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VARIATIONAL PROBLEMS CONCERNING SUB-FINSLER METRICS IN CARNOT GROUPS

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Abstract. This paper is devoted to the study of geodesic distances defined on a subdomain of a given Carnot group, which are bounded both from above and from below by fixed multiples of the Carnot–Carathéodory distance. We show that the uniform convergence (on compact sets) of these distances can be equivalently characterized in terms of Γ-convergence of several kinds of variational problems. Moreover, we investigate the relation between the class of intrinsic distances, their metric derivatives and the sub-Finsler convex metrics defined on the horizontal bundle.

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1. Introduction

For three decades, many classical metric problems in Riemannian geometry have been translated to the context of sub-Riemannian geometry and, in particular, to Carnot groups $\mathbb{G}$, which possess a rich geometry. Indeed, they are connected and simply connected Lie groups whose associated Lie algebra admits a finite-step stratification (see [6, 18, 22, 23]).

One of these problems is the study of a particular class of geodesic distances, bounded from above and below, which has been deeply studied in Euclidean spaces and in the setting of Lipschitz manifolds (see [11–13, 27]). Our purpose is to generalize this class and various related results to Carnot groups. We thus consider all those redgeodesic distances $d : \Omega \times \Omega \to \mathbb{R}$ (where $\Omega \subset \mathbb{G}$ is an open set) satisfying

$$\frac{1}{\alpha} d_{cc}(x, y) \leq d(x, y) \leq \alpha d_{cc}(x, y) \qquad \forall x, y \in \Omega, \tag{1.1}$$

for some $\alpha \geq 1$, where d_{cc} stands for the so-called Carnot–Carathéodory distance. We denote by $\mathcal{D}_{cc}(\Omega)$ the family of all such distances. According to [12, 27], it is quite natural to construct, on the so-called horizontal bundle $H\mathbb{G}$, the family of metrics $\varphi_d : H\mathbb{G} \to [0, +\infty)$ associated to $d \in \mathcal{D}_{cc}(\Omega)$ by differentiation:

$$\varphi_d(x, v) \coloneqq \limsup_{t \searrow 0} \frac{d\big(x, x \cdot \delta_t \exp(\mathrm{d}_x \tau_{x^{-1}}[v])\big)}{t} \quad \text{ for every } (x, v) \in H\mathbb{G}.$$

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Inspired by the notation for horizontal curves introduced by Pauls [24], in the previous definition we denote with $x \cdot \delta_t \exp(\mathrm{d}_x\tau_{x^{-1}}[v])$ the dilation curve starting from the point $x \in \mathbb{G}$ with direction given by the left translation at the identity of a horizontal vector defined on the fiber $H_x\mathbb{G}$. In particular, it turns out that φ_d is a *sub-Finsler convex metric*. These objects play an important role in the setting of the so-called *sub-Finsler Carnot groups* (for a reference see for example [17], Sect. 6), playing the same role as Finsler metrics with the difference that they are defined only on the horizontal bundle (see Definition 2.9). At the same time, it is also natural to consider the length functional induced by the metric derivative φ_d. We will show that we can reconstruct the distance d by minimizing the corresponding functional (Thm. 3.5).

The first purpose of this paper is to compare the asymptotic behaviour of different kinds of functionals involving distances defined on a given open subset Ω of a Carnot group $\mathbb{G}$. Indeed, the most common approach in order to study the following variational problems relies on Γ-convergence of the corresponding functionals (see Sect. 4). In particular, inspired by the proof contained in [9], we state the equivalence between the Γ-convergence of the functionals L_n and J_n associated to a sequence of distances $(d_n)_{n\in\mathbb{N}} \subset \mathcal{D}_{cc}(\Omega)$ respectively through

$$L_n(\gamma) = \int_0^1 \varphi_{d_n}(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \quad \text{and} \quad J_n(\mu) = \int_{\Omega\times\Omega} d_n(x,y)\,\mathrm{d}\mu(x,y),$$

where $\gamma : [0,1] \to \Omega$ is a horizontal curve (Def. 2.1) and μ is a positive and finite Borel measure on $\Omega \times \Omega$. This kind of result has been already studied in the literature, especially for what concerns the homogenization of Riemannian and Finsler metrics ([1, 3]). Moreover, we show an additional characterization when Ω is bounded (Thm. 4.4, point (iv)).

The second purpose is to study a different application: the intrinsic analysis of sub-Finsler metrics. More precisely, under suitable regularity assumptions on the metric under consideration, we prove the following result (Thm. 5.13):

$$\varphi(x, \nabla_{\mathbb{G}} f(x)) = \mathrm{Lip}_{\delta_\varphi} f(x) \quad \text{for a.e. } x \in \mathbb{G}, \tag{1.2}$$

where $f : \Omega \subset \mathbb{G} \to \mathbb{R}$ is a Pansu-differentiable function, δ_φ is the distance defined in (1.3) below, $\varphi \in \mathcal{M}^\alpha_{cc}(\Omega)$ is a sub-Finsler convex metric, and the pointwise Lipschitz constant of f is given by

$$\mathrm{Lip}_{\delta_\varphi} f(x) = \limsup_{y\to x} \frac{|f(y) - f(x)|}{\delta_\varphi(x,y)} \quad \text{for every } x \in \mathbb{G}.$$

The equality (1.2) may be regarded as a generalization of a result achieved in [26], and further generalized by Chang Y. Guo to admissible Finsler metrics defined on open subsets of $\mathbb{R}^n$, in [15]. Then, in order to prove (1.2), we crucially observe that the quantity

$$\delta_\varphi(x,y) \coloneqq \sup\left\{|f(x) - f(y)| \,\middle|\, f\colon \mathbb{G} \to \mathbb{R} \text{ Lipschitz}, \ \|\varphi(\cdot, \nabla_{\mathbb{G}} f(\cdot))\|_\infty \le 1\right\} \tag{1.3}$$

coincides with the intrinsic distance $d_{\varphi^\star}$, induced by the dual metric, where $\|\cdot\|_\infty$ refers to the L^∞-norm. This happens, for instance, when we assume that the sub-Finsler metric φ is either lower semicontinuous or upper semicontinuous on the horizontal bundle (see Thm. 5.11 and Cor. 5.12). These results are a generalization of the analogous statement in [13], due to De Cecco and Palmieri. The proof of Theorem 5.11 heavily relies on two results contained in [19]. The first one allows us to approximate an upper semicontinuous sub-Finsler metric with a family of Finsler metrics. The second result lets us approximate from below the sub-Finsler distance with a family of induced Finsler distances. Finally, we show that in many cases the distance δ_φ, albeit defined as a supremum among Lipschitz functions, is actually already determined by smooth functions (*cf.* Prop. 5.15). An important step in proving this fact is to approximate (say, uniformly on compact sets) any 1-Lipschitz function with a sequence of smooth 1-Lipschitz functions; here, the key point is that the Lipschitz constant is preserved. Since this approximation result holds in much greater generality (for instance, on possibly rank-varying sub-Finsler structures) and might be of independent interest, we will treat it in Appendix A.

We now give a short descriptive plan of the paper. In Section 2 we collect some of the basic geometric facts about Carnot groups and we present all preliminaries about horizontal bundles, horizontal curves, Pansu differentiability for Lipschitz functions, and the main concepts related to sub-Finsler convex metrics. In particular, we prove some properties about dual metrics associated to sub-Finsler metrics. In Section 3 we introduce the main concept of metric derivative and we show its convexity on the horizontal fibers. Moreover, we prove a classical length representation theorem through a distance reconstruction result. Section 4 contains the equivalence theorem between the uniform convergence of distances in $\mathcal{D}_{cc}(\Omega)$ and Γ-convergence of the length and energy functionals. Finally, in Section 5 we introduce the main concepts of intrinsic distance and metric density and we prove the main theorems of this paper.

2. Preliminaries

2.1. Carnot groups

A connected and simply connected Lie group $\mathbb{G}$ is said to be a *Carnot group of step k* if its Lie algebra $\mathfrak{g}$ admits a *step k stratification*, *i.e.*, there exist linear subspaces $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k, \quad [\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}, \quad \mathfrak{g}_k \neq \{0\}, \quad [\mathfrak{g}_1, \mathfrak{g}_k] = \{0\} \quad \text{for } j < k, \tag{2.1}$$

where $[\mathfrak{g}_1, \mathfrak{g}_j]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_j$. Let $m \coloneqq \dim \mathfrak{g}_1 \leq n = \dim \mathfrak{g}$, where $\mathfrak{g}_1$ is called the *first stratum* of the stratification. Choose a basis $e_1, \dots, e_n$ of $\mathfrak{g}$ adapted to the stratification, that is such that

$$e_{h_{j-1}+1}, \dots, e_{h_j} \quad \text{is a basis of } \mathfrak{g}_j \quad \text{for each} \quad j = 1, \dots, k.$$

Let $X_1, \dots, X_n$ be the family of left invariant vector fields such that at the identity element e of $\mathbb{G}$ we have $X_i(e) = e_i$ for every $i = 1, \dots, n$. Given (2.1), the subset $X_1, \dots, X_m$ generates by commutation all the other vector fields; we will refer to $X_1, \dots, X_m$ as generating horizontal vector fields of the group. Given an element $x \in \mathbb{G}$, we denote by $\tau_x : \mathbb{G} \to \mathbb{G}$ the *left translation* by x, which is given by

$$\tau_x z \coloneqq x \cdot z \quad \text{for every } z \in \mathbb{G},$$

where $\cdot$ is the group law in $\mathbb{G}$. Moreover, it holds that the map τ_x is a smooth diffeomorphism, thus we can consider its differential $\mathrm{d}_y \tau_x : T_y\mathbb{G} \to T_{x\cdot y}\mathbb{G}$ at any point $y \in \mathbb{G}$.

2.2. Exponential map and sub-Riemannian structures

We recall that the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is defined as follows. First, we identify the Lie algebra $\mathfrak{g}$ with the tangent space at the identity $T_e\mathbb{G}$. Given any vector $v \in T_e\mathbb{G}$ and denoting by $\gamma\colon [0,1] \to \mathbb{G}$ the (unique) smooth curve satisfying the ODE

$$\begin{cases} \dot{\gamma}(t) = \mathrm{d}_e \tau_{\gamma(t)}[v] & \text{for every } t \in [0,1], \\ \gamma(0) = e, \end{cases} \tag{2.2}$$

we define $\exp(v) = e^v := \gamma(1)$, where $\mathrm{d}_e\tau_{\gamma(t)}[v]$ is a left-invariant vector field. It holds that exp is a diffeomorphism and any $p \in \mathbb{G}$ can be written in a unique way as

$$p = \exp(p_1X_1 + \cdots + p_nX_n) = e^{p_1X_1+\cdots+p_nX_n}, \quad \text{where } v = \sum_{i=1}^{n} p_iX_i.$$

This expression is equivalent to (2.2), since the Lie algebra $\mathfrak{g}$ of a Lie group $\mathbb{G}$ can be characterized as the set of left-invariant vector fields. Indeed, $\mathfrak{g}$ is a vector space, closed under the Lie bracket $[\cdot,\cdot]$ defined on smooth functions by

$$[X, Y](f) = X(Y(f)) - Y(X(f)),$$

and it is canonically isomorphic to the tangent space of $\mathbb{G}$ at the origin via the identification of X and $X(e)$ (see *e.g.* [28]). Moreover, we can identify p with the n-tuple $(p_1, \ldots, p_n) \in \mathbb{R}^n$ and $\mathbb{G}$ with $\mathbb{R}^n$ where the group operation $\cdot$ satisfies (see [5] and [22], Sect. 7)

$$x \cdot y = \exp\left(\exp^{-1}(x) \star \exp^{-1}(y)\right) \quad \text{for every } x, y \in \mathbb{G},$$

where $\star$ denotes the group operation determined by the Campbell–Baker–Hausdorff formula, see *e.g.* [6, 20].

The subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by the vector fields $X_1, \ldots, X_m$ plays a particularly important role in the theory, it is called the *horizontal bundle* $H\mathbb{G}$; the fibers of $H\mathbb{G}$ are

$$H_x\mathbb{G} = \mathrm{span}\,\{X_1(x), \ldots, X_m(x)\}.$$

A sub-Riemannian structure can be defined on $\mathbb{G}$ in the following way. Consider a scalar product $\langle\cdot,\cdot\rangle_e$ on $\mathfrak{g}_1 = H_e\mathbb{G}$ that makes $\{X_1, \ldots, X_m\}$ an orthonormal basis. Moreover, by left translating the horizontal fiber in the identity, we obtain that $H_x\mathbb{G} = \mathrm{d}_e\tau_x(\mathfrak{g}_1)$ and, by Lemma 7.48 of [2], the map

$$T\mathbb{G} \ni (x, v) \mapsto (x, \mathrm{d}_x\tau_{x^{-1}}[v]) \in \mathbb{G} \times T_e\mathbb{G}$$

is an isomorphism between $T\mathbb{G}$ and $\mathbb{G} \times \mathfrak{g}$, in other words, the tangent bundle is trivial. This allows us to define the scalar product $\langle\cdot,\cdot\rangle_x$ on $H_x\mathbb{G}$ as

$$\langle v, w\rangle_x := \left\langle \mathrm{d}_x\tau_{x^{-1}}[v], \mathrm{d}_x\tau_{x^{-1}}[w]\right\rangle_e \quad \text{for every } v, w \in H_x\mathbb{G}.$$

Notice that $\{X_1(x), \ldots, X_m(x)\}$ is an orthonormal basis of $H_x\mathbb{G}$ with respect to $\langle\cdot,\cdot\rangle_x$. We denote by $\|\cdot\|_x$ the norm induced by $\langle\cdot,\cdot\rangle_x$, namely $\|v\|_x := \sqrt{\langle v, v\rangle_x}$ for every $v \in H_x\mathbb{G}$. By the left invariance of the sub-Riemannian structure, for every $v \in H_x\mathbb{G}$ there exists a unique vector $\bar{v} \in H_e\mathbb{G}$ such that $v = \mathrm{d}_e\tau_x[\bar{v}]$ and we get that

$$\|\bar{v}\|_e = \|\mathrm{d}_e\tau_x[\bar{v}]\|_x = \|v\|_x \quad \text{for every } (x, v) \in H\mathbb{G}. \tag{2.3}$$

If $y = (y_1, \ldots, y_n) \in \mathbb{G}$ and $x \in \mathbb{G}$ are given, we set the *projection map* as:

$$\pi_x : \mathbb{G} \to H_x\mathbb{G} \quad \text{as} \quad \pi_x(y) = \sum_{j=1}^{m} y_jX_j(x).$$

The map $y \mapsto \pi_x(y)$ is a smooth section of $H_x\mathbb{G}$ and it is linear in y. Furthermore, if $v \in \mathfrak{g}_1$, by exponential coordinates it holds that

$$\pi_x(e^v) = \pi_x(v_1, \dots, v_m, 0 \dots, 0) = \sum_{i=1}^m v_i X_i(x) = \mathrm{d}_e\tau_x[v] \quad \text{for all } x \in \mathbb{G} \text{ and } v \in g_1. \tag{2.4}$$

2.3. Dilations and Carnot–Carathéodory distance

For any $\lambda > 0$, we denote the unique linear map by $\delta^\star_\lambda : \mathfrak{g} \to \mathfrak{g}$ and such that

$$\delta^\star_\lambda X = \lambda^i X, \qquad \forall X \in \mathfrak{g}_i.$$

The maps $\delta^\star_\lambda : \mathfrak{g} \to \mathfrak{g}$ are Lie algebra automorphisms, *i.e.*, $\delta^\star_\lambda([X,Y]) = [\delta^\star_\lambda X, \delta^\star_\lambda Y]$ for all $X, Y \in \mathfrak{g}$. For every $\lambda > 0$, the map $\delta^\star_\lambda$ naturally induces an automorphism on the group $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ by the identity

$$\delta_\lambda(x) = (\exp \circ \delta^\star_\lambda \circ \log)(x), \tag{2.5}$$

where with log we denote the inverse map of exp. It is easy to verify that both the families $(\delta^*_\lambda)_{\lambda>0}$ and $(\delta_\lambda)_{\lambda>0}$ are one-parameter groups of automorphisms (of Lie algebras and of groups, respectively), namely, $\delta^\star_\lambda \circ \delta^\star_\eta = \delta^\star_{\lambda\eta}$ and $\delta_\lambda \circ \delta_\eta = \delta_{\lambda\eta}$ for all $\lambda, \eta > 0$. The maps $\delta^\star_\lambda, \delta_\lambda$ are both called *dilations of factor* λ.

Let us remark that, since $\delta^\star_\lambda v = \lambda v$ for every $v \in \mathfrak{g}_1 = H_e\mathbb{G}$ and thanks to (2.5), one can easily realize that

$$\delta_\lambda \exp(v) = \exp(\lambda v) \quad \text{ for every } v \in H_e\mathbb{G} \text{ and } \lambda > 0. \tag{2.6}$$

According to [24], we can extend dilations also to negative parameters $\lambda < 0$, denoting $\delta^\star_{|\lambda|}(X) = \delta^\star_\lambda(-X) = |\lambda|^i(-X)$ for $X \in \mathfrak{g}_i$ and, in the present paper, we exploit this fact only on the fist layer $\mathfrak{g}_1$. Indeed, it holds that $\pi_x(\delta_\lambda e^w) = \lambda\pi_x(e^w)$ for every $w \in \mathfrak{g}_1$ and $\lambda > 0$. Since the dilations are defined only on the Lie algebra $\mathfrak{g}$, we extend them, by left translations, on the entire $T_x\mathbb{G}$, for every $x \in \mathbb{G}$. This allows us to write $\delta^*_\lambda v = \lambda v$ for every $\lambda > 0$, $x \in \mathbb{G}$ and $v \in H_x\mathbb{G}$.

