ESAIM: COCV 29 (2023) 6
https://doi.org/10.1051/cocv/2022082

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

RISK-SENSITIVE MEAN FIELD GAMES WITH MAJOR AND MINOR PLAYERS*

Yan Chen[1], Tao Li[1,2,**] and Zhixian Xin[1]

Abstract. We investigate a class of mean field games containing a large number of major and minor players. Each player minimizes a quadratic-tracking type risk-sensitive cost functional, where the reference signal is a function of the state average term of the major and minor players. To reduce the complexity for solving the problem, we design a sequence of decentralized strategies by the Nash certainty equivalence principle. Firstly, for the optimal control problems with quadratic type risk-sensitive cost functionals, we propose a new verification theorem. Secondly, we apply the two-layer state aggregation method to construct the fixed-point equations for the estimations of the state average terms and give the conditions for the existence and uniqueness of the fixed points. Then, we design a sequence of decentralized strategies by the estimations of the state average terms based on local information. It is shown that the estimations of the state average terms are consistent with the true values for the closed-loop systems, and the sequence of strategies designed is a decentralized asymptotic Nash equilibrium. Finally, the effectiveness of the theoretical analysis is demonstrated by a numerical example.

Mathematics Subject Classification. 91A10, 91A25, 93C05, 93E03.

Received March 24, 2022. Accepted December 1, 2022.

1. Introduction

The theory of mean field games is developed by introducing the mean-field method widely applied in chemistry and physics into dynamic games. In mean field games, there are important coupling terms of states or controls called mean field terms, which model the complex interactions among players. Specifically, the mean field terms exist in dynamic equations or cost functionals. These kinds of models can be found in applications ranging from engineering, economics and biology, such as demand dispatch for regulation of the power grid and electricity price design [7, 8, 24], electric vehicle charging control [32], communication resource allocation for Internet of Things [25], tracking the data of infectious diseases and effective separation of infected patients [29, 42].

Mean field game theory was proposed by Huang *et al.* [19–22] and Lasry and Lions [26–28] independently. The main idea of this theory is to replace the overall effect of all agents on a single agent by an aggregation

* This work was supported in part by the National Natural Science Foundation of China under Grant 61977024, in part by the Basic Research Project of Shanghai Science and Technology Commission under Grant 20JC1414000 and in part by NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai. The authors are listed in alphabetical order.

Keywords and phrases: Risk-sensitive cost functional, major and minor players, mean field game, decentralized strategy, decentralized asymptotic Nash equilibrium.

[1] Shanghai Key Laboratory of Pure Mathematics and Mathematical Practice, School of Mathematical Sciences, East China Normal University, Shanghai 200241, China.
[2] NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai, 3663 Zhongshan Road North, Shanghai 200062, China.

** Corresponding author: tli@math.ecnu.edu.cn

effect and thus to transform a multi-agent game into an optimal control of a single agent. Huang *et al.* [19–22] studied mean field games with the fixed point (top–down) method and proposed the Nash certainty equivalence principle. By using an auxiliary system with a continuum of players, they constructed a mean field estimation and designed a sequence of decentralized strategies, then they proved that the sequence of strategies designed is a decentralized asymptotic Nash equilibrium. Lasry and Lions [26–28] studied mean field games with the direct (bottom-up) method. Firstly, they considered the games with N players and obtained a large scale system of coupled equations by dynamic programming, then they analyzed the existence of the solution as N goes to infinity. Weintraub *et al.* [44, 45] proposed a mean field approximation method for discrete-time stochastic games and introduced the concept of oblivious equilibrium. In addition to the pioneering works mentioned above, mean field game theory has been developed further in recent years. For linear-quadratic mean field games, Li and Zhang [30] employed the fixed point method to study mean field games with time-averaged stochastic cost functionals and introduced the notion of decentralized asymptotic Nash equilibrium in the sense of probability; Bardi [2] discussed the case with time-averaged deterministic cost functionals by the direct method developed in [26–28]; Huang and Zhou [23] studied the asymptotic solvability of linear-quadratic mean field games by establishing a scale reset method; Bensoussan *et al.* [3] considered linear-quadratic mean field games based on the stochastic maximum principle. For nonlinear mean field games, Anahtarci *et al.* [1] developed value iteration algorithms to investigate mean field games with discounted cost functionals and time-averaged cost functionals respectively, then they proved that the sequence of strategies designed is a decentralized asymptotic Nash equilibrium.

It is worth noting that for the literature mentioned above, the roles of players are equivalent. However, in practical systems, the roles of players are often different. For example, in price decision-making, the decisions of minor companies are always affected by some major companies. When the major companies implement some regulatory policies, all the minor companies will be affected by those policies in making production plans. There have been lots of results on mean field games with major and minor players [6, 15, 18, 31, 36–39]. Nourian *et al.* [37, 39] considered a class of mean field games with a large number of major and minor players by using maximum likelihood estimation. Caines and Huang *et al.* [6, 15, 18, 31, 36, 38] investigated a class of mean field games with one major player and a large number of minor players. Huang [18] divided minor players into K classes according to the different values of parameters for LQG systems with discount factor, obtained a sequence of decentralized strategies by the Nash certainty equivalence principle and proved that the sequence of strategies is a decentralized asymptotic Nash equilibrium; Nguyen and Huang [36] extended the minor players in [18] from discrete finite classes to a continuum of classes; Caines and Kizilkale [6] generalized the model in [18] to the case with incomplete observation; Nourian and Caines [38] studied nonlinear mean field games with one major player and a large number of minor players, then they proved that the sequence of strategies designed is a decentralized asymptotic Nash equilibrium; Ma and Huang [31] analyzed the asymptotic solvability of mean field games by dynamic programming and scale reset. Huang *et al.* [17] studied social optimal control problems based on the variational method and the mean field uniform approximation method, then they proved that the decentralized strategies designed are asymptotic social optimal strategies.

In a financial market, transaction strategy design problems related to asset management may come down to optimal decision problems with risk-sensitive cost functionals [4, 10, 11, 13]. According to the preferences of players for risk, these problems can be divided into three types: risk-neutral, risk-averse and risk-seeking. Tembine *et al.* [43] characterized the equilibrium solution of the system by a forward Fokker-Planck-Kolmogorov equation and a backward Hamilton-Jacobi-Bellman equation; Moon and Başar [34, 35] considered risk-sensitive linear mean field games with time-averaged cost functionals and nonlinear mean field games by the stochastic maximum principle; Saldi *et al.* [41] studied discrete-time risk-sensitive mean field games and proved that the sequence of strategies designed is a decentralized asymptotic Nash equilibrium. So far, the research on risk-sensitive mean field games has been limited to those with equivalent players, while, some problems may be modeled as games with major and minor players in financial markets. It is well-known that the risk-sensitive utility functional is always used to describe the reward of stock investment [5, 9, 13]. In a financial market with a large number of major and minor stock investment companies, the average wealth of the minor companies has no influence on the major companies, but the average wealth of the major companies has influence on the minor companies. Each company designs an appropriate investment strategy to maximize its investment reward. Inspired by the above consideration, we study a kind of risk-sensitive mean field games with major and minor players.

The contributions of this paper are summarized as follows.

(i) For the optimal control problems with quadratic type risk-sensitive cost functionals, we propose a new verification theorem. Compared with the verification theorem in [12], we do not assume that the admissible control laws are Markov strategies with linear growth and local Lipchitz conditions.

(ii) The state average terms of the major and minor players co-exist in the cost functionals of the minor players. However, they are unavailable for each player to design its decentralized strategy. To this end, we use the two-layer state aggregation method [37, 39] to construct the estimations of the state average terms, that is, by the state aggregation method, firstly, we construct an estimation of the state average term for the major players and design the strategies for the major players by this estimation, then we replace the state average term of the major players in the cost functionals of the minor players by its estimation, and construct an estimation of the state average term for the minor players, at last, we replace the state average terms of both the major and minor players by their estimations, respectively, to design the strategies for the minor players. In fact, for the case without major players, the problem degenerates to the linear quadratic risk-sensitive mean field game in [35].

(iii) The cost of each player is a quadratic-tracking type risk-sensitive functional, which makes the concept of consistency of estimations for the state average terms in [23, 30, 36] no longer applicable. To this end, we define a new concept of consistency: the integrals of norm-square of the differences between the estimations and the true values vanish in mean square as the players increase. Then, we prove that the sequence of strategies designed is a decentralized asymptotic Nash equilibrium through the following steps. Firstly, we construct tracking-type optimal control Auxiliary problems (I) and (II). They are derived by replacing the state average terms of the major and minor players by the corresponding estimations in the cost functionals, respectively. By Dyson expansion and Cauchy-Schwarz inequality, we transform the differences between the costs of the original games and those of Auxiliary problems (I) and (II) into the products of the mean square of the differences between two classes of quadratic functions and the mean square of the integrals of exponential functions. Secondly, by the consistency of the estimations, we prove that the differences between the two classes of quadratic functions vanish in mean square as the players increase. Thirdly, by the calculation of the expectation of exponential functions related to Brownian motion, we obtained the mean square boundedness of the integrals of exponential functions. This implies the vanishing of the differences between the two classes of costs, based on which we prove that the sequence of strategies designed is a decentralized asymptotic Nash equilibrium.

The rest of this paper is organized as follows. In Section 2, we give the state equations and the cost functionals of risk-sensitive mean field games with a large number of major and minor players. In Section 3, we use the state aggregation method to construct the fixed point equations on the estimations of the state average terms for the major and minor players. Then we design the decentralized strategies. In Section 4, we prove that the estimations of the state average terms are consistent with the state average terms for the closed-loop systems. In Section 5, by the consistency of the estimations, we prove that the sequence of strategies designed is a decentralized asymptotic Nash equilibrium. In Section 6, a numerical simulation example is given to demonstrate the theoretical results. Finally, some conclusions and the future research direction are given.

The following notations will be used throughout the paper. We use the subscript L and subscript F as the label of major and minor players. For a vector or matrix X, $X^\top$ denotes the transposition of X, $\|X\|$ denotes the 2-norm of vector or Frobenius norm of matrix, and $\mathrm{Tr}(X)$ denotes the trace of X; I_n denotes the $n\times n$-dimensional identity matrix, $\mathbb{R}^n$ denotes the set of n-dimensional real column vectors, and $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$-dimensional real matrices; $C^{1,2}([0,T]\times\mathbb{R}^n,\mathbb{R})=\big\{V\big|V(t,x):[0,T]\times\mathbb{R}^n\to\mathbb{R},$ and $\frac{\partial V(t,x)}{\partial t}, \frac{\partial V(t,x)}{\partial x}$ and $\frac{\partial^2 V(t,x)}{\partial x_i x_j}$ are continuous with respect to $(t,x)\in[0,T]\times\mathbb{R}^n, i,j=1,\ldots,n\big\}$; $C([0,T],\mathbb{R}^n)=\big\{f\big|f(t):[0,T]\to\mathbb{R}^n,$ and f is continuous with respect to $t\in[0,T]\big\}$; for a vector-valued function $f\in C([0,T],\mathbb{R}^n)$, $\|f\|_\infty\triangleq\sup_{0\le t\le T}\|f(t)\|$;

for vectors $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and matrix $Q \in \mathbb{R}^{n\times n}$, $\|x\|_Q^2 \triangleq x^\top Qx$, $Q > 0$ ($Q \geq 0$) means that Q is positive definite (positive semi-definite), and $x \leq y$ means that each component of x is not bigger than the corresponding component of y; $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ denotes a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$; for a given random variable X, the mathematical expectation of X is denoted by $\mathbb{E}[X]$, and the variance of X is denoted by $\mathrm{var}[X]$.

2. Problem formulation

We consider the following mean field games with N_L major players and N_F minor players. The dynamics of the major and minor players are given by

$$\mathrm{d}x_{L_i}(t) = [A_L(\theta_{L_i})x_{L_i}(t) + B_L(\theta_{L_i})u_{L_i}(t)]\,\mathrm{d}t + D_L(\theta_{L_i})\mathrm{d}W_{L_i}(t), \quad t \geq 0, i = 1, 2, \dots, N_L, \tag{2.1}$$

and

$$\mathrm{d}x_{F_j}(t) = \left[A_F(\theta_{F_j})x_{F_j}(t) + B_F(\theta_{F_j})u_{F_j}(t)\right]\mathrm{d}t + D_F(\theta_{F_j})\mathrm{d}W_{F_j}(t), \quad t \geq 0, j = 1, 2, \dots, N_F, \tag{2.2}$$

respectively, where $x_{L_i}(t) \in \mathbb{R}^n$ and $x_{F_j}(t) \in \mathbb{R}^n$ represent the states of the major player i and minor player j respectively, the initial states $x_{L_i}(0) \in X_L \subseteq \mathbb{R}^n$ and $x_{F_j}(0) \in X_F \subseteq \mathbb{R}^n$, $u_{L_i}(t) \in \mathbb{R}^m$ and $u_{F_j}(t) \in \mathbb{R}^m$ represent the control inputs of the major player i and minor player j respectively, $\theta_{L_i} = (\theta^1_{L_i}, \theta^2_{L_i}, \dots, \theta^{l_1}_{L_i}) \in \Theta_L \subseteq \mathbb{R}^{l_1}$ and $\theta_{F_j} = (\theta^1_{F_j}, \theta^2_{F_j}, \dots, \theta^{l_2}_{F_j}) \in \Theta_F \subseteq \mathbb{R}^{l_2}$ are parameters, $A_L(\cdot)$, $A_F(\cdot)$, $B_L(\cdot)$, $B_F(\cdot)$, $D_L(\cdot)$ and $D_F(\cdot)$ are matrix-valued functions with compatible dimensions, and $\{W_{L_i}(t), W_{F_j}(t), 0 \leq t \leq T, i = 1, 2, \dots, N_L, j = 1, 2, \dots, N_F\}$ are p-dimensional independent standard Brownian motions defined on the complete probability space $(\Omega, \mathcal{F}, P)$ and adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.

The objective of each player is to minimize its cost functional by designing an appropriate strategy. The risk-sensitive cost functionals of the major and minor players are given by

$$J_{L_i}\left(u_{L_i}(\cdot), u_{L_{-i}}(\cdot)\right) = \gamma \ln \mathbb{E}\left[\exp\left(\frac{1}{\gamma}\Phi_L\left(x_{L_i}(\cdot), x_L^{(N_L)}(\cdot), u_{L_i}(\cdot)\right)\right)\right], \quad i = 1, 2, \dots, N_L, \tag{2.3}$$

and

$$J_{F_j}\left(u_{F_j}(\cdot), u_{F_{-j}}(\cdot), u_L(\cdot)\right) = \gamma \ln \mathbb{E}\left[\exp\left(\frac{1}{\gamma}\Phi_F\left(x_{F_j}(\cdot), x_F^{(N_F)}(\cdot), x_L^{(N_L)}(\cdot), u_{F_j}(\cdot)\right)\right)\right], \quad j = 1, 2, \dots, N_F, \tag{2.4}$$

respectively, where $x_L^{(N_L)}(\cdot) = \frac{1}{N_L}\sum_{i=1}^{N_L} x_{L_i}(\cdot)$, $x_F^{(N_F)}(\cdot) = \frac{1}{N_F}\sum_{j=1}^{N_F} x_{F_j}(\cdot)$, $u_L = \left(u_{L_1}(\cdot), u_{L_2}(\cdot), \dots, u_{L_{N_L}}(\cdot)\right)$, $u_{L_{-i}}(\cdot) = \left(u_{L_1}(\cdot), \cdots, u_{L_{i-1}}(\cdot), u_{L_{i+1}}(\cdot), \cdots, u_{L_{N_L}}(\cdot)\right)$, $u_{F_{-j}}(\cdot) = \left(u_{F_1}(\cdot), \cdots, u_{F_{j-1}}(\cdot), u_{F_{j+1}}(\cdot), \dots, u_{F_{N_F}}(\cdot)\right)$,

$$\Phi_L\left(x_{L_i}(\cdot), x_L^{(N_L)}(\cdot), u_{L_i}(\cdot)\right) = \int_0^T \left(\left\|x_{L_i}(t) - H_L x_L^{(N_L)}(t) - g_L\right\|^2_{Q_L} + \|u_{L_i}(t)\|^2_{R_L}\right)\mathrm{d}t,$$

$$\Phi_F\left(x_{F_j}(\cdot), x_F^{(N_F)}(\cdot), x_L^{(N_L)}(\cdot), u_{F_j}(\cdot)\right) = \int_0^T \left(\left\|x_{F_j}(t) - H_F x_F^{(N_F)}(t) - H x_L^{(N_L)}(t) - g_F\right\|^2_{Q_F} + \left\|u_{F_j}(t)\right\|^2_{R_F}\right)\mathrm{d}t,$$

H_L, H_F, H, Q_L, Q_F, R_L, R_F, g_L, g_F are matrices or vectors with compatible dimensions, $Q_L \geq 0$, $Q_F \geq 0$, $R_L > 0$, $R_F > 0$, and γ is the risk-sensitive parameter. The cases with $\gamma > 0$, $\gamma < 0$ and $\gamma \to \infty$ correspond respectively to risk-averse, risk-seeking and risk-neutral attitudes of the players.

From (2.3) and (2.4), we can see that the state average term of the major players has influence on the major and minor players, and the state average term of the minor players only has influence on the minor players.

Remark 2.1. Let $\lambda = \frac{1}{\gamma}$. By using the Taylor expansion to (2.3) and (2.4) around $\lambda = 0$, one arrives at the following relation

$$
\begin{aligned}
J_{L_i}\left(u_{L_i}(\cdot), u_{L_{-i}}(\cdot)\right) &= \mathbb{E}\left[\Phi_L\left(x_{L_i}(\cdot), x_L^{(N_L)}(\cdot), u_{L_i}(\cdot)\right)\right] + \frac{\lambda}{2}\text{var}\left[\Phi_L\left(x_{L_i}(\cdot), x_L^{(N_L)}(\cdot), u_{L_i}(\cdot)\right)\right] + o(\lambda),\\
J_{F_j}\left(u_{F_j}(\cdot), u_{F_{-j}}(\cdot), u_L(\cdot)\right) &= \mathbb{E}\left[\Phi_F\left(x_{F_j}(\cdot), x_F^{(N_F)}(\cdot), x_L^{(N_L)}(\cdot), u_{F_j}(\cdot)\right)\right]\\
&\quad + \frac{\lambda}{2}\text{var}\left[\Phi_F\left(x_{F_j}(\cdot), x_F^{(N_F)}(\cdot), x_L^{(N_L)}(\cdot), u_{F_j}(\cdot)\right)\right] + o(\lambda).
\end{aligned}
$$

Note that the risk-averse cost functionals degenerate to the risk neutral cases if $\lambda \to 0$ ($\gamma \to \infty$).

For the systems (2.1)–(2.4), we introduce the following assumptions.

Assumption 2.2. The sets Θ_L and Θ_F are both bounded closed sets in $\mathbb{R}^{l_1}$ and $\mathbb{R}^{l_2}$. The sets X_L and X_F are both bounded closed sets in $\mathbb{R}^n$.

Assumption 2.3. There exist distribution functions $\mathbf{F}_L(\theta_L, x_L)$ and $\mathbf{F}_F(\theta_F, x_F)$ such that the sequences of empirical distribution functions $\{\mathbf{F}_{N_L}(\theta_L, x_L) = \frac{1}{N_L}\sum_{i=1}^{N_L} 1_{\{\theta_{L_i} \le \theta_L, x_{L_i}(0) \le x_L\}}, N_L = 1, 2, \cdots\}$ and $\{\mathbf{F}_{N_F}(\theta_F, x_F) = \frac{1}{N_F}\sum_{j=1}^{N_F} 1_{\{\theta_{F_j} \le \theta_F, x_{F_j}(0) \le x_F\}}, N_F = 1, 2, \cdots\}$ weakly converge to $\mathbf{F}_L(\theta_L, x_L)$ and $\mathbf{F}_F(\theta_F, x_F)$, respectively.

Assumption 2.4. The matrix-valued functions $A_L(\theta_L)$, $B_L(\theta_L)$ and $D_L(\theta_L)$ are continuous with respect to $\theta_L \in \Theta_L$. The matrix-valued functions $A_F(\theta_F)$, $B_F(\theta_F)$ and $D_F(\theta_F)$ are continuous with respect to $\theta_F \in \Theta_F$. For any given $\theta_L \in \Theta_L$, $B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L) - \frac{2}{\gamma}D_L(\theta_L)D_L^\top(\theta_L) \ge 0$. For any given $\theta_F \in \Theta_F$, $B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F) - \frac{2}{\gamma}D_F(\theta_F)D_F^\top(\theta_F) \ge 0$.

Under Assumption 2.4, for any given $\theta_L \in \Theta_L$, there exists a unique solution to the generalized matrix-valued Riccati differential equation

$$
\begin{cases}
\dot{P}_{\gamma,\theta_L}(t) = -P_{\gamma,\theta_L}(t)A_L(\theta_L) - A_L^\top(\theta_L)P_{\gamma,\theta_L}(t) + P_{\gamma,\theta_L}(t)\Big[B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\\
\qquad\qquad -\frac{2}{\gamma}D_L(\theta_L)D_L^\top(\theta_L)\Big]P_{\gamma,\theta_L}(t) - Q_L,\\
P_{\gamma,\theta_L}(T) = 0,
\end{cases}
\tag{2.5}
$$

and for any given $\theta_F \in \Theta_F$, there exists a unique solution to the generalized matrix-valued Riccati differential equation

$$
\begin{cases}
\dot{P}_{\gamma,\theta_F}(t) = -P_{\gamma,\theta_F}(t)A_F(\theta_F) - A_F^\top(\theta_F)P_{\gamma,\theta_F}(t)\\
\qquad\qquad +P_{\gamma,\theta_F}(t)\left[B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F) - \frac{2}{\gamma}D_F(\theta_F)D_F^\top(\theta_F)\right]P_{\gamma,\theta_F}(t) - Q_F,\\
P_{\gamma,\theta_F}(T) = 0.
\end{cases}
\tag{2.6}
$$

In this paper we require that the risk-sensitive parameter γ in (2.3) and (2.4) is large enough.

Assumption 2.5. For any given $T \ge 0$, $\gamma > \overline{\gamma}_T$, where $\overline{\gamma}_T$ is defined in Lemma A.2.

Remark 2.6. For the systems (2.1)–(2.4) with $\gamma > 0$, Assumption 2.5 guarantees the boundedness of the terms $\sup_{0\le t\le T}\max_{i=1,2,\ldots,N_L}\mathbb{E}\left[\exp\left(\frac{\max\{T^3,1\}\beta^2}{\gamma}\|W_{L_i}(t)\|^2\right)\right]$ and $\sup_{0\le t\le T}\max_{j=1,2,\ldots,N_F}\mathbb{E}\left[\exp\left(\frac{\max\{T^3,1\}\beta^2}{\gamma}\left\|W_{F_j}(t)\right\|^2\right)\right]$ (the expectation of exponential functions related to Brownian motions), where β is a constant related to $A_L(\theta_{L_i})$, $B_L(\theta_{L_i})$, $D_L(\theta_{L_i})$, Q_L, R_L, H_L, T, $A_F(\theta_{F_j})$, $B_F(\theta_{F_j})$, $D_F(\theta_{F_j})$, Q_F, R_F, H and H_F. In fact, if γ is too small, then the above terms are unbounded. For the case $\gamma < 0$, it is obvious that the above terms are bounded.

We define the decentralized admissible control sets of the major and minor players in terms of local information for the systems (2.1)–(2.4) as

$$\begin{aligned}\mathcal{U}^l_{L_i}=\bigg\{u_{L_i}(\cdot)\bigg|&u_{L_i}(t)\text{ is adapted to }\sigma(x_{L_i}(s),s\le t),t\in[0,T],\\&\text{the system (2.1) has a unique strong solution }x_{L_i}(\cdot),\\&\mathbb{E}\bigg[\int_0^T\bigg(\exp\bigg(\frac{\alpha M_{\gamma,L}(\theta_{L_i})}{\gamma}\|x_{L_i}(t)\|^2\bigg)\bigg)\|x_{L_i}(t)\|^2\mathrm{d}t\bigg]<\infty,\ \exists\ \alpha>2,\ \text{and}\\&\mathbb{E}\bigg[\exp\bigg(\frac{32}{\gamma}\int_0^T\bigg(\|x_{L_i}(t)\|^2_{Q_L}+\bigg\|\frac{1}{N_L}H_Lx_{L_i}(t)\bigg\|^2_{Q_L}+\|u_{L_i}(t)\|^2_{R_L}\bigg)\mathrm{d}t\bigg)\bigg]<\infty\bigg\},\quad i=1,2,\dots,N_L,\end{aligned}$$

and

$$\begin{aligned}\mathcal{U}^l_{F_j}=\bigg\{u_{F_j}(\cdot)\bigg|&u_{F_j}(t)\text{ is adapted to }\sigma(x_{F_j}(s),s\le t),t\in[0,T],\\&\text{the system (2.2) has a unique strong solution }x_{F_j}(\cdot),\\&\mathbb{E}\bigg[\int_0^T\bigg(\exp\bigg(\frac{\alpha M_{\gamma,F}(\theta_{F_j})}{\gamma}\|x_{F_j}(t)\|^2\bigg)\bigg)\|x_{F_j}(t)\|^2\mathrm{d}t\bigg]<\infty,\ \exists\ \alpha>2,\ \text{and}\\&\mathbb{E}\bigg[\exp\bigg(\frac{40}{\gamma}\int_0^T\bigg(\|x_{F_j}(t)\|^2_{Q_F}+\bigg\|\frac{1}{N_F}H_Fx_{F_j}(t)\bigg\|^2_{Q_F}+\|u_{F_j}(t)\|^2_{R_F}\bigg)\mathrm{d}t\bigg)\bigg]<\infty\bigg\},\quad j=1,2,\dots,N_F,\end{aligned}$$

where $M_{\gamma,L}(\theta_{L_i})=M_{\gamma,L}(\theta_L)|_{\theta_L=\theta_{L_i}}$, $M_{\gamma,L}(\theta_L)=\sup_{t\in[0,T]}\|P_{\gamma,\theta_L}(t)\|$, $M_{\gamma,F}(\theta_{F_j})=M_{\gamma,F}(\theta_F)|_{\theta_F=\theta_{F_j}}$ and $M_{\gamma,F}(\theta_F)=\sup_{t\in[0,T]}\|P_{\gamma,\theta_F}(t)\|$. For comparison, we define

$$\begin{aligned}\mathcal{U}^g_{L_i}=\bigg\{u_{L_i}(\cdot)\bigg|&u_{L_i}(t)\text{ is adapted to }\sigma(x_{L_{i'}}(s),i'=1,2,\dots,N_L,s\le t),t\in[0,T],\\&\text{the system (2.1) has a unique strong solution }x_{L_i}(\cdot),\\&\mathbb{E}\bigg[\int_0^T\bigg(\exp\bigg(\frac{\alpha M_{\gamma,L}(\theta_{L_i})}{\gamma}\|x_{L_i}(t)\|^2\bigg)\bigg)\|x_{L_i}(t)\|^2\mathrm{d}t\bigg]<\infty,\ \exists\ \alpha>2,\ \text{and}\\&\mathbb{E}\bigg[\exp\bigg(\frac{32}{\gamma}\int_0^T\bigg(\|x_{L_i}(t)\|^2_{Q_L}+\bigg\|\frac{1}{N_L}H_Lx_{L_i}(t)\bigg\|^2_{Q_L}+\|u_{L_i}(t)\|^2_{R_L}\bigg)\mathrm{d}t\bigg)\bigg]<\infty\bigg\},\quad i=1,2,\dots,N_L,\end{aligned}$$

and

$$\begin{aligned}\mathcal{U}^g_{F_j}=\bigg\{u_{F_j}(\cdot)\bigg|&u_{F_j}(t)\text{ is adapted to }\sigma(x_{L_{i'}}(s),x_{F_{j'}}(s),i'=1,2,\dots,N_L,j'=1,2,\dots,N_F,s\le t),t\in[0,T],\\&\text{the system (2.2) has a unique strong solution }x_{F_j}(\cdot),\\&\mathbb{E}\bigg[\int_0^T\bigg(\exp\bigg(\frac{\alpha M_{\gamma,F}(\theta_{F_j})}{\gamma}\|x_{F_j}(t)\|^2\bigg)\bigg)\|x_{F_j}(t)\|^2\mathrm{d}t\bigg]<\infty,\ \exists\ \alpha>2,\ \text{and}\\&\mathbb{E}\bigg[\exp\bigg(\frac{40}{\gamma}\int_0^T\bigg(\|x_{F_j}(t)\|^2_{Q_F}+\bigg\|\frac{1}{N_F}H_Fx_{F_j}(t)\bigg\|^2_{Q_F}+\|u_{F_j}(t)\|^2_{R_F}\bigg)\mathrm{d}t\bigg)\bigg]<\infty\bigg\},\quad j=1,2,\dots,N_F.\end{aligned}$$

The objectives of the major and minor players are to minimize their cost functionals by designing appropriate strategies over the admissible control sets $\mathcal{U}^l_{L_i}$ and $\mathcal{U}^l_{F_j}$.

Definition 2.7. A sequence of strategies $\{\{v^*_{L_i}(\cdot) \in \mathcal{U}^l_{L_i}, v^*_{F_j}(\cdot) \in \mathcal{U}^l_{F_j}, i = 1, 2, \ldots, N_L, j = 1, 2, \ldots, N_F\}, N_L = 1, 2, \ldots, N_F = 1, 2, \cdots\}$ for the systems (2.1)-(2.4) is called a decentralized asymptotic Nash equilibrium with respect to the sequence of the cost functionals $\{\{J_{L_i}, J_{F_j}, i = 1, 2, \ldots, N_L, j = 1, 2, \ldots, N_F\}, N_L = 1, 2, \ldots, N_F = 1, 2, \cdots\}$, if there exist two nonnegative sequences $\{\eta_{N_L}, N_L = 1, 2, \cdots\}$ and $\{\eta_{N_L,N_F}, N_L = 1, 2, \ldots, N_F = 1, 2, \cdots\}$ satisfying $\lim_{N_L\to\infty} \eta_{N_L} = 0$ and $\lim_{N_L\to\infty, N_F\to\infty} \eta_{N_L,N_F} = 0$ and such that

$$J_{L_i}\big(v^*_{L_i}(\cdot), v^*_{L_{-i}}(\cdot)\big) \le \inf_{v_{L_i}(\cdot)\in\mathcal{U}^g_{L_i}} J_{L_i}\big(v_{L_i}(\cdot), v^*_{L_{-i}}(\cdot)\big) + \eta_{N_L}, \qquad i = 1, 2, \ldots, N_L,$$

$$J_{F_j}\big(v^*_{F_j}(\cdot), v^*_{F_{-j}}(\cdot), v^*_L(\cdot)\big) \le \inf_{v_{F_j}(\cdot)\in\mathcal{U}^g_{F_j}} J_{F_j}\big(v_{F_j}(\cdot), v^*_{F_{-j}}(\cdot), v^*_L(\cdot)\big) + \eta_{N_L,N_F}, \qquad j = 1, 2, \ldots, N_F.$$

3. Design of strategies

For the systems (2.1)–(2.4), we need to consider optimal control problems with the reference trajectories $(H_L x^{(N_L)}_L + g_L)$ and $(H_F x^{(N_F)}_F + H x^{(N_L)}_L + g_F)$ for the major and minor players, respectively. However, since $u_{L_i}(\cdot) \in \mathcal{U}^l_{L_i}$ and $u_{F_j}(\cdot) \in \mathcal{U}^l_{F_j}$, the signals $x^{(N_L)}_L$ and $x^{(N_F)}_F$ are unavailable. Therefore, we solve the problem by the Nash certainty equivalence principle [18, 21]. The designing process of the decentralized strategies can be divided into the following three steps.

(i) To solve the quadratic-tracking type risk-sensitive optimal control problem with a deterministic tracking signal.
(ii) To construct the estimation z^*_L of the state average term for the major players, and to design the decentralized strategies for the major players by z^*_L.
(iii) To construct the estimation z^*_F of the state average term for the minor players based on z^*_L, and to design the decentralized strategies for the minor players by z^*_L and z^*_F.

3.1. Optimal control with quadratic-tracking type risk-sensitive cost functional

For any $\tau \in [0, T)$ and $x \in \mathbb{R}^n$, consider the following system

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\mathrm{d}t + D\mathrm{d}W(t), \tag{3.1}$$

with $x(\tau) = x$, where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ represent the state and control input respectively, $W(t)$ is a p-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ and adapted to $\{\mathcal{F}_t\}_{t\ge0}$, A, B and D are matrices with compatible dimensions. The objective is to minimize the cost functional given by

$$J(\tau, x; u(\cdot)) = \gamma \ln L(\tau, x; u(\cdot)), \tag{3.2}$$

where

$$L(\tau, x; u(\cdot)) = \mathbb{E}\Big\{\Big[\exp\Big(\frac{1}{\gamma}\int_\tau^T \Big(\|x(t) - y(t)\|^2_Q + \|u(t)\|^2_R\Big)\mathrm{d}t\Big)\Big]\Big| x(\tau) = x\Big\}, \tag{3.3}$$

$y(t) \in \mathbb{R}^n$ is a deterministic function, Q and R are matrices with compatible dimensions, $Q \ge 0$ and $R > 0$. Moreover, the systems (3.1)–(3.3) satisfy that $BR^{-1}B^\top - \frac{2}{\gamma}DD^\top \ge 0$.

The admissible control set of the systems (3.1)–(3.3) is given by

$$\mathcal{U}_\tau = \Big\{u(\cdot)\Big| u(t) \text{ is adapted to } \sigma(x(s), \tau \le s \le t), t \in [\tau, T], \text{ the system (3.1) has a unique strong solution } x(\cdot),$$

$$\mathbb{E}\bigg[\int_\tau^T\bigg(\exp\bigg(\frac{\alpha M_{\gamma,1}}{\gamma}\|x(t)\|^2\bigg)\bigg)\|x(t)\|^2\mathrm{d}t\bigg]<\infty,\ \exists\ \alpha>2,\ \text{and}$$
$$\mathbb{E}\bigg[\exp\bigg(\frac{4}{\gamma}\int_\tau^T\Big(\big\|x(t)\big\|_Q^2+\|u(t)\|_R^2\Big)\mathrm{d}t\bigg)\bigg]<\infty\bigg\}, \tag{3.4}$$

where $M_{\gamma,1}=\sup_{t\in[0,T]}\|P_\gamma(t)\|$ and $P_\gamma(t)$ is the unique solution of the generalized matrix-valued Riccati differential equation

$$\begin{cases} \dot{P}_\gamma(t)=-P_\gamma(t)A-A^\top P_\gamma(t)+P_\gamma(t)\left[BR^{-1}B^\top-\frac{2}{\gamma}DD^\top\right]P_\gamma(t)-Q,\\ P_\gamma(T)=0. \end{cases} \tag{3.5}$$

It is known that for a family of risk-neutral optimal control problems with different initial times and states, the relationship among them is always established *via* a HJB equation. If the HJB equation is solvable, then one can obtain an optimal feedback strategy by minimising the Hamiltonian involved in the HJB equation [46]. Similarly, for the family of risk-sensitive optimal control problems above, we have the following result.

Theorem 3.1. (Verification theorem) *Suppose that $G\in C^{1,2}([0,T]\times\mathbb{R}^n,\mathbb{R})$ is a solution of the HJB equation*

$$\begin{cases} -\frac{\partial V(\tau,x)}{\partial\tau}=\min\limits_{v\in\mathbb{R}^m}\Big\{\frac{\partial V(\tau,x)}{\partial x}(Ax+Bv)+\frac{1}{\gamma}\left[\|x-y(\tau)\|_Q^2+\|v\|_R^2\right]V(\tau,x)\Big\}\\ \qquad\qquad\qquad +\frac{1}{2}\operatorname{Tr}\left(\frac{\partial^2V(\tau,x)}{\partial x^2}DD^\top\right), & (\tau,x)\in[0,T)\times\mathbb{R}^n,\\ V(T,x)=1, & x\in\mathbb{R}^n, \end{cases} \tag{3.6}$$

such that

$$\left\|\frac{\partial G(\tau,x)}{\partial x}\right\|\leq C_0\bigg[\exp\bigg(\frac{1}{\gamma}\Big(M_{\gamma,1}\|x\|^2+M_{\gamma,2}\|x\|\Big)\bigg)\bigg]\Big(\|x\|+1\Big), \tag{3.7}$$

for some constant $C_0>0$, where A,B,D,Q,R and T are given in the systems (3.1) and (3.3), $M_{\gamma,2}=\sup_{t\in[0,T]}\|\xi_\gamma(t)\|$, $\xi_\gamma(t)$ is the unique solution of the linear differential equation

$$\begin{cases} \dot{\xi}_\gamma(t)=-\left[A^\top-P_\gamma(t)BR^{-1}B^\top+\frac{2}{\gamma}P_\gamma(t)DD^\top\right]\xi_\gamma(t)-Qy(t),\\ \xi_\gamma(T)=0. \end{cases} \tag{3.8}$$

Then

(a) $G(\tau,x)\leq\inf_{\bar{v}(\cdot)\in\mathcal{U}_\tau}L(\tau,x;\bar{v}(\cdot))$ for any $(\tau,x)\in[0,T]\times\mathbb{R}^n$, where $L(\tau,x;\bar{v}(\cdot))$ is defined by (3.3).

(b) If there exists $v^*(\cdot)\in\mathcal{U}_\tau$ such that

$$v^*(t)\in\arg\min_{v\in\mathbb{R}^m}\left\{\frac{\partial G(\tau,x)}{\partial x}\bigg|_{\tau=t,x=x^*(t)}(Ax^*(t)+Bv)+\frac{1}{\gamma}\left[\|x^*(t)-y(t)\|_Q^2+\|v\|_R^2\right]G(t,x^*(t))\right\} \tag{3.9}$$

for Lebesgue×$\mathbb{P}$-almost all $(t,\omega)\in[\tau,T]\times\Omega$, where $\mathcal{U}_\tau$ is defined by (3.4) and $x^*(\cdot)$ is the strong solution of the system (3.1) under the control $v^*(\cdot)$, then $G(\tau,x)=L(\tau,x;v^*(\cdot))=\inf_{\bar{v}(\cdot)\in\mathcal{U}_\tau}L(\tau,x;\bar{v}(\cdot))$ for any $(\tau,x)\in[0,T]\times\mathbb{R}^n$.

Proof. See Appendix B. □

Remark 3.2. For a risk-sensitive optimal control problem, Fleming and Soner proposed a verification theorem (Chap. VI, Thm. 8.2 in [12]), under the conditions that the admissible control laws are Markov strategies with linear growth and local Lipchitz conditions, and the gradient vector of the solution of the HJB equation with respect to x satisfies linear growth condition, they obtained the optimal feedback strategy. Although the systems (3.1)–(3.3) considered in Theorem 3.1 is a special case of [12], we do not assume that the admissible control laws in (3.4) are Markov strategies, besides, the linear growth condition is weakened to (3.7).

For the systems (3.1)–(3.3) and the HJB equation (3.6), we have the following result.

