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# CONTROLLABILITY AND OBSERVABILITY FOR SOME FORWARD STOCHASTIC COMPLEX DEGENERATE/SINGULAR GINZBURG–LANDAU EQUATIONS*

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**Abstract.** This paper is addressed to establishing controllability and observability for some forward linear stochastic complex degenerate/singular Ginzburg–Landau equations. It is sufficient to establish appropriate observability inequalities for the corresponding backward and forward equations. The key is to prove the Carleman estimates of the forward and backward linear stochastic complex degenerate/singular Ginzburg–Landau operators. Compared with the existing deterministic results, it is necessary to overcome the difficulties caused by some complex coefficients and random terms. The results obtained cover those of deterministic cases and generalize those of stochastic degenerate parabolic equations. Moreover, the limit behavior of the coefficients in the equation is discussed.

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## 1. Introduction and main results

The Ginzburg–Landau equation was first given in [16], which is a typical nonlinear equation in the physics community. It may describe a variety of phenomena including light propagation in nonlinear fibers, phenomena related to pulse formation and superconductivity, and plays an important role in the theory of amplitude equations. Real-valued Ginzburg–Landau equations were first derived as long-wave amplitude equations in [22, 26]. The complex Ginzburg–Landau equation was established as a standard 1-$D$ model for some fluid flows (see [27]). The deterministic complex Ginzburg–Landau equation is one of the most frequently studied equations in physics and mathematics. For instance, the Cauchy problems, numerical methods to establish approximate solutions and control problems for the deterministic Ginzburg–Landau equations have been extensively researched (see [1, 7, 10, 23–25]).

And, the uniformly parabolic equations without degeneracies or singularities have been developed in various directions. However, more recently, several situations where the equation is not uniformly parabolic have been

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investigated. Indeed, many problems coming from physics (see [19]), biology (see [4, 8]) and mathematical finance (see [17]) are described by parabolic equations which admit some kind of degeneracy. Another inspiring situation is the case of parabolic equations with singular lower order terms. The corresponding cases arise in quantum mechanics (see [2]), or in combustion problems (see [3]).

In practice, due to the interference of random factors, stochastic processes give a natural replacement for deterministic functions in mathematical descriptions. Compared with deterministic case, some substantially difficulties arise in the study of the stochastic partial differential equations. For example, the solution to a stochastic partial differential equation is non-differentiable with respect to noise variable, and the usual compactness embedding result is not valid for solution spaces of the stochastic evolution equations. Further, the "time" in the stochastic setting is not reversible. Indeed, many tools and methods, which are effective in the deterministic case, do not work anymore in the stochastic setting.

Recently, stochastic complex Ginzburg–Landau equations have received more and more attention, see for example [11, 12, 18]. In this paper, we will study some linear stochastic complex Ginzburg–Landau equations with both degeneracies and singularities. In fact, the linearized complex Ginzburg–Landau equation also models some different phenomena, such as the amplitude equation in pattern formation and the reaction diffusion of two chemicals in one dimension [6]. It is noted that many properties of Ginzburg–Landau equations are between that of parabolic equations and Schrödinger equations. Therefore, this paper is also devoted to considering its limit behavior where degenerate Ginzburg–Landau equations, degenerate parabolic equations and degenerate Schrödinger equations are considered simultaneously. In the following, the problem of this paper is stated in detail.

Let $T > 0$ and $Q = (0, 1) \times (0, T)$. Assume $G_0 = (x_1, x_2)$ to be a given nonempty open subset of $(0, 1)$, and denote by $\chi_{G_0}$ the characteristic function of the set $G_0$. Fix a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$, on which a one-dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined such that $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by $B(\cdot)$, augmented by all the $\mathcal{P}$-null sets in $\mathcal{F}$. Let $\mathcal{H}$ be a Banach space, and let $C([0, T]; \mathcal{H})$ be the Banach space of all $\mathcal{H}$-valued strongly continuous abstract functions defined on $[0, T]$. We denote by $L^2_{\mathbb{F}}(0, T; \mathcal{H})$ the Banach space consisting of all $\mathcal{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;\mathcal{H})}) < \infty$, with the canonical norm; by $L^\infty_{\mathbb{F}}(0, T; \mathcal{H})$ the Banach space consisting of all $\mathcal{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted essentially bounded processes; and by $L^2_{\mathbb{F}}(\Omega; C([0, T]; \mathcal{H}))$ the Banach space consisting of all $\mathcal{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C([0,T];\mathcal{H})}) < \infty$. Similarly, one can define $L^\infty_{\mathbb{F}}(\Omega; C^m([0, T]; \mathcal{H}))$ for any positive integer $m$. Moreover, denote by $\boldsymbol{i}$ the imaginary unit, and for any complex number $\mathbf{c}$, we denote by $\bar{\mathbf{c}}$, $\operatorname{Re}\mathbf{c}$ and $\operatorname{Im}\mathbf{c}$, its complex conjugate, real part and imaginary part, respectively.

Consider the following forward linear stochastic complex degenerate/singular Ginzburg–Landau equation:

$$\begin{cases} \mathrm{d}y - (a + \boldsymbol{i}b)(x^\alpha y_x)_x \mathrm{d}t - (c + \boldsymbol{i}d)\dfrac{\mu}{x^\beta} y \mathrm{d}t = (c_1 y + \chi_{G_0} u)\mathrm{d}t + (c_2 y + v)\mathrm{d}B(t) & \text{in } Q, \\ y(t, 1) = 0 & \text{on } (0, T), \\ \begin{cases} y(t, 0) = 0 & \text{if } 0 \leq \alpha < 1, \\ (x^\alpha y_x)(t, 0) = 0 & \text{if } 1 \leq \alpha < 2, \end{cases} & \text{on } (0, T), \\ y(0, x) = y_0(x) & \text{in } (0, 1), \end{cases} \tag{1.1}$$

where complex-valued coefficients $c_1 \in L^\infty_{\mathbb{F}}(0, T; L^\infty((0, 1); \mathbb{C}))$, and $c_2 \in L^\infty_{\mathbb{F}}(0, T; W^{1,\infty}((0, 1); \mathbb{C}))$. Also, $\alpha \in [0, 2)$, $a, b, c, d, \mu, \beta \in \mathbb{R}$ satisfy some conditions which will be given later. In (1.1), $(u, v)$ is the control variable, $y$ is the state variable, and $y_0 \in L^2((0, 1); \mathbb{C})$ is a given initial value. Unless otherwise stated, we assume that all functions mentioned in this paper are complex-valued. Next, we assume that exponents $\alpha, \beta$ and parameter $\mu$ satisfy the following conditions:

- *sub-critical potentials*:

$$\begin{cases} \alpha \in [0,2), & 0<\beta<2-\alpha, \ \text{no condition on } \mu; \quad (1.2a) \\ \alpha \in [0,2)\backslash\{1\}, & \beta = 2-\alpha, \ \mu < \mu(\alpha) := \dfrac{(1-\alpha)^2}{4}; \quad (1.2b) \end{cases}$$

- *critical potentials*:

$$\alpha \in [0,2)\backslash\{1\}, \ \beta = 2-\alpha, \ \mu = \mu(\alpha). \tag{1.3}$$

We separate the case where both the exponent $\beta$ and the parameter $\mu$ are critical. In the case of (1.3), the potential is called *critical*, and otherwise it is called *sub-critical*. As we shall show later, the case of a *critical potential* requires a specific functional setting and a special care in the derivation of Carleman estimates. The Carleman-type estimate was first introduced by Carleman to study the uniqueness for elliptic equations in [5], which has become an important tool in studying controllability for stochastic partial differential equations.

In what follows, we define some functional spaces that will be used in this paper. In the case of *sub-critical* *i.e.*, (1.2), for $0 \le \alpha < 1$, define the Hilbert space $H^1_\alpha(0,1)$ as follows:

$$H^1_\alpha(0,1) = \Big\{ \ y \in L^2(0,1) \ \Big| \ y \text{ is absolutely continuous in } [0,1], \\ x^{\frac{\alpha}{2}} y_x \in L^2(0,1) \text{ and } y(0)=y(1)=0 \ \Big\}.$$

For $1 \le \alpha < 2$, $H^1_\alpha(0,1)$ is defined as follows:

$$H^1_\alpha(0,1) = \Big\{ \ y \in L^2(0,1) \ \Big| \ y \text{ is locally absolutely continuous in } (0,1], \\ x^{\frac{\alpha}{2}} y_x \in L^2(0,1) \text{ and } y(1)=0 \ \Big\}.$$

In the case of *critical* *i.e.*, (1.3), the functional setting requires some modifications. Instead of $H^1_\alpha(0,1)$, for $0 \le \alpha < 1$, we define $H^*_\alpha(0,1)$ as follows:

$$H^*_\alpha(0,1) = \Big\{ \ y \in L^2(0,1) \cap H^1_{loc}((0,1])\Big| \ \int_0^1 \Big( x^\alpha |y_x|^2 - \frac{\mu(\alpha)}{x^{2-\alpha}}|y|^2 \Big) \mathrm{d}x < +\infty \\ \text{and } y(0)=y(1)=0 \Big\},$$

and for $1 \le \alpha < 2$, $H^*_\alpha(0,1)$ is defined as follows:

$$H^*_\alpha(0,1) = \Big\{ \ y \in L^2(0,1) \cap H^1_{loc}((0,1])\Big| \ \int_0^1 \Big( x^\alpha |y_x|^2 - \frac{\mu(\alpha)}{x^{2-\alpha}}|y|^2 \Big) \mathrm{d}x < +\infty \text{ and } y(1)=0 \Big\}.$$

We first recall the definition of the solution to equation (1.1) and give the well-posedness results. Put

$$\mathcal{H}_1 = L^2_{\mathbb{F}}(0,T;H^1_\alpha((0,1);\mathbb{C})) \cap L^2_{\mathbb{F}}(\Omega;C([0,T];L^2((0,1);\mathbb{C}))),$$

$$\mathcal{H}_2 = L^2_{\mathbb{F}}(0,T;H^*_\alpha((0,1);\mathbb{C})) \cap L^2_{\mathbb{F}}(\Omega;C([0,T];L^2((0,1);\mathbb{C}))).$$

**Definition 1.1.** In the case of (1.2), a process $y \in \mathcal{H}_1$ is said to be a weak solution to equation (1.1), if for any $t \in [0,T]$ and any $\varrho \in H^1_\alpha(0,1)$, it holds that

$$\begin{aligned}\int_0^1 y(x,t)\varrho(x)\mathrm{d}x - \int_0^1 y(x,0)\varrho(x)\mathrm{d}x = \int_0^t\int_0^1 \Big[ &-(a+\boldsymbol{i}b)x^\alpha y_x\varrho_x \\ +(c+\boldsymbol{i}d)\frac{\mu}{x^\beta}y\varrho + (c_1 y + \chi_{G_0}u)\varrho\Big]\mathrm{d}x\mathrm{d}s &+ \int_0^t\int_0^1 (c_2 y + v)\varrho\mathrm{d}x\mathrm{d}B(s).\end{aligned} \tag{1.4}$$

In the case of (1.3), $y \in \mathcal{H}_2$ is said to be a weak solution to equation (1.1), if for any $t \in [0,T]$ and any $\varrho \in H^*_\alpha(0,1)$, (1.4) holds.

By [21, 29], it is easy to check that for any $y_0 \in L^2((0,1);\mathbb{C})$ and $(u,v) \in L^2_{\mathbb{F}}(0,T;L^2(G_0;\mathbb{C})) \times L^2_{\mathbb{F}}(0,T;L^2((0,1);\mathbb{C}))$, equation (1.1) admits a unique weak solution $y \in \mathcal{H}_1$ in the case of (1.2), or $y \in \mathcal{H}_2$ in the case of (1.3).

The main purpose of this paper is to study the null controllability and observability for forward linear stochastic complex degenerate/singular Ginzburg–Landau equation (1.1). The null controllability of (1.1) is formulated as follows. For any $y_0 \in L^2((0,1);\mathbb{C})$, one can find a pair of control $(u,v) \in L^2_{\mathbb{F}}(0,T;L^2(G_0;\mathbb{C})) \times L^2_{\mathbb{F}}(0,T;L^2((0,1);\mathbb{C}))$, such that the solution $y$ to (1.1) satisfies $y(T) = 0$ in $(0,1)$, $\mathcal{P}$-*a.s.*

On the other hand, the observability for (1.1) is stated as follows. If $u = v = 0$ in (1.1), find (if possible) a positive generic constant $\mathcal{C}_1 = \mathcal{C}_1(a,b,c,d)$ such that for any $y_0 \in L^2((0,1);\mathbb{C})$, the solution $y$ to (1.1) satisfies that

$$\mathbb{E}\int_0^1 |y(T,x)|^2\mathrm{d}x \le \mathcal{C}_1(a,b,c,d)\mathbb{E}\int_0^T\int_{G_0}|y(x,t)|^2\mathrm{d}x\mathrm{d}t. \tag{1.5}$$

The observability is one of the most important properties in structural theory. The observability inequality (1.5) means that the terminal value can be dominated by its local information of any solution to (1.1) in $G_0 \times (0,T)$. Such kind of inequalities are closely related to control problems, unique continuation properties and inverse problems.

In the last decades, the theory of controllability and observability for deterministic and stochastic uniformly parabolic equations has been largely developed (see [11, 13, 14, 28] and the references therein). More recently, there are several papers which are concerned with the control problems for deterministic and stochastic degenerate equations (see [4, 20, 31]). In addition, parabolic equations with singular potentials have also been extensively studied. In this aspect, we refer to [9, 29, 30] for deterministic system, and [15, 32] for stochastic system.

There are also some known controllability and observability results about the deterministic and stochastic complex Ginzburg–Landau equations (see [10–12, 23, 24] and the references therein). However, as far as we know, nothing about the null controllability and observability are known for stochastic complex degenerate/singular Ginzburg–Landau equations. In this paper, we study the controllability and observability problems of the general forward linear stochastic complex degenerate/singular Ginzburg–Landau equations for different critical cases of the exponents $\alpha, \beta$ and parameter $\mu$.

Choosing appropriate coefficients in (1.1), one can get some classical deterministic/stochastic partial differential equations. For example, if choose $a = c = 1$, $b = d = 0$, and all functions are real-valued, then (1.1) reduces to a real-valued stochastic degenerate/singular parabolic equation. The corresponding deterministic controllability problem is studied in [29] for the cases of *sub-critical i.e.*, (1.2) and *critical i.e.*, (1.3). If choose $a = 1$, $b = c = d = 0$, and all functions are real-valued, then (1.1) reduces to a stochastic degenerate parabolic equation, and the corresponding Carleman estimate is established in [20]. In what follows, let us introduce some assumptions on the exponents $\alpha$, $\beta$, the coefficients $a$, $b$, $c$, $d$, and the parameter $\mu$ in this paper:

($\mathbf{H_1}$): Assume that (1.2$a$) holds, $bc = ad$, $c \ge 0$ and $a > 0$;

($\mathbf{H_2}$): Assume that (1.2$b$) or (1.3) holds, $bc = ad$, $c \ge 0$, $a > 2c$ and $a^2 + b^2 \ge ac + bd$;

($\mathbf{H_3}$): Assume that (1.2$b$) holds, $bc = ad$, $c \geq 0$, $a > 0$, $a \geq c$ and $a^2 + b^2 \geq ac + bd$.

The controllability result for forward linear stochastic complex degenerate Ginzburg–Landau equation (1.1) can be stated as follows.

**Theorem 1.2.** *Assume that* ($\mathbf{H_1}$) *or* ($\mathbf{H_2}$) *holds. Then, the system* (1.1) *is null controllable.*

**Remark 1.3.** The assumption condition $bc = ad$ is technical, which shows up in some cross terms of the weighted identify, and we now do not know how to drop it.

**Remark 1.4.** The assumption condition $a > 2c$ in ($\mathbf{H_2}$) is not needed in deterministic case. The detailed explanations can refer to Remark 3.8.

**Remark 1.5.** In the case of deterministic degenerate/singular equation, *i.e.*, $a = c = 1$, $b = d = 0$, $c_2 = v = 0$, and choosing all functions to be real-valued in (1.1), then one can get the null controllability result for forward degenerate/singular parabolic equations from Theorem 1.2. This is the main result in [29].

By the standard duality argument, the controllability of (1.1) is transformed to prove the observability estimate of the following backward linear stochastic complex degenerate/singular Ginzburg–Landau equation:

$$\begin{cases} \mathrm{d}w + (a - \boldsymbol{i}b)(x^\alpha w_x)_x \mathrm{d}t + (c - \boldsymbol{i}d)\dfrac{\mu}{x^\beta} w \mathrm{d}t = (-\bar{c}_1 w - \bar{c}_2 W)\mathrm{d}t + W \mathrm{d}B(t) & \text{in } Q, \\ w(t,1) = 0 & \text{on } (0,T), \\ \begin{cases} w(t,0) = 0 & \text{if } 0 \leq \alpha < 1, \\ \big(x^\alpha w_x\big)(t,0) = 0 & \text{if } 1 \leq \alpha < 2, \end{cases} & \text{on } (0,T), \\ w(T,x) = w_T(x) & \text{in } (0,1), \end{cases} \tag{1.6}$$

where $w_T \in L^2(\Omega, \mathcal{F}_T, \mathcal{P}; L^2((0,1);\mathbb{C}))$.

