ESAIM: COCV 29 (2023) 63
https://doi.org/10.1051/cocv/2023053

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

ANALYTICITY AND OBSERVABILITY FOR FRACTIONAL ORDER PARABOLIC EQUATIONS IN THE WHOLE SPACE

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Abstract. In this paper, we study the quantitative analyticity and observability inequality for solutions of fractional order parabolic equations with space-time dependent potentials in $\mathbb{R}^n$. We first obtain a uniformly lower bound of analyticity radius of the spatial variable for the above solutions with respect to the time variable. Next, we prove a globally Hölder-type interpolation inequality on a thick set, which is based on a propagation estimate of smallness for analytic functions. Finally, we establish an observability inequality from a thick set in $\mathbb{R}^n$, by utilizing a telescoping series method.

Mathematics Subject Classification. 35K05, 93B07.

Received February 27, 2022. Accepted July 6, 2023.

1. Introduction and main results

It has been proved in [12] and [33] independently that the following observability inequality

$$\forall T>0, \exists C=C(n,T,E)>0 \text{ so that } \int_{\mathbb{R}^n} |u(T,x)|^2 \,\mathrm{d}x \le C\int_0^T \int_E |u(t,x)|^2 \,\mathrm{d}x \,\mathrm{d}t$$

hold for any solution to the heat equation in the whole space

$$\partial_t u - \Delta u = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^n, \quad u(0,\cdot) \in L^2(\mathbb{R}^n),$$

if and only if the observation set $E \subset \mathbb{R}^n$ is thick, *i.e.*, there exists $L>0$ such that

$$\inf_{x\in\mathbb{R}^n} |E \bigcap Q_L(x)| > 0.$$

Here $|\cdot|$ denotes the n-dimensional Lebesgue measure, and $Q_L(x)$ stands for the cube in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with a side $L>0$.

The arguments in [12, 33] are indeed adapted from the Lebeau-Robbiano strategy (see, *e.g.*, [5, 9, 19, 21, 29, 32]). Let us now recall it in an abstract way: Let $\mathbb{H}$ be a self-adjoint operator so that $-\mathbb{H}$ generates a C_0

Keywords and phrases: Fractional order, parabolic equations, analyticity, observability.

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semigroup $\{e^{-t\mathbb{H}}\}_{t\geq 0}$ in $L^2(\mathbb{R}^n)$, and let $\{\pi_N\}_{N\geq 1}$ be a family of orthogonal projection operators on $L^2(\mathbb{R}^n)$. Assume that there are constants $b > a > 0$ and $C > 0$ so that the spectral inequality

$$\|\pi_N f\|_{L^2(\mathbb{R}^n)} \leq Ce^{CN^a}\|\pi_N f\|_{L^2(E)},$$

and the dissipative inequality

$$\|(I-\pi_N)e^{-t\mathbb{H}}f\|_{L^2(\mathbb{R}^n)} \leq Ce^{-CtN^b}\|f\|_{L^2(\mathbb{R}^n)}, \quad \forall t > 0$$

hold for all $N \geq 1$ and $f \in L^2(\mathbb{R}^n)$. Then, the following observability inequality is true:

$$\forall T > 0, \exists C = C(n, T, E) > 0 \text{ so that}$$
$$\int_{\mathbb{R}^n} |e^{-T\mathbb{H}}f(x)|^2 \,\mathrm{d}x \leq C\int_0^T \int_E |e^{-t\mathbb{H}}f(x)|^2 \,\mathrm{d}x\,\mathrm{d}t, \quad \forall f \in L^2(\mathbb{R}^n).$$

Recently, Lebeau and Moyano in [20] have proved a similar spectral inequality for the Schrödinger operator $\Delta + V(\cdot)$ in $\mathbb{R}^n$, with $V(\cdot)$ being a spatially analytic function vanishing at infinity. More precisely, if E is a thick set, then there exists $C > 0$ so that

$$\|\pi_N f\|_{L^2(\mathbb{R}^n)} \leq Ce^{C\sqrt{N}}\|\pi_N f\|_{L^2(E)}, \quad \forall N > 0, f \in L^2(\mathbb{R}^n),$$

where π_N is a spectral projection from the total space to a low-frequency subset (see [20] for the more accurate definition). Thus, the observability inequality from the thick set holds for all solutions of the heat equation with such a kind of potentials

$$\partial_t u - \Delta u = V(x)u \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^n, \quad u(0,\cdot) \in L^2(\mathbb{R}^n).$$

The following question arise naturally: Does the corresponding observability inequality from the thick set hold true for the heat operator with *space-time* dependent potentials

$$\partial_t u - \Delta u = a(t,x)u \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^n, \quad u(0,\cdot) = u_0 \in L^2(\mathbb{R}^n).$$

Up to now, to the best of our knowledge, it is still open in the existing literature.

It is worth mentioning that the observability inequality from the equi-distributed set (which is stronger than the thick set) is proved in [8] for the heat operator with bounded potentials in $\mathbb{R}^n$.

In this paper, we concern the above question in a more general case: We would like to replace the Laplace operator by a fractional one

$$\partial_t u + \Lambda^s u = a(t,x)u \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^n, \quad u(0,\cdot) = u_0 \in L^2(\mathbb{R}^n), \tag{1.1}$$

where $\Lambda^s = (-\Delta)^{s/2}$, $s > 0$, is the fractional Laplacian defined through the Fourier transform

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \widehat{f}(\xi), \ \ \forall \xi \in \mathbb{R}^n.$$

The Fourier transform $\widehat{f}$ is here given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,\mathrm{d}x, \quad \forall \xi \in \mathbb{R}^n.$$

Due to the space-time dependent potential considered in the present paper, it is not clear that how to adapt the Lebeau-Robbiano approach for the equation (1.1). To overcome this difficulty, we will utilize a different strategy developed in [3, 11] for parabolic equations in bounded domains. This new strategy is mainly based on a locally quantitative estimate of propagation of smallness for analytic functions.

In this paper, we try to generalize this approach for the fractional order parabolic equations in the whole space. To this end, we first study the analyticity for solutions of the equation (1.1) under the following spatial analyticity assumption on potentials $a(\cdot,\cdot)$:

(A) *There exist constants C and $R>0$ so that*

$$\sup_{t>0,x\in\mathbb{R}^n}|\partial_x^\alpha a(t,x)|\le\frac{C\alpha!}{R^{|\alpha|}},\quad\forall\alpha\in\mathbb{N}^n.$$

The first result of this paper can be stated as follows.

Theorem 1.1. *Assume that* (**A**) *is true and $s>1$. Then there exist constants $c>0, C>0$ such that for all solutions u of* (1.1) *and for all $t>0$,*

$$\left\|\widehat{u}(t,\cdot)e^{cR|\cdot|}\right\|_{L^2(\mathbb{R}^n)}\le\exp\left\{C\left[1+\left(t^{-1}R^s\right)^{\frac{1}{s-1}}+t\sup_{\tau>0}\|a(\tau,\cdot)\|_{A^{\frac{R}{2}}}\right]\right\}\|u_0\|_{L^2(\mathbb{R}^n)},\tag{1.2}$$

where the norm $\|\cdot\|_{A^R}$ is defined by (2.5) *below.*

The above estimate shows that the L^2-norm of $\widehat{u}(t,\cdot)e^{cR|\cdot|}$ is finite for every $t>0$. This, according to the well-known Paley-Wiener theorem [31], Theorem IX.13, p.18, implies that the solution $u(t,x)$ can be extended to a holomorphic function $u(t,z)$ in the strip $S_\sigma=\{z\in\mathbb{C}^n:|\mathrm{Im}z|<\sigma\}$ with $\sigma=cR$. Thus, $u(t,\cdot)$ has a uniformly lower bound of analyticity radius for any positive time.

Similar results have been proved in [11] for $2m$-order (m is an integer) parabolic equations in bounded domains with zero Dirichlet boundary conditions. The proof in [11] relies on Schauder's estimates, while (1.2) will be proved by certain tools in Fourier analysis, due to the nonlocal nature of the equation (1.1).

Another novelty of the above estimate (1.2) is that it gives an explicit linear dependence for the spatial analyticity radius of the solution in terms of that of potentials $a(\cdot,\cdot)$. This observation seems to be new according to the existing literature.

Next, based on an interpolation inequality on a thick set for a function f satisfying $\|e^{cR|\cdot|}\widehat{f}(\cdot)\|_{L^2(\mathbb{R}^n)}<\infty$, we could prove a Hölder-type interpolation inequalities of unique continuation for solutions of (1.1).

Theorem 1.2. *Let $s>1$,* (**A**) *hold and E be a thick set in $\mathbb{R}^n$. Then, there exist constants C and C' such that for all solutions u of* (1.1),

$$\int_{\mathbb{R}^n}|u(t,x)|^2\,\mathrm{d}x\le C_0\left(\int_E|u(t,x)|^2\,\mathrm{d}x\right)^\theta\|u_0\|_{L^2(\mathbb{R}^n)}^{2(1-\theta)}\tag{1.3}$$

holds for all $t>0$ and $\theta\in(0,e^{-C'\max\{1,R^{-1}\}})$, where

$$C_0=C\exp\left\{C\left[1+\left(t^{-1}R^s\right)^{\frac{1}{s-1}}+t\sup_{\tau>0}\|a(\tau,\cdot)\|_{A^{\frac{R}{2}}}\right]\right\}.$$

Finally, with regards to the observability inequality for solutions to (1.1), we have the following result.

Theorem 1.3. *Let $s > 1$, ($\mathbf{A}$) hold, and $E \subset \mathbb{R}^n$ be a thick set. Then there exists a constant $C > 0$ such that for all $T > 0$ and all solutions of* (1.1)

$$\int_{\mathbb{R}^n} |u(T,x)|^2 \,\mathrm{d}x \le C \exp\left(C \left(T + \frac{1}{T^{\frac{1}{s-1}}} \right) \right) \int_0^T \int_E |u(t,x)|^2 \,\mathrm{d}x \,\mathrm{d}t.$$

We mention that, in the regime $s > 1$, the observability inequality in Theorem 1.3 has been established in [1], Remark 1.13 and [30], Theorem 2.8 for some fractional heat equations (including the particular case that the potential $a = 0$). Theorem 1.3 generalizes the results in [1, 30] to parabolic equations with space-time potentials. Thus, it partially solves the question raised above under the assumption of spatial analyticity, while the complete answer is quite open.