Theorem 3.3. *For the HJB equation (3.6), if $BR^{-1}B^\top - \frac{2}{\gamma}DD^\top \geq 0$, then in the class of $C^{1,2}([0,T]\times\mathbb{R}^n,\mathbb{R})$, there exists a classical solution given by*

$$G(\tau,x) = \exp\left[\frac{1}{\gamma}\left(x^\top P_\gamma(\tau)x - 2x^\top \xi_\gamma(\tau) + \varphi_\gamma(\tau)\right)\right], \quad (\tau,x)\in[0,T]\times\mathbb{R}^n, \tag{3.10}$$

satisfying (3.7), where $P_\gamma(\tau)$ and $\xi_\gamma(\tau)$ are given by (3.5) and (3.8) respectively, and $\varphi_\gamma(t)$ is the unique solution of the following differential equation

$$\begin{cases} \dot{\varphi}_\gamma(t) = \xi^\top(t)BR^{-1}B^\top\xi_\gamma^\top(t) - y^\top(t)Qy(t) - \frac{2}{\gamma}\xi_\gamma^\top(t)DD^\top\xi_\gamma(t) - \mathrm{Tr}\left(P_\gamma(t)DD^\top\right), \\ \varphi_\gamma(T) = 0. \end{cases} \tag{3.11}$$

Moreover, for the system (3.1) and the cost functional (3.2), if $BR^{-1}B^\top - \frac{2}{\gamma}DD^\top \geq 0$ and $\gamma > \gamma_T$, where γ_T is defined in Lemma B.1, then the optimal strategy is given by

$$u^*(t) = -R^{-1}B^\top P_\gamma(t)x(t) + R^{-1}B^\top\xi_\gamma(t), \quad t\in[\tau,T],$$

and the optimal cost is given by

$$J(\tau,x;u^*(\cdot)) = \gamma\ln G(\tau,x) = x^\top P_\gamma(\tau)x - 2x^\top\xi_\gamma(\tau) + \varphi_\gamma(\tau).$$

Proof. See Appendix B. □

Remark 3.4. In Theorem 3.3, for the HJB equation (3.6), under the condition that $BR^{-1}B^\top - \frac{2}{\gamma}DD^\top \geq 0$, we prove the existence of the classical solution in $C^{1,2}([0,T]\times\mathbb{R}^n,\mathbb{R})$ satisfying (3.7) by using a constructive approach. Moreover, under the conditions that $BR^{-1}B^\top - \frac{2}{\gamma}DD^\top \geq 0$ and $\gamma > \overline{\gamma}_T$, if there exist two classical solutions $G(\tau,x)\in C^{1,2}([0,T]\times\mathbb{R}^n,\mathbb{R})$ and $G'(\tau,x)\in C^{1,2}([0,T]\times\mathbb{R}^n,\mathbb{R})$ both satisfying (3.7) to the HJB equation (3.6), then by Theorem 3.1, we have $G(\tau,x) = G'(\tau,x) = \inf_{\bar{v}(\cdot)\in\mathcal{U}_\tau} L(\tau,x;\bar{v}(\cdot))$ for any $(\tau,x)\in[0,T]\times\mathbb{R}^n$, *i.e.* the classical solution to the HJB equation (3.6) is unique, and is given by (3.10).

Remark 3.5. For the quadratic-tracking type risk-sensitive optimal control problem, under the condition that the system has an optimal strategy, Moon and Başar (Thm. A1 and Ex. 5 in [35]) obtained the optimal condition and proved the existence of the solution for the adjoint equation through the stochastic maximum principle, then they designed the optimal strategy accordingly. Although the problem considered in Theorem 3.3 is a special case of that considered in Theorem A1 of [35], different from [35], we design the optimal strategy for the system (3.1) and the cost functional (3.2) by dynamic programming, and we do not assume the existence of the optimal strategy in advance.

3.2. Design of strategies for major players

For the cost functionals (2.3), if $x_L^{(N_L)}(t)$ is replaced by some deterministic vector-valued function $z_L(t)$, then the game of the major players in the models (2.1)–(2.4) is transformed into the following optimal tracking control problem.

Auxiliary problem (I). Minimize $\bar{J}_{L_i,z_L}(u_{L_i}(\cdot))$ over $\mathcal{U}^l_{L_i}$, where

$$\bar{J}_{L_i,z_L}(u_{L_i}(\cdot)) = \gamma \ln \mathbb{E}\left[\exp\left(\frac{1}{\gamma}\Phi_L\left(x_{L_i}(\cdot), z_L(\cdot), u_{L_i}(\cdot)\right)\right)\right], \quad i = 1, 2, \ldots, N_L, \tag{3.12}$$

$x_{L_i}(t)$ is given by (2.1) and $z_L(t) \in \mathbb{R}^n$ is a deterministic function.

Theorem 3.6. *For Auxiliary problem (I), if Assumptions 2.4 and 2.5 hold, then the optimal strategies are given by*

$$\bar{u}_{L_i}(t) = -R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)x_{L_i}(t) + R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L}(t), \quad t \in [0,T], \quad i = 1, 2, \ldots, N_L, \tag{3.13}$$

where $P_{\gamma,\theta_{L_i}}(t) = P_{\gamma,\theta_L}(t)|_{\theta_L=\theta_{L_i}}$, $\xi_{\gamma,\theta_{L_i},z_L}(t) = \xi_{\gamma,\theta_L,z_L}(t)|_{\theta_L=\theta_{L_i}}$, $P_{\gamma,\theta_L}(t)$ *is given by (2.5), and* $\xi_{\gamma,\theta_L,z_L}(t)$ *is the unique solution of the linear differential equation*

$$\begin{cases} \dot{\xi}_{\gamma,\theta_L,z_L}(t) = -\left[A_L(\theta_L) - B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)P_{\gamma,\theta_L}(t) + \frac{2}{\gamma}D_L(\theta_L)D_L^\top(\theta_L)P_{\gamma,\theta_L}(t)\right]^\top \xi_{\gamma,\theta_L,z_L}(t) \\ \qquad\qquad\qquad - \left[Q_L H_L z_L(t) + Q_L g_L\right], \\ \xi_{\gamma,\theta_L,z_L}(T) = 0. \end{cases} \tag{3.14}$$

Proof. See Appendix B. □

Substituting the strategies (3.13) into (2.1), we get the closed-loop system for the major player i as

$$\begin{aligned} \mathrm{d}\bar{x}_{L_i}(t) = &\left[A_L(\theta_{L_i}) - B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\right]\bar{x}_{L_i}(t)\mathrm{d}t + B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L}(t)\mathrm{d}t \\ &+ D_L(\theta_{L_i})\mathrm{d}W_{L_i}(t), \end{aligned} \tag{3.15}$$

which leads to

$$\begin{aligned} \bar{x}_{L_i}(t) = &x_{L_i}(0) + \int_0^t \left[A_L(\theta_{L_i}) - B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(s)\right]\bar{x}_{L_i}(s)\mathrm{d}s \\ &+ \int_0^t B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L}(s)\mathrm{d}s + \int_0^t D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s). \end{aligned}$$

Taking expectation and then taking derivative with respect to t on both sides of the above equation, we have

$$\frac{\mathrm{d}\mathbb{E}\bar{x}_{L_i}(t)}{\mathrm{d}t} = \left[A_L(\theta_{L_i}) - B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\right]\mathbb{E}\bar{x}_{L_i}(t) + B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L}(t).$$

Then, we construct an estimation of the state average term for the major players by the state aggregation method. Firstly, we construct an auxiliary system

$$\begin{cases} \frac{\mathrm{d}\mathbb{E}\bar{x}_{\theta_L}(t)}{\mathrm{d}t} = \left[A_L(\theta_L) - B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)P_{\gamma,\theta_L}(t)\right]\mathbb{E}\bar{x}_{\theta_L}(t) + B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\xi_{\gamma,\theta_L,z_L}(t), \\ \mathbb{E}\bar{x}_{\theta_L}(0) = x_L, \\ z_L(t) = \int_{\Theta_L \times X_L} \mathbb{E}\bar{x}_{\theta_L}(t)\mathrm{d}\mathbf{F}_L(\theta_L, x_L). \end{cases} \tag{3.16}$$

The above system characterizes the limit case of the system (2.1) with the strategies (3.13) as $N_L \to \infty$, which is a continuum of major players, and each player is marked by θ_L and x_L to indicate the distribution of the parameters. By (3.16), we have

$$\mathbb{E}\bar{x}_{\theta_L}(t) = \int_0^t \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\int_{s_2}^T \psi_{\gamma,\theta_L}(s_2,s_1)\left[Q_LH_Lz_L(s_1)+Q_Lg_L\right]\mathrm{d}s_1\mathrm{d}s_2 + \phi_{\gamma,\theta_L}(t,0)x_L, \tag{3.17}$$

where $\phi_{\gamma,\theta_L}(t,s)$, $0\le t,s\le T$, is the solution of

$$\begin{cases} \mathrm{d}\phi_{\gamma,\theta_L}(t,s) = \left[A_L(\theta_L) - B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)P_{\gamma,\theta_L}(t)\right]\phi_{\gamma,\theta_L}(t,s)\mathrm{d}t, \\ \phi_{\gamma,\theta_L}(s,s) = I_n, \end{cases} \tag{3.18}$$

and $\psi_{\gamma,\theta_L}(t,s)$, $0\le t,s\le T$, is the solution of

$$\begin{cases} \mathrm{d}\psi_{\gamma,\theta_L}(t,s) = -\left[A_L(\theta_L) - B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)P_{\gamma,\theta_L}(t) + \frac{2}{\gamma}D_L(\theta_L)D_L^\top(\theta_L)P_{\gamma,\theta_L}(t)\right]^\top \psi_{\gamma,\theta_L}(t,s)\mathrm{d}t, \\ \psi_{\gamma,\theta_L}(s,s) = I_n. \end{cases} \tag{3.19}$$

For any given $y\in C([0,T],\mathbb{R}^n)$, define the operator $\mathcal{T}_L$ by

$$(\mathcal{T}_Ly)(t) = \int_{\Theta_L}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Ly(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_L(\theta_L) + \int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}_L(\theta_L,x_L), \qquad t\in[0,T]. \tag{3.20}$$

Lemma 3.7. *If Assumptions 2.2, 2.3 and 2.4 hold, then $\mathcal{T}_L$ is an operator from $(C([0,T],\mathbb{R}^n),\|\cdot\|_\infty)$ to $(C([0,T],\mathbb{R}^n),\|\cdot\|_\infty)$.*

Proof. By Lemma A.1, we have

$$\sup_{\theta_L\in\Theta_L}\left\|\phi_{\gamma,\theta_L}(t',s)-\phi_{\gamma,\theta_L}(t,s)\right\| \le \sup_{\theta_L\in\Theta_L}\left(\sup_{0\le t,s\le T}\left\|\frac{\partial\phi_{\gamma,\theta_L}(t,s)}{\partial t}\right\|\right)|t'-t| \le C_1|t'-t|, \quad \forall\, t\in[0,T], t'\in[0,T],$$

which leads to

$$\begin{aligned}
&\sup_{\theta_L\in\Theta_L,x_L\in X_L}\Bigg\|\Bigg[\int_0^{t'}\int_{s_2}^T \phi_{\gamma,\theta_L}(t',s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Ly(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2 \\
&+\phi_{\gamma,\theta_L}(t',0)x_L\Bigg] - \Bigg[\int_0^{t}\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Ly(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2 \\
&+\phi_{\gamma,\theta_L}(t,0)x_L\Bigg]\Bigg\| \\
\le{}& \sup_{\theta_L\in\Theta_L}\left\|\int_t^{t'}\int_{s_2}^T \phi_{\gamma,\theta_L}(t',s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Ly(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\right\| \\
&+\sup_{\theta_L\in\Theta_L}\left\|\int_0^{t}\int_{s_2}^T [\phi_{\gamma,\theta_L}(t',s_2)-\phi_{\gamma,\theta_L}(t,s_2)]B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Ly(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\right\| \\
&+\sup_{\theta_L\in\Theta_L,x_L\in X_L}\left\|[\phi_{\gamma,\theta_L}(t',0)-\phi_{\gamma,\theta_L}(t,0)]x_L\right\| \\
\le{}& \sup_{\theta_L\in\Theta_L}\left\|B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\right\|C_1^2\left(T+T^2\right)\sup_{s_1\in[0,T]}\|Q_LH_Ly(s_1)+Q_Lg_L\|(t'-t)+C_1\sup_{x_L\in X_L}\|x_L\||t'-t|
\end{aligned}$$

$$= \left[\sup_{\theta_L \in \Theta_L} \left\| B_L(\theta_L) R_L^{-1} B_L^\top(\theta_L) \right\| C_1^2 (T + T^2) \sup_{s_1 \in [0,T]} \| Q_L H_L y(s_1) + Q_L g_L \| \right.$$
$$\left. + C_1 \sup_{x_L \in X_L} \| x_L \| \right] |t' - t|, \quad \forall\, y \in C([0,T], \mathbb{R}^n).$$

This together with (3.20) implies that

$$\| (\mathcal{T}_L y)(t') - (\mathcal{T}_L y)(t) \| \le \left[\sup_{\theta_L \in \Theta_L} \left\| B_L(\theta_L) R_L^{-1} B_L^\top(\theta_L) \right\| C_1^2 \left(T + T^2 \right) \sup_{s_1 \in [0,T]} \| Q_L H_L y(s_1) + Q_L g_L \| \right.$$
$$\left. + C_1 \sup_{x_L \in X_L} \| x_L \| \right] |t' - t|, \ \forall\, t \in [0,T], \ \forall\, t' \in [0,T].$$

Combining the continuity of $y(t)$ with respect to $t \in [0,T]$ with the compactness of Θ_L and X_L, we have $\| (\mathcal{T}_L y)(t') - (\mathcal{T}_L y)(t) \| \to 0$ as $|t' - t| \to 0$. Hence, $\mathcal{T}_L$ is an operator from $(C([0,T], \mathbb{R}^n), \|\cdot\|_\infty)$ to $(C([0,T], \mathbb{R}^n), \|\cdot\|_\infty)$. □

By the virtue of $\mathcal{T}_L$ and (3.17), the auxiliary system (3.16) can be rewritten as

$$z_L = \mathcal{T}_L z_L. \tag{3.21}$$

Equation (3.21) especially embodies the property that the estimation $z_L(t)$ of the state average term for the major players should possess: if every player views $z_L(t)$ as the estimation of $x_L^{(N_L)}$ and makes the optimal strategy by $z_L(t)$, then the expectation of the state average term of the closed-loop system for the major players ought to approach $z_L(t)$ as N_L goes to infinity. So, if (3.21) has a unique solution, then this solution can be used as the estimation of the state average term for the major players. The sufficient condition for the existence and uniqueness of the solution of (3.21) is given below.

Assumption 3.8. $\|Q_L\| \|H_L\| \int_{\Theta_L} \int_0^t \int_{s_2}^T \left\| \phi_{\gamma,\theta_L}(t, s_2) B_L(\theta_L) R_L^{-1} B_L^\top(\theta_L) \psi_{\gamma,\theta_L}(s_2, s_1) \right\| \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}\mathbf{F}_L(\theta_L) < 1.$

Theorem 3.9. *If Assumptions 2.2–2.4 and 3.8 hold, then (3.21) has a unique solution.*

Proof. By the definition of $\mathcal{T}_L$, it is known that

$$\| (\mathcal{T}_L y_1)(t) - (\mathcal{T}_L y_2)(t) \|_\infty$$
$$\le \| y_1 - y_2 \|_\infty \|Q_L\| \|H_L\| \int_{\Theta_L} \int_0^t \int_{s_2}^T \left\| \phi_{\gamma,\theta_L}(t, s_2) B_L(\theta_L) R_L^{-1} B_L^\top(\theta_L) \psi_{\gamma,\theta_L}(s_2, s_1) \right\| \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}\mathbf{F}_L(\theta_L)$$

for any $y_1 \in C([0,T], \ \mathbb{R}^n)$ and $y_2 \in C([0,T], \ \mathbb{R}^n)$. From Assumptions 2.2, 2.3, 2.4 and 3.8, we know that $\mathcal{T}_L$ is a contraction operator on $(C([0,T], \ \mathbb{R}^n), \ \|\cdot\|_\infty)$. By Banach fixed point theorem, (3.21) has a unique solution. □

In the following section, we denote the unique solution of (3.21) as z_L^* under Assumptions 2.2, 2.3, 2.4 and 3.8. Through the above analysis for the system (2.1), under Assumptions 2.2, 2.3, 2.4, 2.5 and 3.8, each player can view $z_L^*(t)$ as the estimation of $x_L^{(N_L)}(t)$ and then designs the optimal strategy by $z_L^*(t)$. In the following analysis, we construct the decentralized strategies for the major players as

$$u_{L_i}^*(t) = -R_L^{-1} B_L^\top(\theta_{L_i}) P_{\gamma,\theta_{L_i}}(t) x_{L_i}(t) + R_L^{-1} B_L^\top(\theta_{L_i}) \xi_{\gamma,\theta_{L_i}, z_L^*}(t), \quad t \in [0,T], \quad i = 1, 2, \ldots, N_L, \tag{3.22}$$

where $P_{\gamma,\theta_{L_i}}(t) = P_{\gamma,\theta_L}(t)|_{\theta_L=\theta_{L_i}}$, $\xi_{\gamma,\theta_{L_i},z_L^*}(t) = \xi_{\gamma,\theta_L,z_L}(t)|_{\theta_L=\theta_{L_i},z_L=z_L^*}$, $P_{\gamma,\theta_L}(t)$ and $\xi_{\gamma,\theta_L,z_L}(t)$ satisfy (2.5) and (3.14) respectively. Under the strategies (3.22), the closed-loop system obtained from (2.1) is given by

$$\begin{aligned}\mathrm{d}x_{L_i}^*(t) = &\left[A_L(\theta_{L_i}) - B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\right] x_{L_i}^*(t)\mathrm{d}t + B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L^*}(t)\mathrm{d}t \\ &+ D_L(\theta_{L_i})\mathrm{d}W_{L_i}(t), \quad i = 1, 2, \ldots, N_L. \end{aligned} \tag{3.23}$$

3.3. Design of strategies for minor players

Similar to the design of the strategies for the major players, for the cost functionals (2.4), if $x_L^{(N_L)}(t)$ and $x_F^{(N_F)}(t)$ are replaced by $z_L^*(t)$ and some deterministic vector-valued function $z_F(t)$ respectively, then the game of the minor players in the models (2.1)–(2.4) is transformed into the following optimal tracking control problem.

Auxiliary problem (II). Minimize $\bar{J}_{F_j,z_F,z_L^*}(u_{F_j}(\cdot))$ over $\mathcal{U}_{F_j}^l$, where

$$\bar{J}_{F_j,z_F,z_L^*}(u_{F_j}(\cdot)) = \gamma \ln \mathbb{E}\left[\exp\left(\frac{1}{\gamma}\Phi_F\left(x_{F_j}(\cdot), z_F(\cdot), z_L^*(\cdot), u_{F_j}(\cdot)\right)\right)\right], \quad j = 1, 2, \ldots, N_F, \tag{3.24}$$

$x_{F_j}(t)$ satisfies the system (2.2) and $z_F(t) \in \mathbb{R}^n$ is a deterministic function.

Theorem 3.10. *For Auxiliary problem (II), if Assumptions 2.4 and 2.5 hold, then the optimal strategies are given by*

$$\bar{u}_{F_j}(t) = -R_F^{-1}B_F^\top(\theta_{F_j})P_{\gamma,\theta_{F_j}}(t)x_{F_j}(t) + R_F^{-1}B_F^\top(\theta_{F_j})\xi_{\gamma,\theta_{F_j},z_F}(t), \quad t \in [0, T], \quad j = 1, 2, \ldots, N_F, \tag{3.25}$$

where $P_{\gamma,\theta_{F_j}}(t) = P_{\gamma,\theta_F}(t)|_{\theta_F=\theta_{F_j}}$, $\xi_{\gamma,\theta_{F_j},z_F}(t) = \xi_{\gamma,\theta_F,z_F}(t)|_{\theta_F=\theta_{F_j}}$, $P_{\gamma,\theta_F}(t)$ *is given by (2.6), and* $\xi_{\gamma,\theta_F,z_F}(t)$ *is the unique solution of the linear differential equation*

$$\begin{cases} \dot{\xi}_{\gamma,\theta_F,z_F}(t) = -\left[A_F(\theta_F) - B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)P_{\gamma,\theta_F}(t) + \frac{2}{\gamma}D_F(\theta_F)D_F^\top(\theta_F)P_{\gamma,\theta_F}(t)\right]^\top \xi_{\gamma,\theta_F,z_F}(t) \\ \qquad\qquad -[Q_F H_F z_F(t) + Q_F H z_L^*(t) + Q_F g_F], \\ \xi_{\gamma,\theta_F,z_F}(T) = 0. \end{cases} \tag{3.26}$$

Proof. The proof is similar to that of Theorem 3.6. □

Substituting the strategies (3.25) into (2.2), we get the closed-loop system for the minor player j as

$$\begin{aligned}\mathrm{d}\bar{x}_{F_j}(t) = &\left[A_F(\theta_{F_j}) - B_F(\theta_{F_j})R_F^{-1}B_F^\top(\theta_{F_j})P_{\gamma,\theta_{F_j}}(t)\right]\bar{x}_{F_j}(t)\mathrm{d}t + B_F(\theta_{F_j})R_F^{-1}B_F^\top(\theta_{F_j})\xi_{\gamma,\theta_{F_j},z_F}(t)\mathrm{d}t \\ &+ D_F(\theta_{F_j})\mathrm{d}W_{F_j}(t).\end{aligned}$$

Similar to Section 3.2, we construct an auxiliary system

$$\begin{cases} \frac{\mathrm{d}\mathbb{E}\bar{x}_{\theta_F}(t)}{\mathrm{d}t} = \left[A_F(\theta_F) - B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)P_{\gamma,\theta_F}(t)\right]\mathbb{E}\bar{x}_{\theta_F}(t) + B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)\xi_{\gamma,\theta_F,z_F}(t), \\ \mathbb{E}\bar{x}_{\theta_F}(0) = x_F, \\ z_F(t) = \int_{\Theta_F\times X_F}\mathbb{E}\bar{x}_{\theta_F}(t)\mathrm{d}\mathbf{F}_F(\theta_F, x_F). \end{cases} \tag{3.27}$$

By the system (3.27), we have

$$\begin{aligned}\mathbb{E}\bar{x}_{\theta_F}(t) = &\int_0^t \phi_{\gamma,\theta_F}(t, s_2)B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)\int_{s_2}^T \psi_{\gamma,\theta_F}(s_2, s_1)[Q_F H_F z_F(s_1) \\ &+ Q_F H z_L^*(s_1) + Q_F g_F]\mathrm{d}s_1\mathrm{d}s_2 + \phi_{\gamma,\theta_F}(t, 0)x_F, \end{aligned} \tag{3.28}$$

where $\phi_{\gamma,\theta_F}(t,s)$, $0 \le t, s \le T$, is the solution of

$$\begin{cases} \mathrm{d}\phi_{\gamma,\theta_F}(t,s) = \left[A_F(\theta_F) - B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)P_{\gamma,\theta_F}(t)\right]\phi_{\gamma,\theta_F}(t,s)\mathrm{d}t, \\ \phi_{\gamma,\theta_F}(s,s) = I_n, \end{cases} \tag{3.29}$$

and $\psi_{\gamma,\theta_F}(t,s)$, $0 \le t, s \le T$, is the solution of

$$\begin{cases} \mathrm{d}\psi_{\gamma,\theta_F}(t,s) = -\Big[A_F(\theta_F) - B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)P_{\gamma,\theta_F}(t) \\ \qquad\qquad + \frac{2}{\gamma}D_F(\theta_F)D_F^\top(\theta_F)P_{\gamma,\theta_F}(t)\Big]^\top \psi_{\gamma,\theta_F}(t,s)\mathrm{d}t, \\ \psi_{\gamma,\theta_F}(s,s) = I_n. \end{cases} \tag{3.30}$$

For any given $y' \in C([0,T],\mathbb{R}^n)$, define the operator $\mathcal{T}_F$ by

$$\begin{aligned} (\mathcal{T}_F y')(t) = &\int_{\Theta_F}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_F}(t,s_2)B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)\psi_{\gamma,\theta_F}(s_2,s_1)[Q_FH_Fy'(s_1) + Q_FHz_L^*(s_1) \\ &+ Q_Fg_F]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_F(\theta_F) + \int_{\Theta_F\times X_F}\phi_{\gamma,\theta_F}(t,0)x_F\mathrm{d}\mathbf{F}_F(\theta_F,x_F), \qquad t\in[0,T]. \end{aligned} \tag{3.31}$$

Taking a similar proof of Lemma 3.7, we can derive that $\mathcal{T}_F$ is an operator from $(C([0,T],\mathbb{R}^n),\ \|\cdot\|_\infty)$ to $(C([0,T],\mathbb{R}^n),\ \|\cdot\|_\infty)$.

By the virtue of $\mathcal{T}_F$ and (3.28), the auxiliary system (3.27) can be rewritten as

$$z_F = \mathcal{T}_F z_F. \tag{3.32}$$

Assumption 3.11. $\|Q_F\|\|H_F\|\int_{\Theta_F}\int_0^t\int_{s_2}^T\left\|\phi_{\gamma,\theta_F}(t,s_2)B(\theta_F)R_F^{-1}B(\theta_F)^\top\psi_{\gamma,\theta_F}(s_2,s_1)\right\|\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_F(\theta_F) < 1.$

Theorem 3.12. *If Assumptions 2.2, 2.3, 2.4 and 3.11 hold, then* (3.32) *has a unique solution.*

Proof. The proof is similar to that of Theorem 3.9. □

In the following section, we denote the unique solution of (3.32) as z_F^* under Assumptions 2.2, 2.3, 2.4 and 3.11. Through the above analysis for the system (2.2), under Assumptions 2.2, 2.3, 2.4, 2.5, 3.8 and 3.11, each player can view $z_F^*(t)$ as the estimation of $x_F^{(N_F)}(t)$ and then designs the optimal strategy by $z_F^*(t)$. In the following analysis, we construct the decentralized strategies for the minor players as

$$u_{F_j}^*(t) = -R_F^{-1}B_F^\top(\theta_{F_j})P_{\gamma,\theta_{F_j}}(t)x_{F_j}(t) + R_F^{-1}B_F^\top(\theta_{F_j})\xi_{\gamma,\theta_{F_j},z_F^*}(t), \quad t\in[0,T], \quad j=1,2,\ldots,N_F, \tag{3.33}$$

where $P_{\gamma,\theta_{F_j}}(t) = P_{\gamma,\theta_F}(t)|_{\theta_F=\theta_{F_j}}$, $\xi_{\gamma,\theta_{F_j},z_F^*}(t) = \xi_{\gamma,\theta_F,z_F}(t)|_{\theta_F=\theta_{F_j},z_F=z_F^*}$, $P_{\gamma,\theta_F}(t)$ and $\xi_{\gamma,\theta_F,z_F}(t)$ satisfy (2.6) and (3.26) respectively. Under the strategies (3.33), the closed-loop system obtained from (2.2) is given by

$$\begin{aligned} \mathrm{d}x_{F_j}^*(t) = &\left[A_F(\theta_{F_j}) - B_F(\theta_{F_j})R_F^{-1}B_F^\top(\theta_{F_j})P_{\gamma,\theta_{F_j}}(t)\right]x_{F_j}^*(t)\mathrm{d}t + B_F(\theta_{F_j})R_F^{-1}B_F^\top(\theta_{F_j})\xi_{\gamma,\theta_{F_j},z_F^*}(t)\mathrm{d}t \\ &+ D_F(\theta_{F_j})\mathrm{d}W_{F_j}(t), \quad j=1,2,\ldots,N_F. \end{aligned} \tag{3.34}$$

4. Consistency analysis

In this section, we analyze the consistency of $z_L^*(t)$ and $z_F^*(t)$ as the estimations of the state average terms for the major and minor players, respectively. It is known that for systems with risk-neutral cost functionals, the equilibrium property of the strategies designed based on the Nash certainty equivalence principle often depends on whether or not the estimations of the state average terms are convergent to their true values, which is the consistency of the estimations [6, 18, 30, 38]. Similarly, for risk-sensitive mean field games with major and minor players, the equilibrium of the strategies designed based on the Nash certainty equivalence principle also depends on the consistency of the estimations. To derive the equilibrium property of the strategies, we prove the consistency of the estimations in this section.

Firstly, define two auxiliary operators. For any given $y \in C([0,T],\mathbb{R}^n)$ and $y' \in C([0,T],\mathbb{R}^n)$, define

$$\begin{aligned}(\mathcal{T}_{N_L}y)(t) = &\int_{\Theta_L}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Ly(s_1)\\ &+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_{N_L}(\theta_L)+\int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}_{N_L}(\theta_L,x_L), \qquad t\in[0,T], \end{aligned} \tag{4.1}$$

$$\begin{aligned}(\mathcal{T}_{N_F}y')(t) = &\int_{\Theta_F}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_F}(t,s_2)B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)\psi_{\gamma,\theta_F}(s_2,s_1)[Q_FH_Fy'(s_1)+Q_FHz_L^*(s_1)\\ &+Q_Fg_F]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_{N_F}(\theta_F)+\int_{\Theta_F\times X_F}\phi_{\gamma,\theta_F}(t,0)x_F\mathrm{d}\mathbf{F}_{N_F}(\theta_F,x_F), \qquad t\in[0,T]. \end{aligned} \tag{4.2}$$

Similar to the proof of Lemma 3.7, it can be obtained that $\mathcal{T}_{N_L}$ and $\mathcal{T}_{N_F}$ are operators from $(C([0,T],\mathbb{R}^n),\|\cdot\|_\infty)$ to $(C([0,T],\mathbb{R}^n),\|\cdot\|_\infty)$. Let

$$\begin{aligned}\epsilon_{N_L} = \Bigg(\int_0^T \Bigg\| &\int_{\Theta_L}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)(Q_LH_Lz_L^*(s_1)+Q_Lg_L)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_L(\theta_L)\\ &-\int_{\Theta_L}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)(Q_LH_Lz_L^*(s_1)+Q_Lg_L)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_{N_L}(\theta_L)\\ &+\int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}(\theta_L,x_L)-\int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}_{N_L}(\theta_L,x_L)\Bigg\|^2\mathrm{d}t\Bigg)^{\frac{1}{2}}, \end{aligned} \tag{4.3}$$

$$\begin{aligned}\epsilon_{N_F} = \Bigg(\int_0^T \Bigg\| &\int_{\Theta_F}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_F}(t,s_2)B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)\psi_{\gamma,\theta_F}(s_2,s_1)(Q_FH_Fz_F^*(s_1)+Q_FHz_L^*(s_1)\\ &+Q_Fg_F)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_F(\theta_F)-\int_{\Theta_F}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_F}(t,s_2)B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)\\ &\times\psi_{\gamma,\theta_F}(s_2,s_1)(Q_FH_Fz_F^*(s_1)+Q_FHz_L^*(s_1)+Q_Fg_F)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_{N_F}(\theta_F)\\ &+\int_{\Theta_F\times X_F}\phi_{\gamma,\theta_F}(t,0)x_F\mathrm{d}\mathbf{F}(\theta_F,x_F)-\int_{\Theta_F\times X_F}\phi_{\gamma,\theta_F}(t,0)x_F\mathrm{d}\mathbf{F}_{N_F}(\theta_F,x_F)\Bigg\|^2\mathrm{d}t\Bigg)^{\frac{1}{2}}. \end{aligned} \tag{4.4}$$

Lemma 4.1. *For the systems (2.1)–(2.4), if Assumptions 2.2, 2.3, 2.4, 2.5, 3.8 and 3.11 hold, then we have* $\lim_{N_L\to\infty}\epsilon_{N_L}=0$ *and* $\lim_{N_F\to\infty}\epsilon_{N_F}=0$.

Proof. From (3.20), (3.31), (4.1) and (4.2), we know that (4.3) and (4.4) can be rewritten as

$$\epsilon_{N_L}^2=\int_0^T\|(\mathcal{T}_Lz_L^*)(t)-(\mathcal{T}_{N_L}z_L^*)(t)\|^2\mathrm{d}t \ \text{ and } \ \epsilon_{N_F}^2=\int_0^T\|(\mathcal{T}_Fz_F^*)(t)-(\mathcal{T}_{N_F}z_F^*)(t)\|^2\mathrm{d}t.$$

By the definition of $\phi_{\gamma,\theta_L}(t,s)$ and the compactness of Θ_L and X_L, for any given $t\in[0,T]$, $\phi_{\gamma,\theta_L}(t,0)x_L$ is bounded and continuous with respect to $(\theta_L,x_L)\in\Theta_L\times X_L$, and $\int_0^t\int_{s_2}^T\phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}$

$(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2$ is bounded and continuous with respect to $\theta_L\in\Theta_L$. Based on Assumption 2.3, (3.20) and (4.1), we have

$$\lim_{N_L\to\infty}\|(\mathcal{T}_{N_L}z_L^*)(t)-(\mathcal{T}_Lz_L^*)(t)\|=0,\quad \forall\, t\in[0,T], \tag{4.5}$$

which implies that the vector-valued sequence $\{(\mathcal{T}_{N_L}z_L^*)(t),N_L\geq 1\}$ converges pointwise to $(\mathcal{T}_Lz_L^*)(t)$ on $[0,T]$.

By Lemma A.1, we have

$$\sup_{\theta_L\in\Theta_L}\|\phi_{\gamma,\theta_L}(t',s)-\phi_{\gamma,\theta_L}(t,s)\|\leq \sup_{\theta_L\in\Theta_L}\left(\sup_{0\leq t,s\leq T}\left\|\frac{\partial\phi_{\gamma,\theta_L}(t,s)}{\partial t}\right\|\right)|t'-t)|\leq C_1|t'-t|,\quad \forall\, t\in[0,T],\ t'\in[0,T],$$

which leads to

$$\begin{aligned}
&\sup_{\theta_L\in\Theta_L,x_L\in X_L}\Bigg\|\Bigg[\int_0^{t'}\int_{s_2}^T\phi_{\gamma,\theta_L}(t',s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Ly(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\\
&\quad+\phi_{\gamma,\theta_L}(t',0)x_L\Bigg]-\Bigg[\int_0^{t}\int_{s_2}^T\phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Ly(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\\
&\quad+\phi_{\gamma,\theta_L}(t,0)x_L\Bigg]\Bigg\|\\
\leq&\sup_{\theta_L\in\Theta_L}\left\|\int_t^{t'}\int_{s_2}^T\phi_{\gamma,\theta_L}(t',s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Ly(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\right\|\\
&+\sup_{\theta_L\in\Theta_L}\left\|\int_0^{t}\int_{s_2}^T[\phi_{\gamma,\theta_L}(t',s_2)-\phi_{\gamma,\theta_L}(t,s_2)]B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Ly(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\right\|\\
&+\sup_{\theta_L\in\Theta_L,x_L\in X_L}\left\|[\phi_{\gamma,\theta_L}(t',0)-\phi_{\gamma,\theta_L}(t,0)]x_L\right\|\\
\leq&\sup_{\theta_L\in\Theta_L}\left\|B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\right\|C_1^2\left(T+T^2\right)\sup_{s_1\in[0,T]}\|Q_LH_Ly(s_1)+Q_Lg_L\||t'-t|+C_1\sup_{x_L\in X_L}\|x_L\||t'-t|\\
=&\Bigg[\sup_{\theta_L\in\Theta_L}\left\|B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\right\|C_1^2\left(T+T^2\right)\sup_{s_1\in[0,T]}\|Q_LH_Ly(s_1)+Q_Lg_L\|\\
&+C_1\sup_{x_L\in X_L}\|x_L\|\Bigg]|t'-t|,\quad \forall\, y\in C([0,T],\mathbb{R}^n).
\end{aligned}$$

For any given $\epsilon>0$, take

$$\Delta=\epsilon\Bigg/\Bigg[\sup_{\theta_L\in\Theta_L}\|B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\|C_1^2\left(T+T^2\right)\sup_{s_1\in[0,T]}\|Q_LH_Lz_L^*(s_1)+Q_Lg_L\|+C_1\sup_{x_L\in X_L}\|x_L\|\Bigg].$$

It follows that for any given $N_L\geq 1,t'\in[0,T],t\in[0,T]$ and $|t'-t|\leq\Delta$,

$$\begin{aligned}
&\|(\mathcal{T}_{N_L}z_L^*)(t')-(\mathcal{T}_{N_L}z_L^*)(t)\|\\
\leq&\Bigg[\sup_{\theta_L\in\Theta_L}\left\|B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\right\|C_1^2\left(T+T^2\right)\sup_{s_1\in[0,T]}\|Q_LH_Lz_L^*(s_1)+Q_Lg_L\|+C_1\sup_{x_L\in X_L}\|x_L\|\Bigg]|t'-t|\\
\leq&\epsilon,
\end{aligned}\tag{4.6}$$

which implies that the vector-valued function sequence $\{\mathcal{T}_{N_L}z_L^*(t),N_L\geq 1\}$ is equicontinuous on $[0,T]$. Since uniform convergence can be derived from equicontinuity and pointwise convergence, by (4.5) and (4.6), we get

$\sup_{t\in[0,T]}\|(\mathcal{T}_L z_L^*)(t)-(\mathcal{T}_{N_L} z_L^*)(t)\|\to 0$ as $N_L\to\infty$. Therefore, we have

$$\lim_{N_L\to\infty}\epsilon_{N_L}^2=\lim_{N_L\to\infty}\int_0^T\|(\mathcal{T}_L z_L^*)(t)-(\mathcal{T}_{N_L} z_L^*)(t)\|^2\mathrm{d}t=\int_0^T\lim_{N_L\to\infty}\|(\mathcal{T}_L z_L^*)(t)-(\mathcal{T}_{N_L} z_L^*)(t)\|^2\mathrm{d}t=0,$$

which means $\lim_{N_L\to\infty}\epsilon_{N_L}=0$. By a similar proof of $\lim_{N_L\to\infty}\epsilon_{N_L}=0$, we can get $\lim_{N_F\to\infty}\epsilon_{N_F}=0$. This completes the proof. □

Definition 4.2. Define $\tilde{L}^4_{\mathcal{F}}(0,T;\mathbb{R}^n)=\Big\{f\Big|\ f(t)\in\mathbb{R}^n$ and $f(t)$ is $\mathcal{F}_t$-measurable, $\forall\ t\in[0,T]$, $\mathbb{E}\Big[\Big(\int_0^T\|f(t)\|^2\mathrm{d}t\Big)^2\Big]<\infty\Big\}$. For any given $f\in\tilde{L}^4_{\mathcal{F}}(0,T;\mathbb{R}^n)$, define norm $\|f\|_{\tilde{L}^4_{\mathcal{F}}}=\Big\{\mathbb{E}\Big[\Big(\int_0^T\|f(t)\|^2\mathrm{d}t\Big)^2\Big]\Big\}^{\frac{1}{4}}$.

By Lemma C.1, we know that $(\tilde{L}^4_{\mathcal{F}}(0,T;\mathbb{R}^n),\|\cdot\|_{\tilde{L}^4_{\mathcal{F}}})$ is a normed linear space. Denote $x_L^{*,(N_L)}(t)=\frac{1}{N_L}\sum_{i=1}^{N_L}x^*_{L_i}(t)$ and $x_F^{*,(N_F)}(t)=\frac{1}{N_F}\sum_{j=1}^{N_F}x^*_{F_j}(t)$.

Theorem 4.3. *For the systems (2.1) and (2.2), if Assumptions 2.2, 2.3, 2.4, 2.5, 3.8 and 3.11 hold, then under the strategies (3.22) and (3.33), the solutions of the closed-loop systems (3.23) and (3.34) satisfy*

$$\Big\|z_L^*-x_L^{*,(N_L)}\Big\|_{\tilde{L}^4_{\mathcal{F}}}=O\left(\epsilon_{N_L}+\frac{1}{\sqrt{N_L}}\right),\tag{4.7}$$

$$\Big\|z_F^*-x_F^{*,(N_F)}\Big\|_{\tilde{L}^4_{\mathcal{F}}}=O\left(\epsilon_{N_F}+\frac{1}{\sqrt{N_F}}\right).\tag{4.8}$$

Proof. See Appendix C. □

Remark 4.4. It proves the consistency of the estimations for the state average terms in the sense of $\tilde{L}^4_{\mathcal{F}}$-norm in Theorem 4.3. That is, the $\tilde{L}^4_{\mathcal{F}}$-norms of the differences between the estimations of the state average terms and their true values vanish as players increase.

5. Equilibrium analysis

In this section, we prove the equilibrium property of the strategies $\big\{u^*_{L_i}(\cdot)\in\mathcal{U}^l_{L_i},u^*_{F_j}(\cdot)\in\mathcal{U}^l_{F_j},i=1,2,\dots,N_L,j=1,2,\dots,N_F\big\}$ with respect to the corresponding cost functionals $\big\{J_{L_i},J_{F_j},i=1,2,\dots,N_L,j=1,2,\dots,N_F\big\}$. Next, we give two lemmas prior to the equilibrium analysis.