Also, similar to the method used in [33], for any $w_T \in L^2(\Omega, \mathcal{F}_T, \mathcal{P}; L^2((0,1);\mathbb{C}))$, (1.6) admits a unique weak solution $(w, W) \in \mathcal{H}_1 \times L^2_{\mathbb{F}}(0,T;L^2((0,1);\mathbb{C}))$ in the case of (1.2), or $(w, W) \in \mathcal{H}_2 \times L^2_{\mathbb{F}}(0,T;L^2((0,1);\mathbb{C}))$ in the case of (1.3).

The corresponding observability estimates for backward linear stochastic complex degenerate/singular Ginzburg–Landau equation (1.6) are established.

**Proposition 1.6.** *Assume that* ($\mathbf{H_1}$) *or* ($\mathbf{H_2}$) *holds. Then, for any* $w_T \in L^2(\Omega, \mathcal{F}_T, \mathcal{P}; L^2((0,1);\mathbb{C}))$, *the solution to* (1.6) *satisfies*

$$\int_0^1 |w(0,x)|^2 \mathrm{d}x \leq \mathcal{C}_2(a,b,c,d)\Big(\mathbb{E}\int_0^T \int_{G_0} |w(x,t)|^2 \mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q |W(x,t)|^2 \mathrm{d}x\mathrm{d}t\Big). \tag{1.7}$$

*Moreover, in the case of* ($\mathbf{H_1}$), $\mathcal{C}_2(a,b,c,d)$ *is given by*

$$\mathcal{C}_2(a,b,c,d) = \frac{Ce^{C(1+aK_0)}(1 + a^8 + b^8 + c^8 + d^8 + K_0^8)}{a^4(a^2+b^2)}, \tag{1.8}$$

*and the specific form of* $\mathcal{C}_2(a,b,c,d)$ *in* ($\mathbf{H_2}$) *is*

$$\mathcal{C}_2(a,b,c,d) = \frac{C(1 + a^8 + b^8 + c^8 + d^8)}{a^4(a-2c)(a^2+b^2)}, \tag{1.9}$$

where

$$K_0 = C_0\Big(\alpha, \eta, \frac{4c|\mu|}{a} + 1\Big), \tag{1.10}$$

*and $C_0$ will be defined later by (3.4).*

In the rest of paper, unless otherwise stated, we shall denote by $C$ a generic positive constant independent of $a, b, c, d$, which may change from line to line.

**Remark 1.7.** Choosing $\alpha = \mu = 0$ in (1.1), one can obtain the null controllability and observability for the one-dimensional linear stochastic Ginzburg–Landau equation, which is consistent with the results in [11]. Compared with [11], we choose different weighted functions and use the Hardy inequalities to deal with the difficulties caused by degeneracy and singularity. This leads to more complicated assumptions about the coefficients than [11].

On the other hand, the observability estimates for forward equation (1.1) are as follows:

**Theorem 1.8.** *Assume that $u = v = 0$ in (1.1), and $(\mathbf{H_1})$ or $(\mathbf{H_3})$ holds. Then, the observability estimate (1.5) holds for any solution to equation (1.1). Furthermore, in the case of $(\mathbf{H_1})$, $\mathcal{C}_1(a, b, c, d)$ is given by*

$$\mathcal{C}_1(a, b, c, d) = \frac{Ce^{C(1+aK_0)}(1 + a^8 + b^8 + c^8 + d^8 + \widetilde{K}_0^4)}{a^4}, \tag{1.11}$$

*and the specific form of $\mathcal{C}_1(a, b, c, d)$ in $(\mathbf{H_3})$ is*

$$\mathcal{C}_1(a, b, c, d) = \frac{C(1 + a^8 + b^8 + c^8 + d^8)}{a^4}, \tag{1.12}$$

*where $K_0$ is given by (1.10),*

$$\widetilde{K}_0 = C_0(\alpha, \eta, c|\mu| + 1), \tag{1.13}$$

*and $C_0$ will be defined later by (3.4).*

**Remark 1.9.** Notice that Theorem 1.8 is valid only for the *sub-critical* case (*i.e.*, (1.2)). The reason why this result is not available for the *critical* case (*i.e.*, (1.3)) is that the Carleman estimate we established in this case is based on the $H^*_\alpha(0,1)$-norm of the solution instead of $H^1_\alpha(0,1)$-norm.

**Remark 1.10.** When system (1.1) reduces to a real-valued forward stochastic degenerate parabolic equation ($a = 1$, $b = c = d = 0$), the observability estimates for forward stochastic degenerate parabolic equations can be obtained from the above results. These forms are the same as the known one given in [20].

From the observability estimate (1.5), the unique continuation property of the general forward stochastic degenerate/singular parabolic equations can be obtained immediately.

**Corollary 1.11.** *Assume that $(\mathbf{H_1})$ or $(\mathbf{H_3})$ holds. If $b, d = 0$ and $y = 0$ in $G_0 \times (0, T)$, $\mathcal{P}$-a.s., then by Theorem 1.8 and the backward uniqueness of the stochastic parabolic equations, it is clear that $y(t) = 0$ in $(0, 1)$, $\mathcal{P}$-a.s., for all $t \in [0, T]$.*

From the observability constants (1.11) and (1.12) in Theorem 1.8, we have the following limit behavior of coefficients in (1.1).

**Corollary 1.12.** *Assume that* $(\mathbf{H_1})$ *or* $(\mathbf{H_3})$ *holds. If* $b, d \to 0$, *then the observability estimate* (1.5) *also holds for stochastic parabolic equations with degeneracy and singularity.*

**Remark 1.13.** It is obviously that blow-up phenomena for constant $\mathcal{C}_1(a, b, c, d)$ could occur when $a \to 0$, which means that the internal observability estimate cannot be obtained by using our method for stochastic degenerate Schrödinger equations with singularity. However, the corresponding boundary observability estimate can be derived by Theorem 3.2.

The rest of this paper is organized as follows. In Section 2, we give a pointwise weighted identity for linear stochastic complex degenerate/singular Ginzburg–Landau operator. In Section 3, the global Carleman estimates for the forward and backward linear stochastic degenerate/singular Ginzburg–Landau equations are established. Finally, in Section 4, we prove the main results.

## 2. A weighted identity for linear stochastic complex degenerate/singular Ginzburg–Landau operator

In this section, we are devoted to establishing a pointwise weighted identity for the following linear stochastic complex degenerate/singular Ginzburg–Landau operator:

$$\mathcal{L}p = \mathrm{d}p - (a + \boldsymbol{i}b)(x^\alpha p_x)_x \mathrm{d}t - (c + \boldsymbol{i}d)\frac{\mu}{x^\beta} p \mathrm{d}t, \tag{2.1}$$

which will play a crucial role in the sequel.

First, define two unbounded operators $\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subseteq L^2(0,1) \to L^2(0,1)$ and $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq L^2(0,1) \to L^2(0,1)$ as follows:

$$\mathcal{A}_0 y := (a + \boldsymbol{i}b)(x^\alpha y_x)_x, \ \mathcal{D}(\mathcal{A}_0) = \Big\{ \ y \in H^1_\alpha(0,1) \cap H^2_{loc}((0,1]) \ \Big| \ (x^\alpha y_x)_x \in L^2(0,1) \Big\},$$

where $y \in H^2_{loc}((0,1])$ denotes $y_{xx} \in L^2_{loc}((0,1])$, and

$$\mathcal{A}y := (a + \boldsymbol{i}b)(x^\alpha y_x)_x + (c + \boldsymbol{i}d)\frac{\mu}{x^\beta} y.$$

In the case of *sub-critical i.e.*, (1.2),

$$D(\mathcal{A}) = \Big\{ \ y \in H^1_\alpha(0,1) \cap H^2_{loc}((0,1]) \ \Big| \ (x^\alpha y_x)_x + \frac{\mu}{x^\beta} y \in L^2(0,1) \Big\}.$$

In the case of *critical i.e.*, (1.3), for $\alpha \in [0,1)$,

$$D(\mathcal{A}) = \Big\{ \ y \in H^*_\alpha(0,1) \cap H^2_{loc}((0,1]) \ \Big| \ (x^\alpha y_x)_x + \frac{\mu(\alpha)}{x^{2-\alpha}} y \in L^2(0,1) \ \Big\},$$

and for $\alpha \in [1,2)$,

$$D(\mathcal{A}) = \Big\{ \ y \in H^*_\alpha(0,1) \cap H^2_{loc}((0,1]) \ \Big| \ (x^\alpha y_x)_x + \frac{\mu(\alpha)}{x^{2-\alpha}} y \in L^2(0,1) \text{ and } (x^\alpha y_x)(0) = 0 \ \Big\}.$$

We denote

$$\mathcal{L}_1 p = \mathrm{d}p - \mathcal{A}_0 p \mathrm{d}t,$$

then

$$\mathcal{L}p = \mathcal{L}_1 p - (c + \boldsymbol{i}d)\frac{\mu}{x^\beta}p\mathrm{d}t. \tag{2.2}$$

For a fixed weight function $\ell \in C^3(Q;\mathbb{R})$ and auxiliary function $\Phi \in C^1(Q;\mathbb{C})$, we set

$$\theta = e^\ell, \quad z = \theta p.$$

Then, by an elementary calculation, we can get that

$$\theta\mathcal{L}_1 p = I_1 \mathrm{d}t + I_2, \tag{2.3}$$

where

$$\begin{cases} A = x^\alpha \ell_x^2 - (x^\alpha \ell_x)_x, \quad \Lambda = (x^\alpha z_x)_x + Az, \\ I_1 = -a\Lambda + 2\boldsymbol{i}bx^\alpha \ell_x z_x + (\Phi - \ell_t)z, \\ I_2 = \mathrm{d}z - \boldsymbol{i}b\Lambda \mathrm{d}t + 2ax^\alpha \ell_x z_x \mathrm{d}t - \Phi z\mathrm{d}t. \end{cases} \tag{2.4}$$

By (2.2) and (2.3), it is easy to check that

$$\theta\mathcal{L}p = \theta\mathcal{L}_1 p - (c + \boldsymbol{i}d)\frac{\mu}{x^\beta}z\mathrm{d}t = J_1 \mathrm{d}t + J_2, \tag{2.5}$$

where

$$J_1 = I_1 - c\frac{\mu}{x^\beta}z, \quad J_2 = I_2 - \boldsymbol{i}d\frac{\mu}{x^\beta}z\mathrm{d}t. \tag{2.6}$$

We have the following pointwise weighted identity for the operator $\mathcal{L}$ in (2.1).

**Theorem 2.1.** *Suppose that $\ell \in C^3(Q;\mathbb{R})$ and $\Phi \in C^1(Q;\mathbb{C})$. Let $p$ be a $\mathcal{D}(\mathcal{A})$-valued continuous semimartingale. Set $\theta = e^\ell$ and $z = \theta p$. Then, for a.e. $(x,t) \in Q$ and $\mathcal{P}$-a.s. $\omega \in \Omega$, it holds that*

$$\begin{aligned}
&2\mathrm{Re}\,(\theta\overline{J_1}\mathcal{L}p) \\
&= 2|J_1|^2\mathrm{d}t + \Big[V + 2(ad - bc)\frac{\mu}{x^\beta}x^\alpha \mathrm{Im}\,(z_x\overline{z})\mathrm{d}t - 2(ac + bd)\frac{\mu}{x^\beta}x^\alpha \ell_x |z|^2\mathrm{d}t\Big]_x \\
&\quad + \mathrm{d}\Big(M - c\frac{\mu}{x^\beta}|z|^2\Big) + B|z|^2\mathrm{d}t + D|z_x|^2\mathrm{d}t + 2\big[\mathrm{Re}\,(aE\overline{z}z_x) + \mathrm{Im}\,(bFz\overline{z}_x)\big]\mathrm{d}t \\
&\quad - ax^\alpha|\mathrm{d}z_x|^2 + (\ell_t + aA)|\mathrm{d}z|^2 - 2bx^\alpha \ell_x \mathrm{Im}\,(\mathrm{d}z\mathrm{d}\overline{z}_x) + 2\big[b(x^\alpha \ell_x)_x \mathrm{Im}\,(z\mathrm{d}\overline{z}) + \mathrm{Re}\,(\overline{\Phi}\overline{z}\mathrm{d}z)\big] \\
&\quad + c\frac{\mu}{x^\beta}|\mathrm{d}z|^2 + 2(ad - bc)\beta\mu\frac{x^\alpha}{x^{\beta+1}}\mathrm{Im}\,(\overline{z}z_x)\mathrm{d}t + B_1|z|^2\mathrm{d}t,
\end{aligned} \tag{2.7}$$

where $A$, $J_1$ are given by (2.4), (2.6), respectively, and

$$\begin{cases} B = 2(a^2+b^2)(Ax^\alpha \ell_x)_x + aA_t + 2aA\mathrm{Re}\,\Phi - 2bA\mathrm{Im}\,\Phi + 2\mathrm{Re}\,(\Phi\ell_t) - 2|\Phi|^2 + \ell_{tt}, \\ M = -aA|z|^2 + x^\alpha\big[a|z_x|^2 + 2b\ell_x \mathrm{Im}\,(\overline{z}_x z)\big] - \ell_t|z|^2, \\ V = -2ax^\alpha \mathrm{Re}\,(z_x \mathrm{d}\overline{z}) - 2bx^\alpha \ell_x \mathrm{Im}\,(z\mathrm{d}\overline{z}) - 2A(a^2+b^2)x^\alpha \ell_x |z|^2 \mathrm{d}t \\ \qquad +2ax^\alpha \mathrm{Re}\,(\overline{z}_x \Phi z)\mathrm{d}t + 2bx^\alpha \mathrm{Im}\,\big[z_x(\overline{\Phi}-\ell_t)\overline{z}\big]\mathrm{d}t - 2(a^2+b^2)x^{2\alpha}\ell_x|z_x|^2\mathrm{d}t, \\ D = 2bx^\alpha \mathrm{Im}\,\Phi - 2ax^\alpha \mathrm{Re}\,\Phi + 2(a^2+b^2)\big[2x^\alpha(x^\alpha\ell_x)_x - (x^{2\alpha}\ell_x)_x\big], \\ E = x^\alpha\big(2\ell_x\overline{\Phi} - 2\ell_x\ell_t - \overline{\Phi}_x\big), \\ F = x^\alpha\Phi_x - 2x^\alpha\ell_{tx} - 2x^\alpha\ell_x\Phi, \\ B_1 = 2(ac+bd)\dfrac{\mu}{x^\beta}(x^\alpha\ell_x)_x - 2(ac+bd)\beta\mu\dfrac{x^\alpha}{x^{\beta+1}}\ell_x + 2c\dfrac{\mu}{x^\beta}\mathrm{Re}\,\Phi - 2d\dfrac{\mu}{x^\beta}\mathrm{Im}\,\Phi. \end{cases} \tag{2.8}$$

To prove Theorem 2.1, we first recall the following known result.

**Lemma 2.2.** *Suppose that $\ell \in C^3(Q;\mathbb{R})$ and $\Phi \in C^1(Q;\mathbb{C})$. Let $p$ be a $\mathcal{D}(\mathcal{A}_0)$-valued continuous semimartingle. Set $\theta = e^\ell$ and $z = \theta p$. Then, for a.e. $(x,t) \in Q$ and $\mathcal{P}$-a.s. $\omega \in \Omega$, it holds that*

$$\begin{aligned} 2\mathrm{Re}\,(\theta\overline{I_1}\mathcal{L}_1 p) &= 2|I_1|^2\mathrm{d}t + \mathrm{d}M + V_x + B|z|^2\mathrm{d}t + D|z_x|^2\mathrm{d}t \\ &\quad +2\big[\mathrm{Re}\,(aE\overline{z}z_x) + \mathrm{Im}\,(bFz\overline{z}_x)\big]\mathrm{d}t - ax^\alpha|\mathrm{d}z_x|^2 + (\ell_t + aA)|\mathrm{d}z|^2 \\ &\quad -2bx^\alpha\ell_x\mathrm{Im}\,(\mathrm{d}z\mathrm{d}\overline{z}_x) + 2\big[b(x^\alpha\ell_x)_x\mathrm{Im}\,(z\mathrm{d}\overline{z}) + \mathrm{Re}\,(\overline{\Phi}\overline{z}\mathrm{d}z)\big], \end{aligned}$$

where $A$, $I_1$ are given by (2.4), and $M, V, B, D, E, F$ are given by (2.8).

**Proof.** In [11], by choosing $n = 1$, $a^{11} = x^\alpha$, $a_0 = 1$, and $\mathbf{b_0} = \mathbf{0}$, we can get the result immediately. □

From this Lemma, we give the proof of Theorem 2.1.

