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OPTIMAL BOREL MEASURE-VALUED CONTROLS TO THE VISCOUS CAHN–HILLIARD–OBERBECK–BOUSSINESQ PHASE-FIELD SYSTEM ON TWO-DIMENSIONAL BOUNDED DOMAINS

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Abstract. We consider an optimal control problem for the two-dimensional viscous Cahn–Hilliard–Oberbeck–Boussinesq system with controls that take values in the space of regular Borel measures. The state equation models the interaction between two incompressible non-isothermal viscous fluids. Local distributed controls with constraints are applied in either of the equations governing the dynamics for the concentration, mean velocity, and temperature. Necessary and sufficient conditions characterizing local optimality in terms of the Lagrangian will be demonstrated. These conditions will be obtained through regularity results for the associated adjoint system, *a priori* estimates for the solutions of the linearized system in a weaker norm compared to that of the state space, and the Lebesgue decomposition of Borel measures.

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1. Introduction

Separation of phases for binary fluids may occur through nucleation and growth or spinodal decomposition. For nucleation and growth, the phases are separated by disconnected spherical structures with a fixed composition. With spinodal decomposition, the phases are interconnected and the composition changes through time. In certain situations, it is desirable to modify the process of phase separation in order to improve the chemical composition and physical characteristics of the final mixture. As mentioned in [31], the motion of a fluid mixture can be influenced by velocity controls through the placement of a mechanical stirring device or an ultrasound emitter. Alternatively, magnetic fields can be utilized to dictate the velocity of the flow of electrically conducting fluids [33].

It is favorable in some binary alloys to avoid or at least minimize phase separation to increase the strength and lifetime of the alloy. The performance of polymeric membranes obtained from a homogeneous polymeric solution *via* immersion precipitation process depends on the resulting morphological structure. Here, the solution is separated into two components with contrasting densities, to which the denser component solidifies by crystallization while the lighter components turn into pores [52].

Keywords and phrases: Cahn–Hilliard equation, Oberbeck–Boussinesq system, binary fluids, interpolation spaces, Borel measures, sparsity, optimality conditions, Lagrangian.

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Phase separation is also undesirable in glass production as it results in difficulties in the molding procedure and can lead to poor quality of the final glass. For multi-component glass ceramics used for consumer, medical, or biological applications, it is impeccable that the materials have high mechanical strength and low thermal expansion. Glass materials are typically formed by melting a base glass and then by applying a heat treatment to control the nucleation and precipitation of the glass crystals. On the other hand, by adding and removing suitable elements in sodium borosilicate glasses, it was observed that, depending on the composition of the glass and the duration of the heat treatment, the resulting material has increased alkaline resistance. For further details and other relevant resources, we refer the reader to [27, 40, 44, 45, 50] and the references therein. The above examples serve as motivations in considering controls for the composition, temperature, and velocity of binary fluid flows.

In this paper, we analyze an optimal control problem subject to a system of nonlinear partial differential equations that govern the evolution of non-isothermal, incompressible, and viscous binary flows. The controls will be taken from function spaces that have values in the space of regular Borel measures. These controls act on subsets of the fluid domain and will have point-wise in time constraints. To be precise, we will consider a non-convex optimal control problem

$$\min_{(\sigma_{\mathsf{o}},\sigma_{\mathsf{h}},\boldsymbol{\sigma}_{\mathsf{v}})\in\boldsymbol{\mathcal{M}}_{\mathsf{ad}}^{\infty}} G(\phi,\theta,\boldsymbol{u}) \tag{1.1}$$

where, for a given $(\sigma_{\mathsf{o}},\sigma_{\mathsf{h}},\boldsymbol{\sigma}_{\mathsf{v}})$ in the set of admissible controls

$$\begin{aligned}\boldsymbol{\mathcal{M}}_{\mathsf{ad}}^{\infty} := \{&(\sigma_{\mathsf{o}},\sigma_{\mathsf{h}},\boldsymbol{\sigma}_{\mathsf{v}}) \in L^\infty_{\mathrm{w}}(I;M(\omega_{\mathsf{o}}))\times L^\infty_{\mathrm{w}}(I;M(\omega_{\mathsf{h}}))\times L^\infty_{\mathrm{w}}(I;\boldsymbol{M}(\omega_{\mathsf{v}})) :\\ &\|\sigma_{\mathsf{o}}\|_{L^\infty_{\mathrm{w}}(I;M(\omega_{\mathsf{o}}))}\le\gamma_{\mathsf{o}}, \|\sigma_{\mathsf{h}}\|_{L^\infty_{\mathrm{w}}(I;M(\omega_{\mathsf{h}}))}\le\gamma_{\mathsf{h}}, \|\boldsymbol{\sigma}_{\mathsf{v}}\|_{L^\infty_{\mathrm{w}}(I;\boldsymbol{M}(\omega_{\mathsf{v}}))}\le\gamma_{\mathsf{v}}\},\end{aligned} \tag{1.2}$$

the triple $(\phi,\theta,\boldsymbol{u})$ is a suitable weak solution to the following system of nonlinear partial differential equations:

$$\left[\begin{array}{ll}\partial_t\phi + \mathrm{div}\,(\phi\boldsymbol{u}) - m\Delta\mu = f_{\mathsf{o}} + \chi_{\omega_{\mathsf{o}}}\sigma_{\mathsf{o}} & \text{in } Q,\\ \mu = \tau\partial_t\phi - \epsilon\Delta\phi + F(\phi) + l_{\mathsf{c}}\theta + f_{\mathsf{c}} & \text{in } Q,\\ \partial_t\theta - l_{\mathsf{h}}\partial_t\phi + \mathrm{div}\,((\theta - l_{\mathsf{h}}\phi)\boldsymbol{u}) - \kappa\Delta\theta = \alpha\boldsymbol{g}\cdot\boldsymbol{u} + f_{\mathsf{h}} + \chi_{\omega_{\mathsf{h}}}\sigma_{\mathsf{h}} & \text{in } Q,\\ \partial_t\boldsymbol{u} + \mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \nabla\mathfrak{p} = \mathcal{K}(\mu - l_{\mathsf{c}}\theta)\nabla\phi + \ell(\phi,\theta)\boldsymbol{g} + \boldsymbol{f}_{\mathsf{v}} + \chi_{\omega_{\mathsf{v}}}\boldsymbol{\sigma}_{\mathsf{v}} & \text{in } Q,\\ \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } Q,\\ \phi = \Delta\phi = 0, \quad \theta = 0, \quad \boldsymbol{u} = \boldsymbol{0} & \text{on } \Sigma,\\ \phi(0) = \phi_0, \quad \theta(0) = \theta_0, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in } \Omega.\end{array}\right. \tag{1.3}$$

Here, $\gamma_{\mathsf{o}},\gamma_{\mathsf{h}},\gamma_{\mathsf{v}} > 0$, $I := (0,T)$ with $0 < T < \infty$, $Q := I\times\Omega$, and $\Sigma := I\times\Gamma$, where Γ is the boundary of a sufficiently smooth open, bounded, and connected domain Ω in $\mathbb{R}^2$. Further description on the state system and the set of controls will be given below.

In (1.1), we shall take the cost functional

$$\begin{aligned}G(\phi,\theta,\boldsymbol{u}) :=& \frac{1}{2}\int_0^T\int_\Omega \left(\lambda_{\mathsf{o1}}|\phi(t,x)-\phi_d(t,x)|^2 + \lambda_{\mathsf{o2}}|\nabla\phi(t,x)-\boldsymbol{\psi}_d(t,x)|^2\right)\,\mathrm{d}x\,\mathrm{d}t\\ &+\frac{1}{2}\int_0^T\int_\Omega \left(\lambda_{\mathsf{h}}|\theta(t,x)-\theta_d(t,x)|^2 + \lambda_{\mathsf{v}}|\boldsymbol{u}(t,x)-\boldsymbol{u}_d(t,x)|^2\right)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{1.4}$$

where the weights λ_{o1}, λ_{o2}, λ_{h}, and λ_{v} are nonnegative constants, at least one of them is not zero, and the functions $\phi_d,\theta_d : Q\to\mathbb{R}$ and $\boldsymbol{\psi}_d,\boldsymbol{u}_d : Q\to\mathbb{R}^2$ are the desired concentration, temperature, concentration flux, and velocity, respectively. The value of a weight signifies preference to which a state may be steered closer to the desired target, and typically, a larger value of the weight means more priority to have a smaller residual.

For the state system (1.3), given a control triple $(\sigma_{\mathsf{o}}, \sigma_{\mathsf{h}}, \boldsymbol{\sigma}_{\mathsf{v}})$, the state variables $\phi, \mu, \theta, \mathfrak{p} : Q \to \mathbb{R}$ and $\boldsymbol{u} : Q \to \mathbb{R}^2$ are the order parameter, chemical potential, temperature, pressure, and mean velocity for the binary fluid. Controls will act on subsets ω_{o}, ω_{h}, and ω_{v}, which are relatively closed in Ω, with χ_ω denoting the indicator function of $\omega \subset \Omega$. The subscripts o, h, and v represent order parameter, heat, and velocity, respectively. Although the three controls are simultaneously applied, one can also specialize to the case where only one or two of them are present, and the results in this paper can be easily modified to such scenarios.

The known external sources are denoted by f_{o}, f_{c}, f_{h}, and $\boldsymbol{f}_{\mathsf{v}}$, and these designate the chemical concentration source, internal micro-force, heat source, and external body force. The positive constant parameters m, τ, ϵ, κ, ν, $\mathcal{K}$, l_{c}, and l_{h} correspond to the diffusive mobility, order parameter viscosity, interfacial thickness, thermal conductivity, kinematic viscosity, capillarity stress, and the last two being related to the latent heat. Finally, $F(\phi) = \beta_0\phi^3 - \beta_1\phi$ with $\beta_0, \beta_1 > 0$ is the derivative of a double-well potential, $\ell(\phi, \theta) = \alpha_0 + \alpha_1\phi + \alpha_2\theta$ with $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ is a linearized equation of state for the density, $\boldsymbol{g} \in \mathbb{R}^2$ is a gravitational force, and $\alpha \in \mathbb{R}$ is a linearized adiabatic heat.

The system (1.3) is a coupling of the viscous Cahn–Hilliard and Oberbeck–Boussinesq systems and is based on the classical works [5, 6, 41], the addition of viscous term in [26], and the coupling due to surface tension in [37]. We refer the reader to the work of the author in [42] and the relevant references therein for an outline of the derivation to the above system in the case $\tau = 0$.

Let us discuss the notation introduced in (1.2) for the set of admissible controls. For an open and relatively closed subset ω of Ω, $M(\omega)$ is the Banach space of real and regular Borel measures on ω. According to the Riesz Theorem, $M(\omega)$ can be topologically identified with the dual of the Banach space

$$C_0(\omega) = \{\phi \in C(\bar{\omega}) : \varphi = 0 \text{ on } \partial\omega \cap \Gamma\}$$

endowed with the supremum norm $\|\phi\|_{C_0(\omega)} = \sup_{x\in\bar{\omega}} |\varphi(x)|$. The associated dual norm for $M(\omega)$ is given by

$$\|\sigma\|_{M(\omega)} = \sup_{\|\phi\|_{C_0(\omega)}\le 1} \langle \sigma, \phi\rangle_{M(\omega),C_0(\omega)} = \sup_{\|\phi\|_{C_0(\omega)}\le 1} \int_\omega \phi \,\mathrm{d}\sigma = |\sigma|(\omega)$$

where $|\sigma|$ denotes the total variation measure associated to $\sigma \in M(\omega)$. With respect to the product space $\boldsymbol{C}_0(\omega) := C_0(\omega) \times C_0(\omega)$, we shall consider the norm

$$\|\boldsymbol{\phi}\|_{\boldsymbol{C}_0(\omega)} = \|\phi_1\|_{C_0(\omega)} + \|\phi_2\|_{C_0(\omega)} \qquad \forall\, \boldsymbol{\phi} = (\phi_1, \phi_2) \in \boldsymbol{C}_0(\omega).$$

The corresponding dual norm in $\boldsymbol{M}(\omega) := M(\omega) \times M(\omega) = \boldsymbol{C}_0(\omega)'$ is given by

$$\|\boldsymbol{\sigma}\|_{\boldsymbol{M}(\omega)} = \max\{\|\sigma_1\|_{M(\omega)}, \|\sigma_2\|_{M(\omega)}\} \qquad \forall\, \boldsymbol{\sigma} = (\sigma_1, \sigma_2) \in \boldsymbol{M}(\omega)$$

and duality pairing is defined by

$$\langle \boldsymbol{\sigma}, \boldsymbol{\phi}\rangle_{\boldsymbol{M}(\omega),\boldsymbol{C}_0(\omega)} = \int_\omega \boldsymbol{\phi} \,\mathrm{d}\boldsymbol{\sigma} = \int_\omega \phi_1 \,\mathrm{d}\sigma_1 + \int_\omega \phi_2 \,\mathrm{d}\sigma_2.$$

The definition of $L^\infty_{\mathrm{w}}(I; M(\omega))$ and other measure-theoretic results needed in the analysis will be discussed in Sections 2 and 4.

Starting from [1], several papers have addressed optimal control problems for time-dependent flows in fluid mechanics and their various applications in other areas of the sciences. We only mention some works that are closely related to the state system being considered in this paper. Optimal control for the Cahn–Hilliard equation and other phase-field type systems can be found for instance in [15–18, 22, 25, 34, 48]. In the case of the time-dependent and time-discrete Cahn–Hilliard–Navier–Stokes equation, optimal control problems were discussed in [21, 29, 30]. Most of these papers deal with controls that lie in a Hilbert space. In contrast to (1.1), the controls lie in non-reflexive Banach spaces.

Measure-valued controls, both from theoretical and numerical perspectives, have gained interest since they promote sparsity. This means that the supports of the optimal controls are relatively small compared to the domain. For stationary problems, we refer to the papers [7, 8, 11–13]. For evolutionary problems, see [9, 14, 28, 38, 39]. In particular, the recent work of Casas and Kunisch for the Navier–Stokes equation in [9] greatly influenced the current paper. The coupling of the non-isothermal viscous Cahn–Hilliard system with the Navier–Stokes equation makes the analysis more involved.

Let us mention two main challenges in this direction. First, one has to provide a suitable functional analytic framework for the weak solutions to the state system. Second, with the presence of the term $\mathcal{K}\mu\nabla\phi$ in (1.3) due to surface tension, we need to lay down suitable *a priori* estimates that arise from this term in the linearized and adjoint systems, and will enable us to express the second-order conditions for local optimality. To the best knowledge of the author, measure-valued controls to the viscous Cahn–Hilliard equation have not been studied yet. In addition, the results of this paper can be easily adapted to simpler models, namely, the viscous non-isothermal Cahn–Hilliard system (constant $\boldsymbol{u}$), the Oberbeck–Boussinesq system (constant ϕ), and the Cahn–Hilliard–Navier–Stokes system (constant θ).

The first issue mentioned above has been considered in [43], following the strategy in [10] for the Navier–Stokes equation. The main idea there is to split the state system into linear and non-linear parts, proceed with the linear part using extended maximal parabolic regularity, semigroup methods, and interpolation theory. Then, one can consider the nonlinear part with a classical Faedo–Galerkin method for Hilbert spaces. In principle, this paper is a continuation of the work that has been initiated in [43].

With regard to the second issue, the goal is to have a definiteness for the second derivative at a local solution with respect to a norm equivalent to the cost functional. This norm is weaker than that of the solution space for the state system. Following [9], the second-order conditions will be formulated in terms of the Lagrangian, which is the sum of the cost functional and an integral term associated with the control constraints. Observe that the constraints as defined in (1.2) can be viewed as a list of point-wise in time constraints

$$\|\sigma_{\mathsf{o}}(t)\|_{M(\omega_{\mathsf{o}})} \le \gamma_{\mathsf{o}},\ \|\sigma_{\mathsf{h}}(t)\|_{M(\omega_{\mathsf{h}})} \le \gamma_{\mathsf{h}},\ \|\sigma_{\mathsf{v}1}(t)\|_{M(\omega_{\mathsf{v}})} \le \gamma_{\mathsf{v}},\ \|\sigma_{\mathsf{v}2}(t)\|_{M(\omega_{\mathsf{v}})} \le \gamma_{\mathsf{v}}$$

for a.a.(almost all) $t \in I$. In this direction, following the finite-dimensional case, we shall consider the Lagrangian

$$\begin{aligned} L((\sigma_{\mathsf{o}}, \sigma_{\mathsf{h}}, \boldsymbol{\sigma}_{\mathsf{v}}), (m_{\mathsf{o}}, m_{\mathsf{h}}, \boldsymbol{m}_{\mathsf{v}})) := G(\phi, \theta, \boldsymbol{u}) &+ \int_0^T [m_{\mathsf{o}}(\|\sigma_{\mathsf{o}}\|_{M(\omega_{\mathsf{o}})} - \gamma_{\mathsf{o}}) + m_{\mathsf{h}}(\|\sigma_{\mathsf{h}}\|_{M(\omega_{\mathsf{h}})} - \gamma_{\mathsf{h}})]\,\mathrm{d}t \\ &+ \int_0^T [m_{\mathsf{v}1}(\|\sigma_{\mathsf{v}1}\|_{M(\omega_{\mathsf{v}})} - \gamma_{\mathsf{v}}) + m_{\mathsf{v}2}(\|\sigma_{\mathsf{v}2}\|_{M(\omega_{\mathsf{v}})} - \gamma_{\mathsf{v}})]\,\mathrm{d}t \end{aligned}$$

where $m_{\mathsf{o}}, m_{\mathsf{h}}, m_{\mathsf{v}1}, m_{\mathsf{v}2} \in L^1_+(I) := \{m \in L^1(I) : m \ge 0 \text{ a.a. in } I\}$ are the Lagrange multipliers corresponding to the above inequality constraints, respectively.

The structure of the paper is as follows: Section 2 provides a brief introduction to the notation for the various function spaces needed in future discussions. Section 3 presents the analysis of the state system, the existence of optimal controls, and the well-posedness of the adjoint system. Finally, Section 4 deals with the necessary and sufficient conditions for local optimality.

2. Function spaces

We shall follow the standard notation in [2] for the Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{\mathfrak{s},p}(\Omega)$ for $\mathfrak{s} \ge 0$ and $1 \le p \le \infty$. The closure in $W^{\mathfrak{s},p}(\Omega)$ of the set $C_0^\infty(\Omega)$ of all infinitely differentiable functions having compact support in Ω will be denoted by $W_0^{\mathfrak{s},p}(\Omega)$ and its dual by $W^{-\mathfrak{s},p'}(\Omega) := W_0^{\mathfrak{s},p}(\Omega)'$, where $p' = p/(p-1)$ for $1 < p < \infty$ and $p' = \infty$ if $p = 1$. The space of all functions in $L^p(\Omega)$ with vanishing integrals over Ω will be written by $\widehat{L}^p(\Omega)$. A boldface will be used to denote the product of these function spaces with themselves. For instance, $\boldsymbol{L}^p(\Omega) := L^p(\Omega) \times L^p(\Omega)$, $\boldsymbol{W}^{\mathfrak{s},p}(\Omega) := W^{\mathfrak{s},p}(\Omega) \times W^{\mathfrak{s},p}(\Omega)$, and $\boldsymbol{W}_0^{\mathfrak{s},p}(\Omega) := W_0^{\mathfrak{s},p}(\Omega) \times W_0^{\mathfrak{s},p}(\Omega)$. We shall follow this convention in the succeeding discussion without further notice.

For $1 < q < \infty$, let $A_q = -\Delta : D(A_q) \subset L^q(\Omega) \to L^q(\Omega)$ be the Dirichlet Laplacian with the domain $D(A_q) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. Likewise, for $1 < p < \infty$, we introduce the Stokes operator $\boldsymbol{A}_p = -\boldsymbol{P}_p\Delta : D(\boldsymbol{A}_p) \subset \boldsymbol{L}^p_\sigma(\Omega) \to \boldsymbol{L}^p_\sigma(\Omega)$ with domain $D(\boldsymbol{A}_p) = \boldsymbol{W}^{2,p}(\Omega) \cap \boldsymbol{W}_0^{1,p}(\Omega) \cap \boldsymbol{L}^p_\sigma(\Omega)$. Here, $\boldsymbol{L}^p_\sigma(\Omega)$ is the closure of $\{\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega) : \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}$ in $\boldsymbol{L}^p(\Omega)$ and $\boldsymbol{P}_p : \boldsymbol{L}^p(\Omega) \to \boldsymbol{L}^p_\sigma(\Omega)$ is the Leray–Helmholtz projector.

For $\mathfrak{s} \geq 0$ we set $X^{\mathfrak{s},q}(\Omega) := D(A_q^{\mathfrak{s}/2})$ and $\boldsymbol{X}_\sigma^{\mathfrak{s},p}(\Omega) := D(\boldsymbol{A}_p^{\mathfrak{s}/2})$ equipped with the norms $\|\phi\|_{X^{\mathfrak{s},q}(\Omega)} := \|A_q^{\mathfrak{s}/2}\phi\|_{L^q(\Omega)}$ for $\phi \in X^{\mathfrak{s},q}(\Omega)$ and $\|\boldsymbol{u}\|_{\boldsymbol{X}_\sigma^{\mathfrak{s},p}(\Omega)} := \|\boldsymbol{A}_p^{\mathfrak{s}/2}\boldsymbol{u}\|_{\boldsymbol{L}^p_\sigma(\Omega)}$ for $\boldsymbol{u} \in \boldsymbol{X}_\sigma^{\mathfrak{s},p}(\Omega)$. With regard to the domains of the fractional Dirichlet Laplacian and the Stokes operator in L^p-spaces, we refer to [23] and [24]. The dual spaces will be written as $X^{-\mathfrak{s},q'}(\Omega) := X^{\mathfrak{s},q}(\Omega)'$ and $\boldsymbol{X}_\sigma^{-\mathfrak{s},p'}(\Omega) := \boldsymbol{X}_\sigma^{\mathfrak{s},p}(\Omega)'$. In particular, we have $X^{1,q}(\Omega) = W_0^{1,q}(\Omega)$ and $\boldsymbol{X}_\sigma^{1,p}(\Omega) = \boldsymbol{W}_0^{1,p}(\Omega) \cap \boldsymbol{L}^p_\sigma(\Omega)$.