As mentioned before, Theorem 1.3 cannot follow directly from the Lebeau-Robbiano strategy. Here we prove it by the interpolation inequality (1.3). Note that this approach has been used successfully in a recent work [8]. To prove (1.3), we shall establish the following unique continuation inequality for analytic functions (see Sect. 2.1 for a definition of G^σ and Thm. 3.5 below)

$$\|f\|_{L^p(\mathbb{R}^n)} \le C \|f\|_{L^p(E)}^{\theta} \|f\|_{G^\sigma}^{1-\theta}, \quad \theta \sim e^{-C' \max\{1,\sigma^{-1}\}}, 2 \le p \le \infty.$$

The restriction $s > 1$ is essential for our purpose.[1] In fact, if $0 < s \le 1$, then the fractional heat equation is not null controllable on a thick set E (say, E is the complement of a nonempty open set, see [18, 24, 27]). It is well known that the observability and controllability for linear PDEs are equivalent by the duality argument. We also refer the reader to, *e.g.*, [9, 13, 22, 23, 27–29] for null-controllability results for fractional heat equations on bounded domains.

We also note that there is a growing interest on observability inequalities of parabolic equations associated with the Schrödinger operator $-\Delta + V(x)$, where the potential $V(x) \to +\infty$ as $x \to \infty$. In particular, if $V(x) = |x|^{2k}$, k is an integer, then the solution is analytic and decreasing exponentially as $x \to \infty$. It is shown that the observability inequality holds on some sensor sets of decaying density, which are larger set classes than thick sets, see *e.g.* [7, 25, 26].

Throughout the paper, we use $A \lesssim B$ to denote $A \le CB$ for some universal constant $C > 0$. If both $A \lesssim B$ and $B \lesssim A$ hold, then we write $A \sim B$.

This paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we first establish some interpolation inequalities for analytic functions, and then use them to prove Theorem 1.2, with the aid of Theorem 1.1. Finally, we prove Theorem 1.3 in Section 4.

2. Proof of Theorem 1.1

2.1. Auxiliary results

For every $\sigma > 0$, we define the Banach space $G^\sigma = G^\sigma(\mathbb{R}^n)$, consisting of analytic functions in $S_\sigma = \{z \in \mathbb{C}^n : |\mathrm{Im} z| < \sigma\}$, endowed with the norm

$$\|f\|_{G^\sigma} = \sup_{|y|<\sigma} \|f(\cdot + iy)\|_{L^2(\mathbb{R}^n)}.$$

This kind of analytic functions, according to the Paley-Wiener theorem, is related to the function whose Fourier transform decays exponentially at infinity. The proof of the following lemma is inspired by Problem 76 in [31], p. 132.

[1] For the exponential stabilization of the fractional heat equation, this restriction can be removed, see [2] or [17], Lemma 2.2. We also note that if $s = 1$ (resp. with some log improvements), the solution is analytic (resp. quasi-analytic). This, together with unique continuation of analytic functions, implies the approximate null-controllability result, see *e.g.* [2].

Lemma 2.1. *For all $\sigma > 0$ and all $f \in G^\sigma$,*

$$\|e^{\frac{\sigma}{2}|\xi|}\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)} \lesssim \|f\|_{G^\sigma} \lesssim \|e^{\sigma|\xi|}\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)}. \tag{2.1}$$

Proof. We first claim that

$$\|f\|_{G^\sigma} \sim \sup_{|y|<\sigma} \|e^{y\cdot\xi}\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)}. \tag{2.2}$$

In fact, by the Fourier inversion,

$$f(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\widehat{f}(\xi)\,\mathrm{d}\xi, \quad \forall x \in \mathbb{R}^n.$$

Replacing x by $x+iy$, we find

$$f(x+iy) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-y\cdot\xi}\widehat{f}(\xi)\,\mathrm{d}\xi, \quad \forall x \in \mathbb{R}^n,$$

for every $|y| < \sigma$. By the Plancherel theorem,

$$\sup_{|y|<\sigma}\|f(\cdot+iy)\|_{L^2(\mathbb{R}^n)} \sim \sup_{|y|<\sigma}\|e^{-y\cdot\xi}\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)} = \sup_{|y|<\sigma}\|e^{y\cdot\xi}\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)},$$

which proves (2.2).

Next, we prove (2.1). We note that (2.2) implies $\|f\|_{G^\sigma} \lesssim \|e^{\sigma|\xi|}\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)}$, if we use the simple fact that $|y\cdot\xi| \le |y||\xi| \le \sigma|\xi|$. Thus, it remains to show $\|e^{\frac{\sigma}{2}|\xi|}\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)} \lesssim \|f\|_{G^\sigma}$. This, together with (2.2), reduces to prove $\|e^{\frac{\sigma}{2}|\xi|}\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)} \lesssim \sup_{|y|<\sigma}\|e^{y\cdot\xi}\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)}$. By a scaling argument, it suffices to consider the case $\sigma = 1$, namely,

$$\|e^{\frac{1}{2}|\xi|}\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)} \lesssim \sup_{|y|<1}\|e^{y\cdot\xi}\widehat{f}(\xi)\|_{L^2_\xi(\mathbb{R}^n)}. \tag{2.3}$$

In the case that $n = 1$, this holds clearly, see [16], p. 5285. But for the higher dimensional case we need more analysis. In fact, for every $\xi \in \mathbb{R}^n$, the Lebesgue measure

$$\left|\left\{|y| < 1 : y\cdot\frac{\xi}{|\xi|} \ge \frac{1}{2}\right\}\right| \sim 1.$$

This implies that for all $\xi \in \mathbb{R}^n$

$$e^{|\xi|}|\widehat{f}(\xi)|^2 \lesssim \int_{|y|<1} e^{2y\cdot\xi}|\widehat{f}(\xi)|^2\,\mathrm{d}y. \tag{2.4}$$

Integrating (2.4) over $\xi \in \mathbb{R}^n$, and using the Fubini theorem, we infer that

$$\|e^{\frac{1}{2}|\xi|}\widehat{f}(\xi)\|^2_{L^2_\xi(\mathbb{R}^n)} \lesssim \int_{|y|<1}\int_{\mathbb{R}^n} e^{2y\cdot\xi}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi\,\mathrm{d}y \lesssim \sup_{|y|<1}\int_{\mathbb{R}^n} e^{2y\cdot\xi}|\widehat{f}(\xi)|^2\,\mathrm{d}\xi.$$

This proves (2.3) and completes the proof. □

For every $\sigma > 0$, we introduce an analytic function space A^σ endowed with the norm

$$\|f\|_{A^\sigma} = \sum_{\alpha\in\mathbb{N}^n} \frac{\sigma^{|\alpha|}\|\partial_x^\alpha f\|_{L^\infty(\mathbb{R}^n)}}{\alpha!}. \tag{2.5}$$

Remark 2.2. If $a(\cdot,\cdot)$ satisfies the assumption $(\mathbf{A})$, then for all $t \geq 0$, $a(t,\cdot) \in A^{\frac{R}{2}}$. In fact,

$$\|a(t,\cdot)\|_{A^{\frac{R}{2}}} \leq \sum_{\alpha\in\mathbb{N}^n} \frac{(R/2)^{|\alpha|}\|\partial_x^\alpha a(t,\cdot)\|_{L^\infty(\mathbb{R}^n)}}{\alpha!} \leq C\sum_{\alpha\in\mathbb{N}^n} 2^{-|\alpha|} \lesssim 1.$$

Lemma 2.3. *For all $\sigma > 0$ and all $a \in A^\sigma, u \in G^\sigma$,*

$$\|au\|_{G^\sigma} \lesssim \|a\|_{A^\sigma}\|u\|_{G^\sigma}.$$

Proof. By the Taylor expansion

$$a(x+iy) = \sum_{\alpha\in\mathbb{N}^n} \frac{\partial_x^\alpha a(x)}{\alpha!}(iy)^\alpha,$$

we deduce that

$$\sup_{x\in\mathbb{R}^n, |y|<\sigma} |a(x+iy)| \leq \sum_{\alpha\in\mathbb{N}^n} \left|\frac{\partial_x^\alpha a(x)}{\alpha!}(iy)^\alpha\right| \leq \|a\|_{A^\sigma}. \tag{2.6}$$

Recalling the definition of G^σ norm, and using (2.6), we obtain

$$\|au\|_{G^\sigma} \lesssim \sup_{|y|<\sigma} \|(au)(x+iy)\|_{L_x^2(\mathbb{R}^n)} \leq \|a\|_{A^\sigma} \sup_{|y|<\sigma} \|u(x+iy)\|_{L_x^2(\mathbb{R}^n)} = \|a\|_{A^\sigma}\|u\|_{G^\sigma}.$$

This completes the proof. □

Let $\{e^{-t\Lambda^s}\}_{t\geq 0}$ be the semigroup generated by the fractional Laplacian $-\Lambda^s$ in $L^2(\mathbb{R}^n)$. This semigroup can be expressed by the Fourier transform as

$$\widehat{e^{-t\Lambda^s}f}(\xi) = e^{-t|\xi|^s}\widehat{f}(\xi), \quad \forall f \in L^2(\mathbb{R}^n).$$

Lemma 2.4. *Let $s > 1$. Then for all $t \geq 0$ and all $f \in L^2(\mathbb{R}^n)$*

$$\|e^{-t\Lambda^s}f\|_{G^{t^{\frac{1}{s}}}} \lesssim \|f\|_{L^2(\mathbb{R}^n)}.$$

Proof. By (2.1) and the Plancherel theorem, we have

$$\|e^{-t\Lambda^s}f\|_{G^{t^{\frac{1}{s}}}} \lesssim \|e^{t^{\frac{1}{s}}|\xi|-t|\xi|^s}\widehat{f}(\xi)\|_{L^2(\mathbb{R}^n)} \leq \|e^{t^{\frac{1}{s}}|\xi|-t|\xi|^s}\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.$$