Definition 2.1. An absolutely continuous curve $\gamma \colon [a,b] \to \mathbb{G}$ is said to be *horizontal* if there exists a vector of measurable functions $h = (h_1(t), \dots h_m(t)) : [a,b] \to \mathbb{R}^m$ called the vector of canonical coordinates, such that

- $\dot\gamma(t) = \sum_{i=1}^m h_i(t) X_i(\gamma(t))$ for a.e. $t \in [a,b]$;
- $|h| \in L^\infty(a,b)$.

The *length* of such a curve is given by $L_\mathbb{G}(\gamma) = \int_a^b \|\dot\gamma(t)\|_{\gamma(t)}\, \mathrm{d}t$.

Chow–Rashevskii's theorem ([6], Thm. 19.1.3) asserts that any two points in a Carnot group can be connected by a horizontal curve. Hence, the following definition is well-posed.

Definition 2.2. For every $x, y \in \mathbb{G}$, the *Carnot–Carathéodory (CC) distance* is defined by

$$d_{cc}(x,y) = \inf \{L_\mathbb{G}(\gamma) \colon \gamma \text{ is a horizontal curve joining } x \text{ and } y\}.$$

We remark that, by Chow–Rashevskii's Theorem, the Carnot–Carathéodory distance is finite. Moreover, it is homogeneous with respect to dilations and left translations, more precisely, for every $\lambda > 0$ and for every $x, y, z \in \mathbb{G}$ one has

$$d_{cc}(\delta_\lambda x, \delta_\lambda y) = \lambda d_{cc}(x,y), \qquad d_{cc}(\tau_x y, \tau_x z) = d_{cc}(y,z).$$

This immediately implies that $\tau_x(B(y,r)) = B(\tau_x y, r)$ and $\delta_\lambda B(y,r) = B(\delta_\lambda y, \lambda r)$, where

$$B(x,r) = \{y \in \mathbb{G} : d_{cc}(y,x) < r\}$$

is the open ball centered at $x \in \mathbb{G}$ with radius $r > 0$.

In the sequel, we will need the following crucial estimate, proved in Theorem 1.5.1 of [22].

Theorem 2.3. *Let $\mathbb{G}$ be a Carnot group of step k and let $K \subset \mathbb{G}$ be a compact set. Then there exists $C_K = C(K) > 1$ such that*

$$C_K^{-1}|x-y| \le d_{cc}(x,y) \le C_K|x-y|^{\frac{1}{k}}, \qquad \forall x,y \in K. \tag{2.7}$$

The following lemma shows the biLipschitz equivalence between the Carnot–Carathéodory distance and the norm induced by the scalar product.

Lemma 2.4. *There exists a constant $c \ge 1$ such that*

$$\frac{1}{c}\,\|v\|_e \le d_{cc}(e, \exp v) \le c\,\|v\|_e \quad \text{for every } v \in \mathfrak{g}_1. \tag{2.8}$$

Proof. Denote by S the unit sphere of $(H_e\mathbb{G}, \|\cdot\|_e)$, namely $S := \{v \in H_e\mathbb{G} : \|v\|_e = 1\}$. Define the function $\eta : S \to [0, +\infty)$ as $\eta(v) := d_{cc}(e, \exp v)$ for every $v \in S$. By Theorem 2.3, η is continuous on the compact set S. Then we can find $c \ge 1$ such that $1/c \le \eta(v) \le c$ holds for every $v \in S$. We conclude by 1-homogeneity: since $d_{cc}(e, \exp(\lambda v)) = \lambda d_{cc}(e, \exp v)$ for every $\lambda > 0$ and $v \in S$, we deduce that $\eta(v/\|v\|_e) = d_{cc}(e, \exp v)/\|v\|_e$ for every $v \in H_e\mathbb{G} \setminus \{0\}$ and thus

$$\frac{1}{c} \le \frac{d_{cc}(e, \exp v)}{\|v\|_e} \le c \quad \text{for every } v \in H_e\mathbb{G} \setminus \{0\},$$

which yields (2.8). □

2.4. Differentiability in Carnot Groups

We recall some basic definitions regarding differentiability in Carnot groups.

Definition 2.5. A map $L : \mathbb{G} \to \mathbb{R}$ is called a *homogeneous homomorphism* if

$$L(x \cdot y) = L(x) + L(y) \quad \text{and} \quad L(\delta_\lambda(x)) = \lambda L(x) \quad \text{for every } x,y \in \mathbb{G} \quad \text{and} \quad \lambda > 0.$$

Now we are ready to introduce the following fundamental notion of differentiability, see [23].

Definition 2.6. Let $\Omega \subset \mathbb{G}$ be an open subset. A map $f : \Omega \to \mathbb{R}$ is *Pansu differentiable* at $x \in \Omega$ if there exists a homogeneous homomorphism $L : \mathbb{G} \to \mathbb{R}$ such that

$$\lim_{y \to x} \frac{f(y) - f(x) - L(x^{-1} \cdot y)}{d_{cc}(y,x)} = 0.$$

The resulting map $L =: \mathrm{d}_{\mathbb{G}} f(x) : \mathbb{G} \to \mathbb{R}$ is called the Pansu differential of f at x.

Remark 2.7. If $f : \Omega \to \mathbb{R}$ is differentiable at $x \in \Omega$, then the directional derivative $X_j f(x)$ exists for any $j = 1, \dots, m$, and for any $y \in \mathbb{G}$ we have that

$$\mathrm{d}_{\mathbb{G}} f(x)(x^{-1} \cdot y) = \langle \nabla_{\mathbb{G}} f(x), \pi_x(y) \rangle_x,$$

where the horizontal gradient $\nabla_{\mathbb{G}} f(x)$ is defined as

$$\nabla_{\mathbb{G}} f(x) := \sum_{i=1}^{m} X_i f(x) X_i(x).$$

We stress that the notion of the horizontal gradient only depends on the choice of the generating horizontal vector fields and hence it is uniquely determined by the sub-Riemannian metric chosen on $\Omega \subset \mathbb{G}$.

Unless otherwise specified, by a Lipschitz function $f : \Omega \to \mathbb{R}$ we mean a function that is Lipschitz with respect to the Carnot–Carathéodory distance d_{cc}, namely there exists a constant $C \geq 0$ such that $|f(x) - f(y)| \leq C d_{cc}(x, y)$ for every $x, y \in \Omega$. Moreover, $f : \mathbb{G} \to \mathbb{R}$ is said to be a locally Lipschitz function if it is Lipschitz on every open bounded set $\Omega \subset \mathbb{G}$. The notion of Pansu differentiability is motivated by the following result due to Pansu [23] (see also [21] for a similar result in a more general setting). In general, it states that any Lipschitz map between two Carnot groups has almost everywhere a differential which is a homogeneous homomorphism. Below we report the statement of Pansu's result only for real-valued Lipschitz maps, since this is sufficient for our purposes.

Theorem 2.8. *Let $\Omega \subset \mathbb{G}$ be an open subset. Then for every Lipschitz function $f : \Omega \to \mathbb{R}$ we have that f is Pansu differentiable at $\mathcal{L}^n$-a.e. $x \in \Omega$.*

Let $x \in \mathbb{G}$ and $\bar{v} \in \mathfrak{g}_1$, then the map $t \mapsto x \cdot \delta_t \exp(\bar{v})$ is Lipschitz. Hence, if $f : \mathbb{G} \to \mathbb{R}$ is Lipschitz, then the composition $t \mapsto f(x \cdot \delta_t \exp(\bar{v}))$ is a Lipschitz mapping from $\mathbb{R}$ to itself, hence Pansu differentiable. By Definition 2.6, Remark 2.7, and Theorem 4.6 of [25], it is easy to verify that

$$\langle \nabla_{\mathbb{G}} f(x), \pi_x(v) \rangle = \lim_{t \to 0} \frac{f(x \cdot \delta_t e^{\bar{v}}) - f(x)}{t} \quad \text{for a.e. } x \in \mathbb{G} \text{ and for every } v \in H_x\mathbb{G}, \tag{2.9}$$

where $\bar{v} = \mathrm{d}_x \tau_{x^{-1}}[v]$.

2.5. Sub-finsler metrics and duality

Inspired by [27], now we introduce the following definition. Let $\mathbb{G}$ be a Carnot metric group and let $\Omega \subset \mathbb{G}$ be an open set. If $\alpha \geq 1$, we introduce the family $\mathcal{D}_{cc}(\Omega)$ containing all the geodesic distances $d : \Omega \times \Omega \to [0, +\infty)$ verifying

$$\frac{1}{\alpha} d_{cc}(x, y) \leq d(x, y) \leq \alpha d_{cc}(x, y) \qquad \forall x, y \in \Omega. \tag{2.10}$$

Therefore, $\mathcal{D}_{cc}(\Omega)$ depends on α and we omit such dependence for the sake of brevity. Notice that we may have $\mathcal{D}_{cc}(\Omega) = \emptyset$ if the domain $\Omega \subset \mathbb{G}$ is disconnected or it has an irregular boundary. We endow $\mathcal{D}_{cc}(\Omega)$ with the topology of the uniform convergence on compact subsets of $\Omega \times \Omega$. We will see in the Proof of Theorem 4.4 that $\mathcal{D}_{cc}(\Omega)$ is compact with respect to such topology.

Definition 2.9. For $\alpha \geq 1$, we define $\mathcal{M}^{\alpha}_{cc}(\mathbb{G})$ as the family of maps $\varphi : H\mathbb{G} \to [0, +\infty)$, that we will call metrics on $H\mathbb{G}$, verifying the following properties:

i) $\varphi : H\mathbb{G} \to [0, +\infty)$ is Borel measurable, where $H\mathbb{G}$ is endowed with the product σ-algebra;
ii) $\varphi(x, \delta^*_\lambda v) = |\lambda| \varphi(x, v)$ for every $(x, v) \in H\mathbb{G}$ and $\lambda \in \mathbb{R}$;
iii) $\frac{1}{\alpha} \|v\|_x \leq \varphi(x, v) \leq \alpha \|v\|_x$ for every $(x, v) \in H\mathbb{G}$.

Moreover, we will say that $\varphi \in \mathcal{M}^{\alpha}_{cc}(\mathbb{G})$ is a *sub-Finsler convex metric* if

$$\varphi(x, v_1 + v_2) \leq \varphi(x, v_1) + \varphi(x, v_2) \tag{2.11}$$

for every $x \in \mathbb{G}$ and $v_1, v_2 \in H_x\mathbb{G}$ (or equivalently if $\varphi(x,\cdot)$ is a norm for every $x \in \mathbb{G}$).

Let us remark that condition i) is equivalent to the Borel measurability with respect to the product space $\mathbb{G} \times \mathfrak{g}_1$. Our next aim is to introduce the dual of a metric belonging to $\mathcal{M}^{\alpha}_{cc}(\mathbb{G})$.

Definition 2.10 (Dual Metric)**.** Let us consider $\varphi \in \mathcal{M}^{\alpha}_{cc}(\mathbb{G})$. We define the *dual metric* $\varphi^\star : H\mathbb{G} \to [0, +\infty)$ of φ as

$$\varphi^\star(x, v) \coloneqq \sup\left\{ \frac{|\langle v, w\rangle_x|}{\varphi(x, w)} \;:\; w \in H_x\mathbb{G},\ w \neq 0 \right\}. \tag{2.12}$$

In general, the dual metric enjoys many useful properties, as we can see below.

Proposition 2.11. *For any $\varphi \in \mathcal{M}^{\alpha}_{cc}(\mathbb{G})$, it holds that $\varphi^\star$ is a sub-Finsler convex metric, and in particular*

$$\frac{1}{\alpha}\|v\|_x \leq \varphi^\star(x, v) \leq \alpha\|v\|_x \quad \textit{for every } (x, v) \in H\mathbb{G}. \tag{2.13}$$

Proof. It is straightforward to prove property ii) since for every $v, w \in H_x\mathbb{G}$ and $\lambda \in \mathbb{R}$ we have that $\langle \delta^*_\lambda v, w\rangle_x = \lambda\langle v, w\rangle_x$. Dividing by $\varphi(x, w)$ and passing to the supremum over all $w \in H_x\mathbb{G} \setminus \{0\}$, we obtain that $\varphi^\star(x, \delta^*_\lambda v) = |\lambda|\varphi^\star(x, v)$. In order to prove the convexity on the horizontal bundle, let us consider $v_1, v_2 \in H_x\mathbb{G}$, then we obtain:

$$\begin{aligned}\varphi^\star(x, v_1 + v_2) &\leq \sup\left\{ \frac{|\langle v_1, w\rangle_x|}{\varphi(x, w)} + \frac{|\langle v_2, w\rangle_x|}{\varphi(x, w)} \;:\; w \in H_x\mathbb{G},\ w \neq 0 \right\}\\ &\leq \varphi^\star(x, v_1) + \varphi^\star(x, v_2).\end{aligned}$$

Moreover, if we take $w \in H_x\mathbb{G} \setminus \{0\}$ it holds that

$$\frac{1}{\alpha}\frac{|\langle v, w\rangle_x|}{\|w\|_x} \leq \frac{|\langle v, w\rangle_x|}{\varphi(x, w)} \leq \alpha\frac{|\langle v, w\rangle_x|}{\|w\|_x}.$$

By taking the supremum over all $w \in H_x\mathbb{G} \setminus \{0\}$, we obtain (2.13) and accordingly property iii) of Definition 2.9. Therefore, $\varphi^\star(x,\cdot)$ is a norm, thus in particular it is continuous. Finally, chosen a dense sequence $(w_n)_n$ in $\mathfrak{g}_1 \setminus \{0\}$, we have that $(\mathrm{d}_e\tau_x[w_n])_n$ is dense in $H_x\mathbb{G}$ for every $x \in \mathbb{G}$, thus for any $v \in \mathfrak{g}_1$ we can write

$$\varphi^\star(x, \mathrm{d}_e\tau_x[v]) = \sup_{n\in\mathbb{N}} \frac{|\langle \mathrm{d}_e\tau_x[v], \mathrm{d}_e\tau_x[w_n]\rangle_x|}{\varphi(x, \mathrm{d}_e\tau_x[w_n])} = \sup_{n\in\mathbb{N}} \frac{|\langle v, w_n\rangle_e|}{\varphi(x, \mathrm{d}_e\tau_x[w_n])}$$

for every $x \in \mathbb{G}$, which shows that $\mathbb{G} \ni x \mapsto \varphi^\star(x, \mathrm{d}_e\tau_x[v])$ is measurable and accordingly property i) of Definition 2.9 is satisfied. All in all, $\varphi^\star$ is a sub-Finsler convex metric. □

We can characterize sub-Finsler convex metrics φ in terms of the bidual metric $\varphi^{\star\star}$.

Proposition 2.12. *Let us consider $\varphi \in \mathcal{M}^{\alpha}_{cc}(\mathbb{G})$. Then φ is a sub-Finsler convex metric if and only if $\varphi(x, v) = \varphi^{\star\star}(x, v)$ for every $(x, v) \in H\mathbb{G}$.*

Proof.
$\Longleftarrow$: It follows directly from the proof of Proposition 2.11.
$\Longrightarrow$: Given that $\varphi(x,\cdot)$ is convex and 1-homogeneous, $\varphi(x,\cdot)$ is a norm on $H_x\mathbb{G}$. Moreover, since the horizontal fiber $H_x\mathbb{G}$ is finite-dimensional, we have that the bidual of the Banach space $(H_x\mathbb{G}, \varphi(x,\cdot))$ is isometrically

isomorphic to $(H_x\mathbb{G}, \varphi(x,\cdot))$ itself via the canonical embedding map. It is then also clear that (by definition) $\varphi^{\star\star}(x,\cdot)$ coincides with the bidual norm of $\varphi(x,\cdot)$, whence the validity of the identity $\varphi = \varphi^{\star\star}$ follows. □

At the end, we present the following properties that we will need in the last section.

Lemma 2.13. *Let $\varphi : H\mathbb{G} \to [0,+\infty)$ be a sub-Finsler convex metric. Then the following hold:*

i) *If φ is lower semicontinuous, then $\varphi^\star$ is upper semicontinuous.*
ii) *If φ is upper semicontinuous, then $\varphi^\star$ is lower semicontinuous.*

In particular, if φ is continuous, then $\varphi^\star$ is continuous.

Proof. To prove i) suppose φ is lower semicontinuous. Fix $(x,v) \in H\mathbb{G}$ and $(x_n, v_n) \in H\mathbb{G}$ such that $(x_n, v_n) \to (x,v)$, in the sense that $d_{cc}(x_n, x) + \left\|\mathrm{d}_{x_n}\tau_{x_n^{-1}}[v_n] - \mathrm{d}_x\tau_{x^{-1}}[v]\right\|_e \to 0$. Possibly passing to a subsequence, we can assume that $\limsup_n \varphi^\star(x_n, v_n)$ is actually a limit. Given any $n \in \mathbb{N}$, there exists $w_n \in H_{x_n}\mathbb{G}$ with $\varphi(x_n, w_n) = 1$ and $\varphi^\star(x_n, v_n) = |\langle v_n, w_n\rangle_{x_n}|$. Since the unit sphere of each horizontal fiber is compact, there exists $w \in H_x\mathbb{G}$ such that (up to subsequence) $(x_n, w_n) \to (x,w)$. Being φ lower semicontinuous, we deduce that

$$\varphi(x,w) \le \liminf_{n\to\infty} \varphi(x_n, w_n) \le 1.$$

Therefore, we conclude that

$$\varphi^\star(x,v) \ge \frac{|\langle v, w\rangle_x|}{\varphi(x,w)} \ge \lim_{n\to\infty} |\langle v_n, w_n\rangle_{x_n}| = \limsup_{n\to\infty} \varphi^\star(x_n, v_n),$$

which proves that $\varphi^\star$ is upper semicontinuous.