We denote the dual operators of A_q and $\boldsymbol{A}_p$ by $A_q' : L^{q'}(\Omega) \to X^{-2,q'}(\Omega)$ and $\boldsymbol{A}_p' : \boldsymbol{L}_\sigma^{p'}(\Omega) \to \boldsymbol{X}_\sigma^{-2,p'}(\Omega)$, respectively. These dual operators admit unitary extensions such that $A_q' : W_0^{1,q'}(\Omega) \to W^{-1,q'}(\Omega)$ and $\boldsymbol{A}_p' : \boldsymbol{X}_\sigma^{1,p'}(\Omega) \to \boldsymbol{X}_\sigma^{-1,p'}(\Omega)$, see for instance [9].

Given $1 < r < \infty$, we introduce the following real interpolation spaces

$$\begin{aligned} Z_{q,r}^{\mathfrak{s}}(\Omega) &:= (X^{\mathfrak{s}-2,q}(\Omega), X^{\mathfrak{s},q}(\Omega))_{1/r',r} \\ \boldsymbol{V}_{p,r}^{\mathfrak{s}}(\Omega) &:= (\boldsymbol{X}_\sigma^{\mathfrak{s}-2,p}(\Omega), \boldsymbol{X}_\sigma^{\mathfrak{s},p}(\Omega))_{1/r',r}. \end{aligned}$$

The initial data will be taken in certain sums of these function spaces with suitable values of q, p, r, and $\mathfrak{s}$. In particular, classical interpolation theory yields $Z_{2,2}^3(\Omega) = X^{2,2}(\Omega)$, $Z_{2,2}^1(\Omega) = L^2(\Omega)$, and $\boldsymbol{V}_{2,2}^1(\Omega) = \boldsymbol{L}_\sigma^2(\Omega)$. For further details on interpolation theory, we refer to the standard texts [3, 4, 36, 49].

If X is a Banach space and $1 \leq r \leq \infty$, then we denote by $L^r(I;X)$ the Bochner space of equivalence classes of strongly measurable functions $f : I \to X$ such that $\|f\|_{L^r(I;X)} < \infty$, where

$$\|f\|_{L^r(I;X)} := \left(\int_0^T \|f(t)\|_X^r \,\mathrm{d}t\right)^{1/r} \quad \text{for } 1 \leq r < \infty, \tag{2.1}$$

$$\|f\|_{L^\infty(I;X)} := \operatorname*{ess\,sup}_{t\in I} \|f(t)\|_X. \tag{2.2}$$

For a separable Banach space X, $L^r_{\mathrm{w}}(I;X')$ denotes the space of all equivalence classes of X-weakly measurable functions $f : I \to X'$ such that $\|f\|_{L^r_{\mathrm{w}}(I;X')} < \infty$, where the norm is defined as in (2.1) or (2.2) with X replaced by X'. Then, we have $L^r(I;X)' = L^{r'}_{\mathrm{w}}(I;X')$ for $1 \leq r < \infty$. If X and X' are separable, which is the case for instance when X is reflexive and separable, then the notions of weak and strong measurability for functions from I into X' are equivalent due to Pettis Measurability Theorem [19], Theorem II.1.2, and we have $L^r_{\mathrm{w}}(I;X') = L^r(I;X')$. The reader is referred to [20], Sections 12.2 and 12.9 or [32], Chapter 7 for the details, in particular, to the proof of the above duality identification.

Given two Banach spaces X and Y such that $X \hookrightarrow Y$, that is, X is continuously embedded in Y, we let $W^{1,r}(I;X,Y) := \{u \in L^r(I;X) : \partial_t u \in L^r(I;Y)\}$ equipped with the graph norm, $W^{1,r}(I;X) := W^{1,r}(I;X,X)$, and $W_0^{1,r}(I;X) := \{u \in W^{1,r}(I;X) : u(0) = u(T) = 0\}$. Recall that time-evaluations of elements in $W^{1,r}(I;X,Y)$ are well-defined due to $W^{1,r}(I;X,Y) \hookrightarrow C(\bar{I};Y)$. The function spaces for the state variables, except for the pressure and the chemical potential, will be taken in

$$\begin{aligned} \mathcal{Z}_{q,r}^{\mathfrak{s}}(Q) &:= W^{1,r}(I; X^{\mathfrak{s},q}(\Omega), X^{\mathfrak{s}-2,q}(\Omega)) \\ \boldsymbol{\mathcal{V}}_{p,r}^{\mathfrak{s}}(Q) &:= W^{1,r}(I; \boldsymbol{X}_\sigma^{\mathfrak{s},p}(\Omega), \boldsymbol{X}_\sigma^{\mathfrak{s}-2,p}(\Omega)), \end{aligned}$$

under suitable values of q, p, r, and $\mathfrak{s}$. According to [3], Theorem III.4.10.2, we have the following continuous embeddings

$$\mathcal{Z}^{\mathfrak{s}}_{q,r}(Q) \hookrightarrow C(\bar{I}; Z^{\mathfrak{s}}_{q,r}(\Omega)), \qquad \boldsymbol{\mathcal{V}}^{\mathfrak{s}}_{p,r}(Q) \hookrightarrow C(\bar{I}; \boldsymbol{V}^{\mathfrak{s}}_{p,r}(\Omega)).$$

This is consistent on what have been mentioned earlier for the spaces of initial data. We point out that the notations in [43], Sections 2 and 3 are adapted in this paper.

Finally, in relation to the controls, we consider the function spaces

$$\begin{aligned} \boldsymbol{\mathcal{M}}^r &:= L^r_{\mathrm{w}}(I; M(\omega_{\mathsf{o}})) \times L^r_{\mathrm{w}}(I; M(\omega_{\mathsf{h}})) \times L^r_{\mathrm{w}}(I; \boldsymbol{M}(\omega_{\mathsf{v}})) \\ \boldsymbol{\mathcal{N}}^r_{q,s,p}(Q) &:= L^r(I; W^{-1,q}(\Omega)) \times L^r(I; W^{-1,s}(\Omega)) \times L^r(I; \boldsymbol{W}^{-1,p}(\Omega)). \end{aligned}$$

Then, $\boldsymbol{\mathcal{M}}^r \hookrightarrow \boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)$ for $p, q, s \in (1, 2)$ and $1 \le r < \infty$. Indeed, given $\mathfrak{s} \in (1, 2)$ and a relatively closed subset ω in Ω, we have $2 < \mathfrak{s}' < \infty$, and so $W^{1,\mathfrak{s}'}_0(\Omega) \hookrightarrow C_0(\Omega) \hookrightarrow C_0(\omega)$ by the Sobolev embedding theorem. This implies that $M(\omega) \hookrightarrow W^{-1,\mathfrak{s}}(\Omega)$ by duality, and consequently, $L^r_{\mathrm{w}}(I; M(\omega)) \hookrightarrow L^r_{\mathrm{w}}(I; W^{-1,\mathfrak{s}}(\Omega)) = L^r(I; W^{-1,\mathfrak{s}}(\Omega))$ since $W^{1,\mathfrak{s}'}_0(\Omega)$ is separable and reflexive. We equip $\boldsymbol{\mathcal{M}}^\infty$ with the norm

$$\|(\sigma_{\mathsf{o}}, \sigma_{\mathsf{h}}, \boldsymbol{\sigma}_{\mathsf{v}})\|_{\boldsymbol{\mathcal{M}}^\infty} := \max\{\|\sigma_{\mathsf{o}}\|_{L^\infty_{\mathrm{w}}(I; M(\omega_{\mathsf{o}}))}, \|\sigma_{\mathsf{h}}\|_{L^\infty_{\mathrm{w}}(I; M(\omega_{\mathsf{h}}))}, \|\boldsymbol{\sigma}_{\mathsf{v}}\|_{L^\infty_{\mathrm{w}}(I; \boldsymbol{M}(\omega_{\mathsf{v}}))}\}.$$

3. Analysis of the optimal control problem

In this section, we present the well-posedness of the state system (1.3), the existence of solutions to (1.1), the differentiability properties of the associated control-to-state operator, and finally, the well-posedness of the corresponding dual problem.

3.1. Well-posedness of the state system

For the existence and uniqueness of weak solutions to (1.3), and later the existence of optimal controls, we shall assume the following:

$$q \in (1, 2), \quad s, p \in [4/3, 2), \quad r \in [4, \infty), \quad q \le s. \tag{3.1}$$

Let $2 \le \lambda < \infty$. The function space for the sources will be the product space

$$\begin{aligned} \boldsymbol{\mathcal{F}}^{r,\lambda}_{q,s,p}(Q) := {}& [L^r(I; W^{-1,q}(\Omega)) + L^\lambda(I; W^{-1,2}(\Omega))] \times [L^r(I; W^{-1,s}(\Omega)) + L^\lambda(I; W^{-1,2}(\Omega))] \\ & \times [L^r(I; \boldsymbol{W}^{-1,p}(\Omega)) + L^\lambda(I; \boldsymbol{W}^{-1,2}(\Omega))] \times [L^r(I; W^{1,q}_0(\Omega)) + L^\lambda(I; W^{1,2}_0(\Omega))] \end{aligned}$$

while the function space for the initial data is given by

$$\boldsymbol{\mathcal{D}}^{r,\lambda}_{q,s,p}(\Omega) := [Z^3_{q,r}(\Omega) + Z^3_{2,\lambda}(\Omega)] \times [Z^1_{s,r}(\Omega) + Z^1_{2,\lambda}(\Omega)] \times [\boldsymbol{V}^1_{p,r}(\Omega) + \boldsymbol{V}^1_{2,\lambda}(\Omega)].$$

Also, the weak solution space and the space for the associated pressure will be

$$\begin{aligned} \boldsymbol{\mathcal{W}}^{r,\lambda}_{q,s,p}(Q) &:= [\mathcal{Z}^3_{q,r}(Q) + \mathcal{Z}^3_{2,\lambda}(Q)] \times [\mathcal{Z}^1_{s,r}(Q) + \mathcal{Z}^1_{2,\lambda}(Q)] \\ &\qquad \times [\boldsymbol{\mathcal{V}}^1_{p,r}(Q) + \boldsymbol{\mathcal{V}}^1_{2,\lambda}(Q)] \times [L^r(I; W^{1,q}_0(\Omega)) + L^\lambda(I; W^{1,2}_0(\Omega))] \\ \mathcal{P}^{r,\lambda}_p(Q) &:= W^{-1,r}(I; \widehat{L}^p(\Omega)) + W^{-1,\lambda}(I; \widehat{L}^2(\Omega)). \end{aligned}$$

We refer the reader to [4], Lemma 2.3.1 for the definition of the norms for the sum and the intersection of two Banach spaces, both of them being continuously embedded in some Hausdorff topological vector space. In the current section, we will take $\lambda = 2$. In the case of second-order sufficient conditions, higher integrability is needed. More precisely, we shall take $\lambda = r/2$ with $r > 8$ in the succeeding section.

Let us now present the notion of weak solutions to the state system (1.3). Here, we follow the formulation in [43], Section 4.2.

Definition 3.1. Suppose that (3.1) holds and let $(f_{\mathsf{o}}, f_{\mathsf{h}}, \boldsymbol{f}_{\mathsf{v}}, f_{\mathsf{c}}) \in \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q)$, $(\phi_0, \theta_0, \boldsymbol{u}_0) \in \boldsymbol{\mathcal{D}}^{r,2}_{q,s,p}(\Omega)$, and $(\sigma_{\mathsf{o}}, \sigma_{\mathsf{h}}, \boldsymbol{\sigma}_{\mathsf{v}}) \in \boldsymbol{\mathcal{M}}^r$. We say that $(\phi, \theta, \boldsymbol{u}, \mu) \in \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$ is a *weak solution* to (1.3) if the initial condition $(\phi(0), \theta(0), \boldsymbol{u}(0)) = (\phi_0, \theta_0, \boldsymbol{u}_0)$ holds in $\boldsymbol{\mathcal{D}}^{r,2}_{q,s,p}(\Omega)$, the following variational equations

$$\begin{aligned}
&\int_0^T \{\langle \partial_t\phi + \mathrm{div}\,(\phi\boldsymbol{u}), \rho\rangle_{W^{-1,2}(\Omega), W^{1,2}_0(\Omega)} + m\langle A'_{q'}\mu, \rho\rangle_{W^{-1,q}(\Omega), W^{1,q'}_0(\Omega)}\}\,\mathrm{d}t \\
&\qquad = \int_0^T \{\langle f_{\mathsf{o}}, \rho\rangle_{W^{-1,q}(\Omega), W^{1,q'}_0(\Omega)} + \langle \sigma_{\mathsf{o}}, \rho\rangle_{M(\omega_{\mathsf{o}}), C_0(\omega_{\mathsf{o}})}\}\,\mathrm{d}t \\
&\int_0^T \{\langle \partial_t\theta + \kappa A'_{s'}\theta, \varrho\rangle_{W^{-1,s}(\Omega), W^{1,s'}_0(\Omega)} - l_{\mathsf{h}}(\partial_t\phi, \varrho)_{L^2(\Omega)}\}\,\mathrm{d}t \\
&\qquad + \int_0^T \{\langle \mathrm{div}\,((\theta - l_{\mathsf{h}}\phi)\boldsymbol{u}), \varrho\rangle_{W^{-1,2}(\Omega), W^{1,2}_0(\Omega)} - (\alpha\boldsymbol{g}\cdot\boldsymbol{u}, \varrho)_{L^2(\Omega)}\}\,\mathrm{d}t \\
&\qquad = \int_0^T \{\langle f_{\mathsf{h}}, \varrho\rangle_{W^{-1,s}(\Omega), W^{1,s'}_0(\Omega)} + \langle \sigma_{\mathsf{h}}, \varrho\rangle_{M(\omega_{\mathsf{h}}), C_0(\omega_{\mathsf{h}})}\}\,\mathrm{d}t \\
&\int_0^T \{\langle \partial_t\boldsymbol{u} + \nu\boldsymbol{A}'_{p'}\boldsymbol{u}, \boldsymbol{\rho}\rangle_{\boldsymbol{X}^{-1,p}_\sigma(\Omega), \boldsymbol{X}^{1,p'}_\sigma(\Omega)} + \langle \mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u}), \boldsymbol{\rho}\rangle_{\boldsymbol{W}^{-1,2}(\Omega), \boldsymbol{W}^{1,2}_0(\Omega)}\}\,\mathrm{d}t \\
&\qquad = \int_0^T \{(\ell(\phi,\theta)\boldsymbol{g}, \boldsymbol{\rho})_{\boldsymbol{L}^2(\Omega)} + \mathcal{K}\langle(\mu - l_{\mathsf{c}}\theta)\nabla\phi, \boldsymbol{\rho}\rangle_{\boldsymbol{W}^{-1,2}(\Omega), \boldsymbol{W}^{1,2}_0(\Omega)}\}\,\mathrm{d}t \\
&\qquad + \int_0^T \{\langle \boldsymbol{f}_{\mathsf{v}}, \boldsymbol{\rho}\rangle_{\boldsymbol{X}^{-1,p}_\sigma(\Omega), \boldsymbol{X}^{1,p'}_\sigma(\Omega)} + \langle \boldsymbol{\sigma}_{\mathsf{v}}, \boldsymbol{\rho}\rangle_{\boldsymbol{M}(\omega_{\mathsf{v}}), \boldsymbol{C}_0(\omega_{\mathsf{v}})}\}\,\mathrm{d}t
\end{aligned}$$

are satisfied by every $\rho \in L^{r'}(I; W^{1,q'}_0(\Omega)) \cap L^2(I; W^{1,2}_0(\Omega))$, $\varrho \in L^{r'}(I; W^{1,s'}_0(\Omega)) \cap L^2(I; W^{1,2}_0(\Omega))$, $\boldsymbol{\rho} \in L^{r'}(I; \boldsymbol{X}^{1,p'}_\sigma(\Omega)) \cap L^2(I; \boldsymbol{X}^{1,2}_\sigma(\Omega))$, and we have

$$\mu = \tau\partial_t\phi - \epsilon\Delta\phi + F(\phi) + l_{\mathsf{c}}\theta + f_{\mathsf{c}} \qquad \text{a.a. in } Q.$$

Moreover, a function $\mathfrak{p} \in \mathcal{P}^{r,2}_p(Q)$ is said to be an *associated pressure* if the equation involving $\mathfrak{p}$ in (1.3) is satisfied in the distributional sense, that is, it holds that

$$\begin{aligned}
&\langle \partial_t\boldsymbol{u}, \boldsymbol{\varrho}\rangle_{W^{-1,r}(I;W^{-1,p}(\Omega)) + W^{-1,2}(I;W^{-1,2}(\Omega)), W^{1,r'}_0(I;W^{1,p'}_0(\Omega)) \cap W^{1,2}_0(I;W^{1,2}_0(\Omega))} \\
&\qquad + \int_0^T \{\langle \mathrm{div}\,(\boldsymbol{u}\otimes\boldsymbol{u}), \boldsymbol{\varrho}\rangle_{\boldsymbol{W}^{-1,2}(\Omega), \boldsymbol{W}^{1,2}_0(\Omega)} + \nu\langle\nabla\boldsymbol{u}, \nabla\boldsymbol{\varrho}\rangle_{\boldsymbol{L}^p(\Omega)^2, \boldsymbol{L}^{p'}(\Omega)^2}\}\,\mathrm{d}t \\
&\qquad - \langle \mathfrak{p}, \mathrm{div}\,\boldsymbol{\varrho}\rangle_{W^{-1,r}(I;\widehat{L}^p(\Omega)) + W^{-1,2}(I;\widehat{L}^2(\Omega)), W^{1,r'}_0(I;\widehat{L}^{p'}(\Omega)) \cap W^{1,2}_0(I;\widehat{L}^2(\Omega))} \\
&\qquad = \int_0^T \{(\ell(\phi,\theta)\boldsymbol{g}, \boldsymbol{\varrho})_{\boldsymbol{L}^2(\Omega)} + \mathcal{K}\langle(\mu - l_{\mathsf{c}}\theta)\nabla\phi, \boldsymbol{\varrho}\rangle_{\boldsymbol{W}^{-1,2}(\Omega), \boldsymbol{W}^{1,2}_0(\Omega)}\}\,\mathrm{d}t \\
&\qquad + \int_0^T \{\langle \boldsymbol{f}_{\mathsf{v}}, \boldsymbol{\varrho}\rangle_{\boldsymbol{W}^{-1,p}(\Omega), \boldsymbol{W}^{1,p'}_0(\Omega)} + \langle \boldsymbol{\sigma}_{\mathsf{v}}, \boldsymbol{\varrho}\rangle_{\boldsymbol{M}(\omega_{\mathsf{v}}), \boldsymbol{C}_0(\omega_{\mathsf{v}})}\}\,\mathrm{d}t
\end{aligned}$$

for every $\boldsymbol{\varrho} \in W^{1,r'}_0(I; W^{1,p'}_0(\Omega)) \cap W^{1,2}_0(I; W^{1,2}_0(\Omega))$.

We refer to [43], Section 4.2 for the explanation on why the terms that appear in the above variational equations are well-defined. Duality pairings that involve the measure-valued controls have been discussed in the latter part of the previous section.

Theorem 3.2. *Assume that the first statement of Definition 3.1 is satisfied. Then, the state system* (1.3) *admits a unique weak solution* $(\phi,\theta,\boldsymbol{u},\mu) \in \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$ *with an associated pressure* $\mathfrak{p} \in \mathcal{P}^{r,2}_p(Q)$*. Furthermore, there is a monotone increasing and continuous function* $\mathscr{C} : [0,\infty) \to [0,\infty)$ *independent of the solution, initial data, source functions, and controls such that* $\mathscr{C}(0) = 0$ *and*

$$\begin{aligned}&\|(\phi,\theta,\boldsymbol{u},\mu)\|_{\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)} + \|\mathfrak{p}\|_{\mathcal{P}^{r,2}_p(Q)}\\ &\le \mathscr{C}(|\alpha \boldsymbol{g}| + \|(f_{\mathsf{o}}, f_{\mathsf{h}}, \boldsymbol{f}_{\mathsf{v}}, f_{\mathsf{c}})\|_{\boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q)} + \|(\phi_0,\theta_0,\boldsymbol{u}_0)\|_{\boldsymbol{\mathcal{D}}^{r,2}_{q,s,p}(\Omega)} + \|(\sigma_{\mathsf{o}},\sigma_{\mathsf{h}},\boldsymbol{\sigma}_{\mathsf{v}})\|_{\boldsymbol{\mathcal{M}}^r}).\end{aligned}$$

Proof. Since $\boldsymbol{\mathcal{M}}^r \times \{0\} \hookrightarrow \boldsymbol{\mathcal{N}}^r_{q,s,p}(Q) \times \{0\} \hookrightarrow \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q)$, the result follows immediately from [43], Theorem 4.12. ☐

Let us define the control-to-state operator

$$\mathbf{F} : \boldsymbol{\mathcal{M}}^r \to \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$$

as follows: $\mathbf{F}(\sigma_{\mathsf{o}},\sigma_{\mathsf{h}},\boldsymbol{\sigma}_{\mathsf{v}}) = (\phi,\theta,\boldsymbol{u},\mu)$ if and only if $(\phi,\theta,\boldsymbol{u},\mu) \in \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$ is the weak solution of the system (1.3). In what follows, the source terms $(f_{\mathsf{o}}, f_{\mathsf{h}}, \boldsymbol{f}_{\mathsf{v}}, f_{\mathsf{c}}) \in \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q)$ and initial data $(\phi_0,\theta_0,\boldsymbol{u}_0) \in \boldsymbol{\mathcal{D}}^{r,2}_{q,s,p}(\Omega)$ are fixed.

