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# SPDES WITH SPACE INTERACTIONS AND APPLICATION TO POPULATION MODELLING

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**Abstract.** We consider optimal control of a new type of non-local stochastic partial differential equations (SPDEs). The SPDEs have *space interactions*, in the sense that the dynamics of the system at time $t$ and position in space $x$ also depend on the space-mean of values at neighbouring points. This is a model with many applications, *e.g.* to population growth studies and epidemiology. We prove the existence and uniqueness of strong, smooth solutions of a class of SPDEs with space interactions, and we show that, under some conditions, the solutions are positive for all times if the initial values are. Sufficient and necessary maximum principles for the optimal control of such systems are derived. Finally, we apply the results to study an optimal vaccine strategy problem for an epidemic by modelling the population density as a space-mean stochastic reaction-diffusion equation.

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## 1. Introduction

The purpose of this paper is to introduce a new type generalised stochastic heat equation with *space interactions* as a model for population growth. By space interactions we mean that the dynamics of the population density $Y(t,x)$ at a time $t$ and a point $x$ depends not only on its value and derivatives at $x$, but also on its values in a neighbourhood of $x$. For example, define $G$ to be a space-averaging operator of the form

$$G(x,\varphi) = \frac{1}{V(K_r)} \int_{K_r} \varphi(x+y)\mathrm{d}y; \quad \varphi \in L^2(\mathbb{R}^n), \tag{1.1}$$

where $V(\cdot)$ denotes Lebesgue volume and

$$K_r = \{y \in \mathbb{R}^n; |y| < r\}$$

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is the ball of radius $r > 0$ in $\mathbb{R}^n$ centred at 0. Then

$$\overline{Y}_G(t,x) := G(x, Y(t,\cdot))$$

is the average value of $Y(t, x+\cdot)$ in the ball $K_r$.

More generally, if we are given a nonnegative measure (weight) $\rho(\mathrm{d}y)$ of total mass 1, then the $\rho$-weighted average of $Y$ at $x$ is defined by

$$\overline{Y}_\rho(t,x) := \int_D Y(t, x+y)\rho(\mathrm{d}y).$$

We believe that by allowing interactions between populations at different locations, we get a better model for population growth, including the modelling of epidemics. For example, we know that COVID-19 is spreading by close contact in space.

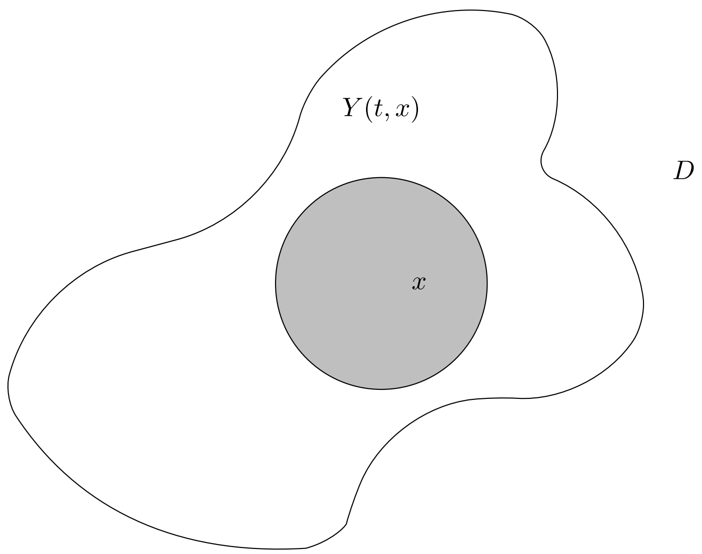
We illustrate the above by the following population growth model:

**Example 1.1.** With $G$ as in (1.1), suppose the density $Y(t,x)$ of a population at the time $t$ and the point $x$ satisfies the following space-interaction version of a reaction-diffusion equation:

$$\begin{cases} \mathrm{d}Y(t,x) &= \Big(\frac{1}{2}\Delta Y(t,x) + \alpha \overline{Y}(t,x) - u(t,x)Y(t,x)\Big)\mathrm{d}t + \beta Y(t,x)\mathrm{d}B(t), \\ Y(0,x) &= \xi(x); \quad x \in D, \\ Y(t,x) &= \eta(t,x); \quad (t,x) \in (0,T)\times\partial D, \end{cases} \tag{1.2}$$

where $\alpha$ is a constant, $\xi, \eta$ are given bounded functions, $\overline{Y}(t,x) = G(x, Y(t,\cdot))$ and $B(t) = B(t,\omega); (t,\omega) \in [0,T]\times\Omega$ is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

Here $u(t,x)$ is our control process, *e.g.* representing our harvesting or vaccine effort.




Then (1.2) is a natural model for population growth in an environment with space interactions.

If $u(t,x)$ represents a vaccination effort rate at $(t,x)$, we define the total expected utility $J_0(u)$ of the harvesting by an expression of the form

$$J_0(u) = \mathbb{E}\Big[\int_D \int_0^T U_1(u(t,x))\mathrm{d}t\mathrm{d}x + \int_D U_2(Y(T,x))\mathrm{d}x\Big],$$

where $U_1$ and $U_2$ are given cost functions. The problem to find the optimal vaccination rate $u^*$ is the following:

**Problem 1.2.** Find $u^* \in \mathcal{U}$ such that

$$J_0(u^*) = \inf_{u\in\mathcal{U}} J_0(u),$$

where $\mathcal{U}$ is a given family of admissible controls.

We will return to the example above after first discussing more general stochastic optimal control models with a system whose state $Y(t,x)$ at time $t$ and at the point $x$ satisfies an SPDE with a non-local space-interaction dynamics of the following type:

$$\begin{cases} \mathrm{d}Y(t,x) & = A_x Y(t,x)\mathrm{d}t + b(t,x,Y(t,x),Y(t,\cdot),u(t,x))\mathrm{d}t \\ & \quad +\sigma(t,x,Y(t,x),Y(t,\cdot),u(t,x))\mathrm{d}B(t), \\ Y(0,x) & = \xi(x); \quad x \in D, \\ Y(t,x) & = \eta(t,x); \quad (t,x) \in (0,T)\times\partial D. \end{cases} \tag{1.3}$$

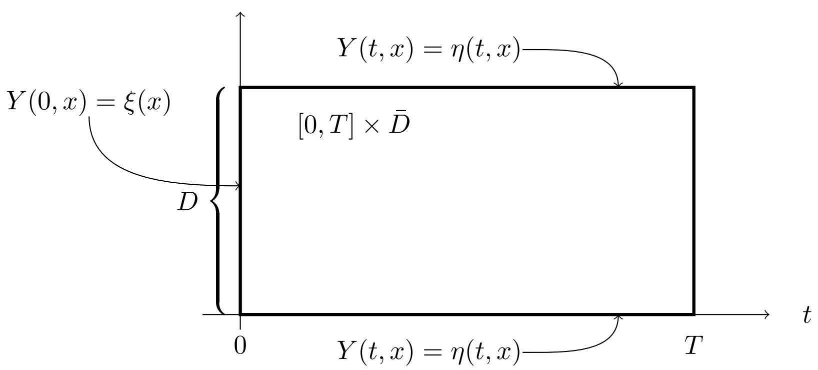
Here $\mathrm{d}Y(t,x)$ denotes the differential with respect to $t$ while $A_x$ is the second order partial differential operator acting on $x$ of the form

$$A_x\phi(x) = \sum_{i,j=1}^n \alpha_{ij}(x)\frac{\partial^2\phi}{\partial x_i \partial x_j} + \sum_{i=1}^n \beta_i(x)\frac{\partial\phi}{\partial x_i}; \quad \phi \in \mathcal{C}_0^2(\mathbb{R}^n). \tag{1.4}$$

Precise conditions on the coefficients will be given in the beginning of Section 4.2.

The domain $D$ is an open set in $\mathbb{R}^n$ with a Lipschitz boundary $\partial D$ and closure $\overline{D}$. We extend $Y(t,x)$ to be a function on all of $[0,T]\times\mathbb{R}^n$ by setting

$$Y(t,x) = 0 \text{ for } x \in \mathbb{R}^n \setminus \overline{D}.$$



**Example 1.3.** In particular, the partial differential operator $A_x$ could be the Laplacian $\Delta$. or more generally an operator of the $div - grad$-form

$$A_x(\varphi) = div(\alpha(x)\nabla\varphi)(x); \quad \varphi \in \mathcal{C}^2(D),$$

where $div$ denotes the divergence operator, $\nabla$ denotes the gradient and

$$\alpha(x) = [\alpha_{i,j}(x)]_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$$

is a nonnegative definite matrix for each $x$. Equations of this type are of interest because they represent important models in many situations, *e.g.* in physics (*e.g.* fluid flow in random media, see *e.g.* Holden *et al* [17]), in epidemiology and in biology, *e.g.* in population growth where $Y(t,x)$ represents the population density at $t, x$.

For more details about the theory of SPDE, we refer for example to Gawarecki and Mandrekar [14], Da Prato and Zabczyk [29], Pardoux [27, 28], Hairer [16], Prévôt and Roeckner [30], Roeckner and Zhang [31] and to the recent book by Lü and Zhang [20].

**Remark 1.4.** With the control $u$ given, sufficient conditions for the existence and uniqueness of a weak solution of the corresponding SPDEs with space interactions are known from general results on SPDEs. See *e.g.* Theorem 3.3 in Gawarecki and Mandrekar [14] or Proposition 12.1 in Lü and Zhang [20]. However, very little seems to be known so far about the properties of such solutions. In Sections 2 and 3 we prove that there is a unique smooth, strong solution of a class of space-interaction reaction-diffusion equations. Moreover, we show that the solution can be obtained as a limit of an iterative procedure, and it is positive if the initial values are. See *e.g.* Theorem 2.1 and Theorem 3.3. This is an important confirmation that such equations are suitable models for population growth in general.