The assertion ii) can be proved noticing that if φ is upper semicontinuous, then $\varphi^\star$ is lower semicontinuous as it can be expressed as a supremum of lower semicontinuous functions. □

Notation. For any $d \in \mathcal{D}_{cc}(\Omega)$ and $a \in \Omega$, we denote by $d_a : \Omega \to [0,+\infty)$ the fixed-point distance map $d_a(x) \coloneqq d(a,x)$. Clearly, d_a is a Lipschitz function and, by Theorem 2.8, d_a is Pansu differentiable for a.e. $x \in \Omega$. We denote by $\mathrm{Lip}([0,1],\Omega)$ the class of all Lipschitz continuous curves, where the target is equipped with the Carnot–Carathéodory distance, and by $\mathcal{H}([0,1],\Omega)$ the class of all horizontal curves. In the sequel, we are going to consider curves defined on the unit interval and we assume that such curves are parametrized with constant speed. Moreover, for every Lebesgue null set $N \subset \Omega$, we set $\mathcal{P}(\Omega, N)$ as the class of all horizontal curves $\gamma : [0,1] \to \Omega$ such that

$$\mathcal{L}^1(\{t \in [0,1] \mid \gamma(t) \in N\}) = 0,$$

where $\mathcal{L}^1$ is the standard 1-dimensional Lebesgue measure. By Lemma 3.7 of [11] we have that $\mathcal{P}(\Omega, N) \neq \emptyset$. Furthermore, with $H\Omega$ we mean the restriction of the horizontal bundle $H\mathbb{G}$ to Ω, *i.e.*,

$$H\Omega \coloneqq \{(x,v) \in H\mathbb{G} : x \in \Omega\}.$$

If not otherwise stated, for every $v \in T_x\mathbb{G}$ and $x \in \mathbb{G}$ sometimes we denote by $\bar{v} \coloneqq \mathrm{d}_x\tau_{x^{-1}}[v]$ the representative vector of $T_x\mathbb{G} \ni v$ in the Lie algebra $\mathfrak{g}$.

3. Metric derivative and length representation theorem

Given a geodesic distance $d \in \mathcal{D}_{cc}(\mathbb{G})$, it is quite natural to consider the associated metric given by differentiation. The next definition is inspired by the ones proposed in [24, 27] but, in our setting, we necessarily have to define it only on the horizontal bundle $H\mathbb{G}$.

Definition 3.1 (Metric derivative)**.** Let $d \in \mathcal{D}_{cc}(\mathbb{G})$. We define $\varphi_d : H\mathbb{G} \to [0, +\infty)$ as

$$\varphi_d(x, v) := \limsup_{t \to 0} \frac{d(x, x \cdot \delta_t \exp \mathrm{d}_x \tau_{x^{-1}}[v])}{|t|} \quad \text{for every } (x, v) \in H\mathbb{G}.$$

Note that we translate the vector $v \in H_x\mathbb{G}$ to e via the differential of the left traslation, because the exponential map is defined on the first stratum $\mathfrak{g}_1 = H_e\mathbb{G}$. The next lemma tells us that the metric derivative is actually a metric.

Lemma 3.2. *For every $d \in \mathcal{D}_{cc}(\mathbb{G})$ we have that $\varphi_d \in \mathcal{M}^{c\alpha}_{cc}(\mathbb{G})$, for some $c \geq 1$ independent of d.*

Proof. Let us show that i), ii), iii) in Definition 2.9 hold. In order to prove i), let just observe that

$$\varphi_d(x, v) = \lim_{n \to \infty} \sup_{\substack{t \in \mathbb{Q}: \\ |t| < 1/n}} \frac{d(x, x \cdot \delta_t \exp \mathrm{d}_x \tau_{x^{-1}}[v])}{|t|} \quad \text{for every } (x, v) \in H\mathbb{G}.$$

Let us verify ii). Pick $x \in \mathbb{G}$, $v \in H_x\mathbb{G}$ and $t, \lambda \in \mathbb{R}$. Since the differential of the left translation is a diffeomorphism, and taking into account also the equality in (2.6), we have that $\delta_t \exp\big(\mathrm{d}_x \tau_{x^{-1}}[\delta^\star_\lambda(v)]\big) = \delta_t \delta_\lambda \exp\big(\mathrm{d}_x \tau_{x^{-1}}[v]\big)$. Therefore

$$\varphi_d(x, \delta^*_\lambda v) = \limsup_{t \to 0} \frac{d(x, x \cdot \delta_t \delta_\lambda e^{\mathrm{d}_x \tau_{x^{-1}}[v]})}{|t|} = |\lambda| \limsup_{t \to 0} \frac{d(x, x \cdot \delta_{t\lambda} e^{\mathrm{d}_x \tau_{x^{-1}}[v]}))}{|t\lambda|} = |\lambda| \varphi(x, v).$$

In order to show iii), fix $x \in \mathbb{G}$ and $v \in H_x\mathbb{G}$. Since $d \in \mathcal{D}_{cc}(\mathbb{G})$ we can write

$$\varphi_d(x, v) \leq \alpha \limsup_{t \to 0} \frac{d_{cc}(x, x \cdot \delta_t e^{\mathrm{d}_x \tau_{x^{-1}}[v]})}{|t|} = \alpha\, d_{cc}(e, \exp \mathrm{d}_x \tau_{x^{-1}}[v]) \leq c\, \alpha \|\mathrm{d}_x \tau_{x^{-1}}[v]\|_e$$

where in the last inequality we applied Lemma 2.4. The estimate from below can be proved similarly. Finally, using the left invariance of the norm, for every $(x, v) \in H\mathbb{G}$, we get that

$$\frac{1}{c\alpha} \|v\|_x \leq \varphi_d(x, v) \leq c\, \alpha \|v\|_x$$

and the conclusion follows. □

The next result comes from Proposition 1.3.3 of [22] and a general proof can be find in Proposition 3.50 of [2]. It says that Lipschitz curves and horizontal ones essentially coincide when the L^∞-norm of the canonical coordinates is finite.

Proposition 3.3. *A curve $\gamma : [a, b] \to \Omega \subset \mathbb{G}$ is Lipschitz if and only if it is horizontal and $\|h\|_{L^\infty(a,b)} \leq L$, where L is the Lipschitz constant.*

In general, if (M, d) is a metric space and $\gamma : [0, 1] \to M$ a Lipschitz curve, then the *classical metric derivative* is defined for almost every $t \in [0, 1]$ by

$$|\dot\gamma(t)|_d := \lim_{s \to 0} \frac{d(\gamma(t + s), \gamma(t))}{|s|}.$$

The existence of the previous limit is a general fact that holds in any metric space (see [8], Thm. 2.7.6); indeed $|\dot\gamma(t)|_d$ exists for a.e. $t \in [0,1]$, it is a measurable function and it satisfies the equality

$$L_d(\gamma) = \int_0^1 |\dot\gamma(t)|_d \,\mathrm{d}t, \tag{3.1}$$

where the *classical length functional* of a rectifiable curve is defined as

$$L_d(\gamma) = \sup_{\{0\le t_1<\ldots<t_k\le 1\}} \sum_{i=1}^{k-1} d(\gamma(t_{i+1}),\gamma(t_i)),$$

and the supremum is taken over all finite partitions of $[0,1]$.

Carnot groups are naturally endowed with sub-Riemannian distances which make them interesting examples of metric spaces $(\mathbb{G}, d_{cc})$. In particular, the metric derivative can be explicitly computed (see [22], Thm. 1.3.5).

Lemma 3.4. *Let $\gamma : [0,1] \to \mathbb{G}$ be a Lipschitz curve and let $h \in L^\infty(0,1)^m$ be its vector of canonical coordinates. Then*

$$|\dot\gamma(t)|_{d_{cc}} = \lim_{s\to 0} \frac{d_{cc}(\gamma(t+s),\gamma(t))}{|s|} = |h(t)| \quad \textit{for a.e. } t \in [0,1]$$
$$\textit{and} \quad \lim_{s\to 0} \delta_{\frac{1}{s}}\big(\gamma(t)^{-1}\cdot\gamma(t+s)\big) = (h_1(t),\ldots,h_m(t),0,\ldots,0) \quad \textit{for a.e. } \ t\in[0,1].$$

The second claim is proved in Lemma 2.1.4 of [22] and it gives a characterization of horizontal curves in terms of canonical coordinates. Therefore, by Proposition 3.3, a Lipschitz curve is horizontal and, with abuse of notation, we set the following quantity:

$$\exp\dot\gamma(t) := \exp \mathrm{d}_{\gamma(t)}\tau_{\gamma(t)^{-1}}[\dot\gamma(t)] = \big(h_1(t),\ldots,h_m(t),0\ldots,0\big), \qquad \text{for a.e. } t\in[0,1].$$

Theorem 3.5. *Let $d \in \mathcal{D}_{cc}(\mathbb{G})$. Then, for every horizontal curve $\gamma : [0,1]\to\mathbb{G}$ we have*

$$L_d(\gamma) = \int_0^1 \varphi_d(\gamma(t),\dot\gamma(t))\,\mathrm{d}t. \tag{3.2}$$

Moreover, for every $x, y \in \mathbb{G}$ we have

$$d(x,y) = \inf\left\{\int_0^1 \varphi_d(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \ :\ \gamma\in\mathcal{H}([0,1],\mathbb{G}),\ \ \gamma(0)=x,\ \gamma(1)=y\right\}. \tag{3.3}$$

The previous result is a consequence of identity (3.1) and the fact that, for any horizontal curve $\gamma : [0,1]\to\mathbb{G}$, we have

$$|\dot\gamma(t)|_d = \varphi_d(\gamma(t),\dot\gamma(t)) \quad \text{for a.e. } t\in[0,1]. \tag{3.4}$$

The identity (3.4) follows the fact that the curve γ is a.e. differentiable, which yields

$$\begin{aligned}|\dot\gamma(t)|_d &= \lim_{s\to0}\frac{d(\gamma(t),\gamma(t+s))}{|s|} = \limsup_{s\to 0}\frac{d\big(\gamma(t),\gamma(t)\cdot\delta_s\exp \mathrm{d}_{\gamma(t)}\tau_{\gamma(t)^{-1}}[\dot\gamma(t)]\big)}{|s|}\\ &= \varphi_d(\gamma(t),\dot\gamma(t))\end{aligned}$$

for a.e. $t \in [0,1]$.

3.1. Convexity of φ_d

The aim of the present section is to prove that if $d \in \mathcal{D}_{cc}(\mathbb{G})$, then φ_d is also a sub-Finsler convex metric. In order to achieve this, first we have to recall some technical results.

Lemma 3.6. *Let $\psi\colon \mathbb{G} \to \mathbb{R}$ be a locally bounded, Borel function and $v \in H_x\mathbb{G} \setminus \{0\}$. Then*

$$\psi(x) = \lim_{t \searrow 0} \frac{1}{t} \int_0^t \psi(x \cdot \delta_s e^{\bar{v}})\, \mathrm{d}s, \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \mathbb{G}. \tag{3.5}$$

Proof. Given any fixed $y \in \mathbb{G}$, we have that $\mathbb{R} \ni t \mapsto \psi(y \cdot \delta_t e^{\bar{v}}) \in \mathbb{R}$ is a locally bounded and Borel function, thus an application of Lebesgue's differentiation theorem guarantees that for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$

$$\psi(y \cdot \delta_r e^{\bar{v}}) = \lim_{t \searrow 0} \frac{1}{t} \int_0^t \psi(y \cdot \delta_{r+s} e^{\bar{v}})\, \mathrm{d}s. \tag{3.6}$$

In particular, an application of Fubini's theorem ensures that the set

$$\Gamma \coloneqq \big\{(y,r) \in \mathbb{G} \times \mathbb{R} \,\big|\, (3.6) \text{ holds}\big\}$$

has $\mathcal{L}^{n+1}$-full measure, thus for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$ we have that (3.6) holds for $\mathcal{L}^n$-a.e. $y \in \mathbb{G}$. Fix any such $r \in \mathbb{R}$ and a $\mathcal{L}^n$-negligible set $N \subset \mathbb{G}$ satisfying (3.6) for every $y \in \mathbb{G} \setminus N$.
Calling $\sigma_z : \mathbb{G} \to \mathbb{G}$ the right-translation map $\sigma_z w \coloneqq w \cdot z$ for every $z, w \in \mathbb{G}$ and defining $N' \coloneqq \sigma_{\delta_r e^{\bar{v}}}(N)$, we thus have that $\psi(x) = \lim_{t \searrow 0} \frac{1}{t} \int_0^t \psi(x \cdot \delta_s e^{\bar{v}})\, \mathrm{d}s$ holds for every $x \in \mathbb{G} \setminus N'$. Here, we also used the fact that $\delta_{r+s} e^{\bar{v}} = \delta_r e^{\bar{v}} \cdot \delta_s e^{\bar{v}}$, which is in turn guaranteed by the fact that $\bar{v}$ belongs to the first layer (see [24], Lem. 2.2). Therefore, in order to prove (3.5) it is only left to check that N' is $\mathcal{L}^n$-negligible. This can be achieved by exploiting the right-invariance of the measure $\mathcal{L}^n$ (see *e.g.* [22], Prop. 1.7.7), namely the fact that $\mathcal{L}^n(E \cdot z) = \mathcal{L}^n(E)$ holds whenever $E \subset \mathbb{G}$ is a Borel set and $z \in \mathbb{G}$. Indeed, this implies that $(\sigma_{\delta_r e^{\bar{v}}})_\# \mathcal{L}^n = \mathcal{L}^n$, because for any Borel set $E \subset \mathbb{G}$ it holds that

$$(\sigma_{\delta_r e^{\bar{v}}})_\# \mathcal{L}^n(E) = \mathcal{L}^n(\sigma^{-1}_{\delta_r e^{\bar{v}}}(E)) = \mathcal{L}^n(\sigma_{\delta_{-r} e^{\bar{v}}}(E)) = \mathcal{L}^n(E \cdot \delta_{-r} e^{\bar{v}}) = \mathcal{L}^n(E).$$

In particular, we conclude that $\mathcal{L}^n(N') = (\sigma_{\delta_r e^{\bar{v}}})_\# \mathcal{L}^n(N') = \mathcal{L}^n(N) = 0$, as required. □

Lemma 3.7. *Let $d \in \mathcal{D}_{cc}(\Omega)$, $\varphi \in \mathcal{M}^\alpha_{cc}(\Omega)$, and $N \subset \Omega$ be such that $|N| = 0$. Suppose that for every $\gamma \in \mathcal{P}(\Omega, N)$ we have that*

$$d(\gamma(0), \gamma(1)) \le \int_0^1 \varphi(\gamma(t), \dot{\gamma}(t))\, \mathrm{d}t.$$

Then for every fixed $a \in \Omega$, for a.e. $x \in \Omega$, and for every $v \in H_x\mathbb{G}$, we have that

$$|\langle \nabla_{\mathbb{G}} d_a(x), v \rangle_x| \le \liminf_{t \to 0} \frac{d(x, x \cdot \delta_t e^{\bar{v}})}{t} \le \limsup_{t \to 0} \frac{d(x, x \cdot \delta_t e^{\bar{v}})}{t} \le \varphi(x, v).$$

Proof. Given any $a \in \Omega$ and $v \in H_e\mathbb{G}$, we define $E(a, v)$ as the set of all $x \in \Omega$ such that:

- d_a is Pansu differentiable at $x \in \Omega$,
- the curve $[0, t_0] \ni t \mapsto x \cdot \delta_t e^v$ belongs to $\mathcal{P}(\Omega, N)$, for some $t_0 > 0$ small enough,

- denoting $v_{x,s} \coloneqq \mathrm{d}_e\tau_{x\cdot\delta_s e^v}[v] \in H_{x\cdot\delta_s e^v}\mathbb{G}$ for $s \geq 0$ and $v_x \coloneqq v_{x,0}$, it holds that

$$\lim_{t\searrow 0}\frac{1}{t}\int_0^t \varphi(x\cdot\delta_s e^v, v_{x,s})\,\mathrm{d}s = \varphi(x, v_x).$$

Thanks to the Pansu–Rademacher theorem and Lemma 3.6 (and the last part of the proof of the latter, for what concerns the second bullet point), we deduce that $|\Omega\setminus E(a,v)| = 0$. Moreover, if $x \in E(a,v)$, then by applying the identity (2.9) we obtain that

$$\lim_{t\to 0}\frac{d_a(x) - d_a(x\cdot\delta_t e^v) - \langle\nabla_{\mathbb{G}} d_a(x), \pi_x(\delta_t e^{-v})\rangle_x}{|t|} = 0.$$

Hence, by the reverse triangle inequality we can assert that

$$\begin{aligned}|\langle\nabla_{\mathbb{G}} d_a(x), v_x\rangle_x| &\leq \left|\liminf_{t\to 0}\frac{d_a(x\cdot\delta_t e^v) - d_a(x)}{t}\right| \leq \liminf_{t\to 0}\frac{d(x, x\cdot\delta_t e^v)}{t}\\ &\leq \limsup_{t\to 0}\frac{d(x, x\cdot\delta_t e^v)}{t} \leq \lim_{t\searrow 0}\frac{1}{t}\int_0^t\varphi(x\cdot\delta_s e^v, v_{x,s})\,\mathrm{d}s = \varphi(x, v_x).\end{aligned}$$

Pick a countable dense subset $F \subset H_x\mathbb{G}$ and put $E(a) = \bigcap_{v\in F} E(a,v)$. Since $|\Omega\setminus E(a)| = 0$, the statement follows. □

We observe that the previous lemma could be proved under more general conditions. Indeed, the distance needs only to be geodesic in its domain.