We also define the operator

$$\mathbf{H} : \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q) \to \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$$

according to $\mathbf{H}(\widetilde{f}_{\mathsf{o}}, \widetilde{f}_{\mathsf{h}}, \widetilde{\boldsymbol{f}}_{\mathsf{v}}, \widetilde{f}_{\mathsf{c}}) = (\phi,\theta,\boldsymbol{u},\mu)$ if and only if $(\phi,\theta,\boldsymbol{u},\mu) \in \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$ is the weak solution of (1.3) with $\sigma_{\mathsf{o}} = \sigma_{\mathsf{h}} = 0$, $\boldsymbol{\sigma}_{\mathsf{v}} = \mathbf{0}$, and $(f_{\mathsf{o}}, f_{\mathsf{h}}, \boldsymbol{f}_{\mathsf{v}}, f_{\mathsf{c}})$ is replaced by $(\widetilde{f}_{\mathsf{o}}, \widetilde{f}_{\mathsf{h}}, \widetilde{\boldsymbol{f}}_{\mathsf{v}}, \widetilde{f}_{\mathsf{c}})$. It is obvious that $\mathbf{F} = \mathbf{H} \circ \mathbf{I}$, where $\mathbf{I} : \boldsymbol{\mathcal{M}}^r \to \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q)$ is given by

$$\mathbf{I}(\sigma_{\mathsf{o}},\sigma_{\mathsf{h}},\boldsymbol{\sigma}_{\mathsf{v}}) = (f_{\mathsf{o}} + \chi_{\omega_{\mathsf{o}}}\sigma_{\mathsf{o}}, f_{\mathsf{h}} + \chi_{\omega_{\mathsf{h}}}\sigma_{\mathsf{h}}, \boldsymbol{f}_{\mathsf{v}} + \chi_{\omega_{\mathsf{v}}}\boldsymbol{\sigma}_{\mathsf{v}}, f_{\mathsf{c}}).$$

Since $\mathbf{I}$ is affine and $\boldsymbol{\mathcal{M}}^r \times \{0\} \hookrightarrow \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q)$, then $\mathbf{I}$ is obviously of class C^∞ and for every $\boldsymbol{s} \in \boldsymbol{\mathcal{M}}^r$, and $\boldsymbol{r} = (\rho_{\mathsf{o}}, \rho_{\mathsf{h}}, \boldsymbol{\rho}_{\mathsf{v}}) \in \boldsymbol{\mathcal{M}}^r$ it holds that

$$\mathrm{D}\mathbf{I}(\boldsymbol{s})\boldsymbol{r} = (\chi_{\omega_{\mathsf{o}}}\rho_{\mathsf{o}}, \chi_{\omega_{\mathsf{h}}}\rho_{\mathsf{h}}, \chi_{\omega_{\mathsf{v}}}\boldsymbol{\rho}_{\mathsf{h}}, 0). \tag{3.2}$$

Due to the fact that the right-hand side of (3.2) is independent of $\boldsymbol{s}$, we shall simply write the left-hand side as $\mathrm{D}\mathbf{I}\boldsymbol{r}$.

Theorem 3.3. *We have* $\mathbf{H} \in C^\infty(\boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q), \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q))$*. The actions of the first and second-order derivatives*

$$\begin{aligned}\mathrm{D}\mathbf{H} &: \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q) \to \mathcal{L}(\boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q), \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q))\\ \mathrm{D}^2\mathbf{H} &: \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q) \to \mathcal{L}(\boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q) \times \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q), \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q))\end{aligned}$$

can be characterized as follows: Given $\boldsymbol{f} \in \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q)$ *and* $\mathbf{g} = (g_{\mathsf{o}}, g_{\mathsf{h}}, \boldsymbol{g}_{\mathsf{v}}, g_{\mathsf{c}}) \in \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q)$*, we have* $\mathrm{D}\mathbf{H}(\boldsymbol{f})\mathbf{g} = (\psi,\zeta,\boldsymbol{w},\xi)$ *if and only if* $(\psi,\zeta,\boldsymbol{w},\xi) \in \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$ *with associated pressure* $\varpi \in \mathcal{P}^{r,2}_p(Q)$ *is the weak solution of* *the linearized system*

$$\left[\begin{array}{ll}\partial_t\psi+\operatorname{div}(\psi\boldsymbol{u})+\operatorname{div}(\phi\boldsymbol{w})-m\Delta\xi=g_{\mathsf{o}} & \textit{in } Q,\\ \xi=\tau\partial_t\psi-\epsilon\Delta\psi+F'(\phi)\psi+l_{\mathsf{c}}\zeta+g_{\mathsf{c}} & \textit{in } Q,\\ \partial_t\zeta-l_{\mathsf{h}}\partial_t\psi+\operatorname{div}((\zeta-l_{\mathsf{h}}\psi)\boldsymbol{u})+\operatorname{div}((\theta-l_{\mathsf{h}}\phi)\boldsymbol{w})-\kappa\Delta\zeta=\alpha\boldsymbol{g}\cdot\boldsymbol{w}+g_{\mathsf{h}} & \textit{in } Q,\\ \partial_t\boldsymbol{w}+\operatorname{div}(\boldsymbol{w}\otimes\boldsymbol{u})+\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{w})-\nu\Delta\boldsymbol{w}+\nabla\varpi & \\ \qquad=\mathcal{K}(\xi-l_{\mathsf{c}}\zeta)\nabla\phi+\mathcal{K}(\mu-l_{\mathsf{c}}\theta)\nabla\psi+(\alpha_1\psi+\alpha_2\zeta)\boldsymbol{g}+\boldsymbol{g}_{\mathsf{v}} & \textit{in } Q,\\ \operatorname{div}\boldsymbol{w}=0 & \textit{in } Q,\\ \psi=\Delta\psi=0,\quad \zeta=0,\quad \boldsymbol{w}=\boldsymbol{0} & \textit{on } \Sigma,\\ \psi(0)=0,\quad \zeta(0)=0,\quad \boldsymbol{w}(0)=\boldsymbol{0} & \textit{in } \Omega,\end{array}\right. \tag{3.3}$$

where $(\phi,\theta,\boldsymbol{u},\mu)=\mathbf{H}(\boldsymbol{f})$. *In addition, given* $\mathbf{g}^k=(g^k_{\mathsf{o}},g^k_{\mathsf{h}},\boldsymbol{g}^k_{\mathsf{v}},g^k_{\mathsf{c}})\in\boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q)$ *for* $k=1,2$, *the equation* $\mathrm{D}^2\mathbf{H}(\boldsymbol{f})(\mathbf{g}^1,\mathbf{g}^2)=(\psi,\zeta,\boldsymbol{w},\xi)$ *holds if and only if* $(\psi,\zeta,\boldsymbol{w},\xi)\in\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$ *with associated pressure* $\varpi\in\mathcal{P}^{r,2}_p(Q)$ *is the weak solution of the linearized system*

$$\left[\begin{array}{ll}\partial_t\psi+\operatorname{div}(\psi\boldsymbol{u})+\operatorname{div}(\phi\boldsymbol{w})-m\Delta\xi=-\operatorname{div}(\psi_1\boldsymbol{w}_2)-\operatorname{div}(\psi_2\boldsymbol{w}_1) & \textit{in } Q,\\ \xi=\tau\partial_t\psi-\epsilon\Delta\psi+F'(\phi)\psi+F''(\phi)\psi_1\psi_2+l_{\mathsf{c}}\zeta & \textit{in } Q,\\ \partial_t\zeta-l_{\mathsf{h}}\partial_t\psi+\operatorname{div}((\zeta-l_{\mathsf{h}}\psi)\boldsymbol{u})+\operatorname{div}((\theta-l_{\mathsf{h}}\phi)\boldsymbol{w})-\kappa\Delta\zeta & \\ \qquad=\alpha\boldsymbol{g}\cdot\boldsymbol{w}-\operatorname{div}((\zeta_1-l_{\mathsf{h}}\psi_1)\boldsymbol{w}_2)-\operatorname{div}((\zeta_2-l_{\mathsf{h}}\psi_2)\boldsymbol{w}_1) & \textit{in } Q,\\ \partial_t\boldsymbol{w}+\operatorname{div}(\boldsymbol{w}\otimes\boldsymbol{u})+\operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{w})-\nu\Delta\boldsymbol{w}+\nabla\varpi & \\ \qquad=\mathcal{K}(\xi-l_{\mathsf{c}}\zeta)\nabla\phi+\mathcal{K}(\mu-l_{\mathsf{c}}\theta)\nabla\psi+(\alpha_1\psi+\alpha_2\zeta)\boldsymbol{g} & \\ \qquad-\operatorname{div}(\boldsymbol{w}_1\otimes\boldsymbol{w}_2)-\operatorname{div}(\boldsymbol{w}_2\otimes\boldsymbol{w}_1)+\mathcal{K}(\xi_1-l_{\mathsf{c}}\zeta_1)\nabla\psi_2+\mathcal{K}(\xi_2-l_{\mathsf{c}}\zeta_2)\nabla\psi_1 & \textit{in } Q,\\ \operatorname{div}\boldsymbol{w}=0 & \textit{in } Q,\\ \psi=\Delta\psi=0,\quad \zeta=0,\quad \boldsymbol{w}=\boldsymbol{0} & \textit{on } \Sigma,\\ \psi(0)=0,\quad \zeta(0)=0,\quad \boldsymbol{w}(0)=\boldsymbol{0} & \textit{in } \Omega,\end{array}\right. \tag{3.4}$$

where $(\phi,\theta,\boldsymbol{u},\mu)=\mathbf{H}(\boldsymbol{f})$ *and* $(\psi_k,\zeta_k,\boldsymbol{w}_k,\xi_k)=\mathrm{D}\mathbf{H}(\boldsymbol{f})\mathbf{g}^k$ *for* $k=1,2$.

Proof. Consider the operator $\mathbf{S}:\boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q)\times\boldsymbol{\mathcal{D}}^{r,2}_{q,s,p}(\Omega)\to\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$ defined by

$$\mathbf{S}((\widetilde{f}_{\mathsf{o}},\widetilde{f}_{\mathsf{h}},\widetilde{\boldsymbol{f}}_{\mathsf{v}},\widetilde{f}_{\mathsf{c}}),(\phi_0,\theta_0,\boldsymbol{u}_0))=(\phi,\theta,\boldsymbol{u},\mu)$$

if and only if $(\phi,\theta,\boldsymbol{u},\mu)$ is the weak solution of (1.3) with $\sigma_{\mathsf{o}}=\sigma_{\mathsf{h}}=0$, $\boldsymbol{\sigma}_{\mathsf{v}}=\boldsymbol{0}$, and $(f_{\mathsf{o}},f_{\mathsf{h}},\boldsymbol{f}_{\mathsf{v}},f_{\mathsf{c}})$ is replaced by $(\widetilde{f}_{\mathsf{o}},\widetilde{f}_{\mathsf{h}},\widetilde{\boldsymbol{f}}_{\mathsf{v}},\widetilde{f}_{\mathsf{c}})$. From [43], Theorem 5.2, we know that $\mathbf{S}$ is of class C^∞. Thus, for a given $(\phi_0,\theta_0,\boldsymbol{u}_0)\in\boldsymbol{\mathcal{D}}^{r,2}_{q,s,p}(\Omega)$, $\mathbf{H}=\mathbf{S}(\cdot,(\phi_0,\theta_0,\boldsymbol{u}_0))$ is of class C^∞. The representations for the first-order and second-order derivatives of $\mathbf{F}$ follow from those of the operator $\mathbf{S}$ provided in [43], Section 5. □

We also note that for each $\boldsymbol{f}\in\boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q)$, we have

$$\mathrm{D}\mathbf{H}(\boldsymbol{f})^{-1}\in\mathcal{L}({}_0\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q),\boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q)), \tag{3.5}$$

where

$${}_0\boldsymbol{\mathcal{W}}^{r,\lambda}_{q,s,p}(Q):=\{(\psi,\zeta,\boldsymbol{w},\xi)\in\boldsymbol{\mathcal{W}}^{r,\lambda}_{q,s,p}(Q):\psi(0)=0,\zeta(0)=0,\boldsymbol{w}(0)=\boldsymbol{0}\} \tag{3.6}$$

taken as a closed subspace of $\boldsymbol{\mathcal{W}}^{r,\lambda}_{q,s,p}(Q)$.

Corollary 3.4. *Under the assumptions of Theorem 3.2, we have* $\mathsf{F} \in C^\infty(\boldsymbol{\mathcal{M}}^r, \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q))$ *and the action of the first and second derivatives are given by*

$$\mathrm{D}\mathsf{F}(\boldsymbol{s})\boldsymbol{r} = \mathrm{D}\mathsf{H}(\mathsf{I}(\boldsymbol{s}))\mathrm{D}\mathsf{I}\boldsymbol{r}$$
$$\mathrm{D}^2\mathsf{F}(\boldsymbol{s})(\boldsymbol{r}^1, \boldsymbol{r}^2) = \mathrm{D}^2\mathsf{H}(\mathsf{I}(\boldsymbol{s}))(\mathrm{D}\mathsf{I}\boldsymbol{r}^1, \mathrm{D}\mathsf{I}\boldsymbol{r}^2)$$

for every $\boldsymbol{s}, \boldsymbol{r}, \boldsymbol{r}^1, \boldsymbol{r}^2 \in \boldsymbol{\mathcal{M}}^r$.

Proof. These follow from $\mathsf{F} = \mathsf{H} \circ \mathsf{I}$, Theorem 3.3, and the chain rule. □

3.2. Existence of optimal controls

The existence of optimal controls relies on the following continuity of the control-to-state operator.

Lemma 3.5. *The operator* $\mathsf{F} : \boldsymbol{\mathcal{M}}^r \to \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$ *is weak*-weak sequentially continuous. That is, if* $\boldsymbol{s}^k \overset{*}{\rightharpoonup} \boldsymbol{s}$ *in* $\boldsymbol{\mathcal{M}}^r$*, then* $\mathsf{F}(\boldsymbol{s}^k) \rightharpoonup \mathsf{F}(\boldsymbol{s})$ *in* $\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$.

Proof. Suppose that $\boldsymbol{s}^k \overset{*}{\rightharpoonup} \boldsymbol{s}$ in $\boldsymbol{\mathcal{M}}^r$ as $k \to \infty$. Then, the sequence $\{\boldsymbol{s}^k\}_{k=1}^\infty$ is bounded in $\boldsymbol{\mathcal{M}}^r$. From Corollary 3.4, this implies that $\{\mathsf{F}(\boldsymbol{s}^k)\}_{k=1}^\infty$ is bounded in $\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$. Since $\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$ is reflexive, it follows that there exists a subsequence, using the same superscripts for simplicity, and an element $(\phi, \theta, \boldsymbol{u}, \mu) \in \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$ such that $\mathsf{F}(\boldsymbol{s}^k) \rightharpoonup (\phi, \theta, \boldsymbol{u}, \mu)$ in $\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$. Adapting the passage of limit for the existence of weak solutions, see for instance Step 3 of the proof of [43], Theorem 4.9, it can be deduced that $(\phi, \theta, \boldsymbol{u}, \mu) = \mathsf{F}(\boldsymbol{s})$. Since the weak limit is uniquely determined, we conclude that the whole sequence must converge weakly, that is, $\mathsf{F}(\boldsymbol{s}^k) \rightharpoonup \mathsf{F}(\boldsymbol{s})$ in $\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$. □

Let us write $\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q) = \boldsymbol{\mathcal{U}}^{r,2}_{q,s,p}(Q) \times [L^r(I; W^{1,q}_0(\Omega)) + L^2(I; W^{1,2}_0(\Omega))]$ and define $G : \boldsymbol{\mathcal{U}}^{r,2}_{q,s,p}(Q) \to \mathbb{R}$ by (1.4). Denote by $\mathsf{P} : \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q) \to \boldsymbol{\mathcal{U}}^{r,2}_{q,s,p}(Q)$ the projection onto the first three components. We then introduce the reduced cost functional $J : \boldsymbol{\mathcal{M}}^r \to \mathbb{R}$ given by

$$J = G \circ \mathsf{P} \circ \mathsf{F}.$$

In this way, the original optimal control problems (1.1)–(1.4) is equivalent to the following constrained infinite-dimensional optimization problem:

$$\min_{\boldsymbol{s} \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}} J(\boldsymbol{s}). \tag{3.7}$$

Let $\boldsymbol{s}^* \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$. We say that $\boldsymbol{s}^*$ is a *global solution* to (3.7) if $J(\boldsymbol{s}^*) \le J(\boldsymbol{s})$ for every $\boldsymbol{s} \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$. If there exists $\varepsilon > 0$ such that $J(\boldsymbol{s}^*) \le J(\boldsymbol{s})$ for every $\boldsymbol{s} \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}} \cap \boldsymbol{\mathcal{B}}^\infty_\varepsilon(\boldsymbol{s}^*)$, where $\boldsymbol{\mathcal{B}}^\infty_\varepsilon(\boldsymbol{s}^*)$ is the open ball in $\boldsymbol{\mathcal{M}}^\infty$ at $\boldsymbol{s}^*$ with radius ε, then $\boldsymbol{s}^*$ is called a *local solution* to (3.7). In addition, if $J(\boldsymbol{s}^*) < J(\boldsymbol{s})$ for any $\boldsymbol{s} \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}} \cap \boldsymbol{\mathcal{B}}^\infty_\varepsilon(\boldsymbol{s}^*) \setminus \{\boldsymbol{s}^*\}$, then we say that $\boldsymbol{s}^*$ is a *strict local solution*. Local solutions with respect to the topology of $\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)$ are defined in a similar way. Since $\boldsymbol{\mathcal{M}}^\infty \hookrightarrow \boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)$, an open ball in $\boldsymbol{\mathcal{M}}^\infty$ is contained in an open ball of $\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)$ with the same center and a scaled radius. Thus, it follows that any local solution in the topology of $\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)$ is also a local solution in the topology of $\boldsymbol{\mathcal{M}}^\infty$.

Theorem 3.6. *Consider the assumptions of Theorem 3.2 and let* $\phi_d, \theta_d \in L^2(Q)$ *and* $\boldsymbol{\psi}_d, \boldsymbol{u}_d \in \boldsymbol{L}^2(Q)$*. Then, the optimization problem* (3.7) *has at least one global solution* $\boldsymbol{s}^* \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$*, that is,* $J(\boldsymbol{s}^*) \le J(\boldsymbol{s})$ *for every* $\boldsymbol{s} \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$.

Proof. First, note that $\boldsymbol{\mathcal{M}}^\infty$ is a Banach space having a separable predual space $L^1(I; C_0(\omega_{\mathsf{o}})) \times L^1(I; C_0(\omega_{\mathsf{h}})) \times L^1(I; \boldsymbol{C}_0(\omega_{\mathsf{v}}))$. Consider a minimizing sequence $\{\boldsymbol{s}^k\}_{k=1}^\infty$ in $\boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$, that is, $J(\boldsymbol{s}^k)$ tends to the infimum of J over $\boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$. This sequence is bounded by the definition of the set of admissible controls, and hence there is a subsequence such that $\boldsymbol{s}^k \overset{*}{\rightharpoonup} \boldsymbol{s}^*$ in $\boldsymbol{\mathcal{M}}^\infty$ for some $\boldsymbol{s}^* \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$, according to the Banach–Alaoglu–Bourbaki

Theorem. This weak*-convergence also holds in $\boldsymbol{\mathcal{M}}^r$ after taking another suitable subsequence since $\boldsymbol{\mathcal{M}}^\infty \subset \boldsymbol{\mathcal{M}}^r$. Thanks to Lemma 3.5, we deduce that if $(\phi^*, \theta^*, \boldsymbol{u}^*, \mu^*) = \mathsf{F}(\boldsymbol{s}^*)$ and $(\phi^k, \theta^k, \boldsymbol{u}^k, \mu^k) = \mathsf{F}(\boldsymbol{s}^k)$, then

$$(\phi^k, \theta^k, \boldsymbol{u}^k, \mu^k) \rightharpoonup (\phi^*, \theta^*, \boldsymbol{u}^*, \mu^*) \text{ in } \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q).$$

Applying the Sobolev embedding theorem, we deduce that $X^{3,q}(\Omega) \overset{c}{\hookrightarrow} X^{2,2}(\Omega) \hookrightarrow X^{1,q}(\Omega)$, $X^{3,2}(\Omega) \overset{c}{\hookrightarrow} X^{2,2}(\Omega) \hookrightarrow X^{1,2}(\Omega)$, $X^{1,s}(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \hookrightarrow X^{-1,s}(\Omega)$, $X^{1,2}(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \hookrightarrow X^{-1,2}(\Omega)$, $\boldsymbol{V}^{1,p}(\Omega) \overset{c}{\hookrightarrow} \boldsymbol{L}^2_\sigma(\Omega) \hookrightarrow \boldsymbol{V}^{-1,p}(\Omega)$, and $\boldsymbol{V}^{1,2}(\Omega) \overset{c}{\hookrightarrow} \boldsymbol{L}^2_\sigma(\Omega) \hookrightarrow \boldsymbol{V}^{-1,2}(\Omega)$. Here and throughout the paper, $\overset{c}{\hookrightarrow}$ denotes a compact embedding. From the Aubin–Lions–Simon Lemma [46], we obtain $\mathcal{Z}^3_{q,r}(Q) + \mathcal{Z}^3_{2,2}(Q) \overset{c}{\hookrightarrow} L^2(I; X^{2,2}(\Omega))$, $\mathcal{Z}^1_{s,r}(Q) + \mathcal{Z}^1_{2,2}(Q) \overset{c}{\hookrightarrow} L^2(I; L^2(\Omega))$, and $\boldsymbol{\mathcal{V}}^1_{p,r}(Q) + \boldsymbol{\mathcal{V}}^1_{2,2}(Q) \overset{c}{\hookrightarrow} L^2(I; \boldsymbol{L}^2_\sigma(\Omega))$. As a result, one can extract a further subsequence such that

$$(\phi^k, \nabla\phi^k, \theta^k, \boldsymbol{u}^k) \to (\phi^*, \nabla\phi^*, \theta^*, \boldsymbol{u}^*) \text{ in } L^2(I; L^2(\Omega))^6.$$

Hence, it follows from the definition of the reduced cost functional J that

$$J(\boldsymbol{s}^*) = \lim_{k\to\infty} J(\boldsymbol{s}^k) = \inf_{\boldsymbol{s} \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}} J(\boldsymbol{s}).$$

Therefore, the minimum is attained at $\boldsymbol{s}^*$, which is then a global solution to the optimization problem (3.7). □

3.3. The adjoint system

We study the dual problem corresponding to the linearized system (3.3). In this direction, let us consider the following backward-in-time linear system of partial differential equations with variable coefficients:

$$\left[\begin{array}{ll} -\partial_t\varphi + l_{\mathsf{h}}\partial_t\vartheta + \tau\partial_t\eta - \boldsymbol{u}\cdot\nabla(\varphi - l_{\mathsf{h}}\vartheta) + \epsilon\Delta\eta = F'(\phi)\eta + \alpha_1\boldsymbol{g}\cdot\boldsymbol{v} - \mathcal{K}\boldsymbol{v}\cdot\nabla(\mu - l_{\mathsf{c}}\theta) + g_{\mathsf{o}} & \text{in } Q,\\ \eta = m\Delta\varphi + \mathcal{K}\boldsymbol{v}\cdot\nabla\phi + g_{\mathsf{c}} & \text{in } Q,\\ -\partial_t\vartheta - \boldsymbol{u}\cdot\nabla\vartheta + \mathcal{K}l_{\mathsf{c}}\boldsymbol{v}\cdot\nabla\phi - \kappa\Delta\vartheta = \alpha_2\boldsymbol{g}\cdot\boldsymbol{v} + l_{\mathsf{c}}\eta + g_{\mathsf{h}} & \text{in } Q,\\ -\partial_t\boldsymbol{v} - \boldsymbol{u}\cdot\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^\top\boldsymbol{u} - \nu\Delta\boldsymbol{v} + \nabla\pi = \alpha\vartheta\boldsymbol{g} + \phi\nabla\varphi + (\theta - l_{\mathsf{h}}\phi)\nabla\vartheta + \boldsymbol{g}_{\mathsf{v}} & \text{in } Q,\\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } Q,\\ \varphi = \Delta\varphi = 0, \quad \vartheta = 0, \quad \boldsymbol{v} = \boldsymbol{0} & \text{on } \Sigma,\\ \varphi(T) = 0, \quad \vartheta(T) = 0, \quad \boldsymbol{v}(T) = \boldsymbol{0} & \text{in } \Omega. \end{array}\right. \tag{3.8}$$

We note that this system, with $\tau = 0$ and homogeneous Neumann boundary conditions for φ and ϑ, has been considered in [42] under the context of very weak solutions. Here, different function spaces for the sources and the weak solutions will be utilized under the presence of the parameter $\tau > 0$ and the different weak solution space for the state system (1.3).

