

DERIVATIVES OF SUB-RIEMANNIAN GEODESICS ARE L_p -HÖLDER CONTINUOUS*

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Abstract. This article is devoted to the long-standing problem on the smoothness of sub-Riemannian geodesics. We prove that the derivatives of sub-Riemannian geodesics are always L_p -Hölder continuous. Additionally, this result has several interesting implications. These include (i) the decay of Fourier coefficients on abnormal controls, (ii) the rate at which they can be approximated by smooth functions, (iii) a generalization of the Poincaré inequality, and (iv) a compact embedding of the set of shortest paths into the space of Bessel potentials.

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1. INTRODUCTION

Mathematics consists in proving the most obvious thing in the least obvious way.

George Pólya

Nonholonomic and non-Riemannian problems are currently being extensively studied by numerous mathematicians. The interest in these problems started to rapidly grow in the mid-20th century, following the works of famous mathematicians such as E. Cartan, Carathéodory, Gromov, Montgomery, Mitchell, Vershik, Agrachev, and many others. The most mysterious and intriguing object in the theory is abnormal geodesics, which arise as critical points of the end-point map. The question of how to investigate these geodesics has been present since the inception of the theory, and in particular, the question on their smoothness has not yet been resolved.

In Riemannian geometry, analyzing geodesic smoothness properties is relatively straightforward. Riemannian geodesics satisfy the Euler–Lagrange equation and hence they are as smooth (or analytic) as the right-hand side of the equation. However, sub-Riemannian geodesics do not always satisfy the system of Euler–Lagrange equations. Instead, they satisfy a system of differential inclusions that comes from the Pontryagin maximum principle (PMP). A sub-Riemannian geodesic is called normal if it satisfies PMP with a non-degenerate Hamiltonian, and

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in this case, it satisfies the Hamiltonian version of the Euler–Lagrange equation and hence it has to be smooth. However, strictly abnormal geodesics do not satisfy PMP with a non-degenerate Hamiltonian. In this case, the corresponding system of differential inclusions is not equivalent to any system of ordinary differential equations, and hence the classical Riemannian approach on investigating geodesics smoothness suffers almost complete failure.

The Pontryagin maximum principle still holds some potential for studying abnormal geodesics. However, investigations based on the principle typically depend on the step value of the sub-Riemannian manifold, as they all rely on the Goh condition. For example, geodesics on sub-Riemannian manifolds of step $s = 2$ are smooth because the Goh condition implies that there are no abnormal geodesics in this case (see [3]). Even in the case $s = 3$, it is very difficult to study smoothness of abnormal geodesics using this method (though there are some results in this direction [9, 15]). In the case $s = 4$, there are almost no general results. The only exception is work [4], where the authors study smoothness of abnormal geodesics on sub-Riemannian manifolds of rank 2 and step 4.

In 2016, Le Donne and Hakavuori achieved a remarkable result in [8]. They proved that sub-Riemannian abnormal geodesics cannot have corners. Notably, their proof does not rely on the Pontryagin maximum principle, and instead employs an idea of using an appropriate blow-up procedure. This result has been used by several other mathematicians, including [5], where authors establish that geodesics on an analytic three-dimensional sub-Riemannian manifold must be C^1 -smooth, regardless of the step value. The above mentioned work [4] also draws on their result. Nevertheless, it should be noted that the absence of corners does not necessarily imply the smoothness of abnormal geodesics on general sub-Riemannian manifolds.

Another interesting result is presented in [11]. The authors prove that the derivatives of geodesics satisfy the Hölder condition in the special case of left-invariant sub-Riemannian structures on Carnot groups of rank 2 that additionally satisfy the following strong restriction: $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ for $i, j \geq 2$ and $i + j > 4$ (here $\mathfrak{g} = \bigoplus_{j=1}^s \mathfrak{g}_j$ is the corresponding Carnot stratification of the Lie algebra \mathfrak{g}).

However, despite the constant efforts of leading mathematicians, smoothness of abnormal sub-Riemannian geodesics has not yet been proven. There are well-known examples [3], Section 12.6.1, of abnormal sub-Riemannian extremals (but not geodesics) that are non-smooth. Nonetheless, to date, there are not a single example of a non-smooth abnormal geodesic. The present work provides an explanation for this gap.

We develop an interpolational estimate on abnormal controls and apply convex duality theory to the estimate, which allows us to prove the following result:

Theorem 1.1. *Any arc length parametrized geodesic on a sub-Riemannian manifold of constant rank has L_p -Hölder continuous derivative for any $1 \leq p < \infty$.*

Therefore, in order to find a non-smooth abnormal geodesic, it is necessary to search within a narrow class of curves that have derivatives which are L_p -Hölder continuous with an exponent of $0 < \alpha \leq 1$, and are not C^1 -smooth. It should be noted that this class of curves is indeed very narrow, as for example, if $\alpha > 1/p$, the class is in fact empty.

Theorem 1.1 is formulated in a qualitative form. We state a quantitative formulation of this result in Theorem 4.1 (see Sect. 4). Namely, Theorem 4.1 gives exact bounds on the exponent α . These bounds depend only on p and the step s of the sub-Riemannian manifold. Some interesting corollaries of the main Theorem 4.1 are given in Section 5. They concern

- the decay of the Fourier coefficients of controls on abnormal geodesics;
- the rate of the approximation of abnormal controls by smooth functions;
- the lower estimate of the coefficient in L_p -Hölder condition with an exponent α , which is similar to the Poincaré inequality.
- the compactness of the set of geodesics in the space of Bessel’s potentials H_p^α .

2. THANKS

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3. BESOV SPACES AND L_p -HÖLDER CONDITION

We start with a brief explanation on the concept of L_p -Hölder condition. Choose an arbitrary function $u \in L_p([t_1; t_2] \rightarrow \mathbb{R}^k)$. In that follows, we write $L_p(t_1; t_2) = L_p([t_1; t_2] \rightarrow \mathbb{R}^k)$ for short. Put $(\sigma_h u)(t) = u(t+h)$.

Definition 3.1. Let $1 \leq p \leq \infty$, $\delta > 0$ be sufficiently small. The following function on $-\delta \leq h \leq \delta$ is called L_p -modulus of continuity of u over $[t_1; t_2]$:

$$\omega_p(h, u) = \|\sigma_h u - u\|_{L_p(I_{\text{sign } h})}.$$

where $I_+ = (t_1; t_2 - \delta)$ and $I_- = (t_1 + \delta; t_2)$.

It is evident that, $\omega_p(0, u) = 0$ and $\omega_p(h, u) \rightarrow 0$ for $h \rightarrow 0$ (if $p \neq \infty$). In the case of smooth manifolds, one can introduce modulus of continuity as follows. Let M be a smooth manifold and g be a Riemannian metric on it. Consider an absolutely continuous curve $x : [t_1; t_2] \rightarrow M$ and put $u(t) = \dot{x}(t)$. In the definition of L_p -modulus of continuity, it is sufficient to replace σ_h by the parallel transport along the curve $x(t)$ over time h with the help of Levi-Civita connection, and to define the norm with the help of g . Let us remark that this definition depends on the choice of g but not moderately high. Namely, let g and \tilde{g} be two Riemannian metrics on M . Denote by $\omega_p(h, \cdot)$ and $\tilde{\omega}_p(h, \cdot)$ the corresponding L_p -moduli of continuity. In this case, for any compact $\mathcal{K} \subset M$, there exists a constant $C \geq 1$ such that if $x(t) \in \mathcal{K}$ and $u(t) = \dot{x}(t)$, then

$$\frac{1}{C} \omega_p(h, u) \leq \tilde{\omega}_p(h, u) \leq C \omega_p(h, u).$$

Definition 3.2. A function u is called L_p -Hölder continuous on $[t_1; t_2]$ if there exist $\alpha > 0$ and $c > 0$ such that the following inequality (called L_p -Hölder condition) is fulfilled for all $0 < |h| \leq \delta$:

$$\omega_p(h, u) \leq c|h|^\alpha.$$

The number α is called the exponent of the L_p -Hölder condition. The minimal possible constant c we denote by $c_p^\alpha(u)$:

$$c_p^\alpha(u) = \sup_{0 < |h| \leq \delta} \frac{\omega_p(h, u)}{|h|^\alpha}.$$

It is easy to check that the definition of L_p -Hölder continuity with a given exponent α does not depend on the choice of $\delta > 0$ (but constant $c_p^\alpha(u)$ does) if δ is sufficiently small. It is evident that in the case $p = \infty$, we obtain usual definitions of the modulus of continuity and Hölder continuous functions. The moduli of continuity are directly related to the Besov and Sobolev spaces (see [14], Sect. 5.3), which will be repeatedly used in this paper.

Definition 3.3. The Nikolski–Besov (or simply Besov¹) space $B_{p,\infty}^\alpha(t_1, t_2)$ (where $0 < \alpha < 1$) consists of functions $u \in L_p(t_1, t_2)$ that satisfy

$$\|u\|_{B_{p,\infty}^\alpha(t_1, t_2)} \stackrel{\text{def}}{=} \|u\|_p + c_p^\alpha(u) < \infty.$$

Let us remark that the different choices of $\delta > 0$ produces different norms on $B_{p,\infty}^\alpha(t_1, t_2)$, but all of them are pairwise equivalent if δ is sufficiently small (see [16], Sect. 4.4.1). Thus, the Nikolski–Besov space $B_{p,\infty}^\alpha(t_1, t_2)$ consists of functions $L_p(t_1, t_2)$ that are L_p -Hölder continuous with the exponent $\alpha \in (0; 1)$.

In what follows, we use interpolation theory of Banach spaces and theory of Besov spaces very often. For the lack of the possibility to give a short introduction into so solid part of mathematics, we refer the reader to the excellent book [16] for details.

4. THE MAIN RESULT

Let (M, Δ, g) be a smooth sub-Riemannian manifold of constant rank, *i.e.* M is a smooth manifold, $\dim M = n$, $\Delta(x) \subset T_x M$ is a smooth distribution with a fixed dimension $\dim \Delta(x) \equiv k < n$, and $g(x)$ is a dot product on $\Delta(x)$, which smoothly depends on x . Besides, following [3], we include into the definition the demand that the distribution $\Delta(x)$ meets the Hörmander condition (the bracket-generating condition). A Lipschitz continuous curve $x(t) \in M$, $t \in [0; 1]$, is called horizontal (or admissible) if $\dot{x}(t) \in \Delta(x(t))$ for a.e. t . The length of a (horizontal) curve is defined in the standard way:

$$l(x) = \int_0^1 |\dot{x}(t)|_{g(x(t))} dt.$$

Let us remind shortly various definitions of local minima (see [2]). Let $x(t)$ be a horizontal curve.

- (G) Curve $x(t)$ is called sub-Riemannian shortest path if any other horizontal curve $y(t)$ that has $x(0) = y(0)$ and $x(1) = y(1)$ meets the condition $l(x) \leq l(y)$.
- (C⁰) Curve $x(t)$ is called C⁰-local minimum if there exists a neighborhood $U \subset M$ of the curve $x(t)$ such that any horizontal curve $y(t)$ that lies in U and has $x(0) = y(0)$, $x(1) = y(1)$ meets $l(x) \leq l(y)$.
- (W_p¹) Curve $x(t)$ is called W_p¹-local minimum for $p = 1$ or $p = \infty$ if there exists $\varepsilon > 0$ such that any horizontal curve $y(t)$ that satisfies $x(0) = y(0)$, $x(1) = y(1)$, and² $\|\dot{x} - \dot{y}\|_p < \varepsilon$ meets $l(x) \leq l(y)$.

Sometimes C⁰-local minimum is called strong, W_∞¹ — weak, and W₁¹ — Pontryagin's. It is evident that there are the following implications: $G \Rightarrow C^0 \Rightarrow W_1^1 \Rightarrow W_\infty^1$ (see [3]).

Theorem 4.1. *Let $\mathcal{K} \subset M$ be a compact set and distribution Δ at points of \mathcal{K} meet the Hörmander condition of step that is not greater than s .*

- (W) *If $x(t)$ is a W₁¹-local minimum of the sub-Riemannian distance, $x(t) \in \mathcal{K}$, and $|\dot{x}(t)|_g \equiv \text{const}$, then derivative \dot{x} is L_2 -Hölder continuous with exponent $\alpha = \frac{1}{s-1}$, *i.e.**

$$\dot{x} \in B_{2,\infty}^{\frac{1}{s-1}}(0; 1).$$

¹According to the authors knowledge, spaces $B_{p,\infty}^\alpha$ were first introduced and studied by Nikolski. Then his student Besov introduced spaces $B_{p,q}^\alpha$ for arbitrary $1 \leq q \leq \infty$ and developed the corresponding theory. Then spaces $B_{p,q}^\alpha$ become much more convenient to work with than just the Nikolski spaces $B_{p,\infty}^\alpha$ because of the very nice interpolational properties of the Besov spaces $B_{p,q}^\alpha$ with $q \neq 1, \infty$.

²Here we use any finite trivialization TM in a neighborhood of curve $x(t)$. The choice of trivialization can affect on ε , but it does not affect on the definition of the local minimum itself.

Moreover, for any $2 \leq p < \infty$, the derivative \dot{x} is L_p -Hölder continuous with any exponent $\alpha < \frac{2}{p(s-1)}$, i.e. $\dot{x} \in B_{p,\infty}^\alpha(0;1)$.

(6) For any $2 + \frac{1}{s-1} \leq p < \infty$, $0 < \beta < \frac{1}{p(s-1)}$, and $0 < \kappa < 1 - \beta(s-2)$, there exists a constant³ $C > 0$ such that any sub-Riemannian shortest path $x(t) \in \mathcal{K}$ with $|\dot{x}(t)|_g \equiv l(x)$ satisfies

$$\|\dot{x}\|_{B_{p,\infty}^\beta(0;1)} \leq Cl(x)^\kappa.$$

The full proof is given in Section 6. It is based on a new approach on how to investigate smoothness of sub-Riemannian optimal controls. We call it **the duality method of interpolational estimates**. Namely, we intend to construct an upper estimate on any optimal control u of the following form:

$$\exists q, r \geq 1 \exists \theta \in (0;1) \exists c > 0 \quad : \quad \forall \varphi \in C_0^\infty(0;1) \quad \int_0^1 u \dot{\varphi} dt \leq c \|\varphi\|_r^{1-\theta} \|\dot{\varphi}\|_q^\theta.$$

This estimate appears to be dual to an estimate on K -functional of an interpolation of the corresponding Sobolev spaces⁴. This leads to a possibility to prove that optimal control u does satisfy L_p -Hölder condition.

5. COROLLARIES OF THE MAIN THEOREM ON L_p -HÖLDER CONTINUITY OF SUB-RIEMANNIAN GEODESICS DERIVATIVES

Theorem 4.1 has many convenient consequences. For instance, Theorem 1.1 that is formulated in the introduction. Here we give its proof.

Proof of Theorem 1.1. Consider a SR-geodesic on (M, Δ, g) . Let $\mathcal{K} \subset M$ be its graph. The Hörmander condition guarantees that $s < \infty$ on compact \mathcal{K} since the step is a lower semi-continuous function and, hence, it reaches its minimum on compact \mathcal{K} . Further, a geodesic can be divided into a finite number of parts, and it is a shortest path on each of them. Hence, according to Theorem 4.1 (namely, the second part of item (W)), the velocity on each of these parts satisfies L_p -Hölder condition for any $2 \leq p < \infty$. It remains to note that L_p -Hölder condition for some p implies L_q -Hölder condition for any $1 \leq q \leq p$. □

Now, let us suppose for the sake of statements simplicity that

$$\Delta(x) = \text{span}(f_1(x), \dots, f_k(x)).$$

In this case, horizontal curves have velocities of the form

$$\dot{x}(t) = \sum_{j=1}^k u_j(t) f_j(x(t)),$$

where $u = (u_1, \dots, u_k)$ is a control.