There are two well-known approaches to solve stochastic control problems: The Bellman dynamic programming method and the Pontryagin maximum principle. Because of the space-mean dependence in our model, the system is not Markovian, and it is not clear how to apply a dynamic programming approach. In stead we will use a stochastic version of the Pontryagin maximum principle, which involves a coupled system of a forward/backward SPDEs.

Stochastic control of SPDEs has been studied widely in the literature. For example, we refer to Bensoussan [3–6], Hu and Peng [19], Zhou [34], Øksendal [23], Fuhrman *et al.* [12], Øksendal *et al.* [20, 24–26] and the references therein. In the fundamental papers [3, 19] it is assumed that the diffusion coefficient of the system does not depend on the control, and in [3, 19], there is no space-mean dependence so they do not cover our situation.

In [20], a general maximum principle of optimal control of SPDEs is proved, with an adjoint equation (BSPDE) formulated in a weak setting. The general setting in [20] covers the situation we consider, except that in [20] only the case with the underlying space $D$ being all of $\mathbb{R}^n$ is considered. Our approach deals with general $D$ and is directly focused on the effect of the space interaction, with application to population modelling in mind. Moreover, for our type of equation we prove the smoothness and positivity of the solution. Specifically, in our case of a control problem for an SPDE with space-interaction in a subset $D$ of $\mathbb{R}^n$ we derive an explicit adjoint equation, which is a BSPDE, also with space-interaction dependence. We derive both sufficient and necessary maximum principles for this type of stochastic control problem. For related singular stochastic control with space-interaction, we refer to Agram *et al.* [1].

Here is a summary of the content of this paper:

- In Section 2 we prove the existence and uniqueness of a strong, smooth solution of a class of space-interaction SPDEs, including the application studied in Section 5, and we give an iterative procedure for finding the solution (Thm. 2.1). This result is new.

- In Section 3 we use white noise theory to prove a positivity theorem for a class of SPDEs with space interactions (Thm. 3.1), and we prove that the solution is positive if the initial values are (Thm. 3.2).These results are also new and of independent interest.
- Subsequently, in Section 4 we study the general optimization problem for such a system. We derive both sufficient and necessary maximum principles for the optimal control. See Theorem 4.6 and Theorem 4.7.
- Finally, as an illustration of our results, in Section 5 we study an example about optimal vaccination strategy for an epidemics modelled as an SPDE with space-interactions.

## 2. Solutions of SPDEs with space interactions, and positivity

In this section we prove the existence and uniqueness of a strong, smooth solution of SPDEs with space interactions. We are not aiming at proving this for the most general SPDE of this type, but we settle for a class of SPDEs which includes the application in Section 5. Thus, for simplicity we consider only the case when $A_x = L$ given by

$$L = \frac{1}{2}\Delta := \frac{1}{2}\sum_{k=1}^{k=n} \frac{\partial^2}{\partial x_k^2}, \text{ and } D = \mathbb{R}^n,$$

but it is clear that our method can also be applied to more general situations.

Fix $t > 0$, and let $k \in \mathbb{N}_0 = \{0, 1, 2, \dots, \dots\}$ , $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbb{N}_0^m; m = 1, 2, \dots.$
For functions $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (the family of functions in $\mathcal{C}(\mathbb{R}^n)$ with compact support), we define the Sobolev norm

$$|f|_k = \sum_{|\alpha| \le k} \big(\int_{\mathbb{R}^n} |\partial^\alpha f(x)|^2 \mathrm{d}x\big)^{\frac{1}{2}}; \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}_0^n,$$

and we define the Sobolev space $\mathbb{H}^k$ to be the closure of $\mathcal{C}_0^\infty(\mathbb{R}^n)$ in this norm.

Note that $\mathbb{H}^k$ is a Hilbert space for all $k$.

Also, note that if $f \in \mathbb{H}^{k+2}$ then $Lf \in \mathbb{H}^k$, because

$$|Lf|_k = \sum_{|\alpha| \le k} \big(\int_{\mathbb{R}^n} |\partial^\alpha Lf(x)|^2 \mathrm{d}x\big)^{\frac{1}{2}} \le \frac{1}{2} \sum_{|\alpha| \le k+2} \big(\int_{\mathbb{R}^n} |\partial^\alpha f(x)|^2 \mathrm{d}x\big)^{\frac{1}{2}} = \frac{1}{2}|f|_{k+2}. \tag{2.1}$$

Let $\mathcal{Y}_k^{(t)}$ denote the family of adapted random fields $Y(s,x) = Y(s,x,\omega)$ , such that $||Y||_{t,k} < \infty$ where

$$||Y||_{t,k} = \mathbb{E}\left[\sup_{s \le t} \left\{ |Y(s,.)|_k^2 \right\}\right]^{\frac{1}{2}}, \tag{2.2}$$

and let $\mathcal{Y}^{(t)}$ be the intersection of all the spaces $\mathcal{Y}_k^{(t)}; k \in \mathbb{N}_0$, with the norm

$$||Y||_t^2 := \sum_{k=1}^{\infty} 2^{-k} ||Y||_{t,k}^2. \tag{2.3}$$

In the following we let

$$\varphi \mapsto \overline{\varphi}(x)$$

be any averaging operator such that there exists a constant $C_1$ such that

$$|\overline{\varphi}|_k \leq C_1 |\varphi|_k \text{ for all } \varphi, k. \tag{2.4}$$

This holds, for example, if $\overline{\varphi}(x) = \int \varphi(x+y)\rho(\mathrm{d}y)$ for some measure $\rho$ of total mass 1.

We can now prove the following:

**Theorem 2.1.** *Let $\xi \in \mathcal{Y}^{(T)}$ be deterministic and let $h : [0,T] \mapsto \mathbb{R}$ be bounded and deterministic.*

*(i) Then there exists a unique solution $Y(t,x) \in \mathcal{Y}^{(T)}$ of the following SPDE with space interactions:*

$$Y(t,x) = \xi(x) + \int_0^t LY(s,x)\mathrm{d}s \qquad + \int_0^t \overline{Y}(s,x)\mathrm{d}s + \int_0^t h(s)Y(s,x)\mathrm{d}B(s); \quad t \in [0,T]. \tag{2.5}$$

*(ii) Moreover, the solution $Y(t,x)$ can be found by iteration, as follows:*
*Choose $Y_0 \in \mathcal{Y}^{(T)}$ arbitrary deterministic and define inductively $Y_m$ to be the solution of*

$$\begin{aligned} Y_m(t,x) = \xi(x) + \int_0^t LY_m(s,x)\mathrm{d}s + \int_0^t \overline{Y}_{m-1}(s,x)\mathrm{d}s \\ + \int_0^t h(s)Y_m(s,x)\mathrm{d}B(s); \quad t \in [0,T]; m = 1,2,.... \end{aligned} \tag{2.6}$$

*Then*

$$Y_m \to Y \text{ in } \mathcal{Y}^{(T)} \text{ when } m \to \infty.$$

*Proof.* **(i)** In the first part of the proof, we are concerned on proving the existence and the uniqueness of the solution of $Y(t,x) \in \mathcal{Y}^{(T)}$.
Define the operator $F : \mathcal{Y}^{(T)} \mapsto \mathcal{Y}^{(T)}$ by $F(Z) = Y^Z$, where $Y^Z$ is the solution of the equation

$$Y^Z(t,x) = \xi(x) + \int_0^t LY^Z(s,x)\,\mathrm{d}s + \int_0^t \overline{Z}(s,x)\,\mathrm{d}s + \int_0^t Y^Z(s,\cdot)\,h(s)\,\mathrm{d}B(s)$$

Note that here $Z$ (and hence $\overline{Z}$) is given. Therefore the existence and uniqueness of the solution $Y^Z$ follows by the general existence and uniqueness theorems for solutions of SPDEs. *e.g.* as given in Theorem 3.3 in [14].