**Proof of Theorem 2.1.** By (2.5), (2.6), and $\mathrm{Re}\,(\boldsymbol{i}\mathbf{c}) = \mathrm{Im}\,\overline{\mathbf{c}}$, it is easy to see that

$$\begin{aligned} 2\mathrm{Re}\,(\theta\overline{J_1}\mathcal{L}p) &= 2|J_1|^2\mathrm{d}t + 2\mathrm{Re}\,(\overline{J_1}J_2) \\ &= 2|J_1|^2\mathrm{d}t + 2\mathrm{Re}\,(\overline{I_1}I_2) - 2c\frac{\mu}{x^\beta}\mathrm{Re}\,(\overline{z}I_2) - 2d\frac{\mu}{x^\beta}\mathrm{Im}\,(\overline{z}I_1)\mathrm{d}t. \end{aligned} \tag{2.9}$$

By (2.3), we can obtain that

$$2\mathrm{Re}\,(\overline{I_1}I_2) = 2\mathrm{Re}\,(\overline{I_1}\theta\mathcal{L}_1 p) - 2|I_1|^2\mathrm{d}t. \tag{2.10}$$

Next, we compute the last two terms in the right side of sign of the equality (2.9). By the definition of $I_2$ in (2.4), and a simple calculation, we have that

$$-2c\frac{\mu}{x^\beta}\mathrm{Re}\,(\overline{z}\mathrm{d}z) = -\mathrm{d}\Big(c\frac{\mu}{x^\beta}|z|^2\Big) + c\frac{\mu}{x^\beta}|\mathrm{d}z|^2, \tag{2.11}$$

and

$$
\begin{aligned}
&2bc\frac{\mu}{x^\beta}\mathrm{Re}\,(\boldsymbol{i}\Lambda\overline{z}\mathrm{d}t) = 2bc\frac{\mu}{x^\beta}\mathrm{Re}\left[\boldsymbol{i}(x^\alpha z_x)_x\overline{z}\mathrm{d}t\right]\\
&= -2bc\frac{\mu}{x^\beta}\mathrm{Im}\left[(x^\alpha z_x)_x\overline{z}\mathrm{d}t\right]\\
&= -2bc\Big[\frac{\mu}{x^\beta}x^\alpha\mathrm{Im}\,(\overline{z}z_x)\mathrm{d}t\Big]_x - 2bc\beta\mu\frac{x^\alpha}{x^{\beta+1}}\mathrm{Im}\,(\overline{z}z_x)\mathrm{d}t.
\end{aligned}
\tag{2.12}
$$

Further,

$$
\begin{aligned}
&-2c\frac{\mu}{x^\beta}\mathrm{Re}\left(2ax^\alpha\ell_x z_x\overline{z}\mathrm{d}t - \Phi z\overline{z}\mathrm{d}t\right)\\
&= -2ac\Big(\frac{\mu}{x^\beta}x^\alpha\ell_x|z|^2\mathrm{d}t\Big)_x + 2ac\frac{\mu}{x^\beta}(x^\alpha\ell_x)_x|z|^2\mathrm{d}t - 2ac\beta\mu\frac{x^\alpha}{x^{\beta+1}}\ell_x|z|^2\mathrm{d}t + 2c\frac{\mu}{x^\beta}\mathrm{Re}\,\Phi|z|^2\mathrm{d}t.
\end{aligned}
\tag{2.13}
$$

Similarly, by the definition of $I_1$ in (2.4) and a simple calculation, it holds that

$$
\begin{aligned}
-2\frac{\mu}{x^\beta}\mathrm{Im}\,(\overline{z}I_1) = {}& 2a\Big[\frac{\mu}{x^\beta}x^\alpha\mathrm{Im}\,(\overline{z}z_x)\Big]_x + 2a\beta\frac{\mu}{x^{\beta+1}}x^\alpha\mathrm{Im}\,(\overline{z}z_x) - 2b\Big(\frac{\mu}{x^\beta}x^\alpha\ell_x|z|^2\Big)_x\\
&+2b\frac{\mu}{x^\beta}(x^\alpha\ell_x)_x|z|^2 - 2b\beta\frac{\mu}{x^{\beta+1}}x^\alpha\ell_x|z|^2 - 2\frac{\mu}{x^\beta}\mathrm{Im}\,\Phi|z|^2.
\end{aligned}
\tag{2.14}
$$

Finally, combining (2.9)–(2.14) with Lemma 2.2, we can get (2.7). □

## 3. Carleman estimates

### 3.1. The global Carleman estimates for forward linear stochastic complex degenerate/singular Ginzburg–Landau operator

In this subsection, based on the pointwise weighted identity in Theorem 2.1, we will derive global Carleman estimates for the following forward linear stochastic complex degenerate/singular Ginzburg–Landau equation:

$$
\begin{cases}
\mathrm{d}p - (a+\boldsymbol{i}b)(x^\alpha p_x)_x\mathrm{d}t - (c+\boldsymbol{i}d)\dfrac{\mu}{x^\beta}p\mathrm{d}t = F_1\mathrm{d}t + F_2\mathrm{d}B(t) & \text{in } Q,\\
p(t,1) = 0 & \text{on } (0,T),\\
\begin{cases} p(t,0) = 0 & \text{if } 0\le\alpha<1,\\ (x^\alpha p_x)(t,0) = 0 & \text{if } 1\le\alpha<2,\end{cases} & \text{on } (0,T),\\
p(0,x) = p_0(x) & \text{in } (0,1),
\end{cases}
\tag{3.1}
$$

where $F_1 \in L^2_{\mathbb{F}}(0,T;L^2((0,1);\mathbb{C}))$, $F_2 \in L^2_{\mathbb{F}}(0,T;H^1_\alpha((0,1);\mathbb{C}))$, and $p_0 \in L^2((0,1);\mathbb{C})$.

Actually, this problem is strongly related to Hardy's inequality:

$$
\mu(\alpha)\int_0^1 \frac{|z|^2}{x^{2-\alpha}}\mathrm{d}x \le \int_0^1 x^\alpha|z_x|^2\mathrm{d}x, \quad \forall z\in C_0^\infty(0,1).
\tag{3.2}
$$

In order to deal with singular terms of the degenerate/singular operator (2.1), we introduce the following improved Hardy-Poincaré inequalities (see [29]):

**Lemma 3.1.** *Let $\alpha \in [0,2)$ be given. For all $m > 0$ and $\eta < 2-\alpha$, there exists a positive constant $C_0 = C_0(\alpha,\eta,m)$, such that for all $z \in C_0^\infty(0,1)$, the following inequality holds:*

$$\int_0^1 x^\alpha z_x^2 \mathrm{d}x + C_0 \int_0^1 z^2 \mathrm{d}x \geq \mu(\alpha) \int_0^1 \frac{z^2}{x^{2-\alpha}} \mathrm{d}x + m \int_0^1 \frac{z^2}{x^\eta} \mathrm{d}x, \tag{3.3}$$

where $\mu(\alpha)$ is given by (1.2). Besides, $C_0$ is explicitly given by

$$C_0 = C_0(\alpha,\eta,m) = (m+1)^{\frac{2-\alpha+\eta}{2-\alpha-\eta}} \frac{2-\alpha+\eta}{2-\alpha-\eta} \Big[ \frac{4\eta}{(2-\alpha)^2 - \eta^2} \Big]^{\frac{2\eta}{2-\alpha-\eta}}. \tag{3.4}$$

For any positive parameter $s$, write

$$\begin{aligned} &\gamma(t) = \frac{1}{t^k (T-t)^k}, \ \phi(x) = \frac{x^{2-\alpha} - 2}{(2-\alpha)^2}, \ \varphi(t,x) = \gamma(t)\phi(x), \\ &\ell = s\varphi(t,x), \ \theta = e^\ell, \ \text{and } k = 1 + \frac{2-\alpha}{\eta}. \end{aligned} \tag{3.5}$$

In the sequel, for any $n \in \mathbb{N}$, we denote by $\mathcal{O}(s^n)$ a function of order $s^n$, for sufficiently large $s$. Then, we give the following Carleman estimates.

**Theorem 3.2.** *Let $\eta$ be given such that $0 < \eta < 2-\alpha$, $F_1 \in L^2_{\mathbb{F}}(0,T;L^2((0,1);\mathbb{C}))$, $F_2 \in L^2_{\mathbb{F}}(0,T;H^1_\alpha((0,1);\mathbb{C}))$, and $p_0 \in L^2((0,1);\mathbb{C})$.*

**(i)** *Assume that $(\mathbf{H_1})$ holds. Then, there exist two positive constants $s_0 = s_0(\alpha,\eta,\mu,a,b,c,d)$ and $C$, such that for all $s \geq s_0$, every solution $p$ to (3.1) satisfies*

$$\begin{aligned} &\frac{1}{(2-\alpha)^2} \mathbb{E} \int_Q \theta^2 s^3 \gamma^3 x^{2-\alpha} |p|^2 \mathrm{d}x\mathrm{d}t + \mathbb{E} \int_Q \theta^2 s\gamma \Big( x^\alpha |p_x|^2 + \frac{1}{x^\eta} |p|^2 \Big) \mathrm{d}x\mathrm{d}t \\ &+ (1-\alpha)^2 \mathbb{E} \int_Q \theta^2 s\gamma \frac{1}{x^{2-\alpha}} |p|^2 \mathrm{d}x\mathrm{d}t \\ &\leq C_1 \mathbb{E} \int_Q \theta^2 \big( |F_1|^2 + s^2\gamma^2 x^{2-\alpha} |F_2|^2 + s\gamma^{1+\frac{1}{k}} |F_2|^2 + x^\alpha |F_{2,x}|^2 \big) \mathrm{d}x\mathrm{d}t \\ &\quad + C\mathbb{E} \int_0^T s\gamma\theta^2(t,1) |p_x(t,1)|^2 \mathrm{d}t, \end{aligned} \tag{3.6}$$

*where*

$$C_1 = \frac{C(1 + a + |b| + \widetilde{K}_0)}{a^2 + b^2}, \tag{3.7}$$

*and $\widetilde{K}_0$ is given by (1.13).*

**(ii)** *Assume that* ($\mathbf{H_3}$) *holds. Then, there exist two positive constants* $s_1 = s_1(\alpha, \eta, a, b, c, d)$ *and* $C$*, such that for all* $s \geq s_1$*, every solution* $p$ *to* (3.1) *satisfies*

$$\begin{aligned}&\frac{1}{(2-\alpha)^2}\mathbb{E}\int_Q \theta^2 s^3\gamma^3 x^{2-\alpha}|p|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2 s\gamma\Big(x^\alpha|p_x|^2 + \frac{1}{x^\eta}|p|^2\Big)\mathrm{d}x\mathrm{d}t\\ &\leq C_2\mathbb{E}\int_Q \theta^2\big(|F_1|^2 + s^2\gamma^2x^{2-\alpha}|F_2|^2 + s\gamma^{1+\frac{1}{k}}|F_2|^2 + x^\alpha|F_{2,x}|^2\big)\mathrm{d}x\mathrm{d}t\\ &\quad + C\mathbb{E}\int_0^T s\gamma\theta^2(t,1)|p_x(t,1)|^2\mathrm{d}t,\end{aligned}\tag{3.8}$$

*where*

$$C_2 = \frac{C(1+a+|b|+c)}{a^2+b^2}.\tag{3.9}$$

**Remark 3.3.** The condition $a \geq c$ is not needed in the case of (**ii**), but it is necessary in the observability inequality. To avoid confusion about the conditions, we relax the conditions here to be consistent with those for the observability estimates.

**Remark 3.4.** In this theorem, we assume that ($\mathbf{H_3}$) holds, that is, $\alpha, \beta, \mu$ satisfy $\alpha \in [0,2)\backslash\{1\}, \beta = 2-\alpha, \mu < \mu(\alpha)$. However, from the proof of **(ii)**, for the critical case (1.3), *i.e.*, $\alpha \in [0,2)\backslash\{1\}, \beta = 2-\alpha, \mu = \mu(\alpha)$, we can also obtain the following Carleman estimate:

$$\begin{aligned}&\frac{1}{(2-\alpha)^2}\mathbb{E}\int_Q \theta^2 s^3\gamma^3 x^{2-\alpha}|p|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2 s\gamma\frac{1}{x^\eta}|p|^2\mathrm{d}x\mathrm{d}t\\ &\quad+\mathbb{E}\int_Q \theta^2 s\gamma\Big[x^\alpha|p_x|^2 - \frac{\mu(\alpha)}{x^{2-\alpha}}|p|^2\Big]\mathrm{d}x\mathrm{d}t\\ &\leq C_2\mathbb{E}\int_Q \theta^2\big(|F_1|^2 + s^2\gamma^2x^{2-\alpha}|F_2|^2 + s\gamma^{1+\frac{1}{k}}|F_2|^2 + x^\alpha|F_{2,x}|^2\big)\mathrm{d}x\mathrm{d}t\\ &\quad + C\mathbb{E}\int_0^T s\gamma\theta^2(t,1)|p_x(t,1)|^2\mathrm{d}t.\end{aligned}$$

**Remark 3.5.** The condition $a^2 + b^2 \geq ac + bd$ in (**ii**) is used to estimate the following term:

$$2(a^2+b^2)\mathbb{E}\int_Q s\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t - 2(ac+bd)\mathbb{E}\int_Q \frac{\mu}{x^{2-\alpha}}s\gamma|z|^2\mathrm{d}x\mathrm{d}t,$$

whose detailed calculation is presented in the **Second case** of **Step 4** in the proof below. Moreover, this condition is not inconsistent with the existing results. In the deterministic case of (**ii**), *i.e.*, choosing $a = c = 1$, $b = d = 0$, $F_2 = 0$, and all functions to be real-valued in (3.1), the corresponding Carleman estimates are the same as the known result in [29].

**Remark 3.6.** From the above results, if choose $a, c \to 0$, $b = d = 1$ in (3.1), the Carleman estimates of stochastic degenerate/singular Schrödinger equations are established. Then, we can obtain null controllability of backward stochastic Schrödinger equations where the control is acted on the boundary. Moreover, choosing $F_2 = 0$ in (3.1), the corresponding results about the deterministic equations can be obtained.

**Proof of Theorem 3.2.** In Theorem 2.1, we choose $\ell$ and $\theta$ as in (3.5). Also, set

$$\Phi = -(a - \boldsymbol{i}b)(x^\alpha \ell_x)_x = -(a - \boldsymbol{i}b)\frac{s\gamma}{2-\alpha}. \tag{3.10}$$

Noting that $\theta(0,x) = \theta(T,x) = 0$ in $[0,1]$, $ad = bc$, integrating (2.7) on $Q$ and taking expectation, one obtains that

$$\mathbb{E}\int_Q 2\mathrm{Re}\left(\theta\overline{J_1}\mathcal{L}p\right)\mathrm{d}x = \mathbb{E}\int_Q 2|J_1|^2\mathrm{d}x\mathrm{d}t + A_1 + A_2 + A_3 + A_4, \tag{3.11}$$

where

$$\begin{aligned}
A_1 &= \mathbb{E}\int_Q \Big[V - 2(ac+bd)\frac{\mu}{x^\beta}x^\alpha \ell_x |z|^2\mathrm{d}t\Big]_x \mathrm{d}x,\\
A_2 &= -a\mathbb{E}\int_Q x^\alpha |\mathrm{d}z_x|^2\mathrm{d}x + \mathbb{E}\int_Q (\ell_t + aA)|\mathrm{d}z|^2\mathrm{d}x - 2b\mathbb{E}\int_Q x^\alpha \ell_x \mathrm{Im}\left(\mathrm{d}z\mathrm{d}\overline{z}_x\right)\mathrm{d}x\\
&\quad + c\mathbb{E}\int_Q \frac{\mu}{x^\beta}|\mathrm{d}z|^2\mathrm{d}x + 2\mathbb{E}\int_Q \left[b(x^\alpha\ell_x)_x \mathrm{Im}\left(z\mathrm{d}\overline{z}\right) + \mathrm{Re}\left(\overline{\Phi}\overline{z}\mathrm{d}z\right)\right]\mathrm{d}x,\\
A_3 &= \mathbb{E}\int_Q B|z|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q D|z_x|^2\mathrm{d}x\mathrm{d}t + 2\mathbb{E}\int_Q \left[\mathrm{Re}\left(aE\overline{z}z_x\right) + \mathrm{Im}\left(bFz\overline{z}_x\right)\right]\mathrm{d}x\mathrm{d}t,\\
A_4 &= \mathbb{E}\int_Q B_1|z|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

In the following, we estimate the terms on the right-hand side of (3.11) one by one.

**Step 1.** Let us estimate $A_1$ in (3.11). Recalling $z(t,1) = 0$ on $(0,T)$ and the definition of $V$ in (2.8), we have

$$\begin{aligned}
A_1 &= \mathbb{E}\int_Q \Big[V - 2(ac+bd)\frac{\mu}{x^\beta}x^\alpha \ell_x |z|^2\mathrm{d}t\Big]_x \mathrm{d}x\\
&= -2(a^2+b^2)\mathbb{E}\int_0^T x^{2\alpha}\ell_x|z_x|^2\mathrm{d}t\Big|_{x=1} - \mathbb{E}\int_0^T \Big[V - 2(ac+bd)\frac{\mu}{x^\beta}x^\alpha\ell_x|z|^2\mathrm{d}t\Big]\Big|_{x=0}.
\end{aligned}$$

The reasonableness of the computations may be delicate since we work in non-standard weighted spaces, specially in the *critical potentials*. For this reason, we make computations that may be justified by the regularization process described in [29]. In order to understand the computations related to $A_1$, it helps to replace $z$ by $z^n := \theta p^n$, where $p^n$ is the solution to the regularized problem in which the potential $\frac{\mu}{x^\beta}$ has been replaced by $\frac{\mu}{(x+\frac{1}{n})^\beta}$. Therefore, the quantity that we actually need to compute is the following one:

$$\mathbb{E}\int_0^T \Big[V^n - 2(ac+bd)\frac{\mu}{(x+\frac{1}{n})^\beta}x^\alpha\ell_x|z^n|^2\mathrm{d}t\Big]\Big|_{x=0},$$

where $V^n$ is obtained by replacing $z$ in $V$ by $z^n$.

