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FLATNESS OF NETWORKS OF SYNAPTICALLY COUPLED EXCITATORY-INHIBITORY NEURAL MODULES

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Abstract. In this paper, we consider networks of N synaptically coupled excitatory-inhibitory neural modules, with N arbitrary. It has been argued that the connection strengths may slowly vary with respect to time and that they can actually be considered as inputs of the network. The problem that we are studying is which connection strengths should be modified (in other words, which connection strengths should be considered as inputs) in order to achieve flatness for the resulting control system. Flatness of the control network depends on the number of inputs and we show that if enough connection strengths (at least N) can be considered as inputs, then the control network is flat without structural conditions. If the number of inputs is smaller than N, then flatness imposes particular configurations of the interactions between the subnetworks. Our main contribution is to identify, analyze and characterize several flat configurations in the latter case.

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1. Introduction

In this paper, we consider networks of N synaptically coupled excitatory-inhibitory modules or subnetworks (simply, EI modules) of the following form [1, 2] (see also [3, 4] and the references therein):

$$\begin{aligned} \tau_e \dot{x}_e^n &= -x_e^n + F_e\left(\textstyle\sum_{m=1}^N (w_{ee}(n,m)x_e^m - w_{ei}(n,m)x_i^m)\right) \\ \tau_i \dot{x}_i^n &= -x_i^n + F_i\left(\textstyle\sum_{m=1}^N (w_{ie}(n,m)x_e^m - w_{ii}(n,m)x_i^m)\right), \end{aligned} \tag{1.1}$$

with $x_e^n, x_i^n \in \mathbb{R}$, and $1 \le n \le N$. Each subnetwork is labeled by the discrete index n and contains a pair of mutually coupled local populations of excitatory and inhibitory neurons. The equations describing the dynamics of each subnetwork are usually called Wilson-Cowan equations for cortical dynamics and have been an important source of inspiration in the computational neuroscience community. They focus on the overall neuronal activity of the brain structure, rather than on the detailed evolution of the electric response of single neurons, and allow to adequately model a vast range of brain functions. The Wilson-Cowan model depicts a neuron module (or

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subnetwork) composed of an excitatory and an inhibitory subpopulation. Both of these subgroups represent large-scale neurons that exhibit various oscillation patterns when fired. All cells have a charge difference across their entire membrane which generates potential energy, and according to [5], in most cell types, the resting potential is resistant to chaos. However, in some cell types, like neurons, variations in the cell's charge distribution can lead to large membrane potential changes. Such cells are called excitatory and neurons use their excitable cell membranes to convey information (the action potential is activated in the cell and transmitted down its axon to the next cell). On the other hand, there are receptor channels with inhibitory effects (in that case, ions flowing through the channel help restore the system to its resting potential). Various signaling behaviors are made possible by the combination of these excitatory and inhibitory signals in the cell body. Further diversity can be added by the receptor-channels' dynamic actions, which can react to signals at different times and for different lengths of time (see [5] for an in-depth study of the types and functions of these excitatory and inhibitory cells). Therefore the above modeling of collective behavior within a neuronal population is based on the crucial observations that all nervous processes of any complexity are dependent upon the interaction of excitatory and inhibitory cells and have strongly nonlinear character (as most other biological systems). The spatial interactions are neglected and the above model simply describes the temporal dynamics of the aggregate network. The dot denotes the derivative with respect to time, *i.e.*, $\dot{x} = \mathrm{d}x/\mathrm{d}t$, and the relevant variables (*i.e.*, the states of the system), x_e^n and x_i^n are, resp., the proportion of excitatory and inhibitory cells of the n-th subnetwork which become active per unit time (implying that the relevant aspect of a single cell activity is not the single spike but rather spike frequency). We will always use lower indices $a \in \{e, i\}$ and $b \in \{e, i\}$ to refer to the excitatory or inhibitory character of the considered objects, and integers $1 \leq n \leq N$ and $1 \leq m \leq N$ as upper indicies to label the subnetworks. The gain functions F_e and F_i are nonlinear, typically sigmoidal functions, τ_e and τ_i are time constants, and $w_{ab}(n, m) \in \mathbb{R}_+$, where $a, b \in \{e, i\}$ and $1 \leq n, m \leq N$, denote the strengths of connections between x_a^n and x_b^m ($w_{ab}(n, m)$ enters, with a positive or negative sign, into the dynamics of x_a^n and characterizes the excitatory or inhibitory action of x_b^m on x_a^n).

It has been argued, see for instance [6] and the references therein, that (some of) the connection strengths $w_{ab}(n, m)$ may slowly vary with respect to time and can actually be considered as inputs of the dynamical system (1.1). Indeed, experimental works have revealed that experience and training modify synapses strength, and these modifications change patterns of neuronal firing and affect behavior (for example, London taxi drivers used to have a much more developed posterior hippocampus – which plays important roles in the consolidation of information from short-term memory to long-term memory, and in spatial memory – since they had to know all London streets by heart in order to pass the taxi driver exam, see [7]). Therefore, activity-dependent synaptic plasticity is believed to play a crucial role in the development of neural circuits and it is thus possible to control a network of synaptically coupled excitatory-inhibitory modules through the connections strengths. One would like to transform dynamical system (1.1) into a control system in order to be able to change the dynamical behavior of specific neural populations in the brain. In recent years, several psychological light disorders, including anxiety, depression, repeated anger, have significantly increased (for instance, due to the COVID-19 pandemic). These disorders are directly associated with the modifications of several neural populations within the brain. Non-pharmacological interventions, such as meditation, hypnosis, sophrology, yoga, have been shown to modify various neural populations through plasticity. The objective is to explore the transition from a mild disorder to a healthier psychological state by using the induced neural plasticity through non pharmacological interventions. In this context, an important question is which connection strengths should be modified (equivalently, which connection strengths should be considered as inputs of the network) in order to achieve a desired property or behaviour for the resulting control system. An example of a significant and relevant behaviour (or control problem) is the transition from a pathological limit cycle to a physiological cycle, while satisfying specific constraints.

A property that is very useful in applications, like trajectory generation and trajectory tracking, constructive controllability or how to steer the system (in the neuroscience context, from a pathological limit cycle to a healthier physiological cycle), the reconstruction of non measured variables from the outputs, *etc.*, for both finite and infinite dimensional control systems is that of differential flatness (see, *e.g.*, [8–12] and references therein), that we simply call flatness in the paper. Indeed, when a system with s states and c control inputs (in general, with $s > c$) is flat, it admits a flat output with c components. All system variables can then be expressed as a (in general, nonlinear) function of the flat output components and a finite number of their time-derivatives. These expressions do not involve any differential equation integration. Thus, the original system (of s differential equations) boil down to c equations yielding the flat output dynamics. This can be seen as a model reduction performed by substituting the original system equations with the equivalent flat output dynamics. Note that this model reduction does not alter the physical relevance of the reduced system, since both the original and reduced systems are mathematically equivalent. This preservation of the physical character can be important when considering so-called computational models, obtained usually through mean-field approximation (see, *e.g.*, [13, 14]).

In this context, we study the following questions: which connection strengths among $w_{ab}(n, m)$, for $a, b \in \{e, i\}$, $1 \leq n, m \leq N$, should be modeled as control variables and how should the subnetworks interact in order to obtain flatness for the resulting control system? For flatness, are there subnetworks that cannot interact (translating into vanishing of some $w_{ab}(n, m)$) or, on the contrary, that have to interact (translating into non-vanishing of some $w_{ab}(n, m)$)? In the first case, we will talk about structural conditions, while in the second case we will call them regularity conditions. Of course, flatness of a control network depends on the number of inputs. We will show that if enough connection strengths can be considered as inputs (at least N for a network of N synaptically coupled subnetworks), then the control network is flat (actually static feedback linearizable) without structural conditions. But, in general, flatness imposes particular configurations of the interactions between the subnetworks or even for the local interactions within a subnetwork. A crucial issue is to identify network configurations that are flat. An important class of flat control systems that arises in our study is that of systems that have a triangular structure. If only p connection strengths, with $p < N$, can be modeled as inputs, then we will propose two triangular flat configurations, each of them containing p triangular chains. The triangular structure of each chain means that the action of any subnetwork on its predecessor is minimal (only one of the associated connection strengths is nonzero) while it can act on its successor in any possible way. Interactions between subnetworks of different chains are possible, but they are also constrained to respect a triangular structure.

The problem of deciding which connection strengths have to be considered as inputs in order to achieve flatness for the control network is related to that of constructing flat inputs [15–19] (*i.e.*, placing the actuators in order to render an observed dynamical system flat with the original measurements being a flat output), but several major differences can be listed. First, we do not have the freedom to place the actuators wherever we want them since the connection strengths are already part of the dynamics, and we can only choose which of them are considered as inputs and which of them are not. Second, contrary to the aforementioned papers, here the system is not observed, we do not have any measurement and the flat output of the resulting control system is not given *a priori*. Finally, the number of controls is not imposed either (when constructing flat inputs, the original measurements are required to form a flat output implying that we necessarily have as many inputs as the number of measurements).

The literature on flatness properties of control systems in neuroscience is very limited. The only publications that we are aware of are [20], where two cases of lumped parameter oscillators were studied, and our recent works [21, 22] that study flatness and Liouvillian properties for several existing quantitative models of the hypothalamic-pituitary-adrenal axis. Preliminary results (where flatness of networks of two synaptically coupled EI neural modules is analysed) appeared in [23]. In the recent paper [24], the particular case of $N = 2$ when all interactions are symmetric (i.e, $w_{ab}(n, m) = w_{ba}(m, n)$, $\forall a, b \in \{e, i\}$ and $1 \leq n, m \leq 2$) was discussed.

The paper is organized as follows. In Section 2, we recall the definition of flatness, state the problem that we are studying and the assumptions under which we work, and illustrate them *via* the simplest network (containing one EI module only). In Section 3, we give our main results: for arbitrary values of the number of connection strengths acting as controls of the system, we identify flat configurations for networks of the form (1.1) with N synaptically coupled EI modules, where N is arbitrary. We give conclusions and directions for future work in Section 4. Finally, we provide proofs in Section 5.

2. Definitions, assumptions and problem statement

2.1. Flatness and static feedback linearization

The fundamental property of flat systems is that all their solutions can be parametrized by a finite number of functions and their time-derivatives, [8, 9, 11]. Consider the nonlinear control system

$$\Xi: \quad \dot{x} = f(x,u),$$

where $x = (x_1, \ldots, x_s)$ is the state defined on an open subset X of $\mathbb{R}^s$, $u = (u_1, \ldots, u_c)$ is the control taking values in an open subset U of $\mathbb{R}^c$. The dynamics f are smooth (the word smooth will always mean $\mathcal{C}^\infty$-smooth, away from singularities). We denote by $\frac{\partial f}{\partial u}$ the matrix $\left(\frac{\partial f_i}{\partial u_j}\right)$, for $1 \le i \le s$, $1 \le j \le c$, and suppose that rk $\frac{\partial f}{\partial u} = c$. In order to recall the definition of flatness, fix an integer $l \ge -1$ and denote $U^l = U \times \mathbb{R}^{cl}$ and $\bar{u}^l = (u, \dot{u}, \ldots, u^{(l)})$. For $l = -1$, the set U^{-1} is empty and $\bar{u}^{-1}$ in an empty sequence.

Definition 2.1. The system $\Xi : \dot{x} = f(x,u)$ is *flat* at $(x_0, \bar{u}_0^l) \in X \times U^l$, for $l \ge -1$, if there exists a neighborhood $\mathcal{O}^l$ of $(x_0, \bar{u}_0^l)$ and c smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \ldots, u^{(l)})$, $1 \le i \le c$, defined in $\mathcal{O}^l$, having the following property: there exist an integer r and smooth functions γ_i, $1 \le i \le s$, and δ_j, $1 \le j \le c$, such that

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \ldots, \varphi^{(r-1)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \ldots, \varphi^{(r)}) \tag{2.1}$$

for any C^{l+r}-control $u(t)$ and corresponding trajectory $x(t)$ that satisfy $(x(t), u(t), \ldots, u^{(l)}(t)) \in \mathcal{O}^l$, where $\varphi = (\varphi_1, \ldots, \varphi_c)$ and is called a *flat output*.