The desired bound follows from the fact that

$$\|e^{t^{\frac{1}{s}}|\xi|-t|\xi|^s}\|_{L^\infty(\mathbb{R}^n)} = \sup_{\tau\geq 0} e^{\tau-\tau^s} \lesssim 1,$$

where we used $s > 1$ in the last inequality. □

Using the semigroup $\{e^{-t\Lambda^s}\}_{t\geq 0}$, we can rewrite the evolutional differential equation (1.1) as an integral equation

$$u(t) = e^{-t\Lambda^s}u_0 + \int_0^t e^{-(t-\tau)\Lambda^s}(au)(\tau)\,\mathrm{d}\tau. \tag{2.7}$$

Proposition 2.5. *Assume* $(\mathbf{A})$ *holds for some* $R > 0$*. Then for every* $u_0 \in L^2(\mathbb{R}^n)$*, there exists a unique solution* u *of* (2.7) *satisfying*

$$\|u(t,\cdot)\|_{G^{t^{\frac{1}{s}}}} \leq C\exp\left\{Ct\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\right\}\|u_0\|_{L^2(\mathbb{R}^n)}, \quad \forall t \in (0,(R/2)^s]\,,$$

where $C > 0$ *is a constant independent of* u_0*.*

Proof. Thanks to Lemmas 2.3 and 2.4, there exist constants $C_0, C_1 > 0$ such that for any $t > 0$

$$\|e^{-t\Lambda^s}u_0\|_{G^{t^{\frac{1}{s}}}} \leq C_0\|u_0\|_{L^2(\mathbb{R}^n)}, \tag{2.8}$$

$$\|au\|_{G^\sigma} \leq C_1\|a\|_{A^{\frac{R}{2}}}\|u\|_{G^\sigma}, \quad \sigma \in \left[0,\frac{R}{2}\right]. \tag{2.9}$$

Here $\|a\|_{A^{\frac{R}{2}}}$ is finite, see Remark 2.2.

Fix $T \in (0,(R/2)^s]$. Define

$$\mathcal{B} = \left\{u : \|u\|_X \leq M\|u_0\|_{L^2(\mathbb{R}^n)}\right\},$$

where $M = e^{C_2T}$, $C_2 = C_0C_1\sup_{t\geq 0}\|a(t,\cdot)\|_{A^{\frac{R}{2}}}$, and

$$\|u\|_X = C_0^{-1}\sup_{t\in[0,T]} e^{-C_2t}\|u(t,\cdot)\|_{G^{t^{\frac{1}{s}}}}.$$

We now consider the linear mapping

$$\Gamma u = e^{-t\Lambda^s}u_0 + \int_0^t e^{-(t-\tau)\Lambda^s}(au)(\tau)\,\mathrm{d}\tau.$$

We claim that $\Gamma\mathcal{B} \subset \mathcal{B}$. In fact, if $u \in \mathcal{B}$, then by (2.8) and (2.9) we have

$$\begin{aligned}
\|\Gamma u\|_X &\leq C_0^{-1}\sup_{t\in[0,T]} e^{-C_2t}\|e^{-t\Lambda^s}u_0\|_{G^{t^{\frac{1}{s}}}} \\
&\quad + C_0^{-1}\sup_{t\in[0,T]} e^{-C_2t}\int_0^t \|e^{-(t-\tau)\Lambda^s}(au)(\tau)\|_{G^{t^{\frac{1}{s}}}}\,\mathrm{d}\tau \\
&\leq \|u_0\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,T]} e^{-C_2t}\int_0^t C_1\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\|u(\tau,\cdot)\|_{G^{\tau^{\frac{1}{s}}}}\,\mathrm{d}\tau \\
&\leq \|u_0\|_{L^2(\mathbb{R}^n)} + \sup_{t\in[0,T]} e^{-C_2t}\int_0^t C_2e^{C_2\tau}\,\mathrm{d}\tau\|u\|_X \\
&= \|u_0\|_{L^2(\mathbb{R}^n)} + (1-e^{-C_2T})\|u\|_X
\end{aligned}$$

$$\leq \|u_0\|_{L^2(\mathbb{R}^n)} + (1-e^{-C_2 T})M\|u_0\|_{L^2(\mathbb{R}^n)}$$
$$= M\|u_0\|_{L^2(\mathbb{R}^n)}.$$

Note that in the second inequality above, we have used (2.8), (2.9), and the inequality

$$t^{\frac{1}{s}} - (t-\tau)^{\frac{1}{s}} \leq \tau^{\frac{1}{s}}$$

to obtain that

$$\|e^{-(t-\tau)\Lambda^s}(au)\|_{G^{t^{\frac{1}{s}}}} \leq C_0\|au\|_{G^{t^{\frac{1}{s}}-(t-\tau)^{\frac{1}{s}}}} \leq C_0\|au\|_{G^{\tau^{\frac{1}{s}}}} \leq C_0 C_1 \sup_{t\geq 0}\|a\|_{A^{\frac{R}{2}}}\|u\|_{G^{\tau^{\frac{1}{s}}}}.$$

In the third inequality above, we have used the definition of C_2 and the norm of X.

Moreover, if $u, v \in \mathcal{B}$, then

$$\|\Gamma u - \Gamma v\|_X \leq (1-e^{-C_2 T})\|u-v\|_X.$$

Hence, $\Gamma : \mathcal{B} \mapsto \mathcal{B}$ is a contraction mapping, and (2.7) has a unique solution u in $\mathcal{B}$ satisfying the bound

$$C_0^{-1} \sup_{t\in[0,T]} e^{-C_2 t}\|u(t,\cdot)\|_{G^{t^{\frac{1}{s}}}} \leq M\|u_0\|_{L^2(\mathbb{R}^n)},$$

which implies that

$$\|u(T,\cdot)\|_{G^{T^{\frac{1}{s}}}} \leq C_0 e^{2C_2 T}\|u_0\|_{L^2(\mathbb{R}^n)}.$$

This gives the desired bound and completes the proof, since $T \in (0, (R/2)^s]$ is arbitrary. ◻

Proposition 2.6. *Assume* (**A**) *holds for some* $R > 0$*. Let* $b \in [0, \frac{R}{2}]$*. Then there exists a constant* $C > 0$ *so that for all* $u_0 \in G^b$*, the solution of* (2.7) *satisfies*

$$\|u(t,\cdot)\|_{G^{t^{\frac{1}{s}}+b}} \leq C\exp\left\{Ct\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\right\}\|u_0\|_{G^b}, \quad \forall t \in (0, (R/2-b)^s].$$

Proof. The proof is the same as above, and so it is omitted here. In fact, it suffices to use

$$\|e^{-t\Lambda^s}u_0\|_{G^{t^{\frac{1}{s}}+b}} \leq C_0\|u_0\|_{G^b},$$

instead of (2.8) along the above lines. ◻

2.2. Proof of Theorem 1.1

We shall utilize Proposition 2.6 repeatedly to obtain a lower bound of analyticity radius, which is independent of the time variable. We begin with the following lemma.

Lemma 2.7. *Assume that* (**A**) *holds for some* $R > 0$*, then there exist constants* $c, C > 0$ *so that the solution* u *of* (2.7) *satisfies*

$$\|u(t,\cdot)\|_{G^{cR}} \leq \exp\left\{C(t^{-1}R^s)^{\frac{1}{s-1}} + Ct\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\right\}\|u_0\|_{L^2(\mathbb{R}^n)}, \quad t \in (0, (R/2)^s].$$

Proof. Let $t \in (0, T]$ with $T = (\frac{R}{2})^s$. For every $m \geq 1$, we make the decomposition

$$[0,t] = \left[0, \frac{1}{m}t\right] \bigcup \left[\frac{1}{m}t, \frac{2}{m}t\right] \bigcup \cdots \bigcup \left[\frac{(m-1)}{m}t, t\right].$$

Applying Proposition 2.6 on each interval $[\frac{j-1}{m}t, \frac{j}{m}t], j = 1, 2, \cdots, m$, we get

$$\begin{aligned}
\|u(\frac{1}{m}t,\cdot)\|_{G^{(\frac{1}{m}t)^{\frac{1}{s}}}} &\leq C \exp\left\{C\frac{1}{m}t\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\right\} \|u_0\|_{L^2(\mathbb{R}^n)}, \\
\|u(\frac{2}{m}t,\cdot)\|_{G^{2(\frac{1}{m}t)^{\frac{1}{s}}}} &\leq C \exp\left\{C\frac{1}{m}t\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\right\} \|u(\frac{1}{m}t)\|_{G^{(\frac{1}{m}t)^{\frac{1}{s}}}}, \\
&\cdots \\
\|u(\frac{j}{m}t,\cdot)\|_{G^{j(\frac{1}{m}t)^{\frac{1}{s}}}} &\leq C \exp\left\{C\frac{1}{m}t\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\right\} \|u(\frac{j-1}{m}t)\|_{G^{(j-1)(\frac{1}{m}t)^{\frac{1}{s}}}}, \\
&\cdots \\
\|u(t,\cdot)\|_{G^{m(\frac{1}{m}t)^{\frac{1}{s}}}} &\leq C \exp\left\{C\frac{1}{m}t\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\right\} \|u(\frac{m-1}{m}t)\|_{G^{(m-1)(\frac{1}{m}t)^{\frac{1}{s}}}}.
\end{aligned}$$

Combining these inequalities, we infer that

$$\|u(t,\cdot)\|_{G^{m(\frac{1}{m}t)^{\frac{1}{s}}}} \leq C^m \exp\{Ct\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\}\|u_0\|_{L^2(\mathbb{R}^n)}, \tag{2.10}$$

provided that

$$m(\frac{1}{m}t)^{\frac{1}{s}} \leq \frac{R}{2}.$$

Choose $m \in \mathbb{N}$ so that

$$m(\frac{1}{m}t)^{\frac{1}{s}} \sim R,$$

which is in fact equivalent to

$$m \sim \left(t^{-1}R^s\right)^{\frac{1}{s-1}}.$$

Hence, it follows from (2.10) that for some $c, C' > 0$

$$\|u(t,\cdot)\|_{G^{cR}} \leq C^{C'\left(t^{-1}R^s\right)^{\frac{1}{s-1}}} \exp\left\{Ct\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\right\} \|u_0\|_{L^2(\mathbb{R}^n)}, \quad t \in (0, (R/2)^s].$$

