x\in[0,\, L]} \left\{ \tfrac{1}{\sqrt{[d_+(x)]_{\min}}}, \tfrac{1}{\sqrt{[d_-(x)]_{\min}}} \right\} \sqrt{\int_{\mathbb{R}^d} \|F_{(\sigma)} u^{(\sigma)})\|^2_{(L^2(0,\, L))^{2N}}\, \rho(\omega)\, \mathrm{d}\omega} \tag{3.6}$$

$$c_2 = \max_{x\in[0,\, L]} \{\|d_+(x)\|_\infty,\, \|d_-(x)\|_\infty\} \int_{\mathbb{R}^d} \|F^*_{(\sigma)} f_R(u^{(\sigma)},\, F_{(\sigma)}\, (u^{(\sigma)}))\|^2_{\mathbb{R}^{2N}}\, \rho_\omega(\omega)\, \mathrm{d}\omega. \tag{3.7}$$

Note that all constants c_0, c_1 and c_2 are finite and independent of T. Then

$$\sqrt{\zeta_0} = \frac{1}{2\,\kappa} \left(\sqrt{(c_2 + 4\,\hat{L}(c_0 + c_1))^2 + 8\kappa \hat{L}(c_0 + c_1)^2} - 4\,\hat{L}\,(c_0 + c_1) - c_2 \right). \tag{3.8}$$

This will be established in the proof of Theorem 3.1. Note that the example (2.17) fulfills the assumptions of the Theorem.

In order to prepare the proof of Theorem 3.1 we first present some auxiliary results in the next section.

3.1. On the difference of the dynamic and the static adjoint operator

Now we provide a bound for the difference between the dynamic and the static adjoint operator.

Lemma 3.3. *For a constant $h \in \mathbb{R}^{2N}$ that we consider as an element of $\tilde{H}_r$ as defined in (2.37), define $\tilde{h} = \tilde{h}(x, \omega)$ as the solution of system (2.42) with constant boundary data $(z_+, z_-)(\omega) = h$.*

For $t \in [0,\, T]$, let υ be defined as the solution to the terminal boundary value problem

$$\begin{cases} \upsilon(T,\, x,\, \omega) = -\tilde{h}(x, \omega), \\ \upsilon_t(t,\, x,\, \omega) + D\, \upsilon_x(t,\, x,\, \omega) = [M(x,\, \omega) - D'(x)]\; \upsilon(t,\, x,\, \omega), \\ \upsilon_+(t,\, L,\, \omega) = 0, \\ \upsilon_-(t,\, 0,\, \omega) = 0. \end{cases} \tag{3.9}$$

Define the number

$$C_*(h) = \max_{x\in[0,\, L]} \{\|d_+(x)\|_\infty,\, \|d_-(x)\|_\infty\} \int_{\mathbb{R}^d} \int_0^L \left\|\tilde{h}(x,\, \omega)\right\|^2_{\mathbb{R}^{2N}}\, \mathrm{d}x\, \rho_\omega(\omega)\, \mathrm{d}\omega. \tag{3.10}$$

Then for all $T > 0$ we have the inequality

$$\int_{\mathbb{R}^d} \int_0^T \left\| \left(\left(\tilde{F}^*_T - F^*_{(\sigma)} \right) h \right) (\tau, \omega) \right\|^2_{\mathbb{R}^{2N}} \rho_\omega(\omega)\, \mathrm{d}\tau\, d\omega \leq C_*(h). \tag{3.11}$$

Note that the constant $C_*(h)$ depends on the solution $\tilde{h}$ to equation (2.42) which in turn depends linearly on h. Also note, that in order to apply $\tilde{F}^*_T$, we extend h constant in time and ω.

Proof. Due to equation (2.28), (2.29), the definition of $\tilde{F}_T$ in Lemma 2.2, (2.42) and (2.43) we have

$$\left\| \left(\left(\tilde{F}_T^* - F_{(\sigma)}^* \right) h \right) \right\|_{H_r(T)}^2 = \int_{\mathbb{R}^d} \int_0^T \left\| \left(d_+(0)\, \upsilon_+(t, 0, \omega),\ |d_-(L)|\, \upsilon_-(t, L, \omega) \right)^\top \right\|_{\mathbb{R}^{2N}}^2 \rho_\omega(\omega)\, \mathrm{d}t\, \mathrm{d}\omega. \tag{3.12}$$