For $\alpha \in [0,1)$, we use the boundary condition $z^n(t,0)=0$ on $(0,T)$ to obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T \Big[V^n - 2(ac+bd)\frac{\mu}{(x+\frac{1}{n})^\beta}x^\alpha \ell_x|z^n|^2\mathrm{d}t\Big]\Big|_{x=0}\\
&= -2(a^2+b^2)\mathbb{E}\int_0^T x^{2\alpha}\ell_x|z_x^n|^2\mathrm{d}t\Big|_{x=0} - 2(ac+bd)\mu\mathbb{E}\int_0^T \frac{x^\alpha}{(x+\frac{1}{n})^\beta}\ell_x|z^n|^2\mathrm{d}t\Big|_{x=0}\\
&= -2(a^2+b^2)\mathbb{E}\int_0^T s\gamma\frac{x^{1+\alpha}}{2-\alpha}|z_x^n|^2\mathrm{d}t\Big|_{x=0} - 2(ac+bd)\mu\mathbb{E}\int_0^T \frac{s\gamma}{2-\alpha}\frac{x}{(x+\frac{1}{n})^\beta}|z^n|^2\mathrm{d}t\Big|_{x=0}.
\end{aligned}$$

Recall the regular solution to regularization problem, then we can obtain $z^n \in \mathcal{D}(\mathcal{A}_0)$. By $\alpha\in[0,1)$, we have $\big(x^{1+\alpha}|z_x^n|^2\big)(t,0) = \big(x^{1-\alpha}|x^\alpha z_x^n|^2\big)(t,0)=0$ on $(0,T)$. Then

$$\mathbb{E}\int_0^T \Big[V^n - 2(ac+bd)\frac{\mu}{(x+\frac{1}{n})^\beta}x^\alpha \ell_x|z^n|^2\mathrm{d}t\Big]\Big|_{x=0} = 0.$$

For $\alpha\in[1,2)$, we have

$$(x^\alpha z_x^n)(t,0) = \Big(\frac{x}{2-\alpha}s\gamma z^n\Big)(t,0),\ t\in[0,T],$$

and by the Lemma 9.4 in [29], we can obtain

$$(x|z^n|^2)\big|_{x=0} = \mathrm{Re}\,(xz^n\mathrm{d}\overline{z}^n)\big|_{x=0} = \mathrm{Im}\,(xz^n\mathrm{d}\overline{z}^n)\big|_{x=0} = 0.$$

Therefore, by the definitions of $A$ in (2.4) and $\Phi$ in (3.10), we get that

$$\begin{aligned}
&\mathbb{E}\int_0^T \Big[V^n - 2(ac+bd)\frac{\mu}{(x+\frac{1}{n})^\beta}x^\alpha \ell_x|z^n|^2\mathrm{d}t\Big]\Big|_{x=0}\\
&= 2\mathbb{E}\int_0^T \big[-ax^\alpha\mathrm{Re}\,(z_x^n\mathrm{d}\overline{z}^n) - bx^\alpha\ell_x\mathrm{Im}\,(z^n\mathrm{d}\overline{z}^n)\big]\Big|_{x=0} - 2\mathbb{E}\int_0^T\big[A(a^2+b^2)x^\alpha\ell_x|z^n|^2\\
&\quad -ax^\alpha\mathrm{Re}\,(\Phi\overline{z}_x^n z^n) - bx^\alpha\mathrm{Im}\,(\overline{\Phi}z_x^n\overline{z}^n) + bx^\alpha\ell_t\mathrm{Im}\,(z_x^n\overline{z}^n) + (a^2+b^2)x^{2\alpha}\ell_x|z_x^n|^2\big]\mathrm{d}t\Big|_{x=0}\\
&\quad -2(ac+bd)\mu\mathbb{E}\int_0^T \frac{x^\alpha}{(x+\frac{1}{n})^\beta}\ell_x|z^n|^2\mathrm{d}t\Big|_{x=0}\\
&= 2\mathbb{E}\int_0^T s\gamma\frac{x}{2-\alpha}\big[-a\mathrm{Re}\,(z^n\mathrm{d}\overline{z}^n) - b\mathrm{Im}\,(z^n\mathrm{d}\overline{z}^n)\big]\Big|_{x=0}\\
&\quad -4\mathbb{E}\int_0^T (a^2+b^2)s^3\gamma^3\frac{x^{3-\alpha}}{(2-\alpha)^3}|z^n|^2\mathrm{d}t\Big|_{x=0} - 2(ac+bd)\mu\mathbb{E}\int_0^T \frac{s\gamma}{2-\alpha}\frac{x}{(x+\frac{1}{n})^\beta}|z^n|^2\mathrm{d}t\Big|_{x=0}\\
&= 0.
\end{aligned}$$

Hence,

$$A_1 = -2(a^2+b^2)\mathbb{E}\int_0^T \frac{s\gamma}{2-\alpha}|z_x(t,1)|^2\mathrm{d}t. \tag{3.12}$$

**Step 2.** Let us estimate $A_2$ in (3.11). By (3.1) and $a > 0$, we know that

$$\begin{aligned}
&-a\mathbb{E}\int_Q x^\alpha |\mathrm{d}z_x|^2 \mathrm{d}x = -a\mathbb{E}\int_Q x^\alpha \theta^2 |F_{2,x} + F_2\ell_x|^2 \mathrm{d}x\mathrm{d}t \\
&\geq -Ca\mathbb{E}\int_Q \theta^2 \big(x^\alpha |F_{2,x}|^2 + x^{2-\alpha}s^2\gamma^2|F_2|^2\big)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.13}$$

From the definition of $A$ in (2.4) and noting that $|\gamma_t| \leq C\gamma^{1+\frac{1}{k}}$, one can obtain that

$$\begin{aligned}
&\mathbb{E}\int_Q (\ell_t + aA)|\mathrm{d}z|^2\mathrm{d}x = \mathbb{E}\int_Q (\ell_t + aA)\theta^2|F_2|^2\mathrm{d}x\mathrm{d}t \\
&\geq \mathbb{E}\int_Q \theta^2\Big[\mathcal{O}(s)\gamma^{1+\frac{1}{k}} + as^2\gamma^2\frac{x^{2-\alpha}}{(2-\alpha)^2} - a\frac{s\gamma}{2-\alpha}\Big]|F_2|^2\mathrm{d}x\mathrm{d}t \\
&= \mathbb{E}\int_Q \theta^2\big[(a+1)\mathcal{O}(s)\gamma^{1+\frac{1}{k}} + a\mathcal{O}(s^2)\gamma^2x^{2-\alpha}\big]|F_2|^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.14}$$

Further,

$$\begin{aligned}
&-2b\mathbb{E}\int_Q x^\alpha\ell_x \mathrm{Im}\,(\mathrm{d}z\mathrm{d}\overline{z}_x)\mathrm{d}x = -2b\mathbb{E}\int_Q x^\alpha \ell_x\theta^2\mathrm{Im}\,\big[\mathrm{d}p\mathrm{d}(\ell_x\overline{p} + \overline{p}_x)\big]\mathrm{d}x \\
&\geq 2|b|\mathbb{E}\int_Q \theta^2\Big[\frac{-2}{(2-\alpha)^2}s^2\gamma^2x^{2-\alpha}|F_2|^2 + \mathcal{O}(s)\gamma x|F_2F_{2,x}|\Big]\mathrm{d}x\mathrm{d}t \\
&\geq -C|b|\mathbb{E}\int_Q \theta^2\big(s^2\gamma^2x^{2-\alpha}|F_2|^2 + x^\alpha|F_{2,x}|^2\big)\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.15}$$

By $2\mathrm{Im}\,\mathbf{c} = \boldsymbol{i}(\overline{\mathbf{c}} - \mathbf{c})$ and (3.10), we have

$$\begin{aligned}
&2b(x^\alpha\ell_x)_x\mathrm{Im}\,(z\mathrm{d}\overline{z}) + 2\mathrm{Re}\,(\overline{\Phi}\overline{z}\mathrm{d}z) = \boldsymbol{i}b(x^\alpha\ell_x)_x(\overline{z}\mathrm{d}z - z\mathrm{d}\overline{z}) + (\overline{\Phi}\overline{z}\mathrm{d}z + \Phi z\mathrm{d}\overline{z}) \\
&= \boldsymbol{i}b(x^\alpha\ell_x)_x(\overline{z}\mathrm{d}z - z\mathrm{d}\overline{z}) - a(x^\alpha\ell_x)_x(\overline{z}\mathrm{d}z + z\mathrm{d}\overline{z}) + \boldsymbol{i}b(x^\alpha\ell_x)_x(z\mathrm{d}\overline{z} - \overline{z}\mathrm{d}z) \\
&= -a(x^\alpha\ell_x)_x(\overline{z}\mathrm{d}z + z\mathrm{d}\overline{z}) \\
&= \mathrm{d}\big[-a(x^\alpha\ell_x)_x|z|^2\big] + a(x^\alpha\ell_x)_{xt}|z|^2\mathrm{d}t + a(x^\alpha\ell_x)_x|\mathrm{d}z|^2.
\end{aligned}$$

From $z(0,x) = z(T,x) = 0$ in $(0,1)$, one can see that

$$\begin{aligned}
&2\mathbb{E}\int_Q \big[b(x^\alpha\ell_x)_x\mathrm{Im}\,(z\mathrm{d}\overline{z}) + \mathrm{Re}\,(\overline{\Phi}\overline{z}\mathrm{d}z)\big]\mathrm{d}x \\
&= \mathbb{E}\int_Q \Big\{\mathrm{d}\big[-a(x^\alpha\ell_x)_x|z|^2\big] + a(x^\alpha\ell_x)_{xt}|z|^2\mathrm{d}t + a(x^\alpha\ell_x)_x|\mathrm{d}z|^2\Big\}\mathrm{d}x \\
&\geq a\mathbb{E}\int_Q \frac{s\gamma}{2-\alpha}\theta^2|F_2|^2\mathrm{d}x\mathrm{d}t + a\mathbb{E}\int_Q \mathcal{O}(s)\gamma^{1+\frac{1}{k}}|z|^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.16}$$

And then we estimate "$c\mathbb{E}\int_Q \frac{\mu}{x^\beta}|\mathrm{d}z|^2\mathrm{d}x$" in two cases: the case of a sub-critical exponent $0 < \beta < 2-\alpha$ and the case of a critical exponent $\beta = 2-\alpha$.

**First case:** For $\alpha \in [0,2)$, $0 < \beta < 2-\alpha$, and $\mu \in \mathbb{R}$, we consider here $\eta$ satisfies $\beta \leq \eta < 2-\alpha$. It is obvious that if the result holds true for any $\eta$ such that $\beta \leq \eta < 2-\alpha$, then it also holds true for all $\eta$ such that $0 < \eta < 2-\alpha$.

Using (3.3) with $m = c|\mu| + 1 > 0$, we have

$$
\begin{aligned}
&c\mathbb{E}\int_Q \frac{\mu}{x^\beta}|\mathrm{d}z|^2\mathrm{d}x = c\mathbb{E}\int_Q \frac{\mu}{x^\beta}\theta^2|F_2|^2\mathrm{d}x\mathrm{d}t \geq -c\mathbb{E}\int_Q \frac{|\mu|}{x^\eta}\theta^2|F_2|^2\mathrm{d}x\mathrm{d}t \\
&\geq -\mathbb{E}\int_Q x^\alpha|(\theta F_2)_x|^2\mathrm{d}x\mathrm{d}t - \widetilde{K}_0\mathbb{E}\int_Q \theta^2|F_2|^2\mathrm{d}x\mathrm{d}t \\
&\geq -\mathbb{E}\int_Q x^\alpha\theta^2|F_{2,x}|^2\mathrm{d}x\mathrm{d}t - C\mathbb{E}\int_Q \theta^2 x^{2-\alpha}s^2\gamma^2|F_2|^2\mathrm{d}x\mathrm{d}t - \widetilde{K}_0\mathbb{E}\int_Q \theta^2|F_2|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where $\widetilde{K}_0$ is given by (1.13). Combining the above inequality with (3.13)–(3.16), we can get

$$
\begin{aligned}
A_2 \geq& -(1+a+|b|+\widetilde{K}_0)\mathbb{E}\int_Q \theta^2\big(s^2\gamma^2x^{2-\alpha}|F_2|^2 + s\gamma^{1+\frac{1}{k}}|F_2|^2 + x^\alpha|F_{2,x}|^2\big)\mathrm{d}x\mathrm{d}t \\
&+a\mathbb{E}\int_Q \mathcal{O}(s)\gamma^{1+\frac{1}{k}}|z|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.17}
$$

**Second case:** For $\alpha \in [0,2)\setminus\{1\}, \beta = 2-\alpha$, and $\mu < \mu(\alpha)$. In the case of $0 \leq \mu < \mu(\alpha)$, it is easy to show that

$$
c\mathbb{E}\int_Q \frac{\mu}{x^\beta}|\mathrm{d}z|^2\mathrm{d}x \geq 0.
$$

In the case of $\mu < 0$, by Hardy's inequality (3.2) and $\mu(\alpha) = \frac{(1-\alpha)^2}{4}$, one gets that

$$
\begin{aligned}
&c\mathbb{E}\int_Q \frac{\mu}{x^\beta}|\mathrm{d}z|^2\mathrm{d}x = c\mathbb{E}\int_Q \frac{\mu}{x^{2-\alpha}}\theta^2|F_2|^2\mathrm{d}x\mathrm{d}t \geq c\frac{\mu}{\mu(\alpha)}\mathbb{E}\int_Q x^\alpha|(\theta F_2)_x|^2\mathrm{d}x\mathrm{d}t \\
&\geq -C\mathbb{E}\int_Q c\theta^2\big(x^\alpha|F_{2,x}|^2 + s^2\gamma^2x^{2-\alpha}|F_2|^2\big)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Combining the above inequality with (3.13)–(3.16), we can get

$$
\begin{aligned}
A_2 \geq& -(1+a+|b|+c)\mathbb{E}\int_Q \theta^2\big(s^2\gamma^2x^{2-\alpha}|F_2|^2 + s\gamma^{1+\frac{1}{k}}|F_2|^2 + x^\alpha|F_{2,x}|^2\big)\mathrm{d}x\mathrm{d}t \\
&+a\mathbb{E}\int_Q \mathcal{O}(s)\gamma^{1+\frac{1}{k}}|z|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{3.18}
$$

**Step 3.** Let us estimate $A_3$ in (3.11). By the definitions of $A$ and $\Phi$, we can get

$$
\begin{aligned}
B &= 2(a^2+b^2)(Ax^\alpha\ell_x)_x + aA_t + 2aA\mathrm{Re}\,\Phi - 2bA\mathrm{Im}\,\Phi + 2\mathrm{Re}\,(\Phi\ell_t) - 2|\Phi|^2 + \ell_{tt} \\
&= 2(a^2+b^2)A_xx^\alpha\ell_x + 2ax^\alpha\ell_x\ell_{xt} - a(x^\alpha\ell_{xt})_x - 2a(x^\alpha\ell_x)_x\ell_t \\
&\quad -2(a^2+b^2)\big[(x^\alpha\ell_x)_x\big]^2 + \ell_{tt}.
\end{aligned}
$$

From (3.10), it is easy to see that

$$
\begin{aligned}
D &= 2bx^{\alpha}\mathrm{Im}\,\Phi - 2ax^{\alpha}\mathrm{Re}\,\Phi + 2(a^2+b^2)\big[2x^{\alpha}(x^{\alpha}\ell_x)_x - (x^{2\alpha}\ell_x)_x\big]\\
&= 6(a^2+b^2)x^{\alpha}(x^{\alpha}\ell_x)_x - 2(a^2+b^2)(x^{2\alpha}\ell_x)_x\\
&= 6(a^2+b^2)\frac{1}{2-\alpha}x^{\alpha}s\gamma - 2(a^2+b^2)\frac{1+\alpha}{2-\alpha}x^{\alpha}s\gamma\\
&= 2(a^2+b^2)x^{\alpha}s\gamma.
\end{aligned}
$$