With regard to the source terms g_{o}, g_{h}, $\boldsymbol{g}_{\mathsf{v}}$, and g_{c} in the system (3.8), we shall consider the function space $\boldsymbol{\mathcal{G}}^2_2(Q)$, where

$$\boldsymbol{\mathcal{G}}^r_{\mathfrak{s}}(Q) := L^r(I; W^{-1,\mathfrak{s}}(\Omega)) \times L^r(I; L^{\mathfrak{s}}(\Omega)) \times L^r(I; \boldsymbol{L}^{\mathfrak{s}}(\Omega)) \times \mathcal{Z}^1_{\mathfrak{s},r}(Q). \tag{3.9}$$

The weak solution $(\varphi, \vartheta, \boldsymbol{v}, \eta, \pi)$ will be sought in the product space

$$\boldsymbol{\mathcal{Y}}^2_2(Q) \times L^2(I; \widehat{W}^{1,2}(\Omega)),$$

where

$$\boldsymbol{\mathcal{Y}}^r_{\mathfrak{s}}(Q) := \mathcal{Z}^3_{\mathfrak{s},r}(Q) \times \mathcal{Z}^2_{\mathfrak{s},r}(Q) \times \boldsymbol{\mathcal{V}}^2_{\mathfrak{s},r}(Q) \times \mathcal{Z}^1_{\mathfrak{s},r}(Q) \tag{3.10}$$

and $\widehat{W}^{1,\mathfrak{s}}(\Omega) := W^{1,\mathfrak{s}}(\Omega) \cap \widehat{L}^{\mathfrak{s}}(\Omega)$. For local second-order sufficient conditions, we shall take $\mathfrak{s} = 4$ and r replaced by $r/4$ with $r > 8$ (see Lem. 4.12 below). At this point, let us impose the condition

$$q, s, p \in [4/3, 2), \quad r \in [4, \infty), \quad q \leq s. \tag{3.11}$$

Theorem 3.7. *Let (3.11) hold, $(\phi, \theta, \boldsymbol{u}, \mu) \in \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$ and $(g_{\mathsf{o}}, g_{\mathsf{h}}, \boldsymbol{g}_{\mathsf{v}}, g_{\mathsf{c}}) \in \boldsymbol{\mathcal{G}}^2_2(Q)$. Then, the adjoint problem (3.8) admits a unique weak solution $(\varphi, \vartheta, \boldsymbol{v}, \eta) \in \boldsymbol{\mathcal{Y}}^2_2(Q)$ with an associated pressure $\pi \in L^2(I; \widehat{W}^{1,2}(\Omega))$. Moreover, there is a continuous function $\mathscr{C} : [0,\infty) \to [0,\infty)$ such that*

$$\|(\varphi, \vartheta, \boldsymbol{v}, \eta)\|_{\boldsymbol{\mathcal{Y}}^2_2(Q)} + \|\pi\|_{L^2(I;\widehat{W}^{1,2}(\Omega))} \leq \mathscr{C}(\|(\phi, \theta, \boldsymbol{u}, \mu)\|_{\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)})\|(g_{\mathsf{o}}, g_{\mathsf{h}}, \boldsymbol{g}_{\mathsf{v}}, g_{\mathsf{c}})\|_{\boldsymbol{\mathcal{G}}^2_2(Q)}. \tag{3.12}$$

Proof. We will only proceed by formally deriving *a priori* estimates. Nonetheless, the proof can be made rigorous by using a standard Faedo–Galerkin method. In what follows, $\delta > 0$ is a constant to be chosen at each step, c is a generic positive constant, and $\mathscr{C} : [0,\infty) \to [0,\infty)$ will denote a generic continuous function. Both c and $\mathscr{C}$ depend on q, s, p, r, Ω, T, and the parameters appearing in (3.8). The derivation will be split into several parts.

• *Estimate for ϑ in $\mathcal{Z}^1_{2,2}(Q) = W^{1,2}(I; X^{2,2}(\Omega), L^2(\Omega))$.* Multiplying the third equation in (3.8) by $-(\partial_t\vartheta + \Delta\vartheta)$ and then integrating by parts over Ω for the term involving the time derivative, one has

$$\begin{aligned} &-\frac{\kappa+1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\vartheta\|^2_{\boldsymbol{L}^2(\Omega)} + (\boldsymbol{u}\cdot\nabla\vartheta, \partial_t\vartheta + \Delta\vartheta)_{L^2(\Omega)} - \mathcal{K}l_{\mathsf{c}}(\boldsymbol{v}\cdot\nabla\phi, \partial_t\vartheta + \Delta\vartheta)_{L^2(\Omega)} \\ &\quad + \|\partial_t\vartheta\|^2_{L^2(\Omega)} + \kappa\|\Delta\vartheta\|^2_{L^2(\Omega)} = -(\alpha_2\boldsymbol{g}\cdot\boldsymbol{v} + l_{\mathsf{c}}\eta + g_{\mathsf{h}}, \partial_t\vartheta + \Delta\vartheta)_{L^2(\Omega)}. \end{aligned} \tag{3.13}$$

Let us estimate the inner products appearing in this equation. Using the Hölder, Poincaré, and Young inequalities, we have

$$\begin{aligned} &|(\alpha_2\boldsymbol{g}\cdot\boldsymbol{v} + l_{\mathsf{c}}\eta + g_{\mathsf{h}}, \partial_t\vartheta + \Delta\vartheta)_{L^2(\Omega)}| \\ &\quad \leq c\{\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\eta\|_{L^2(\Omega)} + \|g_{\mathsf{h}}\|_{L^2(\Omega)}\}\{\|\partial_t\vartheta\|_{L^2(\Omega)} + \|\Delta\vartheta\|_{L^2(\Omega)}\} \\ &\quad \leq \delta\|\partial_t\vartheta\|^2_{L^2(\Omega)} + \delta\|\Delta\vartheta\|^2_{L^2(\Omega)} + c_\delta\{\|\eta\|^2_{L^2(\Omega)} + \|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2} + \|g_{\mathsf{h}}\|^2_{L^2(\Omega)}\}. \end{aligned} \tag{3.14}$$

Similarly, the inner products on the left-hand side of (3.13) can be bounded by

$$\begin{aligned} |(\boldsymbol{u}\cdot\nabla\vartheta, \partial_t\vartheta + \Delta\vartheta)_{L^2(\Omega)}| &\leq \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}\{\|\nabla\vartheta\|_{\boldsymbol{L}^4(\Omega)}\|\partial_t\vartheta\|_{L^2(\Omega)} + \|\nabla\vartheta\|_{\boldsymbol{L}^4(\Omega)}\|\Delta\vartheta\|_{L^2(\Omega)}\} \\ &\leq \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}\{\|\nabla\vartheta\|^{1/2}_{\boldsymbol{L}^2(\Omega)}\|\Delta\vartheta\|^{1/2}_{L^2(\Omega)}\|\partial_t\vartheta\|_{L^2(\Omega)} + \|\nabla\vartheta\|^{1/2}_{\boldsymbol{L}^2(\Omega)}\|\Delta\vartheta\|^{3/2}_{L^2(\Omega)}\} \\ &\leq \delta\|\partial_t\vartheta\|^2_{L^2(\Omega)} + \delta\|\Delta\vartheta\|^2_{L^2(\Omega)} + c_\delta\|\boldsymbol{u}\|^4_{\boldsymbol{L}^4(\Omega)}\|\nabla\vartheta\|^2_{\boldsymbol{L}^2(\Omega)} \end{aligned} \tag{3.15}$$

$$\begin{aligned} |\mathcal{K}l_{\mathsf{c}}(\boldsymbol{v}\cdot\nabla\phi, \partial_t\vartheta + \Delta\vartheta)_{L^2(\Omega)}| &\leq c\|\boldsymbol{v}\|_{\boldsymbol{L}^4(\Omega)}\|\nabla\phi\|_{\boldsymbol{L}^4(\Omega)}\{\|\partial_t\vartheta\|_{L^2(\Omega)} + \|\Delta\vartheta\|_{L^2(\Omega)}\} \\ &\leq \delta\|\partial_t\vartheta\|^2_{L^2(\Omega)} + \delta\|\Delta\vartheta\|^2_{L^2(\Omega)} + c_\delta\|\phi\|^2_{W^{1,4}(\Omega)}\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2}. \end{aligned} \tag{3.16}$$

Here, we used the Gagliardo–Nirenberg inequality in (3.15).

Let $h_{\mathsf{h}} := \|\phi\|^2_{W^{1,4}(\Omega)} + \|\boldsymbol{u}\|^4_{\boldsymbol{L}^4(\Omega)} + 1$. According to the continuous embeddings $\mathcal{Z}^3_{q,r}(Q) + \mathcal{Z}^3_{2,2}(Q) \hookrightarrow L^2(I; W^{1,4}(\Omega))$ and $\boldsymbol{\mathcal{V}}^1_{p,r}(Q) + \boldsymbol{\mathcal{V}}^1_{2,2}(Q) \hookrightarrow L^4(I; \boldsymbol{L}^4(\Omega))$, we see that $h_{\mathsf{h}} \in L^1(I)$ having the norm bound

$$\|h_{\mathsf{h}}\|_{L^1(I)} \leq \mathscr{C}(\|\phi\|_{\mathcal{Z}^3_{q,r}(Q)+\mathcal{Z}^3_{2,2}(Q)} + \|\boldsymbol{u}\|_{\boldsymbol{\mathcal{V}}^1_{p,r}(Q)+\boldsymbol{\mathcal{V}}^1_{2,2}(Q)}). \tag{3.17}$$

Substituting the inequalities (3.14)–(3.16) in (3.13), and then taking $\delta > 0$ small enough so that $1 - 3\delta > \frac{1}{2}$ and $\kappa - 3\delta > \frac{\kappa}{2}$, we deduce the *a priori* estimate

$$-\frac{\kappa+1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\vartheta\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{1}{2}\|\partial_t\vartheta\|^2_{L^2(\Omega)} + \frac{\kappa}{2}\|\Delta\vartheta\|^2_{L^2(\Omega)} - c\|\eta\|^2_{L^2(\Omega)} \\ \le c\|g_{\mathsf{h}}\|^2_{L^2(\Omega)} + ch_{\mathsf{h}}\{\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2} + \|\nabla\vartheta\|^2_{\boldsymbol{L}^2(\Omega)}\}. \tag{3.18}$$

• *Estimate for $\boldsymbol{v}$ in $\boldsymbol{\mathcal{V}}^1_{2,2}(Q) = W^{1,2}(I;\boldsymbol{X}^{2,2}_\sigma(\Omega),\boldsymbol{L}^2_\sigma(\Omega))$.* We shall take the test function $-(\partial_t\boldsymbol{v} + \Delta\boldsymbol{v})$ in the fourth equation of (3.8). To eliminate the pressure, we use the divergence theorem and the fact that $\operatorname{div}(\partial_t\boldsymbol{v} + \Delta\boldsymbol{v}) = (\partial_t + \Delta)\operatorname{div}\boldsymbol{v} = 0$. Let us point out that this argument is valid at the discrete level in the Faedo–Galerkin method. Integration by parts over Ω leads to the equation

$$-\frac{\nu+1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2} + (\boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\top\boldsymbol{u}, \partial_t\boldsymbol{v} + \Delta\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} + \|\partial_t\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \nu\|\Delta\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \\ = -(\alpha\vartheta\boldsymbol{g} + \boldsymbol{g}_{\mathsf{v}}, \partial_t\boldsymbol{v} + \Delta\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)} - (\phi\nabla\varphi + (\theta - l_{\mathsf{h}}\phi)\nabla\vartheta, \partial_t\boldsymbol{v} + \Delta\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}. \tag{3.19}$$

Next, we estimate the inner products in this equation. On the left-hand side, we apply the Hölder, Gagliardo–Nirenberg, and Young inequalities to obtain

$$|(\boldsymbol{u}\cdot\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\top\boldsymbol{u}, \partial_t\boldsymbol{v} + \Delta\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}| \le c\|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}\{\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^4(\Omega)^2}\|\partial_t\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^4(\Omega)^2}\|\Delta\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\} \\ \le c\|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}\{\|\nabla\boldsymbol{v}\|^{1/2}_{\boldsymbol{L}^2(\Omega)^2}\|\Delta\boldsymbol{v}\|^{1/2}_{\boldsymbol{L}^2(\Omega)}\|\partial_t\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\nabla\boldsymbol{v}\|^{1/2}_{\boldsymbol{L}^2(\Omega)^2}\|\Delta\boldsymbol{v}\|^{3/2}_{\boldsymbol{L}^2(\Omega)}\} \\ \le \delta\|\partial_t\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\Delta\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\|\boldsymbol{u}\|^4_{\boldsymbol{L}^4(\Omega)}\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2} \tag{3.20}$$

and the first inner product on the right-hand side is bounded by

$$|(\alpha\vartheta\boldsymbol{g} + \boldsymbol{g}_{\mathsf{v}}, \partial_t\boldsymbol{v} + \Delta\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}| \le c\{\|\vartheta\|_{L^2(\Omega)} + \|\boldsymbol{g}_{\mathsf{v}}\|_{\boldsymbol{L}^2(\Omega)}\}\{\|\partial_t\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\Delta\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\} \\ \le \delta\|\partial_t\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\Delta\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\{\|\nabla\vartheta\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{g}_{\mathsf{v}}\|^2_{\boldsymbol{L}^2(\Omega)}\}. \tag{3.21}$$

Finally, we split the second inner product on the right-hand side of (3.19) into two parts, use the Sobolev embedding $W^{1,4}(\Omega) \hookrightarrow L^\infty(\Omega)$ to the first part, and apply the Gagliardo–Nirenberg inequality to the second part so that

$$|(\phi\nabla\varphi, \partial_t\boldsymbol{v} + \Delta\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}| \le c\|\phi\|_{L^\infty(\Omega)}\|\nabla\varphi\|_{\boldsymbol{L}^2(\Omega)}\{\|\partial_t\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\Delta\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\} \\ \le \delta\|\partial_t\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\Delta\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\|\phi\|^2_{W^{1,4}(\Omega)}\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)} \tag{3.22}$$

$$|((\theta - l_{\mathsf{h}}\phi)\nabla\vartheta, \partial_t\boldsymbol{v} + \Delta\boldsymbol{v})_{\boldsymbol{L}^2(\Omega)}| \le c\|\theta - l_{\mathsf{h}}\phi\|_{L^4(\Omega)}\|\nabla\vartheta\|^{1/2}_{\boldsymbol{L}^2(\Omega)}\|\Delta\vartheta\|^{1/2}_{L^2(\Omega)}\{\|\partial_t\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|\Delta\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\} \\ \le \delta\|\partial_t\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\Delta\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\Delta\vartheta\|^2_{L^2(\Omega)} + c_\delta\{\|\theta\|^4_{L^4(\Omega)} + \|\phi\|^4_{L^4(\Omega)}\}\|\nabla\vartheta\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{3.23}$$

Let $h_{\mathsf{v}} := \|\phi\|^2_{W^{1,4}(\Omega)} + \|\phi\|^4_{L^4(\Omega)} + \|\theta\|^4_{L^4(\Omega)} + \|\boldsymbol{u}\|^4_{\boldsymbol{L}^4(\Omega)} + 1$. Taking into account the continuous embeddings $\mathcal{Z}^3_{q,r}(Q) + \mathcal{Z}^3_{2,2}(Q) \hookrightarrow L^4(I;W^{1,4}(\Omega))$, $\mathcal{Z}^1_{s,r}(Q) + \mathcal{Z}^1_{2,2}(Q) \hookrightarrow L^4(I;L^4(\Omega))$, and $\boldsymbol{\mathcal{V}}^1_{p,r}(Q) + \boldsymbol{\mathcal{V}}^1_{2,2}(Q) \hookrightarrow L^4(I;\boldsymbol{L}^4(\Omega))$, we get $h_{\mathsf{v}} \in L^1(I)$ and

$$\|h_{\mathsf{v}}\|_{L^1(I)} \le \mathscr{C}(\|\phi\|_{\mathcal{Z}^3_{q,r}(Q)+\mathcal{Z}^3_{2,2}(Q)} + \|\theta\|_{\mathcal{Z}^1_{s,r}(Q)+\mathcal{Z}^1_{2,2}(Q)} + \|\boldsymbol{u}\|_{\boldsymbol{\mathcal{V}}^1_{p,r}(Q)+\boldsymbol{\mathcal{V}}^1_{2,2}(Q)}). \tag{3.24}$$

Plugging (3.20)–(3.23) in (3.19), and then taking $\delta > 0$ small enough in such a way that $1-4\delta > \frac{1}{2}$, $\delta < \frac{\kappa}{4}$, and $\nu - 4\delta > \frac{\nu}{2}$, we obtain the *a priori* estimate

$$\begin{aligned} -\frac{\nu+1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2} + \frac{1}{2}\|\partial_t \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\nu}{2}\|\Delta \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{\kappa}{4}\|\Delta\vartheta\|^2_{L^2(\Omega)} \\ \le c\|\boldsymbol{g}_{\mathsf{v}}\|^2_{\boldsymbol{L}^2(\Omega)} + ch_{\mathsf{v}}\{\|\nabla\vartheta\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2}\}. \end{aligned} \tag{3.25}$$

• *Estimate for η in $L^2(I;W_0^{1,2}(\Omega))$.* From the equation for η in (3.8) and the continuous embedding $W^{2,4}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, we immediately get

$$\|\eta\|^2_{L^2(\Omega)} - c\|\Delta\varphi\|^2_{L^2(\Omega)} \le c\{h_{\mathsf{c}}\|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2} + \|g_{\mathsf{c}}\|^2_{L^2(\Omega)}\} \tag{3.26}$$

where $h_{\mathsf{c}} := \|\phi\|^2_{W^{2,4}(\Omega)} \in L^1(I)$ since $\mathcal{Z}^3_{q,r}(Q) + \mathcal{Z}^3_{2,2}(Q) \hookrightarrow L^2(I;W^{2,4}(\Omega))$, and thus

$$\|h_{\mathsf{c}}\|_{L^1(I)} \le \mathscr{C}(\|\phi\|_{\mathcal{Z}^3_{q,r}(Q)+\mathcal{Z}^3_{2,2}(Q)}). \tag{3.27}$$

Using $\nabla(\boldsymbol{v}\cdot\nabla\phi) = (\nabla \boldsymbol{v})\nabla\phi + (\nabla^2\phi)\boldsymbol{v}$ along with the embeddings $W^{2,4}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ and $\boldsymbol{W}^{1,2}_0(\Omega) \hookrightarrow \boldsymbol{L}^4(\Omega)$, we deduce that

$$\begin{aligned} \|\nabla(\boldsymbol{v}\cdot\nabla\phi)\|^2_{\boldsymbol{L}^2(\Omega)} &\le c\{\|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2}\|\nabla\phi\|^2_{\boldsymbol{L}^\infty(\Omega)} + \|\nabla^2\phi\|^2_{\boldsymbol{L}^4(\Omega)^2}\|\boldsymbol{v}\|^2_{\boldsymbol{L}^4(\Omega)}\} \\ &\le c\|\phi\|^2_{W^{2,4}(\Omega)}\|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2}. \end{aligned} \tag{3.28}$$

Taking the gradient of η in the second equation of (3.8) and using (3.28) yield

$$\|\nabla\eta\|^2_{\boldsymbol{L}^2(\Omega)} - c\|\nabla\Delta\varphi\|^2_{\boldsymbol{L}^2(\Omega)} \le c\{\|\phi\|^2_{W^{2,4}(\Omega)}\|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2} + \|\nabla g_{\mathsf{c}}\|^2_{\boldsymbol{L}^2(\Omega)}\}. \tag{3.29}$$

• *Estimate for φ in $L^\infty(I;W_0^{1,2}(\Omega)) \cap L^2(I;X^{2,2}(\Omega))$.* Using the test function φ in the first equation of (3.8), applying Green's second identity for the term involving $\epsilon\Delta\eta$, and noting $(\boldsymbol{u}\cdot\nabla\varphi,\varphi)_{L^2(\Omega)} = 0$ and $(\boldsymbol{v}\cdot\nabla(\mu - l_{\mathsf{c}}\theta),\varphi)_{L^2(\Omega)} = -(\boldsymbol{v}\cdot\nabla\varphi, \mu - l_{\mathsf{c}}\theta)_{L^2(\Omega)}$, we obtain

$$\begin{aligned} -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi\|^2_{L^2(\Omega)} + \tau(\partial_t\eta,\varphi)_{L^2(\Omega)} + l_{\mathsf{h}}(\boldsymbol{u}\cdot\nabla\vartheta,\varphi)_{L^2(\Omega)} + \epsilon(\eta,\Delta\varphi)_{L^2(\Omega)} \\ = (F'(\phi)\eta + \alpha_1\boldsymbol{g}\cdot\boldsymbol{v} - l_{\mathsf{h}}\partial_t\vartheta,\varphi)_{L^2(\Omega)} + \langle g_{\mathsf{o}},\varphi\rangle_{W^{-1,2}(\Omega),W^{1,2}_0(\Omega)} + \mathcal{K}(\boldsymbol{v}\cdot\nabla\varphi, \mu - l_{\mathsf{c}}\theta)_{L^2(\Omega)}. \end{aligned} \tag{3.30}$$