This implies the desired bound and completes the proof. □

Proof of Theorem 1.1. Let $t_0 = (\frac{R}{2})^s$. It follows from Lemma 2.7 that

$$\begin{aligned}
\|u(t_0,\cdot)\|_{G^{cR}} &\leq \exp\left\{C(t_0^{-1}R^s)^{\frac{1}{s-1}} + Ct_0\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\right\} \|u_0\|_{L^2(\mathbb{R}^n)} \\
&\leq \exp\left\{C(1 + t_0\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})})\right\} \|u_0\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Similarly, we have for all $\tau \geq 0$ that

$$\|u(\tau+t_0,\cdot)\|_{G^{cR}} \leq \exp\left\{C(1+t_0\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})})\right\}\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}. \tag{2.11}$$

By the classical energy estimate

$$\|u(\tau,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \exp\{C\tau\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\}\|u_0\|_{L^2(\mathbb{R}^n)},$$

we deduce from (2.11) that for all $t \geq t_0$

$$\|u(t,\cdot)\|_{G^{cR}} \leq \exp\left\{C(1+t\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})})\right\}\|u_0\|_{L^2(\mathbb{R}^n)}. \tag{2.12}$$

By Lemma 2.7 again, we have for all $t \in (0,t_0]$

$$\|u(t,\cdot)\|_{G^{cR}} \leq \exp\left\{C\left[(t^{-1}R^s)^{\frac{1}{s-1}}+t\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\right]\right\}\|u_0\|_{L^2(\mathbb{R}^n)}. \tag{2.13}$$

Together with (2.12) and (2.13), we infer that for any $t>0$

$$\|u(t,\cdot)\|_{G^{cR}} \leq \exp\left\{C\left[1+(t^{-1}R^s)^{\frac{1}{s-1}}+t\|a\|_{L^\infty(0,\infty;A^{\frac{R}{2}})}\right]\right\}\|u_0\|_{L^2(\mathbb{R}^n)}.$$

This, along with Lemma 2.1, indicates the desired bound (1.2), and completes the proof. □

3. Proof of Theorem 1.2

We first prove a global interpolation inequality on thick sets for analytic functions, and then prove Theorem 1.2. To this end, we first recall a local interpolation inequality for analytic functions.

Lemma 3.1 ([3], Thm. 1.3, [10], Lem. 1, [11], Lem. 1.9)**.** *Let $R>0$ and let $f: B_{2R}\subset\mathbb{R}^n\to\mathbb{R}$ be analytic in B_{2R} verifying*

$$|\partial_x^\alpha f(x)| \leq M(\rho R)^{-|\alpha|}|\alpha|!, \quad \textit{when } x\in B_{2R} \textit{ and } \alpha\in\mathbb{N}^n$$

for some positive numbers M and $\rho\in(0,1]$. Let $\omega\subset B_R$ be a subset of positive measure. Then there are constants $C=C(\rho,|\omega|/|B_R|)>0$ and $\theta=\theta(\rho,|\omega|/|B_R|)\in(0,1)$ so that

$$\|f\|_{L^\infty(B_R)} \leq CM^{1-\theta}\Big(\frac{1}{|\omega|}\int_\omega |f(x)|\,\mathrm{d}x\Big)^\theta. \tag{3.1}$$

Here and in the sequel, we use B_R to denote a ball in $\mathbb{R}^n$ with a radius R.

Remark 3.2. As a consequence of (3.1), we could derive its L^p-version ($1\leq p<\infty$), which will be useful later. Let $1\leq p<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. By the Hölder inequality, we have

$$|B_R|^{\frac{1}{p}}\|f\|_{L^\infty(B_R)} \geq \|f\|_{L^p(B_R)}, \quad \frac{1}{|\omega|}\int_\omega |f(x)|\,\mathrm{d}x \leq |\omega|^{\frac{1}{p'}-1}\|f\|_{L^p(\omega)}.$$

Inserting them into (3.1) we obtain

$$\|f\|_{L^p(B_R)} \leq C|B_R|^{\frac{1}{p}}|\omega|^{\theta(\frac{1}{p'}-1)}M^{1-\theta}\|f\|^\theta_{L^p(\omega)}. \tag{3.2}$$

Clearly, (3.2) also holds in the case that $p = \infty$.

Based on Lemma 3.1, we establish an interpolation inequality of unique continuation for functions in G^σ, with an explicit interpolation index.

In the sequel, let Q_L be a closed cube with a side $L > 0$ centered at some point in $\mathbb{R}^n$. We do not specify the center point, since the estimates below are uniformly for cubes with different centers.

Lemma 3.3. *Let $2 \le p \le \infty, L, \sigma > 0$ and $\omega \subset Q_L$ be a subset of positive measure. Then there exist two constants $C = C(p, n, |\omega|, L, \sigma) > 0$ and $C' = C'(n, |\omega|, L) > 0$ so that*

$$\|f\|_{L^p(Q_L)} \le C\|f\|^{\theta}_{L^p(\omega)} M^{1-\theta}$$

holds for all $f \in G^\sigma$ and $\theta \in (0, e^{-C' \max\{1, \frac{L}{\sigma}\}})$, where

$$M = \sup_{\alpha \in \mathbb{N}^n} \frac{\sigma^{|\alpha|}}{|\alpha|!} \|\partial_x^\alpha f\|_{L^\infty(Q_{2L})}.$$

Proof. Arbitrarily take $f \in G^\sigma$ with some $\sigma > 0$. Clearly, f is analytic on $\mathbb{R}^n$. Also, by the definition of M, we have

$$\|\partial_x^\alpha f\|_{L^\infty(Q_{2L})} \le M \left(\frac{1}{\sigma}\right)^{|\alpha|} |\alpha|!, \quad \text{for all } \alpha \in \mathbb{N}^n. \tag{3.3}$$

If $\sigma \ge L$, then (3.3) holds with $\left(\frac{1}{\sigma}\right)^{|\alpha|}$ replaced by $\left(\frac{1}{L}\right)^{|\alpha|}$. Since $\omega \subset Q_L$ has a positive measure, we can apply Lemma 3.1 and Remark 3.2 (with $\rho = 1$, $R = L$) to obtain that

$$\|f\|_{L^p(Q_L)} \le C\|f\|^{\theta_0}_{L^p(\omega)} M^{1-\theta_0}$$

for some $\theta_0 \in (0, 1)$. Thus the lemma holds in this case.

Now we consider the reminder case that $\sigma < L$. We first claim that there exists a point $x_0 \in Q_L$ so that $Q_{\frac{\sigma}{3}}(x_0) \subset Q_L$ and

$$\frac{|\omega \bigcap Q_{\frac{\sigma}{3}}(x_0)|}{|Q_{\frac{\sigma}{3}}(x_0)|} \ge c_0 \tag{3.4}$$

for some $c_0 = c_0(n, L, |\omega|) > 0$. In fact, let $k \ge 1$ be an integer, we split Q_L as disjoint (except for the boundary) small cubes $Q_{\frac{L}{k}}(x_i)$, $i = 1, 2, \cdots, k^n$. Note that if the center of Q_L is given, then x_i are determined uniquely. Then

$$|\omega| = \sum_{1 \le i \le k^n} |\omega \bigcup Q_{\frac{L}{k}}(x_i)|.$$

Choose $x_0 \in \{x_1, x_2, \cdots, x_{k^n}\}$ so that $|\omega \bigcup Q_{\frac{L}{k}}(x_0)| = \max_{1 \le i \le k^n} |\omega \bigcup Q_{\frac{L}{k}}(x_i)|$. Since the number of small cubes is k^n, we infer that

$$|\omega \bigcup Q_{\frac{L}{k}}(x_0)| \ge k^{-n}|\omega|. \tag{3.5}$$

Set $k=[\frac{3L}{\sigma}]+1$. Then $\frac{L}{k}\le\frac{\sigma}{3}$, and of course $Q_{\frac{L}{k}}(x_0)\subset Q_{\frac{\sigma}{3}}$. Thus (3.5) becomes

$$|\omega\bigcup Q_{\frac{\sigma}{3}}(x_0)|\ge k^{-n}|\omega|,$$

which, noting that $k\le\frac{4L}{\sigma}$, implies that

$$\frac{|\omega\bigcap Q_{\frac{\sigma}{3}}(x_0)|}{|Q_{\frac{\sigma}{3}}(x_0)|}\ge\frac{k^{-n}|\omega|}{(\frac{\sigma}{3})^n}\ge\left(\frac{3}{4L}\right)^n|\omega|.$$

This proves the claim (3.4).

Since the proof for the case $p=\infty$ is similar, without loss of generality, we can now assume that $2\le p<\infty$. From (3.4) and the bound (3.3), we apply Lemma 3.1 and Remark 3.2 (with $\rho=1$, $R=\sigma$, ω replaced by $\omega\bigcap Q_{\frac{\sigma}{3}}$) to obtain that

$$\int_{Q_\sigma(x_0)}|f(x)|^p\,\mathrm{d}x\le C\left(\int_{\omega\bigcap Q_{\frac{\sigma}{3}}(x_0)}|f(x)|^p\,\mathrm{d}x\right)^\delta M^{p(1-\delta)}. \tag{3.6}$$

Similarly, for all $Q_\sigma(y)\subset Q_L$

$$\int_{Q_\sigma(y)}|f(x)|^p\,\mathrm{d}x\le C\left(\int_{Q_{\frac{\sigma}{3}}(y)}|f(x)|^p\,\mathrm{d}x\right)^\delta M^{p(1-\delta)}, \tag{3.7}$$

where $\delta=\delta(n,L,|\omega|)\in(0,1)$ and $C=C(p,n,L,|\omega|,\sigma)>0$. We point out that δ is independent of σ, which is important in our proof.