Since $x(t) \in W_\infty^1(0;1)$, one obtains $f_j(x(t)) \in W_\infty^1(0;1)$ and $g(x(t)) \in W_\infty^1(0;1)$. So control

$$u(t) = g(x(t))[f_i(x(t)), f_j(x(t))]^{-1} g(x(t))[\dot{x}(t), f_j(x(t))]$$

³Constant C depends on \mathcal{K} , p , β , and κ , but it does not depend on $x(\cdot)$.

⁴Detailed definitions and properties of K -functional in general theory of interpolation can be found in [14, 16]. The K -functional corresponding to our case is defined in (6.13).

appears to satisfy L_p -Hölder condition with the same exponent α as $\dot{x}(t)$. Moreover, the part (\mathfrak{G}) of Theorem 4.1 remains true with u in place of \dot{x} (only constant C may differ) as fields f_j and their derivatives $\frac{\partial}{\partial x} f_j$ are bounded on compact \mathcal{K} .

The fact that control u is L_p -Hölder continuous makes it possible to estimate rate of decay for Fourier series coefficients. Let $\ell_\gamma = \ell_\gamma(\mathbb{R}^k)$ for $\gamma \geq 1$ denote the space of sequences $\{v^m\}_{m=-\infty}^\infty$, $v^m \in \mathbb{R}^k$, such that $\sum_m |v^m|^\gamma < \infty$.

Corollary 5.1 (on the Fourier coefficients). *Let \hat{u}^m , $m \in \mathbb{Z}$, be the Fourier coefficients of the control $u \in L_\infty(0; 1)$ and S_N be the partial Fourier sum, $S_N = \sum_{m=-N}^N \hat{u}^m e^{2\pi i m t}$.*

(\mathfrak{W}) *Let $x(t)$ be either a geodesic or W_1^1 -local minimum and $|u(t)|_g \equiv l(x)$ be the corresponding control. Then for any⁵ $\alpha \leq \min\{\frac{1}{s-1}, \frac{1}{2}\}$, $\alpha \neq \frac{1}{s-1}$ and $\gamma > \max\{1, 2 - \frac{4}{s+1}\}$ one has⁶*

$$\sum_{m \in \mathbb{Z}} |m|^{2\alpha} |\hat{u}^m|^2 < \infty; \quad \hat{u}^m \in \ell_\gamma; \quad \text{and} \quad \exists C > 0: \forall N \quad \|u - S_N\|_2 \leq CN^{-\alpha}.$$

(\mathfrak{G}) *If $\mathcal{K} \subset M$ is compact, then for any $\beta < \frac{1}{2s-1}$, $0 < \kappa < 1 - \beta(s-2)$, and $\delta > \frac{2}{1+2\beta}$, there exists a constant C such that for any shortest path x lying in \mathcal{K} with the control $|u(t)|_g \equiv l(x)$, one has*

$$\sum_{m \in \mathbb{Z}} |m|^{2\beta} |\hat{u}^m|^2 < Cl(x)^\kappa; \quad \|\hat{u}^m\|_{\ell_\delta} \leq Cl(x)^\kappa; \quad \text{and} \quad \forall N \quad \|u - S_N\|_2 \leq Cl(x)^\kappa N^{-\beta}.$$

Proof. These are well known properties of the Fourier coefficients of functions being L_p -Hölder continuous. Let us prove, say, (\mathfrak{W}) (the proof of (\mathfrak{G}) is fully similar). Control u satisfies L_2 -Hölder condition with any index $0 < \alpha < \frac{1}{s-1}$ due to Theorem 4.1 (in the case (\mathfrak{W}) , one can always choose, say, the graph of x as \mathcal{K}). We start with proving the third assertion. The main tool here is [13], Theorem 1.3.9. Pinsky uses a slightly different shifting operator $\hat{\sigma}_h$ (instead of σ_h used in the present paper) to define L_p -Hölder continuity. Namely, he defines the function $\hat{\sigma}_h u$ to be the periodic shift of u :

$$\hat{\sigma}_h u(t) = u(\{t+h\}) \text{ for a.e. } t \in [0; 1]$$

(here $\{\cdot\}$ stand for the fractional part of a number). We will say that u is periodically L_p -Hölder continuous with exponent $\hat{\alpha}$, if it is L_p -Hölder continuous wrt periodic shift $\hat{\sigma}_h$, *i.e.* if there exists $c > 0$ such that

$$\hat{\omega}_p(h, u) \stackrel{\text{def}}{=} \|\hat{\sigma}_h u - u\|_{L_p(0;1)} \leq \hat{c}|h|^{\hat{\alpha}}.$$

It is easy to see that if $u \in L_\infty(0; 1)$ and $p < \infty$, then these two definitions of L_p -Hölder continuity are equivalent, but the exponents may differ. Indeed, on the one hand, $\omega_p(h, u) \leq \hat{\omega}_p(h, u)$. So, if u is periodically L_p -Hölder continuous with an exponent $\hat{\alpha}$, then it is L_p -Hölder continuous with the same exponent $\hat{\alpha}$ (or greater). On the other hand, for $h > 0$, we have

$$\begin{aligned} \hat{\omega}_p(-h, u)^p &= \hat{\omega}_p(h, u)^p = \int_0^{\frac{1}{2}} |\sigma_h u - u|^p dt + \int_{\frac{1}{2}+h}^1 |u - \sigma_{-h} u|^p dt + \int_{1-h}^1 |\sigma_{-1+h} u - u|^p dt \\ &= \omega_p(h, u)^p + \omega_p(-h, u)^p + (\|\sigma_{-1+h} u - u\|_{L_p(1-h,1)})^p \\ &\leq \omega_p(h, u)^p + \omega_p(-h, u)^p + (2\|u\|_\infty h^{1/p})^p \leq \omega_p(h, u)^p + \omega_p(-h, u)^p + 2^p \|u\|_\infty^p h \end{aligned}$$

⁵Here $s \geq 2$ is the maximal step at points of the curve x .

⁶Authors would like to express their gratitude to professor L. Rizzi for a very useful discussion about the case $\alpha > \frac{1}{2}$.

So if $u \in L_\infty(0; 1)$ is L_p -Hölder continuous with an exponent α , then it is periodically L_p -Hölder continuous with the exponent $\min\{\alpha, \frac{1}{p}\}$ (or greater). Summarizing,

$$\min\left\{\alpha, \frac{1}{p}\right\} \leq \hat{\alpha} \leq \alpha.$$

[13], Theorem 1.3.9 states that if u is periodically L_2 -Hölder continuous with an exponent $\hat{\alpha}$, then $\|u - S_N\|_2 \leq CN^{-\hat{\alpha}}$ and the third assertion is proved. The first one follows from the third by using [13], Exercise 1.3.15. The second assertion can be found, for instance, in [12] by using the inclusion $B_{p,\infty}^\alpha(0; 1) \subset B_{p,q}^{\alpha-\varepsilon}(0; 1)$, which is valid for all $\varepsilon > 0$ and $1 \leq q \leq \infty$ (see [16], Sect. 4.6.1).

The case (G) has a very similar proof. The only differences are the following: (i) since $u \in B_{2+\frac{1}{s-1},\infty}^\beta(0; 1)$ for any $\beta < \frac{1}{2s-1}$, we have $u \in B_{2,\infty}^\beta(0; 1)$; (ii) since $s \geq 2$, we have $\min\{\frac{1}{2s-1}, \frac{1}{2}\} = \frac{1}{2s-1}$. \square

Another property of L_p -Hölder continuity of the control is a possibility to approximate it well by smooth functions. This property is non-trivial. There are a lot of functions in $L_\infty(0; 1)$ that can not be approximated well by smooth functions. Namely, there exists $u \in L_\infty(0; 1)$ such that for any series of functions $w_\varepsilon \in C^1$ approximating u , $\|u - w_\varepsilon\|_2 < \varepsilon$, and any $\gamma > 0$ one has $\varepsilon^\gamma \|\dot{w}_\varepsilon\|_\infty \rightarrow \infty$ as $\varepsilon \rightarrow +0$. In other words, while one is approximating u by smooth functions, their derivatives grow faster than any negative power of ε . However, for functions that are L_p -Hölder continuous, it is possible to select smooth approximating functions that have much smaller derivatives. In the corollary below, we suggest one possible way of choosing norms, but it is possible to use other norms with the help of interpolation methods (see [16]).

Corollary 5.2 (on the approximation rate by smooth functions). *Let $x(t)$ be either a geodesic or W_1^1 -local minimum and $|u(t)|_g \equiv l(x)$ be the corresponding control. If $\gamma > s - 1$,⁷ then there exists a constant $C > 0$ such that, for any $\varepsilon > 0$, there is a function $w \in C^\infty(\mathbb{R})$ such that*

$$\|w - u\|_{L_2(0;1)} \leq C\varepsilon \quad \text{and} \quad \varepsilon^\gamma \|\dot{w}\|_{L_\infty(0;1)} \leq C.$$

Proof. This approximation is obtained by the following chain of inclusions. We first fix $0 < \alpha < 1$. Then for $\delta > 0$ and $2 \leq r < \infty$, we have

$$(L_2(0; 1), W_\infty^1(0; 1))_{\alpha,\infty} \supset (B_{2,2}^\delta(0; 1), B_{r,r}^{1+\frac{1}{r}}(0; 1))_{\tilde{\alpha},p} = B_{p,p}^{(1-\tilde{\alpha})\delta + \tilde{\alpha}(1+\frac{1}{r})}(0; 1) \supset B_{p,\infty}^{(2-\tilde{\alpha})\delta + \tilde{\alpha}(1+\frac{1}{r})}(0; 1).$$

The first inclusion is fulfilled for $\alpha < \tilde{\alpha}$ and any $1 \leq p \leq \infty$ (see [16], Sect. 1.3.3). The second equality is fulfilled for $\frac{1}{p} = \frac{1-\tilde{\alpha}}{2} + \frac{\tilde{\alpha}}{r}$ according to [16], 4.3.1 (and so $p \geq 2$ automatically) — this is so called diagonal case of interpolation of Besov spaces. The last embedding is of Sobolev type (see [16], Sect. 4.6.2) since $\delta > 0$. Now we apply the main Theorem 4.1: the control u belongs to the last space if

$$(2 - \tilde{\alpha})\delta + \tilde{\alpha} \left(1 + \frac{1}{r}\right) < \frac{2}{p(s-1)} = \frac{2}{s-1} \left(\frac{1-\tilde{\alpha}}{2} + \frac{\tilde{\alpha}}{r}\right)$$

or, equivalently,

$$\tilde{\alpha} \left(1 + \frac{1}{s-1} + \frac{1}{r} - \delta + \frac{2}{r(s-1)}\right) < \frac{1}{s-1} - 2\delta.$$

For $\delta \rightarrow +0$ and $r \rightarrow \infty$, the factor in the left-hand side tends to $1 + \frac{1}{s-1}$ and the right-hand side tends to $\frac{1}{s-1}$. Therefore, this inequality is compatible with $\tilde{\alpha} > \alpha$ if and only if $\alpha(1 + \frac{1}{s-1}) < \frac{1}{s-1}$ or $\alpha < \frac{1}{s}$.

⁷Where s is the sub-Riemannian step as always.

Under this choice of α , one obtains $u \in (L_2(0; 1), W_\infty^1(0; 1))_{\alpha, \infty}$. It means that the K -functional of the interpolation (see [16], Sect. 1.3) satisfies the following estimate:

$$\sup_{M>0} (M^{-\alpha} K(M, u)) = \sup_{M>0} \inf_{w \in W_\infty^1(0; 1)} (M^{-\alpha} \|w - u\|_2 + M^{1-\alpha} \|w\|_{W_\infty^1}) = C < \infty.$$

Space $C^\infty(\mathbb{R}) \cap W_p^1(0; 1)$ is dense in $W_p^1(0; 1)$ (see [16], Sect. 2.3.2). Let us denote $M^\alpha = \varepsilon$. Then for any $\varepsilon > 0$, there exists a function $w \in C^\infty(\mathbb{R})$ such that $\|w - u\|_{L_2(0; 1)} \leq 2C\varepsilon$ and $\|w\|_{L_\infty(0; 1)} \leq 2C/\varepsilon^{(1-\alpha)/\alpha}$. \square

Theorem 4.1 also implies an estimate on constant $c_p^\alpha(u)$ in L_p -Hölder condition (see Def. 3.2), which is similar to the Poincaré inequality

Corollary 5.3 (an analog of the Poincaré inequality). *For any p and α from item (W) of Theorem 4.1, there exists a constant $C > 0$ such that for any geodesic (or W_1^1 -local minimum) $x(t)$ lying in \mathcal{K} with control $|u(t)|_g \equiv l(x)$, one has*

$$c_p^\alpha(u) \geq C \|u - \bar{u}\|_p$$

where $\bar{u} = \int_0^1 u(t) dt \in \mathbb{R}^k$ is the mean value of the control $u(t)$ on the interval $t \in [0; 1]$.

In particular, if the system of fields f_j is orthonormal, then for $p = 2$ and any $0 < \alpha < \frac{1}{s-1}$, there exists $C > 0$ such that

$$c_2^\alpha(u) \geq C \sqrt{l(x)^2 - \bar{u}^2}$$

since, in this case,

$$\|u - \bar{u}\|_2^2 = \int_0^1 (u(t) - \bar{u})^2 dt = \int_0^1 (u^2(t) - 2\bar{u}u(t) + \bar{u}^2) dt = l(x)^2 - \bar{u}^2.$$

The proof of Corollary 5.3 is similar to the proof of the classic Poincaré inequality. Let us give it shortly.

Proof of Corollary 5.3. Let $Y \subset B_{p, \infty}^\alpha(0; 1)$ consists of (not necessarily optimal) controls with zero average: $Y = \{u : \bar{u} = 0\}$. First, we prove the inequality for $u \in Y$. Suppose the opposite: for any constant $C > 0$ there exists $u \in Y$ such that $c_p^\alpha(u) \leq C \|u\|_p$. In this case, there exists a sequence $u^m \in Y$ such that $c_p^\alpha(u^m) / \|u^m\|_p \rightarrow 0$ as $m \rightarrow \infty$. Without loss of generality, we assume that $c_p^\alpha(u^m) + \|u^m\|_p = 1$ for all m .

The space $B_{p, \infty}^\alpha(0; 1)$ can be represented as a dual space $B_{p, \infty}^\alpha(0; 1) = (\tilde{B}_{p^*, 1}^{-\alpha}(0; 1))'$ where $\frac{1}{p} + \frac{1}{p^*} = 1$ (see [16], Sect. 4.8.1). Therefore, due to Banach–Alaoglu theorem, it is possible to choose a weak*-convergent subsequence of the sequence u^m . We keep indices m for the subsequence without loss of generality. Space $B_{p, \infty}^\alpha(0; 1)$ is compactly embedded into the space $L_p(0; 1)$ according to the Rellich–Kondrachev theorem (see the proof of Thm. 5.4 for details). Hence, there is a subsequence that converges strongly in $L_p(0; 1)$. We also keep indices m for it.

Thus, we obtain a sequence u^m that converges to u strongly in L_p and weakly* in $B_{p, \infty}^\alpha(0; 1)$ as $m \rightarrow \infty$. It is evident that $u \in Y$.

Recall that $c_p^\alpha(u^m) / \|u^m\|_p \rightarrow 0$ as $m \rightarrow \infty$. Therefore, $\|u^m\|_p \leq 1$ implies $c_p^\alpha(u^m) \rightarrow 0$. So $\|u^m\|_p \rightarrow 1$ and $\|u\|_p = 1$. Moreover, u is the weak*-limit of u^m in $B_{p, \infty}^\alpha(0; 1)$, which implies

$$\|u\|_p + c_p^\alpha(u) = \|u\|_{B_{p, \infty}^\alpha} \leq \limsup_{m \rightarrow \infty} \|u^m\|_{B_{p, \infty}^\alpha} = 1.$$

So $c_p^\alpha(u) = 0$ and the function u is constant, which is impossible since $\bar{u} = 0$ and $\|u\|_p = 1$. Thus, the statement is proved for any $u \in Y$ by contradiction.