Similar as in the proof of Lemma 2.2, since $|D|\, D = \begin{pmatrix} d_+^2 & 0 \\ 0 & -d_-^2 \end{pmatrix}$ we obtain

$$\begin{aligned} &\left\| \begin{pmatrix} d_+(L)\, \upsilon_+(t, L) \\ d_-(0)\, \upsilon_-(t,\, 0) \end{pmatrix} \right\|_{H_r(T)}^2 - \left\| \begin{pmatrix} d_+(0)\, \upsilon_+(t,\, 0) \\ d_-(L)\, \upsilon_-(t,\, L) \end{pmatrix} \right\|_{H_r(T)}^2 \\ &= \int_{\mathbb{R}^d} \int_0^T \int_0^L \left(\upsilon^\top |D|\, D\, \upsilon \right)_x \mathrm{d}x\, \mathrm{d}t\, \rho_\omega(\omega)\, \mathrm{d}\omega \\ &= \int_{\mathbb{R}^d} \int_0^T \int_0^L 2\, \upsilon^\top |D|\, D\, \upsilon_x + 2\, \upsilon^\top |D|\, D'\, \upsilon\, \mathrm{d}x\, \mathrm{d}t\, \rho_\omega(\omega)\, \mathrm{d}\omega. \end{aligned}$$

Due to (3.9) this yields

$$\begin{aligned} &\left\| \begin{pmatrix} d_+(L)\, \upsilon_+(t, L,\, \omega) \\ d_-(0)\, \upsilon_-(t,\, 0,\, \omega) \end{pmatrix} \right\|_{H_r(T)}^2 - \left\| \begin{pmatrix} d_+(0)\, \upsilon_+(t,\, 0,\, \omega) \\ d_-(L)\, \upsilon_-(t,\, L,\, \omega) \end{pmatrix} \right\|_{H_r(T)}^2 \\ &= \int_{\mathbb{R}^d} \int_0^T \int_0^L -2\, \upsilon^\top |D|\, \upsilon_t + 2\, \upsilon^\top |D|\, [M - D']\, \upsilon + 2\, \upsilon^\top |D|\, D'\, \upsilon\, \mathrm{d}x\, \mathrm{d}t\, \rho_\omega(\omega)\, \mathrm{d}\omega \\ &= \int_{\mathbb{R}^d} \int_0^L - \left(\upsilon^\top |D|\, \upsilon \right)\Big|_{\tau=0}^T \mathrm{d}x + 2 \int_0^T \int_0^L \upsilon^\top |D|\, M\, \upsilon\, \mathrm{d}x\, \mathrm{d}t\, \rho_\omega(\omega)\, \mathrm{d}\omega \\ &= \int_{\mathbb{R}^d} \int_0^L \left(\upsilon^\top |D|\, \upsilon \right)(0,\, x,\, \omega)\, \mathrm{d}x - \int_0^L \left(\upsilon^\top |D|\, \upsilon \right)(T,\, x,\, \omega)\, \mathrm{d}x \\ &\quad + \int_0^T \int_0^L 2\, \upsilon^\top |D|\, M \upsilon\, \mathrm{d}x\, \mathrm{d}t\, \rho_\omega(\omega)\, \mathrm{d}\omega \\ &= \int_{\mathbb{R}^d} \int_0^L \left(\upsilon^\top |D|\, \upsilon \right)(0,\, x,\, \omega)\, \mathrm{d}x - \int_0^L \left(\upsilon^\top |D|\, \upsilon \right)(T,\, x,\, \omega)\, \mathrm{d}x \\ &\quad + \int_0^T \int_0^L \upsilon^\top \left[(|D|\, M + M\, |D|) \right] \upsilon\, \mathrm{d}x\, \mathrm{d}t\, \rho_\omega(\omega)\, \mathrm{d}\omega \\ &\geq - \int_{\mathbb{R}^d} \int_0^L \left(\upsilon^\top |D|\, \upsilon \right)(T,\, x,\, \omega)\, \mathrm{d}x\, \rho_\omega(\omega)\, \mathrm{d}\omega \end{aligned}$$

where the last inequality follows with (2.22). Since $\upsilon_+(t,\, L,\, \omega) = 0 = \upsilon_-(t,\, 0,\, \omega)$, due to (3.12) this implies

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_0^T \left\| \left(\left(\tilde{F}_T^* - F_{(\sigma)}^* \right) h \right)(\tau, \omega) \right\|_{\mathbb{R}^{2N}}^2 \mathrm{d}\tau\, \rho_\omega(\omega)\, \mathrm{d}\omega \\ &= \int_{\mathbb{R}^d} \int_0^T \left\| \left(d_+(0)\, \upsilon_+(t, 0,\, \omega),\ |d_-(L)|\, \upsilon_-(t, L,\, \omega) \right)^\top \right\|_{\mathbb{R}^{2N}}^2 \rho_\omega(\omega)\, \mathrm{d}t\, \mathrm{d}\omega. \\ &\leq \max_{x \in [0,\, L]} \{ \|d_+(x)\|_\infty,\ \|d_-(x)\|_\infty \} \int_{\mathbb{R}^d} \int_0^L \|\upsilon(T,\, x,\, \omega)\|_{\mathbb{R}^{2N}}^2 \, \mathrm{d}x\, \rho_\omega(\omega)\, \mathrm{d}\omega. \end{aligned}$$

