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CLASSICAL SOLUTIONS TO LOCAL FIRST-ORDER EXTENDED MEAN FIELD GAMES*

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Abstract. We study the existence of classical solutions to a broad class of local, first order, forward-backward extended mean field games systems, that includes standard mean field games, mean field games with congestion, and mean field type control problems. We work with a strictly monotone cost that may be fully coupled with the Hamiltonian, which is assumed to have superlinear growth. Following previous work on the standard first order mean field games system, we prove the existence of smooth solutions under a coercivity condition that ensures a positive density of players, assuming a strict form of the uniqueness condition for the system. Our work relies on transforming the problem into a partial differential equation with oblique boundary conditions, which is elliptic precisely under the uniqueness condition.

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1. Introduction

In this paper, we prove existence of classical solutions to a broad class of first order mean field games systems (MFG for short) with a local coupling, which includes standard MFG, MFG with congestion, and mean field type control problems. For this purpose, we study the MFG system:

$$\begin{cases} -u_t + H(x, D_x u, m) = 0 & (x,t) \in Q_T = \mathbb{T}^d \times (0,T), \\ m_t - \operatorname{div}(B(x, D_x u, m)) = 0 & (x,t) \in Q_T, \\ m(0,x) = m_0(x),\ u(x,T) = g(x, m(x,T)) & x \in \mathbb{T}^d, \end{cases} \tag{EMFG}$$

where $-H(x,p,m) : \mathbb{T}^d \times \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ and $g(x,m) : \mathbb{T}^d \times (0,\infty) \to \mathbb{R}$ are strictly increasing in m, and $m_0 : \mathbb{T}^d \to \mathbb{R}$ is a positive probability density.

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MFG were introduced by Lasry and Lions [16, 19], and at the same time, in a particular setting, by Caines, Huang, and Malhamé [5]. They are non-cooperative differential games with infinitely many players, in which the players find an optimal strategy by observing the distribution of the others.

The system (EMFG) was introduced by Lions and Souganidis in [20], who coined the term extended MFG, to simultaneously study several MFG type problems for which, in contrast to the case of standard MFG, the vector field B does not necessarily equal mD_pH. It was shown in [20] that (EMFG) has at most one classical solution if

$$-4H_m D_p B > (B_m - D_pH) \otimes (B_m - D_pH). \tag{1.1}$$

In the case of standard MFG with a *separated* Hamiltonian, $H \equiv H(x,p) - f(x,m)$, (1.1) simply reduces to the standard convexity assumption for $H(x,p)$ in the second variable and the monotonicity of f.

Before stating our results, we briefly describe some of the existing work on the well-posedness and regularity of (EMFG). For the case of standard MFG with a separated Hamiltonian, there exists a complete theory of weak solutions (developed by Cardaliaguet, Graber, Porretta, and Tonon [6–8] in the degenerate case $g_m \equiv 0$, that is, when g is independent of m, and by the author [22] in the non-degenerate case considered here). Moreover, it was shown by the author, in [22], that the solutions are classical under the coercivity assumption

$$\lim_{m\to 0^+} H(x,p,m) = +\infty. \tag{1.2}$$

From the optimal control perspective, (1.2) corresponds to placing a very strong incentive for players to occupy low-density regions, and this forces $m > 0$. For first order MFG systems with congestion, weak solutions were shown to exist by Porretta and Achdou [2], but classical solutions had not been obtained so far. Second order MFG systems with congestion were also studied in [1, 9, 13, 15], where weak solutions and short-time existence result in the smooth setting have been obtained. Finally, for second order mean field type control problems with congestion, weak solutions were obtained in [3]. To put our results in context, it is important to observe that assumption (1.2), and it was not made in [1, 2]. This assumption is a significant restriction, and it is critical to our methods, as it ensures the strict positivity of the density, and hence the classical solutions.

In this paper, we follow the same methodology used in [22]. Our contribution is stated below. It is a general result that yields classical solutions to (EMFG), assuming the strict form of the uniqueness condition (1.1), as well as growth assumptions on H and B which are modeled by Hamiltonians of the type

$$H = \psi(m)|p|^\gamma, \ \ \gamma > 1, \ \psi > 0, \ \psi' \le 0. \tag{1.3}$$

We remark that, in particular, we do not require the Hamiltonian to be quadratic or separated, and one of the applications of Theorem 1.1 is the existence of classical solutions to MFG systems with congestion. We refer to Section 2 for the exact assumptions (M), (H), (B) , (G), and (E).

Theorem 1.1. *Let $0 < s < 1$, and assume that (M), (H), (B), (G), and (E) hold. Then there exists a unique classical solution $(u,m) \in C^{3,s}(\overline{Q_T}) \times C^{2,s}(\overline{Q_T})$ to* (EMFG)*.*

An important natural setting, covered by Theorem 1.1 in full generality, is when the derivatives H_m and D_xH satisfy growth conditions that are compatible with (1.3), which we write symbolically as

$$mH_m \sim H \sim \psi(m)|p|^\gamma, \text{ and} \tag{1.4}$$

$$|D_xH| \lesssim \psi(m)|p|^\gamma. \tag{1.5}$$

When $\lim_{m\to\infty}\psi(m)=0$, these correspond to MFG systems with congestion, of which a typical example is

$$\begin{cases}-u_t+\frac{|D_xu|^2}{2(m+c_0)^\alpha}-V(x)=f(m), & u(x,T)=g(x,m(x,T))\\ m_t-\operatorname{div}(\frac{m}{(m+c_0)^\alpha}D_xu)=0, & m(0,x)=m_0(x)\end{cases}\tag{1.6}$$

where the conditions $0<\alpha<2$ and $c_0\geq 0$ ensure that (1.1) and, hence, uniqueness, holds for (1.6) (see [1]). As was mentioned above, for our results to apply, condition (1.2) is essential, and in this example it amounts to requiring that $\lim_{m\to 0}f(m)=-\infty$. In particular, Theorem 1.1 does not apply in several important examples such as when $f\equiv 0$ or $f\equiv m^k$, $k>0$, which illustrates the restrictive character of (1.2), in contrast to the results of [1, 2].

Now, Theorem 1.1 also allows for a more general growth behavior than (1.4), namely, for a constant $0\leq\gamma_1\leq\gamma$,

$$mH_m\sim\psi(m)|p|^{\gamma_1}.$$

One reason for working under such generality is that, despite (1.4) being the natural condition for fully general Hamiltonians $H(x,p,m)$ satisfying (1.3), it is not satisfied by the important example of MFG systems with a separated Hamiltonian. Such systems are nevertheless covered in Theorem 1.1, by setting $\gamma_1=0$.

There are, however, two assumptions that must be strengthened when straying from (1.4). The first is that, whereas in the case $\gamma_1=\gamma$, our result allows for "congestions" in which $\lim_{m\to\infty}\psi(m)=0$, when $\gamma_1<\gamma$ we must instead assume this limit to be positive. The second assumption is the control required for the x-dependence, and can be explained by comparing Theorem 1.1 with Theorem 1.1 of [22]. In the latter work, we obtained classical solutions in the case of separated Hamiltonians, only requiring for $D_xH(x,p)$ the condition

$$|D_xH(x,p)|\lesssim|p|^{\gamma-\epsilon},\quad \epsilon>0,$$

which allows the growth of D_xH to be arbitrarily close to the natural one (1.5). This was achieved by exploiting in a crucial way the separated structure of the system. On the other hand, in (EMFG), no such structure is available, and therefore treating fully coupled Hamiltonians with $0\leq\gamma_1<\gamma$ forces us to impose the stricter control

$$|D_xH|\lesssim\psi(m)|p|^{\gamma_2},$$

where γ_2 must satisfy $\gamma_2<2\gamma_1-\gamma+2$. In other words, in the absence of additional structural assumptions, the more the growth γ_1 of H_m deviates from its natural value γ, the more we must restrict the growth γ_2 of the space oscillation D_xH.