Lemma 5.1. *For the mean field game (2.1)–(2.4) and Auxiliary problems (I) and (II) (corresponding to (2.1), (2.2), (3.12) and (3.24)), if Assumptions 2.2, 2.3, 2.4, 2.5, 3.8 and 3.11 hold, then under the strategies (3.22) and (3.33), the solutions of the closed-loop systems (3.23) and (3.34) satisfy*

$$\mathbb{E}\left[\exp\left(\frac{4}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),x_L^{*,(N_L)}(\cdot),u^*_{L_i}(\cdot)\right)\right)\right]<\infty,\tag{5.1}$$

$$\mathbb{E}\left[\exp\left(\frac{4}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),z_L^*(\cdot),u^*_{L_i}(\cdot)\right)\right)\right]<\infty,\tag{5.2}$$

$$\mathbb{E}\left[\left(\int_0^1\exp\left(\frac{(1-s)}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),x_L^{*,(N_L)}(\cdot),u^*_{L_i}(\cdot)\right)\right)\exp\left(\frac{s}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),z_L^*(\cdot),u^*_{L_i}(\cdot)\right)\mathrm{d}s\right)^2\right]<\infty,\tag{5.3}$$

$$\mathbb{E}\left[\exp\left(\frac{4}{\gamma}\Phi_F\left(x^*_{F_j}(\cdot),x_F^{*,(N_F)}(\cdot),x_L^{*,(N_L)}(\cdot),u^*_{F_j}(\cdot)\right)\right)\right]<\infty,\tag{5.4}$$

$$\mathbb{E}\left[\exp\left(\frac{4}{\gamma}\Phi_F\left(x^*_{F_j}(\cdot), z^*_F(\cdot), z^*_L(\cdot), u^*_{F_j}(\cdot)\right)\right)\right] < \infty, \tag{5.5}$$

$$\mathbb{E}\left[\left(\int_0^1 \exp\left(\frac{(1-s)}{\gamma}\Phi_F\left(x^*_{F_j}(\cdot), x_F^{*,(N_F)}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{F_j}(\cdot)\right)\right)\right.\right.$$
$$\left.\left.\times\exp\left(\frac{s}{\gamma}\Phi_F\left(x^*_{F_j}(\cdot), z^*_F(\cdot), z^*_L(\cdot), u^*_{F_j}(\cdot)\right)\right)\mathrm{d}s\right)^2\right] < \infty \tag{5.6}$$

for any $i = 1, 2, \ldots, N_L$ *and* $j = 1, 2, \ldots, N_F$.

Proof. See Appendix D. □

Lemma 5.2. *For the mean field games (2.1)–(2.4) and Auxiliary problems (I) and (II), if Assumptions 2.2, 2.3, 2.4, 2.5, 3.8 and 3.11 hold, then*

$$\left|J_{L_i}\left(u^*_{L_i}(\cdot), u^*_{L_{-i}}(\cdot)\right) - \bar{J}_{L_i,z^*_L}\left(u^*_{L_i}(\cdot)\right)\right| = O\Bigg(\epsilon_{N_L} + \frac{1}{\sqrt{N_L}}\Bigg), i = 1, 2, \ldots, N_L, \tag{5.7}$$

$$\left|J_{F_j}\left(u^*_{F_j}(\cdot), u^*_{F_{-j}}(\cdot), u^*_L(\cdot)\right) - \bar{J}_{F_j,z^*_F,z^*_L}\left(u^*_{F_j}(\cdot)\right)\right| = O\Bigg(\epsilon_{N_L} + \frac{1}{\sqrt{N_L}}\Bigg) + O\Bigg(\epsilon_{N_F} + \frac{1}{\sqrt{N_F}}\Bigg), j = 1, 2, \ldots, N_F. \tag{5.8}$$

Proof. We prove $\left|J_{L_i}\left(u^*_{L_i}(\cdot), u^*_{L_{-i}}(\cdot)\right) - \bar{J}_{L_i,z^*_L}\left(u^*_{L_i}(\cdot)\right)\right| = O\left(\epsilon_{N_L} + \frac{1}{\sqrt{N_L}}\right)$ first.

Since $\gamma > 0$, $\Phi_L\big(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\big) \geq 0$ and $\Phi_L\big(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\big) \geq 0$, by the property of exponential functions and Lemma 5.1, we obtain $1 \leq \mathbb{E}\left[\exp\left(\frac{1}{\gamma}\Phi_L\big(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\big)\right)\right] < \infty$ and $1 \leq \mathbb{E}\left[\exp\left(\frac{1}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\right)\right)\right] < \infty$. Thus, by the mean value theorem of differentiation, we know that there exists a constant $a' \geq 1$ and $a' \in \Big[\min\Big\{\mathbb{E}\Big[\exp\Big(\frac{1}{\gamma}\Phi_L\big(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\big)\Big)\Big], \mathbb{E}\Big[\exp\Big(\frac{1}{\gamma}\Phi_L\big(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\big)\Big)\Big]\Big\}, \max\Big\{\mathbb{E}\Big[\exp\Big(\frac{1}{\gamma}\Phi_L\big(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\big)\Big)\Big], \mathbb{E}\Big[\exp\Big(\frac{1}{\gamma}\Phi_L\big(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\big)\Big)\Big]\Big\}\Big]$ such that

$$\begin{aligned}
&\left|J_{L_i}(u^*_{L_i}(\cdot), u^*_{L_{-i}}(\cdot)) - \bar{J}_{L_i,z^*_L}(u^*_{L_i}(\cdot))\right| \\
=&\gamma\left|\ln\mathbb{E}\left[\exp\left(\frac{1}{\gamma}\Phi_L\Big(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\Big)\right)\right] - \ln\mathbb{E}\left[\exp\left(\frac{1}{\gamma}\Phi_L\big(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\big)\right)\right]\right| \\
=&\gamma\left|\frac{1}{a'}\left\{\mathbb{E}\left[\exp\left(\frac{1}{\gamma}\Phi_L\Big(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\Big)\right)\right] - \mathbb{E}\left[\exp\left(\frac{1}{\gamma}\Phi_L\big(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\big)\right)\right]\right\}\right| \\
=&\frac{\gamma}{a'}\mathbb{E}\left|\exp\left[\frac{1}{\gamma}\Phi_L\Big(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\Big)\right] - \exp\left[\frac{1}{\gamma}\Phi_L\big(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\big)\right]\right|.
\end{aligned}$$

By the above equation, Dyson expansion $\exp(a+b) - \exp(a) = b\int_0^1 \exp[(1-s)a]\exp[s(a+b)]\mathrm{d}s$ and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left|J_{L_i}\left(u^*_{L_i}(\cdot), u^*_{L_{-i}}(\cdot)\right) - \bar{J}_{L_i,z^*_L}\left(u^*_{L_i}(\cdot)\right)\right| \\
=&\frac{1}{a'}\mathbb{E}\left[\int_0^1 \exp\left(\frac{(1-s)}{\gamma}\Phi_L\Big(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\Big)\right)\right. \\
&\left.\times\left|\Phi_L\big(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\big) - \Phi_L\big(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\big)\right|\exp\left(\frac{s}{\gamma}\Phi_L\big(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\big)\right)\mathrm{d}s\right]
\end{aligned}$$

$$\leq\frac{1}{a'}\left\{\mathbb{E}\left[\left|\Phi_L\Big(x^*_{L_i}(\cdot),x^{*,(N_L)}_L(\cdot),u^*_{L_i}(\cdot)\Big)-\Phi_L\big(x^*_{L_i}(\cdot),z^*_L(\cdot),u^*_{L_i}(\cdot)\big)\right|^2\right]\right\}^{\frac12}$$
$$\times\left\{\mathbb{E}\left[\left(\int_0^1\exp\left(\frac{(1-s)}{\gamma}\Phi_L\Big(x^*_{L_i}(\cdot),x^{*,(N_L)}_L(\cdot),u^*_{L_i}(\cdot)\Big)\right)\exp\left(\frac{s}{\gamma}\Phi_L\big(x^*_{L_i}(\cdot),z^*_L(\cdot),u^*_{L_i}(\cdot)\big)\right)\mathrm{d}s\right)^2\right]\right\}^{\frac12}. \quad (5.9)$$

By Lemma 5.1, we know

$$\left\{\mathbb{E}\left[\left(\int_0^1\exp\left(\frac{(1-s)}{\gamma}\Phi_L\Big(x^*_{L_i}(\cdot),x^{*,(N_L)}_L(\cdot),u^*_{L_i}(\cdot)\Big)\right)\exp\left(\frac{s}{\gamma}\Phi_L\big(x^*_{L_i}(\cdot),z^*_L(\cdot),u^*_{L_i}(\cdot)\big)\right)\mathrm{d}s\right)^2\right]\right\}^{\frac12}<\infty. \quad (5.10)$$

By $a^2-b^2\leq(a-b)^2+2|b||a-b|$, C_r inequality, Cauchy-Schwarz inequality and Theorem 4.3, we get

$$\begin{aligned}
&\mathbb{E}\left[\left|\Phi_L\Big(x^*_{L_i}(\cdot),x^{*,(N_L)}_L(\cdot),u^*_{L_i}(\cdot)\Big)-\Phi_L\big(x^*_{L_i}(\cdot),z^*_L(\cdot),u^*_{L_i}(\cdot)\big)\right|^2\right]\\
=&\mathbb{E}\left[\left(\int_0^T\left(\left\|x^*_{L_i}(t)-H_Lx^{*,(N_L)}_L(t)-g_L\right\|^2_{Q_L}-\|x^*_{L_i}(t)-H_Lz^*_L(t)-g_L\|^2_{Q_L}\right)\mathrm{d}t\right)^2\right]\\
\leq&\mathbb{E}\Bigg[\Bigg(\int_0^T\|Q_L\|\left\|\Big(x^*_{L_i}(t)-H_Lx^{*,(N_L)}_L(t)-g_L\Big)-\big(x^*_{L_i}(t)-H_Lz^*_L(t)-g_L\big)\right\|\\
&\times\left\|\Big(x^*_{L_i}(t)-H_Lx^{*,(N_L)}_L(t)-g_L\Big)-\big(x^*_{L_i}(t)-H_Lz^*_L(t)-g_L\big)+2\big(x^*_{L_i}(t)-H_Lz^*_L(t)-g_L\big)\right\|\mathrm{d}t\Bigg)^2\Bigg]\\
\leq&\mathbb{E}\Bigg[\Bigg(\int_0^T\|Q_L\|\Bigg(\left\|(x^*_{L_i}(t)-x^*_{L_i}(t))-H_L\left(x^{*,(N_L)}_L(t)-z^*_L(t)\right)\right\|^2\\
&+2\|x^*_{L_i}(t)-H_Lz^*_L(t)-g_L\|\left\|(x^*_{L_i}(t)-x^*_{L_i}(t))-H_L\left(x^{*,(N_L)}_L(t)-z^*_L(t)\right)\right\|\Bigg)\mathrm{d}t\Bigg)^2\Bigg]\\
\leq&\mathbb{E}\Bigg[\Bigg(\|Q_L\|\|H_L\|^2\int_0^T\left\|z^*_L(t)-x^{*,(N_L)}_L(t)\right\|^2\mathrm{d}t+2\|Q_L\|\|H_L\|\left(\int_0^T\|x^*_{L_i}(t)-H_Lz^*_L(t)-g_L\|^2\mathrm{d}t\right)^{\frac12}\\
&\times\left(\int_0^T\left\|z^*_L(t)-x^{*,(N_L)}_L(t)\right\|^2\mathrm{d}t\right)^{\frac12}\Bigg)^2\Bigg]\\
\leq&2\mathbb{E}\left[\|Q_L\|^2\|H_L\|^4\left(\int_0^T\left\|z^*_L(t)-x^{*,(N_L)}_L(t)\right\|^2\mathrm{d}t\right)^2\right]\\
&+8\mathbb{E}\left[\|Q_L\|^2\|H_L\|^2\int_0^T\left\|x^*_{L_i}(t)-H_Lz^*_L(t)-g_L\right\|^2\mathrm{d}t\int_0^T\left\|z^*_L(t)-x^{*,(N_L)}_L(t)\right\|^2\mathrm{d}t\right]\\
\leq&2\mathbb{E}\left[\left(\int_0^T\left\|z^*_L(t)-x^{*,(N_L)}_L(t)\right\|^2\mathrm{d}t\right)^2\right]\|Q_L\|^2\|H_L\|^4+8\|Q_L\|^2\|H_L\|^2\\
&\times\left\{\mathbb{E}\left[\left(\int_0^T\|x^*_{L_i}(t)-H_Lz^*_L(t)-g_L\|^2\mathrm{d}t\right)^2\right]\right\}^{\frac12}\left\{\mathbb{E}\left[\left(\int_0^T\left\|z^*_L(t)-x^{*,(N_L)}_L(t)\right\|^2\mathrm{d}t\right)^2\right]\right\}^{\frac12}\\
=&O\left(\epsilon^4_{N_L}+\frac{1}{N_L^2}\right)+O\left(\epsilon^2_{N_L}+\frac{1}{N_L}\right),
\end{aligned}$$

which together with C_r inequality leads to

$$\left\{\mathbb{E}\left[\left|\Phi_L\left(x^*_{L_i}(\cdot),x^{*,(N_L)}_L(\cdot),u^*_{L_i}(\cdot)\right)-\Phi_L\left(x^*_{L_i}(\cdot),z^*_L(\cdot),u^*_{L_i}(\cdot)\right)\right|^2\right]\right\}^{\frac{1}{2}}\le O\left(\epsilon_{N_L}+\frac{1}{\sqrt{N_L}}\right). \tag{5.11}$$

By (5.9)–(5.11) we obtain (5.7). Similar to the proof of (5.7), we can get (5.8). □

Remark 5.3. Lemma 5.2 proves the vanishing of the differences between the costs based on the decentralized strategies and those of Auxiliary problems (I) and (II) as the players increase, which is an important link between the consistency of estimations for the state average terms and the equilibrium property of the decentralized strategies.

Theorem 5.4. *For the systems (2.1)–(2.4), if Assumptions 2.2, 2.3, 2.4, 2.5, 3.8 and 3.11 hold, then*

$$J_{L_i}\left(u^*_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\right)\le\inf_{u_{L_i}\in\mathcal{U}^g_{L_i}}J_{L_i}\left(u_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\right)+O\left(\epsilon_{N_L}+\frac{1}{\sqrt{N_L}}\right),\quad i=1,2,\dots,N_L, \tag{5.12}$$

$$\begin{aligned}J_{F_j}\left(u^*_{F_j}(\cdot),u^*_{F_{-j}}(\cdot),u^*_L(\cdot)\right)\le&\inf_{u_{F_j}\in\mathcal{U}^g_{F_j}}J_{F_j}\left(u_{F_j}(\cdot),u^*_{F_{-j}}(\cdot),u^*_L(\cdot)\right)+O\left(\epsilon_{N_L}+\frac{1}{\sqrt{N_L}}\right)\\&+O\left(\epsilon_{N_F}+\frac{1}{\sqrt{N_F}}\right),\quad j=1,2,\dots,N_F,\end{aligned} \tag{5.13}$$

i.e. the sequence of $\{\{u^*_{L_i}(\cdot),u^*_{F_j}(\cdot),i=1,2,\dots,N_L,j=1,2,\dots,N_F\},\ N_L=1,2,\dots,N_F=1,2,\cdots\}$ *is a decentralized asymptotic Nash equilibrium with respect to* $\{\{J_{L_i},\ J_{F_j},\ i=1,2,\dots,N_L,\ j=1,2,\cdots,N_F\},\ N_L=1,2,\cdots,N_F=1,2,\cdots\}$.

Proof. See Appendix D. □

6. Numerical example

Consider the scalar case of (2.1)–(2.4), where $A_L(\theta_{L_i})=\theta_{L_i}$ and $A_F(\theta_{F_j})=\theta_{F_j}$ are uniformly distributed on the intervals $[\underline{\theta_L},\overline{\theta_L}]$ and $[\underline{\theta_F},\overline{\theta_F}]$ respectively, $B_L(\theta_{L_i})=B_L$, $D_L(\theta_{L_i})=D_L$, $B_F(\theta_{F_j})=B_F$, $D_F(\theta_{F_j})=D_F$, the initial states $x_{L_i}(0)=x_L(0)$, $x_{F_j}(0)=x_F(0)$, $i=1,2,\cdots,N_L$ and $j=1,2,\dots,N_F$. At the same time, $\left|\frac{T^2}{2R_L}B_L^2Q_LH_L\right|\int_{\underline{\theta_L}}^{\overline{\theta_L}}\exp\{\max\{2T\theta_L,0\}\}\mathrm{d}\mathbf{F}_L(\theta_L)<1$ and $\left|\frac{T^2}{2R_F}B_F^2Q_FH_F\right|\int_{\underline{\theta_F}}^{\overline{\theta_F}}\exp\{\max\{2T\theta_F,0\}\}\mathrm{d}\mathbf{F}_F(\theta_F)<1$, where $\mathbf{F}_L(\theta_L)$ and $\mathbf{F}_F(\theta_F)$ are uniform distribution functions on $[\underline{\theta_L},\overline{\theta_L}]$ and $[\underline{\theta_F},\overline{\theta_F}]$ respectively.

By (2.5), (3.18) and (3.19), we have

$$\begin{aligned}P_{\gamma,\theta_{L_i}}(t)=&Q_L\Big\{\exp[\mu_{\gamma,L_i}(T-t)]-\exp[\mu_{\gamma,L_i}(t-T)]\Big\}\Big/\Big\{(\mu_{\gamma,L_i}+\theta_{L_i})\exp[\mu_{\gamma,L_i}(t-T)]\\&+(\mu_{\gamma,L_i}-\theta_{L_i})\exp[\mu_{\gamma,L_i}(T-t)]\Big\},\quad 0\le t\le T,\\\psi_{\gamma,\theta_{L_i}}(t,s_1)=&\exp\left(-\int_{s_1}^t\left(\theta_{L_i}-\frac{1}{R_L}P_{\gamma,\theta_{L_i}}(s)B_L^2+\frac{2}{\gamma}P_{\gamma,\theta_{L_i}}(s)D_L^2\right)\mathrm{d}s\right),\\\phi_{\gamma,\theta_{L_i}}(t,s_2)=&\exp\left(\int_{s_2}^t\left(\theta_{L_i}-\frac{1}{R_L}P_{\gamma,\theta_{L_i}}(s_1)B_L^2\right)\mathrm{d}s_1\right),\end{aligned}$$

where $\mu_{\gamma,L_i}=\sqrt{\theta_{L_i}^2+(\frac{1}{R_L}B_L^2-\frac{2}{\gamma}D_L^2)Q_L}$. By (2.6), (3.29) and (3.30), we have

$$P_{\gamma,\theta_{F_j}}(t)=Q_F\Big\{\exp\{\mu_{\gamma,F_j}(T-t)\}-\exp[\mu_{\gamma,F_j}(t-T)]\Big\}\Big/\Big\{(\mu_{\gamma,F_j}+\theta_{F_j})\exp[\mu_{\gamma,F_j}(t-T)]$$

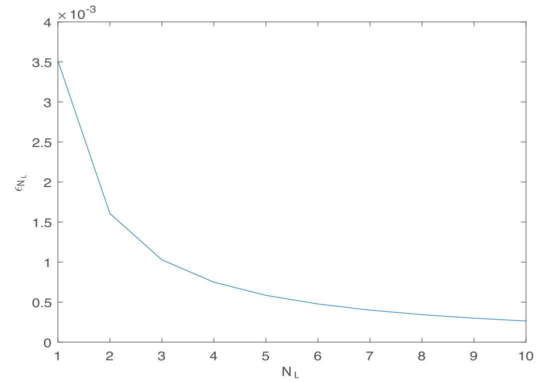

FIGURE 1. The curve of ϵ_{N_L} with respect to N_L.

$$
\begin{aligned}
&+ (\mu_{\gamma,F_j} - \theta_{F_j}) \exp[\mu_{\gamma,F_j}(T-t)]\Big\}, \quad 0 \le t \le T, \\
\psi_{\gamma,\theta_{F_j}}(t,s_1) &= \exp\left(- \int_{s_1}^{t} \left(\theta_{F_j} - \frac{1}{R_F} P_{\gamma,\theta_{F_j}}(s) B_F^2 + \frac{2}{\gamma} P_{\gamma,\theta_{F_j}}(s) D_F^2 \right) \mathrm{d}s \right), \\
\phi_{\gamma,\theta_{F_j}}(t,s_2) &= \exp\left(\int_{s_2}^{t} \left(\theta_{F_j} - \frac{1}{R_F} P_{\gamma,\theta_{F_j}}(s_1) B_F^2 \right) \mathrm{d}s_1 \right),
\end{aligned}
$$

where $\mu_{\gamma,F_j} = \sqrt{\theta_{F_j}^2 + \left(\frac{1}{R_F}B_F^2 - \frac{2}{\gamma}D_F^2\right)Q_F}$. Since $P_{\gamma,\theta_L}(t) \ge 0$ and $P_{\gamma,\theta_F}(t) \ge 0$ for $t \in [0,T]$, we have $\theta_L - \frac{1}{R_L}P_{\gamma,\theta_L}(t)B_L^2 \le \theta_L$ and $\theta_F - \frac{1}{R_F}P_{\gamma,\theta_F}(t)B_F^2 \le \theta_F$. From Assumption 2.4, we know $\theta_L - \frac{1}{R_L}P_{\gamma,\theta_L}(t)B_L^2 + \frac{2}{\gamma}P_{\gamma,\theta_L}(t)D_L^2 \le \theta_L$ and $\theta_F - \frac{1}{R_F}P_{\gamma,\theta_F}(t)B_F^2 + \frac{2}{\gamma}P_{\gamma,\theta_F}(t)D_F^2 \le \theta_F$ for $t \in [0,T]$. Therefore,

$$
\begin{aligned}
&\left|\frac{1}{R_L}B_L^2Q_LH_L\right| \int_{\underline{\theta_L}}^{\overline{\theta_L}} \left\| \int_0^t \int_{s_2}^T \exp\left\{ \int_{s_2}^t \left[\theta_L - \frac{1}{R_L}P_{\gamma,\theta_L}(s_3)B_L^2 \right] \mathrm{d}s_3 \right\} \exp\left\{ - \int_{s_1}^{s_2} \left[\theta_L - \frac{1}{R_L}P_{\gamma,\theta_L}(s_4)B_L^2 \right.\right.\right. \\
&\left.\left.\left. + \frac{2}{\gamma}P_{\gamma,\theta_L}(s_4)D_L^2 \right] ds_4 \right\} \mathrm{d}s_1 \mathrm{d}s_2 \right\|_\infty \mathrm{d}\mathbf{F}(\theta_L) \le \left| \frac{T^2}{2R_L}B_L^2Q_LH_L \right| \int_{\underline{\theta_L}}^{\overline{\theta_L}} \exp\{\max\{2T\theta_L, 0\}\} \mathrm{d}\mathbf{F}(\theta_L) < 1, \\
&\left|\frac{1}{R_F}B_F^2Q_FH_F\right| \int_{\underline{\theta_F}}^{\overline{\theta_F}} \left\| \int_0^t \int_{s_2}^T \exp\left\{ \int_{s_2}^t \left[\theta_F - \frac{1}{R_F}P_{\gamma,\theta_F}(s_3)B_F^2 \right] \mathrm{d}s_3 \right\} \exp\left\{ - \int_{s_1}^{s_2} \left[\theta_F - \frac{1}{R_F}P_{\gamma,\theta_F}(s_4)B_F^2 \right.\right.\right. \\
&\left.\left.\left. + \frac{2}{\gamma}P_{\gamma,\theta_F}(s_4)D_F^2 \right] ds_4 \right\} \mathrm{d}s_1 \mathrm{d}s_2 \right\|_\infty \mathrm{d}\mathbf{F}(\theta_F) \le \left| \frac{T^2}{2R_F}B_F^2Q_FH_F \right| \int_{\underline{\theta_F}}^{\overline{\theta_F}} \exp\{\max\{2T\theta_F, 0\}\} \mathrm{d}\mathbf{F}(\theta_F) < 1.
\end{aligned}
$$

From the above we know that Assumptions 3.8 and 3.11 hold.

For the systems (2.1)–(2.4), we take parameters $[\underline{\theta_L}, \overline{\theta_L}, B_L, D_L, Q_L, R_L, H_L, g_L] = [0.3, 0.6, 1, 0.1, 1, 2.5, 0.8, 0.5]$, $[\underline{\theta_F}, \overline{\theta_F}, B_F, D_F, Q_F, R_F, H_F, H, g_F] = [0.2, 0.5, 1, 0.05, 1, 2, 0.5, 0.3, 0.5]$, the initial states $x_L(0) = 0$ and $x_F(0) = 0$, $T = 1$h and $\gamma = 40$. It can be proved that Assumptions 2.2, 2.3, 2.4, 2.5, 3.8 and 3.11 hold.

We take a step size of 0.01 to discretize t, and a step size of 0.005 to discretize θ_L and θ_F. By using Banach's iterative method of step-by-step, the numerical solutions of $z_L^*(t)$ and $z_F^*(t)$ are obtained. Figure 1 and 2 show the curves of ϵ_{N_L} and ϵ_{N_F} respectively. It is shown that both ϵ_{N_L} and ϵ_{N_F} vanish as N_L and N_F increase.

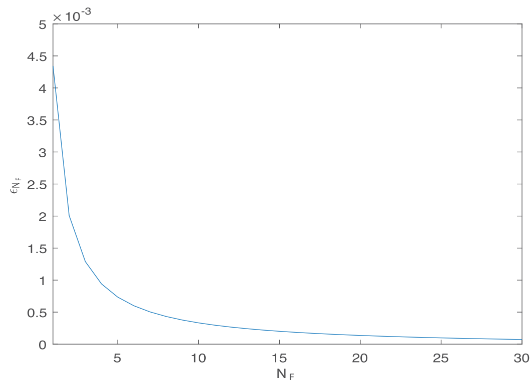

FIGURE 2. The curve of ϵ_{N_F} with respect to N_F.

7. Conclusion and future research

In this paper, we studied a class of risk-sensitive mean field games. The systems considered contain a large number of major and minor players. The state average term of the major players has influence on the major and minor players, and the state average term of the minor players only has influence on the minor players. We constructed the estimations of the state average terms for the major and minor players by the two-layer state aggregation method, and gave the conditions for the existence and uniqueness of the fixed points. Then, we designed a sequence of decentralized strategies. We proved that the estimations of the state average terms are consistent with their true values for the closed-loop systems, and the sequence of strategies designed is a decentralized asymptotic Nash equilibrium.

The control inputs we considered are unconstrained. There have been some works on optimal control and mean field games with input constraints [14–16]. So far, the works on the mean field games with control inputs constraints have assumed that there are finite classes of minor players [14, 15]. The method used in [14, 15] is to obtain the consistency condition of the mean-field system by determining the existence condition of the solution for a set of (finite) coupled forward-backward stochastic differential equations with projection operators. Different from [14, 15], there is a continuum of minor players in the systems (2.1)–(2.4). For the system with a continuum of minor players, the problem of risk-sensitive mean field games with control inputs constraints can not be solved by the existing method. This problem deserves further study in the future.

Appendix A. Proofs in Section 2

For the system (2.1)–(2.4), under Assumptions 2.2 and 2.4, for each $\theta_L \in \Theta_L$, let $\phi_{\gamma,\theta_L}(t,s)$, $0 \le t, s \le T$, be the solution of

$$\left\{ \begin{array}{l} \mathrm{d}\phi_{\gamma,\theta_L}(t,s) = \left[A_L(\theta_L) - B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)P_{\gamma,\theta_L}(t)\right]\phi_{\gamma,\theta_L}(t,s)\mathrm{d}t, \\ \phi_{\gamma,\theta_L}(s,s) = I_n, \end{array} \right. \tag{A.1}$$

and $\psi_{\gamma,\theta_L}(t,s)$, $0 \le t, s \le T$, be the solution of

$$\left\{ \begin{array}{l} \mathrm{d}\psi_{\gamma,\theta_L}(t,s) = -\left[A_L(\theta_L) - B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)P_{\gamma,\theta_L}(t) + \frac{2}{\gamma}D_L(\theta_L)D_L^\top(\theta_L)P_{\gamma,\theta_L}(t)\right]^\top \psi_{\gamma,\theta_L}(t,s)\mathrm{d}t, \\ \psi_{\gamma,\theta_L}(s,s) = I_n, \end{array} \right. \tag{A.2}$$

and for each $\theta_F \in \Theta_F$, let $\phi_{\gamma,\theta_F}(t,s)$, $0 \le t,s \le T$, be the solution of

$$\begin{cases} \mathrm{d}\phi_{\gamma,\theta_F}(t,s) = \left[A_F(\theta_F) - B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)P_{\gamma,\theta_F}(t)\right]\phi_{\gamma,\theta_F}(t,s)\mathrm{d}t, \\ \phi_{\gamma,\theta_F}(s,s) = I_n, \end{cases} \tag{A.3}$$

and $\psi_{\gamma,\theta_F}(t,s)$, $0 \le t,s \le T$, be the solution of

$$\begin{cases} \mathrm{d}\psi_{\gamma,\theta_F}(t,s) = -\Big[A_F(\theta_F) - B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)P_{\gamma,\theta_F}(t) \\ \qquad\qquad + \frac{2}{\gamma}D_F(\theta_F)D_F^\top(\theta_F)P_{\gamma,\theta_F}(t)\Big]^\top \psi_{\gamma,\theta_F}(t,s)\mathrm{d}t, \\ \psi_{\gamma,\theta_F}(s,s) = I_n. \end{cases} \tag{A.4}$$

Then we have the following lemmas.

Lemma A.1. *For the systems (A.1)–(A.4), if Assumptions 2.2 and 2.4 hold, then there exists a constant C_1 such that*

$$\sup_{\theta_L\in\Theta_L}\sup_{\theta_F\in\Theta_F}\sup_{0\le t,s\le T}\max\Bigg\{\left\|\frac{\partial\phi_{\gamma,\theta_L}(t,s)}{\partial t}\right\|, \left\|\frac{\partial\phi_{\gamma,\theta_L}(t,s)}{\partial s}\right\|, \left\|\frac{\partial\phi_{\gamma,\theta_F}(t,s)}{\partial t}\right\|, \left\|\frac{\partial\phi_{\gamma,\theta_F}(t,s)}{\partial s}\right\|, \\ \|\phi_{\gamma,\theta_L}(t,s)\|, \|\psi_{\gamma,\theta_L}(t,s)\|, \|\phi_{\gamma,\theta_F}(t,s)\|, \|\psi_{\gamma,\theta_F}(t,s)\|\Bigg\} \le C_1.$$

Proof. Firstly, by Bernoulli replacement method [40], we transform (2.5) into the following linear form

$$\begin{cases} \dot H_{\gamma,\theta_L}(t) = \Big[A_L(\theta_L) - \Big(B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L) - \frac{2}{\gamma}D_L(\theta_L)D_L^\top(\theta_L)\Big)P_{\gamma,\theta_L}(t)\Big]H_{\gamma,\theta_L}(t), \\ H_{\gamma,\theta_L}(T) = I_n, \\ K_{\gamma,\theta_L}(t) = P_{\gamma,\theta_L}(t)H_{\gamma,\theta_L}(t), \\ K_{\gamma,\theta_L}(T) = 0. \end{cases}$$

Let $\overline{\phi}_{\gamma,\theta_L}(t,s)$, $0 \le t,s \le T$ be the solution of

$$\begin{cases} \mathrm{d}\overline{\phi}_{\gamma,\theta_L}(t,s) = \Big[A_L(\theta_L) - \Big(B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L) - \frac{2}{\gamma}D_L(\theta_L)D_L^\top(\theta_L)\Big)P_{\gamma,\theta_L}(t)\Big]\overline{\phi}_{\gamma,\theta_L}(t,s)\mathrm{d}t, \\ \overline{\phi}_{\gamma,\theta_L}(s,s) = I_n. \end{cases} \tag{A.5}$$

For any given $(t,\theta_L) \in [0,T]\times\Theta_L$, by (A.5), we have $H_{\gamma,\theta_L}(t) = \overline{\phi}_{\gamma,\theta_L}(t,T)H_{\gamma,\theta_L}(T) = \overline{\phi}_{\gamma,\theta_L}(t,T)$. Thus, the existence of $\left(\overline{\phi}_{\gamma,\theta_L}\right)^{-1}(t,T)$ implies that $H^{-1}_{\gamma,\theta_L}(t)$ exists. We know that $(H_{\gamma,\theta_L}(t), K_{\gamma,\theta_L}(t))$ satisfies the equation

$$\begin{bmatrix}\dot H_{\gamma,\theta_L}(t)\\ \dot K_{\gamma,\theta_L}(t)\end{bmatrix} = \begin{bmatrix} A_L(\theta_L) & -\Big(B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L) - \frac{2}{\gamma}D_L(\theta_L)D_L^\top(\theta_L)\Big) \\ -Q_L & -A_L^\top(\theta_L)\end{bmatrix}\begin{bmatrix}H_{\gamma,\theta_L}(t)\\ K_{\gamma,\theta_L}(t)\end{bmatrix}, \qquad \begin{bmatrix}H_{\gamma,\theta_L}(T)\\ K_{\gamma,\theta_L}(T)\end{bmatrix} = \begin{bmatrix}I_n\\ 0\end{bmatrix}.$$

Because $A_L(\theta_L)$, $B_L(\theta_L)$ and $D_L(\theta_L)$ are continuous with respect to $\theta_L \in \Theta_L$, therefore, $H_{\gamma,\theta_L}(t)$ and $K_{\gamma,\theta_L}(t)$ are continuous with respect to $(t,\theta_L) \in [0,T]\times\Theta_L$. This together with the definition of the inverse matrix implies that $H^{-1}_{\gamma,\theta_L}(t)$ is continuous with respect to $(t,\theta_L) \in [0,T]\times\Theta_L$, then $P_{\gamma,\theta_L}(t) = K_{\gamma,\theta_L}(t)H^{-1}_{\gamma,\theta_L}(t)$ is continuous with respect to $(t,\theta_L) \in [0,T]\times\Theta_L$. Next, by the compactness of Θ_L, we can find a constant C_1' such that $\sup_{\theta_L\in\Theta_L}\sup_{0\le t\le T}\|P_{\gamma,\theta_L}(t)\| \le C_1'$, which together with (A.1) yields

$$\sup_{\theta_L\in\Theta_L}\sup_{0\le t,s\le T}\max\left\{\left\|\frac{\partial\phi_{\gamma,\theta_L}(t,s)}{\partial t}\right\|, \|\phi_{\gamma,\theta_L}(t,s)\|\right\} < \infty. \tag{A.6}$$

Similar to the proof of (A.6), we have

$$\max\left\{\sup_{\theta_L\in\Theta_L}\sup_{0\le t,s\le T}\|\psi_{\gamma,\theta_L}(t,s)\|\right\}<\infty. \tag{A.7}$$

Noting that $\mathrm{d}\phi_{\gamma,\theta_L}(t,s)=-\phi_{\gamma,\theta_L}(t,s)\left[A_L(\theta_L)-B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)P_{\gamma,\theta_L}(t)\right]\mathrm{d}s$. Then similar to the proof of (A.6), we have

$$\sup_{\theta_L\in\Theta_L}\sup_{0\le t,s\le T}\max\left\{\left\|\frac{\partial\phi_{\gamma,\theta_L}(t,s)}{\partial s}\right\|\right\}<\infty. \tag{A.8}$$

Combining (A.6)–(A.8), we obtain

$$\sup_{\theta_L\in\Theta_L}\sup_{0\le t,s\le T}\max\left\{\left\|\frac{\partial\phi_{\gamma,\theta_L}(t,s)}{\partial t}\right\|,\left\|\frac{\partial\phi_{\gamma,\theta_L}(t,s)}{\partial s}\right\|,\|\phi_{\gamma,\theta_L}(t,s)\|,\|\psi_{\gamma,\theta_L}(t,s)\|\right\}<\infty. \tag{A.9}$$

Similar to the proof of (A.9), we get

$$\sup_{\theta_F\in\Theta_F}\sup_{0\le t,s\le T}\max\left\{\left\|\frac{\partial\phi_{\gamma,\theta_F}(t,s)}{\partial t}\right\|,\left\|\frac{\partial\phi_{\gamma,\theta_F}(t,s)}{\partial s}\right\|,\|\phi_{\gamma,\theta_F}(t,s)\|,\|\psi_{\gamma,\theta_F}(t,s)\|\right\}<\infty.$$

Combining the above inequality with (A.9), we know that there exists a constant C_1 such that Lemma A.1 holds. □

Lemma A.2. *Let*

$$\overline{\gamma}_T=\inf\{\gamma\mid\gamma>\overline{S}(\gamma)\}, \tag{A.10}$$

where

$$\begin{aligned}
\overline{S}(\gamma)=\sup_{0\le t,s\le T}\sup_{\theta_L\in\Theta_L}\sup_{\theta_F\in\Theta_F}\max\Bigg\{&1280\|H^\top Q_FH\|\left\|\frac{\partial\phi_{\gamma,\theta_L}(t,s)}{\partial s}\right\|^2\|D_L(\theta_L)\|^2\max\left\{T^4,1\right\},\\
&1280\|H^\top Q_FH\|\,\|\phi_{\gamma,\theta_L}(t,s)\|^2\,\|D_L(\theta_L)\|^2\max\left\{T^4,1\right\},\\
&1024\Big(\|Q_L\|+\|H_L^\top Q_LH_L\|+\big(M_{\gamma,L}(\theta_L)\big)^2\|B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\|\Big)\left\|\frac{\partial\phi_{\gamma,\theta_L}(t,s)}{\partial s}\right\|^2\|D_L(\theta_L)\|^2\max\left\{T^4,1\right\},\\
&1024\Big(\|Q_L\|+\|H_L^\top Q_LH_L\|+\big(M_{\gamma,L}(\theta_L)\big)^2\|B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\|\Big)\|\phi_{\gamma,\theta_L}(t,s)\|^2\,\|D_L(\theta_L)\|^2\max\left\{T^4,1\right\},\\
&1280\Big(\|Q_F\|+\|H_F^\top Q_FH_F\|+\big(M_{\gamma,F}(\theta_F)\big)^2\|B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)\|\Big)\left\|\frac{\partial\phi_{\gamma,\theta_F}(t,s)}{\partial s}\right\|^2\|D_F(\theta_F)\|^2\max\left\{T^4,1\right\},\\
&1280\Big(\|Q_F\|+\|H_F^\top Q_FH_F\|+\big(M_{\gamma,F}(\theta_F)\big)^2\|B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)\|\Big)\|\phi_{\gamma,\theta_F}(t,s)\|^2\,\|D_F(\theta_F)\|^2\max\left\{T^4,1\right\},\\
&48\alpha M_{\gamma,L}(\theta_L)\left\|\frac{\partial\phi_{\gamma,\theta_L}(t,s)}{\partial s}\right\|^2\|D_L(\theta_L)\|^2\max\left\{T^4,1\right\},\ 96\alpha M_{\gamma,L}(\theta_L)\|\|\phi_{\gamma,\theta_L}(t,s)\|^2\,\|D_L(\theta_L)\|^2\max\{T^4,1\},\\
&48\alpha M_{\gamma,F}(\theta_F)\left\|\frac{\partial\phi_{\gamma,\theta_F}(t,s)}{\mathrm{d}s}\right\|^2\|D_F(\theta_F)\|^2\max\left\{T^4,1\right\},\ 96\alpha M_{\gamma,F}(\theta_F)\|\|\phi_{\gamma,\theta_F}(t,s)\|^2\,\|D_F(\theta_F)\|^2\max\left\{T^4,1\right\}\Bigg\}.
\end{aligned}$$

Then we have $\overline{\gamma}_T<\infty$.