Observe that flatness is a local and generic property, that is, the desired description (2.1) is local and holds out of singular states and singular values of controls. Moreover, even if the flat output φ is globally defined, in general, φ guarantees local flatness only since the map $(x,u) \to (\varphi_1, \dot{\varphi}_1, \ldots, \varphi_1^{(r_1)}, \ldots, \varphi_c, \dot{\varphi}_c, \ldots, \varphi_c^{(r_c)})$, where r_i is the order of the highest derivative of φ_i involved in (2.1), need not be globally invertible.

Flatness is closely related to the notion of feedback linearization and flat systems can be seen as a generalization of linear systems. Namely they are linearizable *via* dynamic, invertible and endogenous feedback, see, *e.g.*, [25]. A particular class of flat systems is that of control systems linearizable *via* invertible static feedback. The control system $\Xi : \dot{x} = f(x,u)$, $x \in X \subset \mathbb{R}^s$, $u \in U \subset \mathbb{R}^c$, is locally linearizable by static feedback if it is equivalent[1] *via* a local diffeomorphism $z = \phi(x)$ and an invertible feedback transformation, $u = \psi(x,v)$, to a linear controllable system $\Lambda : \dot{z} = Az + Bv$. The problem of static feedback linearization was solved by Brockett [26] (for a smaller class of transformations) and then by Jakubczyk and Respondek [27] and, independently, by Hunt and Su [28], who gave geometric necessary and sufficient conditions that we recall next, see also [29, 30]. Denote $f_u = f(\cdot, u)$ and define $\mathcal{F} = \{f_u, u \in U\}$, the family of all vector fields corresponding to constant controls of Ξ. Define the following sequence of distributions

$$\mathfrak{D}^0(x,u) = \mathrm{Im}\frac{\partial f}{\partial u}(x,u) \quad \text{and} \quad \mathfrak{D}^j(x,u) = \mathfrak{D}^{j-1}(x,u) + \mathrm{span}\,\{[f_u, g](x,u), f_u \in \mathcal{F}, g \in \mathfrak{D}^{j-1}\},\ j \ge 1, \tag{2.2}$$

where the bracket represents the Lie bracket. If Ξ is a control-affine system, *i.e.*, of the form $\dot{x} = f(x) + \sum_{i=1}^c u_i g_i(x)$, we actually have

$$\mathfrak{D}^0 = \mathrm{span}\,\{g_1, \ldots, g_c\} \quad \text{and} \quad \mathfrak{D}^{j+1} = \mathfrak{D}^j + [f, \mathfrak{D}^j] = \mathrm{span}\,\{g_\ell, ad_f g_\ell, \ldots, ad_f^{j+1} g_\ell, 1 \le \ell \le c\},$$

where we define $ad_f g = [f,g]$ and, recursively, $ad_f^{j+1} g = [f, ad_f^j g]$, for $j \ge 1$.

[1]Two general control systems $\Xi : \dot{x} = f(x,u)$, $x \in X$, $u \in U$, and $\tilde{\Xi} : \dot{\tilde{x}} = \tilde{f}(\tilde{x}, \tilde{u})$, $\tilde{x} \in \tilde{X}$, $\tilde{u} \in \tilde{U}$, are said to be static feedback equivalent if there exists a diffeomorphism $(\tilde{x}, \tilde{u}) = (\phi(x), \psi(x,u))$ of $X \times U$ onto $\tilde{X} \times \tilde{U}$, which transforms Ξ into $\tilde{\Xi}$, *i.e.*, $\frac{\partial \phi(x)}{\partial x} f(x,u) = \tilde{f}(\phi(x), \psi(x,u))$.

Theorem 2.2 (Static feedback linearization)**.** *The following conditions are equivalent:*

(FL1) $\Xi : \dot{x} = f(x,u)$ *is locally static feedback linearizable, around* $x_0 \in X$*;*
(FL2) $\Xi : \dot{x} = f(x,u)$ *is locally static feedback equivalent, around* $x_0 \in X$*, to the Brunovský canonical form*

$$(Br) : \left\{ \begin{array}{lcl} \dot{z}_\ell^j & = & z_\ell^{j+1} \\ \dot{z}_\ell^{\rho_\ell} & = & v_\ell, \end{array} \right.$$

where $1 \leq \ell \leq c$*,* $1 \leq j \leq \rho_\ell - 1$*, and* $\sum_{\ell=1}^{c} \rho_\ell = s$*;*
(FL3) For any $j \geq 0$*, the distributions* $\mathfrak{D}^j$ *do not depend on* u*, are of constant rank, around* $x_0 \in X$*, involutive, and* $\mathfrak{D}^{s-1} = TX$*.*

Another important class of flat systems is that of systems that are static feedback equivalent to the following triangular form (where for a fixed integer $j \geq 1$, we denote $\bar{z}^j = (z_1^1, \ldots, z_1^j, \ldots, z_c^1, \ldots, z_c^j)$, with z_ℓ^q, for $\rho_\ell + 1 \leq q \leq j$, missing if $j > \rho_\ell$):

$$\Delta : \left\{ \begin{array}{lcl} \dot{z}_\ell^j & = & f_\ell^j(\bar{z}^{j+1}), \\ \dot{z}_\ell^{\rho_\ell} & = & v_\ell, \end{array} \right. \tag{2.3}$$

where $1 \leq \ell \leq c$, $1 \leq j \leq \rho_\ell - 1$, $\sum_{\ell=1}^{c} \rho_\ell = s$, and for each $1 \leq j \leq \max_{1 \leq \ell \leq c} \rho_\ell$, the matrix $\left(\frac{\partial f_\ell^j}{\partial z_k^{j+1}} \right)$, for $1 \leq \ell, k \leq c$, is of full rank (with the partial derivatives of f_ℓ^j and/or those with respect to z_k^{j+1} missing if $j > \rho_\ell$ and/or if $j + 1 > \rho_k$).

Both the Brunovský canonical form (Br) and the triangular form Δ consist of c chains $z_\ell = (z_\ell^1, \ldots, z_\ell^{\rho_\ell})$ (of pure integrators for (Br), resp., triangular chains for Δ) of lengths, resp., $\rho_1, \rho_2, \ldots, \rho_c$, and are flat with $(\varphi_1, \ldots, \varphi_c) = (z_1^1, \ldots, z_c^1)$, the top variables of each z_ℓ-chain, being a flat output. These two forms will play an important role in our study. Indeed, we will identify flat excitatory-inhibitory networks that are either static feedback linearizable (and thus static feedback equivalent to the Brunovský canonical form) or that can be transformed (*via* an invertible feedback transformation) into the triangular form Δ. Notice that, in general, the triangular form Δ is not static feedback linearizable, but becomes feedback linearizable by successive pre-integrations or prolongations of some controls v_ℓ (see Sect. 3 for more details).

2.2. Assumptions and problem statement

Consider a network of synaptically coupled excitatory-inhibitory modules or subnetworks of the form (1.1). Following [1, 2], we adopt the convention that $x_e^n = 0$, $x_i^n = 0$, $1 \leq n \leq N$, is the resting state, and is assumed to reflect low-level background activity and therefore, small negative values of x_e^n and x_i^n are allowed, they have physiological significance and represent depression or inhibition of resting activity, see [1, 2] for more details. For simplicity, from now on, we suppose that all time constants are equal to one, *i.e.*, $\tau_e = \tau_i = 1$, and that the gain functions F_e and F_i are equal to the shifted logistic sigmoidal function[2] (which is a classical assumption when mathematically modeling excitatory and inhibitory activity in localized populations of neurons [1, 2]):

$$F(y) = F_e(y) = F_i(y) = \mathcal{S}(y) - \mathcal{S}(0), \text{ with } \mathcal{S}(y) = \frac{1}{1 + \mathrm{e}^{-y}}, \tag{2.4}$$

[2]Since $x_e^n = 0$, $x_i^n = 0$, $1 \leq n \leq N$, are chosen to be the state of low-level background activity, they must be a steady-state solution of (1.1), implying that the logistic sigmoidal function has to be shifted downward by a constant amount so that $F(0) = 0$.

where $y \in \mathbb{R}$, that is, we work with systems of the form

$$
\begin{array}{rcl}
\dot{x}_e^n & = & -x_e^n + F\left(\sum_{m=1}^N (w_{ee}(n,m)x_e^m - w_{ei}(n,m)x_i^m)\right)\\
\dot{x}_i^n & = & -x_i^n + F\left(\sum_{m=1}^N (w_{ie}(n,m)x_e^m - w_{ii}(n,m)x_i^m)\right),
\end{array}
\tag{2.5}
$$

with $1 \leq n \leq N$ and F given by (2.4). It is worth noting that the early study [1] has shown that the qualitative properties of solutions of the above equations (such as the number and stability of steady states, hysteresis effects, presence of limit cycles, *etc.*) that are of interest for the neuroscience community, are independent of the particular choice of the logistic curve for the population response functions, and actually any function belonging to the class of sigmoid functions (*i.e.*, monotonically increasing on $\mathbb{R}$, with a lower asymptote of 0 and an upper asymptote of 1, and with only one inflection point) adequately describes the desired behavior. Denote by $f = (f_e^1(x,w), f_i^1(x,w), \ldots, f_e^N(x,w), f_i^N(x,w))$ the right-hand side of (2.5), *i.e.*, we have

$$
\dot{x} = f(x,w), \tag{2.6}
$$

where $x = (x_e^1, x_i^1, \ldots, x_e^N, x_i^N)$ and $w = (w_{ee}(1,1), \ldots, w_{ii}(N,N))$. The problem that we are studying in this paper can be summarized as follows:

Problem. Consider the EI network given by (2.5) and for a given integer $1 \leq c \leq 2N$, define c controls among $w_{ab}(n,m)$, for $a, b \in \{e, i\}$ and $1 \leq n, m \leq N$, leading to a control system $\dot{x} = f(x,u)$, with $\mathrm{rk}\,\frac{\partial f}{\partial u} = c$, for which the connection strengths not defined as inputs are assumed constant, and study how the subnetworks have to interact (equivalently, which are the conditions that the (constant) configuration strengths have to verify) such that the obtained control system $\dot{x} = f(x,u)$ is flat.

From now on, we make the following assumption:

Assumption 2.3. Consider the EI network given by (2.5). We assume the following:

(a) There is no relation between the connection strengths. In particular, we do not suppose any symmetry (for instance, of the form $w_{ab}(n,m) = w_{ba}(m,n)$) between them; some of the w's may be zero ($w_{ab}(n,m) = 0$ means that x_b^m does not directly impact the dynamics of x_a^n), and we can have $w_{ab}(n,m) = 0$, but $w_{ba}(m,n) \neq 0$.

(b) Any connection strength $w_{ab}(n,m)$, for $a, b \in \{e, i\}$, $1 \leq n, m \leq N$, may be modeled as an input for the control EI network and the corresponding input will be denoted by u_a^n (recall that $w_{ab}(n,m)$ appears into the dynamics of x_a^n and describes the action of x_b^m on x_a^n).

(c) Control variables act independently on the system, *i.e.*, for fixed $1 \leq n \leq N$ and $a \in \{e, i\}$, among all connection strengths $w_{ab}(n,m)$, for $b \in \{e, i\}$ and $1 \leq m \leq N$, involved in the expression of $\dot{x}_a^n$, only one of them can be considered as an input for the control system.

(d) Suppose that c connection strengths (where the integer $1 \leq c \leq 2N$ is fixed) $u_a^n = w_{ab}(n,m)$ are modeled as inputs and denote by $\mathcal{C}$ the set of all corresponding pairs (a, n). We will work locally, around a nominal point $x_0 = (x_{e,0}^1, x_{i,0}^1, \ldots, x_{e,0}^N, x_{i,0}^N)$, such that for the obtained control EI network $\dot{x} = f(x,u)$, we have

$$
\mathrm{rk}\,\frac{\partial f}{\partial u}(x_0) = c, \tag{2.7}
$$

implying that for each $u_a^n = w_{ab}(n,m)$, we have $x_{b0}^m \neq 0$, see (2.5)-(2.6), and, in particular, that $x_0 \neq 0$. For a fixed input $u_a^n = w_{ab}(n,m)$, define the set of singularities $S_{a,b}^{n,m} = \{x \in \mathbb{R}^{2N} : x_b^m = 0\}$. Then the set of singularities for the control EI network $\dot{x} = f(x,u)$, for which c connection strengths $u_a^n = w_{ab}(n,m)$,

$(a, n) \in \mathcal{C}$, are modeled as inputs, is

$$S = \bigcup_{(a,n)\in\mathcal{C}} S^{n,m}_{a,b}. \tag{2.8}$$

The control EI network will always be considered around regular points $x_0 \in \mathbb{R}^{2N} \setminus S$.