We will discuss now our methods of proof. The key insight that allows us to obtain classical solutions to this first order system is the observation of Lions that, due to the strict monotonicity of H with respect to m, one can eliminate the variable m and transform (EMFG) into a second order quasilinear equation in u with an oblique, non-linear boundary condition,

$$\begin{cases}Qu:=-\operatorname{Tr}(A(x,Du)D^2u)+b(x,Du)=0 & \text{in } Q_T,\\ Nu:=N(x,t,u,Du)=0 & \text{on } \partial Q_T,\end{cases}\tag{Q}$$

where $Du=(D_xu,u_t)$ and, for $(x,z,p,s)\in\mathbb{T}^d\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$,

$$A(x,p,s)=\left(\frac{B_m+D_pH}{2},-1\right)\otimes\left(\frac{B_m+D_pH}{2},-1\right)-\begin{pmatrix}\frac{B_m-D_pH}{2}\otimes\frac{B_m-D_pH}{2}+H_mD_pB & 0\\ 0 & 0\end{pmatrix},\tag{Q1}$$

$$b(x,p,s) = -D_xH(x,p,H^{-1})\cdot B_m(x,p,H^{-1}) + H_m(x,p,H^{-1})\mathrm{div}_x B(x,p,H^{-1}), \tag{Q2}$$

$$N(x,0,z,p,s) = -s + H(x,p,m_0(x)),\ N(x,T,z,p,s) = -g(x,H^{-1}(x,p,s)) + z, \tag{N}$$

and the function $H^{-1}(x,p,s)$ is the inverse of H with respect to m, defined by

$$H^{-1}(x,p,H(x,p,m)) = m.$$

An important observation that can be seen directly from the definition of A is that this equation is elliptic precisely when (1.1) holds. For this reason, it is to be expected that the methods of quasilinear elliptic equations with oblique boundary conditions, which were successful in obtaining classical solutions to standard MFG systems in [19, 22], may also be applied in this more general setting. This is in fact the approach that we follow here. Namely, we obtain a priori estimates for $||u||_{C^0(\overline{Q_T})}$ and $||Du||_{C^0(\overline{Q_T})}$, and conclude the existence of smooth solutions from the classical $C^{1,\alpha}$ estimates for oblique derivative problems (see [17]), the Schauder theory for linear oblique problems (see [12, 18]), and the non-linear method of continuity (see [12]).

Finally, we discuss some of the newer results that have been obtained after the first version of this work, as well as possible future directions. In [23], A. Porretta showed that one may still obtain classical solutions to standard MFG with a separated Hamiltonian, when $\mathbb{T}^d$ is replaced by a region in $\mathbb{R}^d$, in the setting of the so-called planning problem, where the terminal density is a prescribed function. On the other hand, in [21], N. Mimikos and the author showed that, when $d = 1$, the key coercivity assumption (1.2) may be removed, and classical solutions are obtained both in the setting of (EMFG) and the planning problem. It was also shown that one may weaken the assumption that m_0 be strictly bounded away from 0, and still obtain instant regularization for times $t > 0$, despite the loss of ellipticity at the initial time. However, it remains an open question whether the results of [21] may be extended to dimensions greater than 1 and, even in the case $d = 1$, whether one may allow m_0 to vanish in a set of positive measure.

Remark 1.2. We note here that there is some ambiguity with the term *extended mean field games*, because it is also used to refer to standard MFG systems in which the Hamiltonian depends on the acceleration of the players (see, for instance, [14]). This setting is unrelated to the one present in this work and in [20].

Notation

Let $n, k \in \mathbb{N}$. Given $x, y \in \mathbb{R}^n$, x and y will always be understood to be row vectors, and their scalar product xy^T will be denoted by $x \cdot y$. For $0 < s < 1$, $C^{k,s}(\overline{Q_T})$ refers to the space of k times differentiable real-valued functions with s–Hölder continuous k^{th} order derivatives. If $u \in C^1(\overline{Q_T})$, the notation Du will always refer to the full gradient in all variables, whereas D_xu denotes the gradient in the space variable only. We write $C = C(K_1, K_2, \dots, K_M)$ for a positive constant C depending monotonically on the non-negative quantities $K_1, \dots, K_M$.

2. Assumptions

In what follows, $C_0, \gamma, \gamma_1, \gamma_2$ are fixed constants satisfying

$$C_0 > 0,\ \ \gamma > 1,\ \ \gamma_1 \geq 0,\ \ \gamma_2 \leq \gamma_1 \leq \gamma,\ \ \text{and}\ \ \gamma_2 < 2\gamma_1 - \gamma + 2. \tag{2.1}$$

The continuous functions $\overline{C},\ \psi : (0,\infty) \to (0,\infty)$ are also fixed, with ψ being non-increasing. If $\gamma_1 < \gamma$, we further require that

$$\lim_{m\to\infty} \psi(m) > 0. \tag{2.2}$$

We note that the case $\gamma_1 = 0$, $\psi \equiv 1$, corresponds to a standard MFG system with a separated Hamiltonian, and the case

$$\gamma_1 = \gamma, \;\; \psi(m) \equiv \frac{1}{(m + c_0)^\alpha}$$

corresponds to a MFG system with congestion.

(M) (Assumptions on m_0) The initial density m_0 satisfies

$$m_0 \in C^4(\mathbb{T}^d),\; m_0 > 0, \;\text{and}\; \int_{\mathbb{T}^d} m_0 = 1. \tag{M1}$$

(H) (Assumptions on H) The function $H : \mathbb{T}^d \times \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ is four times continuously differentiable and satisfies $H_m < 0$. Moreover, for $(x, p, m) \in \mathbb{T}^d \times \mathbb{R}^d \times (0, \infty)$,

$$\frac{1}{C_0}\psi(m)(1 + |p|)^{\gamma-2} I \le D^2_{pp} H \le C_0\psi(m)(1 + |p|)^{\gamma-2} I, \tag{H1}$$

$$|D_p H| \le C_0\psi(m)(1 + |p|)^{\gamma-1}, \quad D_p H \cdot p \ge \left(1 + \frac{1}{C_0}\right) H - \overline{C}(m), \tag{H2}$$

$$\frac{1}{C_0}\psi(m)|p|^{\gamma_1} \le -mH_m \le C_0\psi(m)|p|^{\gamma_1} + \overline{C}(m), \tag{HM1}$$

$$|mH_{mm}| \le -C_0 H_m, \quad |p||D_p H_m| \le C_0\psi(m)(1 + |p|)^{\gamma_1}, \tag{HM2}$$

$$|D_x H|,\; |D^2_{xx} H| \le C_0\psi(m)(1 + |p|)^{\gamma_2},\;\; |D^2_{xp} H| \le C_0\psi(m)(1 + |p|)^{\gamma_2-1}, \tag{HX1}$$

$$m|D_x H_m| \le C_0\psi(m)(1 + |p|)^{\gamma_2}, \tag{HX2}$$

$$|D_x H(x, 0, m)| \le C_0. \tag{HX3}$$

(B) (Assumptions on B) The function $B : \mathbb{T}^d \times \mathbb{R}^d \times [0, \infty) \to \mathbb{R}^d$ is four times continuously differentiable, $B(\cdot, \cdot, 0) \equiv 0$ and, mirroring the assumptions on H, B satisfies, for $(x, p, m) \in \mathbb{T}^d \times \mathbb{R}^d \times (0, \infty)$,

$$\frac{1}{C_0} m\psi(m)|p|^{\gamma-2} I \le D_p B \le C_0 m\psi(m)(1 + |p|)^{\gamma-2} I, \tag{B1}$$

$$|B_m| \le C_0\psi(m)(1 + |p|)^{\gamma_1-1}, \quad |p||D_p B_m| \le C_0\psi(m)(1 + |p|)^{\gamma_1-2}, \quad |D^2_{pp} B| \le C_0 m\psi(m)(1 + |p|)^{\gamma-3}, \tag{B2}$$

$$(1 + |p|)|B_{mm}| \le -C_0 H_m, \tag{BM}$$

$$|D_x B|, |D^2_{xx} B| \le mC_0\psi(m)(1 + |p|)^{\gamma_2-1},\;\; |D_x B_m| \le C_0\psi(m)(1 + |p|)^{\gamma_2-1}, \tag{BX1}$$

$$|D^2_{xp} B| \le C_0 m\psi(m)(1 + |p|)^{\gamma_2-2}, \tag{BX2}$$

$$|D_x B(x,0,m)| \leq C_0 m. \tag{BX3}$$

(G) (Assumptions on g) The function $g : \mathbb{T}^d \times (0,\infty) \to \mathbb{R}$ is four times continuously differentiable and satisfies $g_m > 0$. Furthermore, for each $x \in \mathbb{T}^d$,

$$\lim_{m\to\infty} g(x,m) = \sup_{\mathbb{T}^d\times[0,\infty)} g, \text{ and } \lim_{m\to 0^+} g(x,m) = \inf_{\mathbb{T}^d\times[0,\infty)} g. \tag{GX}$$

(E) (Strict ellipticity of the system) The functions H and B satisfy the conditions

$$\lim_{m\to 0^+} H(x,p,m) = +\infty \text{ uniformly in } (x,p) \in \mathbb{T}^d \times \mathbb{R}^d, \tag{E1}$$

$$\lim_{m\to\infty} H(x,p,m) - C_0\psi(m)|p|^\gamma = -\infty \text{ uniformly in } (x,p) \in \mathbb{T}^d \times \mathbb{R}^d, \tag{E2}$$

$$-4H_m D_p B \geq \left(1 + \frac{1}{C_0}\right)(B_m - D_p H) \otimes (B_m - D_p H). \tag{E3}$$

Remark 2.1. In view of (H1), up to increasing the values of C_0 and $\overline{C}$, we may assume, with no loss of generality, that, for $(x,p,m) \in \mathbb{T}^d \times \mathbb{R}^d \times (0,\infty)$,

$$\frac{\psi(m)}{C_0}|p|^\gamma - \overline{C}(m) \leq H(x,p,m) \leq \psi(m)C_0|p|^\gamma + \overline{C}(m). \tag{2.3}$$

$$\max_{[\min m_0, \max m_0]} |H(x,0,\cdot)| \leq C_0, \quad \max_{[\min m_0, \max m_0]} |B(x,0,\cdot)| \leq C_0. \tag{2.4}$$

Moreover, in the case that (2.2) holds, we may also write

$$\psi(m) \geq \frac{1}{C_0}. \tag{2.5}$$

We also note that there is certainly room for weaking the regularity assumptions on the data, at the expense of further technical complications. We refer to [21] for an illustration of this.