On one hand, it follows from (3.6) that

$$\int_{Q_{\frac{\sigma}{3}}(x_0)}|f(x)|^p\,\mathrm{d}x\le C\left(\int_\omega|f(x)|^p\,\mathrm{d}x\right)^\delta M^{p(1-\delta)}. \tag{3.8}$$

On the other hand, it follows from (3.7) that

$$\int_{Q_{\frac{\sigma}{3}}(y)}|f(x)|^p\,\mathrm{d}x\le C\left(\int_{Q_{\frac{\sigma}{3}}(y')}|f(x)|^p\,\mathrm{d}x\right)^\delta M^{p(1-\delta)} \tag{3.9}$$

for all $y,y'\in\mathbb{R}^n$ satisfying $|y-y'|\le\frac{\sigma}{3}$ and $Q_{\frac{\sigma}{3}}(y),Q_{\frac{\sigma}{3}}(y')\subset Q_L$. With (3.9) in hand, for every $m\in\mathbb{N}$, we can use the Harnack chain argument to prove that

$$\int_{Q_{\frac{\sigma}{3}}(y)}|f(x)|^p\,\mathrm{d}x\le C\left(\int_{Q_{\frac{\sigma}{3}}(y')}|f(x)|^p\,\mathrm{d}x\right)^{\delta^m} M^{p(1-\delta^m)} \tag{3.10}$$

for all $y,y'\in\mathbb{R}^n$ satisfying $|y-y'|\le\frac{m\sigma}{3}$ and $Q_{\frac{\sigma}{3}}(y),Q_{\frac{\sigma}{3}}(y')\subset Q_L$. Indeed, if $|y-y'|\le\frac{m\sigma}{3}$, then there exists a sequence $\{y_i\}_{i=0}^{m+1}$ in $\mathbb{R}^n$ such that $Q_{\frac{\sigma}{3}}(y_i)\subset Q_L$ and

$$y_0=y,\quad y_{m+1}=y',\quad\text{and}\quad |y_i-y_{i+1}|\le\frac{\sigma}{3},\ \forall i=0,1,\cdots,m.$$

For every $i = 0, 1, \cdots, m$, applying (3.9) to cubes $Q_{\frac{\sigma}{3}}(y_i)$ and $Q_{\frac{\sigma}{3}}(y_{i+1})$, we obtain

$$\int_{Q_{\frac{\sigma}{3}}(y_i)} |f(x)|^p \, \mathrm{d}x \leq C \left(\int_{Q_{\frac{\sigma}{3}}(y_{i+1})} |f(x)|^p \, \mathrm{d}x \right)^{\delta} M^{p(1-\delta)}. \tag{3.11}$$

Iterating (3.11) m times gives (3.10) with a new constant C.

Since $x_0 \in Q_L$, we have

$$|y - x_0| \leq \sqrt{n}L, \quad \text{for all } y \in Q_L.$$

This, together with (3.10) (setting $y' = x_0$), implies that for all $Q_\sigma(y) \subset Q_L$

$$\int_{Q_{\frac{\sigma}{3}}(y)} |f(x)|^p \, \mathrm{d}x \leq C \left(\int_{Q_{\frac{\sigma}{3}}(x_0)} |f(x)|^p \, \mathrm{d}x \right)^{\delta^m} M^{p(1-\delta^m)}, \tag{3.12}$$

where $m = [\frac{3\sqrt{n}L}{\sigma}] + 1$. Integrating (3.12) over $\{y \in Q_L : Q_\sigma(y) \subset Q_L\}$, we infer that

$$\int_{Q_L} |f(x)|^p \, \mathrm{d}x \leq C \left(\int_{Q_{\frac{\sigma}{3}}(x_0)} |f(x)|^p \, \mathrm{d}x \right)^{\delta^m} M^{p(1-\delta^m)}, \tag{3.13}$$

for some constant $C > 0$.

Finally, combining (3.8) and (3.13), we get

$$\int_{Q_L} |f(x)|^p \, \mathrm{d}x \leq C \left(\int_{\omega} |f(x)|^p \, \mathrm{d}x \right)^{\delta^{m+1}} M^{p(1-\delta^{m+1})}. \tag{3.14}$$

Recall that $m \leq \frac{L}{\sigma}(3\sqrt{n} + 1)$, we have for some $C' = C(n, L, |\omega|) > 0$

$$\delta^{m+1} = e^{-(m+1)\log \delta^{-1}} \geq e^{-C' \frac{L}{\sigma}} := \theta.$$

This, together with (3.14) and the trivial bound $\int_\omega |f(x)|^p \, \mathrm{d}x \leq M^p$, gives that

$$\int_{Q_L} |f(x)|^p \, \mathrm{d}x \leq C \left(\int_{\omega} |f(x)|^p \, \mathrm{d}x \right)^{\theta} M^{p(1-\theta)}.$$

This completes the proof. □

For our purpose, we need to bound the quantity M in Lemma 3.3 in terms of $\|f\|_{G^\sigma}$. To this end, for every $j \in \mathbb{Z}^n$, we define

$$M_j = \sup_{\alpha \in \mathbb{N}^n} \frac{\sigma^{|\alpha|}}{|\alpha|!} \|\partial_x^\alpha f\|_{L^\infty(Q_{2L}(jL))}. \tag{3.15}$$

Here we use the convention notation $jL = (j_1 L, j_2 L, \cdots, j_n L) \in \mathbb{R}^n$.

Lemma 3.4. *Let $p \geq 2$ and $\sigma, L > 0$. Then there exists $C = C(n, \sigma, L) > 0$ so that*

$$\left(\sum_{j\in\mathbb{Z}^n} M_j^p\right)^{\frac{1}{p}} \leq C\|f\|_{G^{4\sigma}}, \quad \text{for all } f \in G^{4\sigma}.$$

Proof. Thanks to the inequality

$$\left(\sum_{j\in\mathbb{Z}^n} M_j^p\right)^{\frac{1}{p}} \leq \left(\sum_{j\in\mathbb{Z}^n} M_j^2\right)^{\frac{1}{2}}, \text{ when } p \geq 2,$$

it suffices to consider the case $p = 2$. Using the definition (3.15), we have

$$\sum_{j\in\mathbb{Z}^n} M_j^2 = \sum_{j\in\mathbb{Z}^n}\left(\sup_{\alpha\in\mathbb{N}^n}\frac{\sigma^{|\alpha|}}{|\alpha|!}\|\partial_x^\alpha f\|_{L^\infty(Q_{2L}(jL))}\right)^2. \tag{3.16}$$

We claim that there exists $C = C(n, L) > 0$ so that

$$\sup_{\alpha\in\mathbb{N}^n}\frac{\sigma^{|\alpha|}}{|\alpha|!}\|\partial_x^\alpha f\|_{L^\infty(Q_{2L})} \leq C(1+\sigma^{-n})\sup_{\alpha\in\mathbb{N}^n}\frac{(2\sigma)^{|\alpha|}}{|\alpha|!}\|\partial_x^\alpha f\|_{L^2(Q_{2L})}. \tag{3.17}$$

In fact, since Q_{2L} satisfies the cone property, by the Sobolev embeding $\|f\|_{L^\infty(Q_{2L})} \leq C\|f\|_{H^n(Q_{2L})}$, we have

$$\|\partial_x^\alpha f\|_{L^\infty(Q_{2L})} \leq C\sum_{\beta\in\mathbb{N}^n, |\beta|\leq n}\|\partial_x^{\alpha+\beta} f\|_{L^2(Q_{2L})}. \tag{3.18}$$

For $\beta \in \mathbb{N}^n, |\beta| \leq n$

$$\begin{aligned}\|\partial_x^{\alpha+\beta} f\|_{L^2(Q_{2L})} &\leq (2\sigma)^{-|\alpha+\beta|}(\alpha+\beta)!\sup_{\alpha\in\mathbb{N}^n}\frac{(2\sigma)^{|\alpha|}}{|\alpha|!}\|\partial_x^\alpha f\|_{L^2(Q_{2L})}\\ &\leq C(1+\sigma^{-n})\sigma^{-|\alpha|}|\alpha|!\sup_{\alpha\in\mathbb{N}^n}\frac{(2\sigma)^{|\alpha|}}{|\alpha|!}\|\partial_x^\alpha f\|_{L^2(Q_{2L})}.\end{aligned}$$

Here we used the facts that $\sigma^{-|\alpha+\beta|} \leq \sigma^{-|\alpha|}(1+\sigma^{-n})$ and $2^{-|\alpha|}(\alpha+\beta)! \leq C|\alpha|!$. This, together with (3.18), gives the bound (3.17).

Note that (3.17) holds for all $Q_{2L}(jL), j \in \mathbb{Z}^n$, we deduce from (3.16) that

$$\begin{aligned}\sum_{j\in\mathbb{Z}^n} M_j^2 &\lesssim \sum_{j\in\mathbb{Z}^n}\left(\sup_{\alpha\in\mathbb{N}^n}\frac{(2\sigma)^{|\alpha|}}{|\alpha|!}\|\partial_x^\alpha f\|_{L^2(Q_{2L}(jL))}\right)^2\\ &\lesssim \sup_{\alpha\in\mathbb{N}^n}\left(\frac{(2\sigma)^{|\alpha|}}{|\alpha|!}\right)^2\sum_{j\in\mathbb{Z}^n}\|\partial_x^\alpha f\|^2_{L^2(Q_{2L}(jL))}\\ &\lesssim \left(\sup_{\alpha\in\mathbb{N}^n}\frac{(2\sigma)^{|\alpha|}}{|\alpha|!}\|\partial_x^\alpha f\|_{L^2(\mathbb{R}^n)}\right)^2.\end{aligned}$$

Here, the implicit constant depends only on n, σ, L. Then the lemma follows if we can show that

$$\sup_{\alpha\in\mathbb{N}^n}\frac{(2\sigma)^{|\alpha|}}{|\alpha|!}\|\partial_x^\alpha f\|_{L^2(\mathbb{R}^n)}\lesssim\|f\|_{G^{4\sigma}}. \tag{3.19}$$

In fact, by the Plancherel theorem, we have

$$\sup_{\alpha\in\mathbb{N}^n}\frac{(2\sigma)^{|\alpha|}}{|\alpha|!}\|\partial_x^\alpha f\|_{L^2(\mathbb{R}^n)}\sim\sup_{\alpha\in\mathbb{N}^n}\frac{(2\sigma)^{|\alpha|}}{|\alpha|!}\|(i\xi)^\alpha\widehat{f}\|_{L^2(\mathbb{R}^n)}\lesssim\|e^{2\sigma|\xi|}\widehat{f}(\xi)\|_{L^2(\mathbb{R}^n)}, \tag{3.20}$$

where we have used $(2\sigma)^{|\alpha|}|(i\xi)^\alpha|\leq(2\sigma|\xi|)^{|\alpha|}\leq|\alpha|!e^{2\sigma|\xi|}$. The bound (3.20) and Lemma 2.1 imply (3.19), and thus the proof is completed. □

We present the following Hölder-type inequality of unique continuation on thick sets for analytic functions in G^σ.