(e) We suppose that the number c of connection strengths that can be modeled as inputs is strictly less than $2N$, the state dimension; (if $c = 2N$, the fact that the control variables act independently on the system implies that flatness at $x_0 \in \mathbb{R}^{2N} \setminus S$ is trivial).

(f) The connection strengths that are not modeled as controls will always be supposed constant (they may be zero or nonzero).

The above assumptions imply that any state variable x^n_a can be affected by one single input (denoted u^n_a) and, conversely, that an input can act directly on a single state. It follows that the constructed control system will be nonlinear with respect to the control (because of the nonlinear sigmoidal function F). The case $c = 2N$ is excluded by Assumption 2.3(e), so the considered control network will have at most $2N - 1$ controls. Finally, remark that due to the independence assumption (both for the relation between the connection strengths and for the control variables), it can be easily seen that the distribution $\mathfrak{D}^0$ (see Sect. 2.1 for its definition) associated to the control network is always independent on u and involutive. Moreover, we can always transform the nonlinear-control system into a control-affine one *via* a (nonlinear) invertible static feedback transformation, see, for instance, the illustrative example of Section 2.3 (in general, this is no longer the case if a symmetry between the connection strengths is assumed, see [24]). When we say rk $\frac{\partial f}{\partial u} = c$, we mean that in any neighborhood of x_0 there exists a point x at which rk $\frac{\partial f}{\partial u}(x) = c$ but this rank can, *a priori*, drop at x_0 (*i.e.*, we can have rk $\frac{\partial f}{\partial u}(x_0) < c$). This singular case is excluded by Assumption 2.3(d), and flatness of the control EI network will always be studied around regular points[3].

Conditions of the type $w_{ab}(n, m) \neq 0$ will be called *regularity conditions*, they do not restrict the interactions between different subnetworks or between excitatory and inhibitory populations. On the other hand, conditions of the type $w_{ab}(n, m) = 0$ will be called *structural conditions* because they impose certain constraints on the interactions (*i.e.*, the EI network is flat for certain configurations for the interactions between subnetworks only). By *interaction configuration* between two subnetworks n and m, we will mean all possible values of the associated connection strengths $w_{ab}(n, m)$, $w_{ab}(m, n)$, for $a, b \in \{e, i\}$.

The following observation, see [3], can be taken into account when deciding which connection strengths may play the role of an input:

Remark 2.4. Synaptic interactions within a local EI subnetwork (that are described by connection strengths of the form $w_{ab}(n, n)$, $a, b \in \{e, i\}$, where n refers to the n-th subnetwork) can be assumed to be stronger than those between EI networks. It follows that it may by more natural to control a state x^n_a, where $1 \le n \le N$, $a \in \{e, i\}$, with the help of one of the connection strengths $w_{ab}(n, n)$, $b \in \{e, i\}$, rather than with $w_{ab}(n, m)$, with $m \neq n$.

[3]Notice that the condition rk $\frac{\partial f}{\partial u}(x_0) = c$ is not necessary for flatness at x_0. Indeed, consider the simplest case of a two-input EI network with $N = 1$ (although excluded from the paper, to illustrate this statement, we suppose $c = 2N = 2$ since the presented situation cannot appear in the single-input case):

$$\begin{aligned} \dot{x}^1_e &= f^1_e(x, u^1_e) = -x^1_e + F(w_{ee}(1,1)x^1_e - u^1_e x^1_i) \\ \dot{x}^1_i &= f^1_i(x, u^1_i) = -x^1_i + F(u^1_i x^1_e - w_{ii}(1,1)x^1_i), \end{aligned}$$

around (x_0, u_0), where $x^1_{e,0} = 0$, $x^1_{i,0} \neq 0$, $u^1_{i,0} \neq 0$, and $u^1_{e,0}$ any. Although rk $\frac{\partial f}{\partial u}(x_0) = 1 < 2$, it is easy to see that the above system is flat at (x_0, u_0) with the flat output $(\varphi_1, \varphi_2) = (x^1_i, u^1_i)$: we have $\dot{\varphi}_1 = f^1_i(\varphi_1, \varphi_2, x^1_e)$, from which around (x_0, u_0), we can express x^1_e *via* the implicit function theorem as $x^1_e = \gamma^1_e(\varphi_1, \dot{\varphi}_1, \varphi_2)$, and then from $\dot{x}^1_e$, we compute $u^1_e = \delta^1_e(\varphi_1, \dot{\varphi}_1, \ddot{\varphi}_1, \varphi_2, \dot{\varphi}_2)$. In the paper, we do not discuss this kind of singularities, which are excluded by Assumption 2.3(d).

We end this section by the following remark discussing how the fact that connection strengths (interpreted as controls) are slowly varying influence the control problems:

Remark 2.5. The fact that $u(t)$ evolves slowly, through neural plasticity, induces a constraint on the derivative of the available controls and in practice, one should choose reference trajectories for the flat output components such that $\dot{u}(t)$ fulfills $\dot{u}(t) < \epsilon$, with epsilon small with respect to the EI network dynamics time constants. This restrains the set of admissible flat output trajectories.

2.3. Illustration of the problem: the simplest case, network with one module

Before giving our main results, let us illustrate the above problem and assumptions by the simplest population-based EI network: a dynamical system describing a pair of mutually coupled local populations of excitatory and inhibitory neurons, that is, a system of the form (2.5) with $N = 1$, and thus with two states x_e^1 and x_i^1 only. Observe that all connection strengths are of the form $w_{ab}(1,1)$, $a, b \in \{e, i\}$, and each of them may be a possible input, so we have four candidates for the control but only one can be modeled as an input (the control system can have at most $c = 2N$ inputs, but recall that we only treat the nontrivial case $c < 2N$, here corresponding to $c = 1$). We will thus define a single-input control system of the form $\dot{x} = f(x, u)$, where $x \in \mathbb{R}^2$, $u \in \mathbb{R}$. Several very similar (actually feedback equivalent) subcases can be considered: either the input acts on x_e^1 (and is defined either by $u_e^1 = w_{ee}(1,1)$ or by $u_e^1 = w_{ei}(1,1)$), or the input acts on x_i^1 (and is defined either by $u_i^1 = w_{ie}(1,1)$ or by $u_i^1 = w_{ii}(1,1)$). We study only one of them. Define $u_i^1 = w_{ie}(1,1)$. Then the associated control system is of the form:

$$\begin{aligned} \dot{x}_e^1 &= f_e^1(x) &&= -x_e^1 + F(w_{ee}(1,1)x_e^1 - w_{ei}(1,1)x_i^1) \\ \dot{x}_i^1 &= f_i^1(x, u_i^1) &&= -x_i^1 + F(u_i^1 x_e^1 - w_{ii}(1,1)x_i^1), \end{aligned} \tag{2.9}$$

with $w_{ee}(1,1)$, $w_{ei}(1,1)$, and $w_{ii}(1,1)$ assumed constant. System (2.9) is considered around points $x_0 \in \mathbb{R}^2 \setminus S$, with $S = S_{i,e}^{1,1} = \{x \in \mathbb{R}^2 : x_e^1 = 0\}$, and can be transformed *via* the invertible (nonlinear) feedback transformation $\tilde{u}_1 = f_i^1(x, u_i^1)$ into the control-affine system $\dot{x}_e^1 = f_e^1(x)$, $\dot{x}_i^1 = \tilde{u}_1$.

Proposition 2.6. *Consider the single-input control EI network given by* (2.9)*, around any $x_0 \in \mathbb{R}^2 \setminus S$. The following conditions are equivalent:*

(i) System (2.9) *is flat at x_0;*
(ii) $\frac{\partial f_e^1}{\partial x_i^1}(x_0) \neq 0$;
(iii) $w_{ei}(1,1) \neq 0$;
(iv) System (2.9) *is locally static feedback equivalent around x_0 to the linear system $\dot{z}_1 = z_2$, $\dot{z}_2 = v_1$.*
(v) System (2.9) *is locally static feedback linearizable around x_0.*

Moreover, if one of the above equivalent conditions is satisfied, then x_e^1 is a flat output at x_0.

Proof. Immediate. □

According to the above result, for flatness we necessarily need an inhibitory action of x_i^1 on x_e^1, and if $w_{ei}(1,1) = 0$, then system (2.9) is never flat. Observe that we recover the results of [31] according to which in the single-input case, flatness is equivalent to static feedback linearization. In the case when the input acts on x_e^1, a similar result holds and the corresponding control system (considered at a suitable x_0) is flat (but now, with x_i^1 a flat output) if and only if $w_{ie}(1,1) \neq 0$ (or, equivalently, $\frac{\partial f_i^1}{\partial x_e^1}(x_0) \neq 0$). To sum up, the corresponding single-input control system is always flat without structural conditions ("equal identically to zero"-type conditions), only a regularity condition is needed ("non zero"-type conditions) which, in the particular case of a neural network formed by two populations, of excitatory and inhibitory neurons, resp., means that one population necessarily needs to act on the second one. On the contrary, a structural condition would mean that the interactions are restricted (one population should not affect the second one).

3. Main results: flatness of networks of N synaptically coupled EI modules

In this section we give our main results: for arbitrary values of c, the number of connection strengths acting as controls of the system, we identify flat configurations for networks of form (2.5) with N synaptically coupled EI modules, where N is arbitrary. Recall that we work under the assumptions that there is no relation between different connection strengths and that the inputs act independently on the system. Each connection strength may be a possible input, so we have $2N \times 2N$ candidates for the controls. We will distinguish two cases: first, we study the case when $N+p$ inputs, with $0 \leq p \leq N-1$, act on the system, and second, the case when the network is affected by p inputs only, with $1 \leq p \leq N-1$.

3.1. Control network with $N+p$ inputs, where $0 \leq p \leq N-1$

In this section we propose a flat control network configuration for the case when we can model $N+p$ connection strengths as controls, where p is such that $0 \leq p \leq N-1$. To make matters more definite, suppose that we control each state x_i^n, for $1 \leq n \leq N$, by an input u_i^n (which can be any connection strength among $w_{ie}(n,m)$ and $w_{ii}(n,m)$, for $1 \leq m \leq N$) and each state x_e^n, for $1 \leq n \leq p$, by an input u_e^n (which can be any connection strength among $w_{ee}(n,m)$ and $w_{ei}(n,m)$, for $1 \leq m \leq N$). To each input we associate its corresponding set of singularities defined according to Section 2.2. The control network takes the form:

$$\begin{array}{rcllrcll} \dot{x}_e^n &=& f_e^n(x,u_e^n) & & \dot{x}_e^n &=& f_e^n(x) & \\ \dot{x}_i^n &=& f_i^n(x,u_i^n), & \text{for } 1 \leq n \leq p, & \dot{x}_i^n &=& f_i^n(x,u_i^n), & \text{for } p+1 \leq n \leq N, \end{array} \tag{3.1}$$

and is considered around $x_0 \in \mathbb{R}^{2N} \setminus S$, where S is defined by (2.8). Let $D(x)$ denote the $(N-p)\times(N+p)$ matrix whose lines are given by

$$\begin{pmatrix} \frac{\partial f_e^n}{\partial x_i^1}(x) & \ldots & \frac{\partial f_e^n}{\partial x_i^N}(x) & \frac{\partial f_e^n}{\partial x_e^1}(x) & \ldots & \frac{\partial f_e^n}{\partial x_e^p}(x) \end{pmatrix}, \text{ for } p+1 \leq n \leq N. \tag{3.2}$$

Similarly, denote by W the constant $(N-p)\times(N+p)$ matrix whose lines are given by

$$\begin{pmatrix} w_{ei}(n,1) & \ldots & w_{ei}(n,N) & w_{ee}(n,1) & \ldots & w_{ee}(n,p) \end{pmatrix}, \text{ for } p+1 \leq n \leq N. \tag{3.3}$$

If $p=0$ (*i.e.*, the control network has N inputs), then we simply have:

$$\begin{array}{rcl} \dot{x}_e^n &=& f_e^n(x) \\ \dot{x}_i^n &=& f_i^n(x,u_i^n), \quad 1 \leq n \leq N, \end{array} \tag{3.4}$$

and the matrices $D(x)$ and W are $N \times N$ matrices.