3. A priori estimates and classical solutions

3.1. Derivation of the quasilinear equation

We begin by briefly showing the equivalence between the first-order system (EMFG) and the elliptic equation (Q), since the latter will be our main object of analysis in the following sections.

Proposition 3.1. *Let $(u,m) \in C^2(\overline{Q_T}) \times C^1(\overline{Q_T})$. Then (u,m) is a solution to (EMFG) if and only if u is a solution to (Q), and m is given by*

$$m = H^{-1}(x, D_x u, u_t). \tag{3.1}$$

Proof. The Hamilton-Jacobi equation

$$-u_t + H(x, D_x u, m) = 0$$

may be rewritten as (3.1). We thus need to show that, after substituting (3.1) in the continuity equation

$$m_t - \operatorname{div}(B(x, D_x u, m)) = 0,$$

one obtains (Q). Indeed, the substitution yields

$$\begin{aligned} 0 = \frac{1}{H_m}(u_{tt} - D_p H \cdot D_x u_t) - \operatorname{div}_x B - \operatorname{Tr}(D_p B D^2_{xx} u) - B_m \cdot \operatorname{div}_x(H^{-1}) \\ = \frac{1}{H_m}(u_{tt} - D_p H \cdot D_x u_t) - \operatorname{div}_x B - \operatorname{Tr}(D_p B D^2_{xx} u) - \frac{1}{H_m} B_m \cdot (-D_x H + D_p H D^2_{xx} u + D_x u_t), \end{aligned}$$

that is,

$$R + b(x, Du) = 0, \tag{3.2}$$

where the first order term $b(x, Du)$ is given by (Q2), and the second order term R is

$$R = -u_{tt} + (B_m + D_p H) \cdot D_x u_t - B_m D^2_{xx} u \cdot D_p H + H_m \operatorname{Tr}(D_p B D^2_{xx} u).$$

Now, R may be rewritten as

$$\begin{aligned} R = -u_{tt} + 2\frac{B_m + D_p H}{2} \cdot D_x u_t - \frac{B_m + D_p H}{2} D^2_{xx} u \cdot \frac{B_m + D_p H}{2} \\ + \frac{B_m - D_p H}{2} D^2_{xx} u \cdot \frac{B_m - D_p H}{2} + H_m \operatorname{Tr}(D_p B D^2_{xx} u) = -\left(\frac{B_m + D_p H}{2}, -1\right) D^2 u \cdot \left(\frac{B_m + D_p H}{2}, -1\right) \\ + \frac{B_m - D_p H}{2} D^2_{xx} u \cdot \frac{B_m - D_p H}{2} + H_m \operatorname{Tr}(D_p B D^2_{xx} u) = -\operatorname{Tr}(A D^2 u). \end{aligned} \tag{3.3}$$

where A is given by (Q1). Substituting (3.3) in (3.2) thus yields the desired elliptic equation. As for the boundary conditions, we may rewrite the initial and terminal conditions in (EMFG) as

$$H^{-1}(x, D_x u(x,0), u_t(x,0)) = m_0(x), \;\; g(x, H^{-1}(x, D_x u(x,T), u_t(x,T))) = u(x,T),$$

that is,

$$N(x, t, u, D_x u, u_t) = 0, \;\; (x,t) \in \mathbb{T}^d \times \{0, T\},$$

where N is given by (N). $\square$

In view of Proposition 3.1, (EMFG) and (Q) will be treated tacitly as the same problem throughout the rest of the paper.

3.2. Estimates for the solution and the terminal density

In the first result of this section, Lemma 3.2, we will estimate the L^∞ norms of u and the terminal density $m(\cdot, T)$, where (u,m) is a solution to (EMFG). In order to provide an explicit form for the estimates of this section, we consider the continuous, strictly increasing functions $f_0, f_1, g_0, g_1 : (0,\infty) \to \mathbb{R}$ defined by

$$f_0(m) = \min_{x\in\mathbb{T}^d} \left(-H(x,0,m)\right), \quad f_1(m) = \max_{x\in\mathbb{T}^d} \left(-H(x,0,m)\right),$$
$$g_0(m) = \min_{\mathbb{T}^d} g(\cdot,m), \quad g_1(m) = \max_{\mathbb{T}^d} g(\cdot,m),$$

and the non-decreasing function $h : (0,\infty) \to [0,\infty)$ by

$$h(s) = \sup\{m > 0 : \sup_{(x,p)\in\mathbb{R}^d} H(x,p,m) - C_0|p|^\gamma \psi(m) \geq -s\}, \tag{3.4}$$

which is well-defined in view of (E2).

Lemma 3.2. *There exists $C = C(C_0)$ such that, for any solution $(u,m) \in C^2(\overline{Q_T}) \times C^1(\overline{Q_T})$ of (EMFG), and every $(x,t) \in \overline{Q_T}$,*

$$g_0 f_1^{-1}(-C) - C(e^{CT} - e^{Ct}) \leq u(x,t) \leq g_1 f_0^{-1}(C) + C(e^{CT} - e^{Ct}), \text{ and} \tag{3.5}$$

$$0 < g_1^{-1} g_0 f_1^{-1}(-C) \leq m(x,T) \leq g_0^{-1} g_1 f_0^{-1}(C), \tag{3.6}$$

Proof. The proof of this statement is analogous to Lemma 3.1, Corollary 3.2 of [22]. We modify the function u in a way that ensures that its maximum value is achieved at $t = T$. This will allow us to conclude by exploiting the fact that the boundary condition (N) that holds at the terminal time is of "Robin type". For this purpose, we set $v(x,t) = u(x,t) + \zeta(t)$, where $\zeta(t) = M(e^{Mt} - e^{MT})$, for a large parameter $M > 1$. Conditions (HM2) and (BM) imply, respectively, the existence of a uniform Lipschitz bound for the maps $w \mapsto H^{-1}(x,0,w)H_m(x,0,H^{-1}(x,0,w))$ and $w \mapsto B_m(x,0,H^{-1}(x,0,w))$. Therefore, using (2.4), we obtain

$$|mH_m(x,0,m)| \leq C(1 + |H(x,0,m)|) \text{ and } |B_m(x,0,m)| \leq C(1 + |H(x,0,m)|).$$

In view of this, (Q), (Q2), (HX3), and (BX3) yield that, at any interior critical point (x,t) of v,

$$\begin{aligned} &-\mathrm{Tr}(A(x,Du)D^2v) = -\mathrm{Tr}(A(x,Du)D^2u) - \zeta''(t) = -b(x,Du) - \zeta''(t) = D_xH(x,0,m)\cdot B_m(x,0,m) \\ &- H_m(x,0,m)\mathrm{div}_x B(x,0,m) - \zeta''(t) \leq C(1+|u_t|) - \zeta''(t) = C(1+\zeta'(t)) - \zeta''(t) \leq C(1 + M^2e^{Mt}) - M^3e^{Mt}. \end{aligned}$$

Thus, if $M > \max(1,2C)$, one has $-\mathrm{Tr}(A(x,Du)D^2v) < 0$ at all interior critical points of v, and therefore v must achieve its maximum value on the boundary ∂Q_T. If the maximum is achieved at $t = 0$, then $D_xv = 0$, $v_t \leq 0$, and so

$$M^2 = \zeta'(0) \leq -u_t = -H(x,0,m_0(x)) \leq C_0.$$

Consequently, a sufficiently large value of M forces the maximum to be achieved at $\{t = T\}$. At such a point (x,T), $D_xv = 0$, $v_t \geq 0$, and, thus, since $\zeta(T) = 0$,

$$M^2e^{MT} = \zeta'(T) \geq -u_t = -H(x,0,m(x,T)) \geq f_0(m(x,T)) = f_0(g^{-1}(x,u(x,T)) = f_0(g^{-1}(x,v(x,T)),$$

which yields $v(x,T) \leq g_1 f_0^{-1}(M^2 e^{MT})$. Since (x,T) is a maximum point of $v = u + \zeta$, this proves the upper bound in (3.5), with the lower bound being obtained through the same reasoning.