By the definitions of $E$ and $F$ in (2.8), and noting that $\Phi_x=0$, we have

$$
\begin{aligned}
&2\mathrm{Re}\,(aE\overline{z}z_x) + 2\mathrm{Im}\,(bFz\overline{z}_x)\\
&= 4ax^{\alpha}\ell_x\mathrm{Re}\,(\overline{\Phi}\overline{z}z_x) - 4ax^{\alpha}\ell_x\ell_t\mathrm{Re}\,(\overline{z}z_x) - 4bx^{\alpha}\ell_{xt}\mathrm{Im}\,(z\overline{z}_x) - 4bx^{\alpha}\ell_x\mathrm{Im}\,(\Phi z\overline{z}_x)\\
&= 2(a-\boldsymbol{i}b)x^{\alpha}\ell_x\overline{\Phi}\overline{z}z_x + 2(a+\boldsymbol{i}b)x^{\alpha}\ell_x\Phi z\overline{z}_x - 4ax^{\alpha}\ell_x\ell_t\mathrm{Re}\,(\overline{z}z_x) - 4bx^{\alpha}\ell_{xt}\mathrm{Im}\,(z\overline{z}_x)\\
&= -4(a^2+b^2)x^{\alpha}\ell_x(x^{\alpha}\ell_x)_x\mathrm{Re}\,(\overline{z}z_x) - 4ax^{\alpha}\ell_x\ell_t\mathrm{Re}\,(\overline{z}z_x) - 4bx^{\alpha}\ell_{xt}\mathrm{Im}\,(z\overline{z}_x)\\
&= \big[-2(a^2+b^2)x^{\alpha}\ell_x(x^{\alpha}\ell_x)_x|z|^2 - 2ax^{\alpha}\ell_x\ell_t|z|^2\big]_x + 2(a^2+b^2)\big[(x^{\alpha}\ell_x)_x\big]^2|z|^2\\
&\quad +2a(x^{\alpha}\ell_x)_x\ell_t|z|^2 + 2ax^{\alpha}\ell_x\ell_{xt}|z|^2 - 4bx^{\alpha}\ell_{xt}\mathrm{Im}\,(z\overline{z}_x).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&B|z|^2 + D|z_x|^2 + 2a\mathrm{Re}\,(E\overline{z}z_x) + 2b\mathrm{Im}\,(Fz\overline{z}_x)\\
&= \big[-2(a^2+b^2)x^{\alpha}\ell_x(x^{\alpha}\ell_x)_x|z|^2 - 2ax^{\alpha}\ell_x\ell_t|z|^2\big]_x + 2(a^2+b^2)A_x x^{\alpha}\ell_x|z|^2\\
&\quad +4ax^{\alpha}\ell_x\ell_{xt}|z|^2 - a(x^{\alpha}\ell_{xt})_x|z|^2 + \ell_{tt}|z|^2 - 4bx^{\alpha}\ell_{xt}\mathrm{Im}\,(z\overline{z}_x) + 2(a^2+b^2)x^{\alpha}s\gamma|z_x|^2.
\end{aligned}
\tag{3.19}
$$

By $z(t,1)=0$ on $(0,T)$, one can get that

$$
\begin{aligned}
&\mathbb{E}\int_Q \big[-2(a^2+b^2)x^{\alpha}\ell_x(x^{\alpha}\ell_x)_x|z|^2 - 2ax^{\alpha}\ell_x\ell_t|z|^2\big]_x \mathrm{d}x\mathrm{d}t\\
&= \mathbb{E}\int_0^T \big[2(a^2+b^2)x^{\alpha}\ell_x(x^{\alpha}\ell_x)_x|z|^2 - 2ax^{\alpha}\ell_x\ell_t|z|^2\big]\Big|_{x=0}\mathrm{d}t\\
&= \mathbb{E}\int_0^T \Big[2(a^2+b^2)\frac{x}{(2-\alpha)^2}s^2\gamma^2|z|^2 - 2a\frac{x}{2-\alpha}s^2\gamma\gamma_t\phi(x)|z|^2\Big]\Big|_{x=0}\mathrm{d}t\\
&= 0.
\end{aligned}
\tag{3.20}
$$

It is easy to check that

$$
A_x = s^2\gamma^2\frac{x^{1-\alpha}}{2-\alpha},\quad x^{\alpha}\ell_x = s\gamma\frac{x}{2-\alpha}.
$$

Therefore, it holds that

$$
2(a^2+b^2)A_x x^{\alpha}\ell_x = 2(a^2+b^2)s^3\gamma^3\frac{x^{2-\alpha}}{(2-\alpha)^2}.
\tag{3.21}
$$

By observing that $|\gamma_t| \le C\gamma^{1+\frac{1}{k}}$, $|\gamma\gamma_t| \le C\gamma^3$, and $|\gamma_{tt}| \le C\gamma^{1+\frac{2}{k}}$, one can conclude that

$$4ax^\alpha \ell_x \ell_{xt}|z|^2 - a(x^\alpha \ell_{xt})_x|z|^2 + \ell_{tt}|z|^2 = a\mathcal{O}(s^2)\gamma^3 x^{2-\alpha}|z|^2 + (a+1)\mathcal{O}(s)\gamma^{1+\frac{2}{k}}|z|^2. \tag{3.22}$$

Further,

$$-4bx^\alpha \ell_{xt}\mathrm{Im}\,(z\overline{z}_x) = |b|\mathcal{O}(s^2)\gamma^3 x^{2-\alpha}|z|^2 + |b|\mathcal{O}(1)\gamma x^\alpha |z_x|^2. \tag{3.23}$$

By (3.19)–(3.23), we obtain that

$$\begin{aligned} A_3 = {} & \mathbb{E}\int_Q 2(a^2+b^2)s^3\gamma^3\frac{x^{2-\alpha}}{(2-\alpha)^2}|z|^2\mathrm{d}x\mathrm{d}t + (a+|b|)\mathbb{E}\int_Q \mathcal{O}(s^2)\gamma^3 x^{2-\alpha}|z|^2\mathrm{d}x\mathrm{d}t \\ & + |b|\mathbb{E}\int_Q \mathcal{O}(1)\gamma x^\alpha |z_x|^2\mathrm{d}x\mathrm{d}t + (a+1)\mathbb{E}\int_Q \mathcal{O}(s)\gamma^{1+\frac{2}{k}}|z|^2\mathrm{d}x\mathrm{d}t \\ & + 2(a^2+b^2)\mathbb{E}\int_Q x^\alpha s\gamma|z_x|^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.24}$$

**Step 4.** In this part, we compute the last term $A_4$ in (3.11). By the definition of $B_1$ in (2.8), we have

$$\begin{aligned} A_4 &= \mathbb{E}\int_Q B_1|z|^2\mathrm{d}x\mathrm{d}t \\ &= \mathbb{E}\int_Q \Big[2(ac+bd)\mu\frac{s\gamma}{2-\alpha}\frac{1}{x^\beta} - 2(ac+bd)\mu\beta\frac{s\gamma x}{2-\alpha}\frac{1}{x^{\beta+1}} - 2(ac+bd)\frac{\mu}{x^\beta}\frac{s\gamma}{2-\alpha}\Big]|z|^2\mathrm{d}x\mathrm{d}t \\ &= -2(ac+bd)\mathbb{E}\int_Q \mu\beta\frac{s\gamma}{2-\alpha}\frac{1}{x^\beta}|z|^2\mathrm{d}x\mathrm{d}t. \end{aligned}$$

We produce estimate of term $A_4$ in two cases: the case of a sub-critical exponent $0<\beta<2-\alpha$ and the case of a critical exponent $\beta = 2-\alpha$.

**First case:** For $\alpha\in[0,2), 0<\beta<2-\alpha$ and $\mu\in\mathbb{R}$. Similarly, we only need to consider here $\eta$ satisfies $\beta\le\eta<2-\alpha$.

By $ad=bc$, $a>0$ and $c\ge 0$, we know that $ac+bd\ge 0$. Applying (3.3) with

$$m = \frac{2(ac+bd)|\mu|\beta}{(2-\alpha)(a^2+b^2)} + 1 \le \frac{2|\mu|(ac+bd)}{a^2+b^2} + 1$$

gives

$$-\frac{2(ac+bd)|\mu|\beta}{(2-\alpha)}\int_0^1 \frac{|z|^2}{x^\eta}\mathrm{d}x \ge (a^2+b^2)\int_0^1 \Big(\mu(\alpha)\frac{|z|^2}{x^{2-\alpha}} - x^\alpha|z_x|^2 + \frac{|z|^2}{x^\eta}\Big)\mathrm{d}x - K_1\int_0^1 |z|^2\mathrm{d}x,$$

where $K_1 = K_1(a,b,c,d) > 0$ is given by

$$K_1 := (a^2+b^2)C_0\Big(\alpha,\eta,\frac{2|\mu|(ac+bd)}{a^2+b^2}+1\Big).$$

Therefore, one can get

$$\begin{aligned} A_4 &= -2(ac+bd)\mathbb{E}\int_Q \mu\beta \frac{s\gamma}{2-\alpha}\frac{1}{x^\beta}|z|^2 \mathrm{d}x\mathrm{d}t \\ &\geq -\frac{2(ac+bd)|\mu|\beta}{2-\alpha}\mathbb{E}\int_Q s\gamma\frac{|z|^2}{x^\eta}\mathrm{d}x\mathrm{d}t \\ &\geq (a^2+b^2)\mathbb{E}\int_Q s\gamma\Big(\mu(\alpha)\frac{|z|^2}{x^{2-\alpha}} - x^\alpha|z_x|^2 + \frac{|z|^2}{x^\eta}\Big)\mathrm{d}x\mathrm{d}t - K_1\mathbb{E}\int_Q s\gamma|z|^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{3.25}$$

**Second case:** For $\alpha \in [0,2)\setminus\{1\}, \beta = 2-\alpha$ and $\mu < \mu(\alpha)$. Let us fix $0 < \eta < 2-\alpha$. In the present case, we observe that

$$A_4 = \mathbb{E}\int_Q B_1|z|^2\mathrm{d}x\mathrm{d}t = -2(ac+bd)\mathbb{E}\int_Q \frac{\mu}{x^{2-\alpha}}s\gamma|z|^2\mathrm{d}x\mathrm{d}t.$$

Since $\eta < 2-\alpha$, applying (3.3) with $m = 1$ gives

$$\begin{aligned} -(a^2+b^2)\mu(\alpha)\int_0^1 \frac{|z|^2}{x^{2-\alpha}}\mathrm{d}x \geq (a^2+b^2)\int_0^1 \frac{|z|^2}{x^\eta}\mathrm{d}x - (a^2+b^2)\int_0^1 x^\alpha|z_x|^2\mathrm{d}x \\ -K_3\int_0^1 z^2\mathrm{d}x, \end{aligned} \tag{3.26}$$

where $K_3 = K_3(a,b)$ is given by

$$K_3 = (a^2+b^2)C_0(\alpha,\eta,1).$$

Now, we estimate the following term

$$2(a^2+b^2)\mathbb{E}\int_Q s\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t - 2(ac+bd)\mathbb{E}\int_Q \frac{\mu}{x^{2-\alpha}}s\gamma|z|^2\mathrm{d}x\mathrm{d}t.$$

If $\mu \leq 0$, then by (3.26), we have

$$\begin{aligned} &2(a^2+b^2)\mathbb{E}\int_Q s\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t - 2(ac+bd)\mathbb{E}\int_Q \frac{\mu}{x^{2-\alpha}}s\gamma|z|^2\mathrm{d}x\mathrm{d}t \\ &\geq 2(a^2+b^2)\mathbb{E}\int_Q s\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t \\ &\geq (a^2+b^2)\mathbb{E}\int_Q s\gamma\Big(x^\alpha|z_x|^2 + \frac{|z|^2}{x^\eta}\Big)\mathrm{d}x\mathrm{d}t - K_3\mathbb{E}\int_Q s\gamma|z|^2\mathrm{d}x\mathrm{d}t. \end{aligned}$$

If $0<\mu<\mu(\alpha)$, then by (3.26) and noting that $a^2+b^2\geq ac+bd$, one can get

$$\begin{aligned}
&2(a^2+b^2)\mathbb{E}\int_Q s\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t-2(ac+bd)\mathbb{E}\int_Q \frac{\mu}{x^{2-\alpha}}s\gamma|z|^2\mathrm{d}x\mathrm{d}t\\
&\geq (a^2+b^2)\mathbb{E}\int_Q s\gamma\Big(x^\alpha|z_x|^2+\frac{\mu(\alpha)}{x^{2-\alpha}}s\gamma|z|^2\Big)\mathrm{d}x\mathrm{d}t+(a^2+b^2)\mathbb{E}\int_Q s\gamma\frac{|z|^2}{x^\eta}\mathrm{d}x\mathrm{d}t\\
&\quad -2(ac+bd)\mathbb{E}\int_Q s\gamma\frac{\mu}{x^{2-\alpha}}|z|^2\mathrm{d}x\mathrm{d}t-K_3\mathbb{E}\int_Q s\gamma|z|^2\mathrm{d}x\mathrm{d}t\\
&\geq (a^2+b^2)\mathbb{E}\int_Q s\gamma\Big(x^\alpha|z_x|^2-\frac{\mu}{x^{2-\alpha}}s\gamma|z|^2\Big)\mathrm{d}x\mathrm{d}t+(a^2+b^2)\mathbb{E}\int_Q s\gamma\frac{|z|^2}{x^\eta}\mathrm{d}x\mathrm{d}t\\
&\quad +2\big[a^2+b^2-(ac+bd)\big]\mathbb{E}\int_Q s\gamma\frac{\mu}{x^{2-\alpha}}|z|^2\mathrm{d}x\mathrm{d}t-K_3\mathbb{E}\int_Q s\gamma|z|^2\mathrm{d}x\mathrm{d}t\\
&\geq (a^2+b^2)\mathbb{E}\int_Q s\gamma\Big(x^\alpha|z_x|^2-\frac{\mu}{x^{2-\alpha}}|z|^2\Big)\mathrm{d}x\mathrm{d}t+(a^2+b^2)\mathbb{E}\int_Q s\gamma\frac{|z|^2}{x^\eta}\mathrm{d}x\mathrm{d}t\\
&\quad -K_3\mathbb{E}\int_Q s\gamma|z|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Notice that

$$\mathbb{E}\int_Q s\gamma\Big(x^\alpha|z_x|^2-\frac{\mu}{x^{2-\alpha}}s\gamma|z|^2\Big)\mathrm{d}x\mathrm{d}t\geq\Big(1-\frac{\mu}{\mu(\alpha)}\Big)\mathbb{E}\int_Q s\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t.$$

Therefore, combining the above two cases, if $\mu<\mu(\alpha)$, we deduce that

$$\begin{aligned}
A_4\geq\ &\mathbb{E}\int_Q C(a^2+b^2)\mathbb{E}\int_Q s\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t+(a^2+b^2)\mathbb{E}\int_Q s\gamma\frac{|z|^2}{x^\eta}\mathrm{d}x\mathrm{d}t\\
&-2(a^2+b^2)\mathbb{E}\int_Q s\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t-K_3\mathbb{E}\int_Q s\gamma|z|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{3.27}$$

**Step 5.** Now, we estimate $\mathbb{E}\int_Q 2\mathrm{Re}\,(\theta\overline{J_1}\mathcal{L}p)\mathrm{d}x$ in two cases: the case of a sub-critical exponent $0<\beta<2-\alpha$ and the case of a critical exponent $\beta=2-\alpha$.

**First case:** For $\alpha\in[0,2),0<\beta<2-\alpha$, and $\mu\in\mathbb{R}$. Similarly, we only need to consider here $\eta$ satisfies $\beta\leq\eta<2-\alpha$. Combining (3.12), (3.17), (3.24), (3.25) with (3.11), we arrive at

$$\begin{aligned}
&\mathbb{E}\int_Q 2\mathrm{Re}\,(\theta\overline{J_1}\mathcal{L}p)\mathrm{d}x\\
&\geq \mathbb{E}\int_Q 2|J_1|^2\mathrm{d}x\mathrm{d}t+2\mathbb{E}\int_Q (a^2+b^2)s^3\gamma^3\frac{x^{2-\alpha}}{(2-\alpha)^2}|z|^2\mathrm{d}x\mathrm{d}t+(a^2+b^2)\mathbb{E}\int_Q s\gamma\frac{|z|^2}{x^\eta}\mathrm{d}x\mathrm{d}t\\
&\quad +\mathbb{E}\int_Q (a^2+b^2)s\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t+|b|\mathbb{E}\int_Q \mathcal{O}(1)\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&-C\big(1+a+|b|+\widetilde{K}_0\big)\mathbb{E}\int_Q \theta^2\big(s^2\gamma^2x^{2-\alpha}|F_2|^2+s\gamma^{1+\frac{1}{k}}|F_2|^2+x^\alpha|F_{2,x}|^2\big)\mathrm{d}x\mathrm{d}t\\
&+\frac{(a^2+b^2)(1-\alpha)^2}{4}\mathbb{E}\int_Q s\gamma\frac{|z|^2}{x^{2-\alpha}}\mathrm{d}x\mathrm{d}t-\frac{2(a^2+b^2)}{2-\alpha}\mathbb{E}\int_0^T s\gamma(t)|z_x(t,1)|^2\mathrm{d}t\\
&+(a+|b|)\mathbb{E}\int_Q \mathcal{O}(s^2)\gamma^3x^{2-\alpha}|z|^2\mathrm{d}x\mathrm{d}t-\big(C+aC+K_1\big)\mathbb{E}\int_Q s\gamma^{1+\frac{2}{k}}|z|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Next, we need to estimate the last term in the above inequality. By Young's inequality, for any $\varepsilon>0$, we have

$$
\begin{aligned}
\mathbb{E}\int_Q \gamma^{1+\frac{2}{k}}|z|^2\mathrm{d}x\mathrm{d}t&=\mathbb{E}\int_Q \big(C\varepsilon^{1-k}\gamma^3x^{2-\alpha}|z|^2\big)^{\frac{1}{k}}\big(\frac{k}{k-1}\varepsilon\gamma x^{\frac{\alpha-2}{k-1}}|z|^2\big)^{\frac{k-1}{k}}\mathrm{d}x\mathrm{d}t\\
&\le C\varepsilon^{1-k}\mathbb{E}\int_Q \gamma^3x^{2-\alpha}|z|^2\mathrm{d}x\mathrm{d}t+\varepsilon\mathbb{E}\int_Q \gamma x^{\frac{\alpha-2}{k-1}}|z|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Since

$$\eta=\frac{k-1}{2-\alpha}.$$

Then, we obtain that

$$\mathbb{E}\int_Q s\gamma^{1+\frac{2}{k}}|z|^2\mathrm{d}x\mathrm{d}t\le C\varepsilon^{1-k}\mathbb{E}\int_Q s\gamma^3x^{2-\alpha}|z|^2\mathrm{d}x\mathrm{d}t+\varepsilon\mathbb{E}\int_Q s\gamma\frac{|z|^2}{x^\eta}\mathrm{d}x\mathrm{d}t. \tag{3.28}$$

Taking $\varepsilon=\frac{a^2+b^2}{2(C+aC+K_1)}$ in (3.28), we know

$$
\begin{aligned}
&\mathbb{E}\int_Q 2\mathrm{Re}\,(\theta\overline{J_1}\mathcal{L}p)\mathrm{d}x\\
&\ge\mathbb{E}\int_Q 2|J_1|^2\mathrm{d}x\mathrm{d}t+2\mathbb{E}\int_Q (a^2+b^2)s^3\gamma^3\frac{x^{2-\alpha}}{(2-\alpha)^2}|z|^2\mathrm{d}x\mathrm{d}t+K_2\mathbb{E}\int_Q \mathcal{O}(s^2)\gamma^3x^{2-\alpha}|z|^2\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{(a^2+b^2)}{2}\mathbb{E}\int_Q s\gamma\frac{|z|^2}{x^\eta}\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q (a^2+b^2)s\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t+|b|\mathbb{E}\int_Q \mathcal{O}(1)\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t\\
&\quad-C\big(1+a+|b|+\widetilde{K}_0\big)\mathbb{E}\int_Q \theta^2\big(s^2\gamma^2x^{2-\alpha}|F_2|^2+s\gamma^{1+\frac{1}{k}}|F_2|^2+x^\alpha|F_{2,x}|^2\big)\mathrm{d}x\mathrm{d}t\\
&\quad+\frac{(a^2+b^2)(1-\alpha)^2}{4}\mathbb{E}\int_Q s\gamma\frac{|z|^2}{x^{2-\alpha}}\mathrm{d}x\mathrm{d}t-\frac{2(a^2+b^2)}{2-\alpha}\mathbb{E}\int_0^T s\gamma(t)|z_x(t,1)|^2\mathrm{d}t,
\end{aligned}
$$

where $K_2$ is given by

$$K_2=C\Big[a+|b|+(a^2+b^2)^{1-k}\big(C+aC+K_1\big)^k\Big].$$

Combine the above inequality with (3.1) and notice that $z=\theta p$, then for sufficiently large $s>0$, we can get (3.6).