For an arbitrary function $u \in B_{p,\infty}^\alpha(0; 1)$, one has $u - \bar{u} \in Y$ and $c_p^\alpha(u - \bar{u}) = c_p^\alpha(u)$. It remains to note that $x(t)$ is a geodesic or W_1^1 -local minimum. Hence, its control belongs to space $B_{p,\infty}^\alpha(0; 1)$ by Theorem 4.1. \square

Theorem 4.1 also gives us a possibility to prove a reinforced version of the Agrachev result on compactness of the set of shortest paths [1]. Denote the set of all shortest paths lying in \mathcal{K} by \mathfrak{X} . Namely,

$$\mathfrak{X} \subset \{(x^0, x^1, u) \in \mathcal{K} \times \mathcal{K} \times L_\infty(0; 1)\},$$

where a triplet (x^0, x^1, u) belongs to \mathfrak{X} if and only if there exists a shortest path $x(t) \in W_\infty^1(0; 1)$ lying in \mathcal{K} such that (i) it has a constant velocity $|\dot{x}(t)|_g \equiv \text{const}$; (ii) $x^0 = x(0)$ and $x^1 = x(1)$ are the initial and final points of $x(t)$; and (iii) $u(t)$ is its control. So \mathfrak{X} consists of all triplets (x^0, x^1, u) corresponding to all shortest paths $x(\cdot)$ in \mathcal{K} .

Theorem 5.4 (on compactness of the set of shortest paths in the space of the Bessel potentials). *Let $\mathcal{K} \subset M$ be a compact set and f_j form an orthonormal frame. Then \mathfrak{X} is a compact subset in $\mathcal{K} \times \mathcal{K} \times H_p^\beta(0; 1)$ where $H_p^\beta(0; 1)$ is the space of Bessel potentials⁸ and p and β are from item (G) of Theorem 4.1 or $1 \leq p < 2 + \frac{1}{s-1}$, $\beta < \frac{1}{2s-1}$.*

Proof. Let us note that one can assume that fields f_j vanish outside a compact without loss of generality. Indeed, in the opposite case, we proceed as follows. Denote by V an open neighborhood of \mathcal{K} such that its closure is compact. Take a smooth function ψ such that $\psi(x) = 1$ for $x \in \mathcal{K}$, $\psi(x) = 0$ for $x \notin V$ and $0 \leq \psi(x) \leq 1$ for all x . We exchange fields f_j by that of $\tilde{f}_j = \psi f_j$ and exchange g appropriately. Under such a change (i) all fields become complete; (ii) the rank becomes nonconstant, but remains constant in a neighborhood of \mathcal{K} (which is obviously enough for Theorem 4.1); (iii) horizontal curves in \mathcal{K} remain the horizontal ones; (iv) a part of horizontal curves lying out of \mathcal{K} cease to be horizontal; and (v) no new horizontal curves arise. Moreover, lengths of horizontal curves in \mathcal{K} remain unchanged, as for the rest, their lengths may only increase. The estimate in item (G) of Theorem 4.1 does not altered since the shortest curves in \mathcal{K} do not altered.

Without loss of generality, assume that p and β are from from item (G) of Theorem 4.1. Indeed, the statement for $p < 2 + \frac{1}{s-1}$ and $\beta < \frac{1}{2s-1}$ follows from the continuity of the inclusion $B_{2+\frac{1}{s-1},\infty}^\beta(0; 1) \subset B_{p,\infty}^\beta(0; 1)$.

At first, let us show that the set of all controls on shortest paths lying in \mathcal{K} belongs to $H_p^\beta(0; 1)$ and forms a precompact set in it. The embedding of $W_p^1(0; 1)$ into $L_p(0; 1)$ is compact by the Rellich–Kondrachev theorem (see [6], Sect. 5.7). Take some numbers β_j , $j = 1, 2$, such that $\beta < \beta_1 < \beta_2 < \frac{1}{p(s-1)}$. Then the Sobolev–Slobodeckij spaces $W_p^{\beta_j}(0; 1) = B_{p,p}^{\beta_j}(0; 1)$ (see [16], Sect. 2.4.1) can be represented as the following interpolational ones (see [16], Sect. 4.3.1):

$$W_p^{\beta_j}(0; 1) = (L_p(0; 1), W_p^1(0; 1))_{\beta_j, p}.$$

Since $W_p^1(0; 1) \Subset L_p(0; 1)$, we have $W_p^{\beta_2}(0; 1) \Subset W_p^{\beta_1}(0; 1)$ (see [16], Sect. 1.16.4). The set of controls on shortest paths lying in \mathcal{K} is bounded in $W_p^{\beta_2}(0; 1)$ by Theorem 4.1 (since $B_{p,\infty}^{\beta_2+\varepsilon}(0; 1) \subset B_{p,p}^{\beta_2}(0; 1)$ for any $\varepsilon > 0$, [16], Sect. 4.6.2) and thus it is precompact in $W_p^{\beta_1}(0; 1)$. Now we recall that $W_p^{\beta_1}(0; 1) \subset H_p^\beta(0; 1)$ by the corresponding embedding of Sobolev type (see [16], Sect. 4.6.2). Thus, the set of controls on shortest paths in \mathcal{K} is precompact in $H_p^\beta(0; 1)$.

⁸The assertion of the theorem remains true if one exchange the set $H_p^\beta(0; 1)$ by the Besov space $B_{p,q}^\beta(0; 1)$ $1 \leq q \leq \infty$ or by the Sobolev–Slobodeckij space $W_p^\beta(0; 1)$ due to the continuity of the Sobolev embeddings $H_p^{\beta+\varepsilon}(0; 1) \subset B_{p,q}^\beta(0; 1)$ and $H_p^{\beta+\varepsilon}(0; 1) \subset W_p^\beta(0; 1)$ for any $\varepsilon > 0$ (see [16], Sect. 4.6.2).

So, the set \mathfrak{X} is precompact in $\mathcal{K} \times \mathcal{K} \times H_p^\beta(0; 1)$. In order to show that it is compact, let us consider two maps

$$\begin{aligned} e, i : \mathcal{K} \times \mathcal{K} \times L_p(0; 1) &\rightarrow \mathcal{K} \times \mathcal{K} \times \mathbb{R}, \\ e(x^0, x^1, u) &= (\tilde{x}^0, \tilde{x}^1, \|u(t)\|_p), \\ i(x^0, x^1, u) &= (x^0, x^1, d_{SR}(x^0, x^1)). \end{aligned}$$

Here $d_{SR}(x^0, x^1)$ is the sub-Riemannian distant between x^0 and x^1 ; \tilde{x}^1 is the value of the unique solution to the differential equation $\dot{x} = \sum_j u_j f_j(x)$ with the initial condition $x(0) = x^0$ at $t = 1$; and \tilde{x}^0 is the value of the unique solution to the same equation, but with the initial condition $x(1) = x^1$ and at $t = 0$.

We claim that since $p > 1$ (in fact, $p > 2 + \frac{1}{s-1}$), we have

$$\mathfrak{X} = \{(x^0, x^1, u) \mid e(x^0, x^1, u) = i(x^0, x^1, u)\}.$$

Indeed, inclusion \subset is evident and inclusion \supset is also not so difficult to obtain. If the first two components of e and i coincide, then control u leads from x^0 to x^1 . If also the last components of e and i coincide, then length of any horizontal curve $x(t)$ having $u(t)$ as a control (that is $\|u(t)\|_1$) is not greater than $d_{SR}(x^0, x^1)$ (since $\|u(t)\|_1 \leq \|u(t)\|_p = d_{SR}(x^0, x^1)$). In addition, $d_{SR}(x^0, x^1) \leq \|u(t)\|_1$ according to the definition of the sub-Riemannian distance. So the equality can happen only if $\|u(t)\|_1 \equiv d_{SR}(x^0, x^1)$ for a.e. t as $p > 1$.

Map i is defined for all (x^0, x^1, u) and it is continuous with respect to any topology on $L_p(0; 1)$, and map e is defined for all (x^0, x^1, u) since fields f_j vanish outside a compact and e is continuous with respect to the strong topology on $L_p(0; 1)$. Therefore, set \mathfrak{X} (as the set of points where $e = i$) is strongly closed in the $\mathcal{K} \times \mathcal{K} \times L_p(0; 1)$. Hence, set \mathfrak{X} is closed in $\mathcal{K} \times \mathcal{K} \times H_p^\beta(0; 1)$ due to continuity of the embedding $H_p^\beta(0; 1) \subset L_p(0; 1)$ of Sobolev type. □

The previous theorem is formulated in terms of controls. But it can be easily reformulated it terms of the shortest paths.

Corollary 5.5. *Let $\mathcal{K} \subset M$ be a compact set. Then the set of shortest paths $x(t)$ in \mathcal{K} with $|\dot{x}|_g \equiv l(x)$ is compact in $H_p^{1+\beta}(0; 1)$ where p and β are from item (\mathfrak{G}) of Theorem 4.1 or $1 \leq p < 2 + \frac{1}{s-1}$ and $\beta < \frac{1}{2s-1}$.*

Proof. Indeed, consider the following map

$$\mathfrak{X} \rightarrow W_\infty^1(0; 1); \quad (x^0, x^1, u) \mapsto x(\cdot)$$

where $x(t)$ is the unique solution to the equation $\dot{x} = \sum u_j f_j(x)$ with the initial condition $x(0) = x^0$ (we simply forget about point x^1). This map is surjective on the set of shortest paths in \mathcal{K} by the definition of \mathfrak{X} . Since $\mathfrak{X} \subset H_p^\beta(0; 1)$ by Theorem 5.4, the image of \mathfrak{X} belongs to $H_p^{1+\beta}(0; 1)$. Moreover, the map is continuous as a map from $\mathcal{K} \times \mathcal{K} \times H_p^\beta(0; 1)$ to $H_p^{1+\beta}(0; 1)$. It remain to note that the continuous image of any compact set (e.g. \mathfrak{X}) is also compact. □

6. PROOF OF THEOREM 4.1

Proofs of the both items (\mathfrak{W}) and (\mathfrak{G}) in Theorem 4.1 are very close, so we write a united proof for both of them and make special notes in places where they are different. For convenience of writing the united proof, in the case (\mathfrak{W}) , we denote by L the given curve length, $L = l(x)$; and in the case (\mathfrak{G}) , lengths of shortest paths in \mathcal{K} may differ, but all of them are bounded by a constant L (i.e. $l(x) \leq L$) since \mathcal{K} is compact and sub-Riemannian distance is continuous. In other words, L is a constant such that $l(x) \leq L$ in both cases (\mathfrak{W}) and (\mathfrak{G}) .

Outline of the proof: The proof of Theorem 4.1 is divided into five main parts. Firstly, in Section 6.1, we present a precise reduction of the subriemannian problem to an optimal control one. This reduction is technical and somewhat standard. Readers interested solely in the optimal control aspect may skip this section.

Secondly, in Section 6.2, we introduce additional assumptions on the optimal control problem that can be assumed without loss of generality. These assumptions simplify greatly the rest of the proof.

Thirdly, in Section 6.3, we present the heart of the proof: a variation on an optimal trajectory that allows us to establish a key interpolational estimate on the optimal control.

Fourthly, in Section 6.4, we demonstrate that the key interpolation estimate is dual to the problem of approximating optimal controls by smooth functions.

Finally, in Section 6.5, we use the approximation to obtain estimates on the K -functional, which leads to an interpolation of Besov spaces.

6.1. Reduction to an optimal control problem

Before we start the proof of Theorem 4.1, let us reduce it to the simplest possible form. We begin with reducing the general problem to the optimal control one. So we want to prove that, without loss of generality, one can assume that there exist vector fields $f_1, \dots, f_k \in \text{Vect}(M)$ such that

$$\Delta = \text{span}(f_1, \dots, f_k); \quad (OC_1)$$

$$M \subset \mathbb{R}^n \text{ and for each } i \text{ there exist some global coordinates on } M \text{ such that } f_i(x) \text{ is constant in these coordinates}; \quad (OC_2)$$

$$g[f_i, f_j] = \delta_{ij}. \quad (OC_3)$$

Proof of (OC₁), (OC₂) and (OC₃). Obviously, any $x \in M$ has a neighborhood where vector fields satisfying (OC₁) can be easily found. Moreover, vector fields f_i are linearly independent since there are exactly $k = \dim \Delta$ of them. In particular, $f_i \neq 0$. Hence, each point $x \in M$ has a neighborhood diffeomorphic to a ball in \mathbb{R}^n such that each vector field f_i is constant in some local coordinates on that neighborhood. Moreover, since f_i are linearly independent, we may assume that f_i are orthonormal by using the standard Gram–Schmidt process.

Thus, any point $x \in M$ has a neighborhood U^x (which is diffeomorphic to a ball in \mathbb{R}^n) and defined on U^x vector fields f_i such that (OC₁), (OC₂), and (OC₃) are fulfilled in $\tilde{M} = U^x$. Let us now show that it is sufficient to prove the theorem assuming $M = U^x$.

Denote $B = \{\xi \in \mathbb{R}^n : |\xi| < 1\}$ and let $\phi^x : B \rightarrow U^x$ be the corresponding diffeomorphism with $\phi^x(0) = x$. Let us cover compact \mathcal{K} by a finite number of neighborhoods of the form $\phi^{x_m}(\frac{1}{3}B)$. Put $\phi^m = \phi^{x_m}$ for short. Further, using this atlas, for any horizontal curve $x(t)$ lying in the union of these neighborhoods, it is possible to construct in standard manner a partition of $[0; 1]$ of the form $0 = t_0 < t_1 < \dots < t_N = 1$ in such a way that, for any $0 \leq j < N$, there exists a (minimal) index m_j such that $x(t_j) \in \phi^{m_j}(\frac{1}{3}B)$; $x(t) \in \phi^{m_j}(\frac{2}{3}B)$ for $t \in [t_j; t_{j+1})$; and if $j + 1 \neq N$, then $x(t_{j+1}) \in \phi^{m_j}(\frac{2}{3}\partial B)$.

Let us now use this partition to construct a number of overlapping intervals that cover $[0; 1]$. Denote $s_j^+ = \max\{t \in [t_j; t_{j+1}) : x(t) \in \phi^{m_j}(\frac{1}{2}\partial B)\}$ for $j < N - 1$. Note that the motion time from a given point on $\phi^m(\frac{1}{3}\partial B)$ to a given point on $\phi^m(\frac{1}{2}\partial B)$ along any horizontal curve with bounded speed $|\dot{x}(t)|_g \leq L$ is separated from 0. A similar statement holds for the motion time from $\phi^m(\frac{1}{2}\partial B)$ to $\phi^m(\frac{2}{3}\partial B)$. Hence, there exists a constant $S^+ > 0$ such that, for any horizontal curve $x(t)$ in \mathcal{K} with $|\dot{x}|_g \leq L$, we have $s_j^+ - t_j \geq S^+$ and $t_{j+1} - s_j^+ \geq S^+$ for $j < N - 1$. For $j < N - 1$, let us now put $s_j^- = \max\{t \in [t_j; t_{j+1}) : x(t) \notin \phi^{m_{j+1}}(\frac{1}{2}B)\}$ if the written set is not empty, and $s_j^- = t_j$ otherwise. Similarly, there exists a constant $S^- > 0$ such that $t_{j+1} - s_j^- \geq S^-$ for any horizontal curve in \mathcal{K} with $|\dot{x}|_g \leq L$. Put $t_{j+1/2} = \max\{s_j^+, s_j^-\}$. Obviously, $x(t) \in \phi^{m_j}(\frac{2}{3}B)$ for $t \in [t_j; t_{j+1/2}]$ (as $[t_j; t_{j+1/2}] \subset [t_j; t_{j+1})$) and $x(t) \in \phi^{m_{j+1}}(\frac{2}{3}B)$ for $t \in [t_{j+1/2}; t_{j+1}]$ (as $x(t) \in \phi^{m_{j+1}}(\frac{1}{2}B)$ for $t \in (s_j^-; t_{j+1})$).