Theorem 3.5. *Let $2\leq p\leq\infty, \sigma>0$ and E be a thick set in $\mathbb{R}^n$. Then there exist two constants $C=C(p,n,E,\sigma)>0$ and $C'=C'(n,E)>0$ so that*

$$\|f\|_{L^p(\mathbb{R}^n)}\leq C\|f\|_{L^p(E)}^{\theta}\|f\|_{G^\sigma}^{1-\theta} \tag{3.21}$$

holds for all $f\in G^\sigma$ and all $\theta\in(0,e^{-C'\max\{1,\sigma^{-1}\}})$.

In particular, by letting $p=2$, we obtain the following result.

Corollary 3.6. *Let $\sigma>0$ and E be a thick set in $\mathbb{R}^n$. Then there exist two constants $C=C(n,E,\sigma)>0$ and $C'=C'(n,E)>0$ so that*

$$\int_{\mathbb{R}^n}|f(x)|^2\,\mathrm{d}x\leq C\left(\int_E|f(x)|^2\,\mathrm{d}x\right)^\theta\|f\|_{G^\sigma}^{2(1-\theta)} \tag{3.22}$$

holds for all $f\in G^\sigma$ and $\theta\in(0,e^{-C'\max\{1,\sigma^{-1}\}})$.

Before proving Theorem 3.5, we give two remarks below.

Remark 3.7. The inequality (3.21) fails in the case that $1\leq p<2$. Indeed, giving now $1\leq p<2$, since $\|f\|_{L^p(E)}\leq\|f\|_{L^p(\mathbb{R}^n)}$, this claim follows if we can disprove

$$\|f\|_{L^p(\mathbb{R}^n)}\leq C\|f\|_{G^\sigma},\quad\forall f\in G^\sigma. \tag{3.23}$$

To this end, for every $s>0$, we define a function

$$f_s(x)=(1+|x|^2)^{-\frac{s}{2}},\quad x\in\mathbb{R}^n.$$

Then $f_s\in L^p(\mathbb{R}^n)$ if and only if $s>\frac{n}{p}$. Moreover, by [14], Proposition 6.1.5, p. 6, if $s<n$, then the Fourier transform $\widehat{f_s}$ satisfies that

$$|\widehat{f}_s(\xi)|\leq\begin{cases}Ce^{-\frac{1}{2}|\xi|}, & |\xi|\geq 2,\\ C|\xi|^{s-n}, & |\xi|\leq 2.\end{cases}$$

Then $f_s \in G^{\frac{1}{4}}$ if $s \in (\frac{n}{2}, n)$. Since $p < 2$, we can always choose s_0 so that $s_0 \le \frac{n}{p}$ and $s_0 \in (\frac{n}{2}, n)$. Then $f_{s_0} \in G^{\frac{1}{4}}$ but $f_{s_0} \notin L^p(\mathbb{R}^n)$, this shows that (3.23) fails to hold for $\sigma = \frac{1}{4}$. We conclude the same result for the general $\sigma > 0$ after a scaling argument.

Remark 3.8. We recall here some previous works on the interpolation inequality (3.22).

- In [20], E is thick, but no explicit dependence of θ on σ, proved by Carleman estimates.
- In [34], E is the complement set of a ball, $\theta \sim e^{-1/\sigma}$, proved by the three-ball inequality of analytic functions.
- In [6, 15], E is a Borel set satisfying the thickness condition, $\theta \sim e^{-1/\sigma}$, proved by harmonic measure estimate.

Note that the complement set of every ball is a thick set, and that every Borel set is a Lebesgue measurable set (but the converse is not true). Thus, Corollary 3.6 recovers all the results in [6, 15, 20, 34] in a unified way.

Proof of Theorem 3.5. Let E be a thick set. Then there exists $L > 0$ so that

$$\inf_{j \in \mathbb{Z}^n} |E \bigcap Q_L(jL)| = C_0 > 0.$$

Let $\sigma > 0$. Since the lower bound C_0 is independent of j, we apply Lemma 3.3 (with $\omega = E \bigcap Q_L(jL)$) to find that for all $j \in \mathbb{Z}^n$

$$\|f\|_{L^p(Q_L(jL))} \le C \|f\|^{\theta}_{L^p(E \bigcap Q_L(jL))} M_j^{1-\theta}, \tag{3.24}$$

where $\theta = e^{-C' \max\{1, L/\sigma\}} \in (0,1)$, $C = C(p, n, |\omega|, L, \sigma) > 0$, $C' = C'(n, |\omega|, L) > 0$ and

$$M_j = \sup_{\alpha \in \mathbb{N}^n} \frac{\sigma^{|\alpha|}}{|\alpha|!} \|\partial_x^\alpha f\|_{L^\infty(Q_{2L}(jL))}.$$

The reminding proof splits into two cases.

Case (1): $p = \infty$**.** By Lemma 3.4, we know

$$\sup_{j \in \mathbb{Z}^n} M_j \le C \|f\|_{G^{4\sigma}},$$

which, together with (3.24), shows that

$$\|f\|_{L^\infty(\mathbb{R}^n)} \le \sup_{j \in \mathbb{Z}^n} \|f\|_{L^\infty(Q_L(jL))} \le C \sup_{j \in \mathbb{Z}^n} \|f\|^{\theta}_{L^\infty(E \bigcap Q_L(jL))} M_j^{1-\theta} \le \|f\|^{\theta}_{L^\infty(E)} \|f\|^{1-\theta}_{G^{4\sigma}}.$$

Note that this holds for all $\sigma > 0$, replacing 4σ by σ, we conclude (3.21) in this case.

Case (2): $2 \le p < \infty$**.** Taking the p-th power of (3.24) we obtain

$$\int_{Q_L(jL)} |f(x)|^p \,\mathrm{d}x \le C \left(\int_{E \bigcap Q_L(jL)} |f(x)|^p \,\mathrm{d}x \right)^{\theta} M_j^{p(1-\theta)}, \quad \forall j \in \mathbb{Z}^n. \tag{3.25}$$

Using the decomposition

$$\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \sim \sum_{j \in \mathbb{Z}^n} \int_{Q_L(jL)} |f(x)|^p \,\mathrm{d}x$$

and the bound (3.25), by Hölder's inequality we deduce

$$
\begin{aligned}
\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x &\le \sum_{j\in\mathbb{Z}^n} C\left(\int_{E\bigcap Q_L(jL)} |f(x)|^p \,\mathrm{d}x\right)^{\theta} M_j^{p(1-\theta)} \\
&\le C\left(\varepsilon \sum_{j\in\mathbb{Z}^n} M_j^p + \varepsilon^{-\frac{1-\theta}{\theta}} \sum_{j\in\mathbb{Z}^n} \int_{E\bigcap Q_L(jL)} |f(x)|^p \,\mathrm{d}x\right) \qquad (3.26)
\end{aligned}
$$

for all $\varepsilon > 0$.

On one hand,

$$
\sum_{j\in\mathbb{Z}^n} \int_{E\bigcap Q_L(jL)} |f(x)|^p \,\mathrm{d}x \sim \int_E |f(x)|^p \,\mathrm{d}x. \qquad (3.27)
$$

On the other hand, thanks to Lemma 3.4, we have

$$
\sum_{j\in\mathbb{Z}^n} M_j^p \le C\|f\|_{G^{4\sigma}}^p. \qquad (3.28)
$$

Inserting (3.27) and (3.28) into (3.26), we have

$$
\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \le C\left(\varepsilon\|f\|_{G^{4\sigma}}^p + \varepsilon^{-\frac{1-\theta}{\theta}} \int_E |f(x)|^p \,\mathrm{d}x\right) \qquad (3.29)
$$

for all $\varepsilon > 0$.

By taking $\varepsilon = \varepsilon_0$ so that

$$
\varepsilon_0\|f\|_{G^{4\sigma}}^p = \varepsilon_0^{-\frac{1-\theta}{\theta}} \int_E |f(x)|^p \,\mathrm{d}x,
$$

we deduce from (3.29) that

$$
\int_{\mathbb{R}^n} |f(x)|^p \,\mathrm{d}x \le C\left(\int_E |f(x)|^p \,\mathrm{d}x\right)^{\theta} \|f\|_{G^{4\sigma}}^{p(1-\theta)}.
$$

This shows that (3.21) holds true for $2 \le p < \infty$. It completes the proof. □

Proof of Theorem 1.2. Let E be a thick set. It follows from Corollary 3.6 and Lemma 2.1 that there exist two constants $C = C(n, E, \sigma) > 0$ and $C' = C'(n, E) > 0$ so that

$$
\int_{\mathbb{R}^n} |f(x)|^2 \,\mathrm{d}x \le C\left(\int_E |f(x)|^2 \,\mathrm{d}x\right)^{\theta} \|e^{\sigma|\xi|}\widehat{f}(\xi)\|_{L^2(\mathbb{R}^n)}^{2(1-\theta)} \qquad (3.30)
$$

holds for all $\sigma > 0$ and $\theta \in (0, e^{-C'\max\{1,\sigma^{-1}\}})$. Hence, Theorem 1.2 follows from Theorem 1.1 and (3.30) immediately. □

4. Proof of Theorem 1.3

In this section, we first present an abstract approach to ensure an observability inequality (which is actually motivated from [4], Thm. 11). Then we apply it to solutions of (1.1) (*i.e.*, giving the proof of Theorem 1.3).