Proof. We know that the generalized matrix Riccati differential equations (2.5) and (2.6) with $\gamma \to \infty$ are given by

$$\begin{cases} \dot{P}_{\theta_L}(t) = -P_{\theta_L}(t)A_L(\theta_L) - A_L^\top(\theta_L)P_{\theta_L}(t) + P_{\theta_L}(t)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)P_{\theta_L}(t) - Q_L, \\ P_{\theta_L}(T) = 0, \end{cases}$$

and

$$\begin{cases} \dot{P}_{\theta_F}(t) = -P_{\theta_F}(t)A_F(\theta_F) - A_F^\top(\theta_F)P_{\theta_F}(t) + P_{\theta_F}(t)B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)P_{\theta_F}(t) - Q_F, \\ P_{\theta_F}(T) = 0. \end{cases}$$

The state transition matrices (A.1) and (A.3) with $\gamma \to \infty$ are given by

$$\begin{cases} \mathrm{d}\phi_{\theta_L}(t,s) = [A_L(\theta_L) - B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)P_{\theta_L}(t)]\phi_{\theta_L}(t,s)\mathrm{d}t, \\ \phi_{\theta_L}(s,s) = I_n, \end{cases}$$

and

$$\begin{cases} \mathrm{d}\phi_{\theta_F}(t,s) = [A_F(\theta_F) - B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)P_{\theta_F}(t)]\phi_{\theta_F}(t,s)\mathrm{d}t, \\ \phi_{\theta_F}(s,s) = I_n, \end{cases}$$

which leads to

$$\begin{cases} \frac{\partial \phi_{\theta_L}(t,s)}{\partial s} = -\phi_{\theta_L}(t,s)[A_L(\theta_L) - B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)P_{\theta_L}(s)], \\ \phi_{\theta_L}(s,s) = I_n, \end{cases} \tag{A.11}$$

and

$$\begin{cases} \frac{\partial \phi_{\theta_F}(t,s)}{\partial s} = -\phi_{\theta_F}(t,s)[A_F(\theta_F) - B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)P_{\theta_F}(s)], \\ \phi_{\theta_F}(s,s) = I_n. \end{cases} \tag{A.12}$$

Similar to the proof of Lemma A.1, by Bernoulli replacement method [40], we know that $P_{\theta_L}(t)$ and $P_{\theta_F}(t)$ are continuous with respect to $(t,\theta_L) \in [0,T] \times \Theta_L$ and $(t,\theta_F) \in [0,T] \times \Theta_F$ respectively. Combining (A.11), (A.12) and the fact of the continuity of $P_{\theta_L}(t)$ and $P_{\theta_F}(t)$, we know that $\frac{\partial \phi_{\theta_L}(t,s)}{\partial s}$ and $\frac{\partial \phi_{\theta_F}(t,s)}{\partial s}$ are continuous with respect to $(t,s,\theta_L) \in [0,T] \times [0,T] \times \Theta_L$ and $(t,s,\theta_F) \in [0,T] \times [0,T] \times \Theta_F$ respectively, which together with the compactness of Θ_L and Θ_F yields

$$\begin{aligned} \lim_{\gamma \to \infty} \overline{S}(\gamma) = \sup_{0 \le t,s \le T} \sup_{\theta_L \in \Theta_L} \sup_{\theta_F \in \Theta_F} \max \Bigg\{ & 1280 \|H^\top Q_F H\| \left\| \frac{\partial \phi_{\theta_L}(t,s)}{\partial s} \right\|^2 \|D_L(\theta_L)\|^2 \max\left\{T^4, 1\right\}, \\ & 1280\|H^\top Q_F H\| \, \|\phi_{\theta_L}(t,s)\|^2 \, \|D_L(\theta_L)\|^2 \max\{T^4,1\}, \\ & 1024\Big(\|Q_L\| + \|H_L^\top Q_L H_L\| + \big(M_L(\theta_L)\big)^2 \|B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\|\Big) \left\| \frac{\partial \phi_{\theta_L}(t,s)}{\partial s} \right\|^2 \|D_L(\theta_L)\|^2 \max\left\{T^4,1\right\}, \\ & 1024\Big(\|Q_L\| + \|H_L^\top Q_L H_L\| + \big(M_L(\theta_L)\big)^2 \|B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\|\Big) \|\phi_{\theta_L}(t,s)\|^2 \, \|D_L(\theta_L)\|^2 \max\left\{T^4,1\right\}, \\ & 1280\Big(\|Q_F\| + \|H_F^\top Q_F H_F\| + \big(M_F(\theta_F)\big)^2 \|B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)\|\Big) \left\| \frac{\partial \phi_{\theta_F}(t,s)}{\partial s} \right\|^2 \|D_F(\theta_F)\|^2 \max\left\{T^4,1\right\}, \\ & 1280\Big(\|Q_F\| + \|H_F^\top Q_F H_F\| + \big(M_F(\theta_F)\big)^2 \|B_F(\theta_F)R_F^{-1}B_F^\top(\theta_F)\|\Big) \|\phi_{\theta_F}(t,s)\|^2 \, \|D_F(\theta_F)\|^2 \max\left\{T^4,1\right\}, \end{aligned}$$

$$48\alpha M_L(\theta_L)\left\|\frac{\partial \phi_{\theta_L}(t,s)}{\partial s}\right\|^2 \|D_L(\theta_L)\|^2 \max\left\{T^4,1\right\},\ 96\alpha M_L(\theta_L)\|\|\phi_{\theta_L}(t,s)\|^2 \|D_L(\theta_L)\|^2 \max\left\{T^4,1\right\},$$
$$48\alpha M_F(\theta_F)\left\|\frac{\partial \phi_{\theta_F}(t,s)}{\mathrm{d}s}\right\|^2 \|D_F(\theta_F)\|^2 \max\left\{T^4,1\right\},\ 96\alpha M_F(\theta_F)\|\|\phi_{\theta_F}(t,s)\|^2 \|D_F(\theta_F)\|^2 \max\left\{T^4,1\right\}\Bigg\},$$

where $M_L(\theta_L) = \sup_{0\le t\le T} P_{\theta_L}(t)$ and $M_F(\theta_F) = \sup_{0\le t\le T} P_{\theta_F}(t)$. Thus, there exist constants $N > 0$ and $\gamma^* > 0$, such that $\overline{S}(\gamma) < N$ for all $\gamma > \gamma^*$. Taking $\gamma = \max\{\gamma^*, N\}$, which together with (A.10) leads to $\overline{\gamma}_T < \infty$. This completes the proof. □

Appendix B. Proofs in Section 3

Proof of Theorem 3.1. Firstly, we prove (a). For each $(\tau, x) \in [0,T]\times\mathbb{R}^n$ and $v \in \mathbb{R}^m$, by (3.6) we know

$$-\frac{\partial G(\tau,x)}{\partial \tau} \le \frac{\partial G(\tau,x)}{\partial x}(Ax+Bv) + \frac{1}{\gamma}\left[\|x-y(\tau)\|_Q^2 + \|v\|_R^2\right]G(\tau,x) + \frac{1}{2}\operatorname{Tr}\left(\frac{\partial^2 G(\tau,x)}{\partial x^2}DD^\top\right),$$

which implies

$$\frac{1}{\gamma}\left[\|x-y(\tau)\|_Q^2 + \|v\|_R^2\right]G(\tau,x) + \frac{\partial G(\tau,x)}{\partial \tau} + \frac{\partial G(\tau,x)}{\partial x}(Ax+Bv) + \frac{1}{2}\operatorname{Tr}\left(\frac{\partial^2 G(\tau,x)}{\partial x^2}DD^\top\right) \ge 0. \tag{B.1}$$

Let $t \in [\tau, T]$ and $\bar{x}(\cdot)$ be the unique solution of system (3.1) under $\bar{v}(\cdot) \in \mathcal{U}_\tau$. Replacing τ, x and v by $t, \bar{x}(t)$ and $\bar{v}(t)$ in (B.1), we have

$$\begin{aligned}&\frac{1}{\gamma}\left[\|\bar{x}(t)-y(t)\|_Q^2 + \|\bar{v}(t)\|_R^2\right]G(t,\bar{x}(t)) + \frac{\partial G(\tau,x)}{\partial \tau}\Big|_{\tau=t,x=\bar{x}(t)} + \frac{\partial G(\tau,x)}{\partial x}\Big|_{\tau=t,x=\bar{x}(t)}(A\bar{x}(t)+B\bar{v}(t))\\&+\frac{1}{2}\operatorname{Tr}\left(\frac{\partial^2 G(\tau,x)}{\partial x^2}\Big|_{\tau=t,x=\bar{x}(t)}DD^\top\right) \ge 0.\end{aligned} \tag{B.2}$$

For the term $\exp\left[\frac{1}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2 + \|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right]G(t,\bar{x}(t))$, by Itô formula, we obtain

$$\begin{aligned}&\mathrm{d}\left[\exp\left(\frac{1}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2 + \|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)G(t,\bar{x}(t))\right]\\
=&\exp\left(\frac{1}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2 + \|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\mathrm{d}G(t,\bar{x}(t))\\
&+G(t,\bar{x}(t))\mathrm{d}\left[\exp\left(\frac{1}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2 + \|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\right].\end{aligned} \tag{B.3}$$

By integrating the both sides of (B.3) from τ to T, we have

$$\begin{aligned}&\exp\left(\frac{1}{\gamma}\int_\tau^T \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2 + \|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)G(T,\bar{x}(T)) - G(\tau,x)\\
=&\int_\tau^T \exp\left(\frac{1}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2 + \|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\left[\frac{1}{\gamma}\left(\|\bar{x}(t)-y(t)\|_Q^2 + \|\bar{v}(t)\|_R^2\right)G(t,\bar{x}(t))\right.\\
&\left.+\frac{\partial G(\tau,x)}{\partial \tau}\Big|_{\tau=t,x=\bar{x}(t)} + \frac{\partial G(\tau,x)}{\partial x}\Big|_{\tau=t,x=\bar{x}(t)}(A\bar{x}(t)+B\bar{v}(t)) + \frac{1}{2}\operatorname{Tr}\left(\frac{\partial^2 G(\tau,x)}{\partial x^2}\Big|_{\tau=t,x=\bar{x}(t)}DD^\top\right)\right]\mathrm{d}t\end{aligned}$$

$$+\int_\tau^T \exp\left(\frac{1}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\frac{\partial G(\tau,x)}{\partial x}\Big|_{\tau=t,x=\bar{x}(t)} D\mathrm{d}W(t). \tag{B.4}$$

Combining (B.2) with (B.4), we get

$$\begin{aligned}&\exp\left(\frac{1}{\gamma}\int_\tau^T \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)-G(\tau,x)\\ \geq&\int_\tau^T \exp\left(\frac{1}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\frac{\partial G(\tau,x)}{\partial x}\Big|_{\tau=t,x=\bar{x}(t)} D\mathrm{d}W(t).\end{aligned} \tag{B.5}$$

Next, we will prove

$$\mathbb{E}\Big[\int_\tau^T \exp\left(\frac{1}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\frac{\partial G(\tau,x)}{\partial x}\Big|_{\tau=t,x=\bar{x}(t)} D\mathrm{d}W(t)\Big|x(\tau)=x\Big]=0. \tag{B.6}$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\mathbb{E}\Big\{\int_\tau^T \Big\|\exp\left(\frac{1}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\frac{\partial G(\tau,x)}{\partial x}\Big|_{\tau=t,x=\bar{x}(t)} D\Big\|^2\mathrm{d}t\Big\}\\ \leq&\mathbb{E}\Big\{\Big[\int_\tau^T \exp\left(\frac{2}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\mathrm{d}t\Big]^{\frac{1}{2}}\\ &\times\Big[\int_\tau^T \Big\|\frac{\partial G(\tau,x)}{\partial x}\Big|_{\tau=t,x=\bar{x}(t)} D\Big\|^2\mathrm{d}t\Big]^{\frac{1}{2}}\Big\}\\ \leq&\Big\{\mathbb{E}\Big[\int_\tau^T \exp\left(\frac{2}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\mathrm{d}t\Big]\Big\}^{\frac{1}{2}}\\ &\times\Big\{\mathbb{E}\Big[\int_\tau^T \Big\|\frac{\partial G(\tau,x)}{\partial x}\Big|_{\tau=t,x=\bar{x}(t)} D\Big\|^2\mathrm{d}t\Big]\Big\}^{\frac{1}{2}}.\end{aligned} \tag{B.7}$$

By C_r inequality, we obtain

$$\begin{aligned}&\mathbb{E}\Big[\int_\tau^T \exp\left(\frac{2}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)-y(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\mathrm{d}t\Big]\\ \leq&\mathbb{E}\Big[\int_\tau^T \exp\left(\frac{4}{\gamma}\int_\tau^t \left(\|y(s_1)\|_Q^2\mathrm{d}s_1\right)\exp\left(\frac{4}{\gamma}\int_\tau^t \left(\|\bar{x}(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\mathrm{d}t\Big]\\ \leq&(T-\tau)\exp\left(\frac{4}{\gamma}\int_\tau^T \|y(s_1)\|_Q^2\mathrm{d}s_1\right)\mathbb{E}\Big[\exp\left(\frac{4}{\gamma}\int_\tau^T \left(\|\bar{x}(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\Big].\end{aligned} \tag{B.8}$$

Since $y(s_1)$ is continuous with respect to $s_1\in[0,T]$, we have

$$\exp\left(\frac{4}{\gamma}\int_\tau^T \|y(s_1)\|_Q^2\mathrm{d}s_1\right)<\infty. \tag{B.9}$$

By (3.4), we get

$$\mathbb{E}\bigg[\exp\bigg(\frac{4}{\gamma}\int_\tau^T \big(\|\bar{x}(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\big)\mathrm{d}s_1\bigg)\bigg]<\infty,$$

which together with (B.8) and (B.9) leads to

$$\mathbb{E}\bigg[\int_\tau^T \exp\bigg(\frac{2}{\gamma}\int_\tau^t \big(\|\bar{x}(s_1)-y(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\big)\mathrm{d}s_1\bigg)\mathrm{d}t\bigg]<\infty. \tag{B.10}$$

We know

$$\mathbb{E}\bigg[\int_\tau^T \bigg\|\frac{\partial G(\tau,x)}{\partial x}\bigg|_{\tau=t,x=\bar{x}(t)} D\bigg\|^2\mathrm{d}t\bigg]\le\|D\|^2\mathbb{E}\bigg[\int_\tau^T \bigg\|\frac{\partial G(\tau,x)}{\partial x}\bigg|_{\tau=t,x=\bar{x}(t)}\bigg\|^2\mathrm{d}t\bigg]. \tag{B.11}$$

For the α given by (3.4), there exists a constant $\epsilon>0$, such that $2+\epsilon\le\alpha$. Then there exists a constant $N_\epsilon>0$, such that if $\|\bar{x}(s)\|\ge N_\epsilon$, then $M_{\gamma,2}\|\bar{x}(s)\|\le\epsilon M_{\gamma,1}\|\bar{x}(s)\|^2$ and $\|\bar{x}(s)\|+1\le\|\bar{x}(s)\|^2$. For the above, by (3.7), we know that

$$\begin{aligned}
&\mathbb{E}\bigg[\int_\tau^T \bigg\|\frac{\partial G(\tau,x)}{\partial x}\bigg|_{\tau=t,x=\bar{x}(t)}\bigg\|^2\mathrm{d}s\bigg]\\
\le&C_0^2\mathbb{E}\bigg\{\int_\tau^T\bigg[\exp\bigg(\frac{2}{\gamma}\Big(M_{\gamma,1}\|\bar{x}(s)\|^2+M_2\|\bar{x}(s)\|\Big)\bigg)\bigg]\Big(\|\bar{x}(s)\|+1\Big)^2\mathrm{d}s\bigg\}\\
\le&C_0^2\mathbb{E}\bigg\{\int_\tau^T I_{\{\|\bar{x}(s)\|\ge N_\epsilon\}}\bigg[\exp\bigg(\frac{2}{\gamma}\Big(M_{\gamma,1}\|\bar{x}(s)\|^2+M_{\gamma,2}\|\bar{x}(s)\|\Big)\bigg)\bigg]\Big(\|\bar{x}(s)\|+1\Big)^2\mathrm{d}s\bigg\}\\
&+C_0^2\mathbb{E}\bigg\{\int_\tau^T I_{\{\|\bar{x}(s)\|<N_\epsilon\}}\bigg[\exp\bigg(\frac{2}{\gamma}\Big(M_{\gamma,1}\|\bar{x}(s)\|^2+M_{\gamma,2}\|\bar{x}(s)\|\Big)\bigg)\bigg]\Big(\|\bar{x}(s)\|+1\Big)^2\mathrm{d}s\bigg\}\\
\le&2C_0^2\mathbb{E}\bigg\{\int_\tau^T\bigg[\exp\bigg(\frac{2+\epsilon}{\gamma}\Big(M_{\gamma,1}\|\bar{x}(s)\|^2\|\Big)\bigg)\bigg]\Big(\|\bar{x}(s)\|^2+\|\bar{x}(s)\|+1\Big)\mathrm{d}s\bigg\}\\
&+C_0^2\mathbb{E}\bigg\{\int_\tau^T I_{\{\|\bar{x}(s)\|<N_\epsilon\}}\bigg[\exp\bigg(\frac{2}{\gamma}\Big(M_{\gamma,1}\|\bar{x}(s)\|^2+M_{\gamma,2}\|\bar{x}(s)\|\Big)\bigg)\bigg]\Big(\|\bar{x}(s)\|+1\Big)^2\mathrm{d}s\bigg\}\\
\le&4C_0^2\mathbb{E}\bigg\{\int_\tau^T\bigg[\exp\bigg(\frac{2+\epsilon}{\gamma}\Big(M_{\gamma,1}\|\bar{x}(s)\|^2\|\Big)\bigg)\bigg]\Big(\|\bar{x}(s)\|^2\Big)\mathrm{d}s\bigg\}\\
&+C_0^2\mathbb{E}\bigg\{\int_\tau^T I_{\{\|\bar{x}(s)\|<N_\epsilon\}}\bigg[\exp\bigg(\frac{2}{\gamma}\Big(M_{\gamma,1}\|\bar{x}(s)\|^2+M_{\gamma,2}\|\bar{x}(s)\|\Big)\bigg)\bigg]\Big(\|\bar{x}(s)\|+1\Big)^2\mathrm{d}s\bigg\}\\
\le&4C_0^2\mathbb{E}\bigg\{\int_\tau^T\bigg[\exp\bigg(\frac{\alpha}{\gamma}\Big(M_{\gamma,1}\|\bar{x}(s)\|^2\|\Big)\bigg)\bigg]\Big(\|\bar{x}(s)\|^2\Big)\mathrm{d}s\bigg\}\\
&+C_0^2\mathbb{E}\bigg\{\int_\tau^T I_{\{\|\bar{x}(s)\|<N_\epsilon\}}\bigg[\exp\bigg(\frac{2}{\gamma}\Big(M_{\gamma,1}\|\bar{x}(s)\|^2+M_{\gamma,2}\|\bar{x}(s)\|\Big)\bigg)\bigg]\Big(\|\bar{x}(s)\|+1\Big)^2\mathrm{d}s\bigg\}.
\end{aligned}$$

This together with (3.4) and (B.11) leads to

$$\mathbb{E}\bigg[\int_\tau^T \bigg\|\frac{\partial G(\tau,x)}{\partial x}\bigg|_{\tau=t,x=\bar{x}(t)} D\bigg\|^2\mathrm{d}t\bigg]<\infty. \tag{B.12}$$

Combining (B.7), (B.10) with (B.12), we have

$$\mathbb{E}\left[\int_{\tau}^{T}\left\|\exp\left(\frac{1}{\gamma}\int_{\tau}^{t}\left(\|\bar{x}(s_1)-y(s_1)\|_Q^2+\|\bar{v}(s_1)\|_R^2\right)\mathrm{d}s_1\right)\frac{\partial G(\tau,x)}{\partial x}\Big|_{\tau=t,x=\bar{x}(t)}D\right\|^2\mathrm{d}t\right]<\infty.$$

This together with Chapter 1, Theorem 5.9 of [33] leads to (B.6). Taking expectations on both sides of (B.5) and combining this with (B.6), we obtain

$$\mathbb{E}\left[\exp\left(\frac{1}{\gamma}\int_{\tau}^{T}\left(\|\bar{x}(t)-y(t)\|_Q^2+\|\bar{v}(t)\|_R^2\right)\mathrm{d}t\right)\Big|\bar{x}(\tau)=x\right]-G(\tau,x)\geq 0, \tag{B.13}$$

which implies

$$G(\tau,x)\leq\mathbb{E}\left[\exp\left(\frac{1}{\gamma}\int_{\tau}^{T}\left(\|\bar{x}(t)-y(t)\|_Q^2+\|\bar{v}(t)\|_R^2\right)\mathrm{d}t\right)\Big|\bar{x}(\tau)=x\right]. \tag{B.14}$$

This together with (3.3) yields $G(\tau,x)\leq L(\tau,x;\bar{v}(\cdot))$. This proves (a). If there exists $v^*(\cdot)\in\mathcal{U}_\tau$ such that (3.9) holds, then the inequalities (B.2), (B.5), (B.13) and (B.14) become equalities with $\bar{v}(\cdot)=v^*(\cdot)$, thus,

$$G(\tau,x)=L(\tau,x;v^*(\cdot)). \tag{B.15}$$

We know $\inf_{\bar{v}(\cdot)\in\mathcal{U}_\tau}L(\tau,x;\bar{v}(\cdot))\leq L(\tau,x;v^*(\cdot))$, which together with (a) and (B.15) implies $L(\tau,x;v^*(\cdot))=\inf_{\bar{v}(\cdot)\in\mathcal{U}_\tau}L(\tau,x;\bar{v}(\cdot))=G(\tau,x)$. This proves (b). □

Lemma B.1. *Let*

$$\gamma_T=\inf\{\gamma\mid\gamma>S(\gamma)\}, \tag{B.16}$$

where

$$\begin{aligned}S(\gamma)=\sup_{0\leq t,s\leq T}\max\Bigg\{&192\Big(\|Q\|+M_{\gamma,1}^2\|BR^{-1}B^\top\|\Big)\left\|\tfrac{\partial\phi_\gamma(t,s)}{\partial s}\right\|^2\|D\|^2\max\{T^4,1\},\\&384\Big(\|Q\|+M_{\gamma,1}^2\|BR^{-1}B^\top\|\Big)\|\phi_\gamma(t,s)\|^2\|D\|^2\max\{T^4,1\},\\&96\alpha M_{\gamma,1}\|\|\phi_\gamma(t,s)\|^2\|D\|^2\max\{T^4,1\},\ 48\alpha M_{\gamma,1}\left\|\tfrac{\partial\phi_\gamma(t,s)}{\partial s}\right\|^2\|D\|^2\max\{T^4,1\}\Bigg\},\end{aligned}$$

and $\phi_\gamma(t,s)$, $0\leq t,s\leq T$, is the solution of

$$\begin{cases}\mathrm{d}\phi_\gamma(t,s)=\left[A-BR^{-1}B^\top P_\gamma(t)\right]\phi_\gamma(t,s)\mathrm{d}t,\\ \phi_\gamma(s,s)=I_n.\end{cases}$$

Then we have $\gamma_T<\infty$.

Proof. The proof is similar to that of Lemma A.2. □

Proof of Theorem 3.3. Firstly, we will prove that $G(\tau,x)$ given by (3.10) is a classical solution to (3.7) in the class of $C^{1,2}([0,T]\times\mathbb{R}^n,\mathbb{R})$ satisfying (3.7).

From $BR^{-1}B^\top - \frac{2}{\gamma}DD^\top \geq 0$, we know that the system (3.5) has a unique solution. By (3.10), it is obvious that $\frac{\partial G(\tau,x)}{\partial \tau}$ and $\frac{\partial G^2(\tau,x)}{\partial x^2}$ are continuous with respect to $(\tau, x) \in [0,T] \times \mathbb{R}^n$, thus $G(\tau, x) \in C^{1,2}([0,T] \times \mathbb{R}^n, \mathbb{R})$. From (3.5), (3.8), (3.10) and (3.11), the left side of (3.6) with $V(\tau,x) = G(\tau,x)$ is given by

$$\begin{aligned}
-\frac{\partial G(\tau,x)}{\partial \tau} =& \frac{1}{\gamma} G(\tau,x) \left[-x^\top \dot{P}_\gamma(\tau) x + 2x^\top \dot{\xi}_\gamma(\tau) - \dot{\varphi}_\gamma(\tau) \right] \\
=& \frac{1}{\gamma} G(\tau,x) \left\{ x^\top \left[P_\gamma(\tau) A + A^\top P_\gamma(\tau) - P_\gamma(\tau) \left(BR^{-1}B^\top - \frac{2}{\gamma} DD^\top \right) P_\gamma(\tau) + Q \right] x \right. \\
& - 2x^\top \left[\left(A - BR^{-1}B^\top P_\gamma(\tau) + \frac{2}{\gamma} DD^\top P_\gamma(\tau) \right)^\top \xi_\gamma(\tau) + Qy(\tau) \right] \\
& \left. - \xi_\gamma^\top(\tau) BR^{-1}B^\top \xi_\gamma(\tau) + y^\top(\tau) Q y(\tau) + \frac{2}{\gamma} \xi_\gamma^\top(\tau) DD^\top \xi_\gamma(\tau) + \mathrm{Tr}\left(P_\gamma(\tau) DD^\top \right) \right\}.
\end{aligned} \tag{B.17}$$

By (3.10) we know $G(\tau, x) > 0$. This together with the right side of (3.6) yields

$$\arg\min_{v \in \mathbb{R}^m} \left\{ \frac{\partial G(\tau,x)}{\partial x}(Ax + Bv) + \frac{1}{\gamma} \left[\|x - y(\tau)\|_Q^2 + \|v\|_R^2 \right] G(\tau,x) \right\} = -\frac{1}{2} R^{-1} B^\top \left(\frac{\partial G(\tau,x)}{\partial x} \right)^\top \Big/ G(\tau,x),$$

which together with (3.10) implies that the right side of (3.6) with $V(\tau,x) = G(\tau,x)$ is given by

$$\begin{aligned}
& \min_{v \in \mathbb{R}^m} \left\{ \frac{\partial G(\tau,x)}{\partial x}(Ax + Bv) + \frac{1}{\gamma} \left[\|x - y(\tau)\|_Q^2 + \|v\|_R^2 \right] G(\tau,x) \right\} + \frac{1}{2} \mathrm{Tr} \left(\frac{\partial^2 G(\tau,x)}{\partial x^2} DD^\top \right) \\
=& \frac{\partial G(\tau,x)}{\partial x} Ax + \|x - y(\tau)\|_Q^2 - \frac{1}{4} \frac{\partial G(\tau,x)}{\partial x} BR^{-1}B^\top \left(\frac{\partial G(\tau,x)}{\partial x} \right)^\top \Big/ G(\tau,x) \\
& + \frac{1}{2} \mathrm{Tr} \left(\frac{\partial^2 G(\tau,x)}{\partial x^2} DD^\top \right) \\
=& \frac{1}{\gamma} G(\tau,x) \Big\{ 2 \left[x^\top P_\gamma(\tau) - \xi_\gamma^\top(\tau) \right] Ax + x^\top Q x + y^\top(\tau) Q y(\tau) - x^\top Q y(\tau) - y^\top(\tau) Q x \\
& - \left[x^\top P_\gamma(\tau) - \xi_\gamma^\top(\tau) \right] \left(BR^{-1}B^\top - \frac{2}{\gamma} DD^\top \right) \left[P_\gamma(\tau) x - \xi_\gamma(\tau) \right] + \mathrm{Tr}\left(P_\gamma(\tau) DD^\top \right) \Big\} \\
=& \frac{1}{\gamma} G(\tau,x) \Big\{ x^\top \left[P_\gamma(\tau) A + A^\top P_\gamma(\tau) - P_\gamma(\tau) \left(BR^{-1}B^\top - \frac{2}{\gamma} DD^\top \right) P_\gamma(\tau) + Q \right] x \\
& - 2x^\top \left[\left(A - BR^{-1}B^\top P_\gamma(\tau) + \frac{2}{\gamma} DD^\top P_\gamma(\tau) \right)^\top \xi_\gamma(\tau) + Qy(\tau) \right] \\
& - \xi_\gamma^\top(\tau) BR^{-1}B^\top \xi_\gamma(\tau) + y^\top(\tau) Q y(\tau) + \frac{2}{\gamma} \xi_\gamma^\top(\tau) DD^\top \xi_\gamma(\tau) + \mathrm{Tr}\left(P_\gamma(\tau) DD^\top \right) \Big\}.
\end{aligned} \tag{B.18}$$

Combining (3.6), (B.17) with (B.18), we know that $G(\tau,x)$ is a classical solution of (3.6) in $C^{1,2}([0,T] \times \mathbb{R}^n, \mathbb{R})$.
By (3.10), the definition of $M_{\gamma,1}$ and $M_{\gamma,2}$, we obtain

$$\begin{aligned}
\left\| \frac{\partial G(\tau,x)}{\partial x} \right\| =& \frac{2}{\gamma} \left[\exp\left(\frac{1}{\gamma} \left(x^\top P_\gamma(\tau) x - 2x^\top \xi_\gamma(\tau) + \varphi_\gamma(\tau) \right) \right) \right] \left(x^\top P_\gamma(\tau) - \xi_\gamma(\tau) \right) \\
\leq& \frac{2}{\gamma} \sup_{\tau \in [0,T]} \left[\exp\left(\varphi_\gamma(\tau) \right) \right] \left[\exp\left(\frac{1}{\gamma} \left(M_{\gamma,1} \|x\|^2 + M_{\gamma,2} \|x\| \right) \right) \right] \left(M_{\gamma,1} \|x\| + M_{\gamma,2} \right)
\end{aligned}$$

$$\leq \frac{2}{\gamma} \sup_{\tau\in[0,T]} \Big[\exp\Big(\varphi_\gamma(\tau)\Big)\Big] \max\Big\{M_{\gamma,1}, M_{\gamma,2}\Big\} \Big[\exp\Big(\frac{1}{\gamma}\Big(M_{\gamma,1}\|x\|^2 + M_{\gamma,2}\|x\|\Big)\Big)\Big]\Big(\|x\|+1\Big),$$

which together with the continuity of $\varphi_\gamma(\tau)$ with respect to $\tau \in [0,T]$ leads to (3.7).

Secondly, by the sufficient condition of the extreme value of unary functions, we have

$$\begin{aligned}
&\arg\min_{v\in\mathbb{R}^m}\left\{\frac{\partial G(\tau,x)}{\partial x}\Big|_{\tau=t,x=x^*(t)}(Ax^*(t)+Bv)+\frac{1}{\gamma}\left[\|x^*(t)-y(t)\|_Q^2+\|v\|_R^2\right]G(t,x^*(t))\right\}\\
=&-\frac{1}{2G(t,x^*(t))}R^{-1}B^\top\left(\frac{\partial G(\tau,x)}{\partial x}\Big|_{\tau=t,x=x^*(t)}\right)^\top\\
=&-R^{-1}B^\top P_\gamma(t)x^*(t)+R^{-1}B^\top\xi_\gamma(t)
\end{aligned} \tag{B.19}$$

for all $(t,\omega) \in [\tau,T]\times\Omega$, where $x^*(\cdot)$ is the strong solution of the system (3.1) under the control $u(t) = -R^{-1}B^\top P_\gamma(t)x^*(t) + R^{-1}B^\top\xi_\gamma(t)$. Let

$$u^*(t) = -R^{-1}B^\top P_\gamma(t)x^*(t) + R^{-1}B^\top\xi_\gamma(t), \quad t\in[\tau,T]. \tag{B.20}$$

Next, we will prove that $u^*(\cdot)\in\mathcal{U}_\tau$. Let

$$g_1(t) = \begin{cases} 0, & t\in[0,\tau),\\ \frac{4}{\gamma}\big(\|x^*(t)\|_Q^2+\|u^*(t)\|_R^2\big), & t\in[\tau,T]. \end{cases} \tag{B.21}$$

From (B.21) and Jensen inequality, we have

$$\begin{aligned}
\mathbb{E}\Big[\exp\Big(\frac{4}{\gamma}\int_\tau^T\Big(\big\|x^*(t)\big\|_Q^2+\|u^*(t)\|_R^2\Big)\mathrm{d}t\Big)\Big]=&\mathbb{E}\Big[\exp\Big(\frac{1}{T}\int_0^T\frac{4T}{\gamma}g_1(t)\mathrm{d}t\Big)\Big]\\
\leq&\frac{1}{T}\mathbb{E}\Big[\int_0^T\exp\Big(\frac{4T}{\gamma}g_1(t)\Big)\mathrm{d}t\Big]\\
\leq&\frac{\tau}{T}+\frac{1}{T}\int_\tau^T\mathbb{E}\Big[\exp\Big(\frac{4T}{\gamma}\Big(\|x^*(t)\|_Q^2+\|u^*(t)\|_R^2\Big)\Big)\Big]\mathrm{d}t.
\end{aligned} \tag{B.22}$$

Let $\phi_\gamma(t,s)$, $0\leq t,s\leq T$, be the solution of

$$\begin{cases} \mathrm{d}\phi_\gamma(t,s) = \big[A - BR^{-1}B^\top P_\gamma(t)\big]\phi_\gamma(t,s)\mathrm{d}t,\\ \phi_\gamma(s,s) = I_n, \end{cases} \tag{B.23}$$

and $\psi_\gamma(t,s)$, $0\leq t,s\leq T$, be the solution of

$$\begin{cases} \mathrm{d}\psi_\gamma(t,s) = -\Big[A - BR^{-1}B^\top P_\gamma(t) + \frac{2}{\gamma}DD^\top P_\gamma(t)\Big]^\top\psi_\gamma(t,s)\mathrm{d}t,\\ \psi_\gamma(s,s) = I_n. \end{cases} \tag{B.24}$$

Combining (3.1), (3.8), (B.20), (B.23) with (B.24), we get

$$x^*(t) = \int_\tau^t \phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^T\psi_\gamma(s_2,s_1)y(s_1)\mathrm{d}s_1\mathrm{d}s_2 + \phi_\gamma(t,\tau)x$$

$$+\int_\tau^t \phi_\gamma(t,s_2)D\mathrm{d}W(s_2), \quad \forall\, t\in[\tau,T]. \tag{B.25}$$

From (B.20) and (B.22), (B.25), C_r inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(\frac{4}{\gamma}\int_\tau^T\Big(\|x^*(t)\|_Q^2+\|u^*(t)\|_R^2\Big)\mathrm{d}t\bigg)\bigg]\\
\le&\frac{\tau}{T}+\frac{1}{T}\int_\tau^T\mathbb{E}\bigg[\exp\bigg(\frac{4T}{\gamma}\Big(\|x^*(t)\|_Q^2+\|-R^{-1}B^\top P_\gamma(t)x^*(t)+R^{-1}B^\top\xi_\gamma(t)\|_R^2\Big)\bigg)\bigg]\mathrm{d}t\\
\le&\frac{\tau}{T}+\frac{1}{T}\int_\tau^T\mathbb{E}\bigg[\exp\bigg(\frac{8T}{\gamma}\Big(\|x^*(t)\|_Q^2+\|R^{-1}B^\top P_\gamma(t)x^*(t)\|_R^2+\|R^{-1}B^\top\xi_\gamma(t)\|_R^2\Big)\bigg)\bigg]\mathrm{d}t\\
\le&\frac{\tau}{T}+\frac{1}{T}\int_\tau^T\exp\bigg[\frac{16T}{\gamma}\bigg(\int_\tau^t\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^T\psi_\gamma(s_2,s_1)Qy(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)\bigg)^\top Q\\
&\times\bigg(\int_\tau^t\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^T\psi_\gamma(s_2,s_1)Qy(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\bigg)\\
&+\frac{16T}{\gamma}\bigg(\int_\tau^t\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^T\psi_\gamma(s_2,s_1)Qy(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\bigg)^\top P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\\
&\times\bigg(\int_\tau^t\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^T\psi_\gamma(s_2,s_1)Qy(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\bigg)+\frac{8T}{\gamma}\|R^{-1}B^\top\xi_\gamma(t)\|_R^2\bigg]\\
&\times\mathbb{E}\bigg\{\exp\bigg[\frac{16T}{\gamma}\bigg(\int_\tau^t\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg)^\top Q\int_\tau^t\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\\
&+\frac{16T}{\gamma}\bigg(\int_\tau^t\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg)^\top P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\int_\tau^t\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg]\bigg\}\mathrm{d}t\\
\le&\frac{\tau}{T}+\frac{1}{T}J_1^{\frac12}J_2^{\frac12},
\end{aligned}\tag{B.26}$$

where

$$\begin{aligned}
J_1=&\int_\tau^T\exp\bigg[\frac{32T}{\gamma}\bigg(\int_\tau^t\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^T\psi_\gamma(s_2,s_1)Qy(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\bigg)^\top Q_L\\
&\times\bigg(\int_\tau^t\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^T\psi_\gamma(s_2,s_1)Qy(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\bigg)\\
&+\frac{32T}{\gamma}\bigg(\int_\tau^t\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^T\psi_\gamma(s_2,s_1)Qy(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\bigg)^\top P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\\
&\times\bigg(\int_\tau^t\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^T\psi_\gamma(s_2,s_1)Qy(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\bigg)+\frac{16T}{\gamma}\|R^{-1}B^\top\xi_\gamma(t)\|_R^2\bigg]\mathrm{d}t,
\end{aligned}$$

and

$$J_2=\int_\tau^T\bigg\{\mathbb{E}\bigg[\exp\bigg(\frac{16T}{\gamma}\bigg(\int_\tau^t\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg)^\top Q\int_\tau^t\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)$$

$$+\frac{16T}{\gamma}\bigg(\int_\tau^t \phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg)^\top P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\int_\tau^t \phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg)\bigg]\bigg\}^2\mathrm{d}t.$$

Since $\phi_\gamma(t,s)$ and $\psi_\gamma(t,s)$ are continuous with respect to $(t,s)\in[\tau,T]\times[\tau,T]$, $P_\gamma(t)$, $\xi_\gamma(t)$ and $y(t)$ are continuous with respect to $t\in[\tau,T]$, we have

$$J_1^{\frac{1}{2}}<\infty. \tag{B.27}$$

Noting that $\int_\tau^t \phi_\gamma(t,s_2)D\mathrm{d}W(s_2)=DW(t)-\phi_\gamma(t,\tau)DW(\tau)-\int_\tau^t\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\mathrm{d}s_2$, by C_r inequality and Jensen inequality, we have

$$\begin{aligned}&\frac{16T}{\gamma}\bigg[\int_\tau^t \phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg]^\top Q\int_\tau^t \phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\\
\le&\frac{48T}{\gamma}\|Q\|\|DW(t)\|^2+\frac{48T}{\gamma}\|Q\|\|\phi_\gamma(t,\tau)DW(\tau)\|^2+\frac{48T}{\gamma}\|Q\|\left\|\int_\tau^t\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\mathrm{d}s_2\right\|^2\\
\le&\frac{48T}{\gamma}\|Q\|\bigg(\|DW(t)\|^2+\|\phi_\gamma(t,\tau)DW(\tau)\|^2+t\int_\tau^t\left\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\right\|^2\mathrm{d}s_2\bigg),\quad\forall\, t\in[\tau,T].\end{aligned}\tag{B.28}$$

Similar to the above inequality, we obtain

$$\begin{aligned}&\frac{16T}{\gamma}\bigg(\int_\tau^t \phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg)^\top P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\int_\tau^t \phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\\
\le&\frac{48T}{\gamma}\left\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\right\|\bigg(\|DW(t)\|^2+\|\phi_\gamma(t,\tau)DW(\tau)\|^2\\
&+t\int_\tau^t\left\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\right\|^2\mathrm{d}s_2\bigg),\quad\forall\, t\in[\tau,T].\end{aligned}\tag{B.29}$$

From (B.28), (B.29), the definition of J_2 and Cauchy-Schwarz inequality, we have

$$\begin{aligned}J_2\le&\int_\tau^T\bigg\{\mathbb{E}\bigg[\exp\bigg(\frac{48T}{\gamma}\Big(\|Q\|+\left\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\right\|\Big)\|DW(t)\|^2\bigg)\\
&\times\exp\bigg(\frac{48T}{\gamma}\Big(\|Q\|+\left\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\right\|\Big)\|\phi_\gamma(t,\tau)DW(\tau)\|^2\bigg)\\
&\times\exp\bigg(\frac{48T}{\gamma}\Big(\|Q\|+\left\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\right\|\Big)t\int_\tau^T\left\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\right\|^2\mathrm{d}s_2\bigg)\bigg]\bigg\}^2\mathrm{d}t\\
\le&\int_\tau^T\bigg\{\mathbb{E}\bigg[\exp\bigg(\frac{192T}{\gamma}\Big(\|Q\|+\left\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\right\|\Big)\|DW(t)\|^2\bigg)\bigg]\bigg\}^{\frac{1}{2}}\\
&\times\bigg\{\mathbb{E}\bigg[\exp\bigg(\frac{192T}{\gamma}\Big(\|Q\|+\left\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\right\|\Big)\|\phi_\gamma(t,\tau)DW(\tau)\|^2\bigg)\bigg]\bigg\}^{\frac{1}{2}}\\
&\times\mathbb{E}\bigg[\exp\bigg(\frac{96T}{\gamma}\Big(\|Q\|+\left\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\right\|\Big)t\int_\tau^T\left\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\right\|^2\mathrm{d}s_2\bigg)\bigg]\mathrm{d}t.\end{aligned}\tag{B.30}$$