Using the terminal condition for υ, (3.11) follows. $\square$

3.2. Proof of Theorem 3.1

Denote by $u^{(\delta,\,T)}$ and $u^{(\sigma)}$ the unique optimal solution to the dynamic problem (2.20) and the static problem (2.40), respectively. The optimality conditions (2.34) for the dynamic problem (2.20) and the conditions (2.44) of the static problem (2.40) are

$$0 = \int_{\mathbb{R}^d} \Big(f_u(u^{(\delta,\,T)},\, F_T\,(u^{(\delta,\,T)},\, r^0)) + \tilde{F}_T^* f_R(u^{(\delta,\,T)},\, F_T(u^{(\delta,\,T)},\, r^0)) \Big)\, \rho_\omega(\omega)\,\mathrm{d}\omega, \tag{3.13}$$

$$0 = \int_{\mathbb{R}^d} \Big(f_u(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) + F_{(\sigma)}^* f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) \Big)\, \rho_\omega(\omega)\,\mathrm{d}\omega. \tag{3.14}$$

The difference of (3.13) and (3.14) yields the equation

$$\int_{\mathbb{R}^d} \Big[f_u(u^{(\delta,\,T)},\, F_T\,(u^{(\delta,\,T)},\, r^0)) - f_u(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) \Big]\, \rho(\omega)\,\mathrm{d}\omega \tag{3.15}$$

$$= \int_{\mathbb{R}^d} \Big[F_{(\sigma)}^* f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - \tilde{F}_T^* f_R(u^{(\delta,\,T)},\, F_T(u^{(\delta,\,T)},\, r^0)) \Big]\, \rho(\omega)\,\mathrm{d}\omega.$$

Define the numbers

$$\begin{aligned} L_0 &= \int_{\mathbb{R}^d}\int_0^T \Big(u^{(\delta,\,T)}(\tau) - u^{(\sigma)} \Big)^\top \Big(f_u(u^{(\delta,\,T)},\, F_T\,(u^{(\delta,\,T)},\, r^0)) - f_u(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) \Big)\,\mathrm{d}\tau\,\rho(\omega)\,\mathrm{d}\omega, \\ L_1 &= \int_{\mathbb{R}^d}\int_0^T \Big(F_T\,(u^{(\delta,\,T)},\, r^0) - F_{(\sigma)}\,(u^{(\sigma)}) \Big)^\top \\ &\quad \Big(f_R(u^{(\delta,\,T)},\, F_T\,(u^{(\delta,\,T)},\, r^0)) - f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) \Big)\,\mathrm{d}\tau\,\rho(\omega)\,\mathrm{d}\omega. \end{aligned}$$

Condition (2.15) implies the inequality

$$L_0 + L_1 \geq \kappa \int_0^T \|u^{(\delta,\,T)}(\tau) - u^{(\sigma)}\|^2_{\mathbb{R}^{2N}}\,\mathrm{d}\tau. \tag{3.16}$$

Due to equation (3.15)

$$\begin{aligned} L_0 &= \int_{\mathbb{R}^d}\int_0^T \Big(u^{(\delta,\,T)}(\tau) - u^{(\sigma)} \Big)^\top \\ &\quad \Big(F_{(\sigma)}^* f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - \tilde{F}_T^* f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)}) \Big) \\ &\quad + \tilde{F}_T^* f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - \tilde{F}_T^* f_R(u^{(\delta,\,T)},\, F_T(\,u^{(\delta,\,T)},\, r^0)) \Big)\,\mathrm{d}\tau\,\rho_\omega(\omega)\,\mathrm{d}\omega \\ &= \int_{\mathbb{R}^d}\int_0^T \Big(\tilde{F}_T\,\Big(u^{(\delta,\,T)}(\tau) - u^{(\sigma)} \Big)\Big)^\top \\ &\quad \Big(f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - f_R(u^{(\delta,\,T)},\, F_T(\,u^{(\delta,\,T)},\, r^0)) \Big)\,\mathrm{d}\tau\,\rho_\omega(\omega)\,\mathrm{d}\omega \\ &\quad + \int_{\mathbb{R}^d}\int_0^T \Big(u^{(\delta,\,T)}(\tau) - u^{(\sigma)} \Big)^\top \Big(F_{(\sigma)}^* - \tilde{F}_T^* \Big)\, f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)}))\,\mathrm{d}\tau\,\rho(\omega)\,\mathrm{d}\omega. \end{aligned}$$