Lemma 3.8. *Let $\varphi \in \mathcal{M}^{\alpha}_{cc}(\Omega)$ be a sub-Finsler convex metric, let $d \in \mathcal{D}_{cc}(\Omega)$ and $\Theta \subset \Omega$ be a countable dense set of Ω. If*

$$\|\varphi(x, \nabla_{\mathbb{G}} d_a(x))\|_\infty \leq 1 \quad \forall a \in \Theta,$$

then there exists $N \subset \Omega$ such that $|N| = 0$ and for every $\gamma \in \mathcal{P}(\Omega, N)$

$$d(\gamma(0), \gamma(1)) \leq \int_0^1 \varphi^\star(\gamma(t), \dot\gamma(t))\,\mathrm{d}t.$$

Proof. Let E be the subset of $x \in \Omega$ where the function $y \mapsto d_a(y)$ is Pansu-differentiable for every $a \in \Theta$. Since Θ is countable, we have that $N = \Omega\setminus E$ has zero Lebesgue measure. Let $\gamma \in \mathcal{P}(\Omega, N)$, pick $a \in \Theta$ and set $f(t) \coloneqq d_a(\gamma(t))$ for every $t \in [0,1]$, then

$$\begin{aligned}d_a(\gamma(1)) - d_a(\gamma(0)) = f(1) - f(0) &\leq \int_0^1\left|\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right|\mathrm{d}t = \int_0^1\left|\langle\nabla_{\mathbb{G}} d_a(\gamma(t)), h(t)\rangle_{\gamma(t)}\right|\mathrm{d}t\\ &\leq \int_0^1\varphi\big(\gamma(t), \nabla_{\mathbb{G}} d_a(\gamma(t))\big)\varphi^\star(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \leq \int_0^1\varphi^\star(\gamma(t), \dot\gamma(t))\,\mathrm{d}t.\end{aligned}$$

Now, by density of Θ in Ω we can choose a sequence $\{a_k\}_{k\in\mathbb{N}} \subset \Theta$ converging to $\gamma(0)$, obtaining

$$d(\gamma(0), \gamma(1)) = \lim_{k\to\infty}\big(d_{a_k}(\gamma(1)) - d_{a_k}(\gamma(0))\big) \leq \int_0^1\varphi^\star(\gamma(t), \dot\gamma(t))\,\mathrm{d}t.$$

□

Theorem 3.9. *Let $d \in \mathcal{D}_{cc}(\Omega)$. Then φ_d is a sub-Finsler convex metric. In particular, for almost all $x \in \Omega$ and all $v \in H_x\mathbb{G}$*

$$\varphi_d(x,v) = \lim_{t\to 0} \frac{d(x, x\cdot\delta_t e^{\bar{v}})}{|t|}. \tag{3.7}$$

Proof. Take a countable dense subset Θ of Ω and, for each $a \in \Theta$, we consider Σ_a a negligible Borel subset of Ω which contains all points where d_a is not Pansu-differentiable. For every $(x,v) \in H\Omega$ we define

$$\xi(x,v) := \begin{cases} \sup_{a\in\Theta} \left|\langle \nabla_{\mathbb{G}} d_a(x), v\rangle_x\right| & \text{if } x \in \Omega \setminus \bigcup_{a\in\Theta}\Sigma_a; \\ 0 & \text{otherwise.} \end{cases}$$

For $\varepsilon > 0$ we define $\xi_\varepsilon : H\Omega \to [0,+\infty)$ as $\xi_\varepsilon(x,v) := \xi(x,v) + \varepsilon\|v\|_x$, that is simple to show that it is a sub-Finsler convex metric on $H\Omega$. Indeed, if we take $v_1, v_2 \in H_x\Omega$ we can estimate in this way

$$\begin{aligned} \xi_\varepsilon(x, v_1+v_2) &= \sup_{a\in\Theta} \left|\langle \nabla_{\mathbb{G}} d_a(x), v_1+v_2\rangle_x\right| + \varepsilon\|v_1+v_2\|_x \\ &\le \xi(x,v_1) + \xi(x,v_2) + \varepsilon\|v_1+v_2\|_x \le \xi_\varepsilon(x,v_1) + \xi_\varepsilon(x,v_2). \end{aligned}$$

The homogeneity w.r.t. the second variable comes from the equality $\mathrm{d}_e\tau_x[\delta^\star_\lambda\bar{v}] = \lambda\mathrm{d}_e\tau_x[\bar{v}]$ where $\bar{v} = \mathrm{d}_x\tau_{x^{-1}}[v]$. Moreover, if $a \in \Theta$ we get that

$$\left|\langle \nabla_{\mathbb{G}} d_a(x), v\rangle_x\right| \le \xi(x,v) \le \xi_\varepsilon(x,v) \quad \text{for a.e. } x \in \Omega \text{ and } v \in H_x\mathbb{G}.$$

Thus, by definition of dual metric, we have

$$\frac{\left|\langle \nabla_{\mathbb{G}} d_a(x), v\rangle_x\right|}{\xi_\varepsilon(x,v)} \le 1 \quad \Rightarrow \quad \left\|\xi^\star_\varepsilon(x, \nabla_{\mathbb{G}} d_a(x))\right\|_\infty \le 1. \tag{3.8}$$

Being Θ countable, by Lemma 3.8, there exists a Lebesgue null set $N \subset \Omega$ such that, for every $\gamma \in \mathcal{P}(\Omega, N)$ we can infer that

$$d(\gamma(0), \gamma(t)) \le \int_0^t \xi_\varepsilon(\gamma(s), \dot{\gamma}(s))\,\mathrm{d}s.$$

Now, we are in position to apply Lemma 3.7 to the metric ξ_ε: for every fixed $a \in \Omega$, a.e. $x \in \Omega$ and all $v \in H_x\mathbb{G}$

$$\left|\langle \nabla_{\mathbb{G}} d_a(x), v\rangle_x\right| \le \liminf_{t\to 0} \frac{d(x, x\cdot\delta_t e^{\bar{v}})}{t} \le \limsup_{t\to 0} \frac{d(x, x\cdot\delta_t e^{\bar{v}})}{t} \le \xi_\varepsilon(x,v).$$

Then the convexity of $\varphi_d(x,\cdot)$ easily follows by taking the supremum over all $a \in \Theta$ of the left-hand side term. Moreover, if we also let $\varepsilon \to 0$, we obtain that

$$\xi(x,v) \le \liminf_{t\to 0} \frac{d(x, x\cdot\delta_t e^{\bar{v}})}{|t|} \le \limsup_{t\to 0} \frac{d(x, x\cdot\delta_t e^{\bar{v}})}{|t|} \le \xi(x,v),$$

which proves the equality (3.7). Therefore, the proof is complete. ∎

4. Application I: Γ-convergence

We start this section by briefly recalling the notion of Γ-convergence and we refer the interested reader to [10] for a complete overview on the subject. Let (M,τ) be a topological space satisfying the first axiom of countability. A sequence of maps $F_h : M \to \overline{\mathbb{R}}$ is said to $\Gamma(\tau)$-converge to F and we will write $F_h \xrightarrow{\Gamma(\tau)} F$ (or simply $F_h \xrightarrow{\Gamma} F$ if there is no risk of confusion) if the following two conditions hold:

(Γ-lim inf inequality) for any $x \in M$ and for any sequence $(x_h)_h$ converging to x in M one has

$$F(x) \leq \liminf_{h\to\infty} F_h(x_h)\,;$$

(Γ-lim sup inequality) for any $x \in M$, there exists a sequence $(x_h)_h$ converging to x in M such that

$$\limsup_{h\to\infty} F_h(x_h) \leq F(x)\,.$$

To any distance d on Ω with $\alpha^{-1}d_{cc} \leq d \leq \alpha d_{cc}$, we associate two functionals defined respectively on the class $\mathcal{B}(\Omega)$ of all positive finite Borel measures μ on $\Omega\times\Omega$ and on $\mathrm{Lip}([0,1],\Omega)$:

$$J_d(\mu) = \int d(x,y)\,\mathrm{d}\mu(x,y), \quad \mu \in \mathcal{B}(\Omega);$$
$$L_d(\gamma) = \int_0^1 \varphi_d(\gamma(t),\dot\gamma(t))\,\mathrm{d}t, \quad \gamma \in \mathrm{Lip}([0,1],\Omega).$$

As already mentioned in Section 2.5, we equip $\mathcal{D}_{cc}(\Omega)$ with the topology of the uniform convergence on compact subsets of $\Omega\times\Omega$. Moreover, we endow $\mathcal{B}(\Omega)$ and $\mathrm{Lip}([0,1],\Omega)$ with the topology of weak* convergence and of the uniform convergence, respectively.

Remark 4.1. The space $\mathcal{B}(\Omega)$ is a subset of the Banach space $\mathfrak{M}(\Omega\times\Omega)$, which consists of all finite signed Borel measures on $\Omega\times\Omega$ and is endowed with the total variation norm. Since $\Omega\times\Omega$ is locally compact and Hausdorff, it is well-known that $\mathfrak{M}(\Omega\times\Omega)$ is the dual of the Banach space $C_0(\Omega\times\Omega)$, defined as the closure in $C(\Omega\times\Omega)$ (with respect to the supremum norm) of the space $C_c(\Omega\times\Omega)$ of all compactly-supported continuous functions from $\Omega\times\Omega$ to $\mathbb{R}$. By the weak* topology on $\mathcal{B}(\Omega)$ we then mean the restriction to $\mathcal{B}(\Omega)$ of the weak* topology of $\mathfrak{M}(\Omega\times\Omega)$ when viewed as the dual of $C_0(\Omega\times\Omega)$. Let us recall that any weakly* converging sequence $\omega_n \overset{*}{\rightharpoonup} \omega$ in the dual $\mathbb{B}'$ of an arbitrary Banach space $\mathbb{B}$ is norm-bounded: indeed, for any vector $v\in\mathbb{B}$ we have that $\omega_n[v]\to\omega[v]$ and thus $(\omega_n[v])_n$ is bounded, so accordingly an application of the Banach–Steinhaus theorem ensures that $(\omega_n)_n$ is a bounded sequence in $\mathbb{B}'$. If applied to our case of interest, this general functional-analytic fact gives

$$\mathcal{B}(\Omega) \ni \mu_n \overset{*}{\rightharpoonup} \mu \in \mathcal{B}(\Omega) \quad\Longrightarrow\quad \sup_{n\in\mathbb{N}} \mu_n(\Omega\times\Omega) < +\infty.$$

This property will play a role in the proof of Theorem 4.4.

Before passing to the main Γ-convergence result of this section (*i.e.*, Thm. 4.4), we state and prove a couple of auxiliary results that might have some independent interest.

Lemma 4.2. *Let $\Omega\subset\mathbb{G}$ be an open set in a Carnot group $\mathbb{G}$. Let $(d_n)_n$ be a sequence in $\mathcal{D}_{cc}(\Omega)$ converging (uniformly on compact sets) to some limit function $d\colon \Omega\times\Omega\to[0,+\infty)$. Then d is a distance on Ω satisfying $\alpha^{-1}d_{cc}\leq d\leq\alpha d_{cc}$. Moreover, the following hold:*

i) *Letting* L_n, L *be the functionals associated respectively to* d_n, d *as before, it holds that*

$$L_d(\gamma) \leq \liminf_{n\to\infty} L_{d_n}(\gamma_n) \quad \textit{whenever } \gamma_n \to \gamma \textit{ in } \mathrm{Lip}([0,1],\Omega). \tag{4.1}$$

ii) *Given any curve* $\gamma \in \mathrm{Lip}([0,1],\Omega)$*, there exists a sequence* $(\gamma_n)_n \subset \mathrm{Lip}([0,1],\Omega)$ *such that* $(\gamma_n(0),\gamma_n(1)) = (\gamma(0),\gamma(1))$ *for every* $n \in \mathbb{N}$ *and*

$$L_d(\gamma) = \lim_{n\to\infty} L_{d_n}(\gamma_n). \tag{4.2}$$

Proof. First of all, one can immediately check that d, being a pointwise limit of distance functions, is a pseudodistance on Ω. Moreover, by letting $n \to \infty$ in $\alpha^{-1}d_{cc} \leq d_n \leq \alpha d_{cc}$ we obtain the inequality $\alpha^{-1}d_{cc} \leq d \leq \alpha d_{cc}$; in particular, d is actually a distance on Ω (since the lower bound $d \geq \alpha^{-1}d_{cc}$ ensures that $d(x,y) > 0$ whenever $x, y \in \Omega$ are distinct).

i) Fix any γ_n, γ as in (4.1). Notice that, thanks to (3.1) and (3.4), it holds that

$$L_d(\gamma) = \sup_{\{0\leq t_1<\ldots<t_k\leq 1\}} \sum_{i=1}^{k-1} d(\gamma(t_{i+1}),\gamma(t_i)), \tag{4.3}$$

where the supremum is over all finite partitions of the interval $[0,1]$. Similarly for $L_{d_n}(\gamma_n)$. Now fix a partition $0 \leq t_1 < \ldots < t_k \leq 1$. The image of the continuous curve γ is a compact subset of Ω; call it K'. Fix another compact set $K \subset \Omega$ whose interior contains K (for example, one can take as K the closed d_{cc}-neighbourhood of K' of radius ε, for some $\varepsilon > 0$ sufficiently small). Since $\gamma_n \to \gamma$ uniformly, there exists $\bar{n} \in \mathbb{N}$ such that the image of γ_n lies in K for every $n \geq \bar{n}$. Hence, since $d_n \to d$ uniformly on $K \times K$, we deduce that $\lim_n d_n(\gamma_n(t_i),\gamma_n(t_{i+1})) = d(\gamma(t_i),\gamma(t_{i+1}))$ for every $i = 1,\ldots,k-1$. In particular, we have

$$\sum_{i=1}^{k-1} d(\gamma(t_i),\gamma(t_{i+1})) = \lim_{n\to\infty} \sum_{i=1}^{k-1} d_n(\gamma_n(t_i),\gamma_n(t_{i+1})) \leq \liminf_{n\to\infty} L_{d_n}(\gamma_n),$$

whence it follows (thanks to (4.3)) that $L_d(\gamma) \leq \liminf_n L_{d_n}(\gamma_n)$. thus proving (4.1).

ii) Let $\gamma \in \mathrm{Lip}([0,1],\Omega)$ be fixed. Call K the image of γ, which is a compact subset of Ω. Since $d_n \to d$ uniformly on $K \times K$, we can find a strictly increasing sequence $(r(n))_n \subset \mathbb{N}$ with $\lim_n r(n) = +\infty$ such that $\lim_n r(n) \sup_{K\times K} |d_n - d| = 0$. For any $n \in \mathbb{N}$, we pick a curve $\gamma_n\colon [0,1] \to \Omega$ with the following properties: letting $t_n^i \coloneqq i/r(n)$ for every $i = 0,\ldots,r(n)$, the restriction of γ_n to $[t_n^{i-1},t_n^i]$ is a Lipschitz curve with constant speed (with respect to d_{cc}) with $(\gamma_n(t_n^{i-1}),\gamma_n(t_n^i)) = (\gamma(t_n^{i-1}),\gamma(t_n^i))$ and $L_{d_n}(\gamma_n|_{[t_n^{i-1},t_n^i]}) \leq d_n(\gamma(t_n^{i-1}),\gamma(t_n^i)) + 2^{-r(n)}$; here, we are using the assumption that d_n is a geodesic distance. In particular, we can estimate

$$\begin{aligned} L_{d_{cc}}(\gamma_n) &= \sum_{i=1}^{r(n)} L_{d_{cc}}(\gamma_n|_{[t_n^{i-1},t_n^i]}) \leq \alpha \sum_{i=1}^{r(n)} L_{d_n}(\gamma_n|_{[t_n^{i-1},t_n^i]}) \leq \alpha \sum_{i=1}^{r(n)} d_n(\gamma(t_n^{i-1}),\gamma(t_n^i)) + \alpha\frac{r(n)}{2^{r(n)}} \\ &\leq \alpha^2 \sum_{i=1}^{r(n)} d_{cc}(\gamma(t_n^{i-1}),\gamma(t_n^i)) + \alpha \leq \alpha^2 L_{d_{cc}}(\gamma) + \alpha \eqqcolon M, \end{aligned}$$

whence it follows that γ_n is M-Lipschitz. Hence, we have $\sup_{t\in[0,1]} d_{cc}(\gamma_n(t),\gamma(t)) \leq 2M/r(n)$ for every $n\in\mathbb{N}$, which shows that $\gamma_n \to \gamma$ uniformly as $n\to\infty$. Finally, by letting $n\to\infty$ in

$$\begin{aligned}L_d(\gamma) &\geq \sum_{i=1}^{r(n)} d(\gamma(t_n^{i-1}),\gamma(t_n^i)) \geq \sum_{i=1}^{r(n)} d_n(\gamma(t_n^{i-1}),\gamma(t_n^i)) - r(n)\sup_{K\times K}|d_n - d|\\ &\geq L_{d_n}(\gamma_n) - \frac{r(n)}{2^{r(n)}} - r(n)\sup_{K\times K}|d_n-d|\end{aligned}$$

we conclude that $L_d(\gamma) \geq \limsup_n L_{d_n}(\gamma_n)$, whence (4.2) follows (thanks also to (4.1)). □