The second inequality (3.6) then follows immediately by setting $t = T$ in (3.5), using the fact that, by (GX), the functions g_0 and g_1 have the same range. □

3.3. An overview of the Bernstein argument

To obtain the gradient estimate, we will make use of a classical method due to S. Bernstein (see [4]), for which we will need to use the linearization of (Q), namely

$$L_u(v) = -\mathrm{Tr}(A(x,Du)D^2 v) - D_q \mathrm{Tr}(A(x,Du)D^2 u)\cdot Dv + D_q b(x,Du)\cdot Dv, \tag{3.7}$$

where, for $(p,s) \in \mathbb{R}^d \times \mathbb{R}$, we denote $q = (p,s)$. The idea behind this classical method is the following general principle about elliptic equations: convex functions $\phi(u)$ and $\Phi(Du)$ of the solution and its gradient are subsolutions of the linearized equation, up to an error that can often be controlled. More precisely, one has

$$L_u(\phi(u)) = -\phi'' DuA\cdot Du + E_1, \quad L_u(\Phi(Du)) = -\mathrm{Tr}(D^2\Phi D^2 u A D^2 u) + E_2, \tag{3.8}$$

where E_1 and E_2 are regarded as error terms to be estimated. This observation can be exploited to bound $||D_x u||_{C^0(\overline{Q_T})}$ as follows. Since $||u||_{C^0(\overline{Q_T})}$ is already known to be bounded a priori, the problem is equivalent to bounding $v = \phi(u) + \Phi(Du)$, as long as $\Phi : \mathbb{R}^d \to \mathbb{R}$ is coercive. At any interior maximum point (x,t) of v, one thus has

$$0 \leq L_u(v) = -\phi'' DuA\cdot Du - \mathrm{Tr}(D^2\Phi D^2 u A D^2 u) + E_1 + E_2,$$

that is,

$$\phi'' DuA\cdot Du \leq -\mathrm{Tr}(D^2\Phi D^2 u A D^2 u) + E_1 + E_2. \tag{3.9}$$

Thus, up to adequately estimating the error $E_1 + E_2$ in terms of the other two dominant signed terms, (3.9) leads naturally to a gradient bound.

Now, we must note that the argument above applies only to interior maxima, so the possibility of the maximum being achieved on ∂Q_T must be accounted for separately. In the usual case of Dirichlet boundary conditions, the bound would follow automatically since $u|_{\partial Q_T}$ would be an a priori given function, but since (N) defines an oblique boundary condition instead, an additional argument must be made. One can proceed by linearizing the boundary operator N and repeating the Bernstein process for this first order operator in place of L_u. Just like in the case of (3.8), the linearization is computed by differentiating both sides of the boundary equation. Whereas the ellipticity of (Q) is what allows $E_1 + E_2$ in (3.9) to be estimated, the error at the boundary is instead controlled with the dominant signed term $D_p H \cdot D_x u$ by virtue of the superlinear growth (H2) of H, the existing bounds on $m|_{\partial Q_T}$ and the non-degeneracy of the boundary condition. Indeed, bounds on $m(\cdot,0)$ and $D_x m(\cdot,0)$ are available because m_0 is given a priori, and Lemma 3.2 provides bounds for $m(\cdot,T)$, albeit not for $D_x m(\cdot,T)$. The error terms that involve $D_x m(\cdot,T)$ have, however, a favorable sign thanks to the "Robin type" nature of (N) at time T that comes from the strict monotonicity of g.

3.4. Estimates for the space-time gradient

To carry out the strategy described above for the gradient estimate, we will require explicit computations of the error terms E_1 and E_2 described in (3.8), provided by the following lemma. We remind the reader that $\cdot$ denotes the standard dot product, and all vectors are taken to be rows.

Lemma 3.3. *Let $\Phi(p,s)\in C^2(\mathbb{R}^{d+1})$, assume that $(u,m)\in C^3(\overline{Q_T})\times C^2(\overline{Q_T})$ solves (EMFG), and set $v(x,t)=\Phi(Du(x,t))$. For each $q=(p,s)\in\mathbb{R}^{d+1}$, and for each $(x,t)\in Q_T$, define*

$$\zeta(p,s)=-s+D_pH(x,D_xu,m)\cdot p,\;\; Y^+=B_m+D_pH,\;\; Y^-=B_m-D_pH$$

Then the following identities hold:

$$\begin{aligned} D_q\mathrm{Tr}(AD^2u)\cdot q=&(-D_xu_t+\frac{1}{2}Y^+D^2_{xx}u)(D_pY^+p^T-(Y^+_m)^TH_m^{-1}\zeta)+\frac{1}{2}Y^-D^2_{xx}u(D_pY^-p^T-(Y^-_m)^TH_m^{-1}\zeta)\\ &-(D_pH_m\cdot p-H_{mm}H_m^{-1}\zeta)\mathrm{Tr}(D_pBD^2_{xx}u)-H_m(D_p\mathrm{Tr}(D_pBD^2_{xx}u)\cdot p-\mathrm{Tr}(D_pB_mD^2_{xx}u)H_m^{-1}\zeta), \end{aligned} \tag{3.10}$$

$$\begin{aligned} D_qb(x,Du)\cdot q=&-B_m(D^2_{xp}Hp^T-D_xH_m^TH_m^{-1}\zeta)-D_xH(D_pB_mp^T-B_{mm}^TH_m^{-1}\zeta)\\ &+H_m(D_p\mathrm{div}_xB\cdot p-\mathrm{div}_xB_mH_m^{-1}\zeta)+\mathrm{div}_xB(D_pH_m\cdot p-H_{mm}H_m^{-1}\zeta), \end{aligned} \tag{3.11}$$

$$\begin{aligned} \mathrm{Tr}(A_{x_i}D^2u)=&(-D_xu_t+\frac{1}{2}Y^+D^2_{xx}u)\cdot(Y^+_{x_i}-Y^+_mH_m^{-1}H_{x_i})+\frac{1}{2}Y^-D^2_{xx}u\cdot(Y^-_{x_i}-Y^-_mH_m^{-1}H_{x_i})\\ &-(H_{x_im}-H_{mm}H_m^{-1}H_{x_i})\mathrm{Tr}(D_pBD^2_{xx}u)-H_m(\mathrm{Tr}(D_pB_{x_i}D^2_{xx}u)-\mathrm{Tr}(D_pB_mD^2_{xx}u)H_m^{-1}H_{x_i}), \end{aligned} \tag{3.12}$$

$$\begin{aligned} D_xb(x,Du)\cdot p=&-B_m(D^2_{xx}Hp^T-D_xH_m^TH_m^{-1}(D_xH\cdot p))-D_xH(D_xB_mp^T-B_{mm}H_m^{-1}(D_xH\cdot p))\\ &+H_m(D_x\mathrm{div}_x(B)\cdot p-\mathrm{div}_xB_mH_m^{-1}(D_xH\cdot p))+\mathrm{div}_xB(D_xH_m\cdot p-H_{mm}H_m^{-1}(D_xH\cdot p)), \end{aligned} \tag{3.13}$$

$$L_uv=-\mathrm{Tr}(D^2\Phi D^2uAD^2u)+\sum_{i=1}^d\mathrm{Tr}(A_{x_i}D^2u)\Phi_{p_i}-D_p\Phi\cdot D_xb. \tag{3.14}$$

Proof. We derive (3.10) differentiating the expressions (Q1) with respect to $q=(p,s)$. Indeed, (Q1) implies that

$$\begin{aligned} D_q\,Tr(AD^2u)\cdot q=&D_q(\mathrm{Tr}\left(\left(\frac{1}{4}(Y^+-Y^-)\otimes(Y^+-Y^-)-H_mD_pB\right)D^2_{xx}u\right)-Y^+D_xu_t)\cdot(p,s)\\ =&D_q(\frac{1}{4}Y^+D^2_{xx}u\cdot Y^++\frac{1}{4}Y^-D^2_{xx}u\cdot Y^--H_m\mathrm{Tr}(D_pBD^2_{xx}u)-Y^+D_xu_t)\cdot(p,s)\\ =&\frac{1}{2}Y^+D^2_{xx}u(D_pY^+p^T-(Y^+_m)^TH_m^{-1}\zeta)+\frac{1}{2}Y^-D^2_{xx}u(D_pY^-p^T-(Y^-_m)^TH_m^{-1}\zeta)\\ &-(D_pH_m\cdot p-H_{mm}H_m^{-1}\zeta)\mathrm{Tr}(D_pBD^2_{xx}u)-H_m(D_p\mathrm{Tr}(D_pBD^2_{xx}u)\cdot p\\ &-\mathrm{Tr}(D_pB_mD^2_{xx}u)H_{mm}H_m^{-1}\zeta)-D_xu_t(D_pY^+p^T-Y^+_mH_m^{-1}\zeta)\\ =&(-D_xu_t+\frac{1}{2}Y^+D^2_{xx}u)(D_pY^+p^T-(Y^+_m)^TH_m^{-1}\zeta)+\frac{1}{2}Y^-D^2_{xx}u(D_pY^-p^T-(Y^-_m)^TH_m^{-1}\zeta)\\ &-(D_pH_m\cdot p-H_{mm}H_m^{-1}\zeta)\mathrm{Tr}(D_pBD^2_{xx}u)-H_m(D_p\mathrm{Tr}(D_pBD^2_{xx}u)\cdot p-\mathrm{Tr}(D_pB_mD^2_{xx}u)H_m^{-1}\zeta). \end{aligned}$$

Similarly, (3.11) follows by differentiating (Q2) with respect to q, and (3.12) and (3.13) result from differentiating (Q1) and (Q2) with respect to the space variables. Finally, (3.14) is obtained by applying $D\Phi \cdot D$ to both sides of (Q) (see, for instance, [22], Lem. 3.4). □

We can now obtain the a priori gradient bound in terms of bounds for the solution u and the terminal density $m(\cdot, T)$, which were already obtained in Section 3.2.