For the first two terms on the right-hand side of (3.30), we have

$$|\langle g_{\mathsf{o}},\varphi\rangle_{W^{-1,2}(\Omega),W^{1,2}_0(\Omega)}| \le c\{\|g_{\mathsf{o}}\|^2_{W^{-1,2}(\Omega)} + \|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)}\} \tag{3.31}$$

$$\begin{aligned} |(F'(\phi)\eta + \alpha_1\boldsymbol{g}\cdot\boldsymbol{v} - l_{\mathsf{h}}\partial_t\vartheta,\varphi)_{L^2(\Omega)}| &\le \delta\|\partial_t\vartheta\|^2_{L^2(\Omega)} + \delta\|\eta\|^2_{L^2(\Omega)} \\ &+ c_\delta\{(\|F'(\phi)\|^2_{L^\infty(\Omega)} + 1)\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla \boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2}\}. \end{aligned} \tag{3.32}$$

From the definition of F, one has $\|F'(\phi)\|_{L^\infty(\Omega)} \le c(\|\phi\|^2_{L^\infty(\Omega)} + 1)$. With respect to the trilinear terms in (3.30), we estimate them as follows

$$\begin{aligned} |l_{\mathsf{h}}(\boldsymbol{u}\cdot\nabla\vartheta,\varphi)_{L^2(\Omega)}| &\le c\|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}\|\nabla\vartheta\|_{\boldsymbol{L}^2(\Omega)}\|\varphi\|_{L^4(\Omega)} \\ &\le c\{\|\nabla\vartheta\|^2_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u}\|^2_{\boldsymbol{L}^4(\Omega)}\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)}\} \end{aligned} \tag{3.33}$$

$$\begin{aligned}|\mathcal{K}(\boldsymbol{v}\cdot\nabla\varphi,\mu-l_{\mathsf{c}}\theta)_{L^2(\Omega)}| &\le c\|\boldsymbol{v}\|_{\boldsymbol{L}^4(\Omega)}\|\nabla\varphi\|_{\boldsymbol{L}^2(\Omega)}\|\mu-l_{\mathsf{c}}\theta\|_{L^4(\Omega)}\\ &\le c\{\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2}+(\|\mu\|^2_{L^4(\Omega)}+\|\theta\|^2_{L^4(\Omega)})\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)}\}.\end{aligned} \tag{3.34}$$

Using the equation for η in (3.8), the second term on the left-hand side of (3.30) can be written as

$$\begin{aligned}\tau(\partial_t\eta,\varphi)_{L^2(\Omega)} = &-\frac{m\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)}+\tau\mathcal{K}(\partial_t\boldsymbol{v}\cdot\nabla\phi,\varphi)_{L^2(\Omega)}\\ &-\tau\mathcal{K}(\boldsymbol{v}\cdot\nabla\varphi,\partial_t\phi)_{L^2(\Omega)}+\tau\langle\partial_t g_{\mathsf{c}},\varphi\rangle_{W^{-1,2}(\Omega),W^{1,2}_0(\Omega)}.\end{aligned} \tag{3.35}$$

The inner products on the right-hand side of the latter equation satisfy the estimates

$$|\tau\langle\partial_t g_{\mathsf{c}},\varphi\rangle_{W^{-1,2}(\Omega),W^{1,2}_0(\Omega)}| \le c\{\|\partial_t g_{\mathsf{c}}\|^2_{W^{-1,2}(\Omega)}+\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)}\} \tag{3.36}$$

$$\begin{aligned}|\tau\mathcal{K}(\partial_t\boldsymbol{v}\cdot\nabla\phi,\varphi)_{L^2(\Omega)}| &\le c\|\partial_t\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\phi\|_{\boldsymbol{L}^4(\Omega)}\|\varphi\|_{L^4(\Omega)}\\ &\le \delta\|\partial_t\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}+c_\delta\|\phi\|^2_{W^{1,4}(\Omega)}\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)}\end{aligned} \tag{3.37}$$

$$\begin{aligned}|\tau\mathcal{K}(\boldsymbol{v}\cdot\nabla\varphi,\partial_t\phi)_{L^2(\Omega)}| &\le c\|\boldsymbol{v}\|_{\boldsymbol{L}^4(\Omega)}\|\nabla\varphi\|_{\boldsymbol{L}^2(\Omega)}\|\partial_t\phi\|_{L^4(\Omega)}\\ &\le c\{\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2}+\|\partial_t\phi\|^2_{L^4(\Omega)}\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)}\}.\end{aligned} \tag{3.38}$$

Finally, adapting the procedure in the case of η, we get the following lower bound for the remaining inner product in (3.30)

$$\epsilon(\eta,\Delta\varphi)_{L^2(\Omega)} \ge \frac{m\epsilon}{2}\|\Delta\varphi\|^2_{L^2(\Omega)}-c_\delta\{\|\phi\|^2_{W^{2,4}(\Omega)}\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2}+\|g_{\mathsf{c}}\|^2_{L^2(\Omega)}\}. \tag{3.39}$$

Let $h_{\mathsf{o}} := \|F'(\phi)\|^2_{L^\infty(\Omega)}+\|\phi\|^2_{W^{2,4}(\Omega)}+\|\partial_t\phi\|^2_{L^4(\Omega)}+\|\mu\|^2_{L^4(\Omega)}+\|\theta\|^2_{L^4(\Omega)}+\|\boldsymbol{u}\|^2_{\boldsymbol{L}^4(\Omega)}+1$. Then, following the same arguments as before, it is not hard to see that $h_{\mathsf{o}} \in L^1(I)$ and

$$\|h_{\mathsf{o}}\|_{L^1(I)} \le \mathscr{C}(\|(\phi,\theta,\boldsymbol{u},\mu)\|_{\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)}). \tag{3.40}$$

Furthermore, by utilizing (3.31)–(3.39) in (3.30) we deduce the *a priori* estimate

$$\begin{aligned}&-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|\varphi\|^2_{L^2(\Omega)}+m\tau\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)}\}+\frac{m\epsilon}{2}\|\Delta\varphi\|^2_{L^2(\Omega)}-\delta\|\eta\|^2_{L^2(\Omega)}-\delta\|\partial_t\vartheta\|^2_{L^2(\Omega)}-\delta\|\partial_t\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}\\ &\quad\le c_\delta\{\|g_{\mathsf{o}}\|^2_{W^{-1,2}(\Omega)}+\|g_{\mathsf{c}}\|^2_{L^2(\Omega)}+\|\partial_t g_{\mathsf{c}}\|^2_{W^{-1,2}(\Omega)}\}+c_\delta h_{\mathsf{o}}\{\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\vartheta\|^2_{\boldsymbol{L}^2(\Omega)}+\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2}\}.\end{aligned} \tag{3.41}$$

We now combine the above *a priori* estimates with suitable weights. Multiplying the estimate for η in (3.26) by $\delta_1 > 0$, those of ϑ and $\boldsymbol{v}$ in (3.18) and (3.25) by $\delta_2 > 0$, and then taking the sum of the resulting inequalities with (3.41), we obtain the differential inequality

$$-\frac{\mathrm{d}}{\mathrm{d}t}e+b \le c_\delta\{g+he\} \qquad \text{in } I, \tag{3.42}$$

where $e,b,h,g : [0,T] \to \mathbb{R}$ are given by

$$\begin{aligned}h &:= \delta_2 h_{\mathsf{h}}+\delta_2 h_{\mathsf{v}}+\delta_1 h_{\mathsf{c}}+h_{\mathsf{o}}\\ g &:= \|g_{\mathsf{o}}\|^2_{W^{-1,2}(\Omega)}+(\delta_1+1)\|g_{\mathsf{c}}\|^2_{L^2(\Omega)}+\|\partial_t g_{\mathsf{c}}\|^2_{W^{-1,2}(\Omega)}+\delta_2\|g_{\mathsf{h}}\|^2_{L^2(\Omega)}+\delta_2\|\boldsymbol{g}_{\mathsf{v}}\|^2_{\boldsymbol{L}^2(\Omega)}\\ e &:= \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)}+\frac{m\tau}{2}\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{(\kappa+1)}{2}\delta_2\|\nabla\vartheta\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{(\nu+1)}{2}\delta_2\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2}\end{aligned}$$

$$b := \left(\frac{m\epsilon}{2} - \delta_1 c\right)\|\Delta\varphi\|_{L^2(\Omega)}^2 + (\delta_1 - \delta - c\delta_2)\|\eta\|_{L^2(\Omega)}^2 + \left(\frac{\delta_2}{2} - \delta\right)\|\partial_t\vartheta\|_{L^2(\Omega)}^2 + \frac{\delta_2\kappa}{4}\|\Delta\vartheta\|_{L^2(\Omega)}^2 + \left(\frac{\delta_2}{2} - \delta\right)\|\partial_t\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2 + \frac{\delta_2\nu}{2}\|\Delta\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2.$$

Let us take δ_1, δ_2, and δ in succession according to

$$0 < \delta_1 < \frac{m\epsilon}{2c}, \quad 0 < \delta_2 < \frac{\delta_1}{2c}, \quad 0 < \delta < \frac{1}{2}\min\{\delta_1, \delta_2\}.$$

Then, the coefficients on the norms appearing in b are all positive. Moreover, from the definition of g and the inequalities (3.17), (3.24), (3.27), and (3.40), we see that $g, h \in L^1(I)$ and one has

$$\|g\|_{L^1(I)} \le \|(g_{\mathsf{o}}, g_{\mathsf{h}}, \boldsymbol{g}_{\mathsf{v}}, g_{\mathsf{c}})\|_{\boldsymbol{\mathcal{G}}_2^2(Q)} \tag{3.43}$$

$$\|h\|_{L^1(I)} \le \mathscr{C}(\|(\phi, \theta, \boldsymbol{u}, \mu)\|_{\boldsymbol{\mathcal{W}}_{q,s,p}^{r,2}(Q)}). \tag{3.44}$$

Using $-\frac{\mathrm{d}}{\mathrm{d}t}e \le -\frac{\mathrm{d}}{\mathrm{d}t}e + b$ in (3.42), invoking the Gronwall Lemma to the resulting inequality, and then applying the vanishing terminal conditions $\varphi(T) = \vartheta(T) = 0$ and $\boldsymbol{v}(T) = \boldsymbol{0}$ in Ω, we obtain

$$\|e\|_{L^\infty(I)} \le c_\delta\|g\|_{L^1(I)}\mathrm{e}^{c_\delta\|h\|_{L^1(I)}}. \tag{3.45}$$

Consequently, integrating (3.42) over I yields

$$\|b\|_{L^1(I)} \le c_\delta\{\|g\|_{L^1(I)} + \|h\|_{L^1(I)}\|e\|_{L^\infty(I)}\}. \tag{3.46}$$

To simplify the succeeding *a priori* estimates, let us introduce the following notation for the right-hand side of (3.12)

$$R := \mathscr{C}(\|(\phi, \theta, \boldsymbol{u}, \mu)\|_{\boldsymbol{\mathcal{W}}_{q,s,p}^{r,2}(Q)})\|(g_{\mathsf{o}}, g_{\mathsf{h}}, \boldsymbol{g}_{\mathsf{v}}, g_{\mathsf{c}})\|_{\boldsymbol{\mathcal{G}}_2^2(Q)}.$$

As mentioned at the beginning of the proof, $\mathscr{C} : [0,\infty) \to [0,\infty)$ denotes a generic continuous function that can be different at each step. From (3.43)–(3.46) and the definitions of the functionals b and e, we have

$$\|\varphi\|_{L^\infty(I;W_0^{1,2}(\Omega))\cap L^2(I;X^{2,2}(\Omega))} + \|\eta\|_{L^2(I;L^2(\Omega))} + \|\vartheta\|_{\mathcal{Z}_{2,2}^2(Q)} + \|\boldsymbol{v}\|_{\boldsymbol{\mathcal{V}}_{2,2}^2(Q)} \le R. \tag{3.47}$$

The remaining parts of the proof are concerned with additional estimates for φ and η, as well as the estimate for the pressure π.

• *Estimate for $\partial_t\varphi$ and $\Delta\varphi$ in $L^2(I; W_0^{1,2}(\Omega))$.* Taking the test function $-(\partial_t\varphi + \Delta\varphi)$ to the first equation in the dual system (3.8), using the fact that $\boldsymbol{u}$ and $\boldsymbol{v}$ are divergence-free, and applying Green's identity for the term involving $\epsilon\Delta\eta$, one has the equation

$$\begin{aligned}
&\|\partial_t\varphi\|^2_{L^2(\Omega)} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)} - \tau(\partial_t\eta, \partial_t\varphi + \Delta\varphi)_{L^2(\Omega)} \\
&\quad - (\boldsymbol{u}\cdot\nabla(\partial_t\varphi + \Delta\varphi), \varphi - l_{\mathsf{h}}\vartheta)_{L^2(\Omega)} + \epsilon(\nabla\eta, \nabla\partial_t\varphi + \nabla\Delta\varphi)_{\boldsymbol{L}^2(\Omega)} \\
&= -(F'(\phi)\eta, \partial_t\varphi + \Delta\varphi)_{L^2(\Omega)} - (\alpha_1\boldsymbol{g}\cdot\boldsymbol{v} - l_{\mathsf{h}}\partial_t\vartheta, \partial_t\varphi + \Delta\varphi)_{L^2(\Omega)} \\
&\quad - \langle g_{\mathsf{o}}, \partial_t\varphi + \Delta\varphi\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)} - \mathcal{K}(\boldsymbol{v}\cdot\nabla(\partial_t\varphi + \Delta\varphi), \mu - l_{\mathsf{c}}\theta)_{L^2(\Omega)}. \qquad (3.48)
\end{aligned}$$

The first three terms on the right-hand side of (3.48) obey the estimates

$$\begin{aligned}
&|(F'(\phi)\eta, \partial_t\varphi + \Delta\varphi)_{L^2(\Omega)}| \le \|F'(\phi)\|_{L^2(\Omega)}\|\eta\|_{L^4(\Omega)}\|\partial_t\varphi + \Delta\varphi\|_{L^4(\Omega)} \\
&\quad \le \delta\|\nabla\partial_t\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\nabla\Delta\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\nabla\eta\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\{\|\phi\|^8_{L^4(\Omega)} + 1\}\|\eta\|^2_{L^2(\Omega)} \qquad (3.49)
\end{aligned}$$

$$|(\alpha_1\boldsymbol{g}\cdot\boldsymbol{v} - l_{\mathsf{h}}\partial_t\vartheta, \partial_t\varphi + \Delta\varphi)_{L^2(\Omega)}| \le \frac{1}{2}\|\partial_t\varphi\|^2_{L^2(\Omega)} + c\{\|\Delta\varphi\|^2_{L^2(\Omega)} + \|\partial_t\vartheta\|^2_{L^2(\Omega)} + \|\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)}\} \qquad (3.50)$$

$$|\langle g_{\mathsf{o}}, \partial_t\varphi + \Delta\varphi\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}| \le \delta\|\nabla\partial_t\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\nabla\Delta\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\|g_{\mathsf{o}}\|^2_{W^{-1,2}(\Omega)}. \qquad (3.51)$$

With regard to the trilinear terms in (3.48), it holds that

$$\begin{aligned}
&|(\boldsymbol{u}\cdot\nabla(\partial_t\varphi + \Delta\varphi), \varphi - l_{\mathsf{h}}\vartheta)_{L^2(\Omega)}| \le \|\boldsymbol{u}\|_{\boldsymbol{L}^4(\Omega)}\|\nabla(\partial_t\varphi + \Delta\varphi)\|_{\boldsymbol{L}^2(\Omega)}\|\varphi - l_{\mathsf{h}}\vartheta\|_{L^4(\Omega)} \\
&\quad \le \delta\|\nabla\partial_t\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\nabla\Delta\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\|\boldsymbol{u}\|^2_{\boldsymbol{L}^4(\Omega)}\{\|\nabla\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\vartheta\|^2_{\boldsymbol{L}^2(\Omega)}\} \qquad (3.52) \\
&|\mathcal{K}(\boldsymbol{v}\cdot\nabla(\partial_t\varphi + \Delta\varphi), \mu - l_{\mathsf{c}}\theta)_{L^2(\Omega)}| \le c\|\boldsymbol{v}\|_{\boldsymbol{L}^4(\Omega)}\|\nabla(\partial_t\varphi + \Delta\varphi)\|_{\boldsymbol{L}^2(\Omega)}\|\mu - l_{\mathsf{h}}\theta\|_{L^4(\Omega)} \\
&\quad \le \delta\|\nabla\partial_t\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\nabla\Delta\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\{\|\mu\|^2_{L^4(\Omega)} + \|\theta\|^2_{L^4(\Omega)}\}\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2}. \qquad (3.53)
\end{aligned}$$

For the term involving the gradient of η in the equation (3.48), using Young inequality and (3.28), we have

$$\begin{aligned}
\epsilon(\nabla\eta, \nabla\partial_t\varphi + \nabla\Delta\varphi)_{\boldsymbol{L}^2(\Omega)} \ge\ & -\frac{m\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\varphi\|^2_{L^2(\Omega)} + \frac{m\epsilon}{2}\|\nabla\Delta\varphi\|^2_{\boldsymbol{L}^2(\Omega)} - \delta\|\nabla\partial_t\varphi\|^2_{\boldsymbol{L}^2(\Omega)} \\
& - c_\delta\{\|\phi\|^2_{W^{2,4}(\Omega)}\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2} + \|\nabla g_{\mathsf{c}}\|^2_{\boldsymbol{L}^2(\Omega)}\}. \qquad (3.54)
\end{aligned}$$

Taking the time derivative of η given in the second equation of (3.8) and getting the inner product with $-\tau(\partial_t\varphi + \Delta\varphi)$ in $L^2(\Omega)$, one has

$$\begin{aligned}
-\tau(\partial_t\eta, \partial_t\varphi + \Delta\varphi)_{L^2(\Omega)} &= m\tau\|\nabla\partial_t\varphi\|^2_{\boldsymbol{L}^2(\Omega)} - \frac{m\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta\varphi\|^2_{L^2(\Omega)} + \tau\mathcal{K}(\partial_t\boldsymbol{v}\cdot\nabla(\partial_t\varphi + \Delta\varphi), \phi)_{L^2(\Omega)} \\
&\quad + \tau\mathcal{K}(\boldsymbol{v}\cdot\nabla(\partial_t\varphi + \Delta\varphi), \partial_t\phi)_{L^2(\Omega)} - \tau\langle\partial_t g_{\mathsf{c}}, \partial_t\varphi + \Delta\varphi\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}. \qquad (3.55)
\end{aligned}$$

The last three terms on the right-hand side in (3.55) can be estimated according to

$$|\tau\langle\partial_t g_{\mathsf{c}}, \partial_t\varphi + \Delta\varphi\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}| \le \delta\|\nabla\partial_t\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\nabla\Delta\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\|\partial_t g_{\mathsf{c}}\|^2_{W^{-1,2}(\Omega)} \qquad (3.56)$$

$$|\tau\mathcal{K}(\partial_t\boldsymbol{v}\cdot\nabla(\partial_t\varphi + \Delta\varphi), \phi)_{L^2(\Omega)}| \le \delta\|\nabla\partial_t\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\nabla\Delta\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\|\phi\|^2_{L^\infty(\Omega)}\|\partial_t\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)} \qquad (3.57)$$

$$|\tau\mathcal{K}(\boldsymbol{v}\cdot\nabla(\partial_t\varphi + \Delta\varphi), \partial_t\phi)_{L^2(\Omega)}| \le \delta\|\nabla\partial_t\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + \delta\|\nabla\Delta\varphi\|^2_{\boldsymbol{L}^2(\Omega)} + c_\delta\|\partial_t\phi\|^2_{L^4(\Omega)}\|\nabla\boldsymbol{v}\|^2_{\boldsymbol{L}^2(\Omega)^2}. \qquad (3.58)$$

Using the inequalities (3.56)–(3.58) in (3.55), (3.49)–(3.54) in (3.48), taking the sum of the resulting inequalities to that of (3.29) multiplied by $\delta_3 > 0$, and utilizing the *a priori* estimate (3.47), lead to

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{m(\tau+\epsilon)\|\Delta\varphi\|_{L^2(\Omega)}^2+\|\nabla\varphi\|_{\boldsymbol{L}^2(\Omega)}^2\}+(m\tau-8\delta)\|\nabla\partial_t\varphi\|_{\boldsymbol{L}^2(\Omega)}^2+\Big(\frac{m\epsilon}{2}-7\delta-c\delta_3\Big)\|\nabla\Delta\varphi\|_{\boldsymbol{L}^2(\Omega)}^2$$
$$+(\delta_3-\delta)\|\nabla\eta\|_{\boldsymbol{L}^2(\Omega)}^2+\frac{1}{2}\|\partial_t\varphi\|_{L^2(\Omega)}^2\leq\widetilde{g}+\widetilde{h}\{\|\Delta\varphi\|_{L^2(\Omega)}^2+\|\nabla\varphi\|_{\boldsymbol{L}^2(\Omega)}^2\} \tag{3.59}$$

for some $\widetilde{g},\widetilde{h}\in L^1(I)$ such that $\|\widetilde{g}\|_{L^1(I)}\leq R$ and $\|\widetilde{h}\|_{L^1(I)}\leq\mathscr{C}(\|(\phi,\theta,\boldsymbol{u},\mu)\|_{\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)})$. Choosing $0<\delta<\delta_3$ and $\delta_3>0$ small enough so that $m\tau-8\delta>0$ and $\frac{m\epsilon}{2}-7\delta-c\delta_3>0$, integrating the differential inequality (3.59), and then applying the Gronwall Lemma, it can be deduced that

$$\|\partial_t\varphi\|_{L^2(I;W^{1,2}_0(\Omega))}+\|\Delta\varphi\|_{L^2(I;W^{1,2}_0(\Omega))}+\|\nabla\eta\|_{L^2(I;\boldsymbol{L}^2(\Omega))}\leq R. \tag{3.60}$$