Theorem 3.1. *Consider the control EI network, given by (3.1), around any $x_0 \in \mathbb{R}^{2N} \setminus S$, where S is defined by (2.8). The following conditions are equivalent:*

(i) The matrix $D(x_0)$ is of full rank, equal to $N-p$;
(ii) The matrix W is of full rank, equal to $N-p$;
(iii) System (3.1) is locally static feedback equivalent to the following form

$$\begin{array}{rcllrcll} \dot{z}_\ell^1 &=& u_\ell, & 1 \leq \ell \leq 2p, & \dot{z}_\ell^1 &=& z_\ell^2 & \\ & & & & \dot{z}_\ell^2 &=& u_\ell, & 2p+1 \leq \ell \leq N+p; \end{array} \tag{3.5}$$

(iv) System (3.1) is locally static feedback linearizable around x_0 with the distribution $\mathfrak{D}^1$ satisfying $\operatorname{rk}\mathfrak{D}^1 = 2N$.

Moreover, if one of the above equivalent condition is satisfied, then system (3.1) is flat at x_0.

Proof. See Section 5.1. □

First of all, notice that there are no structural conditions (*i.e.*, there are no connections that have to be zero) for flatness of the above multi-input EI network and that only a regularity condition is needed (translating into a certain matrix that has to be of full rank, which is the case for generic systems). Control EI network (3.1) is flat around any nominal state $x_0 \in \mathbb{R}^{2N} \setminus S$ and is locally static feedback linearizable. Indeed, it can be brought into form (3.5) which is the Brunovský form with $2p$ chains of length one and $N-p$ chains of length two (if $p=0$, then (3.5) contains only chains of length two[4]), and a geometric characterization is provided by condition (iv) of the above theorem. Observe that, even if the original control EI newtwork (3.1) is nonlinear with respect to the control, it can be immediately transformed *via* a suitable invertible feedback transformation into a control-affine system $\dot{x} = f(x) + \sum_{\ell=1}^{N+p} \tilde{u}_\ell g_\ell(x)$, where the control vector fields are actually of the form $g_\ell = \frac{\partial}{\partial x_a^n}$, for suitable pairs (a,n), and the distribution $\mathfrak{D}^0$ spanned by the g_ℓ's is clearly independent of $\tilde{u}$ and involutive. Theorem 3.1 gives sufficient conditions for flatness, but it does not completely describe flatness of systems of the form (3.1), *i.e.*, there are flat systems of the form (3.1) for which the equivalent conditions (i)–(iv) do not hold; an example is presented in [23] for an EI newtwork consisting of two subnetworks.

Remark 3.2. There is actually a lot a freedom in choosing the $N+p$ controls (among all possible w's) but also in choosing on which states whey act: for instance, we could have controlled all x_e^n-variables instead of all x_i^n (and only p x_i^n-states); any other combination is actually possible. From a mathematical point of view, the cases when other state variables are directly controlled are alike and can be treated in a similar way (they actually lead to control systems that are static feedback equivalent). What is important for achieving flatness is the fact that the uncontrolled variables depend explicitly on $N-p$ controlled ones.

To illustrate the above result consider the extreme case for which the subnetworks indexed by $p+1 \le n \le N$ contain only internal interactions (*i.e.*, interactions within the local subnetwork), implying $w_{ab}(n,m) = 0$, for all $m \neq n$ and $a, b \in \{e, i\}$, and given by

$$\begin{array}{rclrcl} \dot{x}_e^n & = & f_e^n(x, u_e^n) & \dot{x}_e^n & = & -x_e^n + F(w_{ee}(n,n)x_e^n - w_{ei}(n,n)x_i^n), \\ \dot{x}_i^n & = & f_i^n(x, u_i^n),\ 1 \le n \le p, & \dot{x}_i^n & = & f_i^n(x, u_i^n), \quad p+1 \le n \le N. \end{array}$$

Conditions (i)–(iv) translate into $w_{ei}(n,n) \neq 0$, for all $p+1 \le n \le N$, and from the above form, it is clear that, in that case, the system is flat with $(x_e^1, x_i^1, \dots, x_e^p, x_i^p, x_e^{p+1}, \dots, x_e^N)$ being a flat output.

3.2. Control network with p inputs, with $1 \le p \le N-1$

If only p connection strengths, with $1 \le p \le N-1$, can act as inputs, then we identify below two flat triangular configurations (reminding the triangular form Δ introduced at the end of the Sect. 2.1), each of them imposing several structural conditions. The first form admits a triangular chain of maximal length, while the second form contains p triangular chains of minimal length. For each form, we suppose that the inputs act directly on certain state variables. From a mathematical point of view, the cases when other variables are directly controlled are alike and can be treated in a similar way.

Flat EI network with a maximal triangular chain. For $1 \le n \le N$ fixed, we denote $\bar{x}^n = (x_e^1, x_i^1, \dots, x_e^n, x_i^n)$, *i.e.*, $\bar{x}^n$ regroups all states of the n first subnetworks. Consider, around x_0, the following network for which we control each state x_i^n, for $1 \le n \le p-1$, by the input u_i^n, and, finally, x_i^N by u_i^N; moreover, the p first subnetworks can act on all subnetworks, but all other interactions are restricted in the

[4] In the case when $c = 2N$ (*i.e.*, we have as many inputs as state variables), that we excluded in the paper, we would actually obtain an extreme case of form (3.5) with $2N$ chains of length one.

following way:

$$
\begin{array}{rcl}
\dot x_e^n &=& f_e^n(\bar x^p) \\
\dot x_i^n &=& f_i^n(\bar x^p, u_i^n), \\
& & 1 \le n \le p-1,
\end{array}
\qquad
\begin{array}{rcl}
\dot x_e^p &=& f_e^p(\bar x^p) \\
\dot x_i^p &=& f_i^p(\bar x^p, x_e^{p+1}) \\
& \vdots & \\
\dot x_e^N &=& f_e^N(\bar x^N) \\
\dot x_i^N &=& f_i^N(\bar x^N, u_i^N),
\end{array}
\tag{3.6}
$$

(so, in general, for a fixed $p \le n \le N$, we have $\dot x_e^n = f_e^n(\bar x^n) = f_e^n(x_e^1, x_i^1, \ldots, x_e^n, x_i^n)$ and $\dot x_i^n = f_i^n(\bar x^n, x_e^{n+1}) = f_i^n(x_e^1, x_i^1, \ldots, x_e^n, x_i^n, x_e^{n+1})$, with x_e^{N+1} replaced by u_i^N if $n = N$). To each input we associate its corresponding set of singularities defined according to Section 2.2, and the above control network is considered around $x_0 \in \mathbb{R}^{2N} \setminus S$, where S is defined by (2.8).

Theorem 3.3. *Consider the control EI network, given by (3.6), around any $x_0 \in \mathbb{R}^{2N} \setminus S$, where S is defined by (2.8). The following sets of conditions are equivalent:*

(i) We have
- *(a) the matrix $\left(\frac{\partial f_e^n}{\partial x_i^m}(x_0)\right)$, for $1 \le n, m \le p$, is of full rank;*
- *(b) $\frac{\partial f_e^n}{\partial x_i^n}(x_0) \ne 0$, for $p+1 \le n \le N$;*
- *(c) $\frac{\partial f_i^n}{\partial x_e^{n+1}}(x_0) \ne 0$, for $p \le n \le N-1$;*

(ii) We have
- *(a)' the matrix $\big(w_{ei}(n,m)\big)$, for $1 \le n, m \le p$, is of full rank;*
- *(b)' $w_{ei}(n,n) \ne 0$, for $p+1 \le n \le N$;*
- *(c)' $w_{ie}(n,n+1) \ne 0$, for $p \le n \le N-1$;*

If system (3.6) satisfies the equivalent sets of conditions (i) and (ii), then it is flat at x_0 with $(x_e^1, \ldots, x_e^p)$ a flat output.

Proof. See Section 5.2. □

Observe that after applying the invertible static feedback transformation $v_\ell = f_i^n(\bar x^p, u_i^n)$, $1 \le \ell \le p-1$, and $v_p = f_i^N(\bar x^N, u_i^N)$, and renaming the states, system (3.6) can be transformed into:

$$
\begin{array}{rcl}
\dot z_\ell^1 &=& f_\ell^1(\bar z^2) \\
\dot z_\ell^2 &=& v_\ell, \\
& & 1 \le \ell \le p-1,
\end{array}
\qquad
\begin{array}{rcl}
\dot z_p^1 &=& f_p^1(\bar z^2) \\
\dot z_p^2 &=& f_p^2(\bar z^2, z_p^3) \\
& \vdots & \\
\dot z_p^{\rho-1} &=& f_p^{\rho-1}(\bar z^2, z_p^3, \ldots, z_p^\rho) \\
\dot z_p^\rho &=& v_p,
\end{array}
\tag{3.7}
$$

where $\rho = 2N - 2(p-1)$ and recall that $\bar z^2 = (z_1^1, z_1^2, \ldots, z_p^1, z_p^2)$; the equivalent conditions of Theorem 3.3 rewrites

$$
\operatorname{rk}\left(\frac{\partial f_\ell^1}{\partial z_s^2}(z_0)\right)_{1 \le \ell, s \le p} = p \quad \text{and} \quad \frac{\partial f_p^j}{\partial z_p^{j+1}}(z_0) \ne 0, \ \text{for } 2 \le j \le \rho - 1. \tag{3.8}
$$

Therefore system (3.6) (with the functions f_a^n satisfying Thm. 3.3) is brought (after applying the above feedback transformation) in the control-affine triangular form Δ. The $p-1$ first subnetworks form $p-1$ triangular chains $(z_\ell^1, z_\ell^2) = (x_e^n, x_i^n)$, $1 \le \ell, n \le p-1$, of length two. In terms of connections strengths, they are characterized by:

for any fixed $1 \leq n \leq p-1$, we have

$$w_{ab}(n,m) = 0, \text{ for all } a, b \in \{e, i\} \text{ and } p+1 \leq m \leq N.$$

The dynamics of the remaining subnetworks form the triangular chain $(z_p^1, \ldots, z_\ell^\rho) = (x_e^p, x_i^p, \ldots, x_e^N, x_i^N)$ of length $\rho = 2N - 2(p-1)$, that will be called maximal triangular chain. In terms of connections strengths, the triangular structure of this chain is characterized by: for any fixed $p+1 \leq n \leq N-1$, we have

$$\begin{cases} w_{ea}(n,m) = 0, & \text{for all } a \in \{e,i\} \text{ and } n+1 \leq m \leq N, \\ w_{ia}(n,m) = 0 \text{ and } w_{ii}(n,n+1) = 0, & \text{for all } a \in \{e,i\} \text{ and } n+2 \leq m \leq N. \end{cases}$$

If $p = 1$ (*i.e.*, the control network has only one input), then form (3.6) contains only one triangular chain $(x_e^1, x_i^1, \ldots, x_e^N, x_i^N)$ of length $2N$. For any subnetwork labelled by n, for $p+1 \leq n \leq N-1$, we define its predecessor subnetwork as the subnetwork labelled by $n-1$, and its successor subnetwork as the subnetwork labelled by $n+1$. The triangular structure means that the action of any subnetwork on its predecessor is minimal (only $w_{ie}(n-1,n)$ being nonzero) while it can act on its successor in any possible way (since we have no condition on $w_{ab}(n+1,n)$). Observe that we could have required a triangular structure with a triangular chain of length $2N-(p-1)$ (and with $p-1$ chains of length one), but that would have imposed additional structural conditions between subnetworks. Moreover, it is natural to consider $p-1$ chains of length 2, since, in general, the evolution of the excitatory population of a subnetwork depends on that of the inhibitory population (and vice-versa). Furthermore, the structural conditions imposed by the proposed triangular form do not affect the internal interactions within the subnetworks.