Put $S = \min\{S^+, S^-\} > 0$. Then $t_{j+1/2} - t_j \geq S$ and $t_{j+1} - t_{j+1/2} \geq S$ for $j < N - 1$. In particular, number N never exceeds $1 + 2/S$ on any horizontal curve $x(t)$ in \mathcal{K} with $|\dot{x}(t)|_g \leq L$.

Consider the following system of overlapping intervals:

$$[t_0; t_1], [t_{1/2}; t_2], [t_{3/2}; t_3], \dots, [t_{N-3/2}; t_N].$$

We know that $x(t) \in \phi^{m_j}(\frac{2}{3}\text{cl } B)$ for $t \in [t_{j-1/2}; t_{j+1}]$ if $j < N$ and $x(t) \in \phi^{m_1}(\frac{2}{3}\text{cl } B)$ for $t \in [t_0; t_1]$ by construction. Moreover, each interval length is⁹ $2S$ or greater, and length of any two consequent intervals intersection is S or greater. Therefore, L_p -Hölder continuity of \dot{x} on $[0; 1]$ with an exponent ν is equivalent to L_p -Hölder continuity of \dot{x} on each of these intervals with the same exponent ν . Moreover, for any horizontal curve $x(t) \in \mathcal{K}$ with $|\dot{x}(t)|_g \leq L$, the norm of \dot{x} in $B_{p,\infty}^\nu[0; 1]$ is not greater than the sum of its norms in $B_{p,\infty}^\nu[t_0, t_1]$ and $B_{p,\infty}^\nu[t_{j-1/2}, t_{j+1}]$ for $j < N$.

Thus, it is sufficient to prove Theorem 4.1 for the finite number of manifolds $M_m = \phi^{x_m}(B)$ (with compact $\mathcal{K}_m = \mathcal{K} \cap \phi^m(\frac{2}{3}\text{cl } B)$). Conditions (OC₁) and (OC₂) are fulfilled on any manifold M_m . It remains to take into account the renormalization of time $\tau = (t - t_{j-1/2})/(t_{j-1/2} - t_{j+1})$ (or $\tau = (t - t_0)/(t_1 - t_0)$), which allows one to assume $[t_{j-1/2}; t_{j+1}] = [0; 1]$ (or $[t_0; t_1] = [0; 1]$). If the interval length is σ , then this renormalization changes norm $\|u\|_p$ by a factor of σ^{-p} , and constant $c_p^\alpha(u)$ by a factor of $\sigma^{-p-\alpha}$. Hence, after the renormalization, we obtain a norm that is equivalent to the norm for the Besov space on $[0; 1]$ given in Definition 3.3, and this is of no importance as length σ is bounded from 0 by S and does not exceed 1. □

6.2. A series of simplifications

Assumptions (OC₁) and (OC₃) allow us to reformulate the sub-Riemannian problem on shortest paths as an optimal control problem of the following form:

$$\begin{aligned} l(x) &= \int_0^1 \sqrt{u_1^2 + \dots + u_k^2} \rightarrow \min \\ \dot{x} &= u_1 f_1(x) + \dots + u_k f_k(x) \\ x(0) &= x^0; \quad x(1) = x^1. \end{aligned} \tag{6.1}$$

Here $x : [0; 1] \rightarrow M$ is a Lipschitz continuous curve and $u = (u_1, \dots, u_k) \in L^\infty([0; 1] \rightarrow \mathbb{R}^k)$ is a bounded control.

We believe that in the optimal control setting the term “admissible” is more appropriate than “horizontal”. So pair (x, u) is called admissible if it satisfies both ODE and terminal conditions in (6.1). All previously introduced notions can also be reformulated in the new setting, *e.g.* a pair (x, u) is called W_1^1 -local minimum if there exists an $\varepsilon > 0$ such that, for any admissible pair (y, v) with $\|u - v\|_1 < \varepsilon$, one has $l(x) \leq l(y)$.

Note that condition $|\dot{x}|_g = \text{const}$ in Theorem 4.1 is equivalent to the following one

$$|\dot{x}(t)|_g = |u(t)| \equiv l(x) \neq 0 \tag{6.2}$$

(we assume $l(x) \neq 0$ since $u \equiv 0$ if $l(x) = 0$ and there is nothing to be proved).

We claim that to prove Theorem 4.1 it is sufficient to prove that the control u is L_p -Hölder continuous. Indeed, since $x \in W_\infty^1(0; 1)$, L_p -Hölder continuity of u easily implies L_p -Hölder continuity of $\dot{x}(t) = \sum_j f_j(x(t))u_j(t)$ (for item (G)), one should also use the fact that fields f_j and their derivatives $\frac{\partial}{\partial x} f_j$ are bounded on the compact \mathcal{K} .

Thus, item (W) in Theorem 4.1 states that, for any W_1^1 -local minimum (x, u) in problem (6.1) with $|\dot{x}|_g = \text{const}$ and for all $i \leq k$, one has $\|u_i\|_{B_{p,\infty}^\alpha} < \infty$. Item (G) in the theorem states that there exists a constant $C > 0$ such that, for any two given points $x^0, x^1 \in \mathcal{K}$, any global minimum (x, u) in problem (6.1) with $x(t) \in \mathcal{K}$, and

⁹Except for the last one $[t_{N-3/2}, t_N]$, which length is S or greater.

$|\dot{x}|_g = \text{const}$ and for all $i \leq k$, one has $\|u_i\|_{B_{p,\infty}^\beta} \leq Cl(x)^\kappa$. Without loss of generality, we prove both items assuming $i = 1$ since, for other $i \leq k$, the proof is identical.

Let us make some additional simplifications to problem (6.1). Without loss of generality, we assume that

$$f_1(x) \equiv (1, 0, \dots, 0) \quad (A_1)$$

Proof of (A₁). Indeed, according to (OC₂), each vector field f_i is constant under an appropriate choice of global coordinates on M . Let us take the one that makes f_1 constant. □

Without loss of generality, we also assume that

$$M = \mathbb{R}^n \quad \text{and} \quad \exists c_f > 0 : \sup \left\{ \|f_i\|_{C^0}, \left\| \frac{\partial}{\partial x} f_i \right\|_{C^0}, \left\| \frac{\partial^2}{\partial x^2} f_i \right\|_{C^0} \right\} \leq c_f. \quad (A_2)$$

Proof of (A₂). According to (OC₂), M is an open subset in \mathbb{R}^n . Field f_1 can be trivially prolonged to \mathbb{R}^n by keeping condition (A₁). For f_i having $i \geq 2$, we use the following standard trick. Let \mathcal{K} be embedded into M with its ε -neighborhood. Choose a sufficiently smooth function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies $\psi \equiv 1$ on¹⁰ $\mathcal{K} + \frac{1}{3}\varepsilon B$, $\psi \equiv 0$ outside $\mathcal{K} + \frac{2}{3}\varepsilon B$, and $\psi(x) \in [0; 1]$ for all x . Let us now exchange f_i by $\tilde{f}_i = \psi f_i \in C^\infty(\mathbb{R}^n)$ and M by $\tilde{M} = \mathbb{R}^n$ in problem (6.1). Under such an exchange of vector fields, we have: (i) some admissible curves that leave $\mathcal{K} + \frac{1}{3}\varepsilon B$ may become nonadmissible; (ii) lengths of remaining admissible curves may only become bigger; and (iii) lengths of admissible curves in $\mathcal{K} + \frac{1}{3}\varepsilon B$ stay the same. Hence, any admissible curve in \mathcal{K} cannot lose any of properties of being global minimum or W_1^1 -local minimum. Moreover, the Hörmaner condition at points in $\mathcal{K} + \frac{1}{3}\varepsilon B$ is not affected by this exchange. □

From now on, we denote the standard Euclidean norm of vector $x \in M = \mathbb{R}^n$ by $|x|$. Recall that $d_{SR}(x, y)$ denotes the sub-Riemannian distance between $x, y \in M = \mathbb{R}^n$ (if there is no admissible curves joining x and y , we put $d_{SR}(x, y) = \infty$). So we claim that

$$\exists \sigma, c_{bb} > 0 : \forall x \in \mathcal{K} \forall y \in \mathcal{K} + \sigma B \quad \frac{1}{c_{bb}}|x - y| \leq d_{SR}(x, y) \leq c_{bb}|x - y|^{1/s}. \quad (A_3)$$

Proof of (A₃). It follows directly from the ball-box theorem [7] and the fact that step of the sub-Riemannian manifold at points in \mathcal{K} never exceeds s by the hypothesis of the theorem. We denote the constant by c_{bb} since it comes from the ball-box theorem. □

So, without loss of generality, we assume that conditions (A₁), (A₂), and (A₃) are fulfilled for optimal control problem (6.1).

6.3. An interpolational estimate on the optimal control

We start with a brief explanation of key ideas the proof is based on. Consideration of the interpolational estimate on the optimal control is based on the following well known idea. Denote by \dot{u}_1 the (generalized) derivative of u_1 . Suppose for a moment that $\dot{u}_1 \in L_r(0; 1)$ for some $1 < r < \infty$. Then, for any smooth test function $\varphi \in C_0^\infty(0; 1) = \{\varphi \in C^\infty : \text{supp } \varphi \Subset (0; 1)\}$, integration by parts together with the Hölder inequality

¹⁰Here we use the Minkowski sum $A + B = \{a + b | a \in A, b \in B\}$.

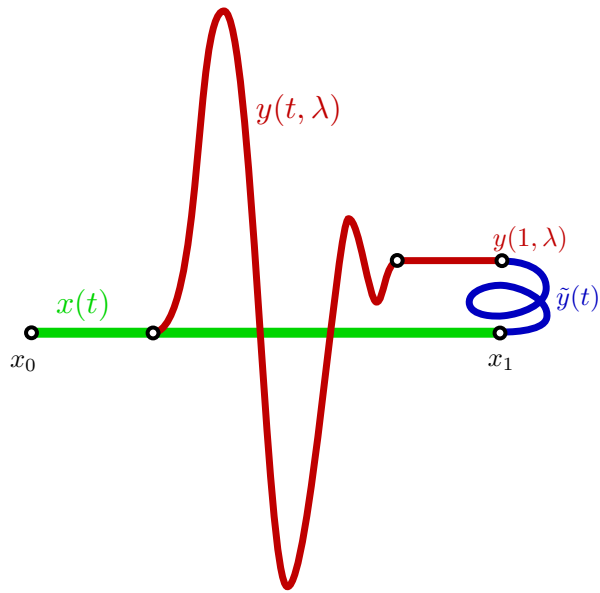


FIGURE 1. “Heartbeat” variation $y(t)$ of the original curve $x(t)$. On the figure, the end points of $\text{supp } \varphi$ are also shown as dots for convenience.

imply

$$\int_0^1 u_1 \dot{\varphi} dt = - \int_0^1 \dot{u}_1 \varphi dt \leq \|\dot{u}_1\|_r \|\varphi\|_{r^*},$$

where $rr^* = r + r^*$. The reversed statement also holds: if $\int_0^1 u_1 \dot{\varphi} dt \leq \text{const} \|\varphi\|_{r^*}$ for any test function $\varphi \in C_0^\infty(0; 1)$, then the generalized derivative \dot{u}_1 is a bounded linear functional on a dense subset of $L_{r^*}(0; 1)$ and, hence, $\dot{u}_1 \in (L_{r^*}(0; 1))^* = L_r(0; 1)$. So in the latter case, u_1 belongs to the Sobolev space $W_r^1(0; 1)$.

At the same time, for an arbitrary function $v \in L_q(0; 1)$, we are only able to guarantee that $\int_0^1 v \dot{\varphi} dt \leq \|v\|_q \|\dot{\varphi}\|_{q^*}$ and one cannot expect $\|\dot{\varphi}\|_{r^*}$ (instead of $\|\dot{\varphi}\|_{q^*}$) in the right-hand side of the inequality. The proof of Theorem 4.1 is based on an interpolational estimate on the optimal control: we will prove that

$$\int_0^1 u_1 \dot{\varphi} dt \leq C \|\varphi\|_r^{1-\theta} \|\dot{\varphi}\|_q^\theta$$

for some $0 < \theta < 1$ and (not necessarily conjugate) q and r . This estimate appears to be dual to an estimate on K -functional of θ -interpolation of spaces $L_{q^*}(0; 1)$ and $W_{r^*}^1(0; 1)$ and allows us to prove that u_1 is L_p -Hölder continuous by proceeding to the Besov spaces.

The heart of the proof is the following variation depicted on Figure 1. Let us choose an arbitrary test function $\varphi \in C_0^\infty(0; 1)$ and consider a new control¹¹ $u_1 - \lambda \dot{\varphi}$ (in the place of u_1), $\lambda \geq 0$, and the corresponding trajectory $y(t, \lambda)$ starting at x^0 , i.e. $y(\cdot, \lambda)$ is the unique solution to the following ODE

$$\dot{y} = (u_1 - \lambda \dot{\varphi})f_1 + u_2 f_2 + \dots u_k f_k$$

¹¹The condition $\lambda \geq 0$ is of no importance and introduced by the authors to simplify further computations.

with the initial state $y(0, \lambda) = x^0$. Since $\varphi(0) = \varphi(1) = 0$, we have $\int_0^1 \dot{\varphi} dt = 0$. Therefore, no matter how large modulo values the function $\dot{\varphi}$ takes, its fluctuations on the interval $[0; 1]$ will compensate each other in the variation $y(t, \lambda)$, and the final distance between x^1 and $y(1, \lambda)$ will be estimated in terms of the norm of function φ itself, and not by its derivative $\dot{\varphi}$ (see Lem. 6.1). Further, if $x^1 \neq y(1, \lambda)$, then we add an admissible curve $\tilde{y}(t)$ that joins $y(1, \lambda)$ and x^1 . Length of that curve can be controlled by the ball-box Theorem (see Lem. 6.2). At the end, we compare lengths of the original curve and the resulting variation, which gives us the desired interpolational estimate on the control u_1 .

So, let us denote by $P(t, \lambda)$ the flow of the linear part of the variational equation along y , i.e. $P(t, \lambda) \in \text{GL}(n)$ and

$$\dot{P} = \left(u_2 \frac{\partial}{\partial x} f_2 + \dots + u_k \frac{\partial}{\partial x} f_k \right) P \quad \text{and} \quad P(0, \lambda) = 1,$$

and put $P_0(t) = P(t, 0)$ for short. The term $u_1 \frac{\partial}{\partial x} f_1$ vanish since f_1 is constant.

As we have mentioned above, the first key idea in the theorem proof is the following: change of the original trajectory is estimated by the norm of φ itself, and not by the norm of its derivative $\dot{\varphi}$.