With the definition of L_1 this implies

$$\begin{aligned} L_0 + L_1 = & \int_{\mathbb{R}^d} \int_0^T \left(\tilde{F}_T \left(u^{(\delta,T)}(\tau) - u^{(\sigma)} \right) - F_T\left(u^{(\delta,T)}, r^0\right) + F_{(\sigma)}\left(u^{(\sigma)}\right) \right)^\top \\ & \left(f_R(u^{(\sigma)}, F_{(\sigma)}(u^{(\sigma)})) - f_R(u^{(\delta,T)}, F_T(u^{(\delta,T)}, r^0)) \right) \mathrm{d}\tau\, \rho_\omega(\omega)\, \mathrm{d}\omega \\ & + \int_{\mathbb{R}^d} \int_0^T \left(u^{(\delta,T)}(\tau) - u^{(\sigma)} \right)^\top \left(F^*_{(\sigma)} - \tilde{F}^*_T \right) f_R(u^{(\sigma)}, F_{(\sigma)}(u^{(\sigma)}))\, \mathrm{d}\tau\, \rho_\omega(\omega)\, \mathrm{d}\omega. \end{aligned}$$

Note that $\tilde{F}_T = F_T(\cdot, 0)$ and due to the linearity of F_T we have

$$\begin{aligned} L_0 + L_1 = & \int_{\mathbb{R}^d} \int_0^T \left(F_{(\sigma)}\left(u^{(\sigma)}\right) - F_T(u^{(\sigma)}, 0) - F_T(0, r_0) \right)^\top \\ & \left(f_R(u^{(\sigma)}, F_{(\sigma)}(u^{(\sigma)})) - f_R(u^{(\delta,T)}, F_T(u^{(\delta,T)}, r^0)) \right) \mathrm{d}\tau\, \rho_\omega(\omega)\, \mathrm{d}\omega \\ & + \int_{\mathbb{R}^d} \int_0^T \left(u^{(\delta,T)}(\tau) - u^{(\sigma)} \right)^\top \left(F^*_{(\sigma)} - \tilde{F}^*_T \right) f_R(u^{(\sigma)}, F_{(\sigma)}(u^{(\sigma)}))\, \mathrm{d}\tau\, \rho_\omega(\omega)\, \mathrm{d}\omega \\ = & \, M_0 + M_1 + M_2 \end{aligned}$$

where

$$\begin{aligned} M_0 &= \int_{\mathbb{R}^d} \int_0^T - \left(F_T(0, r^0) \right)^\top \left(f_R(u^{(\sigma)}, F_{(\sigma)}(u^{(\sigma)})) - f_R(u^{(\delta,T)}, F_T(u^{(\delta,T)}, r^0)) \right) \mathrm{d}\tau \rho_\omega(\omega)\, \mathrm{d}\omega, \\ M_1 &= \int_{\mathbb{R}^d} \int_0^T \left(\left[F_{(\sigma)} - \tilde{F}_T \right] (u^{(\sigma)}) \right)^\top \left(f_R(u^{(\sigma)}, F_{(\sigma)}(u^{(\sigma)})) - f_R(u^{(\delta,T)}, F_T(u^{(\delta,T)}, r^0)) \right) \mathrm{d}\tau \rho_\omega(\omega)\, \mathrm{d}\omega, \\ M_2 &= \int_{\mathbb{R}^d} \int_0^T \left(u^{(\delta,T)}(\tau) - u^{(\sigma)} \right)^\top \left[F^*_{(\sigma)} - \tilde{F}^*_T \right] f_R(u^{(\sigma)}, F_{(\sigma)}(u^{(\sigma)}))\, \mathrm{d}\tau \rho_\omega(\omega)\, \mathrm{d}\omega. \end{aligned}$$

Hence, (3.16) yields the inequality

$$\kappa \int_0^T \|u^{(\delta,T)}(\tau) - u^{(\sigma)}\|^2_{\mathbb{R}^{2N}}\, \mathrm{d}\tau \leq L_0 + L_1 = M_0 + M_1 + M_2. \tag{3.17}$$

The Lipschitz condition (3.3) implies that

$$\begin{aligned} & \left\| f_R(u^{(\sigma)}, F_{(\sigma)}(u^{(\sigma)})) - f_R(u^{(\delta,T)}, F_T(u^{(\delta,T)}, r^0)) \right\|^2_{H_r(T)} \\ & \leq \hat{L}^2 \left(\|u^{(\delta,T)} - u^{(\sigma)}\|^2_{H_r(T)} + \|F_T(u^{(\delta,T)}), r^0) - F_{(\sigma)}(u^{(\sigma)})\|^2_{H_r(T)} \right). \end{aligned}$$