The ensuing result follows from (the proof of) Lemma 4.2 and the Arzelà–Ascoli theorem:

Corollary 4.3. *Let $\Omega\subset\mathbb{G}$ be an open set in a Carnot group $\mathbb{G}$. Then $\mathcal{D}_{cc}(\Omega)$ is compact.*

Proof. Let $(d_n)_n$ be a fixed sequence in $\mathcal{D}_{cc}(\Omega)$. Notice that for any $n\in\mathbb{N}$ the inequalities

$$|d_n(x,y) - d_n(x',y')| \leq d_n(x,x') + d_n(y,y') \leq \alpha(d_{cc}(x,x') + d_{cc}(y,y')),$$

which are valid for every $x,x',y,y'\in\Omega$, show that $d_n\colon\Omega\times\Omega\to[0,+\infty)$ is α-Lipschitz if its domain $\Omega\times\Omega$ is endowed with the distance $((x,y),(x',y'))\mapsto d_{cc}(x,x')+d_{cc}(y,y')$. Now fix an increasing sequence of compact sets $(K_j)_j\subset\Omega$ satisfying $\bigcup_{j\in\mathbb{N}} K_j = \Omega$ and this property:

$$\text{for every } K\subset\Omega \text{ compact, there exists } j\in\mathbb{N} \text{ such that } K_j\subset K. \tag{4.4}$$

Given any $j\in\mathbb{N}$, we thus have that $\{d_n|_{K_j\times K_j} : n\in\mathbb{N}\}$ is a bounded and equicontinuous subset of $C(K_j\times K_j)$, so that an application of the Arzelà–Ascoli theorem ensures that (up to a subsequence) the functions d_n converge uniformly on $K_j\times K_j$ to some limit function. By a diagonalization argument and taking (4.4) into account, we deduce that (up to a subsequence) the functions d_n converge uniformly on compact sets to a limit function $d\colon\Omega\times\Omega\to[0,+\infty)$, which is a distance on Ω satisfying $\alpha^{-1}d_{cc}\leq d\leq\alpha d_{cc}$ by Lemma 4.2. In order to conclude, it remains to check that d is a geodesic distance. To this aim, fix any $x,y\in\Omega$. Define

$$L_d^{x,y}(\gamma) := \begin{cases} L_d(\gamma) & \text{if } \gamma\in\mathrm{Lip}([0,1],\Omega) \text{ and } (\gamma(0),\gamma(1)) = (x,y),\\ +\infty & \text{if } \gamma\in\mathrm{Lip}([0,1],\Omega) \text{ and } (\gamma(0),\gamma(1)) \neq (x,y),\end{cases}$$

$$L_{d_n}^{x,y}(\gamma) := \begin{cases} L_{d_n}(\gamma) & \text{if } \gamma\in\mathrm{Lip}([0,1],\Omega) \text{ and } (\gamma(0),\gamma(1)) = (x,y),\\ +\infty & \text{if } \gamma\in\mathrm{Lip}([0,1],\Omega) \text{ and } (\gamma(0),\gamma(1)) \neq (x,y).\end{cases}$$

We claim that $L_{d_n}^{x,y} \overset{\Gamma}{\to} L_d^{x,y}$ on $\mathrm{Lip}([0,1],\Omega)$. Indeed, the Γ-lim sup inequality readily follows from Lemma 4.2 ii), while for the Γ-lim inf inequality we distinguish two cases: if $\gamma_n\to\gamma$ in $\mathrm{Lip}([0,1],\Omega)$ and $L_d^{x,y}(\gamma)=+\infty$, then $(\gamma(0),\gamma(1))\neq(x,y)$, so that $(\gamma_n(0),\gamma_n(1))\neq(x,y)$ (and thus $L_{d_n}^{x,y}(\gamma_n)=+\infty$) for all n sufficiently large; if $\gamma_n\to\gamma$ in $\mathrm{Lip}([0,1],\Omega)$ and $L_d^{x,y}(\gamma)<+\infty$, then $(\gamma(0),\gamma(1))=(x,y)$ and thus $L_d^{x,y}(\gamma)=L_d(\gamma)\leq\liminf_n L_{d_n}(\gamma_n)\leq \liminf_n L_{d_n}^{x,y}(\gamma_n)$ by Lemma 4.2 i). All in all, the claimed Γ-convergence $L_{d_n}^{x,y}\to L_d^{x,y}$ is proved. It follows that

$$\inf_\gamma L_d^{x,y}(\gamma) = \lim_{n\to\infty}\inf_\gamma L_{d_n}^{x,y}(\gamma) = \lim_{n\to\infty} d_n(x,y) = d(x,y),$$

which proves that d is a geodesic distance. Therefore, $d\in\mathcal{D}_{cc}(\Omega)$ and $d_n\to d$ in $\mathcal{D}_{cc}(\Omega)$. □

We are ready to prove our Γ-convergence result, strongly inspired by Theorem 3.1 of [9]:

Theorem 4.4. *Let $\Omega \subset \mathbb{G}$ be an open set in a Carnot group $\mathbb{G}$. Let $(d_n)_n$, d belong to $\mathcal{D}_{cc}(\Omega)$. If J_n, L_n and J, L are the functionals associated respectively to d_n and d, defined as before, then the following conditions are equivalent:*

i) $d_n \to d$ *in* $\mathcal{D}_{cc}(\Omega)$;
ii) $J_n \xrightarrow{\Gamma} J$ *on* $\mathcal{B}(\Omega)$;
iii) $L_n \xrightarrow{\Gamma} L$ *on* $\mathrm{Lip}([0,1],\Omega)$.

Moreover, if Ω is bounded, then (i), (ii) *and* (iii) *are equivalent to the following condition:*

iv) *J_n continuously converges to J, meaning that $J(\mu) = \lim_n J_n(\mu_n)$ holds whenever a sequence $(\mu_n)_n \subset \mathcal{B}(\Omega)$ weakly* converges to $\mu \in \mathcal{B}(\Omega)$.*

Proof.

i) $\Longrightarrow$ ii) Assuming $d_n \to d$ in $\mathcal{D}_{cc}(\Omega)$, we aim to show that $J_n \xrightarrow{\Gamma} J$ on $\mathcal{B}(\Omega)$. In order to prove the Γ-lim inf inequality, fix $\mu \in \mathcal{B}(\Omega)$ and $(\mu_n)_n \subset \mathcal{B}(\Omega)$ such that μ_n weakly* converges to μ. Fix a sequence $(\eta_k)_k$ of compactly-supported continuous functions $\eta_k : \Omega \times \Omega \to [0,1]$ such that $\eta_k(x) \nearrow 1$ for every $x \in \Omega$. Since $d_n \to d$ in $\mathcal{D}_{cc}(\Omega)$, we deduce that for any $k \in \mathbb{N}$ we have that $\eta_k d_n \to \eta_k d$ uniformly as $n \to \infty$, thus there exists a sequence $(\varepsilon_n^k)_n \subset (0,+\infty)$ such that $\varepsilon_n^k \searrow 0$ as $n \to \infty$ and $|\eta_k d_n - \eta_k d| \le \varepsilon_n^k$ on $\Omega \times \Omega$. Recalling Remark 4.1, we know that $\sup_n \mu_n(\Omega \times \Omega) < +\infty$. Hence, since $\eta_k d$ is continuous and bounded, we get that

$$\begin{aligned}
&\left| \int \eta_k(x,y) d_n(x,y)\,\mathrm{d}\mu_n(x,y) - \int \eta_k(x,y) d(x,y)\,\mathrm{d}\mu(x,y) \right| \\
&\le \varepsilon_n^k\, \mu_n(\Omega \times \Omega) + \left| \int \eta_k(x,y) d(x,y)\,\mathrm{d}\mu_n(x,y) - \int \eta_k(x,y) d(x,y)\,\mathrm{d}\mu(x,y) \right| \to 0
\end{aligned}$$

as $n \to \infty$, for every $k \in \mathbb{N}$. In particular, for any $k \in \mathbb{N}$ we have that

$$\int \eta_k(x,y) d(x,y)\,\mathrm{d}\mu(x,y) = \lim_{n\to\infty} \int \eta_k(x,y) d_n(x,y)\,\mathrm{d}\mu_n(x,y) \le \liminf_{n\to\infty} J_n(\mu_n).$$

By the monotone convergence theorem, we conclude that $J(\mu) \le \liminf_n J_n(\mu_n)$, as desired.

Let us pass to the verification of the Γ-lim sup inequality. Fix any $\mu \in \mathcal{B}(\Omega)$. We aim to show that the sequence constantly equal to μ is a recovery sequence, namely $J(\mu) \ge \limsup_n J_n(\mu)$. If $J(\mu) = +\infty$, then there is nothing to prove. Thus suppose that $J(\mu) < +\infty$. Since $(1/\alpha) d_{cc} \le d$, we deduce that $d_{cc} \in L^1(\mu)$. By combining this information with the fact that $d_n \le \alpha d_{cc}$ for all $n \in \mathbb{N}$ and $d_n \to d$ pointwise on $\Omega \times \Omega$, we are in a position to apply the dominated convergence theorem, obtaining that

$$J(\mu) = \int d(x,y)\,\mathrm{d}\mu(x,y) = \lim_{n\to\infty} \int d_n(x,y)\,\mathrm{d}\mu(x,y) = \lim_{n\to\infty} J_n(\mu).$$

i) $\Longrightarrow$ iii) This implication immediately follows from Lemma 4.2.

ii) $\Longrightarrow$ i) Suppose that $J_n \xrightarrow{\Gamma} J$ on $\mathcal{B}(\Omega)$. Thanks to Corollary 4.3, from any subsequence of $(d_n)_n$ we can extract a further subsequence satisfying $d_n \to \tilde{d}$ in $\mathcal{D}_{cc}(\Omega)$, for some $\tilde{d} \in \mathcal{D}_{cc}(\Omega)$. Let us observe also that the Γ-convergence $J_n \to J = J_d$ on $\mathcal{B}(\Omega)$ is preserved. Moreover, the implication i $\Rightarrow$ ii) proved above ensures that the Γ-convergence $J_n \to J_{\tilde{d}}$ on $\mathcal{B}(\Omega)$ holds as well, whence it follows that $J_d = J_{\tilde{d}}$. In particular, it holds that

$$d(x,y) = J_d(\delta_x \otimes \delta_y) = J_{\tilde{d}}(\delta_x \otimes \delta_y) = \tilde{d}(x,y) \quad \text{for every } x,y \in \Omega,$$

thus showing that $\tilde{d} = d$. Having identified the limit, we conclude that the original sequence $(d_n)_n$ converges to d in the topology of $\mathcal{D}_{cc}(\Omega)$.

iii) $\Longrightarrow$ i) This implication can be proved by arguing as we did in showing that ii) $\Longrightarrow$ i), replacing J_n, J with L_n, L. Once $L_{\tilde{d}} = L_d$ is obtained, the identity $\tilde{d} = d$ stems from (3.3).

iv) $\Longrightarrow$ ii) Trivial (even when Ω is not bounded).

i) $\Longrightarrow$ iv) Assume Ω is bounded and that $d_n \to d$ in $\mathcal{D}_{cc}(\Omega)$. We want to prove iv). Fix any $\mu \in \mathcal{B}(\Omega)$ and $(\mu_n)_n \subset \mathcal{B}(\Omega)$ such that μ_n weakly* converges to μ. Let $\varepsilon > 0$ be fixed. Recalling Remark 4.1, we see that $\sup_n \mu_n(\Omega \times \Omega) < +\infty$. Moreover, we have that $\{\mu_n\}_n$ is weakly* relatively compact by assumption, thus Prokhorov's theorem yields the existence of a compact set $K \subset \Omega \times \Omega$ such that $\mu_n((\Omega \times \Omega) \setminus K) \leq \varepsilon$ for every $n \in \mathbb{N}$. Call D the diameter of Ω with respect to d_{cc}. Since $d : \Omega \times \Omega \to \mathbb{R}$ is bounded and continuous, we deduce that

$$\begin{aligned}\big|J_n(\mu_n) - J(\mu)\big| &\leq \int_K |d_n - d| \,\mathrm{d}\mu_n + \int_{(\Omega\times\Omega)\setminus K} |d_n - d| \,\mathrm{d}\mu_n + \big|J(\mu_n) - J(\mu)\big| \\ &\leq \mu_n(\Omega \times \Omega) \max_K |d_n - d| + 2\alpha D\varepsilon + \left| \int d \,\mathrm{d}\mu_n - \int d \,\mathrm{d}\mu \right|,\end{aligned}$$

whence by letting $n \to \infty$ we get $\limsup_n |J_n(\mu_n) - J(\mu)| \leq 2\alpha D\varepsilon$. By arbitrariness of ε, we finally conclude that $J(\mu) = \lim_n J_n(\mu_n)$, so that iv) is proved. The proof is complete. □

5. Applications II: Intrinsic geometry and sub-Finsler structure

The present section is devoted to generalizing the metric results contained in [15]. To this aim, we introduce two distances which involve the structure of the sub-Finsler metric.

Definition 5.1. If $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ is a sub-Finsler convex metric, for every $x, y \in \mathbb{G}$ we define the following quantity:

$$\delta_\varphi(x, y) := \sup \big\{ |f(x) - f(y)| \,\big|\, f \colon \mathbb{G} \to \mathbb{R} \textit{ Lipschitz}, \, \|\varphi(\cdot, \nabla_{\mathbb{G}} f(\cdot))\|_\infty \leq 1 \big\}. \tag{5.1}$$

Recall that Pansu's theorem assures that $\nabla_{\mathbb{G}} f(x)$ exists at almost every $x \in \mathbb{G}$ and thus the above definition makes sense. From now on, we will say that any Lipschitz function satisfying the conditions in (5.1) is a *competitor* for δ_φ. Moreover, we have that

Lemma 5.2. $\delta_\varphi : \mathbb{G} \times \mathbb{G} \to [0, +\infty)$ *is a distance.*

Proof. Clearly, we have that $\delta_\varphi(x, y) \geq 0$ for every $x, y \in \mathbb{G}$ and $\delta_\varphi(x, y) > 0$ if $x \neq y$. The symmetry comes from the fact that $|f(x) - f(y)| = |f(y) - f(x)|$. Also, δ_φ satisfies the triangle inequality since for every $x, y, z \in \mathbb{G}$ we have

$$\delta_\varphi(x, y) + \delta_\varphi(y, z) \geq |f(x) - f(y)| + |f(y) - f(z)| \geq |f(x) - f(z)|.$$

Passing to the supremum on the right-hand side for every f Lipschitz function such that $\|\varphi(x, \nabla_{\mathbb{G}} f(x))\|_\infty \leq 1$, we get that $\delta_\varphi(x, y) + \delta_\varphi(y, z) \geq \delta_\varphi(x, z)$. □

Under some assumptions, we will show in Theorem 5.11 that δ_φ turns out to be a distance in $\mathcal{D}_{cc}(\mathbb{G})$.

Definition 5.3. The *pointwise Lipschitz constant* of a Lipschitz function $f : \mathbb{G} \to \mathbb{R}$ is defined as

$$\mathrm{Lip}_{\delta_\varphi} f(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{\delta_\varphi(x, y)} \quad \text{for every } x \in \mathbb{G}.$$

We recall now the notion of intrinsic distance that was introduced by De Cecco–Palmieri in Definition 1.4 of [11] in the context of Lipschitz manifold.

Definition 5.4. Given any $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$, we define its induced *intrinsic distance* d_φ as

$$d_\varphi(x,y) \coloneqq \inf_\gamma \int_0^1 \varphi(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \quad \text{for every } x,y \in \mathbb{G},$$

where the infimum is taken over all horizontal curves $\gamma \in \mathcal{H}([0,1],\mathbb{G})$ joining x and y.

The quantity $d_\varphi(x,y)$ is well-defined because the map $t \mapsto (\gamma(t),\dot\gamma(t))$ is Borel measurable on the horizontal bundle.

Let us observe that in Definition 2.9 we are not requiring any regularity on φ besides its Borel measurability. At this level of generality (namely, without semicontinuity assumptions) d_φ might exhibit some 'pathological' behaviour, as we can see in the following example. Others examples of geodesic distances, contained in $\mathcal{D}_{cc}(\mathbb{R}^2)$, which are not intrinsic can be found in Example 1.8 of [14] and Corollary 3.4 of [7].