By the definition of $M_{\gamma,1}$, the condition $\gamma > \gamma_T$ and the calculation of the expectation of the exponential function related to Brownian motion, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(\frac{192T}{\gamma}\Big(\|Q\| + \big\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\big\|\Big)\|DW(t)\|^2\bigg)\bigg]\\
\leq&\int_{\mathbb{R}^p}\exp\bigg(\frac{192T}{\gamma}\Big(\|Q\| + M_{\gamma,1}^2\|BR^{-1}B^\top\|\Big)\|D\|^2x^\top x\bigg)\frac{1}{(\sqrt{2\pi t})^p}\exp\bigg(-\frac{x^\top x}{2t}\bigg)\mathrm{d}x\\
\leq&\bigg[\gamma\Big/\Big(\gamma - 384\sup_{\tau\leq t,s\leq T}\Big(\|Q\| + M_{\gamma,1}^2\|BR^{-1}B^\top\|\Big)\|D\|^2T^2\Big)\bigg]^{\frac{p}{2}}\\
<&\infty, \quad \forall\, t\in[\tau,T].
\end{aligned}\tag{B.31}$$

Similar to the above inequality, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(\frac{192T}{\gamma}\Big(\|Q\|+\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\|\Big)\|\phi_\gamma(t,\tau)DW(\tau)\|^2\bigg)\bigg]\\
\leq&\int_{\mathbb{R}^p}\exp\bigg(\frac{192T}{\gamma}\Big(\|Q\| + M_{\gamma,1}^2\|BR^{-1}B^\top\|\Big)\|\phi_\gamma(t,\tau)D\|^2x^\top x\bigg)\frac{1}{(\sqrt{2\pi t})^p}\exp\bigg(-\frac{x^\top x}{2t}\bigg)\mathrm{d}x\\
\leq&\bigg[\gamma\Big/\Big(\gamma - 384\sup_{\tau\leq t,s\leq T}\Big(\|Q\| + M_{\gamma,1}^2\|BR^{-1}B^\top\|\Big)\|\phi_\gamma(t,\tau)\|\|D\|^2T^2\Big)\bigg]^{\frac{p}{2}}\\
<&\infty, \quad \forall\, t\in[\tau,T].
\end{aligned}\tag{B.32}$$

Let

$$g_2(t) = \begin{cases} 0, & t\in[0,\tau),\\ \Big\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\Big\|^2, & t\in[\tau,T]. \end{cases}\tag{B.33}$$

From (B.33) and Jensen inequality, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(\frac{96T}{\gamma}\Big(\|Q\|+\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\|\Big)t\int_\tau^T\Big\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\Big\|^2\mathrm{d}s_2\bigg)\bigg]\\
=&\mathbb{E}\bigg[\exp\bigg(\frac{1}{T}\int_0^T\frac{96T^2}{\gamma}\Big(\|Q\|+\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\|\Big)tg_2(s_2)\mathrm{d}s_2\bigg)\bigg]\\
\leq&\frac{1}{T}\mathbb{E}\bigg[\int_0^T\exp\bigg(\frac{96T^3}{\gamma}\Big(\|Q\|+\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\|\Big)g_2(s_2)\bigg)\mathrm{d}s_2\bigg]\\
\leq&\frac{\tau}{T}+\frac{1}{T}\int_\tau^T\mathbb{E}\bigg[\exp\bigg(\frac{96T^3}{\gamma}\Big(\|Q\|+\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\|\Big)\Big\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\Big\|^2\bigg)\bigg]\mathrm{d}s_2.
\end{aligned}\tag{B.34}$$

By the definition of $M_{\gamma,1}$, the condition $\gamma > \gamma_T$ and the calculation of the expectation of the exponential function related to Brownian motion, we have

$$\frac{1}{T}\int_\tau^T\mathbb{E}\bigg[\exp\bigg(\frac{96T^3}{\gamma}\Big(\|Q\| + \big\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\big\|\Big)\Big\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\Big\|^2\bigg)\bigg]\mathrm{d}s_2$$

$$
\begin{aligned}
\leq&\frac{1}{T}\int_{\tau}^{T}\int_{\mathbb{R}^p}\exp\left[\frac{96T^3}{\gamma}\Big(\|Q\|+\big\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\big\|\Big)\left\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}\right\|^2\|D\|^2x^\top x\right]\frac{1}{(\sqrt{2\pi s_2})^p}\exp\Big(-\frac{x^\top x}{2s_2}\Big)\mathrm{d}x\mathrm{d}s_2\\
\leq&\frac{1}{T}\int_{\tau}^{T}\left[\gamma\Bigg/\left(\gamma-\sup_{\tau\leq t,s\leq T}\frac{192}{\gamma}\Big(\|Q\|+\big\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\big\|\Big)\left\|\frac{\partial\phi_\gamma(t,s)}{\partial s}\right\|^2\|D\|^2T^4\right)\right]^{\frac{p}{2}}\mathrm{d}s_2\\
<&\infty,\quad \forall\, t\in[\tau,T],
\end{aligned}
$$

which together with (B.34) leads to

$$
\mathbb{E}\left[\exp\left(\frac{96T}{\gamma}\Big(\|Q\|+\|P_\gamma(t)BR^{-1}B^\top P_\gamma(t)\|\Big)t\int_{\tau}^{T}\left\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\right\|^2\mathrm{d}s_2\right)\right]<\infty,\ \forall\, t\in[\tau,T]. \tag{B.35}
$$

Combining (B.30)–(B.32) with (B.35), we get $J_2<\infty$. This together with (B.26) and (B.27) yields

$$
\mathbb{E}\left[\exp\left(\frac{4}{\gamma}\int_{\tau}^{T}\Big(\|x^*(t)\|_Q^2+\|u^*(t)\|_R^2\Big)\mathrm{d}t\right)\right]<\infty.
$$

Moreover, we know

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_{\tau}^{T}\left(\exp\left(\frac{\alpha M_{\gamma,1}}{\gamma}\|x^*(t)\|^2\right)\right)\|x^*(t)\|^2\mathrm{d}t\bigg]\\
\leq&\mathbb{E}\left\{\left[\int_{\tau}^{T}\left(\exp\left(\frac{2\alpha M_{\gamma,1}}{\gamma}\|x^*(t)\|^2\right)\right)\mathrm{d}t\right]^{\frac12}\left[\int_{\tau}^{T}\|x^*(t)\|^4\mathrm{d}t\right]^{\frac12}\right\}\\
\leq&\left\{\mathbb{E}\bigg[\int_{\tau}^{T}\left(\exp\left(\frac{2\alpha M_{\gamma,1}}{\gamma}\|x^*(t)\|^2\right)\right)\mathrm{d}t\bigg]\right\}^{\frac12}\left\{\mathbb{E}\bigg[\int_{\tau}^{T}\|x^*(t)\|^4\mathrm{d}t\bigg]\right\}^{\frac12}\\
=&\left[\int_{\tau}^{T}\mathbb{E}\left(\exp\left(\frac{2\alpha M_{\gamma,1}}{\gamma}\|x^*(t)\|^2\right)\right)\mathrm{d}t\right]^{\frac12}\left[\int_{\tau}^{T}\mathbb{E}\|x^*(t)\|^4\mathrm{d}t\right]^{\frac12}.
\end{aligned} \tag{B.36}
$$

Substituting (B.25) into $\mathbb{E}\Big[\exp\Big(\frac{2\alpha M_{\gamma,1}}{\gamma}\|x^*(t)\|^2\Big)\Big]$, by C_r inequality, we get

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{2\alpha M_{\gamma,1}}{\gamma}\|x^*(t)\|^2\right)\right]\\
=&\mathbb{E}\bigg\{\exp\bigg[\frac{2\alpha M_{\gamma,1}}{\gamma}\bigg(\int_{\tau}^{t}\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^{T}\psi_\gamma(s_2,s_1)y(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\\
&+\int_{\tau}^{t}\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg)^\top\bigg(\int_{\tau}^{t}\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^{T}\psi_\gamma(s_2,s_1)y(s_1)\mathrm{d}s_1\mathrm{d}s_2\\
&+\phi_\gamma(t,\tau)x+\int_{\tau}^{t}\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg)\bigg]\bigg\}\\
\leq&\exp\bigg[\frac{4\alpha M_{\gamma,1}}{\gamma}\bigg(\int_{\tau}^{t}\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^{T}\psi_\gamma(s_2,s_1)y(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\bigg)^\top\\
&\times\bigg(\int_{\tau}^{t}\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^{T}\psi_\gamma(s_2,s_1)y(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\bigg)\bigg]
\end{aligned}
$$

$$\times\mathbb{E}\bigg\{\exp\bigg[\frac{4\alpha M_{\gamma,1}}{\gamma}\bigg(\int_\tau^t\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg)^\top\bigg(\int_\tau^t\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg)\bigg]\bigg\},\quad\forall\ t\in[\tau,T].\tag{B.37}$$

Since $\phi_\gamma(t,s)$ and $\psi_\gamma(t,s)$ are continuous with respect to $(t,s)\in[\tau,T]\times[\tau,T]$, and $y(t)$ is continuous with respect to $t\in[\tau,T]$, we have

$$\begin{aligned}&\exp\bigg[\frac{4\alpha M_{\gamma,1}}{\gamma}\bigg(\int_\tau^t\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^T\psi_\gamma(s_2,s_1)y(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\bigg)^\top\\&\times\bigg(\int_\tau^t\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^T\psi_\gamma(s_2,s_1)y(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\bigg)\bigg]<\infty,\quad\forall\ t\in[\tau,T].\end{aligned}\tag{B.38}$$

Noting that $\int_\tau^t\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)=DW(t)-\phi_\gamma(t,\tau)DW(\tau)-\int_\tau^t\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\mathrm{d}s_2$, by C_r inequality and Jensen inequality, we have

$$\begin{aligned}&\mathbb{E}\bigg\{\exp\bigg[\frac{4\alpha M_{\gamma,1}}{\gamma}\bigg(\int_\tau^t\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg)^\top\bigg(\int_\tau^t\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\bigg)\bigg]\bigg\}\\
\le&\mathbb{E}\bigg[\exp\bigg(\frac{12\alpha M_{\gamma,1}}{\gamma}\bigg(\|DW(t)\|^2+\|\phi_\gamma(t,\tau)DW(\tau)\|^2+\bigg\|\int_\tau^t\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\mathrm{d}s_2\bigg\|^2\bigg)\bigg)\bigg]\\
\le&\mathbb{E}\bigg[\exp\bigg(\frac{12\alpha M_{\gamma,1}}{\gamma}\bigg(\|DW(t)\|^2+\|\phi_\gamma(t,\tau)DW(\tau)\|^2+t\int_\tau^t\bigg\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\bigg\|^2\mathrm{d}s_2\bigg)\bigg)\bigg]\\
\le&\bigg\{\mathbb{E}\bigg[\exp\bigg(\frac{48\alpha M_{\gamma,1}}{\gamma}\|DW(t)\|^2\bigg)\bigg]\bigg\}^{\frac14}\bigg\{\mathbb{E}\bigg[\exp\bigg(\frac{48\alpha M_{\gamma,1}}{\gamma}\|\phi_\gamma(t,\tau)DW(\tau)\|^2\bigg)\bigg]\bigg\}^{\frac14}\\
&\times\bigg\{\mathbb{E}\bigg[\exp\bigg(\frac{24\alpha M_{\gamma,1}t}{\gamma}\int_\tau^t\bigg\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\bigg\|^2\mathrm{d}s_2\bigg)\bigg]\bigg\}^{\frac12},\quad\forall\ t\in[\tau,T].\end{aligned}\tag{B.39}$$

By the condition $\gamma>\gamma_T$ and the calculation of the expectation of the exponential function related to Brownian motion, we have

$$\begin{aligned}\mathbb{E}\bigg[\exp\bigg(\frac{48\alpha M_{\gamma,1}}{\gamma}\|DW(t)\|^2\bigg)\bigg]&\le\int_{\mathbb{R}^p}\bigg[\exp\bigg(\frac{48\alpha M_{\gamma,1}}{\gamma}\|D\|^2x^\top x\bigg)\bigg]\frac{1}{(\sqrt{2\pi t})^p}\bigg[\exp\bigg(-\frac{x^\top x}{2t}\bigg)\bigg]\mathrm{d}x\\
&\le\bigg[\gamma\Big/\Big(\gamma-96\alpha M_{\gamma,1}\|D\|^2T\Big)\bigg]^{\frac p2}\\
&<\infty,\quad\forall\ t\in[\tau,T].\end{aligned}\tag{B.40}$$

Similar to the above inequality, we have

$$\begin{aligned}&\mathbb{E}\bigg[\exp\bigg(\frac{48\alpha M_{\gamma,1}}{\gamma}\|\phi_\gamma(t,\tau)DW(\tau)\|^2\bigg)\bigg]\\
\le&\int_{\mathbb{R}^p}\bigg[\exp\bigg(\frac{48\alpha M_{\gamma,1}}{\gamma}\|\phi_\gamma(t,\tau)D\|^2x^\top x\bigg)\bigg]\frac{1}{(\sqrt{2\pi t})^p}\bigg[\exp\bigg(-\frac{x^\top x}{2t}\bigg)\bigg]\mathrm{d}x\\
\le&\bigg[\gamma\Big/\Big(\gamma-\sup_{0\le s_1,s\le T}96\alpha M_{\gamma,1}\|\phi_\gamma(s_1,s)\|^2\|D\|^2T\Big)\bigg]^{\frac p2}\\
<&\infty,\quad\forall\ t\in[\tau,T].\end{aligned}\tag{B.41}$$

From (B.33) and Jensen inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{24\alpha M_{\gamma,1}t}{\gamma}\int_{\tau}^{T}\left\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\right\|^2\mathrm{d}s_2\right)\right]\\
=&\mathbb{E}\left[\exp\left(\frac{1}{T}\int_0^T\frac{24\alpha M_{\gamma,1}Tt}{\gamma}g_2(s_2)\mathrm{d}s_2\right)\right]\\
\leq&\frac{1}{T}\mathbb{E}\left[\int_0^T\exp\left(\frac{24\alpha M_{\gamma,1}T^2}{\gamma}g_2(s_2)\right)\mathrm{d}s_2\right]\\
\leq&\frac{\tau}{T}+\frac{1}{T}\int_{\tau}^{T}\mathbb{E}\left[\exp\left(\frac{24\alpha M_{\gamma,1}T^2}{\gamma}\left\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\right\|^2\right)\right]\mathrm{d}s_2.
\end{aligned}\tag{B.42}$$

By the condition $\gamma>\gamma_T$ and the calculation of the expectation of the exponential function related to Brownian motion, we have

$$\begin{aligned}
&\frac{1}{T}\int_{\tau}^{T}\mathbb{E}\left[\exp\left(\frac{24\alpha M_{\gamma,1}T^2}{\gamma}\left\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\right\|^2\right)\right]\mathrm{d}s_2\\
\leq&\frac{1}{T}\int_{\tau}^{T}\int_{\mathbb{R}^p}\exp\left(\frac{24\alpha M_{\gamma,1}T^2}{\gamma}\left\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}\right\|^2\|D\|^2x^\top x\right)\frac{1}{(\sqrt{2\pi s_2})^p}\exp\left(-\frac{x^\top x}{2s_2}\right)\mathrm{d}x\mathrm{d}s_2\\
\leq&\frac{1}{T}\int_{\tau}^{T}\left[\gamma\Big/\left(\gamma-\sup_{0\leq s_1,s\leq T}48\alpha M_{\gamma,1}\left\|\frac{\partial\phi_\gamma(s_1,s)}{\partial s}\right\|^2\|D\|^2T^3\right)\right]^{\frac{p}{2}}\mathrm{d}s_2\\
<&\infty,\quad\forall\ t\in[\tau,T],
\end{aligned}$$

which together with (B.42) leads to

$$\mathbb{E}\left[\exp\left(\frac{24\alpha M_{\gamma,1}t}{\gamma}\int_{\tau}^{T}\left\|\frac{\partial\phi_\gamma(t,s_2)}{\partial s_2}DW(s_2)\right\|^2\mathrm{d}s_2\right)\right]<\infty,\ \forall\ t\in[\tau,T].\tag{B.43}$$

Combining (B.39)–(B.41) with (B.43), we have

$$\mathbb{E}\left\{\exp\left[\frac{4\alpha M_{\gamma,1}}{\gamma}\left(\int_{\tau}^{t}\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\right)^\top\left(\int_{\tau}^{t}\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\right)\right]\right\}<\infty,\quad\forall\ t\in[\tau,T],$$

which together with (B.37) and (B.38) leads to

$$\mathbb{E}\left[\exp\left(\frac{2\alpha M_{\gamma,1}}{\gamma}\|x^*(t)\|^2\right)\right]<\infty,\quad\forall\ t\in[\tau,T].\tag{B.44}$$

Substituting (B.25) into $\mathbb{E}\big[\|x^*(t)\|^4\big]$, by C_r inequality, we have

$$\begin{aligned}
\mathbb{E}\big[\|x^*(t)\|^4\big]=&\mathbb{E}\left\|\int_{\tau}^{t}\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^{T}\psi_\gamma(s_2,s_1)y(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x+\int_{\tau}^{t}\phi_\gamma(t,s_2)D\mathrm{d}W(s_2)\right\|^4\\
\leq&8\left\|\int_{\tau}^{t}\phi_\gamma(t,s_2)BR^{-1}B^\top\int_{s_2}^{T}\psi_\gamma(s_2,s_1)y(s_1)\mathrm{d}s_1\mathrm{d}s_2+\phi_\gamma(t,\tau)x\right\|^4
\end{aligned}$$

$$+ 8\mathbb{E}\left[\left\| \int_{\tau}^{t} \phi_\gamma(t,s_2) D\mathrm{d}W(s_2)\right\|^4\right], \quad \forall\ t \in [\tau, T]. \tag{B.45}$$

Since $\phi_\gamma(t,s)$ and $\psi_\gamma(t,s)$ are continuous with respect to $(t,s) \in [\tau,T]\times[\tau,T]$, and $y(t)$ is continuous with respect to $t \in [\tau,T]$, we have

$$\left\| \int_{\tau}^{t} \phi_\gamma(t,s_2) BR^{-1}B^\top \int_{s_2}^{T} \psi_\gamma(s_2,s_1) y(s_1)\mathrm{d}s_1\mathrm{d}s_2 + \phi_\gamma(t,\tau)x\right\|^4 < \infty, \quad \forall\ t \in [\tau,T]. \tag{B.46}$$

Noting that $\int_\tau^t \phi_\gamma(t,s_2)D\mathrm{d}W(s_2) = DW(t) - \phi_\gamma(t,\tau)DW(\tau) - \int_\tau^t \frac{\partial \phi_\gamma(t,s_2)}{\partial s_2} DW(s_2)\mathrm{d}s_2$, by C_r inequality and Jensen inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\left\| \int_{\tau}^{t} \phi_\gamma(t,s_2) D\mathrm{d}W(s_2)\right\|^4\right] \\
=&\mathbb{E}\left[\left\| DW(t) - \phi_\gamma(t,\tau)DW(\tau) - \int_\tau^t \frac{\partial \phi_\gamma(t,s_2)}{\partial s_2} DW(s_2)\mathrm{d}s_2\right\|^4\right] \\
\leq& 27\mathbb{E}\left[\left\|DW(t)\right\|^4\right] + 27\mathbb{E}\left[\left\|\phi_\gamma(t,\tau)DW(\tau)\right\|^4\right] + 27\mathbb{E}\left[\left\| \int_\tau^t \frac{\partial \phi_\gamma(t,s_2)}{\partial s_2} DW(s_2)\mathrm{d}s_2\right\|^4\right] \\
\leq& 27\|D\|^4\mathbb{E}\left[\left\|W(t)\right\|^4\right] + 27 \sup_{0\leq s_1,s\leq T} \left\|\phi_\gamma(s_1,s)D\right\|^4 \mathbb{E}\left[\|W(\tau)\|^4\right] \\
&+ 27 \sup_{0\leq s_1,s\leq T} \left\| \frac{\partial \phi_\gamma(s_1,s)}{\partial s} D\right\|^4 \int_\tau^t \mathbb{E}\left\|W(s_2)\right\|^4 \mathrm{d}s_2 \\
\leq& 27\|D\|^4 p(p+2)t^2 + 27 \sup_{0\leq s_1,s\leq T} \left\|\phi_\gamma(s_1,s)D\right\|^4 p(p+2)\tau^2 \\
&+ 9 \sup_{0\leq s_1,s\leq T} \left\| \frac{\partial \phi_\gamma(s_1,s)}{\partial s} D\right\|^4 p(p+2)\left(t^3 - \tau^3\right) \\
<& \infty, \quad \forall\ t \in [\tau,T].
\end{aligned}$$

This together with (B.45) and (B.46) yields

$$\mathbb{E}\left[\|x^*(t)\|^4\right] < \infty, \quad \forall t \in [\tau,T]. \tag{B.47}$$

Combining (B.36), (B.44) with (B.46), we obtain

$$\mathbb{E}\left[\int_\tau^T \left(\exp\left(\frac{\alpha M_{\gamma,1}}{\gamma}\|x^*(t)\|^2\right)\right)\|x^*(t)\|^2\mathrm{d}t\right] < \infty. \tag{B.48}$$

This completes the proof of $u^*(\cdot) \in \mathcal{U}_\tau$. Then by Theorem 3.1, we obtain $L(\tau,x;u^*(\cdot)) = \inf_{\bar{v}(\cdot)\in\mathcal{U}_\tau} L(\tau,x;\bar{v}(\cdot)) = G(\tau,x) = \exp\left[\frac{1}{\gamma}\left(x^\top P_\gamma(\tau)x - 2x^\top\xi_\gamma(\tau) + \varphi_\gamma(\tau)\right)\right]$. This together with (3.2) leads to that $u^*(t) = -R^{-1}B^\top P_\gamma(t)x(t) + R^{-1}B^\top\xi_\gamma(t)$ is an optimal strategy in $\mathcal{U}_\tau$ for the systems (3.1)–(3.3), and the optimal cost is $J(\tau,x;u^*(\cdot)) = \gamma\ln L(\tau,x;u^*(\cdot)) = \gamma \ln G(\tau,x) = x^\top P_\gamma(\tau)x - 2x^\top\xi_\gamma(\tau) + \varphi_\gamma(\tau)$. This completes the proof. □

Proof of Theorem 3.6. For the convenience of analysis, we introduce a new admissible control set

$$\overline{\mathcal{U}}^l_{L_i}=\Bigg\{u_{L_i}(\cdot)\Bigg|u_{L_i}(t)\text{ is adapted to }\sigma(x_{L_i}(s),s\le t),t\in[0,T],\\ \text{the system (2.1) has a unique strong solution }x_{L_i}(\cdot),\\ \mathbb{E}\Big[\exp\Big(\frac{4}{\gamma}\int_0^T\Big(\big\|x_{L_i}(t)\big\|^2_{Q_L}+\|u_{L_i}(t)\|^2_{R_L}\Big)\mathrm{d}t\Big)\Big]<\infty,\text{ and}\\ \mathbb{E}\Big[\int_0^T\Big(\exp\Big(\frac{\alpha M_{\gamma,L}(\theta_{L_i})}{\gamma}\|x_{L_i}(t)\|^2\Big)\Big)\|x_{L_i}(t)\|^2\mathrm{d}t\Big]<\infty\Bigg\},\quad i=1,2,\dots,N_L,$$

where $M_{\gamma,L}(\theta_{L_i})=M_{\gamma,L}(\theta_L)|_{\theta_L=\theta_{L_i}}$ and α is same with that in $\mathcal{U}^l_{L_i}$.

For the system (2.1) and the cost functionals (3.12), by Assumption 2.5 and Theorem 3.3, the optimal strategies over $\overline{\mathcal{U}}^l_{L_i}, i=1,2,\dots,N_L$, are given by

$$\bar{u}_{L_i}(t)=-R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)x_{L_i}(t)+R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L}(t),\quad t\in[0,T],\quad i=1,2,\dots,N_L,\tag{B.49}$$

where $P_{\gamma,\theta_{L_i}}(t)=P_{\gamma,\theta_L}(t)|_{\theta_L=\theta_{L_i}}$, $\xi_{\gamma,\theta_{L_i},z_L}(t)=\xi_{\gamma,\theta_L,z_L}(t)|_{\theta_L=\theta_{L_i}}$, $P_{\gamma,\theta_L}(t)$ is given by (2.5), and $\xi_{\gamma,\theta_L,z_L}(t)$ is the unique solution of the linear differential equation

$$\begin{cases}\dot{\xi}_{\gamma,\theta_L,z_L}(t)=-\Big[A_L(\theta_L)-B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)P_{\gamma,\theta_L}(t)+\frac{2}{\gamma}D_L(\theta_L)D_L^\top(\theta_L)P_{\gamma,\theta_L}(t)\Big]^\top\xi_{\gamma,\theta_L,z_L}(t)\\ \qquad\qquad\qquad-\left[Q_LH_Lz_L(t)+Q_Lg_L\right],\\ \xi_{\gamma,\theta_L,z_L}(T)=0.\end{cases}\tag{B.50}$$

It is obvious that $\mathcal{U}^l_{L_i}\subset\overline{\mathcal{U}}^l_{L_i}, i=1,2,\dots,N_L$. Next, we will prove that $\bar{u}_{L_i}(t)\in\mathcal{U}^l_{L_i}, i=1,2,\dots,N_L$.

Combining (2.1), (A.1), (A.2), (B.49) with (B.50), we get

$$\begin{aligned}\bar{x}_{L_i}(t)=&\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\\&+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)+\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2),\quad\forall\, t\in[0,T].\end{aligned}$$

Substituting the above equation and (B.49) into $\mathbb{E}\Big[\exp\Big(\frac{32}{\gamma}\int_0^T\big(\|\bar{x}_{L_i}(t)\|^2_{Q_L}+\big\|\frac{1}{N_L}H_L\bar{x}_{L_i}(t)\big\|^2_{Q_L}+\|\bar{u}_{L_i}(t)\|^2_{R_L}\big)\mathrm{d}t\Big)\Big]$, by Jensen inequality, C_r inequality with Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\mathbb{E}\Big[\exp\Big(\frac{32}{\gamma}\int_0^T\Big(\big\|\bar{x}_{L_i}(t)\big\|^2_{Q_L}+\Big\|\frac{1}{N_L}H_L\bar{x}_{L_i}(t)\Big\|^2_{Q_L}+\|\bar{u}_{L_i}(t)\|^2_{R_L}\Big)\mathrm{d}t\Big)\Big]\\=&\mathbb{E}\Big[\exp\Big(\frac{32}{\gamma}\int_0^T\Big(\big\|\bar{x}_{L_i}(t)\big\|^2_{Q_L}+\Big\|\frac{1}{N_L}H_L\bar{x}_{L_i}(t)\Big\|^2_{Q_L}+\|-R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\bar{x}_{L_i}(t)\\&+R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L}(t)\|^2_{R_L}\Big)\mathrm{d}t\Big)\Big]\\\le&\frac{1}{T}\mathbb{E}\Big[\int_0^T\exp\Big(\frac{32T}{\gamma}\Big(\big\|\bar{x}_{L_i}(t)\big\|^2_{Q_L}+\Big\|\frac{1}{N_L}H_L\bar{x}_{L_i}(t)\Big\|^2_{Q_L}+\|-R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\bar{x}_{L_i}(t)\\&+R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L}(t)\|^2_{R_L}\Big)\Big)\mathrm{d}t\Big]\end{aligned}$$

$$
\begin{aligned}
\le&\frac{1}{T}\int_0^T \mathbb{E}\bigg[\exp\bigg(\frac{32T}{\gamma}\Big(\|\bar{x}_{L_i}(t)\|^2_{Q_L}+\Big\|\frac{1}{N_L}H_L\bar{x}_{L_i}(t)\Big\|^2_{Q_L}+\|R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\bar{x}_{L_i}(t)\|^2_{R_L}\\
&+\|R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L}(t)\|^2_{R_L}\Big)\bigg)\bigg]\mathrm{d}t\\
\le&\frac{1}{T}\int_0^T \exp\bigg\{\frac{128T}{\gamma}\bigg[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\\
&+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\bigg]^\top Q_L\bigg[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\\
&\times\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\bigg]\\
&+\frac{128T}{\gamma N_L^2}\bigg[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\\
&+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\bigg]^\top H_L^\top Q_LH_L\bigg[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\\
&\times\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\bigg]+\frac{128T}{\gamma}\bigg[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)\\
&\times B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\bigg]^\top\\
&\times P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\bigg[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)\\
&\times[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\bigg]+\frac{128T}{\gamma}\|R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L}(t)\|^2_{R_L}\bigg\}\\
&\times\mathbb{E}\bigg\{\exp\bigg[\frac{128T}{\gamma}\bigg[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\bigg]^\top Q_L\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\\
&+\frac{128T}{\gamma N_L^2}\bigg[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\bigg]^\top H_L^\top Q_LH_L\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\\
&+\frac{128T}{\gamma}\bigg[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\bigg]^\top P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\\
&\times\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\bigg]\bigg\}\mathrm{d}t\\
\le&\frac{1}{T}J_3^{\frac{1}{2}}J_4^{\frac{1}{2}},
\end{aligned}
\tag{B.51}
$$

where

$$
\begin{aligned}
J_3=&\int_0^T\exp\bigg[\frac{256T}{\gamma}\bigg(\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\\
&+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\bigg)^\top Q_L\bigg(\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\\
&\times\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{256T}{\gamma}\bigg(\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T \psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\\
&+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\bigg)^\top H_L^\top Q_LH_L\bigg(\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\\
&\times\int_{s_2}^T \psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\bigg)+\frac{256T}{\gamma}\bigg(\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)\\
&\times B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T \psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\bigg)^\top\\
&\times P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\bigg(\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T \psi_{\gamma,\theta_{L_i}}(s_2,s_1)\\
&\times[Q_LH_Lz_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\bigg)+\frac{256T}{\gamma}\|R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L}(t)\|_{R_L}^2\Bigg]\mathrm{d}t,
\end{aligned}
$$

and

$$
\begin{aligned}
J_4=\int_0^T\Bigg[&\mathbb{E}\Bigg(\exp\bigg(\frac{128T}{\gamma}\bigg(\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\bigg)^\top Q_L\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\\
&+\frac{128T}{\gamma}\bigg(\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\bigg)^\top H_L^\top Q_LH_L\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\\
&+\frac{128T}{\gamma}\bigg(\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\bigg)^\top P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\\
&\times\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\bigg)\Bigg)\Bigg]^2\mathrm{d}t.
\end{aligned}
$$

Since $\phi_{\gamma,\theta_{L_i}}(t,s)$ and $\psi_{\gamma,\theta_{L_i}}(t,s)$ are continuous with respect to $(t,s)\in[0,T]\times[0,T]$, $z_L(t)$, $\xi_{\gamma,\theta_{L_i},z_L}(t)$ and $P_{\gamma,\theta_{L_i}}(t)$ are continuous with respect to $t\in[0,T]$ and Θ_L is a bounded closed set, we have

$$
J_3^{\frac{1}{2}}<\infty. \tag{B.52}
$$

Noting that $\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s)=D_L(\theta_{L_i})W_{L_i}(t)-\int_0^t \frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}D_L(\theta_{L_i})W_{L_i}(s)\mathrm{d}s$, by C_r inequality and Jensen inequality, we have

$$
\begin{aligned}
&\frac{128T}{\gamma}\bigg(\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\bigg)^\top Q_L\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\\
\le&\frac{256T}{\gamma}\|Q_L\|\Bigg(\|D_L(\theta_{L_i})W_{L_i}(t)\|^2+\bigg\|\int_0^t \frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\mathrm{d}s_2\bigg\|^2\Bigg)\\
\le&\frac{256T}{\gamma}\|Q_L\|\Bigg(\|D_L(\theta_{L_i})W_{L_i}(t)\|^2+t\int_0^t \bigg\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\bigg\|^2\mathrm{d}s_2\Bigg).
\end{aligned} \tag{B.53}
$$

Similar to the above inequality, we obtain

$$\begin{aligned}&\frac{128T}{\gamma}\bigg(\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\bigg)^\top H_L^\top Q_L H_L\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\\ \le&\frac{256T}{\gamma}\|H_L^\top Q_L H_L\|\Bigg(\|D_L(\theta_{L_i})W_{L_i}(t)\|^2+t\int_0^t\bigg\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\bigg\|^2\mathrm{d}s_2\Bigg)\end{aligned}\tag{B.54}$$

and

$$\begin{aligned}&\frac{128T}{\gamma}\bigg(\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\bigg)^\top P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\\ &\times\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\\ \le&\frac{256T}{\gamma}\left\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\right\|\\ &\times\Bigg(\|D_L(\theta_{L_i})W_{L_i}(t)\|^2+t\int_0^t\bigg\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\bigg\|^2\mathrm{d}s_2\Bigg).\end{aligned}\tag{B.55}$$

From (B.53)–(B.55), the definition of J_4, Cauchy-Schwarz inequality and Jensen inequality, we have

$$\begin{aligned}J_4\le&\int_0^T\Bigg\{\mathbb{E}\Bigg[\exp\bigg(\frac{256T}{\gamma}\Big(\|Q_L\|+\|H_L^\top Q_L H_L\|+\left\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\right\|\Big)\\ &\times\|D_L(\theta_{L_i})W_{L_i}(t)\|^2\bigg)\exp\bigg(\frac{384T}{\gamma}\Big(\|Q_L\|+\|H_L^\top Q_L H_L\|+\left\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\right\|\Big)\\ &\times t\int_0^T\bigg\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\bigg\|^2\mathrm{d}s_2\bigg)\Bigg]\Bigg\}^2\mathrm{d}t\\ \le&\frac{1}{T}\int_0^T\mathbb{E}\Bigg[\exp\bigg(\frac{512T}{\gamma}\Big(\|Q_L\|+\|H_L^\top Q_L H_L\|+\left\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\right\|\Big)\\ &\times\|D_L(\theta_{L_i})W_{L_i}(t)\|^2\bigg)\Bigg]\int_0^T\mathbb{E}\Bigg[\exp\bigg(\frac{512T^2}{\gamma}\Big(\|Q_L\|+\|H_L^\top Q_L H_L\|\\ &+\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\|\Big)t\bigg\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\bigg\|^2\bigg)\Bigg]\mathrm{d}s_2\mathrm{d}t.\end{aligned}\tag{B.56}$$

By the definition of $M_{\gamma,L}(\theta_{L_i})$, Assumption 2.5, the compactness of Θ_L and the calculation of the expectation of the exponential function related to Brownian motion, we have

$$\begin{aligned}&\mathbb{E}\bigg[\exp\bigg(\frac{512T}{\gamma}\Big(\|Q_L\|+\|H_L^\top Q_L H_L\|+\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\|\Big)\|D_L(\theta_{L_i})W_{L_i}(t)\|^2\bigg)\bigg]\\ \le&\int_{\mathbb{R}^p}\bigg[\exp\bigg(\frac{512T}{\gamma}\Big(\|Q_L\|+\|H_L^\top Q_L H_L\|+\big(M_{\gamma,L}(\theta_{L_i})\big)^2\|B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\|\Big)\|D_L(\theta_{L_i})\|^2x^\top x\bigg)\bigg]\\ &\times\frac{1}{(\sqrt{2\pi t})^p}\bigg[\exp\bigg(-\frac{x^\top x}{2t}\bigg)\bigg]\mathrm{d}x\end{aligned}$$

$$\begin{aligned}
\leq&\bigg[\gamma\bigg/\bigg(\gamma-1024\sup_{0\leq t\leq T}\sup_{\theta_{L_i}\in\Theta_L}\Big(\|Q_L\|+\|H_L^\top Q_L H_L\|+\big(M_{\gamma,L}(\theta_{L_i})\big)^2\|B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\|\Big)\\
&\times\|D_L(\theta_{L_i})\|^2T^2\bigg)\bigg]^{\frac{p}{2}}\\
<&\infty,\quad \forall\ t\in(0,T]. \qquad\qquad (B.57)
\end{aligned}$$

Similar to the above inequality, we have

$$\begin{aligned}
&\frac{1}{T}\int_0^T\mathbb{E}\Bigg[\exp\Bigg(\frac{512T^2}{\gamma}\Big(\|Q_L\|+\|H_L^\top Q_L H_L\|+\big\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\big\|\Big)\\
&\times t\left\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\right\|^2\Bigg)\Bigg]\mathrm{d}s_2\\
\leq&\frac{1}{T}\int_{0^+}^T\int_{\mathbb{R}^p}\Bigg[\exp\Bigg(\frac{512T^3}{\gamma}\Big(\|Q_L\|+\|H_L^\top Q_L H_L\|+\big\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\big\|\Big)\\
&\times\left\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}\right\|^2\|D_L(\theta_{L_i})\|^2x^\top x\Bigg)\Bigg]\frac{1}{(\sqrt{2\pi s_2})^p}\Bigg[\exp\Bigg(-\frac{x^\top x}{2s_2}\Bigg)\Bigg]\mathrm{d}x\mathrm{d}s_2\\
\leq&\frac{1}{T}\int_0^T\Bigg[\gamma\Bigg/\Bigg(\gamma-1024\sup_{0\leq t,s\leq T}\sup_{\theta_{L_i}\in\Theta_L}\Big(\|Q_L\|+\|H_L^\top Q_L H_L\|\\
&+\big(M_{\gamma,L}(\theta_{L_i})\big)^2\left\|B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\right\|\Big)\left\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}\right\|^2\|D_L(\theta_{L_i})\|^2T^4\Bigg)\Bigg]^{\frac{p}{2}}\mathrm{d}s_2\\
<&\infty,\quad\forall\ t\in[0,T]. \qquad\qquad (B.58)
\end{aligned}$$

Combining (B.56)–(B.58), we get $J_4<\infty$. This together with (B.51) and (B.52) gives $\mathbb{E}\Big[\exp\Big(\frac{32}{\gamma}\int_0^T\big(\big\|\bar{x}_{L_i}(t)\big\|^2_{Q_L}+\big\|\frac{1}{N_L}H_L\bar{x}_{L_i}(t)\big\|^2_{Q_L}+\|\bar{u}_{L_i}(t)\|^2_{R_L}\big)\mathrm{d}t\Big)\Big]<\infty$. Therefore, $\bar{u}_{L_i}(t)\in\mathcal{U}^l_{L_i}, i=1,2,\ldots,N_L$, and then $\bar{u}_{L_i}(t), i=1,2,\ldots,N_L$, are the optimal strategies for Auxiliary problem (I). □

Appendix C. Proofs in Section 4

Lemma C.1. *Define* $\tilde{L}^4_{\mathcal{F}}(0,T;\mathbb{R}^n)=\{f|\ f(t)\in\mathbb{R}^n$ *and* $f(t)$ *is* $\mathcal{F}_t$*-measurable,* $\forall\ t\in[0,T]$, $\mathbb{E}[(\int_0^T\|f(t)\|^2\mathrm{d}t)^2]<\infty\}$. *For any given* $f\in\tilde{L}^4_{\mathcal{F}}(0,T;\mathbb{R}^n)$, *define norm* $\|f\|_{\tilde{L}^4_{\mathcal{F}}}=\{\mathbb{E}[(\int_0^T\|f(t)\|^2\mathrm{d}t)^2]\}^{\frac{1}{4}}$. *We have* $(\tilde{L}^4_{\mathcal{F}}(0,T;\mathbb{R}^n),\|\cdot\|_{\tilde{L}^4_{\mathcal{F}}})$ *is a normed linear space.*

Proof. Firstly, it is easy to prove that $\tilde{L}^4_{\mathcal{F}}(0,T;\mathbb{R}^n)$ is a linear space, and $\|\cdot\|_{\tilde{L}^4_{\mathcal{F}}}$ is non-negative and homogeneous. Next, we will prove that $\|\cdot\|_{\tilde{L}^4_{\mathcal{F}}}$ satisfies triangle inequality.