Proposition 4.1. *Let $\delta > 0$, $0 < \theta < 1, C \geq 1$ and E be a subset of $\mathbb{R}^n$. Assume that u is a function in $[0,1] \times \mathbb{R}^n$ so that $\sup_{0\leq t\leq 1} \|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} < \infty$, and*

$$\int_{\mathbb{R}^n} |u(t_2,x)|^2 \,\mathrm{d}x \leq Ce^{\frac{C}{(t_2-t_1)^\delta}} \left(\int_{t_1}^{t_2}\int_E |u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^\theta \left(\int_{\mathbb{R}^n} |u(t_1,x)|^2 \,\mathrm{d}x\right)^{1-\theta} \tag{4.1}$$

holds for all $0 < t_1 < t_2 \leq 1$. Then for every $T \in (0,1]$, we have

$$\int_{\mathbb{R}^n} |u(T,x)|^2 \,\mathrm{d}x \leq C^{\frac{1}{\theta}} e^{\frac{C+1}{\theta T^\delta}} \int_0^T \int_E |u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{4.2}$$

Proof. Fix $T > 0$. Let $l_1 = T$. For every integer $m \geq 2$, we define $l_m = \lambda^{m-1} l_1$ so that $0 < \cdots < l_{m+1} < l_m < \cdots < l_1$ and

$$\frac{l_m - l_{m+1}}{l_{m+1} - l_{m+2}} = \lambda^{-1} := \left(\frac{C+1}{C+1-\theta}\right)^{\frac{1}{\delta}} > 1. \tag{4.3}$$

Applying (4.1) with $t_2 = l_m$ and $t_1 = l_{m+1}$, we have

$$\int_{\mathbb{R}^n} |u(l_m,x)|^2 \,\mathrm{d}x \leq Ce^{\frac{C}{(l_m-l_{m+1})^\delta}} \left(\int_{l_{m+1}}^{l_m}\int_E |u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^\theta \left(\int_{\mathbb{R}^n} |u(l_{m+1},x)|^2 \,\mathrm{d}x\right)^{1-\theta}. \tag{4.4}$$

Using the inequality $a^\theta b^{1-\theta} \leq \varepsilon^{-(1-\theta)} a + \varepsilon^\theta b$ (when $a, b, \varepsilon > 0$), we deduce from (4.4) that

$$\int_{\mathbb{R}^n} |u(l_m,x)|^2 \,\mathrm{d}x \leq \varepsilon^{-(1-\theta)} C^{\frac{1}{\theta}} e^{\frac{C}{\theta(l_m-l_{m+1})^\delta}} \int_{l_{m+1}}^{l_m}\int_E |u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t + \varepsilon^\theta \int_{\mathbb{R}^n} |u(l_{m+1},x)|^2 \,\mathrm{d}x$$

for all $\varepsilon > 0$, which can be rewritten as

$$\varepsilon^{1-\theta} e^{-\frac{C}{\theta(l_m-l_{m+1})^\delta}} \int_{\mathbb{R}^n} |u(l_m,x)|^2 \,\mathrm{d}x - \varepsilon e^{-\frac{C}{\theta(l_m-l_{m+1})^\delta}} \int_{\mathbb{R}^n} |u(l_{m+1},x)|^2 \,\mathrm{d}x \leq C^{\frac{1}{\theta}} \int_{l_{m+1}}^{l_m}\int_E |u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{4.5}$$

Letting $\varepsilon = e^{-\frac{1}{\theta(l_m-l_{m+1})^\delta}}$ in (4.5) and using (4.3), we infer that

$$e^{-\frac{C+1-\theta}{\theta(l_m-l_{m+1})^\delta}} \int_{\mathbb{R}^n} |u(l_m,x)|^2 \,\mathrm{d}x \quad - \quad e^{-\frac{C+1-\theta}{\theta(l_{m+1}-l_{m+2})^\delta}} \int_{\mathbb{R}^n} |u(l_{m+1},x)|^2 \,\mathrm{d}x \quad \leq \quad C^{\frac{1}{\theta}} \int_{l_{m+1}}^{l_m}\int_E |u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

Taking the sum over $m \geq 1$, we find

$$e^{-\frac{C+1-\theta}{\theta(l_1-l_2)^\delta}} \int_{\mathbb{R}^n} |u(l_1,x)|^2 \,\mathrm{d}x \leq \sum_{m\geq 1} C^{\frac{1}{\theta}} \int_{l_{m+1}}^{l_m} \int_E |u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq C^{\frac{1}{\theta}} \int_0^T \int_E |u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{4.6}$$

where we used the fact that

$$\lim_{m\to\infty} e^{-\frac{C+1-\theta}{\theta(l_{m+1}-l_{m+2})^\delta}} \int_{\mathbb{R}^n} |u(l_{m+1},x)|^2 \,\mathrm{d}x = 0,$$

which follows from $\sup_{0\leq t\leq 1} \|u(t,\cdot)\|_{L^2(\Omega)} < \infty$ and $l_{m+1} - l_{m+2} \to 0$ as $m \to \infty$. Therefore, (4.2) follows from (4.6) clearly. □

We now replace the space-time norm in (4.1) by a space norm at the final time.

Corollary 4.2. *Let $\delta > 0$, $0 < \theta < 1, C \geq 1$ and E be a subset of $\mathbb{R}^n$. Assume that u is a function in $[0,1] \times \mathbb{R}^n$ so that for all $t_1 < t_2$*

$$\int_{\mathbb{R}^n} |u(t_2,x)|^2 \,\mathrm{d}x \leq C \int_{\mathbb{R}^n} |u(t_1,x)|^2 \,\mathrm{d}x, \tag{4.7}$$

$$\int_{\mathbb{R}^n} |u(t_2,x)|^2 \,\mathrm{d}x \leq Ce^{\frac{C}{(t_2-t_1)^\delta}} \left(\int_E |u(t_2,x)|^2 \,\mathrm{d}x\right)^\theta \left(\int_{\mathbb{R}^n} |u(t_1,x)|^2 \,\mathrm{d}x\right)^{1-\theta}. \tag{4.8}$$

Then there exists $C' > 0$ (depending only on C, δ, θ) so that for every $T \in (0,1]$,

$$\int_{\mathbb{R}^n} |u(T,x)|^2 \,\mathrm{d}x \leq C' e^{\frac{C'}{T^\delta}} \int_0^T \int_E |u(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{4.9}$$

Proof. Arbitrarily given $0 \leq t_1 < t_2 \leq 1$. Let $s \in (t_1, t_2]$. By using (4.8), we have

$$\int_{\mathbb{R}^n} |u(s,x)|^2 \,\mathrm{d}x \leq Ce^{\frac{C}{(s-t_1)^\delta}} \left(\int_E |u(s,x)|^2 \,\mathrm{d}x\right)^\theta \left(\int_{\mathbb{R}^n} |u(t_1,x)|^2 \,\mathrm{d}x\right)^{1-\theta}. \tag{4.10}$$

Integrating (4.10) over $s \in [\frac{t_1+t_2}{2}, t_2]$, and then using the Hölder inequality, we infer that

$$\int_{\frac{t_1+t_2}{2}}^{t_2} \int_{\mathbb{R}^n} |u(s,x)|^2 \,\mathrm{d}x\,\mathrm{d}s$$
$$\leq C\left(\frac{t_2-t_1}{2}\right)^{1-\theta} e^{\frac{C2^\delta}{(t_2-t_1)^\delta}} \left(\int_{\frac{t_1+t_2}{2}}^{t_2} \int_E |u(s,x)|^2 \,\mathrm{d}x\,\mathrm{d}s\right)^\theta \left(\int_{\mathbb{R}^n} |u(t_1,x)|^2 \,\mathrm{d}x\right)^{1-\theta}. \tag{4.11}$$

Moreover, it follows from (4.7) that

$$\int_{\mathbb{R}^n} |u(t_2,x)|^2 \,\mathrm{d}x \leq \frac{2C}{t_2-t_1} \int_{\frac{t_1+t_2}{2}}^{t_2} \int_{\mathbb{R}^n} |u(s,x)|^2 \,\mathrm{d}x\,\mathrm{d}s.$$

This, together with (4.11), gives that

$$\int_{\mathbb{R}^n} |u(t_2,x)|^2 \,\mathrm{d}x \le C\left(\frac{2}{t_2-t_1}\right)^\theta e^{\frac{C2^\delta}{(t_2-t_1)^\delta}} \left(\int_{\frac{t_1+t_2}{2}}^{t_2}\int_E |u(s,x)|^2 \,\mathrm{d}x\,\mathrm{d}s\right)^\theta \left(\int_{\mathbb{R}^n} |u(t_1,x)|^2\,\mathrm{d}x\right)^{1-\theta}. \tag{4.12}$$

Absorbing the term $(\frac{2}{t_2-t_1})^\theta$ by the exponential term $e^{\frac{C2^\delta}{(t_2-t_1)^\delta}}$, and enlarging the integral interval $[\frac{t_1+t_2}{2}, t_2]$ to $[t_1,t_2]$ in (4.12), we conclude that

$$\int_{\mathbb{R}^n} |u(t_2,x)|^2 \,\mathrm{d}x \le C_0 e^{\frac{C_0}{(t_2-t_1)^\delta}} \left(\int_{t_1}^{t_2}\int_E |u(s,x)|^2 \,\mathrm{d}x\,\mathrm{d}s\right)^\theta \left(\int_{\mathbb{R}^n} |u(t_1,x)|^2\,\mathrm{d}x\right)^{1-\theta} \tag{4.13}$$

for some constant $C_0 \ge 1$ depending only on C,δ,θ. Since $t_1 < t_2$ can be chosen arbitrarily in (4.13), the inequality (4.9) immediately follows from Proposition 4.1. □

With the aid of Corollary 4.2, we can prove the observability inequality for (1.1) by interpolation inequalities as in Theorem 1.2.