Example 5.5. Let $\mathbb{R}^2$ with the Euclidean structure and consider the Borel set $N \subset \mathbb{R}^2$ as

$$N \coloneqq \bigcup_{x,y\in\mathbb{Q}^2} S_{x,y},$$

where $S_{x,y}$ stands for the segment joining x and y. Notice that N is $\mathcal{L}^2$-negligible and we define the metric $\varphi\colon \mathbb{R}^2 \times \mathbb{R}^2 \to [0,+\infty)$ as

$$\varphi(x,v) \coloneqq \begin{cases} |v|, & \text{if } x \in N,\\ 2|v|, & \text{if } x \notin N. \end{cases}$$

Since $\varphi \in \mathcal{M}^2_{cc}(\mathbb{R}^2)$, for every $x,y \in \mathbb{R}^2$ it holds that $|x-y| \le d_\varphi(x,y) \le 2|x-y|$. In particular $d_\varphi\colon \mathbb{R}^2\times\mathbb{R}^2 \to [0,+\infty)$ is continuous when the domain is endowed with the Euclidean distance. Now observe that for any $x,y \in \mathbb{Q}^2$ we have that $d_\varphi(x,y) = |x-y|$, the shortest path being exactly the segment $S_{x,y}$. By continuity of d_φ and thanks to the density of $\mathbb{Q}^2$ in $\mathbb{R}^2$, we conclude that $d_\varphi(x,y) = |x-y|$ for every $x,y \in \mathbb{R}^2$. This shows that, even if $\varphi(x,\cdot)$ is equal to $2|\cdot|$ for $\mathcal{L}^2$-a.e. $x \in \mathbb{R}^2$, the distance d_φ coincides with the Euclidean distance. In other words, the behaviour of φ on the null set N completely determines the induced distance d_φ.

A further important concept for our treatment is the classical notion of Finsler metrics adapted to Carnot groups (see for instance [4]).

Definition 5.6. We say that a map $F : T\mathbb{G} \to [0,+\infty)$ is a *Finsler metric* if the following properties hold:

- F is continuous on $T\mathbb{G}$ and smooth on $T\mathbb{G}\setminus\{0\}$,
- the Hessian matrix of F^2 is positive definite for any vector $v \in T_x\mathbb{G}\setminus\{0\}$ for every $x \in \mathbb{G}$.

Moreover, we denote by d_F the length distance on $\mathbb{G}$ induced by F, namely we set

$$d_F(x,y) := \inf_\gamma \int_0^1 F(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \quad \text{for every } x,y \in \mathbb{G},$$

where the infimum is taken among all curves $\gamma \in \mathrm{Lip}([0,1],\mathbb{G})$ joining x and y.

Let us observe that the intrinsic distance is induced by a metric on the horizontal bundle while the latter comes from a metric defined on the entire tangent bundle.

Lemma 5.7. *If $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ is a sub-Finsler convex metric, then d_φ is a geodesic distance belonging to $\mathcal{D}_{cc}(\mathbb{G})$.*

Proof. Since $(\mathbb{G}, d_\varphi)$ is a complete, locally compact length space, then d_φ is a geodesic distance, thanks to the general result contained in Theorem 2.5.23 of [8].

To prove the claim, we have that $d_\varphi(x,y) \geq 0$ for every $x, y \in \mathbb{G}$ since the integral of $\varphi(\gamma(\cdot), \dot\gamma(\cdot))$ is non-negative. In order to prove the symmetry, let us consider $\gamma \in \mathcal{H}([0,1], \mathbb{G})$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Set $\xi : [0,1] \to \mathbb{G}$ as $\xi(t) = \gamma(1-t)$, hence this is a horizontal curve in $[0,1]$. By the 1-homogeneity of $\varphi(x,\cdot)$, we get that

$$\int_0^1 \varphi(\xi(t), \dot\xi(t))\,\mathrm{d}t = \int_0^1 \varphi(\gamma(1-t), -\dot\gamma(1-t))\,\mathrm{d}t = \int_0^1 \varphi(\gamma(s), -\dot\gamma(s))\,\mathrm{d}s$$
$$= \int_0^1 \varphi(\gamma(s), \dot\gamma(s))\,\mathrm{d}s.$$

So now, passing to the infimum over $\gamma \in \mathcal{H}([0,1], \mathbb{G})$ we get that $d_\varphi(x,y) = d_\varphi(y,x)$.

To prove the triangle inequality, let $x, y, z \in \mathbb{G}$ and $\gamma_1, \gamma_2 \in \mathcal{H}([0,1], \mathbb{G})$ be such that $\gamma_1(0) = x, \gamma_1(1) = y = \gamma_2(0)$, and $\gamma_2(1) = z$. Let us define the following curve:

$$\eta : [0,1] \to \mathbb{G}, \quad \eta(t) \coloneqq \begin{cases} \gamma_1(2t) & \text{if } t \in [0, \frac{1}{2}]; \\ \gamma_2(2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Then we obtain that

$$d_\varphi(x,z) \leq \int_0^1 \varphi(\eta(t), \dot\eta(t))\,\mathrm{d}t = \int_0^{\frac{1}{2}} \varphi(\gamma_1(2t), 2\dot\gamma_1(2t))\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \varphi(\gamma_2(2t-1), 2\dot\gamma_2(2t-1))\,\mathrm{d}t$$
$$= \int_0^1 \varphi(\gamma_1(s), \dot\gamma_1(s))\,\mathrm{d}s + \int_0^1 \varphi(\gamma_2(s), \dot\gamma_2(s))\,\mathrm{d}s,$$

where we applied a change-of-variable (in both integrals) and the 1-homogeneity of φ. Passing to the infimum respectively over all γ_1, γ_2, we conclude. We are left to prove (2.10). Let us take $x, y \in \mathbb{G}$ and consider the horizontal curve $\gamma : [0,1] \to \mathbb{G}$ s.t. $\gamma(0) = x$ and $\gamma(1) = y$. By definition of $\mathcal{M}^\alpha_{cc}(\mathbb{G})$, we get that

$$\int_0^1 \varphi(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \leq \alpha \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)}\,\mathrm{d}t.$$

Thus, passing to the infimum in the right-hand side we obtain the conclusion and the converse inequality can be achieved by arguing in a similar way. □

5.1. Main results

Before proving one of the main theorems, we recall some basic terminology. Given two Banach spaces $\mathbb{B}_1$ and $\mathbb{B}_2$, and denoting with $\mathrm{L}(\mathbb{B}_1, \mathbb{B}_2)$ the space of all linear and continuous operators $T\colon \mathbb{B}_1 \to \mathbb{B}_2$, it holds that $\mathrm{L}(\mathbb{B}_1, \mathbb{B}_2)$ is a Banach space if endowed with the usual pointwise operations and the operator norm, namely

$$\|T\|_{\mathrm{L}(\mathbb{B}_1,\mathbb{B}_2)} \coloneqq \sup_{v \in \mathbb{B}_1 \setminus \{0\}} \frac{\|T(v)\|_{\mathbb{B}_2}}{\|v\|_{\mathbb{B}_1}} \quad \text{for every } T \in \mathrm{L}(\mathbb{B}_1, \mathbb{B}_2).$$

Remark 5.8. Given a smooth map $\varphi\colon M \to N$ between two smooth manifolds M, N and a point $x \in M$, we denote by $\mathrm{d}_x\varphi\colon T_xM \to T_{\varphi(x)}N$ the differential of φ at x. We recall that if $\gamma\colon [0,1] \to M$ is an absolutely continuous curve in M, then $\sigma := \varphi \circ \gamma$ is an absolutely continuous curve in N and it holds that

$$\dot\sigma(t) = \mathrm{d}_{\gamma(t)}\varphi[\dot\gamma(t)] \quad \text{for a.e. } t \in [0,1]. \tag{5.2}$$

We also point out that

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta_t e^v = \mathrm{d}_e\tau_{\delta_t e^v}[v] \quad \text{for every } v \in H_e\mathbb{G} \text{ and } t \in (0,1). \tag{5.3}$$

Indeed, calling γ the unique curve satisfying (2.2) and defining $\gamma^t(s) := \gamma(ts)$ for all $t \in (0,1)$ and $s \in [0,1]$, we may compute

$$\frac{\mathrm{d}}{\mathrm{d}s}\gamma^t(s) = t\dot\gamma(ts) = t\,\mathrm{d}_e\tau_{\gamma(ts)}[v] = \mathrm{d}_e\tau_{\gamma^t(s)}[tv] \quad \text{for every } s \in (0,1),$$

which shows that γ^t fulfills the ODE defining tv, so that (2.6) yields $\gamma(t) = \gamma^t(1) = e^{tv} = \delta_t e^v$ for every $t \in (0,1)$ and accordingly the identity claimed in (5.3) is proved.

In general, let us observe that, if ψ is a sub-Finsler metric, then the metric derivative with $d = \delta_\psi$, namely φ_{δ_ψ}, could be very different from ψ (see [14], Exam. 1.5). Our purpose is to show a different result for the intrinsic distances. It tells us that, given a sub-Finsler convex metric ψ, the metric derivative with respect to d_ψ is bounded above by ψ almost everywhere. Moreover, we show that the equality holds, for instance, when ψ is lower semicontinuous.

Theorem 5.9. *Let $\psi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ be a sub-Finsler convex metric. Then the following properties are verified:*

i) *It holds that*

$$\text{for a.e. } x \in \mathbb{G}, \quad \varphi_{d_\psi}(x,v) \le \psi(x,v) \quad \text{for every } v \in H_x\mathbb{G}.$$

ii) *If ψ is upper semicontinuous, then*

$$\varphi_{d_\psi}(x,v) \le \psi(x,v) \quad \text{for every } (x,v) \in H\mathbb{G}.$$

iii) *If ψ is lower semicontinuous, then*

$$\varphi_{d_\psi}(x,v) \ge \psi(x,v) \quad \text{for every } (x,v) \in H\mathbb{G}.$$

In particular, for a.e. $x \in \mathbb{G}$ it holds that $\varphi_{d_\psi}(x,v) = \psi(x,v)$ for every $v \in H_x\mathbb{G}$.

Proof.
i) Given $x \in \mathbb{G}$, $v \in H_e\mathbb{G}$, and $t > 0$, we define the curve $\gamma = \gamma_{x,v,t}\colon [0,1] \to \mathbb{G}$ as

$$\gamma(s) := x \cdot \delta_{ts}e^v \quad \text{for every } s \in [0,1].$$

Notice that γ is horizontal and joins x to $x \cdot \delta_t e^v$. We can compute

$$\dot\gamma(s) = \frac{\mathrm{d}}{\mathrm{d}s}\tau_x\big(\delta_{ts}e^{\bar v}\big) \overset{(5.2)}{=} \mathrm{d}_{\delta_{ts}e^v}\tau_x\Big[\frac{\mathrm{d}}{\mathrm{d}s}\delta_{ts}e^v\Big] \overset{(5.3)}{=} \mathrm{d}_e\tau_{x\cdot\delta_{ts}e^v}[tv] \quad \text{for every } s \in (0,1).$$

Therefore, we may estimate

$$\begin{aligned} d_\psi(x, x\cdot\delta_t e^v) &\leq \int_0^1 \psi(\gamma(s),\dot\gamma(s))\,\mathrm{d}s = t\int_0^1 \psi\big(x\cdot\delta_{ts}e^v, \mathrm{d}_e\tau_{x\cdot\delta_{ts}e^v}[v]\big)\,\mathrm{d}s \\ &= \int_0^t \psi\big(x\cdot\delta_s e^v, \mathrm{d}_e\tau_{x\cdot\delta_s e^v}[v]\big)\,\mathrm{d}s. \end{aligned} \tag{5.4}$$

The next argument closely follows along the lines of Lemma 3.6. Fix a dense sequence $(v_i)_i$ in the unit sphere of $H_e\mathbb{G}$ (w.r.t. the norm $\|\cdot\|_e$). Define $v_i(x) \coloneqq \mathrm{d}_e\tau_x[v_i]$ for every $i\in\mathbb{N}$ and $x\in\mathbb{G}$, so that $(v_i(x))_i$ is a dense sequence in the unit sphere of $H_x\mathbb{G}$ (w.r.t. the norm $\|\cdot\|_x$). By using Lebesgue's differentiation theorem and Fubini's theorem, we see that the set Γ_i of all couples $(y,r)\in\mathbb{G}\times\mathbb{R}$ such that

$$\psi\big(y\cdot\delta_r e^{v_i}, \mathrm{d}_e\tau_{y\cdot\delta_r e^{v_i}}[v_i]\big) = \lim_{t\searrow 0}\frac{1}{t}\int_0^t \psi\big(y\cdot\delta_{r+s}e^{v_i}, \mathrm{d}_e\tau_{y\cdot\delta_{r+s}e^{v_i}}[v_i]\big)\,\mathrm{d}s \tag{5.5}$$

has zero $\mathcal{L}^{n+1}$-measure. By using Fubini's theorem again, we can find $r\in\mathbb{R}$ such that for any $i\in\mathbb{N}$ there exists a $\mathcal{L}^n$-null set $N_i\subset\mathbb{G}$ such that (5.5) holds for every point $y\in\mathbb{G}\setminus N_i$. Let us consider the set $N\coloneqq\bigcup_{i\in\mathbb{N}}\sigma_{\delta_r e^{v_i}}(N_i)$, where $\sigma_z\colon\mathbb{G}\to\mathbb{G}$ stands for the right-translation map $\sigma_z w\coloneqq w\cdot z$. The right-invariance of $\mathcal{L}^n$ grants that N is $\mathcal{L}^n$-negligible. Given that

$$\psi(x, v_i(x)) = \lim_{t\searrow 0}\frac{1}{t}\int_0^t \psi\big(x\cdot\delta_s e^{v_i}, \mathrm{d}_e\tau_{x\cdot\delta_s e^{v_i}}[v_i]\big)\,\mathrm{d}s \quad \text{for every } i\in\mathbb{N} \text{ and } x\in\mathbb{G}\setminus N, \tag{5.6}$$

we can conclude that

$$\begin{aligned} \varphi_{d_\psi}(x, v_i(x)) &= \lim_{t\searrow 0}\frac{d_\psi(x, x\cdot\delta_t e^{v_i})}{t} \overset{(5.4)}{\leq} \lim_{t\searrow 0}\frac{1}{t}\int_0^t \psi\big(x\cdot\delta_s e^{v_i}, \mathrm{d}_e\tau_{x\cdot\delta_s e^{v_i}}[v_i]\big)\,\mathrm{d}s \\ &\overset{(5.6)}{=} \psi(x, v_i(x)) \quad \text{for every } i\in\mathbb{N} \text{ and } x\in\mathbb{G}\setminus N. \end{aligned}$$

Since $\psi(x,\cdot)$ is continuous and positively 1-homogeneous, and $(v_i(x))_i$ is dense in the unit $\|\cdot\|_x$-sphere of $H_x\mathbb{G}$, we deduce that $\varphi_{d_\psi}(x,w)\leq\psi(x,w)$ for every $x\in\mathbb{G}\setminus N$ and $w\in H_x\mathbb{G}$.

ii) Suppose ψ is upper semicontinuous. Let $(x,v)\in H\mathbb{G}$ be fixed. Given any $\varepsilon>0$, we can thus find $t_\varepsilon>0$ such that, setting $\bar v\coloneqq \mathrm{d}_x\tau_{x^{-1}}[v]$ for brevity, it holds that

$$\psi\big(x\cdot\delta_t e^{\bar v}, \mathrm{d}_e\tau_{x\cdot\delta_t e^{\bar v}}[\bar v]\big) \leq \psi(x,v)+\varepsilon \quad \text{for every } t\in(0,t_\varepsilon). \tag{5.7}$$

In particular, we may estimate

$$\varphi_{d_\psi}(x,v) = \lim_{t\searrow 0}\frac{d_\psi(x, x\cdot\delta_t e^{\bar v})}{t} \overset{(5.4)}{\leq} \lim_{t\searrow 0}\frac{1}{t}\int_0^t \psi\big(x\cdot\delta_s e^{\bar v}, \mathrm{d}_e\tau_{x\cdot\delta_s e^{\bar v}}[\bar v]\big)\,\mathrm{d}s \overset{(5.7)}{\leq} \psi(x,v)+\varepsilon.$$

Thanks to the arbitrariness of ε, we can conclude that $\varphi_{d_\psi}(x,v)\leq\psi(x,v)$, as desired.

iii) Suppose ψ is lower semicontinuous. First of all, let us extend $\|\cdot\|_e$ to a Hilbert norm (still denoted by $\|\cdot\|_e$) on the whole $T_e\mathbb{G}=\mathfrak{g}$, then by left-invariance we obtain a Hilbert norm $\|\cdot\|_x$ on each tangent space $T_x\mathbb{G}$. Throughout the rest of the proof, we assume that $T_x\mathbb{G}$ is considered with respect to such norm $\|\cdot\|_x$. Moreover, choose any norm $\mathsf{n}:\mathfrak{g}\to[0,+\infty)$ on the Lie algebra which extends $\psi(e,\cdot)$, so that $\mathsf{n}\leq\lambda\|\cdot\|_e$ for some $\lambda>0$.