• *Estimate for $\partial_t\eta$ in $L^2(I;W^{-1,2}(\Omega))$ and for π in $L^2(I;\widehat{W}^{1,2}(\Omega))$.* By taking the time-derivative of η and using the Hölder inequality, we immediately obtain

$$\|\partial_t\eta\|_{L^2(I;W^{-1,2}(\Omega))}\leq c\{\|\partial_t\varphi\|_{L^2(I;W^{1,2}(\Omega))}+\|\phi\|_{L^\infty(I;L^\infty(\Omega))}\|\partial_t\boldsymbol{v}\|_{L^2(I;\boldsymbol{L}^2(\Omega))}$$
$$+\|\partial_t\phi\|_{L^2(I;W^{1,2}(\Omega))}\|\boldsymbol{v}\|_{L^\infty(I;\boldsymbol{W}^{1,2}(\Omega))}+\|\partial_t g_{\mathsf{c}}\|_{L^2(I;W^{-1,2}(\Omega))}\}. \tag{3.61}$$

One can argue the existence and uniqueness of the associated pressure $\pi\in L^2(I;\widehat{W}^{1,2}(\Omega))$ from the de Rham's theorem, see [47] for instance. From the fourth equation in (3.8) and in virtue of the Poincaré–Wirtinger inequality, one has

$$\|\pi\|_{L^2(I;\widehat{W}^{1,2}(\Omega))}\leq c\|\nabla\pi\|_{L^2(I;\boldsymbol{L}^2(\Omega))}\leq c\{\|\partial_t\boldsymbol{v}\|_{L^2(I;\boldsymbol{L}^2(\Omega))}+\|\boldsymbol{u}\|_{L^4(I;\boldsymbol{L}^4(\Omega))}\|\boldsymbol{v}\|_{L^4(I;\boldsymbol{W}^{1,4}(\Omega))}$$
$$+\|\Delta\boldsymbol{u}\|_{L^2(I;\boldsymbol{L}^2(\Omega))}+\|\vartheta\|_{L^2(I;L^2(\Omega))}+\|\phi\|_{L^4(I;L^4(\Omega))}\|\varphi\|_{L^4(I;W^{1,4}(\Omega))}$$
$$+(\|\theta\|_{L^4(I;L^4(\Omega))}+\|\phi\|_{L^4(I;L^4(\Omega))})\|\vartheta\|_{L^4(I;W^{1,4}(\Omega))}+\|\boldsymbol{g}_{\mathsf{v}}\|_{L^2(I;\boldsymbol{L}^2(\Omega))}\}. \tag{3.62}$$

From (3.47), (3.60)–(3.62), and utilizing the continuity of the embeddings $\mathcal{Z}^3_{q,r}(Q)+\mathcal{Z}^3_{2,2}(Q)\hookrightarrow L^\infty(I;L^\infty(\Omega))$, $\mathcal{Z}^1_{s,r}(Q)+\mathcal{Z}^1_{2,2}(Q)\hookrightarrow L^4(I;L^4(\Omega))$, $\boldsymbol{\mathcal{V}}^1_{p,r}(Q)+\boldsymbol{\mathcal{V}}^1_{2,2}(Q)\hookrightarrow L^4(I;\boldsymbol{L}^4(\Omega))$, $\mathcal{Z}^3_{2,2}(Q)\hookrightarrow\mathcal{Z}^2_{2,2}(Q)\hookrightarrow L^4(I;W^{1,4}(\Omega))$, and $\boldsymbol{\mathcal{V}}^2_{2,2}(Q)\hookrightarrow L^\infty(I;\boldsymbol{W}^{1,2}(\Omega))\cap L^4(I;\boldsymbol{W}^{1,4}(\Omega))$, see [43], Section 4.1 for the details on the first embedding, we have

$$\|\partial_t\eta\|_{L^2(I;W^{-1,2}(\Omega))}+\|\pi\|_{L^2(I;\widehat{W}^{1,2}(\Omega))}\leq R. \tag{3.63}$$

Taking the sum of (3.47), (3.60), and (3.63) leads to the desired *a priori* estimate (3.12). We point out here that (3.12) applies to the finite-dimensional approximations that can be constructed from the Faedo–Galerkin method. By pursuing standard weak sequential compactness arguments, the existence of a weak solution to (3.8) can be established. Finally, the fact that this constructed weak solution is the unique one follows from standard arguments, thanks to the linearity of (3.8). □

4. Local optimality conditions

The goal of this section is to present necessary and sufficient conditions for local optimality. We follow the framework developed in [9] for the case of the two-dimensional Navier–Stokes equation. In the context of second-order sufficient conditions, we include the chemical potential and require a norm for the order parameter that is stronger than that of $L^2(Q)$.

4.1. Local first-order optimality condition

Let us introduce the control-to-adjoint operator

$$\mathbf{D} : \boldsymbol{\mathcal{M}}^r \to \boldsymbol{\mathcal{Y}}_2^2(Q)$$

as follows: $\mathbf{D}(\boldsymbol{s}) := (\varphi, \vartheta, \boldsymbol{v}, \eta)$ if and only if the right-hand side is the weak solution of the adjoint system (3.8) with coefficients $(\phi, \theta, \boldsymbol{u}, \mu) = \mathbf{F}(\boldsymbol{s})$ and source functions

$$g_{\mathsf{o}}(\phi) := \lambda_{1\mathsf{o}}(\phi - \phi_d) - \lambda_{2\mathsf{o}}(\Delta\phi - \operatorname{div}\boldsymbol{\psi}_d), \qquad g_{\mathsf{c}} := 0, \tag{4.1}$$

$$g_{\mathsf{h}}(\theta) := \lambda_{\mathsf{h}}(\theta - \theta_d), \qquad \boldsymbol{g}_{\mathsf{v}}(\boldsymbol{u}) := \lambda_{\mathsf{v}}(\boldsymbol{u} - \boldsymbol{u}_d). \tag{4.2}$$

Since $\phi_d, \theta_d \in L^2(I; L^2(\Omega))$ and $\boldsymbol{\psi}_d, \boldsymbol{u}_d \in L^2(I; \boldsymbol{L}^2(\Omega))$, we have

$$(g_{\mathsf{o}}(\phi), g_{\mathsf{h}}(\theta), \boldsymbol{g}_{\mathsf{v}}(\boldsymbol{u}), 0) \in \boldsymbol{\mathcal{G}}_2^2(Q),$$

and hence $\mathbf{D}$ is well-defined thanks to Theorem 3.7. Here, div should be understood in the sense of distributions. More precisely, div $: \boldsymbol{L}^p(\Omega) \to W^{-1,p}(\Omega)$, with $1 < p < \infty$, is given by

$$\langle \operatorname{div}\boldsymbol{\psi}, \phi\rangle_{W^{-1,p}(\Omega), W_0^{1,p'}(\Omega)} = -\langle \boldsymbol{\psi}, \nabla\phi\rangle_{\boldsymbol{L}^p(\Omega), \boldsymbol{L}^{p'}(\Omega)} \quad \forall\, (\boldsymbol{\psi}, \phi) \in \boldsymbol{L}^p(\Omega) \times W_0^{1,p'}(\Omega).$$

In the following theorem, we shall express the first and second derivatives of J in terms of the solutions of the adjoint and linearized state systems.

Theorem 4.1. *The reduced cost functional satisfies $J \in C^\infty(\boldsymbol{\mathcal{M}}^r, \mathbb{R})$. Furthermore, for each $\boldsymbol{s} = (\sigma_{\mathsf{o}}, \sigma_{\mathsf{h}}, \boldsymbol{\sigma}_{\mathsf{v}}) \in \boldsymbol{\mathcal{M}}^r$ and $\boldsymbol{r} = (\rho_{\mathsf{o}}, \rho_{\mathsf{h}}, \boldsymbol{\rho}_{\mathsf{v}}) \in \boldsymbol{\mathcal{M}}^r$, the action of the first and second derivatives of J are given by*

$$\begin{aligned}
\mathrm{D}J(\boldsymbol{s})\boldsymbol{r} &= \int_0^T \left(\int_{\omega_{\mathsf{o}}} \varphi \,\mathrm{d}\rho_{\mathsf{o}} + \int_{\omega_{\mathsf{h}}} \vartheta \,\mathrm{d}\rho_{\mathsf{h}} + \int_{\omega_{\mathsf{v}}} \boldsymbol{v} \,\mathrm{d}\boldsymbol{\rho}_{\mathsf{v}} \right) \mathrm{d}t \\
\mathrm{D}^2 J(\boldsymbol{s})(\boldsymbol{r}, \boldsymbol{r}) &= \int_0^T \int_\Omega \lambda_{\mathsf{o}1}|\psi|^2 + \lambda_{\mathsf{o}2}|\nabla\psi|^2 + \lambda_{\mathsf{h}}|\zeta|^2 + \lambda_{\mathsf{v}}|\boldsymbol{w}|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega 2(\boldsymbol{w}\cdot\nabla\varphi)\psi - 6\beta_0\phi\psi^2\eta + 2(\boldsymbol{w}\cdot\nabla\vartheta)(\zeta - l_{\mathsf{h}}\psi) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega 2(\boldsymbol{w}\cdot\nabla\boldsymbol{v})\cdot\boldsymbol{w} + 2\mathcal{K}(\boldsymbol{v}\cdot\nabla\psi)(\xi - l_{\mathsf{c}}\zeta) \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

where ϕ is the first component of $\mathbf{F}(\boldsymbol{s})$, $(\varphi, \vartheta, \boldsymbol{v}, \eta) = \mathbf{D}(\boldsymbol{s})$, and $(\psi, \zeta, \boldsymbol{w}, \xi) = \mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r}$.

Proof. Since G, $\mathbf{P}$, and $\mathbf{F}$ are of class C^∞, we have $J = G \circ \mathbf{P} \circ \mathbf{F} \in C^\infty(\boldsymbol{\mathcal{M}}^r, \mathbb{R})$ by the chain rule. From the Sobolev embedding theorem and $r' < 2$, we see that

$$\mathcal{Z}_{2,2}^3(Q) \times \mathcal{Z}_{2,2}^2(Q) \times \boldsymbol{\mathcal{V}}_{2,2}^2(Q) \hookrightarrow L^{r'}(I; C_0(\omega_{\mathsf{o}})) \times L^{r'}(I; C_0(\omega_{\mathsf{h}})) \times L^{r'}(I; \boldsymbol{C}_0(\omega_{\mathsf{v}})).$$

This implies that the right-hand side of the above equation for the first derivative is well-defined. The representation of the first-order derivative of J can be derived by following the computations given in the Appendix. Similarly, the second-order derivative can be obtained by following the proof of [42], Section 6.1, Lemma 3. $\square$

Given a regular Borel measure $\sigma \in M(\omega)$, we can write its Hahn–Jordan decomposition as $\sigma = \sigma^+ - \sigma^-$, where σ^+ and σ^- are positive measures. In the following proposition, we characterize the supports of these decompositions, which will be needed in future discussions.

Proposition 4.2. *Let ω be a relatively closed subset of Ω, $\gamma > 0$, $\sigma \in L^\infty_{\mathsf{w}}(I; M(\omega))$, and $y \in L^1(I; C_0(\omega))$. If $\|\sigma\|_{L^\infty_{\mathsf{w}}(I;M(\omega))} \leq \gamma$ and*

$$\int_0^T \int_\omega y \,\mathrm{d}\sigma \,\mathrm{d}t \leq \int_0^T \int_\omega y \,\mathrm{d}\rho \,\mathrm{d}t \quad \forall \|\rho\|_{L^\infty_{\mathsf{w}}(I;M(\omega))} \leq \gamma,$$

then for a.a. $t \in I$, if $\|y(t)\|_{C_0(\omega)} > 0$, then $\|\sigma(t)\|_{M(\omega)} = \gamma$ and

$$\mathrm{Supp}(\sigma^\pm(t)) \subset \{x \in \omega : y(t,x) = \mp\|y(t)\|_{C_0(\omega)}\}.$$

Furthermore, if $\varrho : I \times \omega \to \mathbb{R}$ is defined by

$$\varrho(t,x) = \begin{cases} 1 & \text{if } \|y(t)\|_{C_0(\omega)} = 0, \\ -\|y(t)\|^{-1}_{C_0(\omega)} y(t,x) & \text{otherwise}, \end{cases} \tag{4.3}$$

then $\varrho(t)$ is the Radon–Nikodym derivative of $\sigma(t)$ with respect to the total variation measure $|\sigma(t)|$, that is, $\mathrm{d}\sigma(t) = \varrho(t)\,\mathrm{d}|\sigma(t)|$ for a.a. $t \in I$.

Proof. The proof is contained in the discussion in [9], Section 3. □

To have a more economical way for the statement of the optimality conditions, we write the components of the adjoint states corresponding to the optimal controls according to

$$\boldsymbol{y}^* := (y^*_{\mathsf{o}}, y^*_{\mathsf{h}}, \boldsymbol{y}^*_{\mathsf{v}}) = (\varphi^*, \vartheta^*, \boldsymbol{v}^*) \tag{4.4}$$

and set $\omega_{\mathsf{v1}} = \omega_{\mathsf{v2}} = \omega_{\mathsf{v}}$. The index set for the controls will be denoted by

$$K := \{\mathsf{o}, \mathsf{h}, \mathsf{v1}, \mathsf{v2}\}.$$

Theorem 4.3. *Let $(\sigma^*_{\mathsf{o}}, \sigma^*_{\mathsf{h}}, \boldsymbol{\sigma}^*_{\mathsf{v}}) \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$ be a local solution of (3.7) and $(\varphi^*, \vartheta^*, \boldsymbol{v}^*, \eta^*) = \mathbf{D}(\sigma^*_{\mathsf{o}}, \sigma^*_{\mathsf{h}}, \boldsymbol{\sigma}^*_{\mathsf{v}}) \in \boldsymbol{\mathcal{Y}}^2_2(Q)$ be the associated optimal adjoint state. Then, for every index $\mathsf{k} \in K$ and for a.a. $t \in I$, the following holds:*

$$\left[\begin{array}{l} \text{if } \|y^*_{\mathsf{k}}(t)\|_{C_0(\omega_{\mathsf{k}})} > 0, \text{ then } \|\sigma^*_{\mathsf{k}}(t)\|_{M(\omega_{\mathsf{k}})} = \gamma_{\mathsf{k}}, \text{ and} \\ \mathrm{Supp}(\sigma^{*\pm}_{\mathsf{k}}(t)) \subset \{x \in \omega_{\mathsf{k}} : y^*_{\mathsf{k}}(t,x) = \mp\|y^*_{\mathsf{k}}(t)\|_{C_0(\omega_{\mathsf{k}})}\}. \end{array}\right. \tag{4.5}$$

If ϱ^*_{k} is defined as in (4.3) with $\omega = \omega_{\mathsf{k}}$ and $y = y^*_{\mathsf{k}}$, then $\mathrm{d}\sigma^*_{\mathsf{k}}(t) = \varrho^*_{\mathsf{k}}(t)\,\mathrm{d}|\sigma^*_{\mathsf{k}}(t)|$ for a.a. $t \in I$.

Proof. The differentiability of J and the convexity of the set of admissible controls $\boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$ imply that $\mathrm{D}J(\boldsymbol{s}^*)(\boldsymbol{r} - \boldsymbol{s}^*) \geq 0$, and so by Theorem 4.1,

$$0 \leq \int_0^T \left(\int_{\omega_{\mathsf{o}}} \varphi^* \,\mathrm{d}(\rho_{\mathsf{o}} - \sigma^*_{\mathsf{o}}) + \int_{\omega_{\mathsf{h}}} \vartheta^* \,\mathrm{d}(\rho_{\mathsf{h}} - \sigma^*_{\mathsf{h}}) + \int_{\omega_{\mathsf{v}}} \boldsymbol{v}^* \,\mathrm{d}(\boldsymbol{\rho}_{\mathsf{v}} - \boldsymbol{\sigma}^*_{\mathsf{v}}) \right) \mathrm{d}t$$

for every $\boldsymbol{r} = (\rho_{\mathsf{o}}, \rho_{\mathsf{h}}, \boldsymbol{\rho}_{\mathsf{v}}) \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$. Given $\mathsf{k} \in K$ and $\|\rho\|_{L^\infty_{\mathsf{w}}(I;M(\omega_{\mathsf{k}}))} \leq \gamma_{\mathsf{k}}$, we set $\rho_{\mathsf{k}} = \rho$ and $\rho_{\mathsf{j}} = \sigma^*_{\mathsf{j}}$ for $\mathsf{j} \neq \mathsf{k}$. With these, we have $(\rho_{\mathsf{o}}, \rho_{\mathsf{h}}, \boldsymbol{\rho}_{\mathsf{v}}) \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$, and by substituting to the above inequality, we get

$$\int_0^T \int_{\omega_{\mathsf{k}}} y^*_{\mathsf{k}} \,\mathrm{d}\sigma^*_{\mathsf{k}} \,\mathrm{d}t \leq \int_0^T \int_{\omega_{\mathsf{k}}} y^*_{\mathsf{k}} \,\mathrm{d}\rho \,\mathrm{d}t \quad \forall \|\rho\|_{L^\infty_{\mathsf{w}}(I;M(\omega_{\mathsf{k}}))} \leq \gamma_{\mathsf{k}}.$$

The theorem is now a direct consequence of Proposition 4.2 and the fact that k was arbitrarily chosen in K. □

4.2. Local second-order optimality conditions

Given $\sigma, \rho \in M(\omega)$, we have the Lebesgue decomposition of ρ with respect to $|\sigma|$ as follows:

$$\mathrm{d}\rho = g_\rho \, \mathrm{d}|\sigma| + \mathrm{d}\rho_{\mathsf{s}}. \tag{4.6}$$

Here, $g_\rho \in L^1(\omega, |\sigma|)$ and ρ_{s} are the Radon–Nikodym derivative and the singular part of ρ with respect to $|\sigma|$. Thus, the norm of ρ in $M(\omega)$ can be expressed as

$$\|\rho\|_{M(\omega)} = \int_\omega |g_\rho| \, \mathrm{d}|\sigma| + \|\rho_{\mathsf{s}}\|_{M(\omega)}. \tag{4.7}$$

This follows from the fact that the norm in $M(\omega)$ is equal to the total variation measure and $\mathrm{d}|\rho| = |g_\rho| \, \mathrm{d}|\sigma| + \mathrm{d}|\rho_{\mathsf{s}}|$. The directional derivative of the norm functional $\|\cdot\|_{M(\omega)} : M(\omega) \to \mathbb{R}$ at σ in the direction of ρ, denoted by $\partial\|\sigma\|_{M(\omega)}\rho$, exists and is given by

$$\partial\|\sigma\|_{M(\omega)}\rho = \int_\omega g_\rho \, \mathrm{d}\sigma + \|\rho_{\mathsf{s}}\|_{M(\omega)}, \tag{4.8}$$

see [7], Proposition 3.3. Also, by the convexity of $\|\cdot\|_{M(\omega)}$, we have

$$\partial\|\sigma\|_{M(\omega)}(\rho - \sigma) \le \|\rho\|_{M(\omega)} - \|\sigma\|_{M(\omega)} \qquad \forall \rho, \sigma \in M(\omega). \tag{4.9}$$

Let $L^1_+(I) := \{m \in L^1(I) : m \ge 0 \text{ a.a. in } I\}$ and define $\Lambda : \boldsymbol{\mathcal{M}}^\infty \times L^1_+(I)^4 \to \mathbb{R}$ according to

$$\Lambda(\boldsymbol{s}, \boldsymbol{m}) := \sum_{\mathsf{k} \in K} \int_0^T m_{\mathsf{k}} (\|\sigma_{\mathsf{k}}\|_{M(\omega_{\mathsf{k}})} - \gamma_{\mathsf{k}}) \, \mathrm{d}t$$

for $\boldsymbol{s} = (\sigma_{\mathsf{o}}, \sigma_{\mathsf{h}}, \boldsymbol{\sigma}_{\mathsf{v}}) \in \boldsymbol{\mathcal{M}}^\infty$ and $\boldsymbol{m} = (m_{\mathsf{o}}, m_{\mathsf{h}}, \boldsymbol{m}_{\mathsf{v}}) \in L^1_+(I)^4$. We now introduce the Lagrangian $L : \boldsymbol{\mathcal{M}}^\infty \times L^1_+(I)^4 \to \mathbb{R}$ given by

$$L(\boldsymbol{s}, \boldsymbol{m}) = J(\boldsymbol{s}) + \Lambda(\boldsymbol{s}, \boldsymbol{m}).$$

As in the finite-dimensional case, the first component of saddle points of the Lagrangian are necessarily global solutions to (3.7). We prove this in the following proposition.

Proposition 4.4. *If $(\boldsymbol{s}^*, \boldsymbol{m}^*) \in \boldsymbol{\mathcal{M}}^\infty \times L^1_+(I)^4$ is a saddle point of L, that is,*

$$L(\boldsymbol{s}^*, \boldsymbol{m}) \le L(\boldsymbol{s}^*, \boldsymbol{m}^*) \le L(\boldsymbol{s}, \boldsymbol{m}^*) \quad \forall \, (\boldsymbol{s}, \boldsymbol{m}) \in \boldsymbol{\mathcal{M}}^\infty \times L^1_+(I)^4, \tag{4.10}$$

then $\boldsymbol{s}^*$ is a global solution to (3.7).

Proof. The first inequality in (4.10) implies that for every $\boldsymbol{m} \in L^1_+(I)^4$, we have

$$\sum_{\mathsf{k} \in K} \int_0^T (m_{\mathsf{k}} - m^*_{\mathsf{k}})(\|\sigma^*_{\mathsf{k}}\|_{M(\omega_{\mathsf{k}})} - \gamma_{\mathsf{k}}) \, \mathrm{d}t = L(\boldsymbol{s}^*, \boldsymbol{m}) - L(\boldsymbol{s}^*, \boldsymbol{m}^*) \le 0. \tag{4.11}$$

Let $m \in L^1_+(I)$. Given $\mathsf{k} \in K$, we set $m_\mathsf{k} = m$ and $m_\mathsf{j} = m_\mathsf{j}^*$ for $\mathsf{j} \neq \mathsf{k}$. Taking these as the components of $\boldsymbol{m}$ in (4.11) yields

$$\int_0^T (m - m_\mathsf{k}^*)(\|\sigma_\mathsf{k}^*\|_{M(\omega_\mathsf{k})} - \gamma_\mathsf{k})\,\mathrm{d}t \leq 0. \tag{4.12}$$

Consider a Lebesgue point $t_0 \in I$ of $\|\sigma_\mathsf{k}^*\|_{M(\omega_\mathsf{k})} \in L^\infty(I) \subset L^1(I)$. Choosing $m = m_\mathsf{k}^* + \frac{1}{2\delta}\chi_{[t_0-\delta,t_0+\delta]} \in L^1_+(I)$ with $\delta > 0$ in (4.12) leads to

$$\frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta} \|\sigma_\mathsf{k}^*\|_{M(\omega_\mathsf{k})}\,\mathrm{d}t \leq \gamma_\mathsf{k}.$$

By passing $\delta \to 0$ we deduce that $\|\sigma_\mathsf{k}^*(t_0)\|_{M(\omega_\mathsf{k})} \leq \gamma_\mathsf{k}$ from the Lebesgue differentiation theorem. Since k was an arbitrary element of K and the set of all Lebesgue points has full measure $|I|$, it follows that $\boldsymbol{s}^* \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$.