For any given $f_1\in\tilde{L}^4_{\mathcal{F}}(0,T;\mathbb{R}^n)$ and $f_2\in\tilde{L}^4_{\mathcal{F}}(0,T;\mathbb{R}^n)$, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T\big\|f_1(t)+f_2(t)\big\|^2\mathrm{d}t\right)^2\right]\\
\leq&\mathbb{E}\left[\left(\int_0^T\big\|f_1(t)\big\|\big\|f_1(t)+f_2(t)\big\|\mathrm{d}t+\int_0^T\big\|f_2(t)\big\|\big\|f_1(t)+f_2(t)\big\|\mathrm{d}t\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
\leq & \mathbb{E}\left[\left(\left(\int_0^T \|f_1(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T \|f_1(t)+f_2(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}+\left(\int_0^T \|f_2(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T \|f_1(t)+f_2(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right)^2\right] \\
= & \mathbb{E}\left[\left(\left(\int_0^T \|f_1(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}+\left(\int_0^T \|f_2(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right)^2\left(\int_0^T \|f_1(t)+f_2(t)\|^2 \mathrm{d}t\right)\right] \\
\leq & \left\{\mathbb{E}\left[\left(\left(\int_0^T \|f_1(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}+\left(\int_0^T \|f_2(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right)^4\right]\right\}^{\frac{1}{2}}\left\{\mathbb{E}\left[\left(\int_0^T \|f_1(t)+f_2(t)\|^2 \mathrm{d}t\right)^2\right]\right\}^{\frac{1}{2}}.
\end{aligned}
$$

Dividing both sides of the above inequality into $\{\mathbb{E}[(\int_0^T \|f_1(t)+f_2(t)\|^2 \mathrm{d}t)^2]\}^{\frac{1}{2}}$, we have

$$
\left\{\mathbb{E}\left[\left(\int_0^T \|f_1(t)+f_2(t)\|^2 \mathrm{d}t\right)^2\right]\right\}^{\frac{1}{2}} \leq \left\{\mathbb{E}\left[\left(\left(\int_0^T \|f_1(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}+\left(\int_0^T \|f_2(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right)^4\right]\right\}^{\frac{1}{2}}.
$$

Taking the square root on both sides of the above inequality and combining Minkowski inequality, we have

$$
\begin{aligned}
\left\{\mathbb{E}\left[\left(\int_0^T \|f_1(t)+f_2(t)\|^2 \mathrm{d}t\right)^2\right]\right\}^{\frac{1}{4}} &\leq \left\{\mathbb{E}\left[\left(\left(\int_0^T \|f_1(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}+\left(\int_0^T \|f_2(t)\|^2 \mathrm{d}t\right)^{\frac{1}{2}}\right)^4\right]\right\}^{\frac{1}{4}} \\
&\leq \left\{\mathbb{E}\left[\left(\int_0^T \|f_1(t)\|^2 \mathrm{d}t\right)^2\right]\right\}^{\frac{1}{4}}+\left\{\mathbb{E}\left[\left(\int_0^T \|f_2(t)\|^2 \mathrm{d}t\right)^2\right]\right\}^{\frac{1}{4}},
\end{aligned}
$$

which leads to the triangle inequality. Therefore, $(\tilde{L}^4_{\mathcal{F}}(0,T;\mathbb{R}^n), \|\cdot\|_{\tilde{L}^4_{\mathcal{F}}})$ is a normed linear space. □

Proof of Theorem 4.3. We prove (4.7) first. It follows from (3.20) and (3.21) that

$$
\begin{aligned}
z_L^*(t) = &\int_{\Theta_L}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_L(\theta_L) \\
&+\int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}_L(\theta_L,x_L).
\end{aligned} \tag{C.1}
$$

It follows from (3.14), (3.18), (3.19), (3.22) and (3.23) that

$$
\begin{aligned}
x_L^{*,(N_L)}(t) = &\int_{\Theta_L}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_{N_L}(\theta_L) \\
&+\int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}_{N_L}(\theta_L,x_L)+\frac{1}{N_L}\sum_{i=1}^{N_L}\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s).
\end{aligned} \tag{C.2}
$$

Combining (C.1), (C.2), C_r inequality with (4.3) yields

$$
\int_0^T \left\|z_L^*(t)-x_L^{*,(N_L)}(t)\right\|^2 \mathrm{d}t
$$

$$
\begin{aligned}
\leq & 2\int_0^T \Bigg\| \int_{\Theta_L}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_L(\theta_L) \\
& - \int_{\Theta_L}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_{N_L}(\theta_L) \\
& + \int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}_L(\theta_L,x_L) - \int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}_{N_L}(\theta_L,x_L)\Bigg\|^2\mathrm{d}t \\
& + 2\int_0^T\left\|\frac{1}{N_L}\sum_{i=1}^{N_L}\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s)\right\|^2\mathrm{d}t \\
\leq & 2\left(\epsilon_{N_L}^2 + \int_0^T\left\|\frac{1}{N_L}\sum_{i=1}^{N_L}\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s)\right\|^2\mathrm{d}t\right).
\end{aligned}
$$

Taking mean-square on both sides of the above inequality and using C_r inequality, we have

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_0^T\left\|z_L^*(t)-x_L^{*,(N_L)}(t)\right\|^2\mathrm{d}t\right)^2\right] \\
\leq & 8\epsilon_{N_L}^4 + 8\mathbb{E}\left[\left(\int_0^T\left\|\frac{1}{N_L}\sum_{i=1}^{N_L}\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s)\right\|^2\mathrm{d}t\right)^2\right].
\end{aligned}
\tag{C.3}
$$

Noting that $\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s) = D_L(\theta_{L_i})W_{L_i}(t) - \int_0^t\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}D_L(\theta_{L_i})W_{L_i}(s)\mathrm{d}s$, we have

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_0^T\left\|\frac{1}{N_L}\sum_{i=1}^{N_L}\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s)\right\|^2\mathrm{d}t\right)^2\right] \\
\leq & \mathbb{E}\left[T\int_0^T\left\|\frac{1}{N_L}\sum_{i=1}^{N_L}\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s)\right\|^4\mathrm{d}t\right] \\
= & \mathbb{E}\left[T\int_0^T\left\|\frac{1}{N_L}\sum_{i=1}^{N_L}\left(D_L(\theta_{L_i})W_{L_i}(t)-\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}D_L(\theta_{L_i})W_{L_i}(s)\mathrm{d}s\right)\right\|^4\mathrm{d}t\right] \\
= & T\int_0^T\mathbb{E}\left[\left\|\frac{1}{N_L}\sum_{i=1}^{N_L}\left(D_L(\theta_{L_i})W_{L_i}(t)-\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}D_L(\theta_{L_i})W_{L_i}(s)\mathrm{d}s\right)\right\|^4\right]\mathrm{d}t \\
\leq & \frac{T}{N_L^4}\int_0^T\sum_{i=1}^{N_L}\mathbb{E}\left[\|D_L(\theta_{L_i})W_{L_i}(t)\|^4\right]\mathrm{d}t + \frac{T}{N_L^4}\int_0^T\sum_{i=1}^{N_L}\mathbb{E}\left[\left\|\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}D_L(\theta_{L_i})W_{L_i}(s)\mathrm{d}s\right\|^4\right]\mathrm{d}t \\
& + \frac{3T}{N_L^4}\int_0^T\sum_{i=1}^{N_L}\sum_{i'=1,i'\neq i}^{N_L}\mathbb{E}\left[\|D_L(\theta_{L_i})W_{L_i}(t)\|^2\left\|\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_{i'}}}(t,s)}{\partial s}D_L(\theta_{L_{i'}})W_{L_{i'}}(s)\mathrm{d}s\right\|^2\right]\mathrm{d}t \\
& + \frac{6T}{N_L^4}\int_0^T\sum_{i=1}^{N_L}\mathbb{E}\left[\|D_L(\theta_{L_i})W_{L_i}(t)\|^2\left\|\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}D_L(\theta_{L_i})W_{L_i}(s)\mathrm{d}s\right\|^2\right]\mathrm{d}t \\
& + \frac{3T}{N_L^4}\int_0^T\sum_{i=1}^{N_L}\sum_{i'=1,i'\neq i}^{N_L}\mathbb{E}\left[\|D_L(\theta_{L_i})W_{L_i}(t)\|^2\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2\right]\mathrm{d}t + \frac{3T}{N_L^4}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_0^T \sum_{i=1}^{N_L} \sum_{i'=1, i'\neq i}^{N_L} \mathbb{E}\left[\left\| \int_0^t \frac{\partial \phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s} D_L(\theta_{L_i}) W_{L_i}(s) \mathrm{d}s \right\|^2 \left\| \int_0^t \frac{\partial \phi_{\gamma,\theta_{L_{i'}}}(t,s)}{\partial s} D_L(\theta_{L_{i'}}) W_{L_{i'}}(s) \mathrm{d}s \right\|^2 \right] \mathrm{d}t \\
&+ \frac{4T}{N_L^4} \int_0^T \sum_{i=1}^{N_L} \mathbb{E}\left[\| D_L(\theta_{L_i}) W_{L_i}(t) \|^2 \big(D_L(\theta_{L_i}) W_{L_i}(t) \big)^\top \int_0^t \frac{\partial \phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s} D_L(\theta_{L_i}) W_{L_i}(s) \mathrm{d}s \right] \mathrm{d}t \\
&+ \frac{4T}{N_L^4} \int_0^T \sum_{i=1}^{N_L} \mathbb{E}\Bigg[\left\| \int_0^t \frac{\partial \phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s} D_L(\theta_{L_i}) W_{L_i}(s) \mathrm{d}s \right\|^2 \big(D_L(\theta_{L_i}) W_{L_i}(t) \big)^\top \\
&\times \int_0^t \frac{\partial \phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s} D_L(\theta_{L_i}) W_{L_i}(s) \mathrm{d}s \Bigg] \mathrm{d}t + \frac{12T}{N_L^4} \int_0^T \sum_{i=1}^{N_L} \sum_{i'=1, i'\neq i}^{N_L} \mathbb{E}\Bigg[\big(D_L(\theta_{L_i}) W_{L_i}(t) \big)^\top D_L(\theta_{L_{i'}}) W_{L_{i'}}(t) \\
&\times \left(\int_0^t \frac{\partial \phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s} D_L(\theta_{L_i}) W_{L_i}(s) \mathrm{d}s \right)^\top \int_0^t \frac{\partial \phi_{\gamma,\theta_{L_{i'}}}(t,s)}{\partial s} D_L(\theta_{L_{i'}}) W_{L_{i'}}(s) \mathrm{d}s \Bigg] \mathrm{d}t. \qquad \text{(C.4)}
\end{aligned}
$$

For the first term on the right side of (C.4), we have

$$
\begin{aligned}
\frac{T}{N_L^4} \int_0^T \sum_{i=1}^{N_L} \mathbb{E}\left[\| D_L(\theta_{L_i}) W_{L_i}(t) \|^4 \right] \mathrm{d}t \leq & \frac{T}{N_L^4} \sum_{i=1}^{N_L} \left(\| D_L(\theta_{L_i}) \|^4 \int_0^T \mathbb{E}\left[\| W_{L_i}(t) \|^4 \right] \mathrm{d}t \right) \\
\leq & \frac{p(p+2)T^4}{3N_L^3} \max_{i=1,2,\dots,N_L} \| D_L(\theta_{L_i}) \|^4. \qquad \text{(C.5)}
\end{aligned}
$$

For the second term on the right side of (C.4), we have

$$
\begin{aligned}
& \frac{T}{N_L^4} \int_0^T \sum_{i=1}^{N_L} \mathbb{E}\left[\left\| \int_0^t \frac{\partial \phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s} D_L(\theta_{L_i}) W_{L_i}(s) \mathrm{d}s \right\|^4 \right] \mathrm{d}t \\
\leq & \frac{T}{N_L^4} \int_0^T t^3 \sum_{i=1}^{N_L} \int_0^t \mathbb{E}\left[\left\| \frac{\partial \phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s} D_L(\theta_{L_i}) W_{L_i}(s) \right\|^4 \right] \mathrm{d}s \mathrm{d}t \\
\leq & \frac{C_1^4 p(p+2) T^8}{21 N_L^3} \max_{i=1,2,\dots,N_L} \| D_L(\theta_{L_i}) \|^4. \qquad \text{(C.6)}
\end{aligned}
$$

For the third term on the right side of (C.4), we have

$$
\begin{aligned}
& \frac{3T}{N_L^4} \int_0^T \sum_{i=1}^{N_L} \sum_{i'=1, i'\neq i}^{N_L} \left\{ \mathbb{E}\left[\| D_L(\theta_{L_i}) W_{L_i}(t) \|^2 \right] \mathbb{E}\left[\left\| \int_0^t \frac{\partial \phi_{\gamma,\theta_{L_{i'}}}(t,s)}{\partial s} D_L(\theta_{L_{i'}}) W_{L_{i'}}(s) \mathrm{d}s \right\|^2 \right] \right\} \mathrm{d}t \\
\leq & \frac{3 C_1^2 p^2 T^6 (N_L - 1)}{10 N_L^3} \max_{i=1,2,\dots,N_L} \| D_L(\theta_{L_i}) \|^4. \qquad \text{(C.7)}
\end{aligned}
$$

For the fourth term on the right side of (C.4), we have

$$
\begin{aligned}
& \frac{6T}{N_L^4} \int_0^T \sum_{i=1}^{N_L} \mathbb{E}\left[\| D_L(\theta_{L_i}) W_{L_i}(t) \|^2 \left\| \int_0^t \frac{\partial \phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s} D_L(\theta_{L_i}) W_{L_i}(s) \mathrm{d}s \right\|^2 \right] \mathrm{d}t \\
\leq & \frac{2\sqrt{3} C_1^2 p(p+2) T^6}{5 N_L^3} \max_{i=1,2,\dots,N_L} \| D_L(\theta_{L_i}) \|^4. \qquad \text{(C.8)}
\end{aligned}
$$

For the fifth term on the right side of (C.4), we have

$$\begin{aligned}&\frac{3T}{N_L^4}\int_0^T\sum_{i=1}^{N_L}\sum_{i'=1,i'\neq i}^{N_L}\mathbb{E}\left[\|D_L(\theta_{L_i})W_{L_i}(t)\|^2\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2\right]\mathrm{d}t\\ \leq&\frac{p(p+2)T^4(N_L-1)}{N_L^3}\max_{i=1,2,\ldots,N_L}\|D_L(\theta_{L_i})\|^4.\end{aligned}\tag{C.9}$$

For the sixth term on the right side of (C.4), we have

$$\begin{aligned}&\frac{3T}{N_L^4}\int_0^T\sum_{i=1}^{N_L}\sum_{i'=1,i'\neq i}^{N_L}\mathbb{E}\Bigg[\left\|\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}D_L(\theta_{L_i})W_{L_i}(s)\mathrm{d}s\right\|^2\\ &\times\left\|\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_{i'}}}(t,s)}{\partial s}D_L(\theta_{L_{i'}})W_{L_{i'}}(s)\mathrm{d}s\right\|^2\Bigg]\mathrm{d}t\\ \leq&\frac{3C_1^4p^2T^6(N_L-1)}{10N_L^3}\max_{i=1,2,\ldots,N_L}\|D_L(\theta_{L_i})\|^4.\end{aligned}\tag{C.10}$$

For the seventh term on the right side of (C.4), we have

$$\begin{aligned}&\frac{4T}{N_L^4}\int_0^T\sum_{i=1}^{N_L}\mathbb{E}\left[\|D_L(\theta_{L_i})W_{L_i}(t)\|^2\Big(D_L(\theta_{L_i})W_{L_i}(t)\Big)^\top\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}D_L(\theta_{L_i})W_{L_i}(s)\mathrm{d}s\right]\mathrm{d}t\\ \leq&\frac{\sqrt{2p(p+2)}C_1pT^5}{2N_L^3}\max_{i=1,2,\ldots,N_L}\|D_L(\theta_{L_i})\|^4.\end{aligned}\tag{C.11}$$

For the eighth term on the right side of (C.4), we have

$$\begin{aligned}&\frac{4T}{N_L^4}\int_0^T\sum_{i=1}^{N_L}\mathbb{E}\Bigg[\left\|\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}D_L(\theta_{L_i})W_{L_i}(s)\mathrm{d}s\right\|^2\Big(D_L(\theta_{L_i})W_{L_i}(t)\Big)^\top\\ &\times\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}D_L(\theta_{L_i})W_{L_i}(s)\mathrm{d}s\Bigg]\mathrm{d}t\leq\frac{\sqrt{6p(p+2)}C_1^3pT^7}{9N_L^3}\max_{i=1,2,\ldots,N_L}\|D_L(\theta_{L_i})\|^4.\end{aligned}\tag{C.12}$$

For the ninth term on the right side of (C.4), we have

$$\begin{aligned}&\frac{12T}{N_L^4}\int_0^T\sum_{i=1}^{N_L}\sum_{i'=1,i'\neq i}^{N_L}\mathbb{E}\Bigg[\Big(D_L(\theta_{L_i})W_{L_i}(t)\Big)^\top D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\left(\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}D_L(\theta_{L_i})W_{L_i}(s)\mathrm{d}s\right)^\top\\ &\times\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_{i'}}}(t,s)}{\partial s}D_L(\theta_{L_{i'}})W_{L_{i'}}(s)\mathrm{d}s\Bigg]\mathrm{d}t\leq\frac{6C_1^2p^2T^6(N_L+1)}{5N_L^3}\max_{i=1,2,\ldots,N_L}\|D_L(\theta_{L_i})\|^4.\end{aligned}\tag{C.13}$$

By (C.4)–(C.13) and the compactness of Θ_L, we have

$$\mathbb{E}\left[\left(\int_0^T\left\|\frac{1}{N_L}\sum_{i=1}^{N_L}\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s)\right\|^2\mathrm{d}t\right)^2\right]$$

$$
\begin{aligned}
\leq & \Bigg(\frac{p(p+2)T^4}{3N_L^3}+\frac{C_1^4p(p+2)T^8}{21N_L^3}+\frac{3C_1^2p^2T^6(N_L-1)}{10N_L^3}+\frac{2\sqrt{3}C_1^2p(p+2)T^6}{5N_L^3}+\frac{p(p+2)T^4(N_L-1)}{N_L^3}\\
&+\frac{3C_1^4p^2T^6(N_L-1)}{10N_L^3}+\frac{\sqrt{2p(p+2)}C_1pT^5}{2N_L^3}+\frac{\sqrt{6p(p+2)}C_1^3pT^7}{9N_L^3}+\frac{6C_1^2p^2T^6(N_L+1)}{5N_L^3}\Bigg)\max_{i=1,2,\dots,N_L}\|D_L(\theta_{L_i})\|^4\\
=&O\left(\frac{1}{N_L^2}\right).
\end{aligned}
$$

Combining the above inequality with (C.3), we have $\mathbb{E}\big[\big(\int_0^T\big\|z_L^*(t)-x_L^{*,(N_L)}(t)\big\|^2\mathrm{d}t\big)^2\big]\leq 8\epsilon_{N_L}^4+O\big(\frac{1}{N_L^2}\big)$, which together with C_r inequality leads to (4.7). Similar to the proof of (4.7), we can get (4.8). This completes the proof of Theorem 4.3. □

Appendix D. Proofs in Section 5

Proof of Lemma 5.1. Firstly, we prove (5.1). Combining (3.14), (3.18), (3.19) with (3.23), we obtain

$$
\begin{aligned}
x_{L_i}^*(t)=&\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\\
&+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)+\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2).
\end{aligned}\tag{D.1}
$$

By (D.1) and the distribution of θ_L and x_L, we get

$$
\begin{aligned}
x_L^{*,(N_L)}(t)=&\int_{\Theta_L}\int_0^t\int_{s_2}^T\phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_{N_L}(\theta_L)\\
&+\int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}_{N_L}(\theta_L,x_L)+\frac{1}{N_L}\sum_{i'=1}^{N_L}\int_0^t\phi_{\gamma,\theta_{L_{i'}}}(t,s)D_L(\theta_{L_{i'}})\mathrm{d}W_{L_{i'}}(s).
\end{aligned}\tag{D.2}
$$

Combining (D.1), (D.2), Jensen inequality, C_r inequality with Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{4}{\gamma}\Phi_L\left(x_{L_i}^*(\cdot),x_L^{*,(N_L)}(\cdot),u_{L_i}^*(\cdot)\right)\right)\right]\\
=&\mathbb{E}\bigg[\exp\bigg(\frac{4}{\gamma}\bigg(\int_0^T\Big\|x_{L_i}^*(t)-H_Lx_L^{*,(N_L)}(t)-g_L\Big\|_{Q_L}^2+\|u_{L_i}^*(t)\|_{R_L}^2\mathrm{d}t\bigg)\bigg)\bigg]\\
\leq&\frac{1}{T}\mathbb{E}\bigg[\int_0^T\exp\bigg[\frac{4T}{\gamma}\bigg(\Big\|x_{L_i}^*(t)-H_Lx_L^{*,(N_L)}(t)-g_L\Big\|_{Q_L}^2+\|-R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)x_{L_i}^*(t)\\
&+R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L^*}(t)\|_{R_L}^2\bigg)\bigg]\mathrm{d}t\bigg]\\
\leq&\frac{1}{T}\int_0^T\mathbb{E}\bigg[\exp\bigg(\frac{4T}{\gamma}\bigg(3\|x_{L_i}^*(t)\|_{Q_L}^2+3\Big\|H_Lx_L^{*,(N_L)}(t)\Big\|_{Q_L}^2+3\|g_L\|_{Q_L}^2+3\left\|R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)x_{L_i}^*(t)\right\|_{R_L}^2\\
&+3\left\|R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L^*}(t)\right\|_{R_L}^2\bigg)\bigg)\bigg]\mathrm{d}t\\
\leq&\frac{1}{T}\int_0^T\exp\bigg\{\frac{24T}{\gamma}\bigg[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2
\end{aligned}
$$

$$
\begin{aligned}
&+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\Big]^\top Q_L\Big[\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T \psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L^*(s_1)\\
&+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\Big]+\frac{24T}{\gamma}\Big[\int_{\Theta_L}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)\\
&\times[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_{N_L}(\theta_L)+\int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}_{N_L}(\theta_L,x_L)\Big]^\top H_L^\top Q_LH_L\\
&\times\Big[\int_{\Theta_L}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\\
&+\int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}_{N_L}(\theta_L,x_L)\Big]\mathrm{d}\mathbf{F}_{N_L}(\theta_L)+\frac{24T}{\gamma}\Big[\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\\
&\times\int_{s_2}^T \psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\Big]^\top P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\\
&\times P_{\gamma,\theta_{L_i}}(t)\Big[\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T \psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\\
&+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\Big]+\frac{12T}{\gamma}\|g_L\|_{Q_L}^2+\frac{12T}{\gamma}\left\|R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L^*}(t)\right\|_{R_L}^2\Bigg\}\\
&\times\mathbb{E}\Bigg\{\exp\Bigg[\frac{24T}{\gamma}\Big[\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\Big]^\top Q_L\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\\
&+\frac{24T}{\gamma N_L}\sum_{i'=1}^{N_L}\Big[\int_0^t \phi_{\gamma,\theta_{L_{i'}}}(t,s_2)D_L(\theta_{L_{i'}})\mathrm{d}W_{L_{i'}}(s_2)\Big]^\top H_L^\top Q_LH_L\int_0^t \phi_{\gamma,\theta_{L_{i'}}}(t,s_2)D_L(\theta_{L_{i'}})\mathrm{d}W_{L_{i'}}(s_2)\\
&+\frac{24T}{\gamma}\Big[\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\Big]^\top P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\\
&\times\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\Bigg]\Bigg\}\mathrm{d}t\\
\le&\frac{1}{T}J_5^{\frac12}J_6^{\frac12},
\end{aligned}
\tag{D.3}
$$

where

$$
\begin{aligned}
J_5=\int_0^T\exp\Bigg\{&\frac{48T}{\gamma}\Big[\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T \psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\\
&+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\Big]^\top Q_L\Big[\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T \psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L^*(s_1)\\
&+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\Big]+\frac{48T}{\gamma}\Big[\int_{\Theta_L}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\\
&\times\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_{N_L}(\theta_L)+\int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}_{N_L}(\theta_L,x_L)\Big]^\top H_L^\top Q_L\\
&\times H_L\Big[\int_{\Theta_L}\int_0^t\int_{s_2}^T \phi_{\gamma,\theta_L}(t,s_2)B_L(\theta_L)R_L^{-1}B_L^\top(\theta_L)\psi_{\gamma,\theta_L}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}\mathbf{F}_{N_L}(\theta_L)
\end{aligned}
$$

$$+\int_{\Theta_L\times X_L}\phi_{\gamma,\theta_L}(t,0)x_L\mathrm{d}\mathbf{F}_{N_L}(\theta_L,x_L)\Big]+\frac{48T}{\gamma}\Big[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)$$

$$\times[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\Big]^\top P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)$$

$$\times\Big[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\int_{s_2}^T\psi_{\gamma,\theta_{L_i}}(s_2,s_1)[Q_LH_Lz_L^*(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2$$

$$+\phi_{\gamma,\theta_{L_i}}(t,0)x_{L_i}(0)\Big]+\frac{24T}{\gamma}\|g_L\|_{Q_L}^2+\frac{24T}{\gamma}\|R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L^*}(t)\|_{R_L}^2\Big\}\mathrm{d}t,$$

and

$$J_6=\int_0^T\Bigg\{\mathbb{E}\Bigg[\exp\Bigg(\frac{24T}{\gamma}\Big(\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\Big)^\top Q_L\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)$$

$$+\frac{24T}{\gamma N_L}\Big(\sum_{i'=1}^{N_L}\Big(\int_0^t\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)D_L(\theta_{L_{i'}})\mathrm{d}W_{L_{i'}}(s_2)\Big)^\top H_L^\top Q_LH_L\int_0^t\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)D_L(\theta_{L_{i'}})\mathrm{d}W_{L_{i'}}(s_2)\Big)$$

$$+\frac{24T}{\gamma}\Big(\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\Big)^\top P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)$$

$$\times\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\Bigg)\Bigg]\Bigg\}^2\mathrm{d}t.$$

Since $\phi_{\gamma,\theta_{L_i}}(t,s)$ and $\psi_{\gamma,\theta_{L_i}}(t,s)$ are continuous with respect to $(t,s)\in[0,T]\times[0,T]$, $z_L^*(t)$, $\xi_{\gamma,\theta_{L_i},z_L^*}(t)$ and $P_{\gamma,\theta_{L_i}}(t)$ are continuous with respect to $t\in[0,T]$ and Θ_L is a bounded closed set, we have

$$J_5^{\frac{1}{2}}<\infty. \tag{D.4}$$

Noting that $\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s)=D_L(\theta_{L_i})W_{L_i}(t)-\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}D_L(\theta_{L_i})W_{L_i}(s)\mathrm{d}s$, by C_r inequality and Jensen inequality, we have

$$\begin{aligned}&\frac{24T}{\gamma}\Big[\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\Big]^\top Q_L\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\\ \le&\frac{48T}{\gamma}\|Q_L\|\Bigg(\|D_L(\theta_{L_i})W_{L_i}(t)\|^2+\Big\|\int_0^t\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\mathrm{d}s_2\Big\|^2\Bigg)\\ \le&\frac{48T}{\gamma}\|Q_L\|\Bigg(\|D_L(\theta_{L_i})W_{L_i}(t)\|^2+t\int_0^t\Big\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\Big\|^2\mathrm{d}s_2\Bigg).\end{aligned} \tag{D.5}$$

Similar to the above inequality, we obtain

$$\begin{aligned}&\frac{24T}{\gamma N_L}\Bigg[\sum_{i'=1}^{N_L}\Big(\int_0^t\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)D_L(\theta_{L_{i'}})\mathrm{d}W_{L_{i'}}(s_2)\Big)^\top H_L^\top Q_LH_L\int_0^t\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)D_L(\theta_{L_{i'}})\mathrm{d}W_{L_{i'}}(s_2)\Bigg]\\ \le&\frac{48T}{\gamma N_L}\sum_{i'=1}^{N_L}\|H_L^\top Q_LH_L\|\Bigg(\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2+t\int_0^t\Big\|\frac{\partial\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)}{\partial s_2}D_L(\theta_{L_{i'}})W_{L_{i'}}(s_2)\Big\|^2\mathrm{d}s_2\Bigg)\end{aligned} \tag{D.6}$$

and

$$\begin{aligned}
&\frac{24T}{\gamma}\left(\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\right)^\top P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\\
&\times\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s_2)D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_2)\\
\le&\frac{48T}{\gamma}\left\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\right\|\\
&\times\left(\|D_L(\theta_{L_i})W_{L_i}(t)\|^2+t\int_0^t\left\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\right\|^2\mathrm{d}s_2\right).
\end{aligned}\tag{D.7}$$

From (D.5)–(D.7) and the definition of J_6, we have

$$\begin{aligned}
J_6\le&\int_0^T\Bigg\{\mathbb{E}\Bigg[\exp\Bigg[\frac{48T}{\gamma}\Big(\|Q_L\|+\frac{1}{N_L}\|H_L^\top Q_LH_L\|+\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\|\Big)\\
&\times\|D_L(\theta_{L_i})W_{L_i}(t)\|^2\Bigg]\exp\Bigg(\frac{48T}{\gamma}\Big(\|Q_L\|+\frac{1}{N_L}\|H_L^\top Q_LH_L\|+\left\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\right\|\Big)\\
&\times t\int_0^T\left\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\right\|^2\mathrm{d}s_2\Bigg)\Bigg]\prod_{i'=1,i'\neq i}^{N_L}\mathbb{E}\Bigg[\exp\Bigg(\frac{48T}{\gamma N_L}\|H_L^\top Q_LH_L\|\\
&\times\Bigg(\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2+\int_0^T\left\|\frac{\partial\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)}{\partial s_2}D_L(\theta_{L_{i'}})W_{L_{i'}}(s_2)\right\|^2\mathrm{d}s_2\Bigg)\Bigg)\Bigg]\Bigg\}^2\mathrm{d}t\\
\le&\int_0^T\mathbb{E}\Bigg[\exp\Bigg[\frac{96T}{\gamma}\Big(\|Q_L\|+\frac{1}{N_L}\|H_L^\top Q_LH_L\|+\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\|\Big)\\
&\times\|D_L(\theta_{L_i})W_{L_i}(t)\|^2\Bigg]\Bigg]\frac{1}{T}\int_0^T\mathbb{E}\Bigg[\exp\Bigg(\frac{96T^2}{\gamma}\Big(\|Q_L\|+\frac{1}{N_L}\|H_L^\top Q_LH_L\|\\
&+\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\|\Big)t\left\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\right\|^2\Bigg)\Bigg]\mathrm{d}s_2\\
&\times\prod_{i'=1,i'\neq i}^{N_L}\Bigg\{\mathbb{E}\Bigg[\exp\Bigg(\frac{96T}{\gamma N_L}\|H_L^\top Q_LH_L\|\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2\Bigg)\Bigg]\Bigg\}\\
&\times\prod_{i'=1,i'\neq i}^{N_L}\Bigg\{\frac{1}{T}\int_0^T\mathbb{E}\Bigg[\exp\Bigg(\frac{96T^2t}{\gamma N_L}\|H_L^\top Q_LH_L\|\left\|\frac{\partial\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)}{\partial s_2}D_L(\theta_{L_{i'}})W_{L_{i'}}(s_2)\right\|^2\Bigg)\Bigg]\mathrm{d}s_2\Bigg\}\mathrm{d}t.
\end{aligned}\tag{D.8}$$

By the definition of $M_{\gamma,L}(\theta_{L_i})$, Assumption 2.5, the compactness of Θ_L and the calculation of the expectation of the exponential function related to Brownian motion, we have

$$\begin{aligned}
&\mathbb{E}\Bigg\{\exp\Bigg[\frac{96T}{\gamma}\Big(\|Q_L\|+\frac{1}{N_L}\|H_L^\top Q_LH_L\|+\left\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\right\|\Big)\\
&\times\|D_L(\theta_{L_i})W_{L_i}(t)\|^2\Bigg]\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}^p} \exp\bigg[\frac{96T}{\gamma}\bigg(\|Q_L\| + \frac{1}{N_L}\|H_L^\top Q_L H_L\| + \big(M_{\gamma,L}(\theta_{L_i})\big)^2 \left\|B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\right\|\bigg)\|D_L(\theta_{L_i})\|^2 x^\top x\bigg] \\
&\quad \times \frac{1}{(\sqrt{2\pi t})^p}\exp\bigg(-\frac{x^\top x}{2t}\bigg)\mathrm{d}x \\
&\le \bigg\{\gamma\bigg/\bigg[\gamma - 192 \sup_{0\le t,s\le T}\sup_{\theta_{L_i}\in\Theta_L}\bigg(\|Q_L\| + \frac{1}{N_L}\|H_L^\top Q_L H_L\| + \big(M_{\gamma,L}(\theta_{L_i})\big)^2 \left\|B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\right\|\bigg) \\
&\quad \times \|D_L(\theta_{L_i})\|^2 T^2\bigg]\bigg\}^{\frac{p}{2}} \\
&< \infty, \quad \forall\ t\in(0,T].
\end{aligned} \tag{D.9}
$$

Similar to the above inequality, we have

$$
\begin{aligned}
&\frac{1}{T}\int_0^T \mathbb{E}\bigg[\exp\bigg(\frac{96T^2}{\gamma}\bigg(\|Q_L\| + \frac{1}{N_L}\|H_L^\top Q_L H_L\| + \left\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\right\|\bigg) \\
&\quad \times t\bigg\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}D_L(\theta_{L_i})W_{L_i}(s_2)\bigg\|^2\bigg)\bigg]\mathrm{d}s_2 \\
&\le \frac{1}{T}\int_{0^+}^T\int_{\mathbb{R}^p}\exp\bigg[\frac{96T^3}{\gamma}\bigg(\|Q_L\| + \frac{1}{N_L}\|H_L^\top Q_L H_L\| + \left\|P_{\gamma,\theta_{L_i}}(t)B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\right\|\bigg) \\
&\quad \times \bigg\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s_2)}{\partial s_2}\bigg\|^2\|D_L(\theta_{L_i})\|^2 x^\top x\bigg]\frac{1}{(\sqrt{2\pi s_2})^p}\exp\bigg(-\frac{x^\top x}{2s_2}\bigg)\mathrm{d}x\mathrm{d}s_2 \\
&\le \frac{1}{T}\int_0^T\bigg\{\gamma\bigg/\bigg[\gamma - 192\sup_{0\le t,s\le T}\sup_{\theta_{L_i}\in\Theta_L}\bigg(\|Q_L\| + \frac{1}{N_L}\|H_L^\top Q_L H_L\| \\
&\quad + \big(M_{\gamma,L}(\theta_{L_i})\big)^2\left\|B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\right\|\bigg)\bigg\|\frac{\partial\phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s}\bigg\|^2\|D_L(\theta_{L_i})\|^2T^4\bigg]\bigg\}^{\frac{p}{2}}\mathrm{d}s_2 \\
&< \infty, \quad \forall\ t\in[0,T],
\end{aligned} \tag{D.10}
$$

$$
\begin{aligned}
&\prod_{i'=1,i'\ne i}^{N_L}\mathbb{E}\left[\exp\left(\frac{96T}{\gamma N_L}\|H_L^\top Q_L H_L\|\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2\right)\right] \\
&= \prod_{i'=1,i'\ne i}^{N_L}\left\{\mathbb{E}\left[\exp\left(\frac{96T}{\gamma}\|H_L^\top Q_L H_L\|\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2\right)\right]^{\frac{1}{N_L}}\right\} \\
&\le \prod_{i'=1,i'\ne i}^{N_L}\left\{\mathbb{E}\left[\exp\left(\frac{96T}{\gamma}\|H_L^\top Q_L H_L\|\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2\right)\right]\right\}^{\frac{1}{N_L}} \\
&= \prod_{i'=1,i'\ne i}^{N_L}\left\{\int_{\mathbb{R}^p}\exp\left(\frac{96T}{\gamma}\|H_L^\top Q_L H_L\|\|D_L(\theta_{L_{i'}})\|^2 x^\top x\right)\frac{1}{(\sqrt{2\pi t})^p}\exp\left(-\frac{x^\top x}{2t}\right)\mathrm{d}x\right\}^{\frac{1}{N_L}} \\
&\le \prod_{i'=1,i'\ne i}^{N_L}\left[\gamma\bigg/\left(\gamma - 192\sup_{\theta_{L_{i'}}\in\Theta_L}\|H_L^\top Q_L H_L\|\|D_L(\theta_{L_{i'}})\|^2T^2\right)\right]^{\frac{p}{2N_L}}
\end{aligned}
$$

$$<\infty, \quad \forall\, t \in (0, T], \tag{D.11}$$

and

$$\begin{aligned}
&\prod_{i'=1,i'\neq i}^{N_L}\left\{\frac{1}{T}\int_0^T \mathbb{E}\left[\exp\left(\frac{96T^2 t}{\gamma N_L}\|H_L^\top Q_L H_L\|\left\|\frac{\partial \phi_{\gamma,\theta_{L_{i'}}}(t,s_2)}{\partial s_2}D_L(\theta_{L_{i'}})W_{L_{i'}}(s_2)\right\|^2\right)\right]\mathrm{d}s_2\right\}\\
\le&\prod_{i'=1,i'\neq i}^{N_L}\left\{\frac{1}{T}\int_{0^+}^T\int_{\mathbb{R}^p}\exp\left(\frac{96T^3}{\gamma}\|H_L^\top Q_L H_L\|\left\|\frac{\partial \phi_{\gamma,\theta_{L_{i'}}}(t,s_2)}{\partial s_2}\right\|^2\|D_L(\theta_{L_{i'}})\|^2 x^\top x\right)\right.\\
&\left.\times\frac{1}{(\sqrt{2\pi s_2})^p}\exp\left(-\frac{x^\top x}{2s_2}\right)\mathrm{d}x\mathrm{d}s_2\right\}^{\frac{1}{N_L}}\\
\le&\prod_{i'=1,i'\neq i}^{N_L}\left\{\frac{1}{T}\int_0^T\left[\gamma\Big/\left(\gamma-192\sup_{0\le t,s\le T}\sup_{\theta_{L_{i'}}\in\Theta_L}\|H_L^\top Q_L H_L\|\left\|\frac{\partial \phi_{\gamma,\theta_{L_{i'}}}(t,s)}{\partial s}\right\|^2\right.\right.\right.\\
&\left.\left.\left.\times\|D_L(\theta_{L_{i'}})\|^2T^4\right)\right]^{\frac{1}{2}}\mathrm{d}s_2\right\}^{\frac{p}{N_L}}\\
<&\infty, \quad \forall\, t \in [0, T].
\end{aligned} \tag{D.12}$$

Combining (D.8)–(D.12), we have $J_6 < \infty$. This together with (D.3) and (D.4) leads to (5.1). Similar to the proof of (5.1), we can get (5.2).

Next, we prove (5.3). Firstly, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left\{\mathbb{E}\left[\left(\int_0^1\exp\left(\frac{(1-s)}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),x_L^{*,(N_L)}(\cdot),u^*_{L_i}(\cdot)\right)\right)\exp\left(\frac{s}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),z^*_L(\cdot),u^*_{L_i}(\cdot)\right)\right)\mathrm{d}s\right)^2\right]\right\}^{\frac{1}{2}}\\
\le&\left\{\mathbb{E}\left[\int_0^1\exp\left(\frac{2(1-s)}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),x_L^{*,(N_L)}(\cdot),u^*_{L_i}(\cdot)\right)\right)\mathrm{d}s\int_0^1\exp\left(\frac{2s}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),z^*_L(\cdot),u^*_{L_i}(\cdot)\right)\right)\mathrm{d}s\right]\right\}^{\frac{1}{2}}\\
\le&\left\{\mathbb{E}\left[\left(\int_0^1\exp\left(\frac{2(1-s)}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),x_L^{*,(N_L)}(\cdot),u^*_{L_i}(\cdot)\right)\right)\mathrm{d}s\right)^2\right]\right\}^{\frac{1}{4}}\\
&\times\left\{\mathbb{E}\left[\left(\int_0^1\exp\left(\frac{2s}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),z^*_L(\cdot),u^*_{L_i}(\cdot)\right)\right)\mathrm{d}s\right)^2\right]\right\}^{\frac{1}{4}}.
\end{aligned} \tag{D.13}$$

By Jensen inequality, we get

$$\begin{aligned}
&\left[\int_0^1\exp\left(\frac{2(1-s)}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),x_L^{*,(N_L)}(\cdot),u^*_{L_i}(\cdot)\right)\right)\mathrm{d}s\right]^2\\
\le&\int_0^1\exp\left(\frac{4(1-s)}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),x_L^{*,(N_L)}(\cdot),u^*_{L_i}(\cdot)\right)\right)\mathrm{d}s
\end{aligned} \tag{D.14}$$

and

$$\left[\int_0^1\exp\left(\frac{2s}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),z^*_L(\cdot),u^*_{L_i}(\cdot)\right)\mathrm{d}s\right]^2 \le \int_0^1\exp\left(\frac{4s}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot),z^*_L(\cdot),u^*_{L_i}(\cdot)\right)\right)\mathrm{d}s. \tag{D.15}$$

It follows from (D.13)–(D.15) that

$$\begin{aligned}
&\left\{\mathbb{E}\left[\left(\int_0^1 \exp\left(\frac{(1-s)}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\right)\right)\exp\left(\frac{s}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\right)\right)\mathrm{d}s\right)^2\right]\right\}^{\frac{1}{2}}\\
\le&\left\{\mathbb{E}\left[\int_0^1 \exp\left(\frac{4(1-s)}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\right)\right)\mathrm{d}s\right]\right\}^{\frac{1}{4}}\\
&\times\left\{\mathbb{E}\left[\int_0^1 \exp\left(\frac{4s}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\right)\right)\mathrm{d}s\right]\right\}^{\frac{1}{4}}\\
\le&\left\{\mathbb{E}\left[\exp\left(\frac{4}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\right)\right)\right]\right\}^{\frac{1}{4}}\left\{\mathbb{E}\left[\exp\left(\frac{4}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\right)\right)\right]\right\}^{\frac{1}{4}}.
\end{aligned}$$

This together with (5.1) and (5.2) yields

$$\left\{\mathbb{E}\left[\left(\int_0^1 \exp\left(\frac{(1-s)}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot), x_L^{*,(N_L)}(\cdot), u^*_{L_i}(\cdot)\right)\right)\exp\left(\frac{s}{\gamma}\Phi_L\left(x^*_{L_i}(\cdot), z^*_L(\cdot), u^*_{L_i}(\cdot)\right)\right)\mathrm{d}s\right)^2\right]\right\}^{\frac{1}{2}} < \infty.$$

This completes the proof of (5.3). We can prove (5.4)–(5.6) by a similar proof of (5.1)–(5.3). □

Proof of Theorem 5.4. Firstly, we prove (5.12). The process is similar to the proof of Theorem 4.6 of [36], which can be divided into two steps.