Without loss of generality, up to replacing ψ with the translated metric ψ_x, defined as $\psi_x(y,v)\coloneqq\psi\big(x\cdot y, \mathrm{d}_y\tau_x[v]\big)$ for every $(y,v)\in H\mathbb{G}$, it is sufficient to prove the statement only for $x=e$. Then let $v\in H_e\mathbb{G}$ be fixed. For any $t>0$ we have that the horizontal curve $[0,1]\ni s\mapsto\delta_{st}e^v\in\mathbb{G}$ is a competitor for $d_\psi(e,\delta_t e^v)$, thus we may estimate

$$\begin{aligned}d_\psi(e,\delta_t e^v)&\leq\int_0^1\psi(\delta_{st}e^v,t\,\mathrm{d}_e\tau_{\delta_{st}e^v}[v])\,\mathrm{d}s=t\int_0^1\psi(\delta_{st}e^v,\mathrm{d}_e\tau_{\delta_{st}e^v}[v])\,\mathrm{d}s\\&\leq\alpha t\int_0^1\|\mathrm{d}_e\tau_{\delta_{st}e^v}[v]\|_{\delta_{st}e^v}\,\mathrm{d}s=\alpha t\|v\|_e,\end{aligned}$$

where the last equality comes from the left invariance of the norm. This means that, in order to compute $d_\psi(e,\delta_t e^v)$, it is sufficient to consider those horizontal curves $\gamma\colon\mathbb{G}\to\mathbb{R}$ joining e to $\delta_t e^v$ and satisfying $\int_0^1\|\dot\gamma_s\|_{\gamma_s}\,\mathrm{d}s\leq\alpha\int_0^1\psi(\gamma_s,\dot\gamma_s)\,\mathrm{d}s\leq\alpha^2 t\|v\|_e$. We can also assume without loss of generality that any such curve γ is parametrized by constant speed with respect to the metric $\|\cdot\|_x$. All in all, we have shown that

$$d_\psi(e,\delta_t e^v)=\inf_{\gamma\in\mathcal{C}_t}\int_0^1\psi(\gamma_s,\dot\gamma_s)\,\mathrm{d}s\quad\text{ for every }t>0,\tag{5.8}$$

where the family $\mathcal{C}_t$ of curves is defined as

$$\mathcal{C}_t:=\left\{\gamma\colon[0,1]\to\mathbb{G}\text{ horizontal}\;\middle|\;\gamma_0=e,\,\gamma_1=\delta_t e^v,\,\|\dot\gamma_s\|_{\gamma_s}\equiv\int_0^1\|\dot\gamma_s\|_{\gamma_s}\,\mathrm{d}s\leq\alpha^2 t\|v\|_e\right\}.$$

Now fix any $\varepsilon>0$. Since the map $\exp^{-1}\colon\mathbb{G}\to\mathfrak{g}$ is a diffeomorphism, we can consider its differential $\mathrm{d}_x\exp^{-1}\colon T_x\mathbb{G}\to T_{\exp^{-1}(x)}\mathfrak{g}\cong\mathfrak{g}$ at any point $x\in\mathbb{G}$. Let us observe that $\exp^{-1}$ is smooth, and $\mathrm{d}_e\exp^{-1}=\mathrm{d}_e\tau_{e^{-1}}=\mathrm{id}_{\mathfrak{g}}$.

Since ψ is lower semicontinuous and by the previous argument, we can find $r>0$ such that

$$\psi(x,v)\geq\psi(e,\mathrm{d}_x\tau_{x^{-1}}[v])-\varepsilon\quad\text{ for every }x\in B(e,r)\text{ and }v\in H_x\mathbb{G},\ \|v\|_x\leq 1,\tag{5.9a}$$

$$\big\|\mathrm{d}_x\exp^{-1}-\mathrm{d}_x\tau_{x^{-1}}\big\|_{\mathrm{L}(T_x\mathbb{G},\mathfrak{g})}\leq\varepsilon\quad\text{ for every }x\in B(e,r),\tag{5.9b}$$

where $B(e,r)\equiv B_{d_{cc}}(e,r)$. In particular, given any $t>0$ with $\alpha^2 t\|v\|_e<r$ and $\gamma\in\mathcal{C}_t$, we have that $d_{cc}(e,\gamma_s)\leq s\alpha^2 t\|v\|_e<r$ for every $s\in[0,1]$ and $\|\dot\gamma_s\|_{\gamma_s}\leq\alpha^2 t\|v\|_e$ for a.e. $s\in[0,1]$, thus accordingly (5.9a) and (5.9b) yield

$$\psi(\gamma_s,\dot\gamma_s)\geq\psi(e,\mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}[\dot\gamma_s])-\alpha^2 t\|v\|_e\varepsilon\quad\text{ for a.e. }s\in[0,1],\tag{5.10a}$$

$$\big\|\mathrm{d}_{\gamma_s}\exp^{-1}-\mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}\big\|_{\mathrm{L}(T_{\gamma_s}\mathbb{G},\mathfrak{g})}\leq\varepsilon\quad\text{ for a.e. }s\in[0,1],\tag{5.10b}$$

respectively. Therefore, for any $t>0$ with $\alpha^2 t\|v\|_e<r$ and $\gamma\in\mathcal{C}_t$, we may estimate

$$\begin{aligned}&\left|\psi\bigg(e,\int_0^1\mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}[\dot\gamma_s]\,\mathrm{d}s\bigg)-\mathsf{n}\bigg(\int_0^1\mathrm{d}_{\gamma_s}\exp^{-1}[\dot\gamma_s]\,\mathrm{d}s\bigg)\right|\\\leq\;&\mathsf{n}\bigg(\int_0^1\mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}[\dot\gamma_s]\,\mathrm{d}s-\int_0^1\mathrm{d}_{\gamma_s}\exp^{-1}[\dot\gamma_s]\,\mathrm{d}s\bigg)\leq\lambda\bigg\|\int_0^1\mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}[\dot\gamma_s]-\mathrm{d}_{\gamma_s}\exp^{-1}[\dot\gamma_s]\,\mathrm{d}s\bigg\|_e\\\leq\;&\lambda\int_0^1\big\|(\mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}-\mathrm{d}_{\gamma_s}\exp^{-1})[\dot\gamma_s]\big\|_e\,\mathrm{d}s\leq\lambda\int_0^1\big\|\mathrm{d}_{\gamma_s}\exp^{-1}-\mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}\big\|_{\mathrm{L}(T_{\gamma_s}\mathbb{G},\mathfrak{g})}\|\dot\gamma_s\|_{\gamma_s}\,\mathrm{d}s\\\overset{(5.10b)}{\leq}\;&\lambda\varepsilon\int_0^1\|\dot\gamma_s\|_{\gamma_s}\,\mathrm{d}s\leq\lambda\varepsilon\alpha^2 t\|v\|_e,\end{aligned}$$

whence it follows that

$$\begin{aligned}\int_0^1 \psi(\gamma_s,\dot\gamma_s)\,\mathrm{d}s &\overset{(5.10a)}{\geq} \int_0^1 \psi(e,\mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}[\dot\gamma_s])\,\mathrm{d}s-\alpha^2t\|v\|_e\varepsilon\\ &\geq \psi\bigg(e,\int_0^1 \mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}[\dot\gamma_s]\,\mathrm{d}s\bigg)-\alpha^2t\|v\|_e\varepsilon\\ &\geq \mathsf{n}\bigg(\int_0^1 \mathrm{d}_{\gamma_s}\exp^{-1}[\dot\gamma_s]\,\mathrm{d}s\bigg)-(\lambda+1)\alpha^2t\|v\|_e\varepsilon.\end{aligned}\tag{5.11}$$

where in the second inequality we applied Jensen's inequality to $\psi(e,\cdot)$. Now consider the curve σ in the Hilbert space $(\mathfrak{g},\mathsf{n})$, which is given by $\sigma_s:=\exp^{-1}(\gamma_s)$ for every $s\in[0,1]$. It holds that σ is absolutely continuous and satisfies $\dot\sigma_s=\mathrm{d}_{\gamma_s}\exp^{-1}[\dot\gamma_s]$ for a.e. $s\in[0,1]$, thus

$$\begin{aligned}tv=tv-0_{\mathfrak{g}}&=\exp^{-1}(\delta_te^v)-\exp^{-1}(e)=\exp^{-1}(\gamma_1)-\exp^{-1}(\gamma_0)=\sigma_1-\sigma_0\\ &=\int_0^1\dot\sigma_s\,\mathrm{d}s=\int_0^1 \mathrm{d}_{\gamma_s}\exp^{-1}[\dot\gamma_s]\,\mathrm{d}s.\end{aligned}\tag{5.12}$$

By combining (5.11) and (5.12), we obtain for any $t>0$ with $\alpha^2t\|v\|_e<r$ and $\gamma\in\mathcal{C}_t$ that

$$\int_0^1\psi(\gamma_s,\dot\gamma_s)\,\mathrm{d}s\geq\mathsf{n}(tv)-(\lambda+1)\alpha^2t\|v\|_e\varepsilon=\big[\psi(e,v)-(\lambda+1)\alpha^2\|v\|_e\varepsilon\big]t.\tag{5.13}$$

We are now in a position to conclude the proof of the statement: given $t>0$ with $\alpha^2t\|v\|_e<r$, one has that

$$\frac{d_\psi(e,\delta_te^v)}{t}\overset{(5.8)}{=}\inf_{\gamma\in\mathcal{C}_t}\frac{1}{t}\int_0^1\psi(\gamma_s,\dot\gamma_s)\,\mathrm{d}s\overset{(5.13)}{\geq}\psi(e,v)-(\lambda+1)\alpha^2\|v\|_e\varepsilon.\tag{5.14}$$

By letting $t\searrow 0$, we thus deduce that

$$\varphi_{d_\psi}(e,v)=\limsup_{t\searrow0}\frac{d_\psi(e,\delta_te^v)}{t}\overset{(5.14)}{\geq}\psi(e,v)-(\lambda+1)\alpha^2\|v\|_e\varepsilon.\tag{5.15}$$

Finally, by letting $\varepsilon\searrow0$ in (5.15) we conclude that $\varphi_{d_\psi}(e,v)\geq\psi(e,v)$, as desired. □

Corollary 5.10. *If ψ is a continuous sub-Finsler convex metric, then*

$$\varphi_{d_\psi}(x,v)=\psi(x,v)\quad\text{ for every } (x,v)\in H\mathbb{G}.$$

Proof. It is an immediate consequence of assertions ii) and iii) of Theorem 5.9. □

The crucial observation below states that δ_φ coincides with the intrinsic distance $d_{\varphi^\star}$ when we assume that the sub-Finsler metric is lower semicontinuous. This will allow us to show the same result when φ is upper semicontinuous, thanks to an approximation argument.

Theorem 5.11. *Let $\varphi\in\mathcal{M}^\alpha_{cc}(\mathbb{G})$ be a sub-Finsler convex metric. Then it holds that $\delta_\varphi\leq d_{\varphi^\star}$. Moreover, if φ is lower semicontinuous, then*

$$\delta_\varphi(x,y)=d_{\varphi^\star}(x,y)\quad\text{for every } x,y\in\mathbb{G}.$$

Proof. Let $x, y \in \mathbb{G}$ be fixed. To prove the first part of the statement, pick any Lipschitz function f with $\|\varphi(z, \nabla_{\mathbb{G}} f(z))\|_\infty \leq 1$ and any horizontal curve $\gamma : [0,1] \to \mathbb{G}$ joining x and y such that

$$\mathcal{H}^1\big(\gamma \cap \{z \in \mathbb{G} \,:\, \varphi(z, \nabla_{\mathbb{G}} f(z)) > 1\}\big) = 0.$$

These are competitors for $\delta_\varphi(x,y)$ and $d_{\varphi^\star}(x,y)$, respectively. Then we can estimate

$$\begin{aligned}\big|f(x) - f(y)\big| &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t)))\,\mathrm{d}t\right| = \left|\int_0^1 \langle \nabla_{\mathbb{G}} f(\gamma(t)), \dot\gamma(t)\rangle_{\gamma(t)}\,\mathrm{d}t\right| \\ &\leq \int_0^1 \big|\langle \nabla_{\mathbb{G}} f(\gamma(t)), \dot\gamma(t)\rangle_{\gamma(t)}\big|\,\mathrm{d}t \leq \int_0^1 \varphi(\gamma(t), \nabla_{\mathbb{G}} f(\gamma(t)))\,\varphi^\star(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \\ &\leq \big\|\varphi(\cdot, \nabla_{\mathbb{G}} f(\cdot))\big\|_\infty \int_0^1 \varphi^\star(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \leq \int_0^1 \varphi^\star(\gamma(t), \dot\gamma(t))\,\mathrm{d}t,\end{aligned}$$

whence it follows that $\delta_\varphi(x,y) \leq d_{\varphi^\star}(x,y)$.

Now suppose φ is lower semicontinuous. Define the function $f : \mathbb{G} \to \mathbb{R}$ as $f(\cdot) \coloneqq d_{\varphi^\star}(x, \cdot)$ and since $d_{\varphi^\star}(x,y) \leq \alpha^{-1} d_{cc}(x,y)$ everywhere, we have that f is Lipschitz. Fix any point $z \in \mathbb{G}$ such that $\nabla_{\mathbb{G}} f(z)$ exists and let $v \in H_z\mathbb{G}$. Pick a horizontal curve $\gamma\colon [0,\varepsilon] \to \mathbb{G}$ of class C^1 such that $\gamma(0) = z$ and $\dot\gamma(0) = v$. Thanks to the continuity of $t \mapsto (\gamma(t), \dot\gamma(t))$ and the upper semicontinuity of $\varphi^\star$, granted by Lemma 2.13, we obtain that

$$\limsup_{t \searrow 0} ⨍_0^t \varphi^\star(\gamma(s), \dot\gamma(s))\,\mathrm{d}s \leq \varphi^\star(\gamma(0), \dot\gamma(0)) = \varphi^\star(z, v),$$

whence, by the identities (2.4) and (2.9), it follows that

$$\begin{aligned}\langle \nabla_{\mathbb{G}} f(z), v\rangle_z &= \lim_{t \searrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} \leq \limsup_{t \searrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{d_{\varphi^\star}(\gamma(t), \gamma(0))} \limsup_{t \searrow 0} \frac{d_{\varphi^\star}(\gamma(t), \gamma(0))}{t} \\ &\leq \limsup_{t \searrow 0} \frac{\big|d_{\varphi^\star}(x, \gamma(t)) - d_{\varphi^\star}(x, \gamma(0))\big|}{d_{\varphi^\star}(\gamma(t), \gamma(0))} \limsup_{t \searrow 0} ⨍_0^t \varphi^\star(\gamma(s), \dot\gamma(s))\,\mathrm{d}s \leq \varphi^\star(z, v).\end{aligned}$$

By arbitrariness of $v \in H_z\mathbb{G}$, we deduce that $\varphi(z, \nabla_{\mathbb{G}} f(z)) \leq 1$. Therefore, f is a competitor for $\delta_\varphi(x,y)$. This implies that $\delta_\varphi(x,y) \geq |f(x) - f(y)| = d_{\varphi^\star}(x,y)$. □

In particular, the last part of the proof shows that the supremum appearing in the definition of $\delta_\varphi(x,y)$ is actually a maximum.

The upper semicontinuity of the sub-Finsler metric φ is crucial for our proof, because it allows us to approximate the dual metric $\varphi^\star$ through a family of continuous Finsler metrics.

Corollary 5.12. *Let $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ be a sub-Finsler convex metric. Suppose φ is upper semicontinuous. Then, for every $x, y \in \mathbb{G}$ it holds that $\delta_\varphi(x,y) = d_{\varphi^\star}(x,y)$.*

Proof. Lemma 2.13 ensures that $\varphi^\star$ is lower semicontinuous. We set $\widetilde{\varphi}^\star\colon T\mathbb{G} \to [0, +\infty)$ as

$$\widetilde{\varphi}^\star(x,v) \coloneqq \begin{cases} \varphi^\star(x,v), & \text{if } (x,v) \in H\mathbb{G}, \\ +\infty, & \text{if } (x,v) \in T\mathbb{G} \setminus H\mathbb{G}. \end{cases}$$

Observe that $H\mathbb{G}$ is closed in $T\mathbb{G}$ and thus $\widetilde{\varphi}^\star$ is lower semicontinuous. Thanks to Theorem 3.11 of [19], there exists a sequence $F_n : T\mathbb{G} \to [0, +\infty)$ of Finsler metrics on $\mathbb{G}$ such that $F_n(x,v) \nearrow \tilde{\varphi}^\star(x,v)$ for every $(x,v) \in T\mathbb{G}$.