**Second case:** For $\alpha \in [0,2) \setminus \{1\}, \beta = 2-\alpha$, and $\mu < \mu(\alpha)$. Combining (3.12), (3.18), (3.24), (3.27) with (3.11), we arrive at

$$\begin{aligned}
&\mathbb{E}\int_Q 2\mathrm{Re}\,(\theta\overline{J_1}\mathcal{L}p)\mathrm{d}x\\
&\geq \mathbb{E}\int_Q 2|J_1|^2\mathrm{d}x\mathrm{d}t + 2\mathbb{E}\int_Q (a^2+b^2)s^3\gamma^3\frac{x^{2-\alpha}}{(2-\alpha)^2}|z|^2\mathrm{d}x\mathrm{d}t + (a^2+b^2)\mathbb{E}\int_Q s\gamma\frac{|z|^2}{x^\eta}\mathrm{d}x\mathrm{d}t\\
&\quad + C(a^2+b^2)\mathbb{E}\int_Q s\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t + (a+|b|)\mathbb{E}\int_Q \mathcal{O}(s^2)\gamma^3x^{2-\alpha}|z|^2\mathrm{d}x\mathrm{d}t\\
&\quad - C\big(1+a+|b|+c\big)\mathbb{E}\int_Q \theta^2\big(s^2\gamma^2x^{2-\alpha}|F_2|^2 + s\gamma^{1+\frac{1}{k}}|F_2|^2 + x^\alpha|F_{2,x}|^2\big)\mathrm{d}x\mathrm{d}t\\
&\quad - \frac{2(a^2+b^2)}{2-\alpha}\mathbb{E}\int_0^T s\gamma(t)|z_x(t,1)|^2\mathrm{d}t + |b|\mathbb{E}\int_Q \mathcal{O}(1)\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t\\
&\quad - \big(C+aC+K_3\big)\mathbb{E}\int_Q s\gamma^{1+\frac{2}{k}}|z|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Therefore, choosing $\varepsilon = \frac{a^2+b^2}{2(C+aC+K_3)}$ in (3.28), this implies that

$$\begin{aligned}
&\mathbb{E}\int_Q 2\mathrm{Re}\,(\theta\overline{J_1}\mathcal{L}p)\mathrm{d}x\\
&\geq \mathbb{E}\int_Q 2|J_1|^2\mathrm{d}x\mathrm{d}t + 2\mathbb{E}\int_Q (a^2+b^2)s^3\gamma^3\frac{x^{2-\alpha}}{(2-\alpha)^2}|z|^2\mathrm{d}x\mathrm{d}t + \frac{(a^2+b^2)}{2}\mathbb{E}\int_Q s\gamma\frac{|z|^2}{x^\eta}\mathrm{d}x\mathrm{d}t\\
&\quad + C(a^2+b^2)\mathbb{E}\int_Q s\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t + |b|\mathbb{E}\int_Q \mathcal{O}(1)\gamma x^\alpha|z_x|^2\mathrm{d}x\mathrm{d}t\\
&\quad - C\big(1+a+|b|+c\big)\mathbb{E}\int_Q \theta^2\big(s^2\gamma^2x^{2-\alpha}|F_2|^2 + s\gamma^{1+\frac{1}{k}}|F_2|^2 + x^\alpha|F_{2,x}|^2\big)\mathrm{d}x\mathrm{d}t\\
&\quad - \frac{2(a^2+b^2)}{2-\alpha}\mathbb{E}\int_0^T s\gamma(t)|z_x(t,1)|^2\mathrm{d}t + K_4\mathbb{E}\int_Q \mathcal{O}(s^2)\gamma^3x^{2-\alpha}|z|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

where $K_4 = K_4(a,b,c,d)$ is given by

$$K_4 = C\Big[a+|b|+(a^2+b^2)^{1-k}(C+aC+K_3)^k\Big].$$

Then, for sufficiently large $s>0$, by $z=\theta p$, we can complete the proof of Theorem 3.2. □

### 3.2. The global Carleman estimates for backward linear stochastic complex degenerate/singular Ginzburg–Landau operator

The main purpose of this subsection is to establish global Carleman estimates for the following backward linear stochastic complex degenerate/singular Ginzburg–Landau equation:

$$
\begin{cases}
\mathrm{d}h + (a - \boldsymbol{i}b)(x^\alpha h_x)_x \mathrm{d}t + (c - \boldsymbol{i}d)\dfrac{\mu}{x^\beta} h \mathrm{d}t = F_3 \mathrm{d}t + H \mathrm{d}B(t) & \text{in } Q, \\
h(t,1) = 0 & \text{on } (0,T), \\
\begin{cases} h(t,0) = 0 & \text{if } 0 \le \alpha < 1, \\ \big(x^\alpha h_x\big)(t,0) = 0 & \text{if } 1 \le \alpha < 2, \end{cases} & \text{on } (0,T), \\
h(T,x) = h_T(x) & \text{in } (0,1),
\end{cases}
\tag{3.29}
$$

where $F_3 \in L^2_{\mathbb{F}}(0,T;L^2((0,1);\mathbb{C}))$, and $h_T \in L^2(\Omega, \mathcal{F}_T, \mathcal{P}; L^2((0,1);\mathbb{C}))$.

**Theorem 3.7.** *Let $\eta$ be given such that $0 < \eta < 2 - \alpha$, $F_3 \in L^2_{\mathbb{F}}(0,T;L^2((0,1);\mathbb{C}))$, and $h_T \in L^2(\Omega, \mathcal{F}_T, \mathcal{P}; L^2((0,1);\mathbb{C}))$*

**(i)** *Assume that $(\mathbf{H_1})$ holds. Then, there exist two positive constants $s_3 = s_3(\alpha, \eta, \mu, a, b, c, d)$ and $C$, such that for all $s \ge s_3$, every solution $(h, H)$ to (3.29) satisfies*

$$
\begin{aligned}
&\frac{1}{(2-\alpha)^2}\mathbb{E}\int_Q \theta^2 s^3 \gamma^3 x^{2-\alpha} |h|^2 \mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2 s\gamma \Big(x^\alpha |h_x|^2 + \frac{1}{x^\eta}|h|^2\Big) \mathrm{d}x\mathrm{d}t \\
&\quad + (1-\alpha)^2 \mathbb{E}\int_Q \theta^2 s\gamma \frac{1}{x^{2-\alpha}} |h|^2 \mathrm{d}x\mathrm{d}t \\
&\le C_3 \mathbb{E}\int_Q \theta^2 \big(|F_3|^2 + s^2\gamma^2 x^{2-\alpha}|H|^2 + s\gamma^{1+\frac{1}{k}}|H|^2\big)\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_0^T s\gamma\theta^2(t,1)|h_x(t,1)|^2 \mathrm{d}t,
\end{aligned}
\tag{3.30}
$$

*where*

$$
C_3 = \frac{C(1 + a^2 + b^2 + a^2 K_0)}{a(a^2+b^2)},
\tag{3.31}
$$

*and $K_0$ is given by (1.10).*

**(ii)** *Assume that $(\mathbf{H_2})$ holds. Then, there exist two positive constants $s_4 = s_4(\alpha, \eta, a, b, c, d)$ and $C$, such that for all $s \ge s_4$, every solution $(h, H)$ to (3.29) satisfies*

$$
\begin{aligned}
&\frac{1}{(2-\alpha)^2}\mathbb{E}\int_Q \theta^2 s^3 \gamma^3 x^{2-\alpha} |h|^2 \mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2 s\gamma \frac{1}{x^\eta}|h|^2 \mathrm{d}x\mathrm{d}t \\
&\le C_4 \mathbb{E}\int_Q \theta^2 \big(|F_3|^2 + s^2\gamma^2 x^{2-\alpha}|H|^2 + s\gamma^{1+\frac{1}{k}}|H|^2\big)\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_0^T s\gamma\theta^2(t,1)|h_x(t,1)|^2 \mathrm{d}t,
\end{aligned}
\tag{3.32}
$$

where

$$
C_4 = \frac{C(1 + a^2 + b^2 + c^2)}{(a-2c)(a^2+b^2)}.
\tag{3.33}
$$

**Remark 3.8.** It is worth noting that **(i)** only needs $a > 0$, but **(ii)** needs $a > 2c$. The reason is that in order to remove the term containing $|H_x|^2$ in **(i)**, we can use the improved Hardy-Poincaré inequality (see Lem. 3.2) to choose the desired coefficients. However, the coefficients of the singular terms in **(ii)** can only be $\mu(\alpha)$. If the stochastic equation reduces to the deterministic case, the condition $a > 2c$ is not necessary.

**Remark 3.9.** Similar to **(ii)** in Theorem 3.2, we can also get that, for $\mu < \mu(\alpha)$,

$$\begin{aligned}
&\mathbb{E}\int_Q \theta^2 s\gamma x^\alpha |h_x|^2 \mathrm{d}x\mathrm{d}t \\
&\leq C_4\mathbb{E}\int_Q \theta^2\big(|F_3|^2 + s^2\gamma^2 x^{2-\alpha}|H|^2 + s\gamma^{1+\frac{1}{k}}|H|^2\big)\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_0^T s\gamma\theta^2(t,1)|h_x(t,1)|^2\mathrm{d}t.
\end{aligned}$$

And, for $\mu = \mu(\alpha)$, we have

$$\begin{aligned}
&\mathbb{E}\int_Q \theta^2 s\gamma\Big(x^\alpha|h_x|^2 - \frac{\mu(\alpha)}{x^{2-\alpha}}|h|^2\Big)\mathrm{d}x\mathrm{d}t \\
&\leq C_4\mathbb{E}\int_Q \theta^2\big(|F_3|^2 + s^2\gamma^2 x^{2-\alpha}|H|^2 + s\gamma^{1+\frac{1}{k}}|H|^2\big)\mathrm{d}x\mathrm{d}t + C\mathbb{E}\int_0^T s\gamma\theta^2(t,1)|h_x(t,1)|^2\mathrm{d}t.
\end{aligned}$$

**Proof of Theorem 3.7.** The proof is similar to the proof of Theorem 3.2. The main difference is that $a$, $c$ are replaced by $-a$, $-c$, and **Step 2**. We only prove **Step 2** here.

**Step 2.** Let us estimate $\widetilde{A}_2$, where $\widetilde{A}_2$ is obtained by replacing $a$, $c$ of $A_2$ in (3.11) by $-a$, $-c$. From (3.29) and $a > 0$, we know

$$\begin{aligned}
a\mathbb{E}\int_Q x^\alpha|\mathrm{d}z_x|^2\mathrm{d}x &= a\mathbb{E}\int_Q x^\alpha\theta^2|H_x + H\ell_x|^2\mathrm{d}x\mathrm{d}t \\
&\geq a\mathbb{E}\int_Q \theta^2\Big[\frac{1}{2}x^\alpha|H_x|^2 - \frac{x^{2-\alpha}}{(2-\alpha)^2}s^2\gamma^2|H|^2\Big]\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.34}$$

From the definition of $A$ in (2.4) and $|\gamma_t| \leq C\gamma^{1+\frac{1}{k}}$, it is easy to see that

$$\begin{aligned}
&\mathbb{E}\int_Q (\ell_t - aA)|\mathrm{d}z|^2\mathrm{d}x = \mathbb{E}\int_Q (\ell_t - aA)\theta^2|H|^2\mathrm{d}x\mathrm{d}t \\
&\geq \mathbb{E}\int_Q \theta^2\Big[\mathcal{O}(s)\gamma^{1+\frac{1}{k}} - as^2\gamma^2\frac{x^{2-\alpha}}{(2-\alpha)^2} + a\frac{s\gamma}{2-\alpha}\Big]|H|^2\mathrm{d}x\mathrm{d}t \\
&\geq \mathbb{E}\int_Q \theta^2\Big[(a+1)\mathcal{O}(s)\gamma^{1+\frac{1}{k}} + a\mathcal{O}(s^2)\gamma^2x^{2-\alpha}\Big]|H|^2\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.35}$$

Further, for any $\varepsilon > 0$, one can obtain that

$$\begin{aligned}
&-2b\mathbb{E}\int_Q x^\alpha\ell_x\mathrm{Im}\,(\mathrm{d}z\mathrm{d}\overline{z}_x)\mathrm{d}x = -2b\mathbb{E}\int_Q x^\alpha\ell_x\theta^2\mathrm{Im}\,\big[\mathrm{d}h\mathrm{d}(\ell_x\overline{h} + \overline{h}_x)\big]\mathrm{d}x \\
&\geq |b|\mathbb{E}\int_Q \theta^2\Big[\frac{-2}{(2-\alpha)^2}s^2\gamma^2x^{2-\alpha}|H|^2 + \mathcal{O}(s)\gamma x|HH_x|\Big]\mathrm{d}x\mathrm{d}t \\
&\geq \mathbb{E}\int_Q \theta^2\Big[\Big(|b| + \frac{b^2}{\varepsilon}\Big)\mathcal{O}(s^2)\gamma^2x^{2-\alpha}|H|^2 - \varepsilon x^\alpha|H_x|^2\Big]\mathrm{d}x\mathrm{d}t.
\end{aligned} \tag{3.36}$$

On the other hand, similar to (3.16), one can see that

$$\begin{aligned}&2\mathbb{E}\int_Q \left[b(x^\alpha \ell_x)_x \mathrm{Im}\,(z\mathrm{d}\overline{z})+\mathrm{Re}\,(\overline{\Phi}\overline{z}\mathrm{d}z)\right]\mathrm{d}x\\ &\geq -a\mathbb{E}\int_Q \frac{s\gamma}{2-\alpha}\theta^2|H|^2\mathrm{d}x\mathrm{d}t + a\mathbb{E}\int_Q \mathcal{O}(s)\gamma^{1+\frac{1}{k}}|z|^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.37}$$

And then we estimate "$-c\mathbb{E}\int_Q \frac{\mu}{x^\beta}|\mathrm{d}z|^2\mathrm{d}x$" in two cases: the case of a sub-critical exponent $0<\beta<2-\alpha$ and the case of a critical exponent $\beta = 2-\alpha$.