For $i = 1,2$ choose $Z_i \in \mathcal{Y}^{(T)}$ and define $Y_i = Y^{Z_i} =: F(Z_i)$ to be the solution of the SPDE

$$Y_i(t,x) = \xi(x) + \int_0^t LY_i(s,x)\,\mathrm{d}s + \int_0^t \overline{Z_i}(s,x)\,\mathrm{d}s + \int_0^t Y_i(s,\cdot)\,h(s)\,\mathrm{d}B(s).$$

Note that here $Z_i$ (and hence $\overline{Z}_i$), is given for each $i$. Therefore the existence and uniqueness of the solution $Y_i$ follows by the general existence and uniqueness theorems for solutions of SPDEs. *e.g.* as given in Theorem 3.3 in [14]. Define

$$\begin{aligned} \widetilde{Y} &= Y_1 - Y_2, \\ \widetilde{Z}(t,x) &= Z_1(t,x) - Z_2(t,x) \\ \widetilde{\overline{Z}} &= \overline{Z}_1 - \overline{Z}_2. \end{aligned}$$

Then

$$\widetilde{Y}(t,x)=\int_0^t L\widetilde{Y}(s,x)\,\mathrm{d}s+\int_0^t \widetilde{\overline{Z}}(s,x)\,\mathrm{d}s+\int_0^t \widetilde{Y}(s,x)\,h(s)\,\mathrm{d}B(s)\,.$$

Hence

$$\begin{aligned}\left|\widetilde{Y}(s,.)\right|_k &\le \int_0^s \left|L\widetilde{Y}(r,.)\right|_k \mathrm{d}r+\int_0^s \left|\widetilde{\overline{Z}}(r,.)\right|_k \mathrm{d}r\\ &\quad+\left|\int_0^s \widetilde{Y}(r,\cdot)h(r)\mathrm{d}B(r)\right|_k.\end{aligned}\tag{2.7}$$

By (2.1) we have

$$\left|L\widetilde{Y}(r,.)\right|_k\le\left|\widetilde{Y}(r,.)\right|_{k+2},\tag{2.8}$$

and from (2.4) we get

$$\left|\overline{Z}\right|_k\le C_1\left|Z\right|_k \text{ for all } k.\tag{2.9}$$

Then by (2.7), (2.8) and (2.9), we get

$$\begin{aligned}&\mathbb{E}\left[\sup_{s\le t}\left|\widetilde{Y}(s,.)\right|_k^2\right]\qquad\qquad(2.10)\\ &\le 3\mathbb{E}\left[\sup_{s\le t}\left(\int_0^s\left|\widetilde{Y}(r,.)\right|_{k+2}\mathrm{d}r\right)^2\right]+3C_1\mathbb{E}\left[\sup_{s\le t}\left(\int_0^s\left|\widetilde{Z}(r,.)\right|\mathrm{d}r\right)^2\right]\\ &+3\mathbb{E}\left[\sup_{s\le t}\left|\int_0^s\widetilde{Y}(r,\cdot)\,h(r)\,\mathrm{d}B(r)\right|_k^2\right].\end{aligned}$$

By the Burkholder-Davis-Gundy inequality for Hilbert spaces (see *e.g.* [22]), there exists a constant $C_2$ such that

$$\begin{aligned}\mathbb{E}\left[\sup_{s\le t}\left|\int_0^s\widetilde{Y}(r,\cdot)\,h(r)\,\mathrm{d}B(r)\right|_k^2\right]&\le C_2\mathbb{E}\left[\int_0^t\left|\widetilde{Y}(r,.)\right|_k^2 h^2(r)\,\mathrm{d}r\right]\\ &\le C_2h_0^2t\mathbb{E}\left[\sup_{s\le t}\left|\widetilde{Y}(s,.)\right|_k^2\right];\ \text{where } h_0^2=\sup_{s\in[0,T]}\left|h(s)\right|^2.\end{aligned}$$

Combining the above we get, if $0\le t\le 1$,

$$\mathbb{E}\left[\sup_{s\le t}\left|\widetilde{Y}(s,.)\right|_k^2\right]\tag{2.11}$$

$$\le 3t^2\mathbb{E}\left[\sup_{r\le t}\left|\widetilde{Y}(r,.)\right|_{k+2}^2\right]+3C_1t^2\mathbb{E}\left[\sup_{r\le t}\left|\widetilde{Z}(r,.)\right|_k^2\right]\tag{2.12}$$

$$+3C_2h_0^2t\mathbb{E}\left[\sup_{r\le t}\left|\widetilde{Y}(r,.)\right|_k^2\right].\tag{2.13}$$

In other words,

$$||\widetilde{Y}||_{t,k}^2 \le 3t^2||\widetilde{Y}||_{t,k+2}^2 + 3C_1t^2||\widetilde{Z}||_{t,k}^2 + 3C_2h_0^2t||\widetilde{Y}||_{t,k}^2. \tag{2.14}$$

Note that

$$\sum_{k=1}^{\infty} 2^{-k}||\widetilde{Y}||_{t,k+2}^2 = \sum_{j=3}^{\infty} 2^{-(j-2)}||\widetilde{Y}||_{t,j} \le 4\sum_{j=3}^{\infty} 2^{-j}||\widetilde{Y}||_{t,j} \le 4\sum_{k=1}^{\infty} 2^{-k}||\widetilde{Y}||_{t,k} = 4||\widetilde{Y}||_t.$$

Therefore, by multiplying the terms in (2.14) by $2^{-k}$ and summing over $k$, we get

$$||\widetilde{Y}||_t^2 = \sum_{k=1}^{\infty} 2^{-k}||\widetilde{Y}||_{t,k}^2 \le 12t^2||\widetilde{Y}||_t^2 + 3C_1t^2||\widetilde{Z}||_t^2 + 3C_2h_0^2t||\widetilde{Y}||_t^2,$$

or

$$(1 - 12t^2 - 3C_2h_0^2t)||\widetilde{Y}||_t^2 \le 3C_1t^2||\widetilde{Z}||_t^2.$$

Hence, if $t_0 > 0$ is chosen so small that

$$\frac{3C_1t_0^2}{1 - 12t_0^2 - 3C_2h_0^2t_0} < 1,$$

we obtain that the map

$$Z \to Y^Z = F(Z)$$

is a contraction on $\mathcal{Y}^{(t_0)}$. Therefore, by the Banach fixed point theorem there exists a fixed point $\widehat{Y}$ of this map. Then $\widehat{Y}$ solves the SPDE

$$\begin{cases} \mathrm{d}\widehat{Y}(t,x) = L\widehat{Y}(t,x)\,\mathrm{d}t + \overline{\overline{\widehat{Y}}}(t,x)\,\mathrm{d}t + \widehat{Y}(t,x)\,h(t)\,\mathrm{d}B(t)\,; \quad t \in [0,t_0]\,, \\ \widehat{Y}(0,x) = \xi(x); \quad x \in \mathbb{R}^n. \end{cases}$$

Uniqueness follows by a similar argument.

Since the constants do not depend on $t_0$, we can repeat the argument starting from $t_0$ and hence by induction obtain a solution $Y(t,x) \in \mathcal{Y}^{(2t_0)}$. Repeating this argument we thus obtain a solution $Y \in \mathcal{Y}^{(T)}$.
This proves part (i).

**(ii)** The second part of the theorem follows by the Banach fixed point theorem on the Banach space $\mathcal{Y}^{(T)}$ and therefore it is omitted. □

## 3. The non-homogeneous stochastic heat equation and positivity

In this section we will prove positivity of the solutions $Y(t,x)$ of SPDEs of the form

$$\begin{cases} \mathrm{d}Y(t,x) &= LY\mathrm{d}t + K(t,x)\,\mathrm{d}t + h(t)\,Y(t)\,\mathrm{d}B(t)\,, \\ Y(0,x) &= \xi(x); \quad x \in \mathbb{R}^n, \end{cases}$$

where the function $\xi \in \mathcal{Y}^{(T)}$ is deterministic and positive, $h : [0,T] \mapsto \mathbb{R}$ is bounded and deterministic and $K(t,x) = K(t,x,\omega) : [0,T] \times \mathbb{R}^n \times \Omega \mapsto \mathbb{R}$ is a given positive random field.
To motivate our method, we first recall the following basic results about the classical heat equation:
Let $L = \frac{1}{2}\triangle$ and consider the equation

$$\begin{cases} \mathrm{d}Y(t,x) &= LY\mathrm{d}t + K(t,x)\,\mathrm{d}t, \\ Y(0,x) &= \xi(x); \quad x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $\xi \in \mathcal{Y}^{(T)}$ and $K \in L^2([0,T] \times \mathbb{R}^n)$ are given deterministic functions. Define the operator $P_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$P_t f(x) = \int_{\mathbb{R}^n} (2\pi t)^{-\frac{n}{2}} f(y) \exp\left(-\frac{|x-y|^2}{2t}\right) \mathrm{d}y, \tag{3.2}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t} P_t f = L(P_t f),$$

and if we define

$$Y(t,x) = P_t \xi(x) + \int_0^t P_{t-s}(K(s,.))(x)\,\mathrm{d}s,$$

we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Y(t,x) &= L(P_t\xi)(x) + P_0(K(t,.))(x) + \int_0^t L(P_{t-s}(K(s,.)))(x)\,\mathrm{d}s \\ &= LY(t,x) + K(t,x). \end{aligned}$$

Hence

$$Y(t,x) \text{ solves the heat equation (3.1).}$$

Next, consider the case

$$\mathrm{d}Y(t,x) = LY\mathrm{d}t + K(t,x)\,\mathrm{d}t + \theta(t)\,Y(t,x)\,\mathrm{d}t.$$