Hence we have

$$\begin{aligned} & \left\| f_R(u^{(\sigma)}, F_{(\sigma)}(u^{(\sigma)})) - f_R(u^{(\delta,T)}, F_T(u^{(\delta,T)}, r^0)) \right\|_{H_r(T)} \\ & \leq 2\hat{L} \left(\|u^{(\delta,T)} - u^{(\sigma)}\|_{H_r(T)} + \|\tilde{F}_T(u^{(\delta,T)} - u^{(\sigma)}) + F_T(0, r^0) + \left[\tilde{F}_T - F_{(\sigma)} \right] u^{(\sigma)}\|_{H_r(T)} \right) \\ & \leq 2\hat{L} \left(1 + \|\tilde{F}_T\| \right) \|u^{(\delta,T)} - u^{(\sigma)}\|_{H_r(T)} + 2\hat{L} \left(\|F_T(0, r^0)\|_{H_r(T)} + \| \left[\tilde{F}_T - F_{(\sigma)} \right] u^{(\sigma)}\|_{H_r(T)} \right) \end{aligned}$$

We have $\|F_T(0,\, r^0)\|_{H_r(T)} \leq c_0$ with

$$c_0 := \sup_{x\in[0,\,L]} \left\{ \frac{1}{\sqrt{[d_+(x)]_{\min}}},\, \frac{1}{\sqrt{[d_-(x)]_{\min}}} \right\} \|r^0\|_{(L^2(0,\,L))^{2N}}$$

due to the estimate (2.24). Define $\tilde{r}^0 = (R_+(\cdot,\,\omega),\, R_-(\cdot,\,\omega))$, where (R_+, R_-) solves (2.39) and

$$c_1^2 := \sup_{x\in[0,\,L]} \left\{ \frac{1}{[d_+(x)]_{\min}},\, \frac{1}{[d_-(x)]_{\min}} \right\} \int_{\mathbb{R}^d} \|\tilde{r}^0\|^2_{(L^2(0,\,L))^{2N}}\, \rho_\omega(\omega)\, \mathrm{d}\omega.$$

We have

$$\left[\tilde{F}_T - F_{(\sigma)}\right]\, u^{(\sigma)} = F_T(0,\, -\tilde{r}^0). \tag{3.18}$$

This can be seen as follows. We have $\tilde{F}_T(u^{(\sigma)}) = F_T(u^{(\sigma)}, 0) = (r_+(\cdot, L, \omega),\, r_-(\cdot, 0, \omega))$ where (r_+, r_-) solves (2.5) with the initial state $r^0 = 0$ and the boundary conditions $r_+(\cdot,\, 0, \omega) = u_+^{(\sigma)}$, $r_-(\cdot,\, L, \omega) = u_-^{(\sigma)}$. The definition of $F_{(\sigma)}(u^{(\sigma)})$ implies that it is given by the boundary traces of $(R_+,\, R_-)$ which solves (2.39) but is also a solution of (2.5) with constant boundary data that is independent of t. The boundary conditions are $R_+(0, \omega) = u_+^{(\sigma)}$, $R_-(L, \omega) = u_-^{(\sigma)}$. Since the solution does not depend on time, the initial state is equal to $\tilde{r}^0$. Thus $\left[\tilde{F}_T - F_{(\sigma)}\right]\, u^{(\sigma)}$ can be obtained from (2.5) where the boundary data are equal to zero, since $r_+(\cdot,\, 0, \omega) - R_+(0, \omega) = 0$ and $r_-(\cdot,\, L, \omega) - R_-(L, \omega) = 0$. The corresponding initial data in (2.5) is given by $r^0 - \tilde{r}^0 = -\tilde{r}^0$. Thus we have (3.18).

Hence (2.24) implies $\left\| \left[\tilde{F}_T - F_{(\sigma)}\right]\, u^{(\sigma)} \right\|_{H_r(T)} \leq c_1$. Due to Lemma 2.2 we have that $\|\tilde{F}_T\| \leq 1$ and this yields the inequality

$$\left\| f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) - f_R(u^{(\delta,\,T)},\, F_T(\, u^{(\delta,\,T)},\, r^0)) \right\|_{H_r(T)}$$

$$\leq 2\hat{L}\, \left(2\,\|u^{(\delta,\,T)} - u^{(\sigma)}\|_{H_r(T)} + c_0 + c_1\right)$$

Now we derive an upper bound for M_0. Due to Cauchy–Schwartz we obtain the bound

$$M_0 \leq c_0\; 2\hat{L} \left(2\, \left\| u^{(\delta,\,T)} - u^{(\sigma)} \right\|_{H_r(T)} + c_0 + c_1\right).$$

Similarly, for M_1 we obtain the upper bound

$$M_1 \leq c_1 2\hat{L} \left(2\, \left\| u^{(\delta,\,T)} - u^{(\sigma)} \right\|_{H_r(T)} + c_0 + c_1\right).$$