Lemma 3.4. *Let $(u, m) \in C^3(\overline{Q_T}) \times C^2(\overline{Q_T})$ be a solution to (EMFG), and set*

$$M = ||m||_{C^0(\partial Q_T)} + ||m^{-1}||_{C^0(\partial Q_T)}. \tag{3.15}$$

There exist constants $C, C_1 > 0$, with

$$C = C(C_1, ||(\psi \circ h)^{-1}||_{C^0(0,C_1]}), \quad C_1 = C_1(C_0, T, T^{-1}, ||u||_{C^0(\overline{Q_T})}, M, ||\overline{C}||_{C^0[\frac{1}{M},M]}, ||\psi||_{C^0[\frac{1}{M},M]},$$
$$||\psi^{-1}||_{C^0[\frac{1}{M},M]}, ||D_x g||_{C^0(\mathbb{T}^d \times [\frac{1}{M},M])}, (2\gamma_1 - \gamma + 2 - \gamma_2)^{-1})$$

such that

$$||Du||_{C^0(\overline{Q_T})} \leq C.$$

Proof. We will consider first, for the sake of clarity, the natural case where $\gamma_1 = \gamma_2 = \gamma$. C will denote a constant that is allowed to increase from line to line. First, we verify that it is sufficient to bound the space gradient. Indeed, setting $\Phi(p, s) = s$ in Lemma 3.3 yields

$$L_u(u_t) = L_u(\Phi(Du)) = 0,$$

and, thus, in view of the maximum principle and (2.3),

$$-C \leq u_t \leq C||D_x u||^{\gamma}_{\overline{Q_T}} + C. \tag{3.16}$$

We note that, in (3.16), the constant C already depends on the upper and lower bounds for m on ∂Q_T. Next, we will estimate $||D_x u||_{\overline{Q_T}}$ through the Bernstein method. Let

$$T_u v = -v_t + D_p H(x, D_x u, m) D_x v, \quad \widetilde{u} = u + ||u||_{C^0(\overline{Q_T})} + 1 - \frac{2(||u||_{C^0(\overline{Q_T})} + 1)}{T}(T - t),$$

and note that the function $\tilde{u}$ has been constructed to satisfy

$$|\tilde{u}| \leq C, \quad \widetilde{u}(\cdot, 0) \leq -1, \ \widetilde{u}(\cdot, T) \geq 1. \tag{3.17}$$

Setting

$$k = ||D_x u||^{3/2}_{\overline{Q_T}}, \quad v(x,t) = \frac{k}{2}\widetilde{u}^2 + \frac{1}{2}|D_x u|^2, \tag{3.18}$$

we observe that the quantities $||D_x u||^2_{\overline{Q_T}}$ and $||v||_{\overline{Q_T}}$ are comparable up to constants, so it is therefore sufficient to obtain a bound for the latter.

Let $(x_0,t_0)\in\overline{Q_T}$ be a point where v achieves its maximum value, and set $p=D_xu(x_0,t_0)$. We assume with no loss of generality that

$$|p|\geq 1 \quad\text{and}\quad ||D_xu||^{1/2}_{\overline{Q_T}}\geq 2||\tilde{u}||^2_{\overline{Q_T}}. \tag{3.19}$$

The latter condition ensures that

$$\frac{1}{2}|p|^2\geq\frac{1}{2}||D_xu||^2_{\overline{Q_T}}-\frac{k}{2}||\tilde{u}||^2_{\overline{Q_T}}\geq\frac{1}{4}||D_xu||^2_{\overline{Q_T}}. \tag{3.20}$$

Since the maximum may be achieved at the boundary of Q_T, we must distinguish three cases.

Case 1: $t_0=T$. Then $D_xv=0$, $v_t\geq 0$. Therefore, in view of (2.3), (3.17), (HM1), (H2), (3.20), and the current assumption that $\gamma_1=\gamma_2=\gamma$,

$$\begin{aligned}0\geq T_uv=T_u\left(\frac{1}{2}|D_xu|^2\right)&+k\tilde{u}(-\widetilde{u}_t+D_pH\cdot D_xu)\\
&=-\frac{H_m}{g_m}(|p|^2-D_xg\cdot p)-D_xH\cdot p+k\tilde{u}(-u_t+D_pH\cdot p-C)\geq\frac{\psi(m(T))}{C_0m(T)g_m}|p|^{\gamma+2}\\
&-\left(C_0\frac{\psi(m(T))}{m(T)g_m}|p|^{\gamma}+\overline{C}(m(T))\right)|D_xg||p|-C\psi(m(T))|p|^{\gamma+1}+k\tilde{u}\left(\frac{1}{C}\psi(m(T))|p|^{\gamma}-C\right)\\
&\geq\frac{1}{C}|p|^{\gamma+3/2}+\frac{\psi(m(T))}{C_0m(T)g_m}|p|^{\gamma+2}-C(1+|p|^{\gamma+1}).\end{aligned}$$

Thus, since the second term is non-negative, we obtain

$$\frac{1}{C}|p|^{\gamma+3/2}\leq C(1+|p|^{\gamma+1}),$$

which yields

$$|D_xu|\leq C.$$

Case 2: $t_0=0$. Similarly to the first case, we obtain $D_xv=0$, $v_t\leq 0$, and so

$$\begin{aligned}0\leq T_uv=&T_u\left(\frac{1}{2}|D_xu|^2\right)+k\tilde{u}(-\widetilde{u}_t+D_pH\cdot D_xu)\\
=&-H_mD_xm_0(x)\cdot p-D_xH\cdot p+k\tilde{u}(-u_t+D_pH\cdot p-C)\\
\leq&Cm_0^{-1}(|p|^{\gamma}\psi(m_0)+\overline{C}(m_0))|p|+C\psi(m_0)|p|^{\gamma+1}+k\tilde{u}(\frac{1}{C_0}\psi(m_0)|p|^{\gamma}-C)\\
\leq&-\frac{1}{C}\psi(m_0)|p|^{\gamma+3/2}+C(1+|p|^{\gamma+1}),\end{aligned}$$

and, once more, we conclude that

$$|D_xu|\leq C.$$

Case 3: $0<t_0<T$. Then $Dv=0$, $D^2v\leq 0$, which yields

$$0\leq L_uv. \tag{3.21}$$

By direct computation, we see from (Q) that

$$L_u\left(\frac{1}{2}\tilde{u}^2\right) = -D\tilde{u}AD\tilde{u} + \tilde{u}L_u(\tilde{u}) = -D\tilde{u}AD\tilde{u} - \tilde{u}D_q\mathrm{Tr}(AD^2u)\cdot D\tilde{u} + \tilde{u}D_qb\cdot D\tilde{u} - \tilde{u}b,$$

whereas letting $\Phi(p,s) = \frac{1}{2}|p|^2$ in Lemma 3.3,

$$L_u\left(\frac{1}{2}|D_xu|^2\right) = -\sum_{i=1}^{d} Du_{x_i}A\cdot Du_{x_i} + \sum_{i=1}^{d}\mathrm{Tr}(A_{x_i}D^2u)u_{x_i} - D_xb\cdot p,$$

and thus

$$L_u(v) = -kD\tilde{u}A\cdot D\tilde{u} - \sum_{i=1}^{d} Du_{x_i}A\cdot Du_{x_i} + E, \tag{3.22}$$

where E is the error term, computed as follows. Setting $\Lambda = D_x + \tilde{u}kD_p$ and using Lemma 3.3 once more, we have $E = E_1 + E_2$, with

$$\begin{aligned} E_1 = (-D_xu_t + \frac{1}{2}Y^+D^2_{xx}u)\Lambda Y^+\cdot p + \frac{1}{2}Y^-D^2_{xx}u\Lambda Y^-\cdot p - \mathrm{Tr}(D_pBD^2_{xx}u)\Lambda H_m\cdot p \\ - H_{mm}H_m^{-1}(D_xH\cdot p + \tilde{u}k\zeta)\mathrm{Tr}(D_pBD^2_{xx}u) - H_m(\mathrm{Tr}(\Lambda D_pBD^2_{xx}u)\cdot p \\ - \mathrm{Tr}(D_pB_mD^2_{xx}u)H_m^{-1}(D_xH\cdot p + \tilde{u}k\zeta)), \end{aligned} \tag{3.23}$$

$$\begin{aligned} E_2 = -B_m\cdot(\Lambda D_xHp^T - D_xH_mH_m^{-1}(D_xH\cdot p + \tilde{u}k\zeta)) - D_xH\cdot(\Lambda B_mp^T - B_{mm}H_m^{-1}(D_xH\cdot p + \tilde{u}k\zeta)) \\ + H_m(\Lambda\mathrm{div}_xBp^T - \mathrm{div}_xB_mH_m^{-1}(D_xH\cdot p + \tilde{u}k\zeta)) + \mathrm{div}_xB(\Lambda H_m\cdot p - H_{mm}H_m^{-1}(D_xH\cdot p + \tilde{u}k\zeta)) - k\tilde{u}b. \end{aligned} \tag{3.24}$$