Lemma 6.1. *There exists a constant¹² $c > 0$ such that for all $\lambda \geq 0$ and $\varphi \in C_0^\infty(0; 1)$, one has*

$$|y(t, \lambda) - x(t)| \leq \lambda(|\varphi(t)| + cl(x)\|\varphi\|_1).$$

If additionally test function φ satisfies the following equality¹³

$$\int_0^1 \varphi P_0([f_1, f_2]u_2 + \dots + [f_1, f_k]u_k) dt = 0, \quad (6.3)$$

then

$$|y(1, \lambda) - x^1| \leq c\lambda^2\|\varphi\|_2^2. \quad (6.4)$$

Proof. Let us estimate norms of operator $P \in \text{GL}(n)$ first. Assumption (A₂) implies that $|\frac{\partial}{\partial x} f_j| \leq c_f$ and $\|u_j\|_\infty \leq L$. Therefore, using the Grönwall inequality, we obtain $\|P\| \leq e^{kLc_f}$. Similarly, $\|P^{-1}\| \leq e^{kLc_f}$. In other words, norms of operators P and P^{-1} are bounded by constant e^{kLc_f} for all t, λ , and φ .

Further, using the theorem on smooth dependence of ODE solutions on the parameter, we obtain that $y(t, \lambda)$ is a.e. differentiable in t and smooth in λ . Thus, according to the Lagrange formula, we have

$$|y(t, \lambda) - x(t)| \leq \lambda \sup_{\mu \in [0; \lambda]} |y_\lambda(t, \mu)|,$$

where y_λ stands for the derivative of y w.r.t. λ , $y_\lambda = \frac{\partial}{\partial \lambda} y$. Let us compute y_λ . Assumption (A₁) implies

$$\dot{y}_\lambda = \left(u_2 \frac{\partial}{\partial x} f_2 + \dots + u_k \frac{\partial}{\partial x} f_k \right) y_\lambda - \dot{\varphi} f_1.$$

Hence, using formula for the general solution to affine ODE, we obtain

$$y_\lambda(t, \lambda) = -P(t, \lambda) \int_0^t \dot{\varphi}(s) P^{-1}(s, \lambda) f_1(y(s)) ds,$$

¹²Which does not depend on φ, λ , points $x^0, x^1 \in \mathcal{K}$ and pair (x, u) with $x(t) \in \mathcal{K}$.

¹³As usual, here we compute values of brackets $[f_1, f_j]$ at the corresponding point of the original curve $x(t)$.

which, after an integration by parts, becomes

$$y_\lambda(t, \lambda) = -\varphi(t)f_1(y(t)) + P(t, \lambda) \int_0^t \varphi P^{-1}([f_2, f_1]u_2 + \dots + [f_k, f_1]u_k) ds$$

since f_1 is constant and $\varphi(0) = 0$. Using $|u(t)| \equiv l(x)$, we have

$$|y_\lambda(t, \lambda)| \leq |\varphi(t)| + kLc_f e^{2kLc_f} l(x) \|\varphi\|_1,$$

which proves the first estimate given in the lemma statement.

Now we proceed to the second estimate. Here we restrict ourselves to the right end-point. So let us write down the Taylor expansion up the second order:

$$|y(1, \lambda) - x(1) - \lambda y_\lambda(1, 0)| \leq \frac{1}{2} \lambda^2 \sup_{\mu \in [0; \lambda]} |y_{\lambda\lambda}(1, \mu)|$$

and

$$\dot{y}_{\lambda\lambda} = \left(u_2 \frac{\partial}{\partial x} f_2 + \dots + u_k \frac{\partial}{\partial x} f_k\right) y_{\lambda\lambda} + \left(u_2 \frac{\partial^2}{\partial x^2} f_2 + \dots + u_k \frac{\partial^2}{\partial x^2} f_k\right) (y_\lambda, y_\lambda),$$

where $y_{\lambda\lambda}$ stands for the second derivative of y w.r.t. λ , $y_{\lambda\lambda} = \frac{\partial^2}{\partial \lambda^2} y$.

If φ satisfies the linear equation (6.3), then the linear part in the Taylor expansion disappears, *i.e.* $y_\lambda(1, 0) = 0$. Hence, if we consider only test functions $\varphi \in C^\infty(0; 1)$ that satisfy (6.3), then position of the right end $y(1, \lambda)$ is determined by the second variation. Let us estimate it. Since

$$y_{\lambda\lambda}(1, \lambda) = P(1, \lambda) \int_0^1 P^{-1} \left(u_2 \frac{\partial^2}{\partial x^2} f_2 + \dots + u_k \frac{\partial^2}{\partial x^2} f_k\right) (y_\lambda, y_\lambda) dt,$$

we obtain

$$|y_{\lambda\lambda}(1, \lambda)| \leq kLc_f e^{2kc_f} \int_0^1 |y_\lambda|^2 dt \leq kLc_f e^{2kc_f} (\|\varphi\|_2^2 + 2kLc_f e^{kLc_f} \|\varphi\|_1^2 + k^2 L^2 c_f^2 e^{2kLc_f} \|\varphi\|_1^2).$$

Obviously $\varphi \in L_2(0; 1)$ implies $\|\varphi\|_1 \leq \|\varphi\|_2$ by the Hölder inequality. Hence,

$$|y_{\lambda\lambda}(1, \lambda)| \leq kLc_f e^{2kc_f} (1 + 2kLc_f e^{kLc_f} + k^2 L^2 c_f^2 e^{2kLc_f}) \|\varphi\|_2^2.$$

Thus,

$$c = \max \left\{ 1, kLc_f e^{kLc_f}, kLc_f e^{2kLc_f} (1 + 2kLc_f e^{kLc_f} + k^2 L^2 c_f^2 e^{2kLc_f}) \right\}.$$

□

In the following lemma, we prove that, for some values of parameters r , ζ , and c' , the following implication is fulfilled

$$c' l(x) \lambda \|\varphi\|_1 \leq 1 \implies l(y) - l(x) + c' (l(x) \lambda \|\varphi\|_r)^\zeta \geq 0. \quad (6.5)$$

Note that, using this implication, we then obtain a dual interpolational estimate on the control u_1 in Proposition 6.5.

Lemma 6.2. *There exist*

- (\mathfrak{G}) a constant \tilde{c}' such that, for any sub-Riemannian shortest path (x, u) with $x(t) \in \mathcal{K}$, the material implication (6.5) holds for all $\lambda \geq 0$ and functions $\varphi \in C_0^\infty(0; 1)$ if $r = 1$, $\zeta = 1/s$, and $c' = \tilde{c}'$;
- (\mathfrak{W}) a depending on the choice of W_1^1 -local minimum (x, u) constant $\tilde{c}' = \tilde{c}'(x, u)$ such that the material implication (6.5) holds for all $\lambda \geq 0$ and functions $\varphi \in C_0^\infty(0; 1)$ if $r = 2$, $\zeta = 2/s$, $c' = \tilde{c}'$, and φ and λ satisfy additional restriction $c' l(x) \lambda \|\varphi\|_1 \leq 1$.

Proof. Let us use (A_3). By the hypothesis of Theorem 4.1, $x^1 \in \mathcal{K}$. Hence, Lemma 6.1 implies that if $|x^1 - y(1, \lambda)| < \sigma$, then¹⁴

$$d_\lambda \stackrel{\text{def}}{=} d_{SR}(x^1, y(1, \lambda)) \leq c_{bb} |x^1 - y(1, \lambda)|^{1/s}.$$

Let us consider only $\lambda \geq 0$ such that $c l(x) \lambda \|\varphi\|_1 \leq \sigma$. For these λ , using Lemma 6.1, we obtain

$$d_\lambda \leq c_{bb} (c l(x) \lambda \|\varphi\|_1)^{1/s}. \quad (6.6)$$

If additionally $c l(x) (c_{bb})^s \lambda \|\varphi\|_1 \leq 4^{-s}$, then $d_\lambda \leq \frac{1}{4}$. So, we put

$$c'_0 = c \max\{1/\sigma, (4c_{bb})^s\}$$

and assume both $\hat{c} \geq c'_0$ and $\tilde{c} \geq c'_0$.

By the definition of the sub-Riemannian distance, there exists a pair (\tilde{y}, \tilde{v}) that satisfies the ODE in problem (6.1) with terminal conditions $\tilde{y}(0) = y(1, \lambda)$, $\tilde{y}(1) = x^1$ (see Fig. 1) and has length $\rho \leq 2d_\lambda \leq \frac{1}{2}$. Without loss of generality, we assume $|\tilde{v}(t)| \equiv \rho$. Consider the following composite pair (z, w) , $t \in [0; 1]$:

$$z(t) = \begin{cases} y(t/(1-\rho)), & \text{for } t \in [0; 1-\rho]; \\ \tilde{y}((t-1+\rho)/\rho), & \text{for } t \in [1-\rho; 1]; \end{cases} \quad w(t) = \begin{cases} \frac{1}{1-\rho} v(t/(1-\rho)), & \text{for } t \in [0; 1-\rho]; \\ \frac{1}{\rho} \tilde{v}((t-1+\rho)/\rho), & \text{for } t \in [1-\rho; 1]; \end{cases}$$

where $v = u - \lambda(\dot{\varphi}, 0, \dots, 0)$. Now, the pair (z, w) does satisfy both the ODE and terminal conditions in problem (6.1).

First, we suppose that (x, u) is a global minimum for some given points x^0 and x^1 . In this case, if the estimate $\lambda \|\varphi\|_1 \leq 1/(\tilde{c}' l(x))$ holds, then length of the original curve $x(t)$ cannot be greater than length of the composite curve $z(t)$. Therefore, using Lemma 6.1, we obtain

$$l(x) \leq l(z) \leq l(y) + 2d_\lambda \leq l(y) + 2c_{bb} (c \lambda l(x) \|\varphi\|_1)^{1/s}. \quad (6.7)$$

Thus, item (\mathfrak{G}) in the lemma is proved for $\tilde{c}' = \max\{c'_0, c(c_{bb})^s\} = c'_0$.

Second, we prove item (\mathfrak{W}) in the Lemma. Let (x, u) be not necessarily global but W_1^1 -local minimum in problem (6.1) for some given x^0 and x^1 . In this case, we are able to find a constant \tilde{c}' (stated in the lemma) for any given pair (x, u) , but \tilde{c}' highly depends on (x, u) in general, and it seems that \tilde{c}' cannot be chosen uniformly among all W_1^1 -local minima in \mathcal{K} .

Summarizing, let us fix a W_1^1 -local minimum (x, u) for the rest of the lemma proof. We claim that, for small enough λ , the composite control $w(t)$ belongs to the ε -neighborhood¹⁵ of the original control $u(t)$, i.e. $\|u - w\|_1 < \varepsilon$ (which then implies that the composite pair (z, w) length cannot be less than length of (x, u)).

¹⁴Recall that the constant c_{bb} comes from the ball-box theorem, see (A_3) for details.

¹⁵Here ε is taken from the definition of W_1^1 -local minimum.

In order to prove inequality $\|u - w\|_1 < \varepsilon$, let us estimate the left-hand side:

$$\|u - w\|_1 \leq \|u - v\|_1 + \int_0^{1-\rho} \left| v(t) - \frac{1}{1-\rho} v(t/(1-\rho)) \right| dt + \int_{1-\rho}^1 |v(t)| dt + \frac{1}{\rho} \int_{1-\rho}^1 |\tilde{v}((t-1+\rho)/\rho)| dt.$$

Now we estimate each term in the right-hand side separately:

$$\begin{aligned} \|u - v\|_1 &= \|(\lambda\dot{\varphi}, 0, \dots, 0)\|_1 = \lambda\|\dot{\varphi}\|_1; \\ \int_0^{1-\rho} \left| v(t) - \frac{1}{1-\rho} v(t/(1-\rho)) \right| dt &\leq 2\lambda\|\dot{\varphi}\|_1 + \sum_{i=1}^k \int_0^1 |(1-\rho)u_i((1-\rho)s) - u_i(s)| ds; \\ \int_{1-\rho}^1 |v(t)| dt &\leq \lambda\|\dot{\varphi}\|_1 + \sum_{i=1}^k \int_{1-\rho}^1 |u_i(t)| dt \leq \lambda\|\dot{\varphi}\|_1 + kL\rho; \\ \frac{1}{\rho} \int_{1-\rho}^1 |\tilde{v}((t-1+\rho)/\rho)| dt &= \rho. \end{aligned}$$

Obviously, $L_1(0; 1)$ -norm of each function $(1-\rho)u_i((1-\rho)t) - u_i(t)$ tends to 0 as $\rho \rightarrow +0$. Therefore,

$$\|u - w\|_1 \leq 4\lambda\|\dot{\varphi}\|_1 + \nu(\rho)$$

where $\nu(\rho) \geq 0$ and $\lim_{\rho \rightarrow +0} \nu(\rho) = 0$. Since $\rho \leq d_\lambda$, there exists $\hat{d} > 0$ such that $\nu(\rho) < \varepsilon/2$ if $d_\lambda \leq \hat{d}$.

Thus, in order to fulfill inequality $\|u - w\|_1 < \varepsilon$, it is enough to have $4\lambda\|\dot{\varphi}\|_1 \leq \varepsilon/2$ and $c_{bb}(cl(x)\lambda\|\dot{\varphi}\|_1)^{1/s} \leq \hat{d}$ due to (6.6). The last two inequalities follow from the hypothesis in item (W) of the lemma and the condition in material implication (6.5) if

$$\check{c}'l(x) \geq c'_1 = \max\{8/\varepsilon, cl(x)(c_{bb}/\hat{d})^s\}.$$

So let us choose $\check{c}' \geq \max\{c'_0, c'_1/l(x)\}$ (recall that $l(x) \neq 0$ due to (6.2)). Hence, if estimate $\check{c}'l(x)\lambda\|\dot{\varphi}\|_1 \leq 1$ holds, then $\|u - w\|_1 < \varepsilon$, and length of the original curve $x(t)$ cannot be less than length of the composite curve $z(t)$: $l(x) \leq l(y) + 2d_\lambda$.

Therefore, we obtain the same result as in item (G), but now we are able to use more accurate estimate on the distance between x^1 and $y(1, \lambda)$, which is given in the second part of Lemma 6.1:

$$d_\lambda \leq c_{bb}|x^1 - y(1, \lambda)|^{1/s} \leq c_{bb}c^{1/s}\lambda^{2/s}\|\varphi\|_2^{2/s}.$$

Thus, item (W) of the Lemma is proved for $\check{c}' \geq c_{bb}(c/l(x)^2)^{1/s}$. In other words, item (W) of the lemma is proved by taking

$$\check{c}' = \max\{c'_0, c'_1/l(x), c_{bb}(c/l(x)^2)^{1/s}\}.$$

□

In order to obtain the interpolational estimate on the control u_1 , we construct a lower bound for the left-hand side of (6.5). Vector fields f_j are orthonormal due to assumption (OC₃). Hence,

$$l(y) = \int_0^1 \sqrt{(u_1 - \lambda\dot{\varphi})^2 + u_2^2 + \dots + u_k^2} dt = \int_0^1 \sqrt{l(x)^2 - 2\lambda u_1 \dot{\varphi} + \lambda^2 \dot{\varphi}^2} dt. \quad (6.8)$$

Remark 6.3. Note that the previous equality is the only place in the proof of Theorem 4.1 where we actually use assumption (OC₃). It has not been used before and will not be used further.

It is not so hard to estimate $l(y)$ with the help of the following lemma.