Multiply the equation by

$$Z(t) = \exp\left(-\int_0^t \theta(s)\,\mathrm{d}s\right).$$

Then the equation becomes

$$d(Z(t)\,Y(t,x)) = L(Z(t)\,Y(t,x))\,\mathrm{d}t + Z(t)\,K(t,x)\,\mathrm{d}t.$$

Hence, if we put

$$\widehat{Y} = Z(t)\,Y(t,x),$$

then $\widehat{Y}$ solves the equation

$$\begin{cases} \mathrm{d}\widehat{Y}(t,x) & = L\widehat{Y}\mathrm{d}t + Z(t)K(t,x)\mathrm{d}t, \\ \widehat{Y}(0,x) & = \xi(x), \end{cases}$$

and we are back to the previous case.
Finally, consider the SPDE

$$\mathrm{d}Y(t,x) = LY\mathrm{d}t + K(t,x)\mathrm{d}t + h(t)Y(t)\mathrm{d}B(t), \tag{3.3}$$

where $h$ is a given bounded deterministic function and $K(t,x)$ is stochastic and adapted, and $\mathbb{E}[\int_0^T \int_{\mathbb{R}^n} K^2(t,x)\mathrm{d}t\mathrm{d}x] < \infty$. We handle this case by using white noise calculus on the Hida space $(\mathcal{S})^*$ of stochastic distributions: We introduce *white noise* $W_t \in (\mathcal{S})^*$ defined by

$$W_t = \frac{\mathrm{d}}{\mathrm{d}t}B(t),$$

and then we see that equation (3.3) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t,x) = LY + K(t,x) + Y(t)h(t) \diamond W_t,$$

where $\diamond$ denotes Wick multiplication. We refer to *e.g.* [10] for more information about white noise calculus. If we Wick-multiply this equation by

$$Z_t := \exp^{\diamond}\left(-\int_0^t h(s)\mathrm{d}B(s)\right),$$

where in general $\exp^{\diamond}(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!}\phi^{\diamond n}; \phi \in (\mathcal{S})^*$ is the Wick exponential, we get

$$Z_t \diamond \frac{\mathrm{d}}{\mathrm{d}t}Y(t,x) = L(Y \diamond Z_t) + K \diamond Z_t + Y(t)h(t) \diamond W_t \diamond Z_t. \tag{3.4}$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}t}(Z_t \diamond Y) = Z_t \diamond \frac{\mathrm{d}}{\mathrm{d}t}Y(t) - Y(t) \diamond Z_t \diamond h(t)W_t, \tag{3.5}$$

and hence (3.4) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{(Z_t \diamond Y_t)}_{\widehat{Y}_t} = L\underbrace{(Z_t \diamond Y_t)}_{\widehat{Y}_t} + K(t,x) \diamond Z_t.$$

This has the same form as (3.1). Hence the solution $\widehat{Y}$ is

$$\widehat{Y}(t,x) = P_t\xi(x) + \int_0^t P_{t-s}(K(s,.))(x) \diamond Z_s\mathrm{d}s.$$

Now we go back from $\widehat{Y}$ to $Y$ and get the solution

$$\begin{aligned}Y(t,x) &= \widehat{Y}(t,x) \diamond \exp^{\diamond}\left(\int_0^t h(s)\,\mathrm{d}B(s)\right)\\ &= P_t\xi(x) \diamond \exp^{\diamond}\left(\int_0^t h(s)\,\mathrm{d}B(s)\right)\\ &+ \int_0^t P_{t-s}(K(s,.))(x) \diamond \exp^{\diamond}\left(\int_s^t h(r)\,\mathrm{d}B(r)\right)\mathrm{d}s. \end{aligned} \tag{3.6}$$

Note that

$$\exp^{\diamond}\left(\int_0^t h(s)\,\mathrm{d}B(s)\right) = \exp\left(\int_0^t h(s)\,\mathrm{d}B(s) - \frac{1}{2}\int_0^t h^2(s)\,\mathrm{d}s\right) > 0.$$

Recall the Gjessing-Benth lemma (see [8, 15] or Thm. 2.10.6 in [17] or Prop. 13 in [7]), which states that

$$\phi \diamond \exp^{\diamond}\left(\int_0^t h(s)\,\mathrm{d}B(s)\right) = (\tau_{-h}\phi)\exp^{\diamond}\left(\int_0^t h(s)\,\mathrm{d}B(s)\right),$$

where, for $\phi : \Omega \mapsto \mathbb{R}$, we define $\tau_{-h}\phi(\omega) = \phi(\omega - h); \omega \in \Omega$ to be the shift operator on $\Omega$.

Using this in (3.6) we conclude that if

$$\xi \geq 0 \text{ and } K \geq 0 \text{ then } Y \geq 0.$$

We summarize what we have proved as follows:

**Theorem 3.1.** *Assume that* $\xi \in \mathcal{Y}^{(T)}$ *is deterministic,* $\mathbb{E}[\int_0^T \int_{\mathbb{R}^n} K^2(t,x)\mathrm{d}t\mathrm{d}x] < \infty$ *and let* $h : [0,T] \mapsto [0,T]$ *be bounded deterministic.*

1. *Then the unique solution* $Y(t,x) \in \mathcal{Y}^{(T)}$ *of the non-homogeneous SPDE*

$$\begin{aligned}\mathrm{d}Y(t,x) &= LY\mathrm{d}t + K(t,x)\,\mathrm{d}t + h(t)\,Y(t)\,\mathrm{d}B(t),\\ Y(0,x) &= \xi(x); \quad x \in \mathbb{R}^n\end{aligned}$$

   *is given by*

$$\begin{aligned}Y(t,x) &= (\tau_{-h}P_t\xi)(x)\exp^{\diamond}\left(\int_0^t h(s)\,\mathrm{d}B(s)\right)\\ &+ \int_0^t (\tau_{-h}P_{t-s}(K(s,.))(x)\exp^{\diamond}(\int_s^t h(r)\mathrm{d}B(r))\mathrm{d}s,\end{aligned}$$

   *where* $\exp^{\diamond}(\int_s^t h(r)\mathrm{d}B(r)) = \exp(\int_s^t h(r)\mathrm{d}B(r) - \frac{1}{2}\int_s^t h^2(r)\mathrm{d}r); \quad 0 \leq s \leq t \leq T.$
2. *In particular, if* $\xi(x) \geq 0$ *and* $K(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^n$*, then* $Y(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^n$*.*

Combining this with Theorem 2.1 we get

**Theorem 3.2.** *(Positivity) Assume that $\xi \in \mathcal{Y}^{(T)}$ is deterministic and let $h : [0,T] \mapsto \mathbb{R}$ be bounded and deterministic. Let $Y(t,x) \in \mathcal{Y}^{(T)}$ be the unique solution of the following SPDE with space interactions:*

$$Y(t,x) = \xi(x) + \int_0^t LY(s,x)\mathrm{d}s + \int_0^t \overline{Y}(s,x)\mathrm{d}s + \int_0^t h(s)Y(s,x)\mathrm{d}B(s); \quad t \in [0,T], \tag{3.7}$$

*given by Theorem 3.1.*
*Then if $\xi(x) \geq 0$ for all $x \in \mathbb{R}^n$, we have $Y(t,x) \geq 0$ for all $(t,x) \in [0,T] \times \mathbb{R}^n$.*

*Proof.* By Theorem 2.1 we know that the solution of (3.7) can be obtained as the limit when $m \to \infty$ of the sequence $Y_m(t,x)$ defined recursively by the equation (2.6). Then by Theorem 3.1, part 2, we know that $Y_m(t,x) \geq 0$ for all $t, x, m$. We conclude that $Y(t,x) \geq 0$ for all $(t,x)$. □

**Remark 3.3.** The methods from this and the previous section can be extended to more general equations, with both more general (uniformly elliptic) second order partial differential operator $L$ and more general coefficients satisfying suitable Lipschitz conditions. In particular, they extend to equations of the form

$$\mathrm{d}Y(t,x) = \big[LY(t,x) + \gamma(t,x)Y(t,x)\big]\mathrm{d}t + \overline{Y}(t,x)\mathrm{d}t + h(t)Y(t,x)\mathrm{d}B(t); \quad t \in [0,T], \tag{3.8}$$

for a given adapted process $\gamma \in \mathcal{Y}^{(T)}$. To see this we apply the arguments above with the operator $L$ replaced by the operator $\widehat{L}$ defined by $\widehat{L}\varphi = L\varphi + \gamma\varphi; \varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$. We omit the details.

## 4. The optimization problem

In general, if $\mathcal{X}, \mathcal{Y}$ are two Banach spaces and $F : \mathcal{X} \to \mathcal{Y}$ if Fréchet differentiable at $x \in \mathcal{X}$, then we let $\nabla_x F$ denote the Fréchet derivative of $F$ at $x$. It is a linear operator from $\mathcal{X}$ to $\mathcal{Y}$ and the action of $\nabla_x F$ to $h \in \mathcal{X}$ is denoted by $\nabla_x F(h) = \langle \nabla_x F, h \rangle \in \mathcal{Y}$. Recall that if $F$ is Fréchet differentiable at $x$ with Fréchet derivative $\nabla_x F$, then $F$ has a directional derivative

$$D_x F(h) := \lim_{\epsilon \to 0} \frac{1}{\epsilon}(F(x + \epsilon h) - F(x)) \tag{4.1}$$

in all directions $h \in \mathcal{X}$ and

$$D_x F(h) = \nabla_x F(h) = \langle \nabla_x F, h \rangle. \tag{4.2}$$

In particular, note that if $F$ is a linear operator, then $\nabla_x F = F$ for all $x$.