Proof of Theorem 1.3. Multiplying (1.1) with u and integrating over $x \in \mathbb{R}^n$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} |u(t,x)|^2\,\mathrm{d}x + \int_{\mathbb{R}^n} |\Lambda^{\frac{s}{2}} u(t,x)|^2\,\mathrm{d}x \le \int_{\mathbb{R}^n} |a(t,x)||u(t,x)|^2\,\mathrm{d}x, \quad \forall t>0.$$

Applying Grönwall's inequality, recalling that the potential a is bounded, we see that for some $C>0$

$$\int_{\mathbb{R}^n} |u(t_2,x)|^2\,\mathrm{d}x \le Ce^{C(t_2-t_1)}\int_{\mathbb{R}^n} |u(t_1,x)|^2\,\mathrm{d}x, \quad \forall t_2 \ge t_1. \tag{4.14}$$

We finish the proof by two cases as follows.

Case (1). When $0 < T \le 1$**.** Let $0 \le t_1 < t_2 \le 1$. On one hand, by (4.14) we have

$$\int_{\mathbb{R}^n} |u(t_2,x)|^2\,\mathrm{d}x \le C_1\int_{\mathbb{R}^n} |u(t_1,x)|^2\,\mathrm{d}x \tag{4.15}$$

with $C_1 = Ce^C$. On the other hand, it follows from Theorem 1.2 that

$$\int_{\mathbb{R}^n} |u(t_2,x)|^2\,\mathrm{d}x \le C_2 e^{\frac{C_2}{(t_2-t_1)^\delta}} \left(\int_E |u(t_2,x)|^2\,\mathrm{d}x\,\mathrm{d}t\right)^\theta \left(\int_{\mathbb{R}^n} |u(t_1,x)|^2\,\mathrm{d}x\right)^{1-\theta}, \tag{4.16}$$

where $\delta = 1/(s-1) > 0$, the constants $C_2 \ge 1, \theta \in (0,1)$ depending only on n, s, a, and E. Since (4.15) and (4.16) are both true for all $t_1 < t_2$, according to Corollary 4.2, we conclude that

$$\int_{\mathbb{R}^n} |u(T,x)|^2\,\mathrm{d}x \le C_3 e^{\frac{C_3}{T^\delta}}\int_0^T\int_E |u(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.17}$$

Thus Theorem 1.3 follows in this case.

Case (2). When $T>1$**.** We first apply (4.17) with $T=1$ to find that

$$\int_{\mathbb{R}^n} |u(1,x)|^2\,\mathrm{d}x \le C_3 e^{C_3}\int_0^1\int_E |u(t,x)|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{4.18}$$

And then we apply (4.14) with $t_2 = T$ and $t_1 = 1$ to get

$$\int_{\mathbb{R}^n} |u(T,x)|^2 \,\mathrm{d}x \le Ce^{C(T-1)} \int_{\mathbb{R}^n} |u(1,x)|^2 \,\mathrm{d}x. \tag{4.19}$$

Combining with (4.18) and (4.19), we infer that

$$\int_{\mathbb{R}^n} |u(T,x)|^2 \,\mathrm{d}x \le C_3 e^{C_3} C e^{CT} \int_0^T \int_E |u(t,x)|^2 \,\mathrm{d}x \,\mathrm{d}t.$$

This shows that Theorem 1.3 also holds for $T > 1$. It completes the proof. □

Acknowledgements. The authors wish to thank the anonymous referees for their careful reading and helpful comments to improve the presentation of this paper.

Funding informations. The first author was partially supported by the National Natural Science Foundation of China under grant no.12171442, and the Fundamental Research Funds for the Central Universities, China University of Geosciences (Wuhan) under grants nos. G1323523061 and G1323523041. The second author was partially supported by the National Natural Science Foundation of China under grant no. 11971363, and by the Fundamental Research Funds for the Central Universities under grant No. 2042023kf0193.

References

[1] P. Alphonse and J. Bernier, Smoothing properties of fractional Ornstein-Uhlenbeck semigroups and null-controllability. *Bull. Sci. Math.* **165** (2020) 102914.
[2] P. Alphonse and J. Martin, Stabilization and approximate null-controllability for a large class of diffusive equations from thick control supports. *ESAIM: COCV* **28** (2022) 30.
[3] J. Apraiz and L. Escauriaza, Null-control and measurable sets. *ESAIM: COCV* **19** (2013) 239–254.
[4] J. Apraiz, L. Escauriaza, G. Wang and C. Zhang, Observability inequalities and measurable sets. *J. Eur. Math. Soc.* **16** (2014) 2433–2475.
[5] K. Beauchard, M. Egidi and K. Pravda-Starov, Geometric conditions for the null-controllability of hypoelliptic quadratic parabolic equations with moving control supports. *C. R. Math Acad. Sci. Paris* **358** (2020) 651–700.
[6] J. Bourgain and S. Dyatlov, Spectral gaps without the pressure condition. *Ann. Math.* **187** (2018) 1–43.
[7] A. Dicke, A. Seelmann and I. Veselić, Spectral inequality with sensor sets of decaying density for Schrödinger operators with power growth potentials. Preprint arXiv:2206.08682 (2022).
[8] Y. Duan, L. Wang and C. Zhang, Observability inequalities for the heat equation with bounded potentials on the whole space. *SIAM J. Control Optim.* **58** (2020) 1939–1960.
[9] T. Duyckaerts and L. Miller, Resolvent conditions for the control of parabolic equations. *J. Funct. Anal.* **263** (2012) 3641–3673.
[10] L. Escauriaza, S. Montaner and C. Zhang, Observation from measurable sets for parabolic analytic evolutions and applications. *J. Math. Pures Appl.* **104** (2015) 837–867.
[11] L. Escauriaza, S. Montaner and C. Zhang, Analyticity of solutions to parabolic evolutions and applications. *SIAM J. Math. Anal.* **49** (2017) 4064–4092.
[12] M. Egidi and I. Veselić, Sharp geometric condition for null-controllability of the heat equation on $\mathbb{R}^d$ and consistent estimates on the control cost. *Arch. Math.* **111** (2018) 85–99.
[13] H.O. Fattorini and D. Russell, Exact controllability theorems for linear parabolic equations in one space dimension. *Arch. Rational Mech. Anal.* **43** (1971) 272–292.
[14] L. Grafakos, Modern Fourier Analysis. Springer, New York (2009).
[15] R. Han and W. Schlag, A higher-dimensional Bourgain–Dyatlov fractal uncertainty principle. *Anal. PDE* **13** (2020) 813–863.
[16] J. Huang and M. Wang, New lower bounds on the radius of spatial analyticity for the KdV equation. *J. Differ. Equ.* **266** (2019) 5278–5317.
[17] S. Huang, G. Wang and M. Wang, Characterizations of stabilizable sets for some parabolic equations in $\mathbb{R}^n$. *J. Differ. Equ.* **272** (2021) 255–288.
[18] A. Koenig, Lack of null-controllability for the fractional heat equation and related equations. *SIAM J. Control Optim.* **58** (2020) 3130–3160.
[19] G. Lebeau and L. Robbiano, Contrôle exact de l'équation de la chaleur. *Commun. Partial Differ. Equ.* **20** (1995) 335–356.
[20] G. Lebeau and I. Moyano, Spectral inequalities for the Schrödinger operator. Preprint arXiv:1901.03513 (2019).

[21] J. Le Rousseau and I. Moyano, Null-controllability of the Kolmogorov equation in the whole phase space. *J. Differ. Equ.* **260** (2016) 3193–3233.
[22] P. Lissy, On the cost of fast controls for some families of dispersive or parabolic equations in one space dimension. *SIAM J. Control Optim.* **52** (2014) 2651–2676.
[23] P. Lissy, Construction of Gevrey functions with compact support using the Bray-Mandelbrojt iterative process and applications to the moment method in control theory. *Math. Control Relat. Fields* **7** (2017) 21–40.
[24] P. Lissy, A non-controllability result for the half-heat equation on the whole line based on the prolate spheroidal wave functions and its application to the Grushin equation. Preprint hal-02420212v2 (2020).
[25] J. Martin, Uncertainty principles in Gelfand-Shilov spaces and null-controllability. *J. Funct. Anal.* **283** (2022) 48.
[26] J. Martin, Spectral inequalities for anisotropic Shubin operators. Preprint arXiv:2205.11868 (2022).
[27] S. Micu and E. Zuazua, On the controllability of a fractional order parabolic equation. *SIAM J. Control Optim.* **44** (2006) 1950–1972.
[28] L. Miller, On the controllability of anomalous diffusions generated by the fractional Laplacian. *Math. Control Signals Syst.* **18** (2006) 260–271.
[29] L. Miller, A direct Lebeau–Robbiano strategy for the observability of heat-like semigroups. *Discrete Contin. Dyn. Syst. Ser. B* **14** (2010) 1465–1485.
[30] I. Nakić, M. Täufer, M. Tautenhahn and I. Veselić, Sharp estimates and homogenization of the control cost of the heat equation on large domains. *ESAIM: COCV* **26** (2020) 26.
[31] M. Reed and B. Simon, Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness. Academic Press (1975).
[32] G. Tenenbaum and M. Tucsnak, On the null-controllability of diffusion equations. *ESAIM: COCV* **17** (2011) 1088–1100.
[33] G. Wang, M. Wang, C. Zhang and Y.B. Zhang, Observable set, observability, interpolation inequality and spectral inequality for the heat equation in $\mathbb{R}^n$. *J. Math. Pure Appl.* **126** (2019) 144–194.
[34] G. Wang, M. Wang and Y.B. Zhang, Observability and unique continuation inequalities for the Schrödinger equation. *J. Eur. Math. Soc.* **21** (2019) 3513–3572.