Characterization of the triangular form (3.7) An interesting problem, that we will discuss below, is the geometric characterization of form (3.7), together with conditions (3.8). More precisely, given the control-affine system:

$$\Xi : \begin{cases} \dot{\hat{x}} & = & \hat{f}(\hat{x}, \bar{x}), \\ \dot{\bar{x}} & = & v, \end{cases} \tag{3.9}$$

where $x = (\hat{x}, \bar{x}) \in X \subset \mathbb{R}^{2N}$, $\dim \bar{x} = p$ and $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_p)$, and $v \in \mathbb{R}^p$, we will answer the following question: when can Ξ be transformed *via* a local change of coordinates $z = \phi(x)$ into the triangular form (3.7)? Notice first that the system Ξ is not as general as possible (since the distribution $\mathfrak{D}^0$ is involutive and in coordinates $(\hat{x}, \bar{x})$, it is actually rectified, *i.e.*, we have $\mathfrak{D}^0 = \mathrm{span}\,\{\frac{\partial}{\partial \bar{x}_\ell}, 1 \leq \ell \leq p\}$, that we will simply write $\mathfrak{D}^0 = \mathrm{span}\,\{\frac{\partial}{\partial \bar{x}}\}$). Although not as general as possible, the form Ξ describes however the class of systems into which any control EI network, as constructed in this paper, falls after applying the invertible feedback transformation $\tilde{u}_\ell = f_a^n(x, u_a^n)$, for $1 \leq \ell \leq p$ and suitable pairs (a,n), where p is the number of connection strengths modeled as controls. We denote by f (resp., by g_ℓ, $1 \leq \ell \leq p$) the drift (resp., the control vector fields) of the system Ξ (*i.e.*, we have $f = \sum_{s=1}^{2N-p} \hat{f}_s \frac{\partial}{\partial \hat{x}_s}$, where $\dim \hat{x} = 2N - p$, and $g_\ell = \frac{\partial}{\partial \bar{x}_\ell}$, $1 \leq \ell \leq p$).

Proposition 3.4. *Consider the control system Ξ around x_0. There exists a local change of coordinates transforming Ξ into (3.7), around z_0, with the functions f_ℓ^j satisfying (3.8), if and only if the following conditions are satisfied around x_0:*

(C1) The distribution $\mathfrak{D}^1$ is of constant rank $2p$;
(C2) There exists a vector field $\bar{g}$ among $g_1, \ldots g_p$ such that all distributions $\mathcal{H}^j = \mathfrak{D}^0 + \mathrm{span}\,\{ad_f \bar{g}, \ldots, ad_f^j \bar{g}\}$, for $1 \leq j \leq \rho - 2$, where $\rho = 2N - 2(p-1)$, are involutive, of constant rank $p + j$;
(C3) The distribution $\mathcal{H}^{\rho-1}$ is of constant rank $p + \rho - 1$ and $\mathfrak{D}^1 + \mathcal{H}^{\rho-1} = TX$.

Proof. See Section 5.3. □

From a mathematical point of view, Proposition 3.4 is important because it provides necessary and sufficient conditions to check whether a control system of the form Ξ can be transformed *via* a change of coordinates into the triangular form (3.7). Indeed, the above conditions can be checked easily in terms of vector fields of the original system Ξ, the verification involves differentiation and algebraic operations only (without solving PDE's). Observe that, in general, system (3.7) is not static feedback linearizable, but it becomes static feedback linearizable by prolonging at most $\rho - 2$ times each control v_ℓ, $1 \leq \ell \leq p-1$.

In the context of EI neural networks, Proposition 3.4 is interesting (in particular when the number N of subnetworks is large) since it provides, for all possible combinations (respecting Asm. 2.3) of connection strengths that can be modeled as inputs, an algorithmic way to verify if the network configuration is triangular.

Flat EI network with p minimal triangular chains. Let

$$2N = pq + r', \text{ where } 0 \leq r' \leq p-1,$$

be the Euclidean division of $2N$ and p based on which we will define three integers k, k', and r that will be used in the proposed flat configuration for the control EI network. Notice first that since $1 \leq p \leq N-1$, we have $q \geq 2$. If the quotient q is even, *i.e.*, there exists $k \geq 1$ such that $q = 2k$, then r' is also even and let r be such that $r' = 2r$. If q is odd, that is $q = 2k+1$ for some $k \geq 1$, then r' is odd also and define r by $p + r' = 2r$ (which is indeed even since we have $2N = 2kp + (p + r')$). To sum up, we have defined the integer $k \geq 1$ by

$$k = \begin{cases} q/2, & \text{if } q \text{ is even,} \\ (q-1)/2, & \text{if } q \text{ is odd.} \end{cases} \tag{3.10}$$

and the integer $r \geq 0$ by

$$r = \begin{cases} r'/2, & \text{if } q = 2k, \\ (p+r')/2, & \text{if } q = 2k+1. \end{cases} \tag{3.11}$$

For both cases (q even, resp., q odd), we will construct a flat EI network containing p triangular chains such that the r first chains are of length $2(k+1)$ and the $p-r$ last chains are of length $2k$. We introduce a third index $1 \leq j \leq p$ to distinguish these chains and define the integer $k' \geq 1$, depending on j, by

$$k' = \begin{cases} k+1, & \text{for } 1 \leq j \leq r, \\ k, & \text{for } r+1 \leq j \leq p. \end{cases} \tag{3.12}$$

The states will be denoted by $x_a^{n,j}$, where $a \in \{e, i\}$, $1 \leq j \leq p$ and $1 \leq n \leq k'$ (the integer j referring to the chain and n being the index of the subnetwork in the chain, each chain containing either k or $k+1$ subnetworks). The connection strength $w_{ab}^{j,\ell}(n,m)$, where $a, b \in \{e, i\}$, $1 \leq j, \ell \leq p$, $1 \leq n \leq k'_j$ and $1 \leq m \leq k'_\ell$ (with the lower indices j and ℓ of k' indicating that the upper bound of n depends on j and that of m on ℓ, resp.) corresponds to the coefficient of $x_b^{m,\ell}$ in the expression of $\dot{x}_a^{n,j}$. For fixed $1 \leq n \leq k'$ and $a \in \{e, i\}$, we also introduce the following notations using bold letters:

- $\boldsymbol{x}^n = (x_e^{n,1}, x_i^{n,1}, \ldots, x_e^{n,p}, x_i^{n,p})$ which regroups the states of the subnetworks labeled by n of all p chains;
- $\overline{\boldsymbol{x}}^n = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^n)$ which regroups the states of all subnetworks labeled from 1 to n of all p chains;
- $\boldsymbol{x}_a^n = (x_a^{n,1}, x_a^{n,2}, \ldots, x_a^{n,p})$ which regroups the states of either the excitatory, if $a = e$, or the inhibitory, if $a = i$, populations of the subnetworks labeled by n of all p chains (and that should not be confused with the notation x_a^n – with normal font for x – used in the previous sections for the states of the EI network).

If $n = k+1$ (recall relations (3.10)–(3.12) defining r, k and k'), then we simply have $\boldsymbol{x}^{k+1} = (x_e^{k+1,1}, x_i^{k+1,1}, \ldots, x_e^{k+1,r}, x_i^{k+1,r})$ and $\boldsymbol{x}_a^{k+1} = (x_a^{k+1,1}, \ldots, x_a^{k+1,r})$.

Consider now the following EI network:

$$\begin{array}{lcl}
\dot{x}_e^{1,j} & = & f_e^{1,j}(\overline{\boldsymbol{x}}^1), \\
\dot{x}_i^{1,j} & = & f_i^{1,j}(\overline{\boldsymbol{x}}^1, \boldsymbol{x}_e^2), \\
 & \vdots & \\
\dot{x}_e^{k,j} & = & f_e^{k,j}(\overline{\boldsymbol{x}}^k), \\
\dot{x}_i^{k,j} & = & f_i^{k,j}(\overline{\boldsymbol{x}}^k, \boldsymbol{x}_e^{k+1}), \\
\dot{x}_e^{k+1,j} & = & f_e^{k+1,j}(\overline{\boldsymbol{x}}^{k+1}), \\
\dot{x}_i^{k+1,j} & = & f_i^{k+1,j}(\overline{\boldsymbol{x}}^{k+1}, u_i^j), \quad 1 \leq j \leq r,
\end{array}
\qquad
\begin{array}{lcl}
\dot{x}_e^{1,j} & = & f_e^{1,j}(\overline{\boldsymbol{x}}^1), \\
\dot{x}_i^{1,j} & = & f_i^{1,j}(\overline{\boldsymbol{x}}^1, \boldsymbol{x}_e^2), \\
 & \vdots & \\
\dot{x}_e^{k,j} & = & f_e^{k,j}(\overline{\boldsymbol{x}}^k), \\
\dot{x}_i^{k,j} & = & f_i^{k,j}(\overline{\boldsymbol{x}}^k, u_i^j), \quad r+1 \leq j \leq p.
\end{array}
\tag{3.13}$$

To each input we associate its corresponding set of singularities defined according to Section 2.2, and the above control network is considered around $x_0 \in \mathbb{R}^{2N} \setminus S$, where S is defined by (2.8).

Theorem 3.5. *Consider the control EI network, given by (3.13), around any $x_0 \in \mathbb{R}^{2N} \setminus S$, where S is defined by (2.8). The following sets of conditions are equivalent:*

(i) We have
  - *(a) for fixed $1 \leq n \leq k$, $\mathrm{rk}\left(\frac{\partial f_e^{n,j}}{\partial x_i^{n,\ell}}(x_0)\right) = p$, where $1 \leq j, \ell \leq p$, and for $n = k+1$, $\mathrm{rk}\left(\frac{\partial f_e^{k+1,j}}{\partial x_i^{k+1,\ell}}(x_0)\right) = r$, where $1 \leq j, \ell \leq r$;*
  - *(b) for fixed $1 \leq n \leq k-1$, $\mathrm{rk}\left(\frac{\partial f_i^{n,j}}{\partial x_e^{n+1,\ell}}(x_0)\right) = p$, where $1 \leq j, \ell \leq p$, and for $n = k$, $\mathrm{rk}\left(\frac{\partial f_i^{k,j}}{\partial x_e^{k+1,\ell}}(x_0)\right) = r$, where $1 \leq j, \ell \leq r$;*

(ii) We have
  - *(a)' for fixed $1 \leq n \leq k$, $\mathrm{rk}\left(w_{ei}^{j,\ell}(n,n)\right) = p$, for $1 \leq j, \ell \leq p$, and for $n = k+1$, $\mathrm{rk}\left(w_{ei}^{j,\ell}(k+1,k+1)\right) = r$, for $1 \leq j, \ell \leq r$;*
  - *(b)' for fixed $1 \leq n \leq k-1$, $\mathrm{rk}\left(w_{ie}^{j,\ell}(n,n+1)\right) = p$, for $1 \leq j, \ell \leq p$, and for $n = k$, $\mathrm{rk}\left(w_{ie}^{j,\ell}(k,k+1)\right) = r$, for $1 \leq j, \ell \leq r$;*

If system (3.13) satisfies the equivalent sets of conditions (i) and (ii), then it is flat at x_0 with $(x_e^{1,1}, \ldots, x_e^{1,p})$ a flat output.

Proof. See Section 5.4. □

By applying an invertible static feedback transformation, system (3.13) can be brought into the triangular form Δ with p triangular chains of length $2k$ or $2(k+1)$. In terms of connections strengths, the triangular structure is characterized by the following relations (where the upper-indices $1 \leq j, \ell \leq p$ refer to the number of the triangular chain, while $1 \leq n, m \leq k'$ give the label of the subnetwork in the corresponding chain): for fixed $1 \leq j \leq p$ and $1 \leq n \leq k'$, we have:

$$\begin{cases} w_{ea}^{j,\ell}(n,m) = 0, & \text{for all } n+1 \leq m \leq k', \\ w_{ia}^{j,\ell}(n,m) = 0 \text{ and } w_{ii}^{j,\ell}(n,n+1) = 0, & \text{for all } n+2 \leq m \leq k', \end{cases}$$

for all $a \in \{e, i\}$ and $1 \leq \ell \leq p$ (with the upper bound k', on m, depending on ℓ). If $p = 1$ (*i.e.*, the control network has only one input), then form (3.13) contains only one triangular chain $(x_e^1, x_i^1, \ldots, x_e^N, x_i^N)$ of length $2N$, and actually coincides with system (3.6) (and, in that case, the conditions of Thms. 3.3 and 3.5 are equivalent). Observe that, similarly to form (3.6), we do not impose any structural conditions on the local interactions within subnetworks. On the other hand the interaction between subnetworks are always restricted: the triangular structure of each chain implies that the action of any subnetwork of that chain on its predecessor is minimal (for fixed $1 \leq n \leq k'$ and $1 \leq \ell \leq p$, only $w_{ie}^{\ell,\ell}(n-1,n)$ being nonzero) while it can act on its successor in any possible way (there are no conditions on $w_{ab}^{\ell,\ell}(n+1,n)$). Subnetworks of different chains are also allowed to interact, but again respecting the triangular structure: the n-th subnetwork of the ℓ-th chain never acts on the subnetworks labeled from 1 to $n-2$ of any chain, it can act only on the inhibitory population of the $(n-1)$-th subnetwork of each chain and there are no constraints regarding its action on the subnetworks labeled from $n+1$ to k'.