### 4.1. The Hamiltonian and the adjoint BSPDE

We now give a general formulation of the problem we consider.
Let $A_x$ be a linear second order partial differential operator given by

$$A_x \phi(x) = \sum_{i,j=1}^n \alpha_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^n \beta_i(x) \frac{\partial \phi}{\partial x_i}; \quad \phi \in \mathcal{C}_0^2(\mathbb{R}^n).$$

Let $T > 0$ and assume that the state $Y(t,x)$ at time $t \in [0,T]$ and at the point $x \in \overline{D} := D \cup \partial D$ satisfies the following non-local quasilinear stochastic heat equation:

$$\begin{cases} \mathrm{d}Y(t,x) & = A_x Y(t,x)\mathrm{d}t + b(t,x,Y(t,x),Y(t,\cdot),u(t,x))\mathrm{d}t \\ & +\sigma(t,x,Y(t,x),Y(t,\cdot),u(t,x))\mathrm{d}B(t), \\ Y(0,x) & = \xi(x); \quad x \in D, \\ Y(t,x) & = \eta(t,x); \quad (t,x) \in (0,T) \times \partial D. \end{cases} \tag{4.3}$$

***We make the following assumptions on*** *$(\alpha, \beta, b, \sigma, \xi, \eta)$:*

*(a) $(\alpha_{ij}(x))_{1\le i,j\le n}$ is a given symmetric nonnegative definite $n \times n$ matrix with eigenvalues bounded away from 0 and with entries $\alpha_{ij}(x) \in \mathcal{C}^4(D) \cap \mathcal{C}(\overline{D})$ for all $i,j = 1,2,...,n$.*
*(b) $\beta_i(x) \in \mathcal{C}^3(D) \cap \mathcal{C}(\overline{D})$ for all $i = 1,2,...,n$.*
*(c) The functions $b$ and $\sigma$ are $\mathbb{F}$-adapted, $\mathcal{C}^2$ with respect to $y$ and $u$ and admit uniformly bounded derivatives.*
*(d) $\xi \in L^2(D)$, and $\eta \in L^2([0,T] \times D \times \Omega)$ is $\mathbb{F}$-adapted.*

We call the equation (4.3) a *stochastic partial differential equation with space-interactions.*
In general, the formal adjoint $A^*$ of an operator $A$ is defined by the identity

$$(A\phi, \psi) = (\phi, A^*\psi), \quad \text{for all } \phi, \psi \in \mathcal{C}_0^2(D),$$

where $(\phi_1, \phi_2) := \langle \phi_1, \phi_2 \rangle_{L^2(D)} = \int_D \phi_1(x)\phi_2(x)\mathrm{d}x$ is the inner product in $L^2(D)$ and $\mathcal{C}_0^2(D)$ is the set of twice differentiable functions with compact support in $D$. In our case we have

$$A_x^*\phi(x) = \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j}(\alpha_{ij}(x)\phi(x)) - \sum_{i=1}^n \frac{\partial}{\partial x_i}(\beta_i(x)\phi(x)); \quad \phi \in \mathcal{C}^2(D).$$

We interpret $Y$ as a weak (variational) solution to (4.3), in the sense that

$$\begin{aligned} \langle Y(t), \phi \rangle_{L^2(D)} = \langle \xi(x), \phi \rangle_{L^2(D)} &+ \int_0^t \langle Y(s), A_x^*\phi \rangle_{L^2(D)}\mathrm{d}s \\ &+ \int_0^t \langle b(s,Y(s)), \phi \rangle_{L^2(D)}\mathrm{d}s + \int_0^t \langle \sigma(s,Y(s)), \phi \rangle_{L^2(D)}\mathrm{d}B(s); \phi \in \mathcal{C}_0^2(D). \end{aligned}$$

For simplicity, in the above equation we have not written all the arguments of $b, \sigma$.

In the following we will assume that there is a unique strong solution of (4.3). It is not known to us under what conditions this is the case for general $D$. In the case when $D = \mathbb{R}^n$ it follows by Proposition 12.1 in [20] that there exists a unique weak solution $Y(t,x)$ of (4.3) for all given initial values $\xi \in L^2(D)$. In Section 3, we have proved that there is a unique smooth, strong positive solution of equation (3.8) if $\xi > 0$ and $D = \mathbb{R}^n$.

The process $u(t,x) = u(t,x,\omega)$ is our control process, assumed to have values in a given convex set $U \subset \mathbb{R}^k$.

**Definition 4.1.** We call the control process $u(t,x)$ *admissible* if $u(t,x)$ is $\mathbb{F}$-predictable for all $(t,x) \in (0,T) \times D$ and $u(t,x) \in U$ for all $t,x$. The set of admissible controls is denoted by $\mathcal{U}$.

The performance functional (cost) associated to the control $u$ is assumed to have the form

$$J(u) = \mathbb{E}\Big[\int_0^T \int_D f(t,x,Y(t,x),Y(t,\cdot),u(t,x))\mathrm{d}x\mathrm{d}t + \int_D g(x,Y(T,x),Y(T,\cdot))\mathrm{d}x\Big]; \quad u \in \mathcal{U}. \tag{4.4}$$

**We make the following assumptions on** $(f,g)$:

(e) The function $f(t,x,y,\varphi,u)$ is $\mathbb{F}$-adapted and the function $g(x,y,\varphi)$ is $\mathcal{F}_T$-measurable. They are assumed to be bounded, $\mathcal{C}^2$ with respect to $y,\varphi,u$, with uniformly bounded derivatives.

We consider the following problem of optimal control of a solution of an SPDE:

**Problem 4.2.** Find $\widehat{u} \in \mathcal{U}$ such that

$$J(\widehat{u}) = \inf_{u\in\mathcal{U}} J(u). \tag{4.5}$$

As mentioned in the Introduction (Sect. 1), this type of problem has been studied by many authors, and it may in some sense be considered as a special case of the general problem discussed in [20], except that we are considering strong solutions on $[0,T]\times D$, where $D$ a given open subset of $\mathbb{R}^n$, with given boundary values on $\partial D$. Moreover, our approach is specifically focused on the stochastic reaction-diffusion equation with space interaction presented in Section 2, and therefore gives more explicit results.

To study this problem we define the associated *Hamiltonian* $H : [0,T]\times D\times\mathbb{R}\times L(\mathbb{R}^n)\times U\times\mathbb{R}\times\mathbb{R}\times\Omega\to\mathbb{R}$ by

$$\begin{aligned} H(t,x,y,\varphi,u,p,q) := H(t,x,y,\varphi,u,p,q,\omega) = f(t,x,y,\varphi,u) + b(t,x,y,\varphi,u)p \\ + \sigma(t,x,y,\varphi,u)q. \end{aligned} \tag{4.6}$$

In general, if $h : L^2(D)\mapsto L^2(D)$ is Fréchet differentiable map, then its Fréchet derivative (gradient) at $\varphi\in L^2(D)$ denoted by $\nabla_\varphi h = \nabla h$ is a bounded linear map on the Hilbert space $L^2(D)$, and by the Riesz representation theorem we can represent it by a function $\nabla h(x,y)\in L^2(D\times D)$. We denote the action of $\nabla h$ on a function $\psi\in L^2(D)$ by $\langle\nabla h,\psi\rangle$.

Hence

$$\langle\nabla h,\psi\rangle(x) := \int_D \nabla h(x,y)\psi(y)\mathrm{d}y; \qquad \text{for all } \psi\in L^2(D). \tag{4.7}$$

**Remark 4.3.**

- Note in particular that if $h : L^2(D)\mapsto L^2(D)$ is linear, then

$$\nabla h(x,y) = h(x,y).$$

- Also note that from (4.7) it follows by the Fubini theorem that

$$\begin{aligned} \int_D \langle\nabla h,\psi\rangle(x)\mathrm{d}x &= \int_D\int_D \nabla h(x,y)\psi(y)\mathrm{d}y\mathrm{d}x = \int_D\int_D \nabla h(y,x)\psi(x)\mathrm{d}x\mathrm{d}y \\ &= \int_D\left(\int_D \nabla h(y,x)\mathrm{d}y\right)\psi(x)\mathrm{d}x = \int_D \overline{\nabla} h(x)\psi(x)\mathrm{d}x, \end{aligned}$$

  where

$$\overline{\nabla} h(x) := \int_D \nabla h(y,x)\mathrm{d}y. \tag{4.8}$$

**Example 4.4. a)** Assume that $h : L^2(D) \mapsto L^2(D)$ is given by

$$h(\varphi) = \langle h, \varphi \rangle (x) = G(x, \varphi(\cdot)) = \frac{1}{V(K_r)} \int_{K_r} \varphi(x+y)\mathrm{d}y. \tag{4.9}$$

Then

$$\langle \nabla_\varphi h, \psi \rangle (x) = \langle h, \psi \rangle (x) = \frac{1}{V(K_r)} \int_{K_r} \psi(x+y)\mathrm{d}y.$$

Therefore $\nabla h(x,y)$ is given by the identity

$$\int_D \nabla h(x,y)\psi(y)\mathrm{d}y = \frac{1}{V(K_r)} \int_{K_r} \psi(x+y)\mathrm{d}y; \quad \psi \in L^2(D).$$

Substituting $z = x + y$ this can be written

$$\int_D \nabla^*_\varphi h(x,y)\psi(y)\mathrm{d}y = \frac{1}{V(K_r)} \int_{x+K_r} \psi(z)\mathrm{d}z = \int_D \frac{\mathbf{1}_{x+K_r}(y)}{V(K_r)}\psi(y)\mathrm{d}y.$$

Since this is required to hold for all $\psi$, we conclude the following:

**b)** Suppose that $h$ is given by (4.9). Then

$$\nabla h(x,y) = \frac{\mathbf{1}_{x+K_r}(y)}{V(K_r)},$$

and

$$\begin{aligned}\overline{\nabla}_\varphi h(x) &= \int_D \nabla_\varphi h(y,x)\mathrm{d}y = \frac{1}{V(K_r)} \int_D \mathbf{1}_{y+K_r}(x)\mathrm{d}y = \frac{1}{V(K_r)} \int_D \mathbf{1}_{x-K_r}(y)\mathrm{d}y \\ &= \frac{V((x-K_r)\cap D)}{V(K_r)} = \frac{V((x+K_r)\cap D)}{V(K_r)},\end{aligned}$$

since $K_r = -K_r$.