Lemma 6.4. *Let $l > 0$ and $1 \leq q \leq 2$. Then, there exists a number $c_{SR} > 0$ such that, for any $\psi \in C_0^\infty(0; 1)$ and $u_1 \in L_\infty(0; 1)$ having $\|u_1\|_\infty \leq l$, the following inequality is fulfilled*

$$\int_0^1 \sqrt{l^2 - 2u_1\psi + \psi^2} dt - l \leq c_{SR} l^{1-q} \|\psi\|_q^q - \frac{1}{l} \int_0^1 u_1\psi dt.$$

Proof. First, let $l = 1$. Then

- If $|\psi(t)| \leq 4$, then we are able to estimate the square root in standard manner:

$$\sqrt{1 - 2u_1\psi + \psi^2} - 1 \leq -u_1\psi + \frac{1}{2}\psi^2 \leq -u_1\psi + \psi^2.$$

- If $|\psi(t)| > 4$, then $|u_1\psi| \leq |\psi| \leq \frac{1}{4}\psi^2$ (as $|u_1| \leq 1$) and $1 \leq \frac{1}{16}\psi^2$. Hence,

$$\sqrt{1 - 2u_1\psi + \psi^2} - 1 \leq 2|\psi| \leq -u_1\psi + 4|\psi|.$$

Denote the left-hand side of the inequality in the lemma statement by $F(l, u_1, \psi)$. Then

$$F(1, u_1, \psi) \leq \int_{t:|\psi|\leq 4} \psi^2 dt + 4 \int_{t:|\psi|>4} |\psi| dt - \int_0^1 u_1\psi dt. \quad (6.9)$$

Now we unite both terms in the right-hand side of (6.9) in the following way. Let $1 \leq q \leq 2$ be an arbitrary number. Then, for the first integral, we have

$$\int_{t:|\psi|\leq 4} \psi^2 dt = \int_{t:|\psi|\leq 4} |\psi|^{2-q} |\psi|^q dt \leq 4^{2-q} \int_{t:|\psi|\leq 4} |\psi|^q dt.$$

For the second integral, we have

$$4 \int_{t:|\psi|>4} |\psi| dt = 4^{2-q} \int_{t:|\psi|>4} 4^{q-1} |\psi| dt \leq 4^{2-q} \int_{t:|\psi|>4} |\psi|^{q-1} |\psi| dt = 4^{2-q} \int_{t:|\psi|>4} |\psi|^q dt.$$

Thus, inequality (6.9) implies

$$F(1, u_1, \varphi) \leq 4^{2-q} \int_0^1 |\psi|^q dt - \int_0^1 u_1\psi dt.$$

So

$$F(1, u_1, \psi) \leq 4\|\psi\|_q^q - \int_0^1 u_1\psi.$$

In other words, putting $c_{SR} = 4$ proves the lemma in the case $l = 1$.

For an arbitrary l , the lemma follows from the fact that F is homogeneous: $F(l, u_1, \psi) = lF(1, u_1/l, \psi/l)$. \square

Let us now combine the previous two lemmas and show that they imply the following key interpolational estimate on the control u_1 for some $\zeta > 0$:

$$\int_0^1 u_1 \dot{\varphi} dt \leq c'' l(x)^\kappa \|\varphi\|_r^\theta \|\dot{\varphi}\|_q^{1-\theta}, \quad (6.10)$$

where $\theta = \frac{(q-1)\zeta}{q-\zeta}$ and $\kappa = \frac{1+2q\zeta-3\zeta}{q-\zeta}$.

Here φ is an arbitrary test function that vanishes at $t = 0, 1$, namely

$$\varphi \in \mathring{W}_q^1(0; 1) = \{\varphi \in W_q^1(0; 1) : \varphi(0) = \varphi(1)\}.$$

Proposition 6.5 (interpolational estimate on control).

- (\mathfrak{G}) *There exists a constant \tilde{c}'' such that, for any $1 \leq q \leq 2$, any sub-Riemannian shortest path (x, u) with $x(t) \in \mathcal{K}$, and any test function $\varphi \in \mathring{W}_q^1(0; 1)$, inequality (6.10) holds for $c'' = \tilde{c}''$, $r = 1$, and $\zeta = 1/s$;*
- (\mathfrak{W}) *For any $1 \leq q \leq 2$, there exists a constant \tilde{c}'' (which depends on the choice of W_1^1 -local minimum (x, u) in general, $\tilde{c}'' = \tilde{c}''(x, u, q)$) such that, for any test function $\varphi \in \mathring{W}_q^1(0; 1)$, inequality (6.10) holds for $c'' = \tilde{c}''$, $r = 2$, and $\zeta = 2/s$.*

Proof. If $q = 1$, then inequality (6.10) is trivially fulfilled for $c'' = 1$. So we assume that $q > 1$.

First, let φ be a smooth function, i.e. assume that $\varphi \in C_0^\infty(0; 1)$. If we substitute expression (6.8) for $l(y)$ and the estimate in Lemma 6.4 (with $\psi = \lambda \dot{\varphi}$ and $l = l(x)$) into the inequality in (6.5), we then obtain

$$\lambda \int_0^1 u_1 \dot{\varphi} dt \leq c_{SR} \lambda^q l(x)^{2-q} \|\dot{\varphi}\|_q^q + c' \lambda^\zeta l(x)^{1+\zeta} \|\varphi\|_\zeta. \quad (6.11)$$

Let us divide (6.11) by λ . Two terms will be obtained in the rhs: one has factor λ^{q-1} and the other has factor $\lambda^{\zeta-1}$. The first one tend to ∞ as $\lambda \rightarrow +\infty$ and vanish as $\lambda \rightarrow 0$. The second one tends to ∞ as $\lambda \rightarrow 0$ and vanish as $\lambda \rightarrow \infty$. So an optimal choice of λ is a minimum point of the obtained rhs (which obviously exists due to described behavior of factors λ^{q-1} and $\lambda^{\zeta-1}$). For simplicity put $\lambda_0 = l(x)^{\frac{q+\zeta-1}{q-\zeta}} \|\varphi\|_r^{\frac{\zeta}{q-\zeta}} \|\dot{\varphi}\|_q^{-\frac{q}{q-\zeta}}$ which differs from the optimal value of λ by a complicated constant factor of no importance. So we plug $\lambda = \lambda_0$ into (6.11):

$$\int_0^1 u_1 \dot{\varphi} \leq (c' + c_{SR}) l(x)^\kappa \|\varphi\|_r^\theta \|\dot{\varphi}\|_q^{1-\theta}.$$

Thus, in the case (\mathfrak{G}), the desired inequality is proved for $\varphi \in C_0^\infty(0; 1)$ under restriction $c' l(x) \lambda_0 \|\varphi\|_1 \leq 1$. In the case (\mathfrak{W}), restrictions (6.3) and $c' l(x) \lambda_0 \|\dot{\varphi}\|_1 \leq 1$ should be also fulfilled.

First, let us throw away restriction $c' l(x) \lambda_0 \|\varphi\|_1 \leq 1$. On the one hand, since

$$c' \lambda_0 l(x) \|\varphi\|_1 = c' l(x)^{\frac{2q-1}{q-\zeta}} (\|\varphi\|_1 / \|\dot{\varphi}\|_q)^{\frac{q}{q-\zeta}},$$

the restriction $c' l(x) \lambda_0 \|\varphi\|_1 \leq 1$ is definitely fulfilled if

$$c'' l(x)^{\frac{2q-1}{q}} \|\varphi\|_1 \leq \|\dot{\varphi}\|_{L_q}$$

where $c_0'' \geq (c')^{\frac{q-\zeta}{q}}$. Since $1 \leq q \leq 2$, we may put $c_0'' = \max\{(c')^{1-\frac{\zeta}{2}}, (c')^{1-\zeta}\}$. On the other hand, if the opposite inequality is fulfilled for some test function φ , then $\|u_1\|_\infty \leq l(x)$ implies

$$\begin{aligned} \int_0^1 u_1 \dot{\varphi} dt &\leq l(x) \|\dot{\varphi}\|_1 \leq l(x) \|\dot{\varphi}\|_q \leq l(x) (c_0'' l(x))^{2-\frac{1}{q}} \|\varphi\|_1^\theta \|\dot{\varphi}\|_q^{1-\theta} \\ &= (c_0'')^\theta l(x)^{\kappa+\frac{q-1}{q}} \|\varphi\|_1^\theta \|\dot{\varphi}\|_q^{1-\theta} \leq c_1'' l(x)^\kappa \|\varphi\|_r^\theta \|\dot{\varphi}\|_q^{1-\theta} \end{aligned}$$

where $c_1'' \geq (c_0'')^\theta L^{\frac{q-1}{q}}$. Since $0 \leq \theta \leq \zeta/(2-\zeta)$ and $0 \leq (q-1)/q \leq \frac{1}{2}$, we may put $c_1'' = \max\{c_0'', (c_0'')^{\zeta/(2-\zeta)}\} \max\{1, L^{1/2}\}$.

Thus, we have proved the case (\mathfrak{G}) for $c'' = \hat{c}'' = \max\{\hat{c}'' + c_{SR}, c_0'', c_1''\}$ and $\varphi \in C_0^\infty(0; 1)$. In order to complete the proof of (\mathfrak{G}) , it remain to note that the space $C_0^\infty(0; 1)$ is dense in $\dot{W}_q^1(0; 1)$ (see [16], Sect. 4.2.4) and functionals $\int u_1 \dot{\varphi}$, $\|\varphi\|_r$, and $\|\dot{\varphi}\|_q$ are all continuous on $\dot{W}_q^1(0; 1)$.

Let us now proceed to the case (\mathfrak{W}) . On the one hand, since

$$c' \lambda_0 l(x) \|\dot{\varphi}\|_q = c' l(x)^{\frac{2q-1}{q-\zeta}} (\|\varphi\|_1 / \|\dot{\varphi}\|_q)^{\frac{\zeta}{q-\zeta}},$$

inequality $c' l(x) \lambda_0 \|\dot{\varphi}\|_1 \leq 1$ is definitely fulfilled if

$$c_1'' l(x)^{\frac{2q-1}{\zeta}} \|\varphi\|_1 \leq \|\dot{\varphi}\|_q$$

where $c_1'' = (c')^{-1+\frac{2}{q}}$. On the other hand, if the opposite inequality is fulfilled for some test function φ , then $\|u_1\|_\infty \leq l(x) \leq L$ implies

$$\begin{aligned} \int_0^1 u_1 \dot{\varphi} dt &\leq l(x) \|\dot{\varphi}\|_1 \leq l(x) \|\dot{\varphi}\|_q \leq l(x) (c_1'' l(x))^{\frac{2q-1}{\zeta}} \|\varphi\|_1^\theta \|\dot{\varphi}\|_q^{1-\theta} \\ &= (c_1'')^\theta l(x)^{\kappa+2(q-1)} \|\varphi\|_r^\theta \|\dot{\varphi}\|_q^{1-\theta} = c_2'' l(x)^\kappa \|\varphi\|_r^\theta \|\dot{\varphi}\|_q^{1-\theta} \end{aligned}$$

where $c_2'' = (c_1'')^\theta \max\{1, L^2\}$.

Thus, in the case (\mathfrak{W}) , the desired inequality is proved under two additional restrictions: (i) $\varphi \in C_0^\infty(0; 1)$ and (ii) φ satisfies (6.3). Note that equality (6.3) defines a closed linear subspace in $\dot{W}_q^1(0; 1)$ of the form

$$H = \left\{ \varphi \in \dot{W}_q^1(0; 1) : \int_0^1 \varphi P_0([f_1, f_2]u_2 + \dots + [f_1, f_k]u_k) dt = 0 \right\}.$$

Subspace H has finite codimension n as $u \in L_\infty(0; 1)$ and $P_0 \in L_\infty(0; 1)$. Space $C_0^\infty(0; 1)(0; 1)$ is dense in $\dot{W}_q^1(0; 1)$ and, hence, it is dense in H . Therefore, the desired inequality is fulfilled for all $\varphi \in H$ as both functionals $\mathfrak{p}(\varphi) = c_1'' \|\varphi\|_2^\theta \|\dot{\varphi}\|_q^{1-\theta}$ and $\int_0^1 u_1 \dot{\varphi} dt$ are continuous on $\dot{W}_q^1(0; 1)$.

Generalized derivative \dot{u}_1 can be considered as a linear functional on $\dot{W}_q^1(0; 1)$. Moreover, we have proved that it is dominated by $\mathfrak{p}(\varphi)$ on H . Hence, $\dot{u}_1|_H \leq \text{conv } \mathfrak{p}|_H$ due to linearity of \dot{u}_1 . Functional $\text{conv } \mathfrak{p}$ is positively homogeneous, convex, and nonnegative. Therefore, $\text{conv } \mathfrak{p}$ is a seminorm on $\dot{W}_q^1(0; 1)$. Then, Hahn–Banach separation theorem implies that linear functional \dot{u}_1 has an extension \mathfrak{w} to the whole space $\dot{W}_q^1(0; 1)$ that is also dominated by $\text{conv } \mathfrak{p}$, *i.e.* $\mathfrak{w} \leq \text{conv } \mathfrak{p} \leq \mathfrak{p}$. Since \mathfrak{w} and \dot{u}_1 coincide on H , the H definition implies

$$\dot{u}_1 = \mathfrak{w} + \langle \mu, P_0([f_1, f_2]u_2 + \dots + [f_1, f_k]u_k) \rangle$$

for a tuple of n constants $\mu \in \mathbb{R}^{n*}$. It remains to note that $P_0([f_1, f_2]u_2 + \dots + [f_1, f_k]u_k) \in L_\infty(0; 1)$. So, for some constant c_3'' and any test function $\varphi \in \dot{W}_q^1(0; 1)$, we obtain

$$\int_0^1 \varphi \langle \mu, P_0([f_1, f_2]u_2 + \dots + [f_1, f_k]u_k) \rangle dt \leq c_3'' \|\varphi\|_1 \leq c_3'' \|\varphi\|_1^\alpha \|\dot{\varphi}\|_1^{1-\alpha} \leq c_3'' \|\varphi\|_2^\alpha \|\dot{\varphi}\|_q^{1-\alpha}.$$

Thus, it is sufficient to put $\tilde{c}'' = c_2'' + c_3'' l(x)^{-\kappa}$ (recall that, in the case (\mathfrak{W}) , constant \tilde{c}'' may depend on the choice of local minimum (x, u) in general). □

6.4. Duality and approximation of the optimal control

So, in order to improve length of the original trajectory $x(t)$, we are trying to choose a test function $\varphi(t)$ in such a way that the control variation $u_1 \mapsto u_1 - \lambda \dot{\varphi}$ improves the functional (6.1). It means that the term $-u_1 \dot{\varphi}$ should be as negative as possible. However, the $x(t)$ optimality forbids $-\int_0^1 u_1 \dot{\varphi} dt$ to be too negative. Namely, it is bounded by Proposition 6.5. Nonetheless, let us try to choose $\varphi(t)$ as best as possible assuming that norms $\|\varphi\|_r$ and $\|\dot{\varphi}\|_q$ in the right-hand side of (6.10) are given. To do so, we consider the following convex optimization problem

$$\begin{aligned} -\int_0^1 u_1 \dot{\varphi} dt &\rightarrow \min_{\varphi \in W_q^1[0;1]} \\ \begin{cases} \|\varphi\|_r \leq 1; \\ \|\dot{\varphi}\|_q \leq M; \\ \varphi(0) = \varphi(1) = 0. \end{cases} & \end{aligned} \quad (6.12)$$

Denote by $S_r^q(u_1, M)$ the value (*i.e.* infimum) of problem (6.12). So the $x(t)$ optimality implies the following lower bound on $S_r^q(u_1, M)$:

Corollary 6.6. *If for some $r \geq 1$, $q \geq 1$, $\zeta \leq 1$, and $c'' > 0$, the control u_1 satisfies inequality (6.10), then, for any $M \geq 0$, one has*

$$S_r^q(u_1, M) \geq -c'' l(x)^\kappa M^{1-\theta}.$$

Proof. Indeed, (6.10) implies

$$\int_0^1 u_1 \dot{\varphi} dt \leq c'' l(x)^\kappa \|\varphi\|_r^\theta \|\dot{\varphi}\|_q^{1-\theta} \leq c'' l(x)^\kappa M^{1-\theta}$$

since $0 \leq \theta \leq 1$. □

The value $S_r^q(u_1, M)$ of infimum in problem (6.12) is closely related to the approximation quality of the control u_1 by smooth function. It is well known that the rate of such an approximation is equivalent to an upper bound on K -functional of interpolation of the Lebesgue space $L_{q^*}(0; 1)$ and the Sobolev space $W_{r^*}^1(0; 1)$. We will actively use this interpolation in the next subsection, and now we show duality of K -functional and value $S_r^q(u_1, M)$ of infimum in problem (6.12). Recall that K -functional is defined in our case as follows:

$$K_{q^*}^{r^*}(M, u_1) \stackrel{\text{def}}{=} \inf_w (\|w\|_{W_{r^*}^1} + M \|w - u_1\|_{q^*}). \quad (6.13)$$

where $\frac{1}{r} + \frac{1}{r^*} = \frac{1}{q} + \frac{1}{q^*} = 1$.