After applying the invertible static feedback transformation $v_\ell = f_i^{k+1,\ell}(\bar{\boldsymbol{x}}^{k+1}, u_i^\ell)$, for $1 \le \ell \le r$, and $v_\ell = f_i^{k,\ell}(\bar{\boldsymbol{x}}^{k}, u_i^\ell)$, for $r+1 \le \ell \le p$, and renaming the states, system (3.13) can be transformed into:

$$\begin{array}{rclrcl} \dot z_\ell^j &=& f_\ell^j(\bar z^{j+1}),\ 1\le j\le \rho-1, & \dot z_\ell^j &=& f_\ell^j(\bar z^{j+1}),\ 1\le j\le \rho-1,\\ \dot z_\ell^\rho &=& f_\ell^\rho(\bar z^{\rho+1}) & \dot z_\ell^\rho &=& v_\ell,\\ \dot z_\ell^{\rho+1} &=& f_\ell^{\rho+1}(\bar z^{\rho+2}) & && r+1\le \ell\le p\\ \dot z_\ell^{\rho+2} &=& v_\ell, & &&\\ && 1\le \ell\le r, &&& \end{array} \tag{3.14}$$

where $\rho = 2k$ and $\bar z^j = (z_1^1, \dots, z_1^j, \dots, z_p^1, \dots, z_p^j)$, with $z_\ell^{\rho+1}$ and $z_\ell^{\rho+2}$, for $r+1 \le \ell \le p$, missing, if $j = \rho+1$ or $j = \rho+2$. The equivalent conditions of Theorem 3.5 rewrites

$$\mathrm{rk}\left(\frac{\partial f_\ell^j}{\partial z_s^{j+1}}(z_0)\right)_{1\le \ell,s\le p} = p,\ \text{for } 1\le j\le \rho-1, \quad \text{and} \quad \mathrm{rk}\left(\frac{\partial f_\ell^j}{\partial z_s^{j+1}}\right)_{1\le \ell,s\le r}(z_0) = r,\ \text{for } j=\rho,\rho+1. \tag{3.15}$$

Characterization of the triangular form (3.14). We will give below a geometric characterization of form (3.14). We distinguish two cases: first $r=0$, that is, all p chains are of length equal to $2k$, and second, $r \ge 1$, that is, there is at least one chain among the p chains of length $2(k+1)$. Consider again the control system Ξ, given by (3.9), and compute the corresponding integers k and r as defined above. If $r=0$, then it is easy to see that form (3.14) (with $r=0$) is actually static feedback linearizable and we have the following result:

Proposition 3.6. *Consider the control system Ξ, with $x \in X \subset \mathbb{R}^{2N}$ and $v \in \mathbb{R}^p$, around x_0. There exists a local change of coordinates transforming Ξ into (3.14), with $r=0$, around z_0, with the functions f_ℓ^j satisfying (3.15), if and only if all distributions $\mathfrak{D}^j$, for $0 \le j \le 2k-1$, are involutive and of constant rank equal to $(j+1)p$ around x_0.*

Proof. The proof is immediate and follows from [27, 28]. □

Let us now discuss the case $r \ge 1$.

Proposition 3.7. *Consider the control system Ξ, with $x \in X \subset \mathbb{R}^{2N}$ and $v \in \mathbb{R}^p$, around x_0. Suppose that the integer r, defined by (3.11), is such that $r \ge 1$. There exists a local change of coordinates transforming Ξ into (3.14), with $r \ge 1$, around z_0, with the functions f_ℓ^j satisfying (3.15), if and only if the following conditions are satisfied around x_0:*

(C1) There exist vector fields $\bar g_1, \dots, \bar g_r$ among $g_1, \dots g_p$ such that the distributions $\mathcal{H}^1 = \mathfrak{D}^0 + \mathrm{span}\,\{ad_f \bar g_\ell, 1 \le \ell \le r\}$ and $\mathcal{H}^2 = \mathfrak{D}^0 + \mathrm{span}\,\{ad_f \bar g_\ell, ad_f^2 \bar g_\ell, 1 \le \ell \le r\}$ are involutive, of constant rank $p+r$ and $p+2r$, resp.

(C2) All distributions $\mathcal{H}^{j+1} = \mathcal{H}^j + [f, \mathcal{H}^j]$, for $2 \le j \le 2k$, are involutive, of constant rank $jp+2r$.

Proof. See Section 5.5 □

Similarly to Proposition 3.4, the above proposition gives necessary and sufficient conditions to check whether a control system of the form Ξ can be transformed *via* a change of coordinates into the triangular form (3.14) and the remarks made for Proposition 3.4 apply or can be adapted here. Observe that, under the assumption that $r \geq 1$, system (3.14) is never static feedback linearizable, but it becomes static feedback linearizable by prolonging (at most) twice each control v_ℓ, for $r+1 \leq \ell \leq p$.

4. Conclusions and future work

In this paper, we analyzed flatness of two synaptically coupled excitatory-inhibitory neural modules in function of the connection strengths. We showed that if enough connection strengths (at least as many as the number N of EI modules) can be considered as inputs, then the control network is flat without structural conditions. If the number of inputs is smaller than N, then flatness imposes particular configurations of the interactions between the subnetworks (or even for the local interactions within a subnetwork), and we identified, discussed and characterized several flat configurations. Under the independence assumption of the connections strengths and of the control variable, we distinguished two particular types of flat control neural networks: static feedback linearizable ones, resp., those that exhibit a triangular form.

Our future work includes taking into account possible symmetries or anti-symmetries of the connections strengths as well as adding into the control network dynamics the slow dynamics associated to the connections strengths that are not considered as inputs. Studying flatness of networks with other type of gain functions F or adding background inputs into the dynamics may also be interesting.

5. Proofs

5.1. Proof of Theorem 3.1

Consider the control EI network (3.1), around any $x_0 \in \mathbb{R}^{2N}$. Recall that $f_e^n(x) = -x_e^n + F\big(\sum_{m=1}^N (w_{ee}(n,m)x_e^m - w_{ei}(n,m)x_i^m)\big)$, with F given by (2.4). Thus for any fixed $p+1 \leq n \leq N$, we have

$$\begin{aligned}
\frac{\partial f_e^n(x)}{\partial x_i^m} &= -w_{ei}(n,m)F'\left(\textstyle\sum_{m=1}^N (w_{ee}(n,m)x_e^m - w_{ei}(n,m)x_i^m)\right) &&= -w_{ei}(n,m)\nu^n(x), \quad 1 \leq m \leq N,\\
\frac{\partial f_e^n(x)}{\partial x_e^{m'}} &= w_{ee}(n,m')F'\left(\textstyle\sum_{m=1}^N (w_{ee}(n,m)x_e^m - w_{ei}(n,m)x_i^m)\right) &&= w_{ee}(n,m')\nu^n(x), \quad 1 \leq m' \leq p,
\end{aligned}$$

where $F'(y) = \frac{\mathrm{e}^y}{(1+\mathrm{e}^y)^2}$ and $\nu^n(x) = F'\left(\sum_{m=1}^N (w_{ee}(n,m)x_e^m - w_{ei}(n,m)x_i^m)\right)$. Notice that all above expressions are a product between a connection strength (or the opposite of a connection strength) and the nonzero function $\nu^n(x)$, which is the same for all partial derivatives of $f_e^n(x)$. It follows that the rank of the $(N-p)\times(N+p)$ matrix $D(x)$ given by (3.2) (and whose components are the above partial derivatives of $f_e^n(x)$), always equals that of the matrix W given by (3.3), whose components are $w_{ei}(n,m)$, $w_{ee}(n,m')$, for $p+1 \leq n \leq N$, $1 \leq m \leq N$, and $1 \leq m' \leq p$. This shows the equivalence between (i) and (ii).

Let us now suppose that condition (i) is satisfied and suppose, without loos of generally (permute the states x_i^m, for $1 \leq m \leq N$, and $x_e^{m'}$, for $1 \leq m' \leq p$, if necessary), that the $(N-p)\times(N-p)$ submatrix of $D(x)$ whose components are $\frac{\partial f_e^n(x)}{\partial x_i^m}$, $\frac{\partial f_e^n(x)}{\partial x_e^{m'}}$, for $p+1 \leq n \leq N$, $r_i+1 \leq m \leq N$, and $r_e+1 \leq m' \leq p$, where r_i and r_e are such that $r_i + r_e = 2p$, is invertible around x_0. Then

$$\begin{aligned}
z_\ell^1 &= x_e^\ell,\ 1 \leq \ell \leq r_e, \quad & z_{r_e+\ell}^1 &= x_i^\ell,\ 1 \leq \ell \leq r_i, \quad & z_\ell^1 &= x_e^{\ell-p},\\
& & & & z_\ell^2 &= f_e^{\ell-p}(x), 2p+1 \leq \ell \leq N+p,
\end{aligned}$$

defines a change of coordinates around x_0, in which, after applying a suitable invertible feedback transformation, system (3.1) takes the (linear) form (3.5), proving (iii). It follows that (3.1) is actually static feedback linearizable and a straightforward computation shows that $\mathfrak{D}^1 = TX$, hence $\operatorname{rk}\mathfrak{D}^1 = 2N$. This shows (iv).

Now assume that system (3.1) is locally static feedback linearizable around x_0 with the distributions $\mathfrak{D}^1$ satisfying $\operatorname{rk}\mathfrak{D}^1 = 2N$. We have immediately that $\mathfrak{D}^0 = \operatorname{span}\{\frac{\partial}{\partial x_e^{m'}}, \frac{\partial}{\partial x_i^m}, 1 \leq m' \leq p, 1 \leq m \leq N\}$, and

$\mathfrak{D}^1 = \mathfrak{D}^0 + \mathrm{span}\,\{\sum_{n=p+1}^N \frac{\partial f_e^n}{\partial x_e^{m'}}\frac{\partial}{\partial x_e^n}, \sum_{n=p+1}^N \frac{\partial f_e^n}{\partial x_i^m}\frac{\partial}{\partial x_e^n}, 1 \le m' \le p, 1 \le m \le N\}$, and is clear that $\mathrm{rk}\,\mathfrak{D}^1 = 2N$ implies that $\mathrm{rk}\,D(x) = N - p$ around x_0, showing (i).

5.2. Proof of Theorem 3.3

Consider the control EI network, given by (3.6), around any $x_0 \in \mathbb{R}^{2N}$. For $1 \le n \le p$, we gave $f_e^n(x) = -x_e^n + F\left(\sum_{m=1}^p (w_{ee}(n,m)x_e^m - w_{ei}(n,m)x_i^m)\right)$, and for $1 \le m \le p$

$$\frac{\partial f_e^n(x)}{\partial x_i^m} = -w_{ei}(n,m)F'\left(\sum_{m=1}^p (w_{ee}(n,m)x_e^m - w_{ei}(n,m)x_i^m)\right) = -w_{ei}(n,m)\nu_e^n(x),$$

with $F(y)$ given by (2.4) and $F'(y) = \frac{\mathrm{e}^y}{(1+\mathrm{e}^y)^2}$. The function $\nu_e^n(x)$ (which is the same for all partial derivatives of $f_e^n(x)$) is always nonzero, it follows that $\mathrm{rk}\left(\frac{\partial f_e^n}{\partial x_i^m}(x)\right)_{1\le n,m\le p} = \mathrm{rk}\,(w_{ei}(n,m))_{1\le n,m\le p}$, showing the equivalence between (a) and (a)'.