Setting

$$\varphi_n\colon H\mathbb{G} \to [0+\infty) \quad \text{as } \varphi_n := (F_n|_{H\mathbb{G}})^\star,$$

we obtain that $\varphi_n \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ and $\varphi_n^\star(x,v) \nearrow \varphi^\star(x,v)$ for every $(x,v) \in H\mathbb{G}$. Therefore $\varphi_n(x,v) \searrow \varphi(x,v)$ for every $(x,v) \in H\mathbb{G}$. In particular, the inequality $\varphi_n \geq \varphi$ holds for all $n \in \mathbb{N}$. This implies that any competitor f for δ_{φ_n} is a competitor for δ_φ, so that accordingly

$$\delta_{\varphi_n}(x,y) \leq \delta_\varphi(x,y), \qquad \text{for every } n \in \mathbb{N} \text{ and } x,y \in \mathbb{G}. \tag{5.16}$$

Moreover, since the infimum in the definition of d_{F_n} is computed with respect to all Lipschitz curves, while the infimum in the definition of $d_{\varphi_n^\star}$ is just over horizontal curves, for every $x,y \in \mathbb{G}$ we get that

$$d_{F_n}(x,y) \leq d_{\varphi_n^\star}(x,y) \leq d_{\varphi^\star}(x,y) \quad \text{ for every } n \in \mathbb{N}. \tag{5.17}$$

From the convergence of F_n to $\widetilde{\varphi}^\star$ we deduce that $d_{F_n}(x,y) \to d_{\varphi^\star}(x,y)$ for every $x,y \in \mathbb{G}$ (*cf.* the proof of [19], Thm. 5.1), and thus

$$d_{\varphi^\star}(x,y) = \lim_{n\to\infty} d_{\varphi_n^\star}(x,y) \quad \text{ for every } x,y \in \mathbb{G}. \tag{5.18}$$

Finally, since φ_n is lower semicontinuous (actually, continuous) by Lemma 2.13, we know from the second part of Theorem 5.11 that

$$\delta_{\varphi_n}(x,y) = d_{\varphi_n^\star}(x,y) \quad \text{for every } n \in \mathbb{N}. \tag{5.19}$$

All in all, we obtain that

$$d_{\varphi^\star}(x,y) \overset{(5.18)}{=} \lim_{n\to\infty} d_{\varphi_n^\star}(x,y) \overset{(5.19)}{=} \lim_{n\to\infty} \delta_{\varphi_n}(x,y) \overset{(5.16)}{\leq} \delta_\varphi(x,y) \quad \text{ for every } x,y \in \mathbb{G}.$$

Since the converse inequality $d_{\varphi^\star} \geq \delta_\varphi$ is granted by the first part of Theorem 5.11, we conclude that $\delta_\varphi = d_{\varphi^\star}$, as required. □

Theorem 5.13. *Let $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ be an upper semicontinuous sub-Finsler convex metric. Then for any locally Lipschitz function $f : \mathbb{G} \to \mathbb{R}$ we have that*

$$\varphi(x, \nabla_{\mathbb{G}} f(x)) = \mathrm{Lip}_{\delta_\varphi} f(x) \quad \textit{for a.e. } x \in \mathbb{G}.$$

Proof.
$\underline{\leq}$: Since both sides are positively 1-homogeneous with respect to f, we only need to show that, if $\mathrm{Lip}_{\delta_\varphi} f(x) = 1$, then $\varphi(x, \nabla_{\mathbb{G}} f(x)) \leq 1$ for a.e. $x \in \mathbb{G}$. By Corollary 5.12, we know that $\mathrm{Lip}_{\delta_\varphi} f(x) = \mathrm{Lip}_{d_{\varphi^\star}} f(x)$, hence if we fix $(x,v) \in H\mathbb{G}$, thanks to (2.4) and the expression (2.9) we can write:

$$\begin{aligned}\langle \nabla_{\mathbb{G}} f(x), v\rangle_x &= \lim_{t\to 0} \frac{f(x\cdot\delta_t e^{\bar v}) - f(x)}{t} \leq \limsup_{t\to 0} \frac{d_{\varphi^\star}(x, x\cdot\delta_t e^{\bar v})}{t} \cdot \limsup_{t\to 0} \frac{|f(x\cdot\delta_t e^{\bar v}) - f(x)|}{d_{\varphi^\star}(x, x\cdot\delta_t e^{\bar v})} \\ &\leq \varphi_{d_{\varphi^\star}}(x,v)\, \mathrm{Lip}_{d_{\varphi^\star}} f(x) \leq \varphi^\star(x,v),\end{aligned}$$

where in the last inequality we used item i) of Theorem 5.9. By arbitrariness of $v \in H_x\mathbb{G}$ and the fact that

$$\varphi(x, \nabla_{\mathbb{G}} f(x)) = \varphi^{\star\star}(x, \nabla_{\mathbb{G}} f(x)) \leq 1,$$

we get the conclusion.

$\geq$: Thanks to a convolution argument, we can find a sequence $(f_n)_n \subset C^1(\mathbb{G})$ such that $f_n \to f$ uniformly on compact sets and $\nabla_{\mathbb{G}} f_n \to \nabla_{\mathbb{G}} f$ in the almost everywhere sense. Recall that any C^1-function is locally Lipschitz. Fix any $x \in \mathbb{G}$ such that $\nabla_{\mathbb{G}} f_n(x)$ exists for all $n \in \mathbb{N}$ and $\nabla_{\mathbb{G}} f_n(x) \to \nabla_{\mathbb{G}} f(x)$ as $n \to \infty$. Now let $\varepsilon > 0$ be fixed. Then we can choose $r' > 0$ and $\bar{n} \in \mathbb{N}$ so that

$$\sup_{B(x,2r')} |f_{\bar{n}} - f| \leq \varepsilon \quad \text{and} \quad \varphi(x, \nabla_{\mathbb{G}} f_{\bar{n}}(x) - \nabla_{\mathbb{G}} f(x)) \leq \varepsilon,$$

where the ball is with respect to the distance $d_\varphi^\star$. Calling $g \coloneqq f_{\bar{n}}$ and being $z \mapsto \nabla_{\mathbb{G}} g(z)$ continuous, we deduce that $z \mapsto \varphi(z, \nabla_{\mathbb{G}} g(z))$ is upper semicontinuous, thus there exists $r < r'$ such that

$$\varphi(y, \nabla_{\mathbb{G}} g(y)) \leq \varphi(x, \nabla_{\mathbb{G}} g(x)) + \varepsilon \quad \text{for every } y \in B(x, 2r).$$

Fix any point $y \in B(x,r)$ and consider a horizontal curve $\gamma : [0,1] \to \mathbb{G}$ such that $\gamma(0) = x$, $\gamma(1) = y$ with $\gamma([0,1]) \subset B(x, 2r)$. We can estimate in this way:

$$\begin{aligned}
|f(x) - f(y)| &\leq |g(x) - g(y)| + 2\varepsilon \leq \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} g(\gamma(t))\,\mathrm{d}t + 2\varepsilon \\
&\leq \int_0^1 \varphi(\gamma(t), \nabla_{\mathbb{G}} g(\gamma(t)))\varphi^\star(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t + 2\varepsilon \\
&\leq \Big(\varphi(x, \nabla_{\mathbb{G}} g(x)) + \varepsilon\Big) \int_0^1 \varphi^\star(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t + 2\varepsilon \\
&\leq \Big(\varphi(x, \nabla_{\mathbb{G}} f(x)) + 2\varepsilon\Big) \int_0^1 \varphi^\star(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t + 2\varepsilon.
\end{aligned}$$

By taking the infimum over all $\gamma \in \mathcal{H}([0,1], B(x, 2r))$, we obtain that

$$|f(x) - f(y)| \leq \Big(\varphi(x, \nabla_{\mathbb{G}} f(x)) + 2\varepsilon\Big) d_{\varphi^\star}(x,y) + 2\varepsilon,$$

whence by letting $\varepsilon \to 0$ we obtain that

$$\frac{|f(x) - f(y)|}{d_{\varphi^\star}(x,y)} \leq \varphi(x, \nabla_{\mathbb{G}} f(x)).$$

Finally, by letting $y \to x$ we conclude that

$$\operatorname{Lip}_{\delta_\varphi} f(x) = \operatorname{Lip}_{d_{\varphi^\star}} f(x) \leq \varphi(x, \nabla_{\mathbb{G}} f(x)),$$

as required. □

To conclude, in Proposition 5.15 we prove that in the definition (5.1) of the distance δ_φ it is sufficient to consider smooth functions. Before passing to the proof of this claim, we prove the following technical result. By $\mathrm{LIP}_{d_\varphi^\star}(f) \in [0, +\infty)$ we mean the (global) Lipschitz constant of $f \in \mathrm{LIP}_{d_\varphi^\star}(\mathbb{G})$.

Lemma 5.14. *Let $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ be a sub-Finsler convex metric. Then it holds that*

$$\mathrm{LIP}_{d_{\varphi^\star}}(f) = \operatorname*{ess\,sup}_{x \in \mathbb{G}} \operatorname{Lip}_{d_{\varphi^\star}} f(x) \qquad \textit{for every } f \in \mathrm{LIP}_{d_{\varphi^\star}}(\mathbb{G}). \tag{5.20}$$

Proof. The inequality $(\geq)$ is trivial. To prove the converse inequality, we argue by contradiction: suppose there exist $x, y \in \mathbb{G}$ with $x \neq y$, a negligible Borel set $N \subseteq \mathbb{G}$ and $\delta > 0$ such that

$$\frac{\big|f(x) - f(y)\big|}{d_{\varphi^\star}(x,y)} \geq \sup_{z \in \mathbb{G}\setminus N} \mathrm{Lip}_{d_{\varphi^\star}} f(z) + \delta.$$

Given any $\varepsilon > 0$, we can find $\gamma \in \mathcal{H}([0,1],\mathbb{G})$ such that $\gamma(0) = x$, $\gamma(1) = y$, and

$$\int_0^1 \varphi^*(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \leq d_{\varphi^\star}(x,y) + \varepsilon.$$

Since $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is Lipschitz, so Pansu-differentiable almost everywhere, we deduce that

$$\begin{aligned}
\big|f(x) - f(y)\big| &\leq \int_0^1 \big|(f\circ\gamma)'(t)\big|\,\mathrm{d}t \leq \int_0^1 \varphi(\gamma(t), \nabla_{\mathbb{G}} f(\gamma(t)))\,\varphi^*(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \\
&= \int_0^1 \mathrm{Lip}_{d_{\varphi^\star}} f(\gamma(t))\,\varphi^\star(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \leq \sup_{z\in\mathbb{G}\setminus N} \mathrm{Lip}_{d_{\varphi^\star}} f(z) \int_0^1 \varphi^\star(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \\
&\leq \left[\frac{\big|f(x)-f(y)\big|}{d_{\varphi^\star}(x,y)} - \delta\right]\big(d_{\varphi^\star}(x,y) + \varepsilon\big).
\end{aligned}$$

By letting $\varepsilon \searrow 0$ in the above estimate, we get $0 \leq -\delta\, d_{\varphi^\star}(x,y)$, which leads to a contradiction. Therefore, also the inequality $(\leq)$ in (5.20) is proved, whence the statement follows. □

Proposition 5.15. *Let $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ be a sub-Finsler convex metric. Suppose φ is upper semicontinuous. Then for any $x, y \in \mathbb{G}$ it holds that*

$$\delta_\varphi(x,y) = \sup\Big\{\big|f(x) - f(y)\big| \;\Big|\; f \in C^\infty(\mathbb{G}),\ \big\|\varphi(\cdot, \nabla_{\mathbb{G}} f(\cdot)\big\|_\infty \leq 1\Big\}. \tag{5.21}$$

Proof. Denote by $\tilde\delta_\varphi(x,y)$ the quantity in the right-hand side of (5.21). Since any competitor for $\tilde\delta_\varphi(x,y)$ is a competitor for $\delta_\varphi(x,y)$, we have that $\delta_\varphi(x,y) \geq \tilde\delta_\varphi(x,y)$. To prove the converse inequality, fix any Lipschitz function $f\colon \mathbb{G} \to \mathbb{R}$ such that $\big\|\varphi(\cdot, \nabla_{\mathbb{G}} f(\cdot))\big\|_\infty \leq 1$. Corollary 5.12 and Theorem 5.13 grant that $\operatorname{ess\,sup} \mathrm{Lip}_{d_{\varphi^\star}} f \leq 1$, thus Lemma 5.14 yields $\mathrm{LIP}_{d_{\varphi^\star}}(f) \leq 1$. Given that $d_{\varphi^\star}$ is an increasing, pointwise limit of Finsler distances by Theorem 3.11 of [19], we are in a position to apply Theorem A.1. Therefore, we obtain a sequence $(f_n)_n \subseteq C^\infty(\mathbb{G}) \cap \mathrm{LIP}_{d_{\varphi^\star}}(\mathbb{G})$ such that $\mathrm{LIP}_{d_{\varphi^\star}}(f_n) \leq 1$ for all $n \in \mathbb{N}$ and $f_n \to f$ uniformly on compact sets. Corollary 5.12 and Theorem 5.13 imply that

$$\big\|\varphi(\cdot, \nabla_{\mathbb{G}} f_n(\cdot))\big\|_\infty = \sup \mathrm{Lip}_{d_{\varphi^\star}} f_n \leq 1,$$

thus f_n is a competitor for $\tilde\delta_\varphi(x,y)$. Then we conclude that

$$\big|f(x) - f(y)\big| = \lim_{n\to\infty} \big|f_n(x) - f_n(y)\big| \leq \tilde\delta_\varphi(x,y),$$

whence it follows that $\delta_\varphi(x,y) \leq \tilde\delta_\varphi(x,y)$ by arbitrariness of f. □

Appendix A. Smooth approximation of Lipschitz functions on generalized sub-Finsler manifolds

The aim of this appendix is to prove an approximation result for real-valued Lipschitz functions defined on some very weak kind of sub-Finsler manifold. More precisely, we consider a distance d on a smooth manifold that can be obtained as the monotone increasing limit of Finsler distances; this notion covers the case of generalized (so, possibly rank–varying) sub-Finsler manifolds, thanks to Theorem 3.11 of [19]. In this framework, we prove (see Thm. A.1 below) that any Lipschitz function can be approximated (uniformly on compact sets) by smooth functions having the same Lipschitz constant. This generalizes previous results that were known on 'classical' sub-Riemannian manifolds, *cf.* [16] and the references therein.

Let us fix some notation. Given a metric space (X,d), we denote by $\mathrm{LIP}_d(X)$ the family of real-valued Lipschitz functions on X. For any $f\in\mathrm{LIP}_d(X)$, we denote by $\mathrm{LIP}_d(f)\in[0,+\infty)$ and $\mathrm{Lip}_d f\colon X\to[0,+\infty)$ the (global) Lipschitz constant and the pointwise Lipschitz constant of f, respectively. Moreover, given a Finsler manifold (M,F), we denote by d_F the length distance on M induced by the Finsler metric F.

Theorem A.1. *Let M be a smooth manifold. Let d be a distance on M having the following property: there exists a sequence $(F_i)_i$ of Finsler metrics on M such that*

$$d_{F_i}(x,y)\nearrow d(x,y)\quad \text{for every } x,y\in M.$$

Then for any $f\in\mathrm{LIP}_d(M)$ there exists a sequence $(f_n)_n\subseteq C^\infty(M)\cap\mathrm{LIP}_d(M)$ such that

$$\sup_{n\in\mathbb{N}}\mathrm{LIP}_d(f_n)\le\mathrm{LIP}_d(f),\qquad f_n\to f\ \text{ uniformly on compact sets.}$$

Proof. Denote $L:=\mathrm{LIP}_d(f)$ and $d_i:=d_{F_i}$ for every $i\in\mathbb{N}$. Choose any countable, dense subset $(x_j)_j$ of (M,d). Given any $n\in\mathbb{N}$, we define the function $h_n\in\mathrm{LIP}_d(M)$ as

$$h_n(x):=\big(-L\,d(x,x_1)+f(x_1)\big)\vee\cdots\vee\big(-L\,d(x,x_n)+f(x_n)\big)-\frac{1}{n}\quad\text{for every } x\in M.$$

Observe that $\mathrm{LIP}_d(h_n)\le L$ and that $h_n(x)<h_{n+1}(x)<f(x)$ for every $n\in\mathbb{N}$ and $x\in M$. We claim that $h_n(x)\nearrow f(x)$ for all $x\in M$. In order to prove it, fix any $x\in M$ and $\varepsilon>0$. Pick some $\bar n\in\mathbb{N}$ such that $1/\bar n<\varepsilon$ and $d(x,x_{\bar n})<\varepsilon$. Then for every $n\ge\bar n$ it holds that

$$h_n(x)\ge -L\,d(x,x_{\bar n})+f(x_{\bar n})-\frac{1}{n}\ge -L\,\varepsilon+\big(f(x)-L\,d(x,x_{\bar n})\big)-\frac{1}{\bar n}\ge f(x)-(2L+1)\varepsilon,$$

thus proving the claim. Fix an increasing sequence $(K_n)_n$ of compact sets in M satisfying the following property: given any compact set $K\subseteq M$, there exists $n\in\mathbb{N}$ such that $K\subseteq K_n$. In particular, one has that $\bigcup_n K_n=M$. Notice that $h_n+\frac{1}{n(n+1)}\le h_{n+1}$ on K_n for all $n\in\mathbb{N}$. Since $d_{F_i}\nearrow d$, there exists $i_n\in\mathbb{N}$ such that the function $g_n\colon M\to\mathbb{R}$, given by

$$g_n(x):=\big(-L\,d_{i_n}(x,x_1)+f(x_1)\big)\vee\cdots\vee\big(-L\,d_{i_n}(x,x_n)+f(x_n)\big)-\frac{1}{n}\quad\text{for every } x\in M,$$

satisfies $h_n<g_n<h_n+\frac{1}{n(n+1)}$ on K_n. Note that $g_n\in\mathrm{LIP}_{d_{i_n}}(M)$ and $\mathrm{LIP}_{d_{i_n}}(g_n)=L$. Thanks to a mollification argument, it is possible to build a function $f_n\in C^\infty(M)\cap\mathrm{LIP}_{d_{i_n}}(M)$ such that $\mathrm{LIP}_{d_{i_n}}(f_n)\le L$ and $g_n<f_n<g_{n+1}$ on K_n. Therefore, for any $n\in\mathbb{N}$ and $x\in K_n$ it holds that the sequence $\big(f_j(x)\big)_{j\ge n}$ is strictly increasing and converging to $f(x)$. This grants that $f_j\to f$ uniformly on K_n for any given $n\in\mathbb{N}$. Hence, our specific choice of $(K_n)_n$ implies that $f_n\to f$ uniformly on compact sets. Finally, the inequality $d_{i_n}\le d$ yields $f_n\in\mathrm{LIP}_d(M)$ and $\mathrm{LIP}_d(f_n)\le\mathrm{LIP}_{d_{i_n}}(f_n)\le L$ for all $n\in\mathbb{N}$, whence the statement follows. □

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