The first step. To prove

$$\mathbb{E}\left[\left(\int_0^T \left\|\frac{1}{N_L}x_{L_i}(t) + \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(t) - z^*_L(t)\right\|^2 \mathrm{d}t\right)^2\right] = O\left(\epsilon^4_{N_L} + \frac{1}{N_L^2}\right), \quad i = 1,\dots,N_L. \tag{D.16}$$

Let u_{L_i} be the strategy of the major player i, the strategies of other players are denoted by $u^*_{L_k}, k = 1,\dots,i-1, i+1,\cdots,N_L$. The state equations of the major player i and the major player $k (k \neq i)$ are given by

$$\begin{aligned}
\mathrm{d}x_{L_i}(t) =&[A_L(\theta_{L_i}) - B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)]x_{L_i}(t)\mathrm{d}t + B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z^*_L}(t)\mathrm{d}t\\
&+ D_L(\theta_{L_i})\mathrm{d}W_{L_i}(t) + B_L(\theta_{L_i})(u_{L_i}(t) - u^*_{L_i}(t))\mathrm{d}t, \qquad \text{(D.17)}\\
dx^*_{L_k}(t) =&[A_L(\theta_{L_k}) - B_L(\theta_{L_k})R_L^{-1}B_L^\top(\theta_{L_k})P_{\gamma,\theta_{L_k}}(t)]x^*_{L_k}(t)\mathrm{d}t + B_L(\theta_{L_k})R_L^{-1}B_L^\top(\theta_{L_k})\xi_{\gamma,\theta_{L_k},z^*_L}(t)\mathrm{d}t\\
&+ D_L(\theta_{L_k})\mathrm{d}W_{L_k}(t). \qquad \text{(D.18)}
\end{aligned}$$

The cost function of the major player i is given by

$$J_{L_i}\left(u_{L_i}(\cdot), u^*_{L_{-i}}(\cdot)\right) = \gamma \ln \mathbb{E}\left[\exp\left(\frac{1}{\gamma}\Phi_L\left(x_{L_i}(\cdot), \frac{1}{N_L}x_{L_i}(\cdot) + \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(\cdot), u_{L_i}(\cdot)\right)\right)\right],$$

where $u^*_{L_{-i}}(\cdot) = \left(u^*_{L_1}(\cdot), u^*_{L_2}(\cdot),\dots,u^*_{L_{i-1}}(\cdot), u^*_{L_{i+1}}(\cdot),\dots,u^*_{L_{N_L}}(\cdot)\right)$, $\Phi_L\Big(x_{L_i}(\cdot), \frac{1}{N_L}x_{L_i}(\cdot) + \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(\cdot),$ $u_{L_i}(\cdot)\Big) = \int_0^T \left(\|x_{L_i}(t) - H_L z^*_L(t) + H_L\left(z^*_L(t) - \frac{1}{N_L}x_{L_i}(t) - \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(t)\right) - g_L\|^2_{Q_L} + \|u_{L_i}(t)\|^2_{R_L}\right)\mathrm{d}t$. The admissible control set is $\mathcal{U}^g_{L_i}$, which is defined in Section 2. By lemma 5.1, there exists a constant C_2 such that

$J_{L_i}\big(u^*_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\big)\le C_2$. If $u_{L_i}(\cdot)$ satisfies $J_{L_i}\big(u_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\big)>C_2$, it is obvious that

$$J_{L_i}\big(u^*_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\big)\le J_{L_i}\big(u_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\big)+O\left(\epsilon_{N_L}+\frac{1}{\sqrt{N_L}}\right),\quad i=1,2,\dots,N_L. \tag{D.19}$$

In the following part of step 1 and step 2, we only consider the strategy u_{L_i} satisfying $J_{L_i}\big(u_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\big)\le J_{L_i}\big(u^*_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\big)\le C_2$.

Since $u_{L_i}(\cdot)\in\mathcal{U}^g_{L_i}$, we have $\mathbb{E}\big\{\exp\big[\frac{1}{\gamma}\int_0^T\|u_{L_i}(t)\|^2_{R_L}\mathrm{d}t\big]\big\}<\infty$. This together with Jensen inequality gives

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t\|u_{L_i}(s)\|^2_{R_L}\mathrm{d}s\right)^2\right]=&\gamma^2\mathbb{E}\left[\left(\frac{1}{\gamma}\int_0^t\|u_{L_i}(s)\|^2_{R_L}\mathrm{d}s\right)^2\right]\\ \le&\gamma^2\mathbb{E}\left\{\left[\exp\left(\frac{1}{\gamma}\int_0^t\|u_{L_i}(s)\|^2_{R_L}\mathrm{d}s\right)\right]^2\right\}\\ \le&\gamma^2\mathbb{E}\left\{\left[\exp\left(\frac{1}{\gamma}\int_0^T\|u_{L_i}(s)\|^2_{R_L}\mathrm{d}s\right)\right]^2\right\}\\ <&\infty,\quad\forall\, t\in[0,T].\end{aligned} \tag{D.20}$$

For any given $x\in\mathbb{R}^m$, define $\|x\|_{R_L}=(\|x\|^2_{R_L})^{\frac{1}{2}}$. It is easy to prove that $\|\cdot\|_{R_L}$ is a norm on $\mathbb{R}^m$ over the field $\mathbb{R}$. Because all norms on a finite-dimensional normed linear space are equivalent, so $\|\cdot\|_{R_L}$ is equivalent to $\|\cdot\|$ on $\mathbb{R}^m$. Hence, there exist constants $\alpha>0$ and $\beta>0$ such that $\alpha\|x\|_{R_L}\le\|x\|_2\le\beta\|x\|_{R_L}$, which together with (D.20) leads to

$$\mathbb{E}\left[\left(\int_0^t\|u_{L_i}(s)\|^2\mathrm{d}s\right)^2\right]\le\beta^4\mathbb{E}\left[\left(\int_0^t\|u_{L_i}(s)\|^2_{R_L}\mathrm{d}s\right)^2\right]<\infty,\quad\forall\, t\in[0,T],$$

which together with (3.19), C_r inequality, the compactness of Θ_L and the continuity of $P_{\gamma,\theta_{L_i}}(s)$ and $\xi_{\gamma,\theta_{L_i},z^*_L}(s)$ with respect to $s\in[0,T]$ follows that there exist constants C_3 and C_4 such that

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^t\|u_{L_i}(s)-u^*_{L_i}(s)\|^2\mathrm{d}s\right)^2\right]\\ \le&\mathbb{E}\left[\left(2\int_0^t\|u_{L_i}(s)\|^2\mathrm{d}s+2\int_0^t\|u^*_{L_i}(s)\|^2\mathrm{d}s\right)^2\right]\\ \le&8\mathbb{E}\left[\left(\int_0^t\|u_{L_i}(s)\|^2\mathrm{d}s\right)^2\right]+8\mathbb{E}\left[\left(\int_0^t\|u^*_{L_i}(s)\|^2\mathrm{d}s\right)^2\right]\\ =&8\mathbb{E}\left[\left(\int_0^t\|u_{L_i}(s)\|^2\mathrm{d}s\right)^2\right]+8\mathbb{E}\left[\left(\int_0^t\|-R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(s)x_{L_i}(s)+R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z^*_L}(s)\|^2\mathrm{d}s\right)^2\right]\\ \le&C_3+C_4\mathbb{E}\left[\left(\int_0^t\|x_{L_i}(s)\|^2\mathrm{d}s\right)^2\right],\quad\forall\, t\in[0,T].\end{aligned}$$

By the above inequality and Jensen inequality, we have

$$\begin{aligned}
&\mathbb{E}\left\{\left[\int_0^t\left(\int_0^s\|u_{L_i}(s_1)-u^*_{L_i}(s_1)\|\mathrm{d}s_1\right)^2\mathrm{d}s\right]^2\right\}\\
\leq&\mathbb{E}\left[\left(\int_0^t s\int_0^s\|u_{L_i}(s_1)-u^*_{L_i}(s_1)\|^2\mathrm{d}s_1\mathrm{d}s\right)^2\right]\\
\leq&\mathbb{E}\left[t\int_0^t s^2\left(\int_0^s\|u_{L_i}(s_1)-u^*_{L_i}(s_1)\|^2\mathrm{d}s_1\right)^2\mathrm{d}s\right]\\
\leq&t^3\int_0^t\mathbb{E}\left[\left(\int_0^s\|u_{L_i}(s_1)-u^*_{L_i}(s_1)\|^2\mathrm{d}s_1\right)^2\right]\mathrm{d}s\\
\leq&t^3\int_0^t\left\{C_3+C_4\mathbb{E}\left[\left(\int_0^s\|x_{L_i}(s_1)\|^2\mathrm{d}s_1\right)^2\right]\right\}\mathrm{d}s\\
\leq&C_3t^4+C_4t^3\int_0^t\mathbb{E}\left[\left(\int_0^s\|x_{L_i}(s_1)\|^2\mathrm{d}s_1\right)^2\right]\mathrm{d}s,\quad\forall\ t\in[0,T].
\end{aligned}\tag{D.21}$$

By (3.23), we can rewrite (D.17) and (D.18) as

$$x_{L_i}(t)=x^*_{L_i}(t)+\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_1)B_L(\theta_{L_i})(u_{L_i}(s_1)-u^*_{L_i}(s_1))\mathrm{d}s_1,\tag{D.22}$$

$$x^*_{L_k}(t)=x^*_{L_k}(t),\quad k=1,2,\dots,i-1,i+1,\dots,N_L.\tag{D.23}$$

By adding both sides of (D.22) and (D.23) for $i=1,\dots,N_L$ and normalizing by $\frac{1}{N_L}$, we have

$$\begin{aligned}
&\frac{1}{N_L}x_{L_i}(t)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(t)-z^*_L(t)\\
=&x_L^{*,(N_L)}(t)-z^*_L(t)+\frac{1}{N_L}\int_0^t\phi_{\gamma,\theta_{L_i}}(t,s_1)B_L(\theta_{L_i})(u_{L_i}(s_1)-u^*_{L_i}(s_1))\mathrm{d}s_1,
\end{aligned}$$

which together with Lemma A.1 gives

$$\begin{aligned}
&\left\|\frac{1}{N_L}x_{L_i}(t)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(t)-z^*_L(t)\right\|\\
\leq&\left\|x_L^{*,(N_L)}(t)-z^*_L(t)\right\|+\frac{C_1\|B_L(\theta_{L_i})\|}{N_L}\int_0^t\|u_{L_i}(s)-u^*_{L_i}(s)\|\mathrm{d}s,\quad\forall\ t\in[0,T].
\end{aligned}$$

Combining the above inequality with C_r inequality, we have

$$\mathbb{E}\left[\left(\int_0^T\left\|\frac{1}{N_L}x_{L_i}(t)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(t)-z^*_L(t)\right\|^2\mathrm{d}t\right)^2\right]$$

$$
\begin{aligned}
\leq & \mathbb{E}\left\{\left[2\int_0^T \left\|x_L^{*,(N_L)}(t)-z_L^*(t)\right\|^2 \mathrm{d}t+\frac{2C_1^2\|B_L(\theta_{L_i})\|^2}{N_L^2}\int_0^T\left(\int_0^t \|u_{L_i}(s)-u_{L_i}^*(s)\|\mathrm{d}s\right)^2\mathrm{d}t\right]^2\right\} \\
\leq & 8\mathbb{E}\left[\left(\int_0^T \left\|x_L^{*,(N_L)}(t)-z_L^*(t)\right\|^2 \mathrm{d}t\right)^2\right]+\frac{8C_1^4\|B_L(\theta_{L_i})\|^4}{N_L^4}\mathbb{E}\left\{\left[\int_0^T\left(\int_0^t \|u_{L_i}(s)-u_{L_i}^*(s)\|\mathrm{d}s\right)^2\mathrm{d}t\right]^2\right\}.
\end{aligned}
$$

Combining the above inequality with Theorem 4.3 and (D.21), we know that there exists a constant C_5 such that

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_0^T \left\|\frac{1}{N_L}x_{L_i}(t)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x_{L_{i'}}^*(t)-z_L^*(t)\right\|^2\mathrm{d}t\right)^2\right] \\
\leq & C_5\left(\epsilon_{N_L}^4+\frac{1}{N_L^2}\right)+\frac{8C_1^4C_3T^4\|B_L(\theta_{L_i})\|^4}{N_L^4}+\frac{C_1^4C_4T^3\|B_L(\theta_{L_i})\|^4}{N_L^4}\int_0^T \mathbb{E}\left[\left(\int_0^t \|x_{L_i}(s)\|^2\mathrm{d}s\right)^2\right]\mathrm{d}t.
\end{aligned} \tag{D.24}
$$

By (D.17) and C_r inequality, we get

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_0^t \|x_{L_i}(s)\|^2\mathrm{d}s\right)^2\right] \\
= & \mathbb{E}\Bigg[\Bigg(\int_0^t \Bigg\|x_{L_i}(0)+\int_0^s \big(A_L(\theta_{L_i})-B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(s_1)\big)x_{L_i}(s_1)\mathrm{d}s_1+\int_0^s B_L(\theta_{L_i})R_L^{-1} \\
& \times B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L^*}(s_1)\mathrm{d}s_1+\int_0^s B_L(\theta_{L_i})(u_{L_i}(s_1)-u_{L_i}^*(s_1))\mathrm{d}s_1+\int_0^s D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_1)\Bigg\|^2\mathrm{d}s\Bigg)^2\Bigg] \\
\leq & \mathbb{E}\Bigg\{\Bigg[4\int_0^t\Bigg(\Bigg\|x_{L_i}(0)+\int_0^s B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L^*}(s_1)\mathrm{d}s_1\Bigg\|^2+\Bigg\|\int_0^s \big(A_L(\theta_{L_i})-B_L(\theta_{L_i})R_L^{-1} \\
& \times B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(s_1)\big)x_{L_i}(s_1)\mathrm{d}s_1\Bigg\|^2+\Bigg\|\int_0^s B_L(\theta_{L_i})(u_{L_i}(s_1)-u_{L_i}^*(s_1))\mathrm{d}s_1\Bigg\|^2 \\
& +\Bigg\|\int_0^s D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_1)\Bigg\|^2\Bigg)\mathrm{d}s\Bigg]^2\Bigg\} \\
\leq & 64\left(\int_0^t\left\|x_{L_i}(0)+\int_0^s B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L^*}(s_1)\mathrm{d}s_1\right\|^2\mathrm{d}s\right)^2+64\mathbb{E}\Bigg[\Bigg(\int_0^t\Bigg\|\int_0^s\big(A_L(\theta_{L_i}) \\
& -B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(s_1)\big)x_{L_i}(s_1)\mathrm{d}s_1\Bigg\|^2\mathrm{d}s\Bigg)^2\Bigg]+64\mathbb{E}\Bigg[\Bigg(\int_0^t\Bigg\|\int_0^s B_L(\theta_{L_i}) \\
& \times(u_{L_i}(s_1)-u_{L_i}^*(s_1))\mathrm{d}s_1\Bigg\|^2\mathrm{d}s\Bigg)^2\Bigg]+64\mathbb{E}\left[\left(\int_0^t\left\|\int_0^s D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_1)\right\|^2\mathrm{d}s\right)^2\right], \quad \forall\, t\in[0,T].
\end{aligned} \tag{D.25}
$$

By the compactness of X_L and the continuity of $\xi_{\gamma,\theta_{L_i},z_L^*}(t)$ with respect to $t \in [0,T]$, we have

$$64\left(\int_0^t \left\| x_{L_i}(0) + \int_0^s B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L^*}(s_1)\mathrm{d}s_1 \right\|^2 \mathrm{d}s\right)^2 < \infty, \quad \forall\, t \in [0,T]. \tag{D.26}$$

From Jensen inequality, the compactness of Θ_L and the continuity of $P_{\gamma,\theta_{L_i}}(t)$ with respect to $t \in [0,T]$, we know that there exists a constant C_6 such that

$$\begin{aligned}
&64\mathbb{E}\left[\left(\int_0^t \left\| \int_0^s \big(A_L(\theta_{L_i}) - B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(s_1)\big)\tilde{x}'_{L_i}(s_1)\mathrm{d}s_1 \right\|^2 \mathrm{d}s\right)^2\right] \\
\leq & 64\mathbb{E}\left[\left(\int_0^t s \sup_{s_2\in[0,T]} \left\| A_L(\theta_{L_i}) - B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(s_2)\right\|^2 \int_0^s \left\|\tilde{x}'_{L_i}(s_1)\right\|^2 \mathrm{d}s_1\mathrm{d}s\right)^2\right] \\
\leq & 64t^3 \sup_{s_2\in[0,T]} \left\| A_L(\theta_{L_i}) - B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(s_2)\right\|^4 \int_0^t \mathbb{E}\left[\left(\int_0^s \left\|\tilde{x}'_{L_i}(s_1)\right\|^2 \mathrm{d}s_1\right)^2\right]\mathrm{d}s \\
\leq & C_3 \int_0^t \mathbb{E}\left[\left(\int_0^s \left\|\tilde{x}'_{L_i}(s_1)\right\|^2 \mathrm{d}s_1\right)^2\right]\mathrm{d}s, \quad \forall\, t \in [0,T].
\end{aligned} \tag{D.27}$$

By Jensen inequality and the compactness of Θ_L, we have

$$\begin{aligned}
64\mathbb{E}\left[\left(\int_0^t \left\| \int_0^s D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_1)\right\|^2 \mathrm{d}s\right)^2\right] \leq & 64\mathbb{E}\left[\left(\int_0^t \|D_L(\theta_{L_i})W_{L_i}(s)\|^2\mathrm{d}s\right)^2\right] \\
\leq & 64t\|D_L(\theta_{L_i})\|^2 \int_0^t \mathbb{E}[\|W_{L_i}(s)\|^4]\mathrm{d}s \\
= & \frac{64}{3}t^3\|D_L(\theta_{L_i})\|^2 p(p+2) \\
< & \infty, \quad \forall\, t \in [0,T].
\end{aligned} \tag{D.28}$$

Combining (D.25)–(D.28), we know that there exist constants C_7 and C_8 such that

$$\mathbb{E}\left[\left(\int_0^t \|x_{L_i}(s)\|^2\mathrm{d}s\right)^2\right] \leq C_7 + C_8\int_0^t \mathbb{E}\left[\left(\int_0^s \|x_{L_i}(s_1)\|^2\mathrm{d}s_1\right)^2\right]\mathrm{d}s, \quad \forall\, t \in [0,T].$$

By the above inequality and Gronwall inequality, we obtain

$$\mathbb{E}\left[\left(\int_0^t \|x_{L_i}(s)\|^2\mathrm{d}s\right)^2\right] < \infty, \quad \forall\, t \in [0,T],$$

which together with the compactness of Θ_L leads to

$$\frac{C_1^4 C_4 T^3 \|B_L(\theta_{L_i})\|^4}{N_L^4} \int_0^T \mathbb{E}\left[\left(\int_0^t \|x_{L_i}(s)\|^2\mathrm{d}s\right)^2\right]\mathrm{d}t = O\left(\frac{1}{N_L^4}\right). \tag{D.29}$$

By the compactness of Θ_L, we have $\frac{8C_1^4C_3T^4\|B_L(\theta_{L_i})\|^4}{N_L^4} = O\big(\frac{1}{N_L^4}\big)$, which together with (D.24) and (D.29) gives (D.16).

The second step. To prove

$$J_{L_i}\big(u_{L_i}(\cdot), u^*_{L_{-i}}(\cdot)\big) \geq J_{L_i}\big(u^*_{L_i}(\cdot), u^*_{L_{-i}}(\cdot)\big) - O\left(\epsilon_{N_L} + \frac{1}{\sqrt{N_L}}\right). \tag{D.30}$$

We rewrite (D.17) as

$$\begin{aligned} \mathrm{d}x_{L_i}(t) =& \Big[\big(A_L(\theta_{L_i}) - B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\big)x_{L_i}(t) + B_L u'_{L_i}(t) \\ &+ B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L^*}(t)\Big]\mathrm{d}t + D_L(\theta_{L_i})\mathrm{d}W_{L_i}(t), \end{aligned} \tag{D.31}$$

where $u'_{L_i}(t) = u_{L_i}(t) + R_L^{-1}B_L^\top[P_{\gamma,\theta_{L_i}}(t)x_{L_i}(t) - \xi_{\gamma,\theta_{L_i},z_L^*}(t)]$.

We constructed the following auxiliary problem. Minimize $\tilde{J}_{L_i}(u''_{L_i})$, *i.e.*

$$\min_{u''_{L_i}(\cdot)\in\tilde{\mathcal{U}}_{L_i}} \tilde{J}_{L_i}(u''_{L_i}(\cdot)) = \gamma \ln \mathbb{E}\Big[\exp\Big(\frac{1}{\gamma}\Phi_L\big(\tilde{x}_{L_i}(\cdot), z_L^*(\cdot), u''_{L_i}(\cdot) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot) - \xi_{\gamma,\theta_{L_i},z_L^*}(\cdot))\big)\Big)\Big],$$

s.t.

$$\begin{aligned} \mathrm{d}\tilde{x}_{L_i}(t) =& \Big[\big(A_L(\theta_{L_i}) - B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\big)\tilde{x}_{L_i}(t) + B_L(\theta_{L_i})u''_{L_i}(t) \\ &+ B_L(\theta_{L_i})R_L^{-1}B_L^\top(\theta_{L_i})\xi_{\gamma,\theta_{L_i},z_L^*}(t)\Big]\mathrm{d}t + D_L(\theta_{L_i})\mathrm{d}W_{L_i}(t), \end{aligned} \tag{D.32}$$

where $\tilde{x}_{L_i}(0) = x_L(0)$, $\Phi_L\big(\tilde{x}_{L_i}(\cdot), z_L^*(\cdot), u''_{L_i}(\cdot) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot) - \xi_{\gamma,\theta_{L_i},z_L^*}(\cdot))\big) = \int_0^T \big(\|\tilde{x}_{L_i}(t) - H_L z_L^*(t) - g_L\|_{Q_L}^2 + \|u''_{L_i}(t) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(t)\tilde{x}_{L_i}(t) - \xi_{\gamma,\theta_{L_i},z_L^*}(t))\|_{R_L}^2\big)\mathrm{d}t$, and the admissible control set is defined as

$$\begin{aligned} \tilde{\mathcal{U}}_{L_i} =\Bigg\{ u''_{L_i}(\cdot)\Bigg| & u''_{L_i}(t) \text{is adapted to } \sigma(\tilde{x}_{L_{i'}}(s), i' = 1, 2, \ldots, N_L, s \leq t), t \in [0, T], \\ & \text{the system (D.32) has a unique strong solution } \tilde{x}_{L_i}(\cdot), \\ & \mathbb{E}\Big[\int_0^T \Big(\exp\Big(\frac{\alpha M_{\gamma,L}(\theta_{L_i})}{\gamma}\|\tilde{x}_{L_i}(t)\|^2\Big)\Big)\|\tilde{x}_{L_i}(t)\|^2\mathrm{d}t\Big] < \infty, \text{ and } \mathbb{E}\Big[\exp\Big(\frac{32}{\gamma}\int_0^T \Big(\|\tilde{x}_{L_i}(t)\|_{Q_L}^2 \\ & + \Big\|\frac{1}{N_L}H_L\tilde{x}_{L_i}(t)\Big\|_{Q_L}^2 + \big\|u''_{L_i}(t) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(t)\tilde{x}_{L_i}(t) - \xi_{\gamma,\theta_{L_i},z_L^*}(t))\big\|_{R_L}^2\Big)\mathrm{d}t\Big)\Big] < \infty\Bigg\}, \end{aligned}$$

where α is same with that given in $\mathcal{U}^g_{L_i}$.

By comparing this auxiliary problem with Auxiliary problem (I), we know that $\tilde{J}_{L_i}\big(u''_{L_i}(\cdot)\big)$ reaches the minimum at $u''_{L_i}(\cdot) \equiv 0$. Hence,

$$\tilde{J}_{L_i}\big(u'_{L_i}(\cdot)\big) \geq \tilde{J}_{L_i}(0) = \bar{J}_{L_i,z_L^*}\big(u^*_{L_i}(\cdot)\big). \tag{D.33}$$

Take $u''_{L_i}(\cdot) = u'_{L_i}(\cdot)$ in (D.32) and let $\tilde{x}'_{L_i}(\cdot)$ be the associated solution. We have $\tilde{x}'_{L_i}(t) = x^*_{L_i}(t) + \int_0^t \psi_{\gamma,L_i}(t,s) \times u'_{L_i}(s)\mathrm{d}s$, where $x^*_{L_i}(t)$ is determined by (3.23). By (D.31) and (D.32), we have

$$\tilde{x}'_{L_i}(t) = x_{L_i}(t), \quad \forall\, t \in [0,T]. \tag{D.34}$$

Since $J_{L_i}\big(u_{L_i}(\cdot), u^*_{L_{-i}}(\cdot)\big) \leq C_2$, we have

$$\mathbb{E}\bigg[\exp\bigg(\frac{1}{\gamma}\Phi_L\bigg(x_{L_i}(\cdot), \frac{1}{N_L}x_{L_i}(\cdot) + \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(\cdot), u_{L_i}(\cdot)\bigg)\bigg)\bigg] < \infty. \tag{D.35}$$

By C_r inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\exp\bigg(\frac{1}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot), z^*_L(\cdot), u'_{L_i}(\cdot) - R_L^{-1}B_L^\top(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot) - \xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\bigg)\bigg]\\
\leq&\mathbb{E}\bigg[\exp\bigg(\frac{3}{\gamma}\int_0^T \Big(\|x_{L_i}(t)\|^2_{Q_L} + \|H_L z^*_L(t)\|^2_{Q_L} + \|g_L\|^2_{Q_L}\\
&+ \big\|u'_{L_i}(t) - R_L^{-1}B_L^\top(P_{\gamma,\theta_{L_i}}(t)\tilde{x}_{L_i}(t) - \xi_{\gamma,\theta_{L_i},z^*_L}(t))\big\|^2_{R_L}\Big)\mathrm{d}t\bigg)\bigg]\\
\leq&\mathbb{E}\bigg[\exp\bigg(\frac{3}{\gamma}\int_0^T \Big(\|x_{L_i}(t)\|^2_{Q_L} + \big\|u'_{L_i}(t) - R_L^{-1}B_L^\top(P_{\gamma,\theta_{L_i}}(t)\tilde{x}_{L_i}(t) - \xi_{\gamma,\theta_{L_i},z^*_L}(t))\big\|^2_{R_L}\Big)\,\mathrm{d}t\bigg)\bigg]\\
&\times \exp\bigg(\frac{3}{\gamma}\int_0^T \big(\|z^*_L(t)\|^2_{Q_L} + \|g_L\|^2_{Q_L}\big)\,\mathrm{d}t\bigg).
\end{aligned} \tag{D.36}$$

By the definition of admissible control sets $\mathcal{U}^g_{L_i}$ and the boundedness of $\|g_L\|^2_{Q_L}$, we have

$$\mathbb{E}\bigg[\exp\bigg(\frac{3}{\gamma}\int_0^T \Big(\|x_{L_i}(t)\|^2_{Q_L} + \big\|u'_{L_i}(t) - R_L^{-1}B_L^\top(P_{\gamma,\theta_{L_i}}(t)\tilde{x}_{L_i}(t) - \xi_{\gamma,\theta_{L_i},z^*_L}(t))\big\|^2_{R_L}\Big)\,\mathrm{d}t\bigg)\bigg] < \infty. \tag{D.37}$$

From the continuity of $z^*_L(t)$ with respect to $t \in [0,T]$ and boundedness of $\|g_L\|_{Q_L}$, we get

$$\exp\bigg(\frac{3}{\gamma}\int_0^T \big(\|z^*_L(t)\|^2_{Q_L} + \|g_L\|^2_{Q_L}\big)\,\mathrm{d}t\bigg) < \infty,$$

which together with (D.36) and (D.37) implies

$$\mathbb{E}\left[\exp\left(\frac{1}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot), z^*_L(\cdot), u'_{L_i}(\cdot) - R_L^{-1}B_L^\top(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot) - \xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\right)\right] < \infty. \tag{D.38}$$

Since $\gamma > 0$, we have $\Phi_L\big(x_{L_i}(\cdot), \frac{1}{N_L}x_{L_i}(\cdot) + \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(\cdot), u_{L_i}(\cdot)\big) \geq 0$ and $\Phi_L\big(\tilde{x}'_{L_i}(\cdot), z^*_L(\cdot), u'_{L_i}(\cdot) - R_L^{-1}B_L^\top(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot) - \xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big) \geq 0$. By the properties of the exponential function, (D.35) and (D.38), we have $1 \leq \mathbb{E}\big\{\exp\big[\frac{1}{\gamma}\Phi_L\big(x_{L_i}(\cdot), \frac{1}{N_L}x_{L_i}(\cdot) + \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(\cdot), u_{L_i}(\cdot)\big)\big]\big\} < \infty$ and $1 \leq \mathbb{E}\big[\exp\big(\frac{1}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot), z^*_L(\cdot), u'_{L_i}(\cdot) - R_L^{-1}B_L^\top(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot) - \xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\big)\big] < \infty$, and by the differential mean value theorem, we know that there exists a constant $b' \geq 1$ such that $b' \in \Big[\min\big\{\mathbb{E}\big[\exp\big[\frac{1}{\gamma}\Phi_L\big(x_{L_i}(\cdot), \frac{1}{N_L}x_{L_i}(\cdot) + \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(\cdot), u_{L_i}(\cdot)\big)\big]\big],\ \mathbb{E}\big[\exp\big(\frac{1}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot), z^*_L(\cdot),\ u'_{L_i}(\cdot) - R_L^{-1}B_L^\top(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot) - \xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\big)\big]\big\},$

$\max\big\{\mathbb{E}\big[\exp\big[\frac{1}{\gamma}\Phi_L\big(x_{L_i}(\cdot),\ \frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\big)\big]\big],\mathbb{E}\big[\exp\big(\frac{1}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top (P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\big)\big]\big\}\big]$ and

$$
\begin{aligned}
&\Big|J_{L_i}\big(u_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\big)-\tilde{J}_{L_i}\big(u'_{L_i}(\cdot)\big)\Big|\\
=&\Big|\gamma\ln\mathbb{E}\Big[\exp\Big(\frac{1}{\gamma}\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\Big)\Big]\\
&-\gamma\ln\mathbb{E}\Big[\exp\Big(\frac{1}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\Big)\Big]\Big|\\
=&\Big\{\frac{\gamma}{b'}\Big|\mathbb{E}\Big[\exp\Big(\frac{1}{\gamma}\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\Big)\\
&-\exp\Big(\frac{1}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\Big)\Big]\Big|\Big\}.
\end{aligned}
$$

By the above equation, Dyson expansion and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\Big|J_{L_i}(u_{L_i}(\cdot),u^*_{L_{-i}}(\cdot))-\tilde{J}_{L_i}(u'_{L_i}(\cdot))\Big|\\
=&\frac{1}{b'}\mathbb{E}\Big\{\int_0^1\exp\Big[\frac{(1-s)}{\gamma}\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\Big]\\
&\times\Big[\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\\
&-\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\Big]\\
&\times\exp\Big[\frac{s}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\Big]\mathrm{d}s\Big\}\\
\leq&\frac{1}{b'}\Big\{\mathbb{E}\Big[\Big|\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\\
&-\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\Big|^2\Big]\Big\}^{\frac{1}{2}}\\
&\times\Big\{\mathbb{E}\Big[\Big(\int_0^1\exp\Big(\frac{(1-s)}{\gamma}\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\Big)\\
&\times\exp\Big(\frac{s}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\Big)\mathrm{d}s\Big)^2\Big]\Big\}^{\frac{1}{2}}\\
\leq&\frac{1}{b'}\Big\{\mathbb{E}\Big[\Big|\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\\
&-\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\Big|^2\Big]\Big\}^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\times\bigg\{\mathbb{E}\bigg[\int_0^1 \exp\bigg(\frac{2(1-s)}{\gamma}\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\bigg)\mathrm{d}s\\
&\times\int_0^1 \exp\bigg(\frac{2s}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\bigg)\mathrm{d}s\bigg]\bigg\}^{\frac{1}{2}}\\
\le&\frac{1}{b'}\bigg\{\mathbb{E}\bigg[\bigg|\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\\
&-\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\bigg|^2\bigg]\bigg\}^{\frac{1}{2}}\\
&\times\bigg\{\mathbb{E}\bigg[\bigg(\int_0^1 \exp\bigg(\frac{2(1-s)}{\gamma}\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\bigg)\mathrm{d}s\bigg)^2\bigg]\bigg\}^{\frac{1}{4}}\\
&\times\bigg\{\mathbb{E}\bigg[\bigg(\int_0^1 \exp\bigg(\frac{2s}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\bigg)\mathrm{d}s\bigg)^2\bigg]\bigg\}^{\frac{1}{4}}.
\end{aligned}
$$

From the above inequality and Jensen inequality, we obtain

$$
\begin{aligned}
&\Big|J_{L_i}\big(u_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\big)-\tilde{J}_{L_i}\big(u'_{L_i}(\cdot)\big)\Big|\\
\le&\frac{1}{b'}\bigg\{\mathbb{E}\bigg[\bigg|\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\\
&-\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\bigg|^2\bigg]\bigg\}^{\frac{1}{2}}\\
&\times\bigg\{\mathbb{E}\bigg[\int_0^1 \exp\bigg(\frac{4(1-s)}{\gamma}\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\bigg)\mathrm{d}s\bigg]\bigg\}^{\frac{1}{4}}\\
&\times\bigg\{\mathbb{E}\bigg[\int_0^1 \exp\bigg(\frac{4s}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\bigg)\mathrm{d}s\bigg]\bigg\}^{\frac{1}{4}}\\
\le&\frac{1}{b'}\bigg\{\mathbb{E}\bigg[\bigg|\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\\
&-\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\bigg|^2\bigg]\bigg\}^{\frac{1}{2}}\\
&\times\bigg\{\mathbb{E}\bigg[\exp\bigg(\frac{4}{\gamma}\Phi_L\Big(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\Big)\bigg)\bigg]\bigg\}^{\frac{1}{4}}\\
&\times\bigg\{\mathbb{E}\bigg[\exp\bigg(\frac{4}{\gamma}\Phi_L\big(\tilde{x}'_{L_i}(\cdot),z^*_L(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\bigg)\bigg]\bigg\}^{\frac{1}{4}}.
\end{aligned}
\tag{D.39}
$$

By $a^2 - b^2 \le (a-b)^2 + 2|b||a-b|$, (D.34) and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\mathbb{E}\Bigg[\Bigg|\Phi_L\Big(x_{L_i}(\cdot), \frac{1}{N_L}x_{L_i}(\cdot) + \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(\cdot), u_{L_i}(\cdot)\Big) \\
&- \Phi_L\big(\tilde{x}'_{L_i}(\cdot), z^*_L(\cdot), u'_{L_i}(\cdot) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot) - \xi_{\gamma,\theta_{L_i},z^*_L}(\cdot))\big)\Bigg|^2\Bigg] \\
=&\mathbb{E}\Bigg\{\Bigg[\int_0^T \Bigg(\Big\|x_{L_i}(t) - H_L z^*_L(t) + H_L\Big(z^*_L(t) - \frac{1}{N_L}x_{L_i}(t) - \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(t)\Big) - g_L\Big\|^2_{Q_L} \\
&-\|\tilde{x}'_{L_i}(t) - H_L z^*_L(t) - g_L\|^2_{Q_L}\Bigg)\mathrm{d}t + \int_0^T \Big(\|u'_{L_i}(t) - R_L^{-1}B_L^\top(\theta_{L_i})\big(P_{\gamma,\theta_{L_i}}(t)x_{L_i}(t) - \xi_{\gamma,\theta_{L_i},z^*_L}(t)\big)\|^2_{R_L} \\
&-\|u'_{L_i}(t) - R_L^{-1}B_L^\top(\theta_{L_i})\big(P_{\gamma,\theta_{L_i}}(t)\tilde{x}'_{L_i} - \xi_{\gamma,\theta_{L_i},z^*_L}(t)\big)\|^2_{R_L}\Big)\mathrm{d}t\Bigg]^2\Bigg\} \\
\le&\mathbb{E}\Bigg\{\Bigg[\int_0^T \|Q_L\|\Bigg\|\Big(x_{L_i}(t) - H_L z^*_L(t) + H_L\Big(z^*_L(t) - \frac{1}{N_L}x_{L_i}(t) - \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(t)\Big) - g_L\Big) \\
&- \Big(\tilde{x}'_{L_i}(t) - H_L z^*_L(t) - g_L\Big)\Bigg\|\Bigg\|\Big(x_{L_i} - H_L z^*_L(t) + H_L\Big(z^*_L(t) - \frac{1}{N_L}x_{L_i}(t) - \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(t)\Big) \\
&- g_L\Big) - \Big(\tilde{x}'_{L_i}(t) - H_L z^*_L(t) - g_L\Big) + 2\Big(\tilde{x}'_{L_i}(t) - H_L z^*_L(t) - g_L\Big)\Bigg\|\mathrm{d}t \\
&+ \int_0^T \|R_L\|\Big\|\Big(u'_{L_i}(t) - R_L^{-1}B_L^\top(\theta_{L_i})\big(P_{\gamma,\theta_{L_i}}(t)x_{L_i}(t) - \xi_{\gamma,\theta_{L_i},z^*_L}(t)\big)\Big) \\
&- \Big(u'_{L_i}(t) - R_L^{-1}B_L^\top(\theta_{L_i})\big(P_{\gamma,\theta_{L_i}}(t)\tilde{x}'_{L_i} - \xi_{\gamma,\theta_{L_i},z^*_L}(t)\big)\Big)\Big\|\Big\|\Big(u'_{L_i}(t) - R_L^{-1}B_L^\top(\theta_{L_i}) \\
&\times \big(P_{\gamma,\theta_{L_i}}(t)x_{L_i}(t) - \xi_{\gamma,\theta_{L_i},z^*_L}(t)\big)\Big) - \Big(u'_{L_i}(t) - R_L^{-1}B_L^\top(\theta_{L_i})\big(P_{\gamma,\theta_{L_i}}(t)\tilde{x}'_{L_i} - \xi_{\gamma,\theta_{L_i},z^*_L}(t)\big)\Big) \\
&+ 2\Big(u'_{L_i}(t) - R_L^{-1}B_L^\top(\theta_{L_i})\big(P_{\gamma,\theta_{L_i}}(t)\tilde{x}'_{L_i} - \xi_{\gamma,\theta_{L_i},z^*_L}(t)\big)\Big)\Big\|\mathrm{d}t\Bigg]^2\Bigg\} \\
\le&\mathbb{E}\Bigg\{\Bigg[\int_0^T \Bigg(\|Q_L\|\Big\|x_{L_i}(t) - \tilde{x}'_{L_i}(t) + H_L\Big(z^*_L(t) - \frac{1}{N_L}x_{L_i}(t) - \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(t)\Big)\Big\|^2 + 2\|Q_L\| \\
&\times \Big\|x_{L_i}(t) - \tilde{x}'_{L_i}(t) + H_L\Big(z^*_L(t) - \frac{1}{N_L}x_{L_i}(t) - \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(t)\Big)\Big\|\|\tilde{x}'_{L_i}(t) - H_L z^*_L(t) - g_L\|\Bigg)\mathrm{d}t \\
&+ \int_0^T \Big(\|R_L\|\|R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\|^2\|x_{L_i}(t) - \tilde{x}'_{L_i}(t)\|^2 + 2\|R_L\|\|R_L^{-1}B_L^\top(\theta_{L_i})P_{\gamma,\theta_{L_i}}(t)\| \\
&\times \|x_{L_i}(t) - \tilde{x}'_{L_i}(t)\|\|u'_{L_i}(t) - R_L^{-1}B_L^\top(\theta_{L_i})\big(P_{\gamma,\theta_{L_i}}(t)\tilde{x}'_{L_i}(t) - \xi_{\gamma,\theta_{L_i},z^*_L}(t)\big)\|\Big)\mathrm{d}t\Bigg]^2\Bigg\} \\
=&\mathbb{E}\Bigg\{\Bigg[\int_0^T \Bigg(\|Q_L\|\Big\|H_L\Big(z^*_L(t) - \frac{1}{N_L}x_{L_i}(t) - \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(t)\Big)\Big\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+2\|Q_L\|\bigg\|H_L\Big(z_L^*(t)-\frac{1}{N_L}x_{L_i}(t)-\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x_{L_{i'}}^*(t)\Big)\bigg\|\|\tilde{x}_{L_i}'(t)-H_Lz_L^*(t)-g_L\|\Big)\mathrm{d}t\bigg]^2\bigg\}\\
\le&\mathbb{E}\bigg\{\bigg[\|Q_L\|\int_0^T\bigg\|H_L\Big(z_L^*(t)-\frac{1}{N_L}x_{L_i}(t)-\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x_{L_{i'}}^*(t)\Big)\bigg\|^2\mathrm{d}t+2\|Q_L\|\bigg(\int_0^T\bigg\|H_L\Big(z_L^*(t)\\
&-\frac{1}{N_L}x_{L_i}(t)-\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x_{L_{i'}}^*(t)\Big)\bigg\|^2\mathrm{d}t\bigg)^{\frac{1}{2}}\Big(\int_0^T\|\tilde{x}_{L_i}'(t)-H_Lz_L^*(t)-g_L\|^2\mathrm{d}t\Big)^{\frac{1}{2}}\bigg]^2\bigg\}\\
\le&2\mathbb{E}\bigg[\|Q_L\|^2\bigg(\int_0^T\bigg\|H_L\Big(z_L^*(t)-\frac{1}{N_L}x_{L_i}(t)-\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x_{L_{i'}}^*(t)\Big)\bigg\|^2\mathrm{d}t\bigg)^2\bigg]+8\mathbb{E}\bigg[\|Q_L\|^2\\
&\times\int_0^T\bigg\|H_L\Big(z_L^*(t)-\frac{1}{N_L}x_{L_i}(t)-\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x_{L_{i'}}^*(t)\Big)\bigg\|^2\mathrm{d}t\int_0^T\|\tilde{x}_{L_i}'(t)-H_Lz_L^*(t)-g_L\|^2\mathrm{d}t\bigg]\\
\le&2\|Q_L\|^2\mathbb{E}\bigg[\bigg(\int_0^T\bigg\|H_L\Big(\frac{1}{N_L}x_{L_i}(t)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x_{L_{i'}}^*(t)-z_L^*(t)\Big)\bigg\|^2\mathrm{d}t\bigg)^2\bigg]\\
&+8\|Q_L\|^2\bigg\{\mathbb{E}\bigg[\bigg(\int_0^T\bigg\|H_L\Big(\frac{1}{N_L}x_{L_i}(t)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x_{L_{i'}}^*(t)-z_L^*(t)\Big)\bigg\|^2\mathrm{d}t\bigg)^2\bigg]\bigg\}^{\frac{1}{2}}\\
&\times\bigg\{\mathbb{E}\bigg[\Big(\int_0^T\|\tilde{x}_{L_i}'(t)-H_Lz_L^*(t)-g_L\|^2\mathrm{d}t\Big)^2\bigg]\bigg\}^{\frac{1}{2}}.
\end{aligned}
\tag{D.40}
$$