For $p+1 \le n \le N$, we gave

$$\begin{array}{rcl} f_e^n(x) & = & -x_e^n + F\left(\sum_{m=1}^n (w_{ee}(n,m)x_e^m - w_{ei}(n,m)x_i^m)\right), \\ f_i^n(x) & = & -x_i^n + F\left(\sum_{m=1}^n (w_{ie}(n,m)x_e^m - w_{ii}(n,m)x_i^m) + w_{ie}(n,n+1)x_e^{n+1}\right), \end{array}$$

and

$$\begin{array}{rcl} \frac{\partial f_e^n(x)}{\partial x_i^n} & = & -w_{ei}(n,n)F'\left(\sum_{m=1}^n (w_{ee}(n,m)x_e^m - w_{ei}(n,m)x_i^m)\right) = -w_{ei}(n,n)\nu_e^n(x), \\ \frac{\partial f_i^n(x)}{\partial x_e^{n+1}} & = & w_{ie}(n,n+1)F'\left(\sum_{m=1}^n (w_{ie}(n,m)x_e^m - w_{ii}(n,m)x_i^m) + w_{ie}(n,n+1)x_e^{n+1}\right) \\ & = & w_{ie}(n,n+1)\nu_i^n(x), \end{array}$$

with $\nu_e^n(x) > 0$ and $\nu_i^n(x) > 0$. It follows that $\frac{\partial f_e^n}{\partial x_i^n}(x) \neq 0$ if and only if $w_{ie}(n,n) \neq 0$ and that $\frac{\partial f_i^n(x)}{\partial x_e^{n+1}}(x) \neq 0$ if and only if $w_{ei}(n,n+1) \neq 0$, proving that (b) is equivalent to (b)' and (c) to (c)', resp.

Now suppose that conditions (a)–(c) are satisfied around x_0. We show that the control EI network, given by (3.6), is flat with $\varphi = (\varphi_1, \ldots, \varphi_p) = (x_e^1, \ldots x_e^p)$ being a flat output at x_0. We have $\dot\varphi_n = \dot x_e^n = f_e^n(\varphi, x_i^1, \ldots, x_i^p)$, $1 \le n \le p$. Condition (a) implies that, around x_0, we can express each x_i^n, for $1 \le n \le p$, *via* the implicit function theorem, as $x_i^n = \gamma_i^n(\varphi, \dot\varphi)$, $1 \le n \le p$. From $\dot x_i^n = \dot\gamma_i^n(\varphi, \dot\varphi) = f_i^n(\varphi, \dot\varphi, u_i^n)$, for $1 \le n \le p-1$, we compute $u_i^n = \delta_i^n(\varphi, \dot\varphi, \ddot\varphi)$, $1 \le n \le p-1$. From $\dot x_i^p = \dot\gamma_i^p(\varphi, \dot\varphi) = f_i^p(\varphi, \dot\varphi, x_e^{p+1})$, we get $x_e^{p+1} = \gamma_e^{p+1}(\varphi, \dot\varphi, \ddot\varphi)$, then $\dot x_e^{p+1}$ will give $x_i^{p+1} = \gamma_i^{p+1}(\varphi, \ldots, \varphi^{(3)})$ and so on. Finally, we get $u_i^p = \delta_i^p(\varphi, \ldots, \varphi^{(\mu)})$, where the highest derivative order μ can be expressed in function of N and p.

5.3. Proof of Proposition 3.4

Necessity. Consider the triangular form (3.7) and suppose that the functions f_ℓ^j satisfy (3.8). We have

$$\mathfrak{D}^1 = \mathrm{span}\,\{\frac{\partial}{\partial z_\ell^2}, \frac{\partial}{\partial z_p^\rho}, \sum_{s=1}^p \frac{f_s^1}{z_\ell^2}\frac{\partial}{\partial z_s^1} + \sum_{j=1}^{\rho-1}\frac{\partial f_p^j}{\partial z_\ell^2}\frac{\partial}{\partial z_p^j}, \frac{\partial f_p^{\rho-1}}{\partial z_p^\rho}\frac{\partial}{\partial z_p^{\rho-1}}, 1 \le \ell \le p-1\},$$

and it follows from (3.8), that $\mathfrak{D}^1$ is of constant rank $2p$. It is easy to show that the vector field $\bar g = g_p$ verifies condition (C2). Indeed, for $\bar g = g_p$, we have (recall that, according to (3.8), $\frac{\partial f_p^{\rho-1}}{\partial z_p^\rho}(z_0) \neq 0$, where

$\rho = 2N - 2(p-1)$):

$$\mathcal{H}^1 = \operatorname{span}\{\frac{\partial}{\partial z_\ell^2}, \frac{\partial}{\partial z_p^\rho}, \frac{\partial f_p^{\rho-1}}{\partial z_p^\rho}\frac{\partial}{\partial z_p^{\rho-1}}, 1 \le \ell \le p-1\} = \operatorname{span}\{\frac{\partial}{\partial z_\ell^2}, \frac{\partial}{\partial z_p^{\rho-1}}, \frac{\partial}{\partial z_p^\rho}, 1 \le \ell \le p-1\},$$

which is involutive and of constant rank $p+1$. Recursively, for $1 \le j \le \rho-2$, compute

$$\mathcal{H}^j = \operatorname{span}\{\frac{\partial}{\partial z_\ell^2}, \frac{\partial}{\partial z_p^{\rho-j}}, \dots, \frac{\partial}{\partial z_p^{\rho-1}}, \frac{\partial}{\partial z_p^\rho}, 1 \le \ell \le p-1\},$$

which, similarly to $\mathcal{H}^1$, is involutive and of constant rank $p+j$. Finally

$$\mathcal{H}^{\rho-1} = \mathcal{H}^{\rho-2} + \operatorname{span}\{\sum_{s=1}^{p}\frac{f_s^1}{z_p^2}\frac{\partial}{\partial z_s^1}\},$$

and again, from (3.8), we deduce immediately that $\mathcal{H}^{\rho-1}$ is of constant rank $p+\rho-1$ and that $\mathfrak{D}^1 + \mathcal{H}^{\rho-1} = TX$.

Sufficiency. Consider the control system Ξ, given by (3.9), around x_0 and suppose that conditions (C1)–(C3) are satisfied around x_0. Let us assume without loss of generality that $\bar g = g_p$ (otherwise permute the inputs v_ℓ). By condition (C2), there exists the following nested sequence of involutive distributions:

$$\mathfrak{D}^0 \subset \mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^{\rho-2}.$$

By successive applications of the Frobenius theorem, it can be shown, see [27], that there exists a local change of coordinates $\zeta = \phi(x)$ that simultaneously rectifies all involutive distributions of the above sequence, that is, in new coordinates ζ, we have: $\mathfrak{D}^0 = \operatorname{span}\{\frac{\partial}{\partial \zeta^0}\}$, where $\dim \zeta^0 = p$, $\zeta^0 = (\zeta_1^0, \dots, \zeta_p^0)$ and $\frac{\partial}{\partial \zeta^0}$ simply denotes $\frac{\partial}{\partial \zeta_1^0}, \dots, \frac{\partial}{\partial \zeta_p^0}$, and $\mathcal{H}^j = \operatorname{span}\{\frac{\partial}{\partial \zeta^0}, \frac{\partial}{\partial \zeta^1}, \dots, \frac{\partial}{\partial \zeta^j}\}$, where $\dim \zeta^j = 1$, for $1 \le j \le \rho-2$. Since for the original system Ξ, we have $\mathfrak{D}^0 = \operatorname{span}\{\frac{\partial}{\partial \bar x}\}$, it follows that we can take $\zeta^0 = \bar x$. Now consider the vector field $\frac{\partial}{\partial \zeta^1}$. In coordinates $(\hat x, \bar x)$, it is of the general form $\sum_{s=1}^{2N-p} A_s^1(\hat x, \bar x)\frac{\partial}{\partial \hat x_s} + \sum_{\ell=1}^{p} B_\ell^1(\hat x, \bar x)\frac{\partial}{\partial \bar x_\ell}$, that we will simply write it, in a compact way, $A^1(\hat x, \bar x)\frac{\partial}{\partial \hat x} + B^1(\hat x, \bar x)\frac{\partial}{\partial \bar x}$. We have $[\frac{\partial}{\partial \zeta_\ell^0}, \frac{\partial}{\partial \zeta^1}] \equiv 0$, for $1 \le \ell \le p$, thus in coordinates $(\hat x, \bar x)$:

$$[\frac{\partial}{\partial \bar x_\ell}, A^1(\hat x, \bar x)\frac{\partial}{\partial \hat x} + B^1(\hat x, \bar x)\frac{\partial}{\partial \bar x}] = \frac{\partial A^1}{\partial \bar x_\ell}\frac{\partial}{\partial \hat x} + \frac{\partial B^1}{\partial \bar x_\ell}\frac{\partial}{\partial \bar x} \equiv 0,\ 1 \le \ell \le p,$$

implying that A^1 and B^1 do not depend on $\bar x$, and that we may write in the $(\hat x, \bar x)$-coordinates:

$$\mathcal{H}^1 = \mathfrak{D}^0 + \operatorname{span}\{A^1(\hat x)\frac{\partial}{\partial \hat x} + B^1(\hat x)\frac{\partial}{\partial \bar x}\} = \operatorname{span}\{\frac{\partial}{\partial \bar x}, A^1(\hat x)\frac{\partial}{\partial \hat x}\}.$$

In the same way, for $1 \le j \le \rho-2$, we obtain:

$$\mathcal{H}^j = \mathfrak{D}^0 + \operatorname{span}\{A^{j'}(\hat x)\frac{\partial}{\partial \hat x} + B^{j'}(\hat x)\frac{\partial}{\partial \bar x}, 1 \le j' \le j\} = \operatorname{span}\{\frac{\partial}{\partial \bar x}, A^{j'}(\hat x)\frac{\partial}{\partial \hat x}, 1 \le j' \le j\}.$$

For $1 \le j \le \rho-2$, define the nested sequence of distributions:

$$\mathcal{E}^j = \operatorname{span}\{A^{j'}(\hat x)\frac{\partial}{\partial \hat x}, 1 \le j' \le j\}.$$

Since the distributions $\mathcal{H}^j$, $1 \leq j \leq \rho-2$, are all involutive and of constant rank $p+j$, it follows immediately that the distributions $\mathcal{E}^j$ are also involutive and of constant rank j. Hence using the same argument as before, there exist local coordinates $\hat{z} = \hat{\phi}(\hat{x})$, that we label $\hat{z} = (z_1^1, \dots, z_{p-1}^1, z_p^1, z_p^2, \dots, z_p^{\rho-1})$, such that

$$\mathcal{E}^j = \operatorname{span}\{\frac{\partial}{\partial z_p^{\rho-j}}, \dots, \frac{\partial}{\partial z_p^{\rho-1}}\}, 1 \leq j \leq \rho-2,$$

implying that

$$\mathcal{H}^j = \operatorname{span}\{\frac{\partial}{\partial \bar{x}}\} + \operatorname{span}\{\frac{\partial}{\partial z_p^{\rho-j}}, \dots, \frac{\partial}{\partial z_p^{\rho-1}}\}, 1 \leq j \leq \rho-2.$$

Now recall that we assumed $\bar{g} = g_p$, thus by definition of $\mathcal{H}^j$, we have $ad_f^j g_p \in \mathcal{H}^j$ and $ad_f^j g_p \notin \mathcal{H}^{j-1}$. Writing the drift f in the $(\hat{z}, \bar{x})$-coordinates as $\big((f_\ell^j), 0_p\big)$, where the f_ℓ^j-components correspond to the $\hat{z}$-part of f and are labeled accordingly to the $\hat{z}$-variables, and the 0_p is a zero p-dimensional block corresponding to the $\bar{x}$-part of f (which has not been modified with respect to the original coordinates $x = (\hat{x}, \bar{x})$), we have:

$$ad_f \bar{g} = [f, \frac{\partial}{\partial \bar{x}_p}] = -\sum_{\ell=1}^{p} \frac{\partial f_\ell^1}{\partial \bar{x}_p} \frac{\partial}{\partial z_\ell^1} - \sum_{j=2}^{\rho-1} \frac{\partial f_p^j}{\partial \bar{x}_p} \frac{\partial}{\partial z_p^j} \in \mathcal{H}^1 = \operatorname{span}\{\frac{\partial}{\partial \bar{x}}\} + \operatorname{span}\{\frac{\partial}{\partial z_p^{\rho-1}}\},$$

implying that $\frac{\partial f_\ell^1}{\partial \bar{x}_p} = 0$, for $1 \leq \ell \leq p$, and $\frac{\partial f_p^j}{\partial \bar{x}_p} = 0$, for $2 \leq j \leq \rho-2$, and $\frac{\partial f_p^{\rho-1}}{\partial \bar{x}_p}(\xi_0) \neq 0$, where $\xi_0 = (\hat{z}_0, \bar{x}_0)$. Repeating the same argument for $\mathcal{H}^j$, $2 \leq j \leq \rho-2$, we deduce that $\frac{\partial f_\ell^1}{\partial z_p^j} = 0$, for $1 \leq \ell \leq p$ and $3 \leq j \leq \rho-1$, and $\frac{\partial f_p^j}{\partial z_p^{j'}} = 0$, for $2 \leq j \leq \rho-2$ and $j+2 \leq j' \leq \rho-1$, and $\frac{\partial f_p^j}{\partial z_p^{j+1}}(\xi_0) \neq 0$. Compute

$$\mathcal{H}^{\rho-1} = \operatorname{span}\{\frac{\partial}{\partial \bar{x}}, \frac{\partial}{\partial z_p^2}, \dots, \frac{\partial}{\partial z_p^{\rho-1}}, \sum_{s=1}^{p} \frac{\partial f_s^1}{\partial z_p^2} \frac{\partial}{\partial z_s^1}\},$$

and

$$ad_f g_\ell = [f, \frac{\partial}{\partial \bar{x}_\ell}] = -\sum_{s=1}^{p} \frac{\partial f_s^1}{\partial \bar{x}_\ell} \frac{\partial}{\partial z_s^1} \mod \operatorname{span}\{\frac{\partial}{\partial z_p^2}, \dots, \frac{\partial}{\partial z_p^{\rho-1}}\},\ 1 \leq \ell \leq p-1,$$

and from condition (C3), it follows that $\operatorname{rk}\left(\frac{\partial(f_1^1,\dots,f_p^1)}{\partial(\bar{x}_1,\dots,\bar{x}_{p-1},z_p^2)}\right) = p$. By renaming the variables $z_\ell^2 = \bar{x}_\ell$, for $1 \leq \ell \leq p-1$, and $z_p^\rho = \bar{x}_p$, we find exactly the triangular form (3.7), together with conditions (3.8).