In the following proposition, we use a functional that is very similar to $K_{q^*}^r$: we only change the term $\|w\|_{W_{r^*}^1}$ to the term $\|\dot{w}\|_{r^*}$. In fact, this change does not affect the interpolation procedure, which we explain in the next section.

Proposition 6.7 (dual interpolational estimate on K -functional). *For any $u_1 \in L_\infty(0; 1)$, $1 \leq r < \infty$, $1 \leq q < \infty$, and $M \geq 0$, the following equality holds¹⁶*

$$S_q^r(u_1, M) + \inf_{w \in W_{r^*}^1(0; 1)} (\|\dot{w}\|_{r^*} + M\|u_1 - w\|_{q^*}) = 0 \quad (6.14)$$

Proof. The statement trivially holds for $M = 0$. So we assume $M > 0$.

Problem (6.12) is convex. Hence, it is natural to apply duality theory of convex problems. In order to construct the dual problem, we should include problem (6.12) into a family of convex problems. So let us take arbitrary $\psi = (\psi_0, \psi_1) \in L_r(0; 1) \times L_q(0; 1)$ and $\varphi^* \in W_{q^*}^1(0; 1)$ and consider the following family of convex problems, which depend on parameters φ^* and $\psi = (\psi_0, \psi_1)$:

$$\begin{aligned} - \int_0^1 (\dot{\varphi}^* \dot{\varphi} + \varphi^* \varphi) dt \rightarrow \min_{\varphi \in W_q^1(0; 1)} \\ \begin{cases} \|\varphi - \psi_0\|_r \leq 1; \\ \|\dot{\varphi} - \psi_1\|_q \leq M; \\ \varphi(0) = \varphi(1) = 0. \end{cases} \end{aligned} \quad (6.15)$$

Further, we show that problem (6.12) is in family (6.15) for $\psi_0 = \psi_1 = 0$ and under an appropriate choice of parameter φ^* .

Recall that a dual family to a family of optimization problems given in the following standard form (with parameters ψ and φ^*)

$$F(\varphi, \psi) - \langle \varphi^*, \varphi \rangle \rightarrow \min_{\varphi} \quad (6.16)$$

is a family of optimization problems with the same parameters ψ and φ^* of the form

$$F^*(\varphi^*, \psi^*) - \langle \psi, \psi^* \rangle \rightarrow \min_{\psi^*} \quad (6.17)$$

where F^* is the Legendre–Young–Fenchel convex conjugate to F . We emphasize that parameters φ^* and ψ are common for the both families, but original family problems are minimizing in φ , and dual family problems are minimizing in ψ^* .

Usually, considerations of dual convex optimization problems require a careful choice of pair of spaces in duality. So we assume that (φ, φ^*) and (ψ, ψ^*) belong to some spaces (Φ, Φ^*) and (Ψ, Ψ^*) , correspondingly. Moreover, we equip both pair of spaces (Φ, Φ^*) and (Ψ, Ψ^*) with the weak and weak* topologies, correspondingly (see [10] for details). This careful choice is needed in the first place for proper computation of convex conjugates, e.g. F^* and F^{**} .

The dual family of optimization problems can be very useful as it is usually possible to prove that there is no duality gap between the origin family and the dual one. Namely, it means that the sum of infima in both families identically vanish for any choice of parameters ψ and φ^* . Let us denote by $S_{\varphi^*}(\psi)$ the value of the corresponding problem in the original family (note that φ^* enters the original family linearly, and ψ enters it nonlinearly), and by $D_\psi(\varphi^*)$ the value of the corresponding problem in the dual family (here ψ enters the dual family linearly,

¹⁶The second term is in fact K -functional up to some minor terms, and the first term is its dual interpolation estimate on control u_1

and φ^* enters it nonlinearly). Then, a trivial inequality $F(\varphi, \psi) + F^*(\varphi, \psi) \geq \langle \varphi, \varphi^* \rangle + \langle \psi, \psi^* \rangle$ implies that $S_{\varphi^*}(\psi) + D_{\psi}(\varphi^*) \geq 0$. Moreover, we show below that there is no duality gap, *i.e.* $S_{\varphi^*}(\psi) + D_{\psi}(\varphi^*) = 0$.

Now we compute the dual family to the original one given in (6.15). In order to do so, let us rewrite family (6.15) in the standard form. We have $\Phi = W_q^1(0; 1)$, $\Phi^* = W_{q^*}^1(0; 1)$, $\Psi = L_r[0; 1] \times L_q(0; 1)$, and $\Psi^* = L_{r^*}[0; 1] \times L_{q^*}(0; 1)$. We equip both spaces Φ and Ψ with the weak topologies, and spaces Φ^* and Ψ^* with the weak* topologies. Under this choice of topologies, pairs (Φ, Φ^*) and (Ψ, Ψ^*) become pairs of spaces in duality (see [10]).

Let us now write down a convenient formula for functional F . Obviously, in order to throw out restrictions (6.15) on φ (which are formally not included into the standard form (6.16)), we define F as follows: $F(\varphi, \psi) = 0$ if φ does satisfy restrictions in (6.15), and $F(\varphi, \psi) = \infty$ if it doesn't:

$$F(\varphi, \psi) = \begin{cases} 0, & \text{if } \|\varphi - \psi_0\|_r \leq 1, \|\dot{\varphi} - \psi_1\|_q \leq M \text{ and } \varphi(0) = \varphi(1) = 0; \\ \infty, & \text{otherwise.} \end{cases}$$

Before we proceed to computing the dual family, let us find a particular value of parameter φ^* that includes the original problem (6.12) into family (6.15). In other words, we are seeking for a function $\varphi^* = \hat{\varphi}^*$ such that

$$\forall \varphi \in W_q^1[0; 1] \quad \langle \hat{\varphi}^*, \varphi \rangle \stackrel{\text{def}}{=} \int_0^1 (\varphi \hat{\varphi}^* + \dot{\varphi} \dot{\hat{\varphi}}^*) dt \equiv \int_0^1 u_1 \dot{\varphi} dt.$$

Linear functional $\int_0^1 u_1 \dot{\varphi} dt$ is continuous on $W_q^1(0; 1)$. Hence, the desired function $\hat{\varphi}^*$ exists and belongs to $(W_q^1(0; 1))^* = W_{q^*}^1(0; 1)$. So $\int_0^1 (\dot{\varphi}(\dot{\hat{\varphi}}^* - u_1) + \varphi \hat{\varphi}^*) dt = 0$ for any function $\varphi \in W_q^1(0; 1)$. Therefore, the du Bois–Reymond lemma implies that $\dot{\hat{\varphi}}^* - u_1 \in \dot{W}_{q^*}^1(0; 1)$ and

$$\frac{d}{dt}(\dot{\hat{\varphi}}^* - u_1) = \hat{\varphi}^*. \quad (6.18)$$

Under the described choice of F , original family (6.15) is represented in standard form (6.16), and original problem (6.12) is included into the family when $\psi = 0$ and $\varphi^* = \hat{\varphi}^*$.

Let us now compute the dual family (having the same parameters φ^* and ψ). In order to do so, we first compute F^* :

$$F^*(\varphi^*, \psi^*) = \sup_{\varphi, \psi} (\langle \varphi^*, \varphi \rangle + \langle \psi^*, \psi \rangle - F(\varphi, \psi)) = \sup_{\substack{\|\varphi - \psi_0\|_r \leq 1 \\ \|\dot{\varphi} - \psi_1\|_q \leq M \\ \varphi(0) = \varphi(1) = 0}} \int_0^1 (\dot{\varphi}^* \dot{\varphi} + \varphi^* \varphi + \psi_0^* \psi_0 + \psi_1^* \psi_1) dt.$$

Let us substitute $\eta_0 = \psi_0 - \varphi \in L_r(0; 1)$ and $\eta_1 = \psi_1 - \dot{\varphi} \in L_q(0; 1)$, or $\psi_0 = \varphi + \eta_0$ and $\psi_1 = \dot{\varphi} + \eta_1$:

$$\begin{aligned} F^*(\varphi^*, \psi^*) &= \sup_{\substack{\|\eta_0\|_r \leq 1 \\ \|\eta_1\|_q \leq M \\ \varphi(0) = \varphi(1) = 0}} \int_0^1 (\dot{\varphi}^* \dot{\varphi} + \varphi^* \varphi + \psi_0^*(\varphi + \eta_0) + \psi_1^*(\dot{\varphi} + \eta_1)) dt \\ &= \sup_{\|\eta_0\|_r \leq 1} \int_0^1 \psi_0^* \eta_0 dt + \sup_{\|\eta_1\|_q \leq M} \int_0^1 \psi_1^* \eta_1 dt + \sup_{\varphi(0) = \varphi(1) = 0} \int_0^1 (\dot{\varphi}^*(\dot{\varphi} + \psi_1^*) + \varphi(\varphi^* + \psi_0^*)) dt. \end{aligned}$$

The first supremum in the right-hand side is equal to $\|\psi_0^*\|_{r^*}$, and the second one is equal to $M\|\psi_1^*\|_{q^*}$. So let us compute the third one. Terminal constraints $\varphi(0) = \varphi(1) = 0$ are linear in φ . The inside integral is also linear in φ . Hence, there is an alternative: either the inside integral identically vanishes for all $\varphi \in W_q^1(0; 1)$ having $\varphi(0) = \varphi(1) = 0$ (and, in this case, the third supremum is equal to 0) or there exists a function φ having

$\varphi(0) = \varphi(1) = 0$ such that the inside integral does not vanish (and, in this case, the third supremum is equal to ∞). So, let us find all $\psi^* = (\psi_0^*, \psi_1^*)$ such that

$$\int_0^1 (\dot{\varphi}(\dot{\varphi}^* + \psi_1^*) + \varphi(\varphi^* + \psi_0^*)) dt = 0$$

for all φ having $\varphi(0) = \varphi(1) = 0$. According to the du Bois–Reymond lemma, the latter is equivalent to the following: $\frac{d}{dt}(\dot{\varphi}^* + \psi_1^*) \in L_1(0; 1)$ and $\frac{d}{dt}(\dot{\varphi}^* + \psi_1^*) = \varphi^* + \psi_0^*$, *i.e.* $\dot{\varphi}^* + \psi_1^* \in W_{r^*}^1(0; 1)$.

Thus, dual family (6.17) of convex problems has the following form:

$$\begin{aligned} \|\psi_0^*\|_{r^*} + M\|\psi_1^*\|_{q^*} - \int_0^1 (\psi_0\psi_0^* + \psi_1\psi_1^*) \rightarrow \min_{\psi^* \in L_{r^*}(0;1) \times L_{q^*}(0;1)} \\ \frac{d}{dt}(\dot{\varphi}^* + \psi_1^*) = \varphi^* + \psi_0^*. \end{aligned} \quad (6.19)$$

Recall that we denote by $D_\psi(\varphi^*)$ the value of the corresponding problem in the dual family.

Functional F is convex (as both norms $\|\cdot\|_{r^*}$ and $\|\cdot\|_{q^*}$ are convex, and terminal constraints $\varphi(0) = \varphi(1) = 0$ are linear) and closed (as any convex lower semicontinuous in the strong topology functional is also lower semicontinuous in the weak topology¹⁷). Therefore, the Fenchel–Moreau theorem implies $F = F^{**}$. Hence, the bidual family (*i.e.* dual to dual family (6.19)) coincides with the original one.

Let us now show that there is no duality gap when parameters φ^* and ψ have the desired values, namely, $\psi = 0$ and $\varphi^* = \dot{\varphi}^*$. So let us prove that $S_{\dot{\varphi}^*}(0) + D_0(\dot{\varphi}^*) = 0$. Equalities $D_\psi(\varphi^*) + S_{\varphi^*}^{**}(\psi) = 0$ and $S_{\varphi^*}(\psi) + D_{\psi^*}^{**}(\varphi^*) = 0$ can be proved in standard way (see [10]). Convexity of both functionals F and F^* implies convexity of S_{φ^*} (on ψ) and D_ψ (on φ^*). Therefore, the Fenchel–Moreau theorem implies $S_{\varphi^*}^{**} = \text{cl } S_{\varphi^*}$ and $D_{\psi^*}^{**} = \text{cl } D_\psi$. So it remains to show that functional D_0 is closed.

Let us make a natural change of variables in (6.19) and denote $v = \dot{\varphi}^* + \psi_1^*$. Then $v \in W_{r^*}^1(0; 1)$ and problem (6.19) for $\psi = 0$ takes the following form:

$$G(v) = \|\dot{v} - \varphi^*\|_{r^*} + M\|v - \dot{\varphi}^*\|_{q^*} \rightarrow \min_v.$$

Clearly, the infimum value of G does not exceed $\Delta = \|\varphi^*\|_{r^*} + M\|\dot{\varphi}^*\|_{q^*}$. We claim that function $G(v)$ is coercive. Indeed, if $\|\dot{v} - \varphi^*\|_{r^*} > \Delta$ or $\|v - \dot{\varphi}^*\|_{q^*} > \frac{1}{M}\Delta$, then $G(v) > \Delta$. Therefore, infimum of $G(v)$ over all functions v coincides with that over the set of functions v having $\|v - \dot{\varphi}^*\|_{q^*} \leq \Delta$ and $\|\dot{v} - \varphi^*\|_{r^*} \leq \frac{1}{M}\Delta$. This set is convex, bounded, and strongly closed. Hence, it is weak* compact in $W_{r^*}^1(0; 1)$ by Banach–Alaoglu theorem. Consequently, for any φ^* , there exists a global minimum v_{φ^*} . Moreover, both norms $\|v_{\varphi^*}\|_{q^*}$ and $\|\dot{v}_{\varphi^*}\|_{r^*}$ are bounded as $G(v_{\varphi^*}) \leq \Delta$:

$$\|v_{\varphi^*}\|_{q^*} \leq \frac{1}{M}\|\varphi^*\|_{r^*} + 2\|\dot{\varphi}^*\|_{q^*} \quad \text{and} \quad \|\dot{v}_{\varphi^*}\|_{r^*} \leq 2\|\varphi^*\|_{r^*} + M\|\dot{\varphi}^*\|_{q^*}. \quad (6.20)$$

Now we are able to show that D_0 is weak* lower semicontinuous using existence of a solution and bounds (6.20). Let $\varphi_0^* \in W_{q^*}^1(0; 1)$ be an arbitrary function. Consider a sequence φ_m^* that converges to φ_0^* in the weak* topology on $W_{q^*}^1(0; 1)$ as $m \rightarrow \infty$ and such that the sequence $D_0(\varphi_m^*)$ has a limit. So we need to prove that $D_0(\varphi_0^*) \leq \lim_{m \rightarrow \infty} D_0(\varphi_m^*)$.