By C_r inequality, we have

$$
\begin{aligned}
\mathbb{E}\left[\Big(\int_0^T\|\tilde{x}_{L_i}'(t)-H_Lz_L^*(t)-g_L\|^2\mathrm{d}t\Big)^2\right]\le&\mathbb{E}\left[\Big(\int_0^T\Big(2\|\tilde{x}_{L_i}'(t)\|^2+2\|H_Lz_L^*(t)+g_L\|^2\Big)\mathrm{d}t\Big)^2\right]\\
\le&8\mathbb{E}\left[\Big(\int_0^T\|\tilde{x}_{L_i}'(t)\|^2\mathrm{d}t\Big)^2\right]+8\Big(\int_0^T\Big(\|H_Lz_L^*(t)+g_L\|^2\Big)\mathrm{d}t\Big)^2.
\end{aligned}
\tag{D.41}
$$

By (D.32) and C_r inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left[\Big(\int_0^t\|\tilde{x}_{L_i}'(s)\|^2\mathrm{d}s\Big)^2\right]\\
=&\mathbb{E}\bigg[\bigg(\int_0^t\bigg\|\tilde{x}_{L_i}'(0)+\int_0^sA_L(\theta_{L_i})\tilde{x}_{L_i}'(s_1)\mathrm{d}s_1+\int_0^sB_L(\theta_{L_i})\big(u_{L_i}'(s_1)-R_L^{-1}B_L^\top(\theta_{L_i})\\
&\times(P_{\gamma,\theta_{L_i}}(s_1)-\xi_{\gamma,\theta_{L_i},z_L^*}(s_1))\big)\mathrm{d}s_1+\int_0^sD_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_1)\bigg\|^2\mathrm{d}s\bigg)^2\bigg]\\
\le&\mathbb{E}\bigg\{\bigg[4\int_0^t\bigg(\|\tilde{x}_{L_i}'(0)\|^2+\bigg\|\int_0^sA_L(\theta_{L_i})\tilde{x}_{L_i}'(s_1)\mathrm{d}s_1\bigg\|^2+\bigg\|\int_0^sB_L(\theta_{L_i})\big(u_{L_i}'(s_1)-R_L^{-1}B_L^\top(\theta_{L_i})
\end{aligned}
$$

$$
\begin{aligned}
&\times (P_{\gamma,\theta_{L_i}}(s_1) - \xi_{\gamma,\theta_{L_i},z_L^*}(s_1))\big)\mathrm{d}s_1\Big\|^2 + \Big\|\int_0^s D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_1)\Big\|^2\Big)\mathrm{d}s\Big]^2\Big\}\\
\le& 64\|\tilde{x}'_{L_i}(0)\|^4 t^2 + 64\mathbb{E}\left[\left(\int_0^t \left\|\int_0^s A_L(\theta_{L_i})\tilde{x}'_{L_i}(s_1)\mathrm{d}s_1\right\|^2 \mathrm{d}s\right)^2\right]\\
&+ 64\mathbb{E}\left[\left(\int_0^t \left\|\int_0^s B_L(\theta_{L_i})\left(u'_{L_i}(s_1) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(s_1) - \xi_{\gamma,\theta_{L_i},z_L^*}(s_1))\right)\mathrm{d}s_1\right\|^2 \mathrm{d}s\right)^2\right]\\
&+ 64\mathbb{E}\left[\left(\int_0^t \left\|\int_0^s D_L(\theta_{L_i})\mathrm{d}W_{L_i}(s_1)\right\|^2 \mathrm{d}s\right)^2\right], \quad \forall\, t\in[0,T]. \qquad \text{(D.42)}
\end{aligned}
$$

By Jensen inequality and the compactness of Θ_L, we know that there exists a constant C_9 such that

$$
\begin{aligned}
64\mathbb{E}\left[\left(\int_0^t \left\|\int_0^s A_L(\theta_{L_i})\tilde{x}'_{L_i}(s_1)\mathrm{d}s_1\right\|^2 \mathrm{d}s\right)^2\right] \le& 64\mathbb{E}\left[\left(\int_0^t s\|A_L(\theta_{L_i})\|^2\int_0^s \|\tilde{x}'_{L_i}(s_1)\|^2\mathrm{d}s_1\mathrm{d}s\right)^2\right]\\
\le& 64t^3\|A_L(\theta_{L_i})\|^4\int_0^t \mathbb{E}\left[\left(\int_0^s \|\tilde{x}'_{L_i}(s_1)\|^2\mathrm{d}s_1\right)^2\right]\mathrm{d}s\\
\le& C_9\int_0^t \mathbb{E}\left[\left(\int_0^s \|\tilde{x}'_{L_i}(s_1)\|^2\mathrm{d}s_1\right)^2\right]\mathrm{d}s, \quad \forall\, t\in[0,T]. \qquad \text{(D.43)}
\end{aligned}
$$

Since $u'_{L_i}(\cdot)\in\tilde{\mathcal{U}}_{L_i}$, we know $\mathbb{E}\big[\exp\big(\frac{2}{\gamma}\int_0^T \big\|u'_{L_i}(s_1) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(s_1) - \xi_{\gamma,\theta_{L_i},z_L^*}(s_1))\big\|_{R_L}^2\mathrm{d}s_1\big)\big] < \infty$. This together with Jensen inequality and the compactness of Θ_L gives

$$
\begin{aligned}
&64\mathbb{E}\left[\left(\int_0^t \left\|\int_0^s B_L(\theta_{L_i})\left(u'_{L_i}(s_1) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(s_1) - \xi_{\gamma,\theta_{L_i},z_L^*}(s_1))\right)\mathrm{d}s_1\right\|_{R_L}^2 \mathrm{d}s\right)^2\right]\\
\le& 64\mathbb{E}\left[\left(\int_0^t s\left\|B_L(\theta_{L_i})\right\|^2\int_0^s \left\|u'_{L_i}(s_1) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(s_1) - \xi_{\gamma,\theta_{L_i},z_L^*}(s_1))\right\|_{R_L}^2 \mathrm{d}s_1\mathrm{d}s\right)^2\right]\\
\le& 64t^3\|B_L(\theta_{L_i})\|^4\int_0^t \mathbb{E}\left[\left(\int_0^s \left\|u'_{L_i}(s_1) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(s_1) - \xi_{\gamma,\theta_{L_i},z_L^*}(s_1))\right\|_{R_L}^2 \mathrm{d}s_1\right)^2\right]\mathrm{d}s\\
=& 64t^3\|B_L(\theta_{L_i})\|^4\gamma^2\int_0^t \mathbb{E}\left[\left(\frac{1}{\gamma}\int_0^s \left\|u'_{L_i}(s_1) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(s_1) - \xi_{\gamma,\theta_{L_i},z_L^*}(s_1))\right\|_{R_L}^2 \mathrm{d}s_1\right)^2\right]\mathrm{d}s\\
\le& 64t^3\|B_L(\theta_{L_i})\|^4\gamma^2\int_0^t \mathbb{E}\left[\exp\left(\frac{2}{\gamma}\int_0^T \left\|u'_{L_i}(s_1) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(s_1) - \xi_{\gamma,\theta_{L_i},z_L^*}(s_1))\right\|_{R_L}^2 \mathrm{d}s_1\right)\right]\mathrm{d}s\\
<& \infty, \quad \forall\, t\in[0,T]. \qquad \text{(D.44)}
\end{aligned}
$$

The norm $\|\cdot\|_{R_L}$ is equivalent to $\|\cdot\|$ on $\mathbb{R}^m$. This together with (D.44) implies

$$
64\mathbb{E}\left[\left(\int_0^t \left\|\int_0^s B_L(\theta_{L_i})\left(u'_{L_i}(s_1) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(s_1) - \xi_{\gamma,\theta_{L_i},z_L^*}(s_1))\right)\mathrm{d}s_1\right\|^2 \mathrm{d}s\right)^2\right]
$$

$$<\infty, \forall\ t \in [0,T]. \tag{D.45}$$

We know $64\|\tilde{x}'_{L_i}(0)\|^4 t^2 < \infty$ for any $t \in [0,T]$ due to the compactness of X_L. Combining this with (D.28), (D.42), (D.43) and (D.45), we know that there exist constants C_{10} and C_{11} such that

$$\mathbb{E}\left[\left(\int_0^t \|\tilde{x}'_{L_i}(s)\|^2 \mathrm{d}s\right)^2\right] \leq C_{10} + C_{11}t^3 \int_0^t \mathbb{E}\left[\left(\int_0^s \|\tilde{x}'_{L_i}(s_1)\|^2 \mathrm{d}s_1\right)^2\right] \mathrm{d}s, \quad \forall\ t \in [0,T].$$

This together with Gronwall inequality leads to

$$\mathbb{E}\left[\left(\int_0^t \|\tilde{x}'_{L_i}(s)\|^2 \mathrm{d}s\right)^2\right] < \infty, \quad \forall\ t \in [0,T]. \tag{D.46}$$

We know $8\big(\int_0^T (\|H_L z_L^*(t) + g_L\|^2)\mathrm{d}t\big)^2 < \infty$ since $z_L^*(t)$ is continuous with respect to $t \in [0,T]$. Combining this with (D.41) and (D.46) gives

$$\mathbb{E}\left[\left(\int_0^T \|\tilde{x}'_{L_i}(t) - H_L z_L^*(t) - g_L\|^2 \mathrm{d}t\right)^2\right] < \infty,$$

which together with (D.16), (D.40) and C_r inequality leads to

$$\begin{aligned}
&\mathbb{E}\Bigg[\Bigg|\Phi_L\Bigg(x_{L_i}(\cdot), \frac{1}{N_L}x_{L_i}(\cdot) + \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(\cdot), u_{L_i}(\cdot)\Bigg) \\
&\quad -\Phi_L\big(\tilde{x}'_{L_i}(\cdot), z_L^*(\cdot), u'_{L_i}(\cdot) - R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot) - \xi_{\gamma,\theta_{L_i},z_L^*}(\cdot))\big)\Bigg|^2\Bigg] \\
=&O\left(\epsilon_{N_L}^2 + \frac{1}{N_L}\right).
\end{aligned} \tag{D.47}$$

By C_r inequality and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathbb{E}\Bigg[\exp\Bigg(\frac{4}{\gamma}\Phi_L\Bigg(x_{L_i}(\cdot), \frac{1}{N_L}x_{L_i}(\cdot) + \frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(\cdot), u_{L_i}(\cdot)\Bigg)\Bigg)\Bigg] \\
\leq&\mathbb{E}\Bigg[\exp\Bigg(\frac{16}{\gamma}\int_0^T\Bigg(\|x_{L_i}\|^2_{Q_L} + \left\|\frac{1}{N_L}H_L x_{L_i}(t)\right\|^2_{Q_L} + \left\|\frac{1}{N_L}H_L\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(t)\right\|^2_{Q_L} \\
&\quad + \|g_L\|^2_{Q_L} + \|u_{L_i}(t)\|^2_{R_L}\Bigg)\mathrm{d}t\Bigg)\Bigg] \\
\leq&\left\{\mathbb{E}\left[\exp\left(\frac{32}{\gamma}\int_0^T\left(\|x_{L_i}(t)\|^2_{Q_L} + \left\|\frac{1}{N_L}H_L x_{L_i}(t)\right\|^2_{Q_L} + \|u_{L_i}(t)\|^2_{R_L}\right)\mathrm{d}t\right)\right]\right\}^{\frac{1}{2}} \\
&\quad \times \exp\left(\frac{16}{\gamma}\int_0^T \|g_L\|^2_{Q_L}\,\mathrm{d}t\right)\left\{\mathbb{E}\Bigg[\exp\Bigg(\frac{32}{\gamma}\int_0^T\left\|\frac{1}{N_L}H_L\sum_{i'=1,i'\neq i}^{N_L} x^*_{L_{i'}}(t)\right\|^2_{Q_L}\mathrm{d}t\Bigg)\Bigg]\right\}^{\frac{1}{2}}.
\end{aligned} \tag{D.48}$$

By the definition of admissible control sets $\mathcal{U}^g_{L_i}$ and the boundedness of $\|g_L\|^2_{Q_L}$, we have

$$
\begin{aligned}
&\left\{\mathbb{E}\left[\exp\left(\frac{32}{\gamma}\int_0^T\left(\|x_{L_i}(t)\|^2_{Q_L}+\left\|\frac{1}{N_L}H_L x_{L_i}(t)\right\|^2_{Q_L}+\|u_{L_i}(t)\|^2_{R_L}\right)\mathrm{d}t\right)\right]\right\}^{\frac{1}{2}}\\
&\times\exp\left(\frac{16}{\gamma}\int_0^T\|g_L\|^2_{Q_L}\mathrm{d}t\right)<\infty.
\end{aligned}
\tag{D.49}
$$

Combining (D.18), Jensen inequality with Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{32}{\gamma}\int_0^T\left\|\frac{1}{N_L}H_L\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(t)\right\|^2_{Q_L}\mathrm{d}t\right)\right]\\
\le&\frac{1}{T}\mathbb{E}\left[\int_0^T\exp\left(\frac{32T}{\gamma}\left\|\frac{1}{N_L}H_L\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(t)\right\|_{Q_L}\right)\mathrm{d}t\right]\\
\le&\frac{1}{T}\int_0^T\exp\left[\frac{64T}{\gamma N_L^2}\left(\sum_{i'=1,i'\neq i}^{N_L}\int_0^t\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)B_L(\theta_{L_{i'}})R_L^{-1}B_L^\top(\theta_{L_{i'}})\int_{s_2}^T\psi_{\gamma,\theta_{L_{i'}}}(s_2,s_1)[Q_LH_Lz^*_L(s_1)\right.\right.\\
&\left.+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_{i'}}}(t,0)x_{L_{i'}}(0)\right)^\top H_L^\top Q_LH_L\left(\sum_{k=1,k\neq i}^{N_L}\int_0^t\phi_{\gamma,\theta_{L_k}}(t,s_2)B_L(\theta_{L_k})R_L^{-1}B_L^\top(\theta_{L_k})\right.\\
&\left.\left.\times\int_{s_2}^T\psi_{\gamma,\theta_{L_k}}(s_2,s_1)[Q_LH_Lz^*_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_k}}(t,0)x_{L_k}(0)\right)\right]\mathbb{E}\left[\exp\left(\frac{64T}{\gamma N_L^2}\sum_{i'=1,i'\neq i}^{N_L}\right.\right.\\
&\left.\left.\times\left[\int_0^t\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)D_L(\theta_{L_{i'}})\mathrm{d}W_{L_{i'}}(s_2)\right]^\top H_L^\top Q_LH_L\int_0^t\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)D_L(\theta_{L_{i'}})\mathrm{d}W_{L_{i'}}(s_2)\right)\right]\mathrm{d}t\\
\le&\frac{1}{T}J_7^{\frac{1}{2}}J_8^{\frac{1}{2}},
\end{aligned}
\tag{D.50}
$$

where

$$
\begin{aligned}
J_7=&\int_0^T\exp\left[\frac{128T}{\gamma}\left(\sum_{i'=1,i'\neq i}^{N_L}\int_0^t\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)B_L(\theta_{L_{i'}})R_L^{-1}B_L^\top(\theta_{L_{i'}})\int_{s_2}^T\psi_{\gamma,\theta_{L_{i'}}}(s_2,s_1)[Q_LH_Lz^*_L(s_1)\right.\right.\\
&\left.+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_{i'}}}(t,0)x_{L_{i'}}(0)\right)^\top H_L^\top Q_LH_L\left(\sum_{k=1,k\neq i}^{N_L}\int_0^t\phi_{\gamma,\theta_{L_k}}(t,s_2)B_L(\theta_{L_k})R_L^{-1}B_L^\top(\theta_{L_k})\right.\\
&\left.\left.\times\int_{s_2}^T\psi_{\gamma,\theta_{L_k}}(s_2,s_1)[Q_LH_Lz^*_L(s_1)+Q_Lg_L]\mathrm{d}s_1\mathrm{d}s_2+\phi_{\gamma,\theta_{L_k}}(t,0)x_{L_k}(0)\right)\right]\mathrm{d}t,
\end{aligned}
$$

and

$$
J_8=\int_0^T\left\{\mathbb{E}\left[\exp\left(\frac{64T}{\gamma N_L^2}\sum_{i'=1,i'\neq i}^{N_L}\left(\int_0^t\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)D_L(\theta_{L_{i'}})\mathrm{d}W_{L_{i'}}(s_2)\right)^\top H_L^\top Q_LH_L\right.\right.\right.
$$

$$\times \int_0^t \phi_{\gamma,\theta_{L_{i'}}}(t,s_2) D_L(\theta_{L_{i'}}) \mathrm{d}W_{L_{i'}}(s_2)\Bigg)\Bigg]\Bigg\}^2 \mathrm{d}t.$$

Since $\phi_{\gamma,\theta_{L_i}}(t,s)$ and $\psi_{\gamma,\theta_{L_i}}(t,s)$ are continuous with respect to $(t,s) \in [0,T]\times[0,T]$, $z_L^*(t)$ is continuous with respect to $t \in [0,T]$ and Θ_L is a bounded closed set, we have

$$J_7^{\frac{1}{2}} < \infty. \tag{D.51}$$

Noting that $\int_0^t \phi_{\gamma,\theta_{L_i}}(t,s) D_L(\theta_{L_i}) \mathrm{d}W_{L_i}(s) = D_L(\theta_{L_i})W_{L_i}(t) - \int_0^t \frac{\partial \phi_{\gamma,\theta_{L_i}}(t,s)}{\partial s} D_L(\theta_{L_i}) W_{L_i}(s)\mathrm{d}s$, by C_r inequality and Jensen inequality, we have

$$\begin{aligned}
&\frac{64T}{\gamma N_L}\Bigg[\sum_{i'=1,i'\neq i}^{N_L}\Big(\int_0^t \phi_{\gamma,\theta_{L_{i'}}}(t,s_2) D_L(\theta_{L_{i'}}) \mathrm{d}W_{L_{i'}}(s_2)\Big)^\top H_L^\top Q_L H_L \int_0^t \phi_{\gamma,\theta_{L_{i'}}}(t,s_2) D_L(\theta_{L_{i'}}) \mathrm{d}W_{L_{i'}}(s_2)\Bigg]\\
\leq &\frac{128T}{\gamma N_L^2}\sum_{i'=1,i'\neq i}^{N_L}\|H_L^\top Q_L H_L\|\Bigg(\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2 + t\int_0^t \Big\|\frac{\partial \phi_{\gamma,\theta_{L_{i'}}}(t,s_2)}{\partial s_2} D_L(\theta_{L_{i'}})W_{L_{i'}}(s_2)\Big\|^2 \mathrm{d}s_2\Bigg),
\end{aligned}$$

which together with the definition of J_8 leads to

$$\begin{aligned}
J_8 \leq &\int_0^T \Bigg\{\prod_{i'=1,i'\neq i}^{N_L} \mathbb{E}\Bigg[\exp\Big(\frac{128T}{\gamma N_L^2}\|H_L^\top Q_L H_L\|\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2\Big)\\
&\times \exp\Big(\frac{128Tt}{\gamma N_L^2}\|H_L^\top Q_L H_L\|\int_0^T \Big\|\frac{\partial \phi_{\gamma,\theta_{L_{i'}}}(t,s_2)}{\partial s_2} D_L(\theta_{L_{i'}})W_{L_{i'}}(s_2)\Big\|^2 \mathrm{d}s_2\Big)\Bigg]\Bigg\}^2 \mathrm{d}t\\
\leq &\int_0^T \prod_{i'=1,i'\neq i}^{N_L}\Bigg\{\mathbb{E}\Bigg[\exp\Big(\frac{256T}{\gamma N_L^2}\|H_L^\top Q_L H_L\|\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2\Big)\Bigg]\Bigg\}\\
&\times \prod_{i'=1,i'\neq i}^{N_L}\Bigg\{\frac{1}{T}\int_0^T \mathbb{E}\Bigg[\exp\Big(\frac{256T^2 t}{\gamma N_L^2}\|H_L^\top Q_L H_L\|\Big\|\frac{\partial \phi_{\gamma,\theta_{L_{i'}}}(t,s_2)}{\partial s_2} D_L(\theta_{L_{i'}})W_{L_{i'}}(s_2)\Big\|^2\Big)\Bigg]\mathrm{d}s_2\Bigg\}\mathrm{d}t.
\end{aligned} \tag{D.52}$$

By Assumption 2.5, the compactness of Θ_L and the calculation of the expectation of the exponential function related to Brownian motion, we have

$$\begin{aligned}
&\prod_{i'=1,i'\neq i}^{N_L}\mathbb{E}\Bigg[\exp\Big(\frac{256T}{\gamma N_L}\|H_L^\top Q_L H_L\|\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2\Big)\Bigg]\\
=&\prod_{i'=1,i'\neq i}^{N_L}\Bigg\{\mathbb{E}\Bigg[\exp\Big(\frac{256T}{\gamma}\|H_L^\top Q_L H_L\|\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2\Big)\Bigg]^{\frac{1}{N_L}}\Bigg\}\\
\leq&\prod_{i'=1,i'\neq i}^{N_L}\Bigg\{\mathbb{E}\Bigg[\exp\Big(\frac{256T}{\gamma}\|H_L^\top Q_L H_L\|\|D_L(\theta_{L_{i'}})W_{L_{i'}}(t)\|^2\Big)\Bigg]\Bigg\}^{\frac{1}{N_L}}\\
=&\prod_{i'=1,i'\neq i}^{N_L}\Bigg\{\int_{\mathbb{R}^p}\exp\Big(\frac{256T}{\gamma}\|H_L^\top Q_L H_L\|\|D_L(\theta_{L_{i'}})\|^2 x^\top x\Big)\frac{1}{(\sqrt{2\pi t})^p}\exp\Big(-\frac{x^\top x}{2t}\Big)\mathrm{d}x\Bigg\}^{\frac{1}{N_L}}
\end{aligned}$$

$$
\begin{aligned}
&\le \prod_{i'=1,i'\neq i}^{N_L}\Big[\gamma\Big/\Big(\gamma-512\sup_{\theta_{L_{i'}}\in\Theta_L}\|H_L^\top Q_L H_L\|\|D_L(\theta_{L_{i'}})\|^2T^2\Big)\Big]^{\frac{p}{2N_L}}\\
&<\infty,\quad \forall\, t\in(0,T],
\end{aligned}\tag{D.53}
$$

and

$$
\begin{aligned}
&\prod_{i'=1,i'\neq i}^{N_L}\left\{\frac{1}{T}\int_0^T\mathbb{E}\left[\exp\left(\frac{256T^2t}{\gamma N_L}\|H_L^\top Q_L H_L\|\left\|\frac{\partial\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)}{\partial s_2}D_L(\theta_{L_{i'}})W_{L_{i'}}(s_2)\right\|^2\right)\right]\mathrm{d}s_2\right\}\\
&\le \prod_{i'=1,i'\neq i}^{N_L}\left\{\frac{1}{T}\int_0^T\int_{\mathbb{R}^p}\exp\left(\frac{256T^3}{\gamma}\|H_L^\top Q_L H_L\|\left\|\frac{\partial\phi_{\gamma,\theta_{L_{i'}}}(t,s_2)}{\partial s_2}\right\|^2\|D_L(\theta_{L_{i'}})\|^2x^\top x\right)\right.\\
&\quad\left.\times\frac{1}{(\sqrt{2\pi s_2})^p}\exp\left(-\frac{x^\top x}{2s_2}\right)\mathrm{d}x\mathrm{d}s_2\right\}^{\frac{1}{N_L}}\\
&\le \prod_{i'=1,i'\neq i}^{N_L}\left\{\frac{1}{T}\int_0^T\left[\gamma\Big/\left(\gamma-512\sup_{0\le t,s\le T}\sup_{\theta_{L_{i'}}\in\Theta_L}\|H_L^\top Q_L H_L\|\left\|\frac{\partial\phi_{\gamma,\theta_{L_{i'}}}(t,s)}{\partial s}\right\|^2\|D_L(\theta_{L_{i'}})\|^2T^4\right)\right]^{\frac{1}{2}}\mathrm{d}s_2\right\}^{\frac{p}{N_L}}\\
&<\infty,\quad\forall\, t\in[0,T].
\end{aligned}\tag{D.54}
$$

Combining (D.52)–(D.54), we have $J_8<\infty$. This together with (D.50) and (D.51) leads to

$$
\mathbb{E}\left[\exp\left(\frac{8}{\gamma}\int_0^T\left\|\frac{1}{N_L}H_L\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(t)\right\|^2_{Q_L}\mathrm{d}t\right)\right]<\infty,
$$

which together with (D.48) and (D.49) implies

$$
\mathbb{E}\left[\exp\left(\frac{4}{\gamma}\Phi_L\left(x_{L_i}(\cdot),\frac{1}{N_L}x_{L_i}(\cdot)+\frac{1}{N_L}\sum_{i'=1,i'\neq i}^{N_L}x^*_{L_{i'}}(\cdot),u_{L_i}(\cdot)\right)\right)\right]<\infty.\tag{D.55}
$$

Similar to the proof of (D.55), we can get

$$
\mathbb{E}\left[\exp\left(\frac{4}{\gamma}\Phi_L(\tilde{x}'_{L_i}(\cdot),z_L^*(\cdot),u'_{L_i}(\cdot)-R_L^{-1}B_L^\top(\theta_{L_i})(P_{\gamma,\theta_{L_i}}(\cdot)\tilde{x}_{L_i}(\cdot)-\xi_{\gamma,\theta_{L_i},z_L^*}(\cdot)))\right)\right]<\infty.\tag{D.56}
$$

Combining (D.39), (D.47), (D.55), (D.56) and C_r inequality, we have $\left|J_{L_i}\big(u_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\big)-\tilde{J}_{L_i}\big(u'_{L_i}(\cdot)\big)\right|\le O\left(\epsilon_{N_L}+\frac{1}{\sqrt{N_L}}\right)$, which leads to $J_{L_i}\big(u_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\big)\ge\tilde{J}_{L_i}\big(u'_{L_i}(\cdot)\big)-O\left(\epsilon_{N_L}+\frac{1}{\sqrt{N_L}}\right)$. This together with (D.33) gives

$$
J_{L_i}\big(u_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\big)\ge\bar{J}_{L_i,z_L^*}\big(u^*_{L_i}(\cdot)\big)-O\left(\epsilon_{N_L}+\frac{1}{\sqrt{N_L}}\right).\tag{D.57}
$$

By Lemma 5.2, we have

$$
\bar{J}_{L_i,z_L^*}\big(u^*_{L_i}(\cdot)\big)\ge J_{L_i}\big(u^*_{L_i}(\cdot),u^*_{L_{-i}}(\cdot)\big)-O\left(\epsilon_{N_L}+\frac{1}{\sqrt{N_L}}\right).
$$

Combining the above inequality with (D.57), we have $J_{L_i}\big(u_{L_i}(\cdot), u^*_{L_{-i}}(\cdot)\big) \geq J_{L_i}\big(u^*_{L_i}(\cdot), u^*_{L_{-i}}(\cdot)\big) - O\big(\epsilon_{N_L} + \frac{1}{\sqrt{N_L}}\big)$. This completes the proof of (D.30). Combining (D.19) with (D.30), we have (5.12).

Next, we rewrite $\Phi_F\big(x_{F_j}(\cdot), \frac{1}{N_F}\sum_{j'=1}^{N_F} x_{F_{j'}}(\cdot), \frac{1}{N_L}\sum_{i'=1}^{N_L} x_{L_{i'}}(\cdot), u_{F_j}(\cdot)\big) = \int_0^T (\|x_{F_j}(t) - H_F z^*_F(t) - H z^*_L(t) + H_F(z^*_F(t) - \frac{1}{N_F}\sum_{j'=1}^{N_F} x_{F_{j'}}(t)) + H(z^*_L(t) - \frac{1}{N_L}\sum_{i'=1}^{N_L} x_{L_{i'}}(t)) - g_F\|^2_{Q_F} + \|u_{F_j}(t)\|^2_{R_F})\mathrm{d}t$. Then similar to (5.12), we get (5.13). □

References

[1] B. Anahtarci, C.D. Kariksiz and N. Saldi, Value iteration algorithm for mean-field games. *Syst. Control Lett.* **143** (2020) 1–10.

[2] M. Bardi, Explicit solutions of some linear-quadratic mean field games. *Netw. Heterog. Media* **7** (2012) 243–261.

[3] A. Bensoussan, K.C.J. Sung, S.C.P. Yam and S.P. Yung, Linear-quadratic mean field games. *J. Optim. Theory Appl.* **169** (2016) 496–529.

[4] T.R. Bielecki and S.R. Pliska, Risk-sensitive ICAPM with application to fixed-income management. *IEEE Trans. Autom. Control* **49** (2004) 420–432.

[5] J.R. Birge, L. Bo and A. Capponi, Risk-sensitive asset management and cascading defaults. *Math. Oper. Res.* **43** (2018) 1–28.

[6] P.E. Caines and A.C. Kizilkale, ϵ-Nash equilibria for partially observed LQG mean field games with a major player. *IEEE Trans. Autom. Control* **62** (2017) 3225–3234.

[7] Y. Chen, A. Bušić and S.P. Meyn, State estimation and mean field control with application to demand dispatch, in *Proceedings of IEEE 54th Conference on Decision and Control.* Osaka (2015) 6548–6555.

[8] Y. Chen, A. Bušić and S.P. Meyn, State estimation for the individual and the population in mean field control with application to demand dispatch. *IEEE Trans. Autom. Control* **62** (2017) 1138–1149.

[9] M.K. Das, A. Goswami and N. Rana, Risk sensitive portfolio optimization in a jump diffusion model with regimes. *SIAM J. Control Optim.* **56** (2018) 1550–1576.

[10] W.H. Fleming and S.J. Sheu, Risk-sensitive control and an optimal investment model. *Math. Financ.* **10** (2000) 197–213.

[11] W.H. Fleming and S.J. Sheu, Risk-sensitive control and an optimal investment model II. *Ann. Appl. Probab.* **12** (2002) 730–767.

[12] W.H. Fleming and H.M. Soner, Controlled Markov processes and viscosity solutions. Springer, New York (2006).

[13] H. Hata, Risk sensitive asset management with lognormal interest rates. *Asia-Pac. Financ. Marka.* **28** (2021) 169–206.

[14] Y. Hu, J. Huang and X. Li, Linear quadratic mean field game with control input constraint. *ESAIM: COCV* **24** (2018) 901–919.

[15] Y. Hu, J. Huang and T. Nie, Linear-quadratic-Gaussian mixed mean-field games with heterogeneous input constraints. *SIAM J. Control Optim.* **56** (2018) 2835–2877.

[16] Y. Hu and X.Y. Zhou, Constrained stochastic LQ control with random coefficients, and application to portfolio selection. *SIAM J. Control Optim.* **44** (2005) 444–466.

[17] J. Huang, B.C. Wang and T. Xie, Social optimal in leader-follower mean field linear quadratic control. *ESAIM: COCV* **27** (2021) 1–31.

[18] M. Huang, Large-population LQG games involving a major player: The Nash certainty equivalence principle. *SIAM J. Control Optim.* **48** (2010) 3318–3353.

[19] M. Huang, P.E. Caines and R.P. Malhamé, An invariance principle in large population stochastic dynamic games. *J. Syst. Sci. Complex* **20** (2007) 162–172.

[20] M. Huang, R.P. Malhamé and P.E. Cains, On a class of large-scale cost-coupled Markov games with applications to decentralized power control, in *Proceedings of IEEE 43th Conference on Decision and Control.* Nassau (2004) 2830–2835.

[21] M. Huang, R.P. Malhamé and P.E. Caines, Large population stochastic dynamic games: Closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. *Commu. Inf. Syst.* **6** (2006) 221–252.

[22] M. Huang, R.P. Malhamé and P.E. Caines, Nash cerntainty equivalence in large population stochastic dynamic games: connections with the physics of interacting particle systems, in *Proceedings of IEEE 45th Conference on Decision and Control.* San Diego (2006) 4921–4926.

[23] M. Huang and M. Zhou, Linear quadratic mean field games: Asymptotic solvability and relation to the fixed point approach. *IEEE Trans. Autom. Control* **65** (2020) 1397–1412.

[24] A.C. Kizilkale, R. Salhab and R.P. Malhamé, An integral control formulation of mean field game based large scale coordination of loads in smart grids. *Automatica* **100** (2019) 312–322.

[25] M. Larrañage, J. Denis, M. Assaad and K.D. Turck, Energy-efficient distributed transmission scheme for MTC in dense wireless networks: a mean field approach. *IEEE Internet Things J.* **7** (2020) 477–490.

[26] J.M. Lasry and P.L. Lions, Jeux à champ moyen. I. Le cas stationnaire. *C.R. Math.* **343** (2006) 619–625.

[27] J.M. Lasry and P.L. Lions, Jeux à champ moyen. II. Horizon fini et contrôle optimal. *C.R. Math.* **343** (2006) 679–684.

[28] J.M. Lasry and P.L. Lions, Mean field games. *Jpn. J. Math.* **2** (2007) 229–260.

[29] W. Lee, S. Liu, H. Tembine, W. Li and S. Osher, Controlling propagation of epidemics *via* mean-field control. *SIAM J. Appl. Math.* **81** (2021) 190–207.

[30] T. Li and J.F. Zhang, Asymptotically optimal decentralized control for large population stochastic multiagent systems. *IEEE Trans. Autom. Control* **53** (2008) 1643–1660.

[31] Y. Ma and M. Huang, Linear quadratic mean field games with a major player: the multi-scale approach. *Automatica* **113** (2020) 1–11.
[32] Z. Ma, D.S. Callaway and I.A. Hiskens, Decentralized charging control of large populations of plug-in electric vehicles. *IEEE Trans. Control Syst. Technol.* **21** (2013) 67–78.
[33] X. Mao, The stochastic differential equations and applications. Woodhead Publishing, Philadelphia (2007).
[34] J. Moon and T. Başar, Linear quadratic risk-sensitive and robust mean field games. *IEEE Trans. Autom. Control* **62** (2017) 1062–1077.
[35] J. Moon and T. Başar, Risk-sensitive mean field games *via* the stochastic maximum principle. *Dyn. Games Appl.* **9** (2019) 1100–1125.
[36] S.L. Nguyen and M. Huang, Linear-quadratic-Gaussian mixed games with continuum-parametrized minor players. *SIAM J. Control Optim.* **50** (2012) 2907–2937.
[37] M. Nourian, P.E. Caines, R.P. Malhamé and M. Huang, Mean field LQG control in leader-follower stochastic multi-agent systems: Likelihood ratio based adaptation. *IEEE Trans. Autom. Control* **57** (2012) 2801–2816.
[38] M. Nourian and P.E. Caines, ϵ-Nash mean field game theory for nonlinear stochastic dynamical systems with major and minor agents. *SIAM J. Control Optim.* **51** (2013) 3302–3331.
[39] M. Nourian, R.P. Malhamé, M. Huang and P.E. Caines, Mean field (NCE) formulation of estimation based leader-follower collective dynamics. *Int. J. Robot. Autom.* **26** (2011) 120–129.
[40] L. Ntogramatzidis and A. Ferrante, On the solution of the Riccati differential equation arising from the LQ optimal control problem. *Syst. Control Lett.* **59** (2010) 114–121.
[41] N. Saldi, T. Başar and M. Raginsky, Approximate Markov-Nash equilibria for discrete-time risk-sensitive mean-field games. *Math. Oper. Res.* **45** (2020) 1596–1620.
[42] H. Tembine, COVID-19: data-driven mean-field-type game perspective. *Games* **11** (2020) 1–107.
[43] H. Tembine, Q. Zhu and T. Başar, Risk-sensitive mean-field games. *IEEE Trans. Autom. Control* **59** (2014) 835–850.
[44] G.Y. Weintraub, C.L. Benkard and B.V. Roy, Oblivious equilibrium: a mean field approximation for large-scale dynamic games, in *Proceedings of 18th International Conference on Neural Information Processing Systems*. Vancouver (2005) 1489–1496.
[45] G.Y. Weintraub, C.L. Benkard and B.V. Roy, Markov perfect industry dynamics with many firms. *Econometrica* **76** (2008) 1375–1411.
[46] J. Yong and X.Y. Zhou, Stochastic controls Hamiltonnian systems and HJB equations. Springer, New York (1999).