Finally, we derive an upper bound for M_2. With C_* as in (3.10), define $c_2 = C_*(f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})))$. Due to equation (3.11) of Lemma 3.3 we have

$$\left\| \left(F^*_{(\sigma)} - \tilde{F}^*_T\right) f_R(u^{(\sigma)},\, F_{(\sigma)}\,(u^{(\sigma)})) \right\|^2_{H_r(T)} \leq c_2.$$

Then

$$M_2 \leq c_2 \left\| u^{(\delta,\, T)} - u^{(\sigma)} \right\|_{H_r(T)} .$$

Summarizing the previous estimates yields the inequality

$$\kappa \left\| u^{(\delta,\, T)} - u^{(\sigma)} \right\|^2_{H_r(T)} \tag{3.19}$$

$$\leq 2\hat{L}\ (c_0 + c_1) \left(2 \left\| u^{(\delta,\, T)} - u^{(\sigma)} \right\|_{H_r(T)} + c_0 + c_1 \right) + c_2 \left\| u^{(\delta,\, T)} - u^{(\sigma)} \right\|_{H_r(T)} .$$

Define the polynomial

$$P(x) = \kappa\, x^2 - \left(c_2 + 4\, \hat{L}\ (c_0 + c_1) \right) x - 2\hat{L}\ (c_0 + c_1)^2 .$$

Then the set $\mathcal{S} = \{x \in [0,\, \infty) : P(x) \leq 0\}$ is bounded and there exists a number $\zeta_0 > 0$ such that $\mathcal{S} = [0,\, \sqrt{\zeta_0}]$. It is worth mentioning that ζ_0 is independent of T. Due to the inequality (3.19) we have $\left\| u^{(\delta,\, T)} - u^{(\sigma)} \right\|_{H_r(T)} \in \mathcal{S}$. Thus we obtain the inequality

$$\left\| u^{(\delta,\, T)} - u^{(\sigma)} \right\|_{H_r(T)} \leq \sqrt{\zeta_0}$$

and inequality (3.1) follows. Thus we have proved Theorem 3.1. □

4. Numerical discretization

In this section we consider the numerical discretization of the optimal control problem (2.20) using a generalized polynomial chaos expansion (gPC) in the random variable. We prove that the discretized optimal control problem has the turnpike property for any fixed truncation order of the gPC series. To establish this result we will formulate the problem such that Theorem 3.1 applies. Note that the previous Theorem 3.1 shows the integral turnpike property for random systems of any fixed dimension $2N$. This obviously also includes the deterministic case which will be used to establish the turnpike property of the gPC expanded system later on.

We recall some results on random processes and their approximation. We refer to [10, 32, 43, 46, 50, 54] for a general introduction and to [5, 56] for results on convergence of truncated approximation series. W.l.o.g we assume in the following that $r^{\pm}(t, x, \omega) \in \mathbb{R}$ are square–integrable with respect to ω, *i.e.*, for each $t \in [0, T]$ we have $r(t, \cdot) = (r_+, r_-)(t, \cdot, \cdot) \in L^2_{1\times\rho_\omega}((0, L) \times \mathbb{R}^d; \mathbb{R}^2)$. The gPC expansion constitutes a set of orthogonal subspaces $S_k \subset L^2_{\rho_\omega}(\mathbb{R}^d)$ such that $\mathbf{S}_K := \prod_{k=0}^K S_k$ yields $L^2_{\rho_\omega}(\mathbb{R}^d)$ for $K \to \infty$. The gPC basis of $\mathbf{S}_K$ is denoted by $\{\phi_k\}_{k=0}^K$ where we assume w.l.o.g. that the basis functions are orthonormal, *i.e.*,

$$\langle \phi_i, \phi_j \rangle := \int_{\mathbb{R}^d} \phi_i(\omega)\phi_j(\omega)\rho_\omega(\omega)\mathrm{d}\omega = \delta_{i,j} \tag{4.1}$$

where $\delta_{i,j}$ is the Kronecker-delta. For each fixed (t, x) the Galerkin projection of $r(t, x, \omega)$ on $\mathbf{S}_K$ is given by

$$P_K(r^{\pm}(t, x, \cdot))(\omega) := \sum_{k=0}^{K} r_k^{\pm}(t, x)\phi_k(\omega) \in \mathbf{S}_K \tag{4.2}$$

where for each $k = 0, \dots$

$$r_k^{\pm}(t,x) = \langle r^{\pm}(t,x,\cdot), \phi_k(\cdot)\rangle. \tag{4.3}$$

The truncated series expansion convergences in $L^2_{\rho_\omega}$ towards r, see [5]:

$$\lim_{K\to\infty} \int_{\mathbb{R}^d} \|P_K(r^{\pm}(t,x,\cdot)) - r^{\pm}(t,x,\cdot)\|^2 \rho_\omega(\omega)\mathrm{d}\omega = 0. \tag{4.4}$$