We associate to the Hamiltonian the following BSPDE

$$\mathrm{d}p(t,x) = -\left[A^*_x p(t,x) + \tfrac{\partial H}{\partial y}(t,x) + \overline{\nabla} H(t,x)\right]\mathrm{d}t + q(t,x)\mathrm{d}B(t), \tag{4.10}$$

with boundary/terminal values

$$\begin{cases} p(T,x) &= \tfrac{\partial g}{\partial y}(x) + \overline{\nabla} g(x); \quad x \in D, \\ p(t,x) &= 0; \quad (t,x) \in (0,T) \times \partial D, \end{cases} \tag{4.11}$$

where we have used the simplified notation

$$H(t,x) = H(t,x,y,\varphi,u,p,q)|_{y=Y(t,x),\varphi=Y(t,\cdot),u=u(t,x),p=p(t,x),q=q(t,x)},$$

and similarly we have used the notation $g(x)$ for $g(x, Y(T,x), Y(T,\cdot))$. Here $A^*_x$ denotes the adjoint of the operator $A_x$.

Note that the differential of $p$ in (4.10) can be written explicitly as follows:

$$\begin{aligned}
\mathrm{d}p\,(t,x) = -\Big[ &\sum_{i,j=1}^{n} \alpha_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} p(t,x) \\
&+ \sum_{i=1}^{n} \Big( -\beta_i(x) + 2 \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \alpha_{ij}(x) \Big) \frac{\partial}{\partial x_i} p(t,x) \\
&\Big( -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \beta_i(x) + \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \alpha_{ij}(x) \Big) p(t,x) + \Big( (\frac{\partial}{\partial y} + \overline{\nabla}) b(t,x) \Big) p(t,x) \\
&+ \Big( (\frac{\partial}{\partial y} + \overline{\nabla}) \sigma(t,x) \Big) q(t,x) + \Big( \frac{\partial}{\partial y} + \overline{\nabla} \Big) f(t,x) \Big] \mathrm{d}t + q(t,x) \mathrm{d}B(t).
\end{aligned}$$

To the best of our knowledge, the existence and uniqueness of a solution of (4.10)–(4.11) is not known in general. However, note that (for given $u$) the equation (4.10), regarded as a BSPDE in the unknown $\mathcal{Y}^{(T)} \times \mathcal{Y}^{(T)}$-valued processes $(p,q)$, is linear. Therefore, in view of our general assumptions (a)-(d) above, the existence and uniqueness of solution follows from *e.g.* Theorem 2.1 in [11], provided that the terms $\overline{\nabla} b(t,x), \overline{\nabla}\sigma(t,x)$ and $\overline{\nabla} f(t,x)$ satisfy condition $(F_m)$ in [11]. To this end, it suffices that $b, \sigma$ and $f$ depend linearly on $\varphi$ and in a space-averaging manner, as in the example with $h$ in (4.9) above. In particular, this holds in the application studied in Section 5.

**Remark 4.5.** Here, as in Sections 2 and 3, we are primarily interested in strong solutions $(p,q) \in \mathcal{Y}^{(T)} \times \mathcal{Y}^{(T)}$, but weak solutions are also of interest. A pair $(p,q)$ of random fields is said to be a weak solution to the BSPDE (4.10)–(4.11) if, for all $\phi \in \mathcal{C}_0^2(D)$,

$$\begin{aligned}
\langle p(t,.), \phi(.)\rangle - \langle p(T,.), \phi(.)\rangle = &\int_t^T \langle A_x^* p(s,.), \phi(.)\rangle \mathrm{d}s + \int_t^T \langle \tfrac{\partial H}{\partial y}(t,.) + \overline{\nabla} H(t,.), \phi(.)\rangle \mathrm{d}s \\
&- \int_t^T \langle q(s,.), \phi(.)\rangle \mathrm{d}B(s); \quad \text{a.s. for each t} \in [0,T].
\end{aligned}$$

Hence, we observe that $p$ admits the following mild representation

$$p(t,x) = P_{T-t}\Big(p(T,x)\Big) + \int_t^T P_{s-t}\Big(\tfrac{\partial H}{\partial y}(t,x) + \overline{\nabla} H(t,x)\Big)\mathrm{d}s - \int_t^T P_{s-t}\Big(q(s,x)\Big)\mathrm{d}B(s); \quad 0 \le t \le T,$$

where $P_t$ denotes the semigroup of the operator $A^*$.

## 4.2. A sufficient maximum principle approach (I)

We now formulate a sufficient version ( a verification theorem) of the maximum principle for the optimal control of the problem (4.3)–(4.5). In the special case when $D = \mathbb{R}^n$ the result follows from Theorem 12.21 in [20]. We give a direct proof for our situation, with general $D$.

**Theorem 4.6.** *[Sufficient Maximum Principle (I)]Suppose $\widehat{u} \in \mathcal{U}$, with corresponding $\widehat{Y}(t,x), \widehat{p}(t,x), \widehat{q}(t,x)$. Suppose the functions $(y,\varphi) \mapsto g(x,y,\varphi)$ and $(y,\varphi,u) \mapsto H(t,x,y,\varphi,u,\widehat{p}(t,x),\widehat{q}(t,x))$ are convex for each $(t,x) \in [0,T] \times D$. Moreover, suppose that, for all $(t,x) \in [0,T] \times D$,*

$$\begin{aligned}
&\min_{v \in U} H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),v,\widehat{p}(t,x),\widehat{q}(t,x)) \\
&= H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\widehat{u}(t,x),\widehat{p}(t,x),\widehat{q}(t,x)).
\end{aligned}$$

*Then $\widehat{u}$ is an optimal control.*

*Proof.* Consider

$$J(u) - J(\widehat{u}) = I_1 + I_2,$$

where

$$I_1 = \mathbb{E}\left[\int_0^T \int_D \{f(t,x,Y(t,x),Y(t,\cdot),u(t,x)) - f(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\widehat{u}(t,x))\}\mathrm{d}x\mathrm{d}t\right],$$

and

$$I_2 = \int_D \mathbb{E}\left[g(x,Y(T,x),Y(T,\cdot)) - g(x,\widehat{Y}(T,x),\widehat{Y}(T,\cdot))\right]\mathrm{d}x.$$

By convexity on $g$ together with the identities (4.7)–(4.8) (by putting $\nabla h(x,y) = \nabla_\varphi \widehat{g}(T,x)$ and $\psi = (Y(T,\cdot) - \widehat{Y}(T,\cdot)))$, we get

$$\begin{aligned}
I_2 &\geq \int_D \mathbb{E}\left[\tfrac{\partial \widehat{g}}{\partial y}(T,x)(Y(T,x) - \widehat{Y}(T,x)) + \left\langle \nabla_\varphi \widehat{g}(T,x), (Y(T,\cdot) - \widehat{Y}(T,\cdot))\right\rangle\right]\mathrm{d}x \\
&= \int_D \mathbb{E}\left[\tfrac{\partial \widehat{g}}{\partial y}(T,x)(Y(T,x) - \widehat{Y}(T,x)) + \overline{\nabla}\widehat{g}(T,x)(Y(T,x) - \widehat{Y}(T,x))\right]\mathrm{d}x \\
&= \int_D \mathbb{E}\left[\widehat{p}(T,x)(Y(T,x) - \widehat{Y}(T,x))\right]\mathrm{d}x \\
&= \int_D \mathbb{E}\left[\widehat{p}(T,x)\widetilde{Y}(T,x)\right]\mathrm{d}x,
\end{aligned}$$

where we put

$$\widetilde{Y}(t,x) = Y(t,x) - \widehat{Y}(t,x); (t,x) \in [0,T] \times D. \tag{4.12}$$

Applying the Itô formula to $\widehat{p}(t,x)\widetilde{Y}(t,x)$, we have

$$\begin{aligned}
I_2 \geq \int_0^T \int_D \mathbb{E}\Big[&\widehat{p}(t,x)\{A_x\widetilde{Y}(t,x) + \widetilde{b}(t,x)\} - \widetilde{Y}(t,x)\{A_x^*\widehat{p}(t,x) \\
&+ \frac{\partial \widehat{H}}{\partial y}(t,x) + \overline{\nabla}_\varphi^* \widehat{H}(t,x)\} + \widehat{q}(t,x)\widetilde{\sigma}(t,x)\Big]\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{4.13}$$

where

$$\widetilde{b}(t) = b(t) - \widehat{b}(t),\ \widetilde{\sigma}(t) = \sigma(t) - \widehat{\sigma}(t). \tag{4.14}$$

Since $\widetilde{Y}(t,x) = \widehat{p}(t,x) \equiv 0$, for all $(t,x) \in (0,T) \times \partial D$, we get

$$\int_D \widehat{p}(t,x)A_x\widetilde{Y}(t,x)\mathrm{d}x = \int_D \widetilde{Y}(t,x)A_x^*\widehat{p}(t,x)\mathrm{d}x. \tag{4.15}$$

Substituting (4.15) in (4.13), yields

$$I_2 \geq \int_0^T \int_D \mathbb{E}\left[\widehat{p}(t,x)\widetilde{b}(t,x) - \widetilde{Y}(t,x)\left\{\tfrac{\partial \widehat{H}}{\partial y}(t,x) + \overline{\nabla}\widehat{H}(t,x)\right\} + \widehat{q}(t,x)\widetilde{\sigma}(t,x)\right] \mathrm{d}x\mathrm{d}t. \tag{4.16}$$

Using the definition of the Hamiltonian $H$ in (4.6), and putting

$$\begin{aligned}\widetilde{H}(t,x) &= H(t,x,Y(t,x),Y(t,\cdot),u(t,x),\widehat{p}(t,x),\widehat{q}(t,x))\\ &\quad - H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\widehat{u}(t,x),\widehat{p}(t,x),\widehat{q}(t,x)),\end{aligned} \tag{4.17}$$

we get

$$\begin{aligned}I_1 &= \mathbb{E}\left[\int_0^T \int_D \{\widetilde{H}(t,x) - \widehat{p}(t,x)\widetilde{b}(t,x) - \widehat{q}(t,x)\widetilde{\sigma}(t,x)\}\mathrm{d}x\mathrm{d}t\right]\\ &\geq \mathbb{E}\left[\int_0^T \int_D \left\{\frac{\partial \widehat{H}}{\partial y}(t,x)\widetilde{Y}(t,x) + \left\langle \nabla\widehat{H}(t,x), \widetilde{Y}(t,\cdot)\right\rangle\right.\right.\\ &\quad \left.\left. + \frac{\partial \widehat{H}}{\partial u}(t,x)\widetilde{u}(t,x) - \widehat{p}(t,x)\widetilde{b}(t,x) - \widehat{q}(t,x)\widetilde{\sigma}(t,x)\right\}\mathrm{d}x\mathrm{d}t\right],\end{aligned} \tag{4.18}$$

where the last inequality holds because of the concavity assumption of $H$.