Now suppose that $t_0 \in I$ is a Lebesgue point of $m_\mathsf{k}^*(\|\sigma_\mathsf{k}^*\|_{M(\omega_\mathsf{k})} - \gamma_\mathsf{k}) \in L^1(I)$. Taking $m = m_\mathsf{k}^*(1 - \chi_{[t_0-\delta,t_0+\delta]}) \in L^1_+(I)$ in (4.12) and then dividing by -2δ, we obtain

$$\frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta} m_\mathsf{k}^*(\|\sigma_\mathsf{k}^*\|_{M(\omega_\mathsf{k})} - \gamma_\mathsf{k})\,\mathrm{d}t \geq 0.$$

Sending $\delta \to 0$, and again since Lebesgue points have full measure, we get that $m_\mathsf{k}^*(\|\sigma_\mathsf{k}^*\|_{M(\omega_\mathsf{k})} - \gamma_\mathsf{k}) \geq 0$ a.a. in I. Since m_k^* is almost everywhere non-negative and $\boldsymbol{s}^*$ is admissible, we conclude that $\Lambda(\boldsymbol{s}^*, \boldsymbol{m}^*) = 0$. Using this in the second inequality of (4.10), it is not difficult to see that $J(\boldsymbol{s}^*) \leq J(\boldsymbol{s})$ for every $\boldsymbol{s} \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$, and so $\boldsymbol{s}^*$ is a global solution to (3.7). □

Consider the Lagrange multipliers

$$\boldsymbol{m}^* = (m_\mathsf{o}^*, m_\mathsf{h}^*, \boldsymbol{m}_\mathsf{v}^*) := (\|\varphi^*\|_{C_0(\omega_\mathsf{o})}, \|\vartheta^*\|_{C_0(\omega_\mathsf{h})}, \|v_1^*\|_{C_0(\omega_\mathsf{v})}, \|v_2^*\|_{C_0(\omega_\mathsf{v})}) \in L^1_+(I)^4.$$

From our notation in (4.4), we have $m_\mathsf{k}^* = \|y_\mathsf{k}^*\|_{C_0(\omega_\mathsf{k})}$ a.a. in I for every $\mathsf{k} \in K$. Theorem 4.3 implies that $m_\mathsf{k}^*(t)(\|\sigma_\mathsf{k}^*(t)\|_{M(\omega_\mathsf{k})} - \gamma_\mathsf{k}) = 0$ for a.a. $t \in I$, and hence $\Lambda(\boldsymbol{s}^*, \boldsymbol{m}^*) = 0$. This means that either the Lagrange multiplier vanishes or the inequality constraint is active almost everywhere in I.

For each $\mathsf{k} \in K$ and for almost all $t \in I$, let $\varrho_\mathsf{k}^*(t)$ be the Radon–Nikodymn derivative of $\sigma_\mathsf{k}^*(t)$ with respect to $|\sigma_\mathsf{k}^*(t)|$, as stated in Theorem 4.3. From (4.6)–(4.8) and Theorem 4.1, the derivative of the Lagrangian at $(\boldsymbol{s}^*, \boldsymbol{m}^*)$ with respect to the control in the direction $\boldsymbol{r} = (\rho_\mathsf{o}, \rho_\mathsf{h}, \boldsymbol{\rho}_\mathsf{v}) \in \boldsymbol{\mathcal{M}}^\infty$ can be expressed as

$$\begin{aligned}
\partial_{\boldsymbol{s}} L(\boldsymbol{s}^*, \boldsymbol{m}^*)\boldsymbol{r} &= \mathrm{D}J(\boldsymbol{s}^*)\boldsymbol{r} + \sum_{\mathsf{k}\in K}\int_0^T m_\mathsf{k}^* \partial\|\sigma_\mathsf{k}^*\|_{M(\omega_\mathsf{k})}\rho_\mathsf{k}\,\mathrm{d}t \\
&= \sum_{\mathsf{k}\in K}\int_0^T \left(\int_{\omega_\mathsf{k}} (y_\mathsf{k}^* + m_\mathsf{k}^*\varrho_\mathsf{k}^*)g_{\rho_\mathsf{k}}\,\mathrm{d}|\sigma_\mathsf{k}^*| + \int_{\omega_\mathsf{k}} y_\mathsf{k}^*\,\mathrm{d}\rho_{\mathsf{ks}} + m_\mathsf{k}^*\|\rho_{\mathsf{ks}}\|_{M(\omega_\mathsf{k})}\right)\mathrm{d}t \\
&= \sum_{\mathsf{k}\in K}\int_0^T \left(\int_{\omega_\mathsf{k}} y_\mathsf{k}^*\,\mathrm{d}\rho_{\mathsf{ks}} + m_\mathsf{k}^*\|\rho_{\mathsf{ks}}\|_{M(\omega_\mathsf{k})}\right)\mathrm{d}t
\end{aligned}$$

where $(y_\mathsf{o}^*, y_\mathsf{h}^*, \boldsymbol{y}_\mathsf{v}^*, \eta^*) = \mathbf{D}(\boldsymbol{s}^*)$, since $y_\mathsf{k}^*(t) + m_\mathsf{k}^*(t)\varrho_\mathsf{k}^*(t) = 0$ for a.a. $t \in I$, according to (4.3).

The above expression implies that $\partial_{\boldsymbol{s}} L(\boldsymbol{s}^*, \boldsymbol{m}^*) \in (\boldsymbol{\mathcal{M}}^\infty)'$ admits an extension such that $\partial_{\boldsymbol{s}} L(\boldsymbol{s}^*, \boldsymbol{m}^*) \in (\boldsymbol{\mathcal{M}}^r)'$. Moreover, since $|\int_{\omega_\mathsf{k}} y_\mathsf{k}^*\,\mathrm{d}\rho_{\mathsf{ks}}| \leq m_\mathsf{k}^*\|\rho_{\mathsf{ks}}\|_{M(\omega_\mathsf{s})}$ for a.a. in I, we see that

$$\partial_{\boldsymbol{s}} L(\boldsymbol{s}^*, \boldsymbol{m}^*)\boldsymbol{r} \geq 0 \quad \forall\, \boldsymbol{r} \in \boldsymbol{\mathcal{M}}^r. \tag{4.13}$$

Equality to zero holds if and only if for a.a. $t \in I$, and for all $\mathsf{k} \in K$, if $\|y_\mathsf{k}^*(t)\|_{C_0(\omega_\mathsf{k})} > 0$, then

$$\mathrm{Supp}(\rho_{\mathsf{ks}}^{\pm}(t)) \subset \{x \in \omega_\mathsf{k} : y_\mathsf{k}^*(t,x) = \mp\|y_\mathsf{k}^*(t)\|_{C_0(\omega_\mathsf{k})}\} \setminus \mathrm{Supp}(|\sigma_\mathsf{k}^*(t)|).$$

Indeed, this follows from the fact $\partial_{\boldsymbol{s}} L(\boldsymbol{s}^*, \boldsymbol{m}^*)\boldsymbol{r} = 0$ if and only if we have $\int_{\omega_\mathsf{k}} y_\mathsf{k}^* \,\mathrm{d}\rho_{\mathsf{ks}} = -m_\mathsf{k}^*\|\rho_{\mathsf{ks}}\|_{M(\omega_\mathsf{s})} = -\|y_\mathsf{k}^*\|_{C_0(\omega_\mathsf{k})}\|\rho_{\mathsf{ks}}\|_{M(\omega_\mathsf{s})}$ in I. Applying [13], Lemma 3.4 and recalling that $\rho_{\mathsf{ks}}(t)$ is the singular part of $\rho_\mathsf{k}(t)$ with respect to the total variation measure $|\sigma_\mathsf{k}^*(t)|$ lead to the above claim. These results are the same as those in [9] for the in-stationary Navier–Stokes equation, however, with a slightly different Lagrangian.

Consider the cone of critical directions $\boldsymbol{\mathcal{C}}^r(\boldsymbol{s}^*) \subset \boldsymbol{\mathcal{M}}^r$ given as follows:

$$\boldsymbol{\mathcal{C}}^r(\boldsymbol{s}^*) := \left\{ \boldsymbol{r} \in \boldsymbol{\mathcal{M}}^r \,\middle|\, \begin{array}{l} \partial_{\boldsymbol{s}} L(\boldsymbol{s}^*, \boldsymbol{m}^*)\boldsymbol{r} = 0, \text{ for a.a. } t \in I \text{ and for every } \mathsf{k} \in K \\ \|\sigma_\mathsf{k}^*(t)\|_{M(\omega_\mathsf{k})} = \gamma_\mathsf{k} \text{ implies } (\,\partial\|\sigma_\mathsf{k}^*(t)\|_{M(\omega_\mathsf{k})}\rho_\mathsf{k}(t) = 0 \\ \text{if } \|y_\mathsf{k}^*(t)\|_{C_0(\omega_\mathsf{k})} > 0 \text{ or } \partial\|\sigma_\mathsf{k}^*(t)\|_{M(\omega_\mathsf{k})}\rho_\mathsf{k}(t) \leq 0 \text{ otherwise}\,) \end{array} \right\}.$$

One can easily check that $\boldsymbol{\mathcal{C}}^r(\boldsymbol{s}^*)$ is indeed a cone having an apex at the origin, that is, $\varepsilon\boldsymbol{r} \in \boldsymbol{\mathcal{C}}^r(\boldsymbol{s}^*)$ whenever $\varepsilon > 0$ and $\boldsymbol{r} \in \boldsymbol{\mathcal{C}}^r(\boldsymbol{s}^*)$.

Theorem 4.5. *If $\boldsymbol{s}^* \in \boldsymbol{\mathcal{M}}_{\mathsf{ad}}^\infty$ is a local solution of (3.7), then $\mathrm{D}^2 J(\boldsymbol{s}^*)(\boldsymbol{r},\boldsymbol{r}) \geq 0$ for every $\boldsymbol{r} \in \boldsymbol{\mathcal{C}}^r(\boldsymbol{s}^*)$.*

Proof. Having established Theorem 4.3 and (4.13), one may proceed as in the proof of [9], Theorem 4.1. We do not repeat the arguments here for the sake of brevity. □

Let us now discuss the second-order sufficient condition for local optimality. For the remaining parts of this section, we let $\boldsymbol{s}^* = (\sigma_\mathsf{o}^*, \sigma_\mathsf{h}^*, \boldsymbol{\sigma}_\mathsf{v}^*) \in \boldsymbol{\mathcal{M}}_{\mathsf{ad}}^\infty$ to be a local solution, $(\phi^*, \theta^*, \boldsymbol{u}^*, \mu^*) = \mathbf{F}(\boldsymbol{s}^*) \in \boldsymbol{\mathcal{W}}_{q,s,p}^{r,2}(Q)$ the corresponding optimal state with the associated pressure $\mathfrak{p}^* \in \mathcal{P}_p^{r,2}(Q)$, and $(\varphi^*, \vartheta^*, \boldsymbol{v}^*, \eta^*) = \mathbf{D}(\boldsymbol{s}^*) \in \boldsymbol{\mathcal{Y}}_2^2(Q)$ the optimal adjoint state with the associated pressure $\pi^* \in L^2(I; \widehat{W}^{1,2}(\Omega))$. The largest bound in the definition of admissible controls will be denoted by

$$\gamma := \max\{\gamma_\mathsf{o}, \gamma_\mathsf{h}, \gamma_\mathsf{v}\}.$$

The supremum of the norms for the weak solutions of the state system over the set of admissible controls will be denoted by

$$F_\gamma := \sup_{\boldsymbol{s} \in \boldsymbol{\mathcal{M}}_{\mathsf{ad}}^\infty} \|\mathbf{F}(\boldsymbol{s})\|_{\boldsymbol{\mathcal{W}}_{q,s,p}^{r,2}(Q)}. \tag{4.14}$$

This is finite due to Theorem 3.2 and the boundedness of $\boldsymbol{\mathcal{M}}_{\mathsf{ad}}^\infty$.

The development of the second-order sufficient conditions will be divided into several lemmas. For the first lemma, we establish the stability of the control-to-state operator, where the norm for the controls are taken in the space $\boldsymbol{\mathcal{N}}_{q,s,p}^r(Q)$.

Lemma 4.6. *There exists $c_0 = c_0(\gamma) > 0$ such that*

$$\|\mathbf{F}(\boldsymbol{s}) - \mathbf{F}(\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{W}}_{q,s,p}^{r,2}(Q)} \leq c_0\|\boldsymbol{s} - \boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}_{q,s,p}^r(Q)} \qquad \forall\, \boldsymbol{s}, \boldsymbol{s}^* \in \boldsymbol{\mathcal{M}}_{\mathsf{ad}}^\infty.$$

Proof. Recall from Theorem 3.3 that $\mathbf{H} \in C^\infty(\boldsymbol{\mathcal{F}}_{q,s,p}^{r,2}(Q), \boldsymbol{\mathcal{W}}_{q,s,p}^{r,2}(Q))$. By Corollary 3.4 and the mean value theorem, there exists $0 < \delta < 1$ such that

$$\begin{aligned} \|\mathbf{F}(\boldsymbol{s}) - \mathbf{F}(\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{W}}_{q,s,p}^{r,2}(Q)} &= \|\mathrm{D}\mathbf{F}(\delta\boldsymbol{s} + (1-\delta)\boldsymbol{s}^*)(\boldsymbol{s} - \boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{W}}_{q,s,p}^{r,2}(Q)} \\ &= \|\mathrm{D}\mathbf{H}(\mathbf{I}(\delta\boldsymbol{s} + (1-\delta)\boldsymbol{s}^*))\mathrm{D}\mathbf{I}(\boldsymbol{s} - \boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{W}}_{q,s,p}^{r,2}(Q)}. \end{aligned}$$

Note that $\|\delta \boldsymbol{s} + (1-\delta)\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)} \le c\{\delta\|\boldsymbol{s}\|_{\boldsymbol{\mathcal{M}}^\infty} + (1-\delta)\|\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{M}}^\infty}\} \le c\gamma$ since $\boldsymbol{s}, \boldsymbol{s}^* \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$, where $c > 0$ denotes the operator norm of the continuous embedding $\boldsymbol{\mathcal{M}}^\infty \hookrightarrow \boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)$. Take

$$c_0 := \sup_{\|\boldsymbol{s}\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)} \le c\gamma} \|\mathrm{D}\mathbf{H}(\mathbf{I}(\boldsymbol{s}))\|_{\mathcal{L}(\boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q), \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q))} \|\mathrm{D}\mathbf{I}\|_{\mathcal{L}(\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q), \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q))}.$$

Applying (3.2) proves the desired estimate. □

The next lemma deals with a Lipschitz-type estimate for the action of the first derivative with respect to the norm of the function space

$$\boldsymbol{\mathcal{T}}^2_2(Q) := \mathcal{Z}^2_{2,2}(Q) \times [L^2(I; L^2(\Omega))]^3.$$

It is obvious that $\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q) \hookrightarrow \boldsymbol{\mathcal{T}}^2_2(Q)$, hence the norm of $\boldsymbol{\mathcal{T}}^2_2(Q)$ is weaker than that of the weak solution space $\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)$.

Lemma 4.7. *Let $\boldsymbol{s} \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$ and $\boldsymbol{r} \in \boldsymbol{\mathcal{M}}^r$. Then, there exist constants $c_1 > 0$ and $c_2 > 0$ independent of $\boldsymbol{s}$ and $\boldsymbol{r}$ such that*

$$\|\mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r} - \mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)} \le c_1 \|\boldsymbol{s} - \boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)} \|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)} \tag{4.15}$$

$$\|\mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)} \le c_2 \|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)}. \tag{4.16}$$

Proof. Let us prove the inequality (4.15). This will be done by a duality argument. Let us write $(\psi_{\boldsymbol{r}}, \zeta_{\boldsymbol{r}}, \boldsymbol{w}_{\boldsymbol{r}}, \xi_{\boldsymbol{r}}) = \mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r}$ and $(\psi^*_{\boldsymbol{r}}, \zeta^*_{\boldsymbol{r}}, \boldsymbol{w}^*_{\boldsymbol{r}}, \xi^*_{\boldsymbol{r}}) = \mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}$. Then, the difference

$$(\psi, \zeta, \boldsymbol{w}, \xi) := (\psi_{\boldsymbol{r}}, \zeta_{\boldsymbol{r}}, \boldsymbol{w}_{\boldsymbol{r}}, \xi_{\boldsymbol{r}}) - (\psi^*_{\boldsymbol{r}}, \zeta^*_{\boldsymbol{r}}, \boldsymbol{w}^*_{\boldsymbol{r}}, \xi^*_{\boldsymbol{r}}) = \mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r} - \mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}$$

satisfies the equation $(\psi, \zeta, \boldsymbol{w}, \xi) = \mathrm{D}\mathbf{H}(\mathbf{I}(\boldsymbol{s}))\boldsymbol{f}^*_{\boldsymbol{s}}$, where $\boldsymbol{f}^*_{\boldsymbol{s}} = (f^*_{\mathsf{o}}, f^*_{\mathsf{h}}, \boldsymbol{f}^*_{\mathsf{v}}, f^*_{\mathsf{c}})$ has the following components:

$$\begin{aligned}
f^*_{\mathsf{o}} &:= -\operatorname{div}(\psi^*_{\boldsymbol{r}}(\boldsymbol{u} - \boldsymbol{u}^*)) - \operatorname{div}((\phi - \phi^*)\boldsymbol{w}^*_{\boldsymbol{r}}) \\
f^*_{\mathsf{h}} &:= -\operatorname{div}((\zeta^*_{\boldsymbol{r}} - l_{\mathsf{h}}\psi^*_{\boldsymbol{r}})(\boldsymbol{u} - \boldsymbol{u}^*)) - \operatorname{div}((\theta - l_{\mathsf{c}}\phi - \theta^* + l_{\mathsf{c}}\phi^*)\boldsymbol{w}^*_{\boldsymbol{r}}) \\
\boldsymbol{f}^*_{\mathsf{v}} &:= -\operatorname{div}(\boldsymbol{w}^*_{\boldsymbol{r}} \otimes (\boldsymbol{u} - \boldsymbol{u}^*)) - \operatorname{div}((\boldsymbol{u} - \boldsymbol{u}^*) \otimes \boldsymbol{w}^*_{\boldsymbol{r}}) \\
&\quad + \mathcal{K}(\xi^*_{\boldsymbol{r}} - l_{\mathsf{c}}\zeta^*_{\boldsymbol{r}})\nabla(\phi - \phi^*) + \mathcal{K}(\mu - l_{\mathsf{c}}\theta - \mu^* + l_{\mathsf{c}}\theta^*)\nabla\psi^*_{\boldsymbol{r}} \\
f^*_{\mathsf{c}} &:= 3\beta_0(\phi + \phi^*)(\phi - \phi^*)\psi^*_{\boldsymbol{r}}
\end{aligned}$$

and $(\phi, \theta, \boldsymbol{u}, \mu) = \mathbf{H}(\mathbf{I}(\boldsymbol{s})) = \mathbf{F}(\boldsymbol{s})$. From [43], Corollaries 4.3 and 4.5, we have

$$\boldsymbol{f}^*_{\boldsymbol{s}} \in L^2(I; W^{-1,2}(\Omega)) \times L^2(I; W^{-1,2}(\Omega)) \times L^2(I; \boldsymbol{W}^{-1,2}(\Omega)) \times L^2(I; W^{1,2}_0(\Omega)).$$

Thus, it holds that $(\psi, \zeta, \boldsymbol{w}, \xi) \in \mathcal{Z}^3_{2,2}(Q) \times \mathcal{Z}^1_{2,2}(Q) \times \boldsymbol{\mathcal{V}}^1_{2,2}(Q) \times L^2(I; W^{1,2}_0(\Omega))$ having the associated pressure $\varpi \in L^2(I; \widehat{W}^{1,2}(\Omega))$ by Theorem 3.3 and [43], Theorem 4.9.

Let $\mathcal{Z}^1_{2,2,T}(Q) := \{g_{\mathsf{c}} \in \mathcal{Z}^1_{2,2}(Q) : g_{\mathsf{c}}(T) = 0\}$ and

$$\boldsymbol{\mathcal{G}}^2_{2,T}(Q) := L^2(I; W^{-1,2}(\Omega)) \times L^2(I; L^2(\Omega)) \times L^2(I; \boldsymbol{L}^2(\Omega)) \times \mathcal{Z}^1_{2,2,T}(Q)$$

considered as subspaces of $\mathcal{Z}^1_{2,2}(Q)$ and $\boldsymbol{\mathcal{G}}^2_2(Q)$, respectively. The above time-evaluation is well-defined thanks to $\mathcal{Z}^1_{2,2}(Q) \hookrightarrow C(\bar{I}; L^2(\Omega))$. Suppose that $(g_{\mathsf{o}}, g_{\mathsf{h}}, \boldsymbol{g}_{\mathsf{v}}, g_{\mathsf{c}}) \in \boldsymbol{\mathcal{G}}^2_{2,T}(Q)$ and let $(\varphi, \vartheta, \boldsymbol{v}, \eta) \in \boldsymbol{\mathcal{Y}}^2_2(Q)$ be the solution of the adjoint system (3.8) corresponding to these source functions, see Theorem 3.7. Since $\Delta\varphi(T) = 0$, $\boldsymbol{v}(T) = 0$, and $g_{\mathsf{c}}(T) = 0$, we have

$$\eta(T) = m\Delta\varphi(T) + \mathcal{K}\boldsymbol{v}(T)\cdot\nabla\phi(T) + g_{\mathsf{c}}(T) = 0. \tag{4.17}$$

Furthermore, $(\varphi, \vartheta, \boldsymbol{v}, \eta)$ enjoys the estimate

$$\|(\varphi, \vartheta, \boldsymbol{v}, \eta)\|_{\boldsymbol{\mathcal{Y}}_2^2(Q)} \le c_{F_\gamma}\|(g_{\mathsf{o}}, g_{\mathsf{h}}, \boldsymbol{g}_{\mathsf{v}}, g_{\mathsf{c}})\|_{\boldsymbol{\mathcal{G}}^2_{2,T}(Q)}, \tag{4.18}$$

where $c_{F_\gamma} > 0$ is a constant depending on F_γ.