**First case:** For $\alpha\in[0,2)$, $0<\beta<2-\alpha$, $\mu\in\mathbb{R}$, and $a>0$, we only need to consider here $\eta$ satisfies $\beta\leq\eta<2-\alpha$. Applying (3.3) with $m=\frac{4c|\mu|}{a}+1>0$ gives

$$\begin{aligned}-\frac{4c|\mu|}{a}\int_Q \theta^2\frac{|H|^2}{x^\eta}\mathrm{d}x\mathrm{d}t \geq \mu(\alpha)\int_Q \theta^2\frac{|H|^2}{x^{2-\alpha}}\mathrm{d}x\mathrm{d}t - \int_Q x^\alpha|(\theta H)_x|^2\mathrm{d}x\mathrm{d}t\\ -K_0\int_Q\theta^2|H|^2\mathrm{d}x\mathrm{d}t + \int_Q\theta^2\frac{|H|^2}{x^\eta}\mathrm{d}x\mathrm{d}t,\end{aligned}\tag{3.38}$$

where $K_0$ is given by (1.10). From (3.38), we have

$$\begin{aligned}&-c\mathbb{E}\int_Q\frac{\mu}{x^\beta}|\mathrm{d}z|^2\mathrm{d}x \geq -c\mathbb{E}\int_Q\frac{|\mu|}{x^\beta}\theta^2|H|^2\mathrm{d}x\mathrm{d}t\geq -c\mathbb{E}\int_Q\frac{|\mu|}{x^\eta}\theta^2|H|^2\mathrm{d}x\mathrm{d}t\\ &\geq \frac{a}{4}\mathbb{E}\int_Q\left[\theta^2\mu(\alpha)\frac{|H|^2}{x^{2-\alpha}} - x^\alpha|(\theta H)_x|^2 - K_0\theta^2|H|^2+\theta^2\frac{|H|^2}{x^\eta}\right]\mathrm{d}x\mathrm{d}t\\ &\geq -\frac{a}{4}\mathbb{E}\int_Q x^\alpha\theta^2|H_x|^2\mathrm{d}x\mathrm{d}t - Ca\mathbb{E}\int_Q\theta^2s^2\gamma^2x^{2-\alpha}|H|^2\mathrm{d}x\mathrm{d}t - \frac{a}{4}K_0\mathbb{E}\int_Q\theta^2|H|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Therefore, by above inequality and (3.34)–(3.37), $a>0$, taking $\varepsilon=\frac{a}{8}$ in (3.36), it holds that

$$\begin{aligned}\widetilde{A}_2 \geq &-\Big(1+a+|b|+aK_0+\frac{b^2}{a}\Big)\mathbb{E}\int_Q\theta^2\Big(s^2\gamma^2x^{2-\alpha}|H|^2+s\gamma^{1+\frac{1}{k}}|H|^2\Big)\mathrm{d}x\mathrm{d}t\\ &+\frac{a}{8}\mathbb{E}\int_Q x^\alpha\theta^2|H_x|^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{3.39}$$

**Second case:** For $\alpha\in[0,2)\setminus\{1\}, \beta=2-\alpha$, and $\mu\leq\mu(\alpha)$, by (3.2), one can obtain that

$$\begin{aligned}&-c\mathbb{E}\int_Q\frac{\mu}{x^\beta}|\mathrm{d}z|^2\mathrm{d}x = -c\mathbb{E}\int_Q\frac{\mu}{x^{2-\alpha}}\theta^2|H|^2\mathrm{d}x\mathrm{d}t \geq -c\mathbb{E}\int_Q\frac{\mu(\alpha)}{x^{2-\alpha}}\theta^2|H|^2\mathrm{d}x\mathrm{d}t\\ &\geq -c\mathbb{E}\int_Q\theta^2\left[x^\alpha|H_x|^2+\mathcal{O}(s^2)\gamma^2x^{2-\alpha}|H|^2\right]\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Hence, by $a>2c$ and taking $\varepsilon=\frac{a-2c}{2}$ in (3.36), we have

$$\widetilde{A}_2\geq -\Big(1+a+|b|+c+\frac{b^2}{a-2c}\Big)\mathbb{E}\int_Q\theta^2\Big[s^2\gamma^2x^{2-\alpha}|H|^2+s\gamma^{1+\frac{1}{k}}|H|^2\Big]\mathrm{d}x\mathrm{d}t.\tag{3.40}$$

Combining the proof of Theorem 3.2 with (3.39)–(3.40), we complete the proof of Theorem 3.7. ◻

## 4. Proofs of the main results

In the section, we give proofs of controllability and observability results for forward linear stochastic complex degenerate/singular Ginzburg–Landau equation (1.1), respectively. First, by the standard duality technique ([28]) and observability estimate (1.7), the null controllability result in Theorem 1.2 can be obtained immediately. Therefore, we only need to prove Proposition 1.6.

**Proof of Proposition 1.6.** In the case of $(\mathbf{H_1})$, choose a cut-off function $\xi \in C^\infty(\mathbb{R};[0,1])$ such that

$$\begin{cases} \xi(x) = 1,\ x \le x_1', \\ \xi(x) = 0\ \ x \ge x_2', \end{cases} \tag{4.1}$$

where $x_1' = \frac{2x_1+x_2}{3}, x_2' = \frac{x_1+2x_2}{3}$. Let $G_1 = (x_1', x_2')$, and it is easy to see that $G_1 \subseteq G_0$.

Set $z = \xi w$, $Z = \xi W$, where $(w, W)$ is the solution to (1.6). Then, $(z, Z)$ satisfies

$$\begin{cases} \mathrm{d}z + (a - \boldsymbol{i}b)(x^\alpha z_x)_x \mathrm{d}t + (c - \boldsymbol{i}d)\dfrac{\mu}{x^\beta} z\mathrm{d}t = \widetilde{f}\mathrm{d}t + Z\mathrm{d}B(t) & \text{in } Q, \\ z(t,1) = 0 & \text{on } (0,T), \\ \begin{cases} z(t,0) = 0 & \text{if } 0 \le \alpha < 1, \\ (x^\alpha z_x)(t,0) = 0 & \text{if } 1 \le \alpha < 2, \end{cases} & \text{on } (0,T), \\ z(T,x) = z_T(x) & \text{in } (0,1), \end{cases} \tag{4.2}$$

where

$$\widetilde{f} := (a - \boldsymbol{i}b)\big[2x^\alpha \xi_x w_x + (x^\alpha \xi_x)_x w\big] - \bar{c}_1 z - \bar{c}_2 Z.$$

Then, applying (3.30) to (4.2), we find that for sufficiently large $s > 0$,

$$\begin{aligned} &\mathbb{E}\int_Q s^3\gamma^3 x^{2-\alpha}\theta^2|z|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2 s\gamma\Big(x^\alpha|z_x|^2 + \frac{1}{x^\eta}|z|^2\Big)\mathrm{d}x\mathrm{d}t \\ &\le C_3\mathbb{E}\int_Q \theta^2(|\widetilde{f}|^2 + s^2\gamma^2 x^{2-\alpha}|Z|^2 + s\gamma^{1+\frac{1}{k}}|Z|^2)\mathrm{d}x\mathrm{d}t \\ &\le C_3(a^2+b^2)\mathbb{E}\int_0^T\int_{G_1}\theta^2\big(|w|^2 + |w_x|^2\big)\mathrm{d}x\mathrm{d}t + C_3\mathbb{E}\int_Q |W|^2\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{4.3}$$

where $C_3$ is given by (3.31). Next, we give an estimate on $\mathbb{E}\displaystyle\int_0^T\int_{G_1}\theta^2|w_x|^2\mathrm{d}x\mathrm{d}t$. To this end, choose a function $\zeta \in C^\infty(\mathbb{R};[0,1])$ satisfying $\mathrm{supp}\zeta \subseteq G_0$, and $\zeta(x) = 1$ in $G_1$. Notice that

$$\mathrm{d}(\zeta^2\theta^2|w|^2) = (\zeta^2\theta^2)_t|w|^2\mathrm{d}t + \zeta^2\theta^2(\overline{w}\mathrm{d}w + w\mathrm{d}\overline{w}) + \zeta^2\theta^2|\mathrm{d}w|^2,$$

then, by (1.6), it is easy to show that

$$
\begin{aligned}
&2a\mathbb{E}\int_{G_0} x^\alpha \zeta^2\theta^2|w_x|^2\mathrm{d}x\mathrm{d}t\\
&= -\mathbb{E}\int_{G_0}(\theta^2)_t\zeta^2|w|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_{G_0}\theta^2\zeta^2\Big(2c\frac{\mu}{x^\beta}|w|^2 + 2\mathrm{Re}\, c_1|w|^2 + 2\mathrm{Re}\,\overline{c}_2\overline{w}W - |W|^2\Big)\mathrm{d}x\mathrm{d}t\\
&\quad -\mathbb{E}\int_{G_0} x^\alpha(\theta^2\zeta^2)_x\big[(a-\boldsymbol{i}b)\overline{w}w_x + (a+\boldsymbol{i}b)w\overline{w}_x\big]\mathrm{d}x\mathrm{d}t\\
&\le \frac{C(1+a^2+b^2+c^2)}{a}\mathbb{E}\int_0^T\int_{G_0} s^2\gamma^2\theta^2|w|^2\mathrm{d}x\mathrm{d}t + a\mathbb{E}\int_{G_0} x^\alpha\zeta^2\theta^2|w_x|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

This implies that

$$
\mathbb{E}\int_0^T\int_{G_1}\theta^2|w_x|^2\mathrm{d}x\mathrm{d}t \le \frac{C(1+a^2+b^2+c^2)}{a^2}\mathbb{E}\int_0^T\int_{G_0} s^2\gamma^2\theta^2|w|^2\mathrm{d}x\mathrm{d}t. \tag{4.4}
$$

Combining (4.3) with (4.4), we end up with

$$
\begin{aligned}
&\mathbb{E}\int_0^{x_1'} s^3\gamma^3x^{2-\alpha}\theta^2|w|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^{x_1'}\theta^2 s\gamma\Big(x^\alpha|w_x|^2 + \frac{1}{x^\eta}|w|^2\Big)\mathrm{d}x\mathrm{d}t\\
&\le \widetilde{C}_3\mathbb{E}\int_0^T\int_{G_1}|w|^2\mathrm{d}x\mathrm{d}t + C_3\mathbb{E}\int_Q|W|^2\mathrm{d}x\mathrm{d}t,
\end{aligned} \tag{4.5}
$$

where

$$
\widetilde{C}_3 = \frac{C(1+a^4+b^4+c^4+a^4K_0+b^4K_0+c^4K_0+K_0^2)}{a^3}.
$$

Let $r=\rho w$, $R=\rho W$, where $\rho = 1-\xi$. Then, $(r,R)$ satisfies

$$
\begin{cases}
\mathrm{d}r + (a-\boldsymbol{i}b)(x^\alpha r_x)_x\mathrm{d}t = \widehat{f}\mathrm{d}t + R\mathrm{d}B(t) & \text{in } (0,T)\times(x_1',1),\\
r(t,x_1') = r(t,1) = r_x(t,x_1') = 0 & \text{on } (0,T),
\end{cases} \tag{4.6}
$$

where

$$
\widehat{f} := (a-\boldsymbol{i}b)\big[2x^\alpha\rho_x w_x + (x^\alpha\rho_x)_x w\big] - (c-\boldsymbol{i}d)\frac{\mu}{x^\beta}\rho w - \bar{c}_1 r - \bar{c}_2 R.
$$

Noting that the above equation is uniformly parabolic in $(0,T)\times(x_1',1)$, we can use a known Carleman estimate in [12] for (4.6). By Lemma 2.4 in [12] and an elementary computation, we can obtain that for some constant

$\sigma_1 > 0$,

$$
\begin{aligned}
&\mathbb{E}\int_0^T\int_{x_2'}^1 s^3\gamma^3 e^{-2\sigma_1 s\gamma}|w|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_{x_2'}^1 s\gamma e^{-2\sigma_1 s\gamma}|w_x|^2\mathrm{d}x\mathrm{d}t\\
&\le \mathbb{E}\int_0^T\int_{x_1'}^1 s^3\gamma^3 e^{-2\sigma_1 s\gamma}|r|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_0^T\int_{x_1'}^1 s\gamma e^{-2\sigma_1 s\gamma}|r_x|^2\mathrm{d}x\mathrm{d}t\\
&\le \frac{C(1+a^4+b^4+c^4+d^4)}{a^4}\Big(\mathbb{E}\int_0^T\int_{G_0}|w|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q |W|^2\mathrm{d}x\mathrm{d}t\Big).
\end{aligned}
\tag{4.7}
$$

Combining (4.5) with (4.7), and noting that $x^{2-\alpha}, x^\alpha, \frac{1}{x^\eta}$ are bounded in $(x_2', 1)$, we have

$$
\begin{aligned}
&\mathbb{E}\int_Q e^{-2\sigma_2 s\gamma}\Big[x^{2-\alpha}s^3\gamma^3|w|^2 + s\gamma\Big(x^\alpha|w_x|^2 + \frac{1}{x^\eta}|w|^2\Big)\Big]\mathrm{d}x\mathrm{d}t\\
&\le \hat{C}_3\Big(\mathbb{E}\int_0^T\int_{G_0}|w|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q|W|^2\mathrm{d}x\mathrm{d}t\Big) + C\mathbb{E}\int_0^T\int_{G_1} e^{-2\sigma_2 s\gamma}s\gamma|w_x|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}
$$

where $\sigma_2 = \max\big\{\frac{2}{(2-\alpha)^2}, \sigma_1\big\}$, and

$$
\hat{C}_3 = \frac{C(1+a^8+b^8+c^8+d^8+K_0^8)}{a^4(a^2+b^2)}.
$$

Similar to the proof of (4.4), we can get

$$
\mathbb{E}\int_0^T\int_{G_1} e^{-2\sigma_2 s\gamma}s\gamma|w_x|^2\mathrm{d}x\mathrm{d}t \le \frac{C(1+a^2+b^2+c^2)}{a^2}\mathbb{E}\int_0^T\int_{G_0} s^2\gamma^3 e^{-2\sigma_2 s\gamma}|w|^2\mathrm{d}x\mathrm{d}t.
$$

It follows that

$$
\begin{aligned}
&\mathbb{E}\int_Q e^{-2\sigma_2 s\gamma}\Big[x^{2-\alpha}s^3\gamma^3|w|^2 + s\gamma\Big(x^\alpha|w_x|^2 + \frac{1}{x^\eta}|w|^2\Big)\Big]\mathrm{d}x\mathrm{d}t\\
&\le \hat{C}_3\Big(\mathbb{E}\int_0^T\int_{G_0}|w|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q|W|^2\mathrm{d}x\mathrm{d}t\Big).
\end{aligned}
\tag{4.8}
$$

On the other hand, notice that $\mathrm{d}(|w|^2) = w\mathrm{d}\overline{w} + \overline{w}\mathrm{d}w + |\mathrm{d}w|^2$. Hence, for any $0 \le t_1 \le t_2 \le T$,

$$
\begin{aligned}
&\mathbb{E}\int_0^1 |w(t_2)|^2\mathrm{d}x - \mathbb{E}\int_0^1 |w(t_1)|^2\mathrm{d}x\\
&= \mathbb{E}\int_{t_1}^{t_2}\int_0^1\Big\{w\Big[-(a+\boldsymbol{i}b)(x^\alpha\overline{w}_x)_x - (c+\boldsymbol{i}d)\frac{\mu}{x^\beta}\overline{w} - c_1\overline{w} - c_2\overline{W}\Big]\\
&\qquad + \overline{w}\Big[-(a-\boldsymbol{i}b)(x^\alpha w_x)_x - (c-\boldsymbol{i}d)\frac{\mu}{x^\beta}w - \bar{c}_1 w - \bar{c}_2 W\Big] + |W|^2\Big\}\mathrm{d}x\mathrm{d}t\\
&\ge \mathbb{E}\int_{t_1}^{t_2}\int_0^1\Big(2ax^\alpha|w_x|^2 - 2c\frac{\mu}{x^\beta}|w|^2 - 2\mathrm{Re}\,c_1|w|^2 - |c_2 w|^2\Big)\mathrm{d}x\mathrm{d}t.
\end{aligned}
\tag{4.9}
$$

Similarly, we consider here $\eta$ satisfies $\beta \le \eta < 2-\alpha$. Using (3.3) with $m = \frac{4c|\mu|}{a} + 1 > 0$, we have

$$\begin{aligned}&\mathbb{E}\int_{t_1}^{t_2}\int_0^1 -2c\frac{\mu}{x^\beta}|w|^2\mathrm{d}x\mathrm{d}t \ge \mathbb{E}\int_{t_1}^{t_2}\int_0^1 -2c\frac{|\mu|}{x^\eta}|w|^2\mathrm{d}x\mathrm{d}t\\&\ge \mathbb{E}\int_{t_1}^{t_2}\int_0^1 -(4c|\mu|+a)\frac{1}{x^\eta}|w|^2\mathrm{d}x\mathrm{d}t\\&\ge -a\mathbb{E}\int_{t_1}^{t_2}\int_0^1 x^\alpha|w_x|^2\mathrm{d}x\mathrm{d}t - aK_0\mathbb{E}\int_{t_1}^{t_2}\int_0^1 |w|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Therefore, one can get

$$\mathbb{E}\int_0^1 |w(t_2)|^2\mathrm{d}x - \mathbb{E}\int_0^1 |w(t_1)|^2\mathrm{d}x \ge -C(1+aK_0)\mathbb{E}\int_{t_1}^{t_2}\int_0^1 |w|^2\mathrm{d}x\mathrm{d}t.$$

By Gronwall's inequality, it follows that

$$\mathbb{E}\int_0^1 |w(t_1)|^2\mathrm{d}x \le Ce^{C(1+aK_0)}\mathbb{E}\int_0^1 |w(t_2)|^2\mathrm{d}x, \qquad 0 \le t_1 \le t_2 \le T.$$

Combining the above equality with (4.8), we have

$$\begin{aligned}&\int_0^1 |w(0)|^2\mathrm{d}x \le Ce^{C(1+aK_0)}\mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 |w|^2\mathrm{d}x\mathrm{d}t\\&\le \hat{C}_3e^{C(1+aK_0)}\Big(\mathbb{E}\int_0^T\int_{G_0}|w|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q |W|^2\mathrm{d}x\mathrm{d}t\Big).\end{aligned}$$