Summing (4.16) and (4.18), and using (4.7), (4.8), we end up with

$$I_1 + I_2 \quad \geq \mathbb{E}\left[\int_0^T \int_D \frac{\partial \widehat{H}}{\partial u}(t,x)\widetilde{u}(t,x)\mathrm{d}x\mathrm{d}t\right].$$

By the maximum condition of $H$ we have

$$J(u) - J(\widehat{u}) \geq \mathbb{E}\left[\int_0^T \int_D \tfrac{\partial \widehat{H}}{\partial u}(t,x)\widetilde{u}(t,x)\mathrm{d}x\mathrm{d}t\right] \geq 0.$$

□

### 4.3. A necessary maximum principle approach (I)

We now go to the other version of the necessary maximum principle which can be seen as an extension of Pontryagin's maximum principle to SPDE with space-mean dynamics. In the case when $D = \mathbb{R}^n$ a version of the necessary maximum principle is proved in [20]. Here concavity assumptions are not required . We consider the following:

Given arbitrary controls $u, \widehat{u} \in \mathcal{U}$ with $u$ bounded, we define the following convex perturbation

$$u^\theta := \widehat{u} + \theta u; \quad \theta \in [0,1]. \tag{4.19}$$

Note that, thanks to the convexity of $U$, we also have $u^\theta \in \mathcal{U}$. We denote by $Y^\theta := Y^{u^\theta}$ and by $\widehat{Y} := Y^{\widehat{u}}$ the solution processes of (4.3) corresponding to $u^\theta$ and $\widehat{u}$, respectively.

Define the derivative process $Z(t,x)$ by

$$Z(t,x)=\lim_{\theta\to 0}\frac{1}{\theta}(Y^{\theta}(t,x)-\widehat{Y}(t,x)) \text{ (limit in } \mathcal{Y}^{(T)}). \tag{4.20}$$

Then, by our assumptions on $f,g,b$ and $\sigma$ it is easy to see that $Z(t,x)$ exists and satisfies the following equation:

$$\begin{cases} \mathrm{d}Z(t,x) &= \Big\{A_x Z(t,x)+\frac{\partial b}{\partial y}(t,x)Z(t,x)+\langle\nabla b(t,x),Z(t,\cdot)\rangle+\frac{\partial b}{\partial u}(t,x)u(t,x)\Big\}\mathrm{d}t \\ & \quad +\Big\{\frac{\partial\sigma}{\partial y}(t,x)Z(t,x)+\langle\nabla\sigma(t,x),Z(t,\cdot)\rangle+\frac{\partial\sigma}{\partial u}(t,x)u(t,x)\Big\}\mathrm{d}B(t), \\ Z(t,x) &= 0;\quad (t,x)\in(0,T)\times\partial D, \\ Z(0,x) &= 0;\quad x\in D. \end{cases} \tag{4.21}$$

Note that (4.21), regarded as an SPDE in the unknown $\mathcal{Y}^{(T)}$-valued process $Z$, is linear and hence the existence and uniqueness of solution follows from *e.g.* Theorem 3.3 in [14].

**Theorem 4.7.** *[Necessary Maximum Principle (I)]Let $\widehat{u}(t,x)$ be an optimal control and $\widehat{Y}(t,x)$ the corresponding trajectory and adjoint processes $(\widehat{p}(t,x),\widehat{q}(t,x))$. Then we have*

$$\frac{\partial\widehat{H}}{\partial u}\Big|_{u=\hat{u}}(t,x)=0;\qquad a.s.$$

*Proof.* Since $\widehat{u}$ is optimal we get, by the definition (4.4) of $J$, dominated convergence and the chain rule,

$$\begin{aligned} 0 &\leq \varlimsup_{\theta\to 0}\frac{J(u^{\theta})-J(\widehat{u})}{\theta} \\ &= \varlimsup_{\theta\to 0}\frac{1}{\theta}\mathbb{E}\Big[\int_D\{g(x,Y^{\theta}(T),Y^{\theta}(T,\cdot))-g(x,\widehat{Y}(T,x),\widehat{Y}(T,\cdot))\}\mathrm{d}x \\ &+\int_D\int_0^T\{f(t,x,Y^{\theta}(t,x),Y^{\theta}(t,\cdot),u(t,x))-f(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),u(t,x))\}\mathrm{d}t\mathrm{d}x\Big] \\ &= \mathbb{E}\Big[\int_D\varlimsup_{\theta\to 0}\frac{1}{\theta}\{g(x,Y^{\theta}(T),Y^{\theta}(T,\cdot))-g(x,\widehat{Y}(T,x),\widehat{Y}(T,\cdot))\}\mathrm{d}x \\ &+\int_D\int_0^T\varlimsup_{\theta\to 0}\frac{1}{\theta}\{f(t,x,Y^{\theta}(t,x),Y^{\theta}(t,\cdot),u(t,x))-f(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),u(t,x))\}\mathrm{d}t\mathrm{d}x\Big] \\ &= \mathbb{E}\Big[\int_D\frac{\partial g}{\partial y}(x,\widehat{Y}^{\theta}(T,x),\widehat{Y}^{\theta}(T,\cdot))\varlimsup_{\theta\to 0}\frac{1}{\theta}(Y^{\theta}(t,x)-\widehat{Y}(t,x)) \\ &+\Big\langle\nabla g(x,\widehat{Y}^{\theta}(T,x),\widehat{Y}^{\theta}(T,\cdot)),\varlimsup_{\theta\to 0}\frac{1}{\theta}(Y^{\theta}(t,\cdot)-\widehat{Y}(t,\cdot))\Big\rangle\mathrm{d}x \\ &+\int_D\int_0^T\varlimsup_{\theta\to 0}\frac{1}{\theta}\{f(t,x,Y^{\theta}(t,x),Y^{\theta}(t,\cdot),u(t,x))-f(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),u(t,x))\}\mathrm{d}t\mathrm{d}x\Big]. \end{aligned}$$

Therefore, writing $\frac{\partial\widehat{g}}{\partial y}(T,x)=\frac{\partial g}{\partial y}(x,\widehat{Y}(T,x),\widehat{Y}(T,\cdot))$ and $\frac{\partial\widehat{f}}{\partial y}(t,x)=\frac{\partial f}{\partial y}(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\widehat{u}(t,x))$ and similarly with $\nabla\widehat{g}(T,x),\nabla\widehat{f}(t,x)$ we obtain

$$\begin{aligned} 0 &\leq \varlimsup_{\theta\to 0}\tfrac{J(u^{\theta})-J(\widehat{u})}{\theta} \\ &= \mathbb{E}\left[\int_D\Big\{\frac{\partial\widehat{g}}{\partial y}(T,x)Z(T,x)+\langle\nabla\widehat{g}(T,x),Z(T,\cdot)\rangle\Big\}\mathrm{d}x\right] \\ &+\mathbb{E}\left[\int_0^T\int_D\left\{\frac{\partial\widehat{f}}{\partial y}(t,x)Z(t,x)+\Big\langle\nabla\widehat{f}(t,x),Z(t,\cdot)\Big\rangle+\frac{\partial\widehat{f}}{\partial u}(t,x)u(t,x)\right\}\mathrm{d}x\mathrm{d}t\right]. \end{aligned} \tag{4.22}$$

By (4.7) and the BSPDE for $\widehat{p}(t,x)$, we have

$$\mathbb{E}\left[\int_D \left\{\frac{\partial \widehat{g}}{\partial y}(T,x)Z(T,x) + \langle \nabla \widehat{g}(T,x), Z(T,\cdot)\rangle \right\}\mathrm{d}x\right] = \mathbb{E}\left[\int_D \widehat{p}(T,x)Z(T,x)\mathrm{d}x\right],$$