Before estimating the E_i, compute lower bounds for the dominant signed terms in (3.22), in the following way. Setting $r = (1+\frac{1}{2C_0})^{-1}$, and using (Q1), we may write

$$\begin{aligned} DuA\cdot Du = |-u_t + \frac{1}{2}Y^+\cdot p|^2 - |\frac{1}{2}Y^-\cdot p|^2 - H_mpD_pB\cdot p = |-u_t + \frac{1}{2}Y^+\cdot p|^2 \\ - |\frac{1}{2}Y^-\cdot p|^2 - rH_mpD_pB\cdot p - (1-r)H_mpD_pB\cdot p. \end{aligned}$$

Now, in view of (E3), observing that $r(1+\frac{1}{C_0}) > 1$ and $r<1$, we obtain

$$\begin{aligned} DuA\cdot Du &\geq |-u_t + \frac{1}{2}Y^+\cdot p|^2 - |\frac{1}{2}Y^-\cdot p|^2 + r(1+\frac{1}{C_0})|\frac{1}{2}Y^-\cdot p|^2 - (1-r)H_mpD_pB\cdot p \\ &= |-u_t + \frac{1}{2}Y^+\cdot p|^2 + (r(1+\frac{1}{C_0})-1)|\frac{1}{2}Y^-\cdot p|^2 - (1-r)H_mpD_pB\cdot p \\ &\geq |-u_t + \frac{1}{2}Y^+\cdot p|^2 + \frac{1}{C}|\frac{1}{2}Y^-\cdot p|^2 - \frac{1}{C}H_mpD_pB\cdot p. \end{aligned} \tag{3.25}$$

Similarly, (Q1) and (E3) yield

$$\sum_{i=1}^{d} Du_{x_i}A\cdot Du_{x_i} = |-D_xu_t+\frac{1}{2}Y^+D^2_{xx}u|^2-|\frac{1}{2}Y^-D^2_{xx}u|^2-H_m\mathrm{Tr}D^2_{xx}uD_pB\cdot D^2_{xx}u$$
$$\geq |-D_xu_t+\frac{1}{2}Y^+D^2_{xx}u|^2+\frac{1}{C}|\frac{1}{2}Y^-D^2_{xx}u|^2-\frac{1}{C}H_m\mathrm{Tr}(D^2_{xx}uD_pB\cdot D^2_{xx}u). \quad (3.26)$$

Therefore, (Q1) and (E3) yield

$$D\tilde{u}A\cdot D\tilde{u} = |-\tilde{u}_t+\frac{1}{2}Y^+\cdot p|^2-|\frac{1}{2}Y^-\cdot p|^2-H_mpD_pB\cdot p\geq \frac{1}{2}DuA\cdot Du - C, \quad (3.27)$$

and, on the other hand, since $\frac{1}{2}(Y^++Y^-)=D_pH$,

$$|\zeta|^2 = |-\tilde{u}_t+D_pH\cdot D_x\tilde{u}|^2\leq 2(|-\tilde{u}_t+\frac{1}{2}Y^+\cdot D_x\tilde{u}|^2+|\frac{1}{2}Y^-\cdot D_x\tilde{u}|^2)\leq CD\tilde{u}A\cdot D\tilde{u}. \quad (3.28)$$

Applying Young's inequality and (3.26) in (3.23), we obtain

$$|E_1|\leq \frac{1}{2}\sum_{i=1}^{d} Du_{x_i}A\cdot Du_{x_i}+C[|\Lambda Y^+|^2|p|^2+|\Lambda Y^-|^2|p|^2+|H_m|^{-1}|D_pB||\Lambda H_m|^2|p|^2$$
$$+|H_{mm}|^2|H_m|^{-3}(|D_xH|^2|p|^2+k^2\zeta^2)|D_pB|+|H_m||\Lambda D_pB|^2|D_pB|^{-1}|p|^2$$
$$+|D_pB_m|^2|D_pB|^{-1}|H_m|^{-1}(|D_xH|^2|p|^2+k^2\zeta^2)]. \quad (3.29)$$

Now, the terms in (3.29) may all be estimated with the help of the growth assumptions (H) and (B). Indeed, in view of (H1), (HX1), (HX2), (BX1), (B2), and (3.19), we estimate

$$|\Lambda Y^+|^2, |\Lambda Y^-|^2\leq C(|D^2_{xp}H|^2+|D_xB_m|^2+k^2|D^2_{pp}H|^2+k^2|D_pB_m|^2)\leq C\psi(m)^2(|p|^{2\gamma-2}+|p|^{2\gamma-1}), \quad (3.30)$$

$$|\Lambda H_m|^2\leq C|D_xH_m|^2+Ck^2|D_pH_m|^2\leq C\psi(m)^2m^{-2}(|p|^{2\gamma}+|p|^{2\gamma+1}), \quad (3.31)$$

$$|\Lambda D_pB|^2\leq C|D^2_{xp}B|^2+Ck^2|D^2_{pp}B|^2\leq C\psi(m)^2m^2(|p|^{2\gamma-4}+|p|^{2\gamma-3}). \quad (3.32)$$

Thus, using (HM1), (HM2), (B1), (B2), (HX1), (3.30), (3.31), and (3.32) in (3.29) yields

$$|E_1|\leq \frac{1}{2}\sum_{i=1}^{d} Du_{x_i}A\cdot Du_{x_i}+C\psi(m)^2|p|^{2\gamma+1}+C|p|^{-1/2}k\zeta^2. \quad (3.33)$$

Similarly, for the second error term, we use Young's Inequality and (3.28) in (3.24), obtaining

$$|E_2|\leq \frac{1}{4}kDuA\cdot Du+C[|B_m||\Lambda D_xH||p|+|B_m||D_xH_m||H_m^{-1}||D_xH||p|+k|B_m|^2|D_xH_m|^2|H_m|^{-2}$$
$$+|D_xH|(|\Lambda B_m||p|+|B_{mm}||H_m|^{-1}|D_xH||p|)+k|D_xH|^2|B_{mm}|^2|H_m|^{-2}+|H_m||\Lambda\mathrm{div}_xB||p|$$
$$+|\mathrm{div}_xB_m||D_xH||p|+|\mathrm{div}_xB_m|^2k+|\mathrm{div}_xB|(|\Lambda H_m||p|$$

$$+ |H_{mm}||H_m|^{-1}|D_xH||p|) + k|\mathrm{div}_xB|^2|H_{mm}|^2|H_m|^{-2} + k|\tilde{u}|(|H_m||\mathrm{div}_xB| + |B_m||D_xH|)]. \quad (3.34)$$

In view of (HX1), (B2), (BX2), and (BX1), we obtain

$$|\Lambda D_xH| \leq C(|D^2_{xx}H| + k|D^2_{px}H|) \leq C\psi(m)(|p|^{\gamma_2} + |p|^{\gamma_2+1/2}),$$

$$|\Lambda \mathrm{div}_xB| \leq C(|D^2_{xx}B| + k|D^2_{px}B|) \leq C\psi(m)(|p|^{\gamma_2-1} + |p|^{\gamma_2-1/2}),$$

$$|\Lambda B_m| \leq C(|D_xB_m| + k|D_pB_m|) \leq C\psi(m)(|p|^{\gamma_2-1} + |p|^{\gamma_1-1/2}),$$

$$|\Lambda \mathrm{div}_xB| \leq C(|D^2_{xx}B| + k|D^2_{px}B|) \leq C\psi(m)(|p|^{\gamma_2-1} + |p|^{\gamma_2-1/2}).$$

Consequently, (3.34), (HM1), (HM2), (B2), (HX1), (HX2), (BM), and (BX1) imply

$$|E_2| \leq \frac{1}{4}kDuA \cdot Du + C\psi(m)^2|p|^{2\gamma+1}. \quad (3.35)$$

Having estimated the error terms, (3.22), (3.27), (3.33), (3.28), and (3.35) yield

$$\begin{aligned} L_u(v) &= -kD\tilde{u}A \cdot D\tilde{u} - \sum_{i=1}^{d} Du_{x_i}A \cdot Du_{x_i} + E \\ &\leq -\frac{k}{8}DuA \cdot Du - \frac{1}{C}k\zeta^2 - \frac{1}{2}\sum_{i=1}^{d} Du_{x_i}A \cdot Du_{x_i} + C\psi(m)^2|p|^{2\gamma+1} + C|p|^{-1/2}k\zeta^2 + Ck. \end{aligned}$$