We define the (deterministic) vector of (gPC) coefficients $s = (s_+, s_-)$ by

$$(s_+, s_-)(t,x) = (r_{+,0}, r_{+,1}, \dots, r_{+,K}, r_{-,0}, r_{-,1}, \dots, r_{-,K})(t,x) \in \mathbb{R}^{2(K+1)} \tag{4.5}$$

By Galerkin projection of equation (2.5) on the subspace $\mathbf{S}_K$ we obtain a coupled deterministic system for the coefficients (r_+, r_-). Even so, the results hold true in the general case, we consider equation (2.5) in the case $N = 1$, *i.e.*, $r = r(t,x,\omega)$ has two components. The projection leads then to deterministic system for $s = (s_+, s_-)$ of dimension $2(K+1)$. This leads to the deterministic linear system for $x \in [0,L]$ and $t \geq 0$

$$\partial_t s(t,x) + \mathcal{D}(x)\partial_x s(t,x) = -\overline{\mathcal{M}}(x)s(t,x) \tag{4.6}$$

where for each x, $\mathcal{D}(x) \in \mathbb{R}^{2(K+1)\times 2(K+1)}$ and $\overline{\mathcal{M}}(x) \in \mathbb{R}^{2(K+1)\times 2(K+1)}$ are block matrices obtained as the Galerkin projections of $D(x) \in \mathbb{R}^{2\times 2}$ and $\mathcal{M}(x,\omega) \in \mathbb{R}^{2\times 2}$ on $\mathbf{S}_K$, *i.e.*, for $k, i = 0, \dots, K$

$$\overline{\mathcal{M}}_{k,i}(x) = \int_{\mathbb{R}^d} \mathcal{M}(x,\omega)\phi_k(\omega)\phi_i(\omega)\rho_\omega(\omega)\,\mathrm{d}\omega, \tag{4.7}$$

$$\mathcal{D}_{k,i}(x) = D(x)\delta_{k,i}. \tag{4.8}$$

Since the initial data r_0 in equation (2.5) is deterministic, we obtain the initial data for s

$$s_k(0,x) = r^0(x)\,\delta_{k,0} \tag{4.9}$$

and similarly the boundary conditions are given

$$s_{+,k}(t,0) = u_+(t)\,\delta_{k,0} \text{ and } s_{-,k}(t,L) = u_-(t)\,\delta_{k,0}. \tag{4.10}$$

Note that for a general random variable $\omega \to g(\omega) \in L^2_{\rho_\omega}(\mathbb{R}^d;\mathbb{R})$ the first coefficient g_0 of the gPC expansion is its expectation, *i.e.*, $g_0 = \int_{\mathbb{R}^d} g(\omega)\rho_\omega(\omega)\mathrm{d}\omega = \mathbb{E}_\rho(g)$. In case, of g being deterministic all further coefficients are equal to zero. Therefore, in the previous equations only the zero order coefficient remains.

The system (4.6), (4.9) and (4.10) is of the type of system (2.5) with $N = 2(K+1)$, but without any dependence on ω. Analogously to the existence result for solutions (2.5) we obtain the existence and uniqueness of $s \in C((0,T); L^2((0,L); \mathbb{R}^{2(K+1)})$ for inital and boundary data $r^0 \in L^2((0,L);\mathbb{R}^2)$ and $u = (u_+, u_-) \in L^2((0,T);\mathbb{R}^2)$. Therefore, the result of Theorem 3.1 is applicable to the system (4.6) provided that properties (2.22) and (3.3) hold. In the deterministic case the property (2.22) and (2.2) coincide. Hence, we prove that the property (2.2) holds for system (4.6).

Lemma 4.1. *Assume $\mathcal{D}, \mathcal{M}$ fulfill equation (2.22). Furthermore, define the deterministic coefficients $\overline{\mathcal{M}}$ by (4.7).*

Then for any $K > 0$ and for all $x \in [0,L]$ we have

$$|\mathcal{D}(x)|\,\overline{\mathcal{M}}(x) + \overline{\mathcal{M}}(x)\,|\mathcal{D}(x)| \geq 0. \tag{4.11}$$