Notice that $\hat{C}_3e^{C(1+aK_0)}$ is $\mathcal{C}_1(a,b,c,d)$ in (1.8). Then, we have completed the proof for the case of ($\mathbf{H_1}$). In the case of ($\mathbf{H_2}$), similar to ($\mathbf{H_1}$), we know

$$\mathbb{E}\int_Q e^{-2\sigma_2 s\gamma}\Big(x^{2-\alpha}s^3\gamma^3|w|^2 + s\gamma\frac{1}{x^\eta}|w|^2\Big)\mathrm{d}x\mathrm{d}t \le \widetilde{C}_4\Big(\mathbb{E}\int_0^T\int_{G_0}|w|^2\mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q |W|^2\mathrm{d}x\mathrm{d}t\Big),$$

where

$$\widetilde{C}_4 = \frac{C(1+a^8+b^8+c^8+d^8)}{a^4(a^2+b^2)(a-2c)}.$$

Recalling (3.2), it holds that

$$\begin{aligned}&\mathbb{E}\int_{t_1}^{t_2}\int_0^1 -2c\frac{\mu}{x^\beta}|w|^2\mathrm{d}x\mathrm{d}t = \mathbb{E}\int_{t_1}^{t_2}\int_0^1 -2c\frac{\mu}{x^{2-\alpha}}|w|^2\mathrm{d}x\mathrm{d}t \ge \mathbb{E}\int_{t_1}^{t_2}\int_0^1 -2c\frac{\mu(\alpha)}{x^{2-a}}|w|^2\mathrm{d}x\mathrm{d}t\\&\ge -2c\mathbb{E}\int_{t_1}^{t_2}\int_0^1 x^\alpha|w_x|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

By (4.9) and $a > 2c$, one can get

$$\mathbb{E}\int_0^1 |w(t_1)|^2 \mathrm{d}x \le C\mathbb{E}\int_0^1 |w(t_2)|^2 \mathrm{d}x, \quad 0 \le t_1 \le t_2 \le T.$$

Hence, we obtain that

$$\int_0^1 |w(0)|^2 \mathrm{d}x \le C\mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_0^1 |w|^2 \mathrm{d}x\mathrm{d}t \le \widetilde{C}_4\Big(\mathbb{E}\int_0^T \int_{G_0} |w|^2 \mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q |W|^2 \mathrm{d}x\mathrm{d}t\Big).$$

Notice that $\widetilde{C}_4$ is $\mathcal{C}_2(a,b,c,d)$ in (1.9). The proof of Proposition 1.6 is completed. $\square$

Now, we give a proof of Theorem 1.8.

**Proof of Theorem 1.8.** We first assume that $(\mathbf{H_1})$ holds. Set $\upsilon = \xi y$, where $y$ is the solution to (1.1) and $\xi$ is given by (4.1). Then, $\upsilon$ satisfies

$$\begin{cases} \mathrm{d}\upsilon - (a+\boldsymbol{i}b)(x^\alpha \upsilon_x)_x \mathrm{d}t - (c+\boldsymbol{i}d)\dfrac{\mu}{x^\beta}\upsilon \mathrm{d}t = \widetilde{f}\mathrm{d}t + \widetilde{g}\mathrm{d}B(t) & \text{in } Q, \\ \upsilon(t,1) = 0 & \text{on } (0,T), \\ \begin{cases} \upsilon(t,0) = 0 & \text{if } 0 \le \alpha < 1, \\ (x^\alpha \upsilon_x)(t,0) = 0 & \text{if } 1 \le \alpha < 2, \end{cases} & \text{on } (0,T), \\ \upsilon(0,x) = \upsilon_0(x) & \text{in } (0,1), \end{cases} \tag{4.10}$$

where

$$\widetilde{f} := -(a+\boldsymbol{i}b)\big[2x^\alpha \xi_x y_x + (x^\alpha \xi_x)_x y\big] + c_1\upsilon, \quad \widetilde{g} := c_2\upsilon.$$

Then, applying (3.6) to (4.10), and noting that (3.28), we find that for sufficiently large $s > 0$,

$$\begin{aligned} &\mathbb{E}\int_Q s^3\gamma^3 x^{2-\alpha}\theta^2|\upsilon|^2 \mathrm{d}x\mathrm{d}t + \mathbb{E}\int_Q \theta^2 s\gamma\Big(x^\alpha|\upsilon_x|^2 + \frac{1}{x^\eta}|\upsilon|^2\Big)\mathrm{d}x\mathrm{d}t \\ &\le C_1\mathbb{E}\int_Q \theta^2(|\widetilde{f}|^2 + s^2\gamma^2 x^{2-\alpha}|\widetilde{g}|^2 + s\gamma^{1+\frac{1}{k}}|\widetilde{g}|^2 + x^\alpha|\widetilde{g}_x|^2)\mathrm{d}x\mathrm{d}t \\ &\le C_1(a^2+b^2)\mathbb{E}\int_0^T \int_{G_1} \theta^2(|y|^2 + |y_x|^2)\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{4.11}$$

where $C_1$ is given by (3.7). Similar to (4.4), one can see that

$$\mathbb{E}\int_0^T \int_{G_1} \theta^2|y_x|^2 \mathrm{d}x\mathrm{d}t \le \frac{C(1+a^2+b^2+c^2)}{a^2}\mathbb{E}\int_0^T \int_{G_0} s^2\gamma^2\theta^2|y|^2 \mathrm{d}x\mathrm{d}t. \tag{4.12}$$

Combining (4.11) with (4.12), we find that

$$\begin{aligned}&\mathbb{E}\int_0^T\int_0^{x_1'}s^3\gamma^3x^{2-\alpha}\theta^2|y|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_0^{x_1'}\theta^2s\gamma\Big(x^\alpha|y_x|^2+\frac{1}{x^\eta}|y|^2\Big)\mathrm{d}x\mathrm{d}t\\&\leq\mathbb{E}\int_Q s^3\gamma^3x^{2-\alpha}\theta^2|\upsilon|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_Q\theta^2s\gamma\Big(x^\alpha|\upsilon_x|^2+\frac{1}{x^\eta}|\upsilon|^2\Big)\mathrm{d}x\mathrm{d}t\\&\leq\frac{(1+a^4+b^4+c^4+\widetilde{K}_0^2)}{a^2}\mathbb{E}\int_0^T\int_{G_0}|y|^2\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{4.13}$$

Let $\vartheta=\rho y$, where $\rho=1-\xi$. Then, $\vartheta$ satisfies

$$\begin{cases}\mathrm{d}\vartheta-(a+\boldsymbol{i}b)(x^\alpha\vartheta_x)_x\mathrm{d}t=\widehat{f}\mathrm{d}t+\widehat{g}\mathrm{d}B(t) & \text{in } (0,T)\times(x_1',1),\\ \vartheta(t,x_1')=\vartheta(t,1)=\vartheta_x(t,x_1')=0 & \text{on } (0,T),\end{cases}$$

where

$$\widehat{f}:=-(a+\boldsymbol{i}b)\big[2x^\alpha\rho_xy_x+(x^\alpha\rho_x)_xy\big]-(c-\boldsymbol{i}d)\frac{\mu}{x^\beta}\rho y+c_1\vartheta,\quad \widehat{g}:=c_2\vartheta.$$

Similar to (4.7) and by an elementary computation, for some constant $\sigma_1>0$, we can obtain

$$\begin{aligned}&\mathbb{E}\int_0^T\int_{x_2'}^1 s^3\gamma^3e^{-2\sigma_1s\gamma}|y|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_{x_2'}^1 s\gamma e^{-2\sigma_1s\gamma}|y_x|^2\mathrm{d}x\mathrm{d}t\\&\leq\mathbb{E}\int_0^T\int_{x_1'}^1 s^3\gamma^3e^{-2\sigma_1s\gamma}|\vartheta|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_{x_1'}^1 s\gamma e^{-2\sigma_1s\gamma}|\vartheta_x|^2\mathrm{d}x\mathrm{d}t\\&\leq\frac{C(1+a^4+b^4+c^4+d^4)}{a^4}\mathbb{E}\int_0^T\int_{G_0}|y|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

It follows that, for some constant $\sigma_2>0$,

$$\begin{aligned}&\mathbb{E}\int_Q e^{-2\sigma_2s\gamma}\Big[x^{2-\alpha}s^3\gamma^3|y|^2+s\gamma\Big(x^\alpha|y_x|^2+\frac{1}{x^\eta}|y|^2\Big)\Big]\mathrm{d}x\mathrm{d}t\\&\leq\widetilde{C}_1\mathbb{E}\int_0^T\int_{G_0}|y|^2\mathrm{d}x\mathrm{d}t+\mathbb{E}\int_0^T\int_{G_1}e^{-2\sigma_2s\gamma}s\gamma|y_x|^2\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $\sigma_2=\max\big\{\frac{2}{(2-\alpha)^2},\sigma_1\big\}$, and

$$\widetilde{C}_1=\frac{C(1+a^8+b^8+c^8+d^8+\widetilde{K}_0^4)}{a^4}.$$

Similar to the proof of (4.12), we can get

$$\mathbb{E}\int_0^T\int_{G_1}e^{-2\sigma_2s\gamma}s\gamma|y_x|^2\mathrm{d}x\mathrm{d}t\leq\frac{C(1+a^2+b^2+c^2)}{a^2}\mathbb{E}\int_0^T\int_{G_0}s^2\gamma^3e^{-2\sigma_2s\gamma}|y|^2\mathrm{d}x\mathrm{d}t.$$

It follows that

$$\mathbb{E}\int_Q e^{-2\sigma_2 s\gamma}\Big[x^{2-\alpha}s^3\gamma^3|y|^2+s\gamma\Big(x^\alpha|y_x|^2+\frac{1}{x^\eta}|y|^2\Big)\Big]\mathrm{d}x\mathrm{d}t\le \widetilde{C}_1\mathbb{E}\int_0^T\int_{G_0}|y|^2\mathrm{d}x\mathrm{d}t. \tag{4.14}$$

On the other hand, notice that $\mathrm{d}(|y|^2)=y\mathrm{d}\overline{y}+\overline{y}\mathrm{d}y+|\mathrm{d}y|^2$, then, for any $0\le t_1\le t_2\le T$, by (1.1), it holds that

$$\begin{aligned}
&\mathbb{E}\int_0^1|y(t_2)|^2\mathrm{d}x-\mathbb{E}\int_0^1|y(t_1)|^2\mathrm{d}x\\
&=\mathbb{E}\int_{t_1}^{t_2}\int_0^1\Big\{y\Big[(a-\boldsymbol{i}b)(x^\alpha\overline{y}_x)_x+(c-\boldsymbol{i}d)\frac{\mu}{x^\beta}\overline{y}+\overline{c}_1\overline{y}\Big]\\
&\qquad\qquad+\overline{y}\Big[(a+\boldsymbol{i}b)(x^\alpha y_x)_x+(c+\boldsymbol{i}d)\frac{\mu}{x^\beta}y+c_1y\Big]+|c_2y|^2\Big\}\mathrm{d}x\mathrm{d}t\\
&=\mathbb{E}\int_{t_1}^{t_2}\int_0^1\Big(-2ax^\alpha|y_x|^2+2c\frac{\mu}{x^\beta}|y|^2+2\mathrm{Re}\,c_1|y|^2+|c_2y|^2\Big)\mathrm{d}x\mathrm{d}t.
\end{aligned}\tag{4.15}$$

Similarly, we consider here $\eta$ satisfies $\beta\le\eta<2-\alpha$. Using (3.3) with $m=\frac{4c|\mu|}{a}+1>0$, one can see that

$$\begin{aligned}
&\mathbb{E}\int_{t_1}^{t_2}\int_0^1 2c\frac{\mu}{x^\beta}|y|^2\mathrm{d}x\mathrm{d}t\le\mathbb{E}\int_{t_1}^{t_2}\int_0^1 2c\frac{|\mu|}{x^\eta}|y|^2\mathrm{d}x\mathrm{d}t\le\mathbb{E}\int_{t_1}^{t_2}\int_0^1(4c|\mu|+a)\frac{1}{x^\eta}|y|^2\mathrm{d}x\mathrm{d}t\\
&\le a\mathbb{E}\int_{t_1}^{t_2}\int_0^1 x^\alpha|y_x|^2\mathrm{d}x\mathrm{d}t+aK_0\mathbb{E}\int_{t_1}^{t_2}\int_0^1|y|^2\mathrm{d}x\mathrm{d}t,
\end{aligned}\tag{4.16}$$

where $K_0$ is given by (1.10). Therefore, by (4.15) and (4.16), we can obtain

$$\mathbb{E}\int_0^1|y(t_2)|^2\mathrm{d}x-\mathbb{E}\int_0^1|y(t_1)|^2\mathrm{d}x\le C(1+aK_0)\mathbb{E}\int_{t_1}^{t_2}\int_0^1|y|^2\mathrm{d}x\mathrm{d}t.$$

By Gronwall's inequality, it holds that

$$\mathbb{E}\int_0^1|y(t_2)|^2\mathrm{d}x\le Ce^{(1+aK_0)}\mathbb{E}\int_0^1|y(t_1)|^2\mathrm{d}x,\qquad 0\le t_1\le t_2\le T.$$

Combining the above equality with (4.14), we get

$$\begin{aligned}
&\mathbb{E}\int_0^1|y(T)|^2\mathrm{d}x\le Ce^{C(1+aK_0)}\mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1|y|^2\mathrm{d}x\mathrm{d}t\\
&\le Ce^{C(1+aK_0)}\mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 e^{-2\sigma_2 s\gamma}s\gamma\frac{|y|^2}{x^\eta}\mathrm{d}x\mathrm{d}t\\
&\le Ce^{C(1+aK_0)}\mathbb{E}\int_Q e^{-2\sigma_2 s\gamma}s\gamma\frac{|y|^2}{x^\eta}\mathrm{d}x\mathrm{d}t\le\widetilde{C}_1e^{C(1+aK_0)}\mathbb{E}\int_0^T\int_{G_0}|y|^2\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

Notice that $\widetilde{C}_1e^{C(1+aK_0)}$ is $\mathcal{C}_1(a,b,c,d)$ in (1.11). For the case of $(\mathbf{H_1})$, we have completed the proof.

In the case of ($\mathbf{H_3}$), the proof is similar to ($\mathbf{H_1}$). Here, a brief proof is given. Similar to ($\mathbf{H_1}$), we deduce that

$$\mathbb{E}\int_Q e^{-2\sigma_2 s\gamma}\Big[x^{2-\alpha}s^3\gamma^3|y|^2 + s\gamma\Big(x^\alpha|y_x|^2+\frac{1}{x^\eta}|y|^2\Big)\Big]\mathrm{d}x\mathrm{d}t \le \widetilde{C}_2\mathbb{E}\int_0^T\int_{G_0}|y|^2\mathrm{d}x\mathrm{d}t, \tag{4.17}$$

where

$$\widetilde{C}_2 = \frac{C(1+a^8+b^8+c^8+d^8)}{a^4}.$$

Recalling (3.2), it holds that

$$\begin{aligned}&\mathbb{E}\int_{t_1}^{t_2}\int_0^1 2c\frac{\mu}{x^\beta}|y|^2\mathrm{d}x\mathrm{d}t = \mathbb{E}\int_{t_1}^{t_2}\int_0^1 2c\frac{\mu}{x^{2-\alpha}}|y|^2\mathrm{d}x\mathrm{d}t \le \mathbb{E}\int_{t_1}^{t_2}\int_0^1 2c\frac{\mu(\alpha)}{x^{2-a}}|y|^2\mathrm{d}x\mathrm{d}t\\ &\le 2c\mathbb{E}\int_{t_1}^{t_2}\int_0^1 x^\alpha|y_x|^2\mathrm{d}x\mathrm{d}t.\end{aligned} \tag{4.18}$$

Therefore, by (4.15), (4.18) and $a \ge c$, we know

$$\mathbb{E}\int_0^1|y(t_2)|^2\mathrm{d}x - \mathbb{E}\int_0^1|y(t_1)|^2\mathrm{d}x \le C\mathbb{E}\int_{t_1}^{t_2}\int_0^1|y|^2\mathrm{d}x\mathrm{d}t.$$

Then, by Gronwall's inequality, it follows that

$$\mathbb{E}\int_0^1|y(t_2)|^2\mathrm{d}x \le C\mathbb{E}\int_0^1|y(t_1)|^2\mathrm{d}x, \qquad 0\le t_1\le t_2\le T.$$

Combining the above equality with (4.17), we have that

$$\begin{aligned}&\mathbb{E}\int_0^1|y(T)|^2\mathrm{d}x \le C\mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1|y|^2\mathrm{d}x\mathrm{d}t \le C\mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_0^1 e^{-2\sigma_2 s\gamma}s\gamma\frac{|y|^2}{x^\eta}\mathrm{d}x\mathrm{d}t\\ &\le C\mathbb{E}\int_Q e^{-2\sigma_2 s\gamma}s\gamma\frac{|y|^2}{x^\eta}\mathrm{d}x\mathrm{d}t \le \widetilde{C}_2\mathbb{E}\int_0^T\int_{G_0}|y|^2\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Notice that $\widetilde{C}_2$ is $\mathcal{C}_1(a,b,c,d)$ in (1.12). The proof of Theorem 1.8 is completed. □

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