Therefore, in view of (3.20), (3.25), (B1), and (HM1),

$$\begin{aligned} L_u(v) &\leq \frac{k}{8}H_mpD_pB \cdot p - \frac{1}{2C}k|\zeta|^2 + C\psi(m)^2|p|^{2\gamma+1} + C|p|^{-1/2}k\zeta^2 + C|p|^{3/2} \\ &\leq -\frac{1}{8C_0^2}\psi(m)^2|p|^{2\gamma+3/2} - \frac{\int 1}{2C}k|\zeta|^2 + C\psi(m)^2|p|^{2\gamma+1} + C|p|^{-1/2}k\zeta^2 + C|p|^{3/2} \\ &\leq -\psi(m)^2(\frac{1}{8C_0^2}|p|^{2\gamma+3/2} - C|p|^{2\gamma+1}) - k\zeta^2(\frac{1}{2C} - C|p|^{-1/2}) + C|p|^{3/2}. \end{aligned}$$

So, given that (x_0, t_0) is a maximum point of v, we have $L_u(v) \geq 0$, and, thus,

$$\psi(m)^2(\frac{1}{8C_0^2}|p|^{2\gamma+3/2} - C|p|^{2\gamma+1}) + k\zeta^2(\frac{1}{2C} - C|p|^{-1/2}) \leq C|p|^{3/2}. \quad (3.36)$$

This implies that either $\frac{1}{2C} - C|p|^{-1/2} \leq 0$ or $\psi(m)^2(\frac{1}{8C_0^2}|p|^{2\gamma+3/2} - C|p|^{2\gamma+1}) \leq C|p|^{3/2}$. If the former holds, there is nothing to prove, so we may assume the latter. We may further assume that $|p|$ is large enough that $\frac{1}{8C_0^2}|p|^{2\gamma+3/2} - C|p|^{2\gamma+1} \geq \frac{1}{16C_0^2}|p|^{2\gamma+3/2}$. We thus obtain

$$\psi(m)|p|^{\gamma} \leq C. \quad (3.37)$$

In view of (E2) and the fact that, by (3.16), H is bounded below, we conclude that $m \leq C$. Hence $\psi(m)$ is bounded below, which finally yields $|p| \leq C$, concluding the proof when $\gamma_1 = \gamma$.

Now we describe the necessary changes in the proof to deal with the case in which $\gamma_1 < \gamma$. Setting

$$\eta = (2\gamma_1 - \gamma + 2) - \gamma_2,$$

we see, in view of (2.1), that $\eta > 0$. In (3.18), we replace k by $k' = ||D_x u||^{\kappa}_{Q_T}$, where

$$\kappa = \max\left(\frac{1}{2}(\gamma_2 - \gamma) + 1, \gamma_1 - \gamma + \frac{3}{2}\right).$$

The proofs of Case 1 and Case 2 follow through with no change until the last step, leading in both cases to the inequality

$$\frac{1}{C}|p|^{\gamma+\kappa} \leq C(1 + |p|^{\gamma_1+1} + |p|^{\kappa} + |p|^{\gamma_2+1}). \tag{3.38}$$

By definition, $\kappa \geq \frac{3}{2} + \gamma_1 - \gamma$, so the left hand side of (3.38) has higher degree than the right hand side, and thus

$$|p| \leq C.$$

The proof of Case 3 proceeds analogously as well. (3.29) and (3.34) are obtained with no change. To estimate the errors, instead of (3.30), (3.31), and (3.32), we now have the bounds

$$|\Lambda Y^+|^2, |\Lambda Y^-|^2 \leq C(|D^2_{xp}H|^2 + |D_x B_m|^2 + k^2|D^2_{pp}H|^2 + k^2|D_p B_m|^2) \leq C\psi(m)^2(|p|^{2\gamma_2-2} + |p|^{2\gamma+2\kappa-4}),$$

$$|\Lambda H_m|^2 \leq C|D_x H_m|^2 + Ck^2|D_p H_m|^2 \leq C\psi(m)^2 m^{-2}(|p|^{2\gamma_2} + |p|^{2\gamma_1+2\kappa-2}),$$

$$|\Lambda D_p B|^2 \leq C|D^2_{xp}B|^2 + Ck^2|D^2_{pp}B|^2 \leq C\psi(m)^2 m^2(|p|^{2\gamma_2-4} + |p|^{2\gamma+2\kappa-6}).$$

This allows us to estimate E_1 as before, this time obtaining

$$|E_1| \leq \frac{1}{2}\sum_{i=1}^{d} Du_{x_i} A Du_{x_i} + C\psi(m)^2(|p|^{2\gamma+2\kappa-2} + |p|^{2\gamma_2+\gamma-\gamma_1} + k|p|^{-(2+\gamma_1-\gamma-\kappa)}\zeta^2). \tag{3.39}$$

Since the dominant power of $|p|$ in (3.25) now has the exponent

$$\alpha = \gamma + \gamma_1 + \kappa,$$

we must verify that (3.39) does not have a higher degree. Indeed,

$$\alpha - (2\gamma + 2\kappa - 2) = 2 + \gamma_1 - \gamma - \kappa \geq \min(2 + \gamma_1 - \gamma - (\frac{1}{2}(\gamma_2 - \gamma) + 1), \gamma_1 - \gamma + 2 - (\gamma_1 - \gamma + \frac{3}{2})) = \frac{1}{2}\min(\eta, 1), \tag{3.40}$$

$$\alpha-(2\gamma_2+\gamma-\gamma_1)=2(\gamma_1-\gamma_2)+\kappa\geq\gamma_1-\gamma_2+\kappa\geq\gamma_1-\gamma_2+\frac{1}{2}(\gamma_2-\gamma)+1=\frac{1}{2}\eta, \tag{3.41}$$

Hence, letting $\epsilon=\frac{1}{2}\min(1,\eta)$, it follows from (3.39), (3.40) and (3.41),

$$|E_1|\leq\frac{1}{2}\sum_{i=1}^{d}Du_{x_i}ADu_{x_i}+C\psi(m)^2(|p|^{\alpha-\epsilon}+k|p|^{-\epsilon}\zeta^2). \tag{3.42}$$

Moving on to E_2, we first obtain

$$|\Lambda D_xH|\leq C(|D^2_{xx}H|+k|D^2_{px}H|)\leq C\psi(m)(|p|^{\gamma_2}+|p|^{\gamma_2-1+\kappa}),$$

$$|\Lambda\mathrm{div}_xB|\leq C(|D^2_{xx}B|+k|D^2_{px}B|)\leq C\psi(m)(|p|^{\gamma_2-1}+|p|^{\gamma_2-2+\kappa}),$$

$$|\Lambda B_m|\leq C(|D_xB_m|+k|D_pB_m|)\leq C\psi(m)(|p|^{\gamma_2-1}+|p|^{\gamma_1-2+\kappa}),$$

$$|\Lambda\mathrm{div}_xB|\leq C(|D^2_{xx}B|+k|D^2_{px}B|)\leq C\psi(m)(|p|^{\gamma_2-1}+|p|^{\gamma_2-2+\kappa}),$$

and so, in place of (3.35),

$$|E_2|\leq\frac{1}{4}kDuA\cdot Du+C\psi(m)^2(|p|^{\gamma+\gamma_2}+|p|^{\gamma_2+\gamma+\kappa-1}+|p|^{2\gamma_2+\gamma-\gamma_1}+|p|^{\kappa+2\gamma_2-2+2\gamma-2\gamma_1}). \tag{3.43}$$

We again verify that the exponents do not exceed α,

$$\alpha-(\gamma+\gamma_2)=\gamma_1-\gamma_2+\kappa\geq\gamma_1-\gamma_2+\frac{1}{2}(\gamma_2-\gamma)+1=\frac{1}{2}\eta, \tag{3.44}$$

$$\alpha-(\gamma_2+\gamma+\kappa-1)=\gamma_1-\gamma_2+1\geq 1, \tag{3.45}$$

$$\alpha-(\kappa+2\gamma_2-2+2\gamma-2\gamma_1)=(\gamma_1-\gamma_2)+((2\gamma_1-\gamma+2)-\gamma_2)\geq((2\gamma_1-\gamma+2)-\gamma_2)=\eta. \tag{3.46}$$

and thus, (3.43), (3.41), (3.44), (3.45), and (3.46) yield

$$|E_2|\leq\frac{1}{4}kDuA\cdot Du+C\psi(m)^2|p|^{\alpha-\epsilon}. \tag{3.47}$$

Consequently, in view of (3.42) and (3.47), we obtain, instead of (3.36),

$$\psi(m)^2(\frac{1}{8C_0^2}|p|^{\alpha}-C|p|^{\alpha-\epsilon})+k\zeta^2(\frac{1}{2C}-C|p|^{-\epsilon})\leq C|p|^{\kappa},$$

and, thus, in place of (3.37), this time we conclude

$$\psi(m)|p|^{\frac{\gamma+\gamma_1}{2}} \leq C. \tag{3.48}$$

Since $\gamma_1 < \gamma$, (2.2) holds, and thus we have (2.5). This, together with (3.48), implies that $|p| \leq C$, as wanted. $\square$

The following lemma provides global, positive two-sided bounds for the density in terms of the gradient bound.