The Itô formula applied to the product $\widehat{p}(t,x)\cdot Z(t,x)$, where $\widehat{p}$ and $Z$ are the associated equations (4.21), (4.10)–(4.11), respectively, to the optimal control $\widehat{u}$, combined with the definition of $\widehat{H}$ in (4.6), leads to

$$\begin{aligned}
&\mathbb{E}\left[\int_D \widehat{p}(T,x)Z(T,x)\mathrm{d}x\right] = \mathbb{E}\left[\int_0^T \int_D \Big(\widehat{p}(t,x)\mathrm{d}Z(t,x) + Z(t,x)\mathrm{d}\widehat{p}(t,x)\right.\\
&\left. + \int_0^T \int_D \Big\{\widehat{q}(t,x)\left(\frac{\partial \sigma}{\partial y}(t,x)Z(t,x) + \langle \nabla\sigma(t,x), Z(t,\cdot)\rangle + \frac{\partial \sigma}{\partial u}(t,x)u(t,x)\right)\Big\}\mathrm{d}t\mathrm{d}x\right]\\
&= \mathbb{E}\Big[\int_0^T \int_D \Big\{\widehat{p}(t,x)\Big(A_x Z(t,x) + \frac{\partial b}{\partial y}(t,x)Z(t,x) + \langle \nabla b(t,x), Z(t,\cdot)\rangle\\
&+ \frac{\partial b}{\partial u}(t,x)u(t,x)\Big) + Z(t,x)\Big( - A_x^*\widehat{p}(t,x) - \frac{\partial \widehat{H}}{\partial y}(t,x) - \overline{\nabla}\widehat{H}(t,x)\Big)\\
&+ \Big\{\widehat{q}(t,x)\left(\frac{\partial \sigma}{\partial y}(t,x)Z(t,x) + \langle \nabla\sigma(t,x), Z(t,\cdot)\rangle + \frac{\partial \sigma}{\partial u}(t,x)u(t,x)\right)\Big\}\mathrm{d}t\mathrm{d}x\Big].
\end{aligned}$$

Substituting this in (4.22), we get

$$0 \le \mathbb{E}\left[\int_0^T \int_D \frac{\partial \widehat{H}}{\partial u}(t,x)u(t,x)\mathrm{d}x\mathrm{d}t\right].$$

In particular, if we apply this to

$$u(t,x) = \mathbf{1}_{[s,T]}(t)\alpha(x),$$

where $\alpha(x)$ is bounded and $\mathcal{F}_s$-measurable we get

$$0 \ge \mathbb{E}\left[\int_s^T \int_D \frac{\partial \widehat{H}}{\partial u}(t,x)\alpha(x)\mathrm{d}x\mathrm{d}t\right].$$

Since this holds for all such $\alpha$ (positive or negative) and all $s \in [0,T]$, we conclude that

$$0 = \frac{\partial \widehat{H}}{\partial u}(t,x); \qquad \text{for a.a. } t, x.$$

□

### 4.4. Controls which are independent of $x$

In many situations, for example in connection with partial observation control, it is of interest to study the case when the controls $u(t) = u(t,\omega)$ are not allowed to depend on the space variable $x$. Let us denote the set of such controls $u \in \mathcal{U}$ by $\overline{\mathcal{U}}$. Then the corresponding control problem is to find $\widehat{u} \in \overline{\mathcal{U}}$ such that

$$J(\widehat{u}) = \inf_{u\in\overline{\mathcal{U}}} J(u).$$

The equations for $J$, $Y$, $H$ and $p$ are as before, except that we replace $u(t,x)$ by $u(t)$. We handle this situation by introducing integration with respect to d$x$ in the Hamiltonian. We state the corresponding modified theorems without proofs:

**Theorem 4.8** (Sufficient Maximum Principle (II))**.** *Suppose $\widehat{u} \in \overline{\mathcal{U}}$, with corresponding $\widehat{Y}(t,x), \widehat{p}(t,x), \widehat{q}(t,x)$. Suppose the functions $(y,\varphi) \mapsto g(x,y,\varphi)$ and $(y,\varphi,u) \mapsto H(t,x,y,\varphi,u,\widehat{p}(t,x),\widehat{q}(t,x)$ are convex for each $(t,x) \in [0,T] \times D$. Moreover, suppose the following average minimum condition,*

$$\min_{v\in U}\left\{\int_D H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),v,\widehat{p}(t,x),\widehat{q}(t,x))\mathrm{d}x\right\}$$

$$= \int_D H(t,x,\widehat{Y}(t,x),\widehat{Y}(t,\cdot),\widehat{u}(t),\widehat{p}(t,x),\widehat{q}(t,x))\mathrm{d}x.$$

*Then $\widehat{u}$ is an optimal control.*

**Theorem 4.9** (Necessary Maximum Principle (II))**.** *Let $\widehat{u}(t)$ be an optimal control and $\widehat{Y}(t,x)$ the corresponding trajectory and adjoint processes $(\widehat{p}(t,x),\widehat{q}(t,x))$. Then we have*

$$\int_D \frac{\partial \widehat{H}}{\partial u}\bigg|_{u=\widehat{u}}(t,x)\mathrm{d}x = 0; \qquad a.s.\ \mathrm{d}t\times d\mathbb{P}.$$

## 5. Application to vaccine optimisation

Assume that the *density* $Y(t,x)$ *of infected individuals in a population* in a random/noisy environment changes over time $t$ and space point $x$ according to the following space-interaction reaction-diffusion equation

$$\begin{cases} \mathrm{d}Y(t,x) &= \frac{1}{2}\Delta Y(t,x)\mathrm{d}t + \Big(\alpha\overline{Y}(t,x) - u(t,x)Y(t,x)\Big)\mathrm{d}t + \beta Y(t,x)\mathrm{d}B(t),\\ Y(0,x) &= \xi(x)\geq 0; \quad x\in D,\\ Y(t,x) &= \eta(t,x)\geq 0; \quad (t,x)\in(0,T)\times\partial D,\end{cases}$$

where $\alpha,\beta$ are given constants modelling the effect on the growth $\mathrm{d}Y(t,x)$ of the term $\overline{Y}$ and of the noise, respectively, and $\overline{Y}(t,x) = G(x,Y(t,\cdot))$, where, as before, $G$ is a space-averaging operator of the form

$$G(x,\varphi) = \frac{1}{V(K_r)}\int_{K_r}\varphi(x+y)\mathrm{d}y; \quad \varphi\in L^2(D),$$

with $V(\cdot)$ denoting Lebesgue volume and

$$K_r = \{y\in\mathbb{R}^n; |y|<r\}$$

is the ball of radius $r>0$ in $\mathbb{R}^n$ centered at 0.

By a slight extension of Theorem 3.2 (see Rem. 3.3), we know that $Y(t,x)\geq 0$ for all $t,x$.

If $u(t,x)$ represents our vaccine effort rate at $(t,x)$, we define the total expected cost $J(u)$ of the effort by

$$J(u) = \mathbb{E}\Big[\frac{\rho}{2}\int_D\int_0^T u(t,x)^2 Y(t,x)\mathrm{d}t\mathrm{d}x + \int_D h_0(x)Y(T,x)\mathrm{d}x\Big],$$

where $\rho>0$ is a constant, and $h_0(x)>0$ is a bounded function. Here we may regard the first quadratic term as the cost of the vaccination effort, with unit price $\rho$, and the second term as the cost of having remaining infection at time $T$. In this case the Hamiltonian is

$$H(t,x,y,\overline{y},p,q) = (\alpha\overline{y} - uy)p + \beta yq + \frac{\rho}{2}u^2 y,$$

and the adjoint equation satisfies

$$\begin{cases} \mathrm{d}p(t,x) = -\Big[\frac{1}{2}\Delta p(t,x) - u(t,x)p(t,x) + \overline{\nabla}_{\overline{y}}H(t,x) + \beta q(t,x) + \frac{\rho}{2}u^2(t,x)\Big]\mathrm{d}t + q(t,x)\mathrm{d}B(t),\\ p(T,x) = h_0(x); \quad x\in D\\ p(t,x) = 0; \quad (t,x)\in(0,T)\times\partial D,\end{cases} \tag{5.1}$$

where, by Example 4.4, $\overline{\nabla}_{\overline{y}}H(t,x) = v_D(x)\alpha p(t,x)$, with $v_D(x) := \frac{V((x+K_r)\cap D)}{V(K_r)}$.

The first order condition for an optimal $u=\widehat{u}$ for $H$ together with the requirement that $Y(t,x)>0$. lead to

$$\widehat{u}(t,x) = \frac{p(t,x)}{\rho}.$$

Hence the pair of random fields $(\widehat{p},\widehat{q})$ becomes

$$\begin{cases} \mathrm{d}\widehat{p}(t,x) &= -\Big[\frac{1}{2}\Delta\widehat{p}(t,x) + \frac{1}{2\rho}\widehat{p}^2(t,x) + v_D(x)\alpha\widehat{p}(t,x) + \beta\widehat{q}(t,x)\Big]\mathrm{d}t + \widehat{q}(t,x)\mathrm{d}B(t),\\ \widehat{p}(T,x) &= h_0(x); \quad x\in D,\\ \widehat{p}(t,x) &= 0; \quad (t,x)\in(0,T)\times\partial D.\end{cases} \tag{5.2}$$

Since $h_0$ and all the coefficients of this equation are deterministic, we can conclude that $\widehat{q} = 0$ and (5.2) reduces to the deterministic partial differential equation

$$\begin{cases} \frac{\partial}{\partial t}\widehat{p}(t,x) & = -\left[\frac{1}{2}\Delta\widehat{p}(t,x) + \frac{1}{2\rho}\widehat{p}^2(t,x) + v_D(x)\alpha\widehat{p}(t,x)\right], \\ \widehat{p}(T,x) & = h_0(x); \quad x \in D, \\ \widehat{p}(t,x) & = 0; \quad (t,x) \in (0,T) \times \partial D. \end{cases}$$

This is a (deterministic) Fujita type backward quadratic reaction diffusion equation. We could also from the beginning have allowed $h_0(x)$ to be random and satisfy $\mathbb{E}\left[\int_D h_0^2(x)\mathrm{d}x\right] < \infty$. Then the equation (5.2) would have become a nonlinear backward *stochastic* reaction-diffusion equation. We will not discuss this further here, but refer to Bandle, and Levine [2], Dalang *et al.* [9] and Fujita [13] and the references therein for more information.

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