Since $\psi = 0$, we have

$$D_0(\varphi^*) = \inf_{\psi^*} F^*(\varphi^*, \psi^*) = G(v_{\varphi^*})$$

¹⁷Indeed, any strongly closed convex set is weakly closed by the Hahn–Banach theorem.

as the infimum above is attained at $\psi^* = (\dot{v}_{\varphi^*} - \varphi^*, v - \dot{\varphi}^*(\varphi^*))$ for any choice of φ^* . Consider the following sequence of solutions

$$\psi_m^* = (\dot{v}_{\varphi_m^*} - \varphi_m^*, v_{\varphi_m^*} - \dot{\varphi}_m^*).$$

Since φ_m^* weakly* converges, the φ_m^* norms are bounded by uniform boundedness principle. Therefore, (6.20) implies that the $v_{\varphi_m^*}$ norms are bounded as $m \rightarrow \infty$. So using the Banach–Alaoglu theorem for the second time, we are able to find a subsequence of $v_{\varphi_m^*}$ that weakly* converges. Without loss of generality, we keep indices m for the subsequence. Hence, sequence ψ_m^* also weakly* converges to a function ψ_0^* as $m \rightarrow \infty$. It remain to note that functional F^* is lower weak* semicontinuous (as F^* is convex conjugate to F), and so

$$\lim_{m \rightarrow \infty} D_0(\varphi_m^*) = \lim_{m \rightarrow \infty} F^*(\varphi_m^*, \psi_m^*) \geq F^*(\varphi_0^*, \psi_0^*) \geq D_0(\varphi_0^*).$$

The latter proves that D_0 is weak* lower semicontinuous. Consequently, $S_{\varphi^*}(0) + D_0(\varphi^*) = 0$ for all φ^* . In particular, it is true for $\varphi^* = \hat{\varphi}^*$.

Now, as we have already shown absence of the duality gap, we can construct an estimate on the rate of u_1 approximation by functions from $W_{q^*}^1(0; 1)$. In order to do so, let us simplify the dual family under the required choice of parameters $\psi = 0$ and $\varphi^* = \hat{\varphi}^*$. According to (6.18), ODE in (6.19) becomes $\frac{d}{dt}(u_1 + \psi_1^*) = \psi_0^*$. Obviously, $u_1 + \psi_1^* \in W_{r^*}^1(0; 1)$. So, if $\psi = 0$ and $\varphi^* = \hat{\varphi}^*$, then substitution $w = u_1 + \psi_1^*$ gives the following equivalent problem:

$$\|\dot{w}\|_{r^*} + M\|w - u_1\|_{q^*} \rightarrow \min_{w \in W_{r^*}^1}.$$

Sum of that problem value and $S_r^q(u_1, M)$ is equal to 0 as there is no duality gap. So (6.14) is proved. \square

6.5. Interpolation

Duality proved in Proposition 6.7 together with the interpolational estimate proved in Proposition 6.5 and Corollary 6.6 imply the following estimate on the rate of control u_1 approximation by smooth functions. Namely, we have proved that, for any $M \geq 0$, there exists a function $w \in W_{r^*}^1(0; 1)$ such that

$$\|\dot{w}\|_{r^*} + M\|w - u_1\|_{q^*} \leq 2c''l(x)^\kappa M^{1-\theta} \quad (6.21)$$

where $q^* = q/(q-1) \geq 2$, $r^* = 2$ in the case (\mathfrak{W}) , and $r^* = \infty$ in the case (\mathfrak{G}) , and θ and κ are defined by ζ in (6.10). According to Proposition 6.5, $\zeta = \frac{2}{s}$ in the case (\mathfrak{W}) and $\zeta = \frac{1}{s}$ in the case (\mathfrak{G}) .

The left hand side of (6.21) is very close the K -functional given in (6.13). We want to obtain an estimation on the K -functional. Since we have already proved (6.21), we only need to estimate norm $\|w\|_{W_{r^*}^1(0; 1)}$. There are a lot of different pairwise equivalent norms on space $W_{r^*}^1(0; 1)$. For our purposes, it is convenient to use $\|w\|_{W_{r^*}^1(0; 1)} = \|\dot{w}\|_{r^*} + \|w\|_{q^*}$. The first term is trivially estimated by (6.21). In order to estimate $\|w\|_{q^*}$, let us use inequality $|u_1| \leq l(x)$: in the case (\mathfrak{G}) , we have $0 < \kappa \leq 1$ and so $l(x) \leq l(x)^\kappa L^{1-\kappa}$; and in the case (\mathfrak{W}) , we have $l(x) = l(x)^\kappa L^{1-\kappa}$. Therefore, if $M > 1$, then

$$\|w\|_{q^*} \leq (2c'' + L^{1-\kappa})l(x)^\kappa M^{-\theta} \leq (2c'' + L^{1-\kappa})l(x)^\kappa M^{1-\theta}$$

and

$$\|\dot{w}\|_{W_{r^*}^1} + M\|w - u_1\|_{q^*} \leq (4c'' + L^{1-\kappa})l(x)^\kappa M^{1-\theta}.$$

If the opposite $0 \leq M \leq 1$ holds, then we can always take $w = 0$, which guarantees

$$\|\dot{w}\|_{W_{r^*}^1} + M\|w - u_1\|_{q^*} = M\|u_1\|_{q^*} \leq l(x)M \leq L^{1-\kappa}l(x)^\kappa M^{1-\theta}.$$

Hence, K -functional of interpolation satisfies the following estimate:

$$K_{q^*}^{r^*}(M, u_1) = \inf_w (\|w\|_{W_{r^*}^1} + M\|w - u_1\|_{q^*}) \leq c'''l(x)^\kappa M^{1-\theta}$$

where $c''' \geq 4c'' + L^{1-\kappa}$. Since κ is between 1 and $(1 + \zeta)/(2 - \zeta)$, we can put $c''' = 4c'' + \max\{1, L^{(1-2\zeta)/(2-\zeta)}\}$.

Thus, the control u_1 belongs to the following interpolation space (see [16], Sect. 1.3.1):

$$u_1 \in X = (W_{r^*}^1(0; 1), L_{q^*}(0; 1))_{1-\theta, \infty} = (L_{q^*}(0; 1), W_{r^*}^1(0; 1))_{\theta, \infty}. \quad (6.22)$$

So, it remains to reduce the interpolation space in the right-hand side to a standard one.

Interpolation (6.22) belongs to a so called nondiagonal case as $\frac{1-\theta}{q^*} + \frac{\theta}{r^*} \neq \frac{1}{\infty}$. Unfortunately, no exact formula for the nondiagonal interpolation is known. Nonetheless, we can exchange the nondiagonal interpolation space X by a little bit bigger space, which we are actually able to compute explicitly. So let us consider the cases (\mathfrak{G}) and (\mathfrak{W}) separately.

In the case (\mathfrak{G}) , we have $r^* = \infty$, $q^* \geq 2$, and θ is given in (6.10). Hence, X defined in (6.22) becomes

$$X = (L_{q^*}(0; 1), W_\infty^1(0; 1))_{\theta, \infty}.$$

Let us use embedding theorems of Sobolev type (see [16], Sect. 4.6.2). First, $L_{q^*}(0; 1) = H_{q^*}^0(0; 1) \subset B_{q^*, q^*}^0(0; 1)$ as $q^* \geq 2$. Second, we fix $q^* < a < \infty$ and use embedding $W_\infty^1(0; 1) \subset B_{a, a}^1(0; 1)$. Obviously, $B_{a, a}^1(0; 1) \subset B_{q^*, q^*}^0(0; 1)$. Hence, for any $1 \leq p \leq \infty$ and $0 < \beta < \theta$, we obtain (see [16], Sect. 1.3.3)

$$X \subset (B_{q^*, q^*}^0(0; 1), B_{a, a}^1(0; 1))_{\theta, \infty} \subset (B_{q^*, q^*}^0(0; 1), B_{a, a}^1(0; 1))_{\beta, p}.$$

Let us make a choice of p that leads to the diagonal type interpolation, namely

$$\frac{1}{p} = \frac{1-\beta}{q^*} + \frac{\beta}{a}.$$

Then, the corresponding theorem for the diagonal type interpolation (see [16], Sect. 4.3.1) gives

$$X \subset (B_{q^*, q^*}^0(0; 1), B_{a, a}^1(0; 1))_{\beta, p} = B_{p, p}^\beta(0; 1) \subset B_{p, \infty}^\beta(0; 1).$$

Now, we find values that parameter β can take assuming that a fixed value of p is given. So the following restrictions must be fulfilled: $a > q^* \geq 2$ and

$$0 < \beta = \frac{\frac{1}{q^*} - \frac{1}{p}}{\frac{1}{q^*} - \frac{1}{a}} < \theta = \frac{(q-1)\zeta}{q-\zeta} = \frac{\zeta}{q^*(1-\zeta) + \zeta} = \frac{1}{q^*(s-1) + 1}.$$

The latter equality holds as $\zeta = \frac{1}{s}$ in the case (\mathfrak{G}) . Since a can be taken arbitrary large, the previous pair of inequalities is equivalent to the following:

$$0 < 1 - \frac{q^*}{p} < \frac{1}{q^*(s-1) + 1} \quad \Leftrightarrow \quad p - \frac{1}{s-1} < q^* < p.$$

Consequently, if $p - \frac{1}{s-1} \geq 2$, then q^* can be taken arbitrary close to $p - \frac{1}{s-1}$. So, proceeding to the limit as $a \rightarrow \infty$ and $q^* \rightarrow p - \frac{1}{s-1}$, we obtain $\theta \rightarrow \frac{1}{p(s-1)}$. Note that if $q^* \rightarrow p - \frac{1}{s-1}$, then $\kappa \rightarrow 1 - \frac{s-2}{p(s-1)}$. Since $B_{p,\infty}^\beta \subset B_{p,\infty}^{\beta-\varepsilon}$ for an arbitrary $\varepsilon > 0$, number κ can be taken arbitrary close to $1 - \frac{s-2}{p(s-1)}$ independently of the choice of β . Hence, for any $p \geq 2 + \frac{1}{s-1}$, $0 < \beta < \frac{1}{p(s-1)}$, and $0 < \kappa < 1 - \frac{s-2}{p(s-1)}$, there exists a constant $C > 0$ such that

$$\|u_1\|_{B_{p,\infty}^\beta(0;1)} \leq Cl(x)^\kappa.$$

Now, let us enlarge the interval of possible κ values using embeddings of Sobolev type. Let us find $\tilde{p} > p$ such that we are still able to select a number $\tilde{\beta} < \frac{1}{\tilde{p}(s-1)}$ such that the embedding $B_{\tilde{p},\infty}^{\tilde{\beta}}(0;1) \subset B_{p,\infty}^\beta(0;1)$ holds. Obviously, this embedding holds if the following conditions are satisfied (see [16], Sect. 4.6.2)

$$\beta < \tilde{\beta} \quad \text{and} \quad \beta - \frac{1}{p} < \tilde{\beta} - \frac{1}{\tilde{p}}.$$

Since $\tilde{p} > p$, the second inequality follows from the first one. Thus, we need to satisfy only $\beta < \tilde{\beta} < \frac{1}{\tilde{p}(s-1)}$. Therefore, existence of the required $\tilde{\beta}$ is guaranteed if $\tilde{p} > \frac{1}{\beta(s-1)}$. Note that $\frac{1}{\beta(s-1)} > p$ as $\beta < \frac{1}{p(s-1)}$.

Thus, for some constant $C' > 0$, we have

$$\|u_1\|_{B_{p,\infty}^\beta(0;1)} \leq C' \|u_1\|_{B_{\tilde{p},\infty}^{\tilde{\beta}}(0;1)}.$$

The last norm (as we have already shown) can be estimated by $\tilde{C}l(x)^{\tilde{\kappa}}$ where $\tilde{\kappa} < 1 - \frac{s-2}{\tilde{p}(s-1)}$. Since \tilde{p} can be taken arbitrary close to $\frac{1}{\beta(s-1)}$, $\tilde{\kappa}$ can take values that are arbitrary close to $1 - \beta(s-2)$. In other words, for any $\tilde{\kappa} < 1 - \beta(s-2)$, there exists a constant $C'' = C'\tilde{C}$ such that

$$\|u_1\|_{B_{p,\infty}^\beta(0;1)} \leq C'' l(x)^{\tilde{\kappa}},$$

and item (Ⓔ) in Theorem 4.1 is proved.

Now we proceed to the case (Ⓐ). In that case, we have $r^* = 2$. So space X defined in (6.22) takes the form

$$X = (L_{q^*}(0;1), W_2^1(0;1))_{\theta,\infty}.$$

Since $W_2^1(0;1) = B_{2,2}^1(0;1)$ and $L_{q^*}(0;1) \subset B_{q^*,q^*}^0(0;1)$ (see [16], Sect. 4.6.2), we obtain

$$X \subset (B_{q^*,q^*}^0(0;1), B_{2,2}^1(0;1))_{\theta,\infty}.$$

Again, $B_{2,2}^1(0;1) \subset B_{q^*,q^*}^0(0;1)$ as $1 - \frac{1}{2} > 0 - \frac{1}{q^*}$ (see [16], Sect. 4.6.2). Therefore, for any $0 < \alpha < \theta$ and $1 \leq p \leq \infty$, we have (see [16], Sect. 1.3.3)

$$X \subset (B_{q^*,q^*}^0(0;1), B_{2,2}^1(0;1))_{\alpha,p}.$$

If we take p that satisfies $\frac{1-\alpha}{q^*} + \frac{\alpha}{2} = \frac{1}{p}$, then we obtain the following diagonal type interpolation:

$$(B_{q^*,q^*}^0(0;1), B_{2,2}^1(0;1))_{\alpha,p} = B_{p,p}^\alpha(0;1) \subset B_{p,\infty}^\alpha(0;1).$$

We emphasize that $p = ((q^*)^{-1} + \alpha(\frac{1}{2} - (q^*)^{-1}))^{-1} \rightarrow ((q^*)^{-1} + \theta(\frac{1}{2} - (q^*)^{-1}))^{-1} + 0$ as $\alpha \rightarrow \theta - 0$ since $q^* \geq 2$. Therefore $p \geq 2$. Further, as before, we fix $p \geq 2$ and check values that parameter α can take:

$$0 < \alpha = \frac{\frac{1}{p} - \frac{1}{q^*}}{\frac{1}{2} - \frac{1}{q^*}} < \theta = \frac{\zeta}{q^*(1 - \zeta) + \zeta} = \frac{2}{q^*(s - 2) + 2}$$

The latter equality holds as $\zeta = \frac{2}{s}$ in the case (\mathfrak{W}) . These inequalities are equivalent to the following ones:

$$p < q^* < \frac{p(s - 1) - 2}{s - 2}$$

This pair of inequalities is compatible if $p > 2$. Thus, for any $p > 2$, number q^* can be taken arbitrary close to the right-hand side, and θ becomes arbitrary close to $\frac{2}{p(s-1)}$. Therefore,

$$u_1 \in B_{p,\infty}^\alpha(0; 1) \tag{6.23}$$

for any $p > 2$ and $\alpha < \frac{2}{p(s-1)}$. So we have proved item (\mathfrak{W}) in Theorem 4.1 for $p > 2$.

It remains to consider the case $p = 2$. If $p = 2$, then for $q^* = 2$, $r^* = 2$, and $\theta = \frac{1}{s-1}$ (see (6.10) for $\zeta = 2/s$), definition (6.22) gives

$$X = (L_2(0; 1), W_2^1(0; 1))_{\frac{1}{s-1}, \infty}.$$

Since $L_2(0; 1) = H_2^0(0; 1)$ and $W_2^1(0; 1) = H_2^1(0; 2)$ (see [16], Sects. 2.3.1 and 4.2.1), this interpolation belongs to the diagonal case, and according to [16], Section 4.3.1, we have

$$X = (H_2^0(0; 1), H_2^1(0; 1))_{\frac{1}{s-1}, \infty} = B_{2,\infty}^{\frac{1}{s-1}}(0; 1).$$

It remains to note that $B_{2,\infty}^{\frac{1}{s-1}}(0; 1) \subset B_{2,\infty}^\alpha(0; 1)$ for any $\alpha < \frac{1}{s-1}$ (see [16], Sect. 4.6.2).

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