5.4. Proof of Theorem 3.5

Consider the control EI network, given by (3.13), around any $x_0 \in \mathbb{R}^{2N}$. For $1 \leq j \leq p$, we gave:

$$f_e^{n,j}(x) = -x_e^{n,j} + F\left(\sum_{\ell=1}^{p}\sum_{m=1}^{n}(w_{ee}^{j,\ell}(n,m)x_e^{m,\ell} - w_{ei}^{j,\ell}(n,m)x_i^{m,\ell})\right),\ 1 \leq n \leq k',$$
$$f_i^{n,j}(x) = -x_i^{n,j} + F\left(\sum_{\ell=1}^{p}\left(\sum_{m=1}^{n}(w_{ie}^{j,\ell}(n,m)x_e^{m,\ell} - w_{ii}^{j,\ell}(n,m)x_i^{m,\ell}) + w_{ie}^{j,\ell}(n,n+1)x_e^{n+1,\ell}\right)\right),\ 1 \leq n \leq k'-1,$$

where k' depends on j and is given by (3.12). So for fixed $1 \leq n \leq k'$,

$$\frac{\partial f_e^{n,j}}{\partial x_i^{n,\ell}} = -w_{ei}^{j,\ell}(n,n)F'\left(\sum_{\ell=1}^{p}\sum_{m=1}^{n}(w_{ee}^{j,\ell}(n,m)x_e^{m,\ell} - w_{ei}^{j,\ell}(n,m)x_i^{m,\ell})\right) = -w_{ei}^{j,\ell}(n,n)\nu_e^{n,j}(x),$$

with $1 \leq \ell \leq p$ if $1 \leq n \leq k$ and $1 \leq \ell \leq r$ if $n = k+1$, and the usual expressions for F and F' (implying $\nu_e^{n,j}(x) > 0$). We deduce that for fixed n, the rank of the matrix $\left(\frac{\partial f_e^{n,j}}{\partial x_i^{n,\ell}}(x)\right)_{j,\ell}$ equals that of $\left(w_{ei}^{j,\ell}(n,n)\right)_{j,\ell}$. This shows the equivalence of (a) and (a)'. In a similar way one can show the equivalence of (b) and (b)'.

Let us now suppose that conditions (a) and (b) are satisfied around x_0. We show that the control EI network, given by (3.13), is flat with $\varphi = (\varphi_1, \dots, \varphi_p) = (x_e^{1,1}, \dots x_e^{1,p})$ being a flat output at x_0. We have $\dot{\varphi}_j = \dot{x}_e^{1,j} = f_e^{1,j}(\varphi, x_i^{1,1}, \dots, x_i^{1,p})$, $1 \leq j \leq p$. Condition (a) for $n = 1$ implies that, around x_0, we can express each $x_i^{1,j}$, for $1 \leq j \leq p$, *via* the implicit function theorem, as $x_i^{1,j} = \gamma_i^{1,j}(\varphi, \dot{\varphi})$, $1 \leq j \leq p$. From $\dot{x}_i^{1,j} = \dot{\gamma}_i^{1,j}(\varphi, \dot{\varphi}) = f_i^{1,j}(\varphi, \dot{\varphi}, x_e^{2,1}, \dots, x_e^{2,p})$, for $1 \leq j \leq p$, and using condition (b) for $n = 1$, we compute $x_e^{2,j} = \gamma_e^{2,j}(\varphi, \dot{\varphi}, \ddot{\varphi})$, $1 \leq j \leq p$, and so on. Finally $\dot{x}_i^{k,j}$, for $r+1 \leq j \leq p$, will allow to express $u_i^j = \delta_i^j(\varphi, \dots, \varphi^{(\mu)})$, $r+1 \leq j \leq p$, and $\dot{x}_i^{k+1,j}$, for $1 \leq j \leq r$, will give $u_i^j = \delta_i^j(\varphi, \dots, \varphi^{(\mu+1)})$, $1 \leq j \leq r$, where the highest derivative order μ can be expressed in function of k.

5.5. Proof of Proposition 3.7

Necessity. Consider the triangular form (3.14) and suppose that the functions f_ℓ^j satisfy (3.15). By a straightforward computation, it is easy to show that the vector fields $\bar{g}_\ell = g_\ell$, for $1 \leq \ell \leq r$, verify condition (C1). Indeed, we have $\mathcal{H}^1 = \mathrm{span}\,\{\frac{\partial}{\partial z_\ell^{\rho+1}}, \frac{\partial}{\partial z_\ell^{\rho+2}}, 1 \leq \ell \leq r, \frac{\partial}{\partial z_s^{\rho}}, r+1 \leq s \leq p\}$ and $\mathcal{H}^2 = \mathcal{H}^1 + \mathrm{span}\,\{\frac{\partial}{\partial z_\ell^{\rho}}, 1 \leq \ell \leq r\}$, which are involutive and of constant rank $p+r$ and $p+2r$, resp. Recursively, for $3 \leq j \leq \rho$, where $\rho = 2k$, we compute $\mathcal{H}^{j+1} = \mathcal{H}^j + \mathrm{span}\,\{\frac{\partial}{\partial z_\ell^{\rho-j+1}}, 1 \leq \ell \leq p\}$, which is involutive and of constant rank $jp + 2r$.

Sufficiency. Consider the control system Ξ, given by (3.9), around x_0 and suppose that conditions (C1)–(C2) are satisfied around x_0. Let us assume without loss of generality that $\bar{g}_\ell = g_\ell$, for $1 \leq \ell \leq r$ (otherwise permute the inputs v_ℓ). There exists the following nested sequence of involutive distributions:

$$\mathfrak{D}^0 \subset \mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^{\rho+1} = TX.$$

Similarly to the proof of Proposition 3.4, by successive applications of the Frobenius theorem, there exists a local change of coordinates $\zeta = \phi(x)$ that simultaneously rectifies all involutive distributions of the above sequence, that is, in new coordinates ζ, we have: $\mathfrak{D}^0 = \mathrm{span}\,\{\frac{\partial}{\partial \zeta^0}\}$ and $\mathcal{H}^j = \mathrm{span}\,\{\frac{\partial}{\partial \zeta^0}, \frac{\partial}{\partial \zeta^1}, \dots, \frac{\partial}{\partial \zeta^j}\}$, where $\dim \zeta^0 = p$, $\dim \zeta^1 = \dim \zeta^2 = r$, and $\dim \zeta^j = p$, for $3 \leq j \leq \rho+1$ (and we use the same notation for $\frac{\partial}{\partial \zeta^j}$ as in the proof of Prop. 3.4). Since for the original system Ξ, we have $\mathfrak{D}^0 = \mathrm{span}\,\{\frac{\partial}{\partial \bar{x}}\}$, it follows that we can take $\zeta^0 = \bar{x}$. Using the same arguments as those in the proof of Proposition 3.4, we can show that in coordinates $(\hat{x}, \bar{x})$, all vector fields $\frac{\partial}{\partial \zeta_\ell^j}$ are of the general (compact) form $A_\ell^j(\hat{x})\frac{\partial}{\partial \hat{x}} + B_\ell^j(\hat{x})\frac{\partial}{\partial \bar{x}}$ (*i.e.*, with A_ℓ^j and B_ℓ^j depending on $\hat{x}$ only), and that, for $1 \leq j \leq \rho+1$, we may write:

$$\begin{aligned}\mathcal{H}^j &= \mathfrak{D}^0 + \mathrm{span}\,\{A_\ell^{j'}(\hat{x})\tfrac{\partial}{\partial \hat{x}} + B_\ell^{j'}(\hat{x})\tfrac{\partial}{\partial \bar{x}}, \text{ for } 1 \leq j' \leq j \text{ and suitable } \ell\}\\ &= \mathrm{span}\,\{\tfrac{\partial}{\partial \bar{x}}, A_\ell^{j'}(\hat{x})\tfrac{\partial}{\partial \hat{x}}, \text{ for } 1 \leq j' \leq j \text{ and suitable } \ell\}.\end{aligned}$$

For $1 \leq j \leq \rho+1$, define the nested sequence of distributions:

$$\mathcal{E}^j = \mathrm{span}\,\{A_\ell^{j'}(\hat{x})\frac{\partial}{\partial \hat{x}}, \text{ for } 1 \leq j' \leq j \text{ and suitable } \ell\}.$$

Since the distributions $\mathcal{H}^j = \operatorname{span}\{\frac{\partial}{\partial \bar{x}}\} + \mathcal{E}^j$, $1 \le j \le \rho+1$, are all involutive and of constant rank, it follows immediately that the distributions $\mathcal{E}^j$ are also involutive and of constant rank (with $\operatorname{rk}\mathcal{E}^j = \operatorname{rk}\mathcal{H}^j - p$). Hence, using the same argument as before, there exist local coordinates $\hat{z} = \hat{\phi}(\hat{x})$, with $\dim \hat{z} = \dim \hat{x} = 2N - p$, that we label $\hat{z} = (z_1^1, \ldots, z_p^1, \ldots, z_1^\rho, \ldots, z_p^\rho, z_1^{\rho+1}, \ldots, z_r^{\rho+1})$, such that $\mathcal{E}^1 = \operatorname{span}\{\frac{\partial}{\partial z_\ell^{\rho+1}}, 1 \le \ell \le r\}$, $\mathcal{E}^2 = \operatorname{span}\{\frac{\partial}{\partial z_\ell^{\rho}}, \frac{\partial}{\partial z_\ell^{\rho+1}}, 1 \le \ell \le r\}$, and

$$\mathcal{E}^j = \operatorname{span}\{\frac{\partial}{\partial z_\ell^{\rho-j+2}}, \ldots, \frac{\partial}{\partial z_\ell^{\rho}}, \frac{\partial}{\partial z_s^{\rho+1}}, 1 \le \ell \le p, 1 \le s \le r\}, \text{ for } 3 \le j \le \rho+1.$$

From

$$\mathcal{H}^j = \operatorname{span}\{\frac{\partial}{\partial \bar{x}}\} + \mathcal{E}^j,$$

and using the definition of $\mathcal{H}^j$ with the help of $\bar{g}_\ell = g_\ell$, $1 \le \ell \le r$ (see Prop. 3.7, conditions (C1)–(C2)), we conclude that in the $(\hat{z}, \bar{x})$-coordinates, the drift f is in a triangular form, and by renaming $z_\ell^{\rho+2} = \bar{x}_\ell$, for $1 \le \ell \le r$, and $z_\ell^\rho = \bar{x}_\ell$, for $r+1 \le \ell \le p$, we get exactly the triangular form (3.14), together with conditions (3.15), the latter following from the constant rank of $\mathcal{H}^j$.

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