Lemma 3.5. *Let $(u,m) \in C^3(\overline{Q_T}) \times C^2(\overline{Q_T})$ be a solution to (EMFG), and set, for $K \in \mathbb{R}$,*

$$\delta_K = \inf_{(x,p,s)\in\mathbb{T}^d\times\mathbb{R}^d\times(-\infty,K]} H^{-1}(x,p,s).$$

There exist constants $C, C_1 > 0$, with

$$C = C(C_1, h(C_1)), \quad C_1 = C_1(C_0, ||Du||_{C^0(\overline{Q_T})}, \delta^{-1}_{||Du||}, ||\psi||_{C^0[\delta_{||Du||},\infty)}),$$

such that

$$||m||_{C^0(\overline{Q_T})} + ||m^{-1}||_{C^0(\overline{Q_T})} \leq C.$$

Proof. Due to (E1), $\delta_K > 0$ is well-defined for each $K \in \mathbb{R}$, and we may apply $H^{-1}(x, D_xu, \cdot)$ to both sides of the inequality

$$H(x, D_xu, m) = u_t \leq ||Du||_{C^0(\overline{Q_T})},$$

which yields, for $(x,t) \in \overline{Q_T}$,

$$H^{-1}(x, D_xu, ||Du||_{C^0(\overline{Q_T})}) \leq m(x,t).$$

Letting $\delta = \delta_{||Du||}$, we thus obtain $\delta \leq m(x,t)$ and, hence,

$$||m^{-1}||_{C^0(\overline{Q_T})} \leq \delta^{-1}.$$

On the other hand,

$$H(x,D_xu,m) - C_0\psi(m)|D_xu|^\gamma \geq u_t - C_0\psi(m)||D_xu||^\gamma_{C^0(\overline{Q_T})} \geq -||Du||_{C^0(\overline{Q_T})} - C_0||\psi||_{[\delta,\infty)}||D_xu||^\gamma_{C^0(\overline{Q_T})} \geq -C_1,$$

which, in view of the definition of h (see 3.4), implies that

$$m \leq h(C_1).$$

$\square$

We now summarize all of the a priori bounds obtained in this section.

Theorem 3.6. *Let $(u,m) \in C^3(\overline{Q_T}) \times C^2(\overline{Q_T})$ be a solution to (EMFG), and let δ be defined as in Lemma 3.5. Then there exist constants $M, M_1, L, L_1, K, K_1 > 0$, with*

$$L = \left(L_1, g_1(f_0^{-1}(L_1))^+, g_0(f_1^{-1}(-L_1))^-, g_0^{-1}g_1(f_0^{-1}(L_1)), \frac{1}{g_1^{-1}g_0(f_1^{-1}(-L_1))}\right),\ L_1 = L_1(C_0, T),$$

$$K = K(C_1, ||(\psi \circ h)^{-1}||_{C^0(0,K_1]}),$$

$$K_1 = K_1(L, T^{-1}, ||\overline{C}||_{C^0[\frac{1}{L},L]}, ||\psi||_{C^0[\frac{1}{L},L]}, ||\psi^{-1}||_{C^0[\frac{1}{L},L]}, ||D_x g||_{C^1(\mathbb{T}^d \times [\frac{1}{L},L])}, (2\gamma_1 - \gamma + 2 - \gamma_1)^{-1}),$$

$$M = M(M_1, h(M_1)),\quad M_1 = M_1(K, \delta_K^{-1}, ||\psi||_{C^0[\delta_K,\infty)})$$

such that

$$||u||_{C^0(\overline{Q_T})} \leq L,\ \ ||Du||_{C^0(\overline{Q_T})} \leq K,\ \ \textit{and}\ \ ||m||_{C^0(\overline{Q_T})} + ||m^{-1}||_{C^0(\overline{Q_T})} \leq M.$$

Proof. This result follows from combining Lemma 3.2, Lemma 3.4, and Lemma 3.5. □

3.5. Classical solutions

Having obtained the gradient bound, the existence result follows through the method of continuity.

Proof of Theorem 1.1. We only sketch the proof, which follows the same steps as Theorem 1.1 of [22]. We define, for $\theta \in [0,1]$ and $(x,p,m) \in \mathbb{T}^d \times \mathbb{R}^d \times (0,\infty)$,

$$H^\theta(x,p,m) = \theta H(x,p,m) + (1-\theta)H(0,p,m),\quad B^\theta(x,p,m) = \theta B(x,p,m) + (1-\theta)B(0,p,m),$$
$$g^\theta(x,m) = \theta g(x,m) + (1-\theta)m,\quad m_0^\theta(x) = \theta m_0(x) + (1-\theta),$$

and consider the family of (EMFG) systems

$$\begin{cases} -u_t + H^\theta(x, D_x u, m) = 0 & (x,t) \in Q_T, \\ m_t - \operatorname{div}(B^\theta(x, D_x u, m)) = 0 & (x,t) \in Q_T, \\ m(0,x) = m_0^\theta(x),\ u(x,T) = g^\theta(x, m(x,T)) & x \in \mathbb{T}^d, \end{cases} \tag{EMFG$_\theta$}$$

together with the corresponding elliptic and boundary operators Q^θ and N^θ associated to them, according to (Q). We observe first that for $\theta = 0$, the solution is simply $(u,m) \equiv ((t-T)H(0,0,1)+1, 1)$. Now, by definition of g^θ,

$$g_0^\theta \circ g_0^{-1} \circ g_1 = \theta g_0 \circ g_0^{-1} \circ g_1 + (1-\theta)g_0^{-1} \circ g_1 \geq \theta g_1 + (1-\theta) g_0^{-1} \circ g_0 = g_1^\theta,$$

so we obtain

$$(g_0^\theta)^{-1} g_1^\theta \leq g_0^{-1} g_1, \tag{3.49}$$

and similarly

$$g_1^{-1} g_0 \le (g_1^\theta)^{-1} g_0^\theta. \tag{3.50}$$

Moreover, setting

$$f_0^\theta(m) = \min_{\mathbb{T}^d}(-H^\theta(\cdot, 0, m)) = \theta f_0(m) - (1-\theta)H(0,0,m), \text{ and}$$
$$f_1^\theta(m) = \max_{\mathbb{T}^d}(-H^\theta(\cdot, 0, m)) = \theta f_1(m) - (1-\theta)H(0,0,m),$$

it is readily seen that, by definition,

$$(f_0^\theta)^{-1} \le f_0^{-1}, \quad f_1^{-1} \le (f_1^\theta)^{-1}. \tag{3.51}$$

In view of (3.49), (3.50), and (3.51), Theorem 3.6 yields a constant C, independent of θ, such that

$$||u^\theta||_{C^1(\overline{Q_T})} \le C.$$

Moreover, the classical $C^{1,\alpha}$ estimates for oblique derivative problems (see, for instance, [17], Lem. 2.3) yield

$$||u^\theta||_{C^{1+s'}(\overline{Q_T})} \le C. \tag{3.52}$$

for some s'. Now we define the Banach spaces

$$E = C^{3,s}(\overline{Q_T}), \ F = C^{1,s}(\overline{Q_T}) \times C^{2,s}(\partial Q_T),$$

and the continuously differentiable operator $S : E \times [0,1] \to F$ by

$$S(u,\theta) = (Q^\theta u, N^\theta u), \ (u,\theta) \in E \times [0,1].$$

Direct computation shows that, for fixed $(u,\theta) \in E \times [0,1]$, the linearization S_u of S with respect to u has the form $(L^1_{(u,\theta)} w, L^2_{(u,\theta)} w)$, where $L^1_{(u,\theta)}$ is a linear, uniformly elliptic operator and $L^2_{(u,\theta)}$ is a linear oblique boundary operator. Moreover, the homogeneous problem $(L^1_{(u,\theta)} w, L^2_{(u,\theta)} w) = (0,0)$ has only the trivial solution. The standard Fredholm alternative for linear oblique problems (see [12]) thus implies that S_u is invertible in $C^{3,s}(\overline{Q_T})$. Thus, by the implicit function theorem, the set

$$D = \{\theta \in [0,1] : \text{the equation } S(u,\theta) = (0,0) \text{ has a unique solution } u \in C^{3,s}(\overline{Q_T})\}$$

is open in $[0,1]$. On the other hand, (3.52) together with the Schauder estimates for linear oblique problems imply that D is also closed. Since $0 \in D$, we conclude that $D = [0,1]$, and in particular $1 \in D$, which completes the proof. □

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