Proof. Let $z \in \mathbb{R}^{2(K+1)}$ be any non–zero vector and for $i \in \{0, ..., K\}$ let $z_i = (z_{+,i}, z_{-,i}) \in \mathbb{R}^2$. Then,

$$z^T \left(|\mathcal{D}(x)| \, \overline{\mathcal{M}}(x) + \overline{\mathcal{M}}(x) \, |\mathcal{D}(x)| \right) z = \sum_{i=0}^{K} \sum_{j=0}^{K} z_i^T \left(|D(x)| \, \overline{\mathcal{M}}_{i,j}(x) + \overline{\mathcal{M}}_{i,j}(x) \, |D(x)| \right) z_j,$$

due to the definition of $\mathcal{D}$ and $\mathcal{M}$. For all $i = 0, \ldots, K$ define

$$w_i(x, \omega) := \phi_i(\omega) \, \sqrt{(|D(x)| \, \mathcal{M}(x, \omega) + \mathcal{M}(x, \omega) \, |D(x)|)} \, z_i \in \mathbb{R}^2.$$

The square–root is well–defined a.s. due to assumption (2.22). Using (4.7) we continue the previous computation and obtain

$$\begin{aligned} & z^T \left(|\mathcal{D}(x)| \, \overline{\mathcal{M}}(x) + \overline{\mathcal{M}}(x) \, |\mathcal{D}(x)| \right) z \\ &= \int_{\mathbb{R}^d} \sum_{i=0}^{K} \sum_{j=0}^{K} z_i^T \left(|D(x)| \, \mathcal{M}(x, \omega) + \mathcal{M}(x, \omega) \, |D(x)| \right) z_j \phi_i(\omega) \phi_j(\omega) \rho_\omega(\omega) \mathrm{d}\omega \\ &= \int_{\mathbb{R}^d} \sum_{i=0}^{K} \sum_{j=0}^{K} w_i(x, \omega)^T \, w_j(x, \omega) \rho_\omega(\omega) \mathrm{d}\omega \geq 0, \end{aligned}$$

The last inequality holds true since $\rho_\omega \geq 0$ $\square$

Next, we state the integral turnpike result for the gPC expanded system. The objective function depends on the deterministic control $u = (u_+, u_-)$ and the truncated gPC series defined by equation (4.2).

$$J_T^{(K)}(u) := \int_{\mathbb{R}^d} \int_0^T f(u_+(\tau), \, u_-(\tau), \, P_K(r_+(\tau, \, L, \cdot)(\omega), \, P_K(r_-(\tau, \, 0, \cdot)(\omega)) \, \mathrm{d}\tau \, \rho_\omega(\omega) \, \mathrm{d}\omega. \tag{4.12}$$

The corresponding dynamic control problem reads

$$\min_{u \in L^2((0,T);\mathbb{R}^2)} J_T^{(K)}(u) \;\text{ subject to } (4.6), (4.9), (4.10) \tag{4.13}$$

Analogously to the results of Section 2, a unique solution $u^{(K)} \in L^2((0, T); \mathbb{R}^2)$ exists. The static optimal problem has the same structure as problem (2.40) and it is therefore omitted. The unique solution to the static problem is denoted by $u^{(K,\sigma)} \in \mathbb{R}^2$. The integral turnpike property then reads for some $\zeta_0(K)$ that depends possibly on K :

$$\int_0^T \left\| u^{(K)}(\tau) - u^{(K,\sigma)} \right\|_{\mathbb{R}^2}^2 \mathrm{d}\tau \leq \zeta_0(K). \tag{4.14}$$

The turnpike result for the expanded system is given by the following theorem.

Theorem 4.2. *Assume that the gradients f_R are Lipschitz continuous as in Theorem 3.1. We assume that* (4.11) *holds true.*

Then, for any fixed natural number $K > 0$ there exists a constant $\zeta_0(K) > 0$ such that (4.14) holds for all $T > 0$.

Proof. The proof is analogous to the proof of Theorem 3.1 with $N = 2(K+1)$. However, we replace the control space $L^2((0,T);\mathbb{R}^{2N})$ by

$$L^2((0,T);\mathbb{R}^2) \times \{0\}^{2K}, \tag{4.15}$$

since the control only appears in the boundary conditions for the zero component $(s_{+,0}, s_{-,0})$, see equation (4.10).

Note that $\zeta_0(K)$ is given by equation (3.8) and depends on the Lipschitz constant of f given by $\hat{L}$ and the constants c_0, c_1 and c_2, respectively. Further, it depends on the constant κ from condition (2.15). Note that c_0 defined by (3.5) only depends on the initial data and the entries of $D(x)$ and hence is independent of K. ☐

5. Conclusion

We have shown a turnpike theorem for a problem of optimal boundary control for random systems in Theorem 3.1. Its approximation via generalized polynomial chaos expansion (gPC) also fulfills a similar result, see Theorem 4.2. The turnpike result shows that regardless of the initial state, the dynamic optimal control approaches the corresponding static optimal control with increasing time horizon. The results cover the case of systems of any dimension as well as $d-$dimensional parametric uncertainties. Here, the randomness appears in general form in the source term of the system. It has been shown that the integral turnpike property transfers from the original random system to its gPC counterpart.

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