Note that the solution $(\varphi, \vartheta, \boldsymbol{v}, \eta)$ of the adjoint system can be used as a test function to the linearized system satisfied by $(\psi, \zeta, \boldsymbol{w}, \xi)$. Similarly, the solution $(\psi, \zeta, \boldsymbol{w}, \xi)$ of the linearized system can also be used as a test function to the adjoint system. Integrating by parts, using the boundary conditions and the vanishing terminal and initial conditions for the adjoint and linearized systems, respectively, lead to the equation

$$\begin{aligned}\int_0^T &\langle g_{\mathsf{o}}, \psi\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)} + (g_{\mathsf{h}}, \zeta)_{L^2(\Omega)} + (\boldsymbol{g}_{\mathsf{v}}, \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} + (g_{\mathsf{c}}, \xi)_{L^2(\Omega)}\,\mathrm{d}t \\ &= \int_0^T \langle f_{\mathsf{o}}^*, \varphi\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}\,\mathrm{d}t + \int_0^T \langle f_{\mathsf{h}}^*, \vartheta\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}\,\mathrm{d}t \\ &+ \int_0^T \langle \boldsymbol{f}_{\mathsf{v}}^*, \boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,2}(\Omega), \boldsymbol{W}_0^{1,2}(\Omega)}\,\mathrm{d}t + \int_0^T (f_{\mathsf{c}}^*, \eta)_{L^2(\Omega)}\,\mathrm{d}t. \end{aligned} \tag{4.19}$$

For the sake of the reader, the details are provided in the Appendix.

Applications of Hölder's inequality to the right-hand sides of (4.19) yield the following estimates

$$\begin{aligned}\int_0^T \langle f_{\mathsf{o}}^*, \varphi\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}\,\mathrm{d}t &\le c\|\boldsymbol{u} - \boldsymbol{u}^*\|_{L^4(I;\boldsymbol{L}^4(\Omega))}\|\psi_{\boldsymbol{r}}^*\|_{L^4(I;L^2(\Omega))}\|\varphi\|_{L^2(I;W_0^{1,4}(\Omega))} \\ &+ c\|\boldsymbol{w}_{\boldsymbol{r}}^*\|_{L^2(I;\boldsymbol{L}^2(\Omega))}\|\phi - \phi^*\|_{L^\infty(I;L^\infty(\Omega))}\|\varphi\|_{L^2(I;W_0^{1,2}(\Omega))} \end{aligned} \tag{4.20}$$

$$\begin{aligned}\int_0^T \langle f_{\mathsf{h}}^*, \vartheta\rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)}\,\mathrm{d}t &\le c\|\boldsymbol{u} - \boldsymbol{u}^*\|_{L^4(I;\boldsymbol{L}^4(\Omega))}\{\|\zeta_{\boldsymbol{r}}^*\|_{L^2(I;L^2(\Omega))} + \|\psi_{\boldsymbol{r}}^*\|_{L^2(I;L^2(\Omega))}\}\|\vartheta\|_{L^4(I;W_0^{1,4}(\Omega))} \\ &+ c\|\boldsymbol{w}_{\boldsymbol{r}}^*\|_{L^2(I;\boldsymbol{L}^2(\Omega))}\{\|\theta - \theta^*\|_{L^4(I;L^4(\Omega))} + \|\phi - \phi^*\|_{L^4(I;L^4(\Omega))}\}\|\vartheta\|_{L^4(I;W_0^{1,4}(\Omega))} \end{aligned} \tag{4.21}$$

$$\begin{aligned}\int_0^T \langle \boldsymbol{f}_{\mathsf{v}}^*, \boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,2}(\Omega), \boldsymbol{W}_0^{1,2}(\Omega)}\,\mathrm{d}t &\le c\|\boldsymbol{u} - \boldsymbol{u}^*\|_{L^4(I;\boldsymbol{L}^4(\Omega))}\|\boldsymbol{w}_{\boldsymbol{r}}^*\|_{L^2(I;\boldsymbol{L}^2(\Omega))}\|\boldsymbol{v}\|_{L^4(I;\boldsymbol{W}_0^{1,4}(\Omega))} \\ &+ c\{\|\xi_{\boldsymbol{r}}^*\|_{L^2(I;L^2(\Omega))} + \|\zeta_{\boldsymbol{r}}^*\|_{L^2(I;L^2(\Omega))}\}\|\phi - \phi^*\|_{L^2(I;W_0^{1,4}(\Omega))}\|\boldsymbol{v}\|_{L^\infty(I;\boldsymbol{L}^4(\Omega))} \\ &+ c\{\|\mu - \mu^*\|_{L^2(I;L^4(\Omega))} + \|\theta - \theta^*\|_{L^2(I;L^4(\Omega))}\}\|\psi_{\boldsymbol{r}}^*\|_{L^2(I;W_0^{1,2}(\Omega))}\|\boldsymbol{v}\|_{L^\infty(I;\boldsymbol{L}^4(\Omega))} \end{aligned} \tag{4.22}$$

$$\int_0^T (f_{\mathsf{c}}^*, \eta)_{L^2(\Omega)}\,\mathrm{d}t \le c\|(\phi + \phi^*)(\phi - \phi^*)\|_{L^\infty(I;L^4(\Omega))}\|\psi_{\boldsymbol{r}}^*\|_{L^2(I;L^4(\Omega))}\|\eta\|_{L^2(I;L^2(\Omega))}. \tag{4.23}$$

The first norm on the right-hand side of (4.23) can be bounded from above by

$$\|(\phi + \phi^*)(\phi - \phi^*)\|_{L^\infty(I;L^4(\Omega))} \le c\{\|\phi\|_{L^\infty(I;L^8(\Omega))} + \|\phi^*\|_{L^\infty(I;L^8(\Omega))}\}\|\phi - \phi^*\|_{L^\infty(I;L^8(\Omega))}. \tag{4.24}$$

The preceding inequalities (4.20)–(4.24), along with the various continuous embeddings presented in the proof of Theorem 3.7 and

$$(g_{\mathsf{c}}, \xi)_{L^2(Q)} = \langle \xi, g_{\mathsf{c}}\rangle_{\mathcal{Z}^1_{2,2,T}(Q)', \mathcal{Z}^1_{2,2,T}(Q)},$$

when applied to (4.19) provide us the estimate

$$\begin{aligned}
&\int_0^T \langle g_{\mathsf{o}},\psi\rangle_{W^{-1,2}(\Omega),W_0^{1,2}(\Omega)} + (g_{\mathsf{h}},\zeta)_{L^2(\Omega)} + (\boldsymbol{g}_{\mathsf{v}},\boldsymbol{w})_{\boldsymbol{L}^2(\Omega)}\,\mathrm{d}t + \langle \xi, g_{\mathsf{c}}\rangle_{\mathcal{Z}^1_{2,2,T}(Q)',\mathcal{Z}^1_{2,2,T}(Q)}\\
&\quad\le c\|\mathsf{F}(\boldsymbol{s})-\mathsf{F}(\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)}\|(\psi^*_{\boldsymbol{r}},\zeta^*_{\boldsymbol{r}},\boldsymbol{w}^*_{\boldsymbol{r}},\xi^*_{\boldsymbol{r}})\|_{\boldsymbol{\mathcal{T}}^2_2(Q)}\|(\varphi,\vartheta,\boldsymbol{v},\eta)\|_{\boldsymbol{\mathcal{Y}}^2_2(Q)}\\
&\quad\le cc_0c_{F_\gamma}\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)}\|\mathrm{D}\mathsf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)}\|(g_{\mathsf{o}},g_{\mathsf{h}},\boldsymbol{g}_{\mathsf{v}},g_{\mathsf{c}})\|_{\boldsymbol{\mathcal{G}}^2_{2,T}(Q)}.
\end{aligned}$$

We used Lemma 4.6 and (4.18) in the last inequality. Let $\widetilde{c} := cc_0c_{F_\gamma}$. By duality and the definition of $\boldsymbol{\mathcal{G}}^2_{2,T}(Q)$, this implies

$$\|(\psi,\zeta,\boldsymbol{w},\xi)\|_{L^2(I;W_0^{1,2}(\Omega))\times L^2(I;L^2(\Omega))\times L^2(I;\boldsymbol{L}^2(\Omega))\times\mathcal{Z}^1_{2,2,T}(Q)'} \le \widetilde{c}\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)}\|\mathrm{D}\mathsf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)}. \tag{4.25}$$

From (4.20), (4.23), and (4.24), we can also see that for some constant $c>0$, we have

$$\|f^*_{\mathsf{o}}\|_{L^2(I;X^{-2,2}(\Omega))} + \|f^*_{\mathsf{c}}\|_{L^2(I;L^2(\Omega))} \le c\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)}\|\mathrm{D}\mathsf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)}. \tag{4.26}$$

It remains to establish the estimate for ψ in $\mathcal{Z}^1_{2,2}(Q)$ and ξ in $L^2(I;L^2(\Omega))$. Using Lemma 4.8 below for the estimates of the very weak solution to a linearized viscous Cahn–Hilliard system with source functions $f = f^*_{\mathsf{o}} - \mathrm{div}\,(\phi\boldsymbol{w})$ and $g = l_{\mathsf{c}}\zeta + f^*_{\mathsf{c}}$, we deduce that

$$\|\psi\|_{\mathcal{Z}^2_{2,2}(Q)} + \|\xi\|_{L^2(I;L^2(\Omega))} \le c_{F_\gamma}(\|f^*_{\mathsf{o}}\|_{L^2(I;X^{-2,2}(\Omega))} + \|f^*_{\mathsf{c}}\|_{L^2(I;L^2(\Omega))} + \|\boldsymbol{w}\|_{L^2(I;\boldsymbol{L}^2(\Omega))} + \|\zeta\|_{L^2(I;L^2(\Omega))}).$$

Here, we used the inequality

$$\|\mathrm{div}\,(\phi\boldsymbol{w})\|_{L^2(I;X^{-2,2}(\Omega))} \le c\|\phi\|_{L^\infty(I;L^4(\Omega))}\|\boldsymbol{w}\|_{L^2(I;\boldsymbol{L}^2(\Omega))}.$$

By applying the previous estimates (4.25) and (4.26), one has

$$\|\psi\|_{\mathcal{Z}^2_{2,2}(Q)} + \|\xi\|_{L^2(I;L^2(\Omega))} \le c\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)}\|\mathrm{D}\mathsf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)}. \tag{4.27}$$

Taking the sum of (4.25) and (4.27), we deduce (4.15).

The estimate (4.16) follows from (4.15), the triangle inequality, and the fact that $\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)} \le c\gamma$ for some $c>0$. In particular, one can take $c_2 = c_1c\gamma + 1$. □

Let us prove the following lemma utilized in the preceding proof.

Lemma 4.8. *Suppose that $f\in L^2(I;X^{-2,2}(\Omega))$, $g\in L^2(I;L^2(\Omega))$, $\phi\in L^4(I;L^6(\Omega))$, and $\boldsymbol{u}\in L^2(I;\boldsymbol{L}^2_\sigma(\Omega))$. Then, the linear system*

$$\left[\begin{array}{ll}
\partial_t\psi + \mathrm{div}\,(\psi\boldsymbol{u}) - m\Delta\xi = f & \text{in } Q,\\
\xi = \tau\partial_t\psi - \epsilon\Delta\psi + F'(\phi)\psi + g & \text{in } Q,\\
\psi = \Delta\psi = 0 \quad \text{on } \Sigma, \quad \psi(0) = 0 & \text{in } \Omega.
\end{array}\right. \tag{4.28}$$

admits a unique very weak solution $(\psi,\xi)\in\mathcal{Z}^2_{2,2}(Q)\times L^2(I;L^2(\Omega))$. Moreover, there exists a continuous function $\mathscr{C}:[0,\infty)\to[0,\infty)$ such that

$$\|\psi\|_{\mathcal{Z}^2_{2,2}(Q)} + \|\xi\|_{L^2(I;L^2(\Omega))} \le \mathscr{C}(\|(\phi,\boldsymbol{u})\|_{L^4(I;L^6(\Omega))\times L^2(I;\boldsymbol{L}^2_\sigma(\Omega))})(\|f\|_{L^2(I;X^{-2,2}(\Omega))} + \|g\|_{L^2(I;L^2(\Omega))}). \tag{4.29}$$

Proof. We only formally derive the *a priori* estimates needed for the Faedo–Galerkin approach. Taking the test function ψ to the first equation in (4.28), substituting the formula for ξ, and noting that $(\boldsymbol{u}\cdot\nabla\psi,\psi)_{L^2(\Omega)}=0$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|\psi\|_{L^2(\Omega)}+m\tau\|\nabla\psi\|^2_{\boldsymbol{L}^2(\Omega)}\}+m\epsilon\|\Delta\psi\|^2_{L^2(\Omega)}=m(F'(\phi)\psi+g,\Delta\psi)_{L^2(\Omega)}+\langle f,\psi\rangle_{X^{-2,2}(\Omega),X^{2,2}(\Omega)}. \quad (4.30)$$

With the Hölder and Young inequalities, and recalling $F'(\phi)=3\beta_0\phi^2-\beta_1$, we can estimate the terms on the right-hand side by

$$\begin{aligned}|\langle f,\psi\rangle_{X^{-2,2}(\Omega),X^{2,2}(\Omega)}|&\le\frac{m\epsilon}{4}\|\Delta\psi\|^2_{L^2(\Omega)}+c\|f\|^2_{X^{-2,2}(\Omega)}\\|m(F'(\phi)\psi+g,\Delta\psi)_{L^2(\Omega)}|&\le\frac{m\epsilon}{4}\|\Delta\psi\|^2_{L^2(\Omega)}+c\{\|g\|^2_{L^2(\Omega)}+(\|\phi\|^4_{L^6(\Omega)}+1)\|\psi\|^2_{L^6(\Omega)}\}.\end{aligned}$$

Substituting these in (4.30), using the Sobolev embedding $W^{1,2}_0(\Omega)\hookrightarrow L^6(\Omega)$ and the Poincaré inequality, we obtain the differential inequality

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{\|\psi\|_{L^2(\Omega)}+m\tau\|\nabla\psi\|^2_{\boldsymbol{L}^2(\Omega)}\}+\frac{m\epsilon}{2}\|\Delta\psi\|^2_{L^2(\Omega)}\\&\qquad\le c\{\|f\|^2_{X^{-2,2}(\Omega)}+\|g\|^2_{L^2(\Omega)}+(\|\phi\|^4_{L^6(\Omega)}+1)\|\nabla\psi\|^2_{\boldsymbol{L}^2(\Omega)}\}.\end{aligned} \quad (4.31)$$

Next, by using the test function $-\Delta^{-1}\partial_t\psi$ to the first equation in (4.28) and again substituting the formula for ξ, we have

$$\begin{aligned}&\|(-\Delta)^{-\frac{1}{2}}\partial_t\psi\|^2_{L^2(\Omega)}+\frac{m\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\psi\|^2_{\boldsymbol{L}^2(\Omega)}+m\tau\|\partial_t\psi\|^2_{L^2(\Omega)}+(\psi\boldsymbol{u},\nabla\Delta^{-1}\partial_t\psi)_{\boldsymbol{L}^2(\Omega)}\\&\qquad=-m(F'(\phi)\psi+g,\partial_t\psi)_{L^2(\Omega)}-\langle f,\Delta^{-1}\partial_t\psi\rangle_{X^{-2,2}(\Omega),X^{2,2}(\Omega)}.\end{aligned} \quad (4.32)$$

Note that $(-\Delta)^{-\frac{1}{2}}\in\mathcal{L}(L^2(\Omega),W^{1,2}_0(\Omega))$, where $-\Delta$ denotes the Dirichlet Laplacian. The inner products in this equation obey the following estimates

$$\begin{aligned}|\langle f,\Delta^{-1}\partial_t\psi\rangle_{X^{-2,2}(\Omega),X^{2,2}(\Omega)}|&\le\frac{m\tau}{6}\|\partial_t\psi\|^2_{L^2(\Omega)}+c\|f\|^2_{X^{-2,2}(\Omega)}\\|m(F'(\phi)\psi+g,\partial_t\psi)_{L^2(\Omega)}|&\le\frac{m\tau}{6}\|\partial_t\psi\|^2_{L^2(\Omega)}+c\{\|g\|^2_{L^2(\Omega)}+(\|\phi\|^4_{L^6(\Omega)}+1)\|\psi\|^2_{L^6(\Omega)}\}\\|(\psi\boldsymbol{u},\nabla\Delta^{-1}\partial_t\psi)_{\boldsymbol{L}^2(\Omega)}|&\le\frac{m\tau}{6}\|\partial_t\psi\|^2_{L^2(\Omega)}+c\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2_\sigma(\Omega)}\|\psi\|^2_{L^4(\Omega)}\end{aligned}$$

where we used the fact that $\|\nabla\Delta^{-1}\partial_t\psi\|_{\boldsymbol{L}^4(\Omega)}\le c\|\Delta^{-1}\partial_t\psi\|_{X^{2,2}(\Omega)}=c\|\partial_t\psi\|_{L^2(\Omega)}$ in the last inequality. Plugging these inequalities in (4.32) yields

$$\begin{aligned}&\frac{m\epsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\psi\|^2_{\boldsymbol{L}^2(\Omega)}+\frac{m\tau}{2}\|\partial_t\psi\|^2_{L^2(\Omega)}\\&\qquad\le c\{\|f\|^2_{X^{-2,2}(\Omega)}+\|g\|^2_{L^2(\Omega)}+(\|\phi\|^4_{L^6(\Omega)}+\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2_\sigma(\Omega)}+1)\|\nabla\psi\|^2_{\boldsymbol{L}^2(\Omega)}\}.\end{aligned} \quad (4.33)$$

Getting the sum of and (4.33) and (4.31), and then invoking the Gronwall Lemma, one has (4.29), but without the term ξ, *i.e.*

$$\|\psi\|_{\mathcal{Z}^2_{2,2}(Q)}\le\mathscr{C}(\|(\phi,\boldsymbol{u})\|_{L^4(I;L^6(\Omega))\times L^2(I;\boldsymbol{L}^2_\sigma(\Omega))})(\|f\|_{L^2(I;X^{-2,2}(\Omega))}+\|g\|_{L^2(I;L^2(\Omega))}).$$

However, using this estimate in the second equation of (4.28) will give us the following estimate for ξ

$$\|\xi\|_{L^2(I;L^2(\Omega))} \le c\{\|\psi\|_{\mathcal{Z}^2_{2,2}(Q)} + (\|\phi\|^2_{L^4(I;L^6(\Omega))} + 1)\|\psi\|_{L^2(I;W^{1,2}(\Omega))} + \|g\|_{L^2(I;L^2(\Omega))}\}.$$

This completes the proof of the derivation of the *a priori* estimate (4.29). □

We shall denote the closed ball in $\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)$ with center $\boldsymbol{s}$ and radius ε_0 by $\boldsymbol{\mathcal{B}}^r_{\varepsilon_0}(\boldsymbol{s})$. The succeeding lemma is concerned with the distance between the values of the control-to-state operator and its first-order approximation around a local solution, and the norm is taken with respect to $\boldsymbol{\mathcal{T}}^2_2(Q)$.

By ignoring the last three terms on the right-hand side of the second equation in the linearized system (3.3), we see that ξ and $\tau\partial_t\psi - \epsilon\Delta\psi$ must have the same regularity. Thus, if $\xi \in L^2(I;L^2(\Omega))$, then $\psi \in \mathcal{Z}^2_{2,2}(Q)$, which follows from the classical regularity theory for the heat operator. This is the motivation for the use of the function space $\boldsymbol{\mathcal{T}}^2_2(Q)$ in relation to the order parameter and chemical potential.

Lemma 4.9. *There exists $\varepsilon_0 > 0$ such that for every $\boldsymbol{s} \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}} \cap \boldsymbol{\mathcal{B}}^r_{\varepsilon_0}(\boldsymbol{s}^*)$ we have*

$$\|\mathbf{F}(\boldsymbol{s}) - \mathbf{F}(\boldsymbol{s}^*) - \mathrm{D}\mathbf{F}(\boldsymbol{s}^*)(\boldsymbol{s}-\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{T}}^2_2(Q)} \le \|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)(\boldsymbol{s}-\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{T}}^2_2(Q)}. \tag{4.34}$$

Proof. Let $(\phi,\theta,\boldsymbol{u},\mu) = \mathbf{F}(\boldsymbol{s})$ and recall that $(\phi^*,\theta^*,\boldsymbol{u}^*,\mu^*) = \mathbf{F}(\boldsymbol{s}^*)$. Consider

$$(\psi,\zeta,\boldsymbol{w},\xi) := \mathbf{F}(\boldsymbol{s}) - \mathbf{F}(\boldsymbol{s}^*) - \mathrm{D}\mathbf{F}(\boldsymbol{s}^*)(\boldsymbol{s}-\boldsymbol{s}^*). \tag{4.35}$$

It can be checked that $(\psi,\zeta,\boldsymbol{w},\xi) = \mathrm{D}\mathbf{H}(\mathbf{I}(\boldsymbol{s}^*))\boldsymbol{f}^*_{\boldsymbol{s}}$, where $\boldsymbol{f}^*_{\boldsymbol{s}} = (f^*_{\mathsf{o}}, f^*_{\mathsf{h}}, \boldsymbol{f}^*_{\mathsf{v}}, f^*_{\mathsf{c}})$ is given by

$$\begin{aligned}
f^*_{\mathsf{o}} &:= -\operatorname{div}((\phi-\phi^*)(\boldsymbol{u}-\boldsymbol{u}^*))\\
f^*_{\mathsf{h}} &:= -\operatorname{div}((\theta - l_{\mathsf{c}}\phi - \theta^* + l_{\mathsf{c}}\phi^*)(\boldsymbol{u}-\boldsymbol{u}^*))\\
\boldsymbol{f}^*_{\mathsf{v}} &:= -\operatorname{div}((\boldsymbol{u}-\boldsymbol{u}^*)\otimes(\boldsymbol{u}-\boldsymbol{u}^*)) + \mathcal{K}(\mu - l_{\mathsf{c}}\theta - \mu^* + l_{\mathsf{c}}\theta^*)\nabla(\phi-\phi^*)\\
f^*_{\mathsf{c}} &:= F(\phi) - F(\phi^*) - F'(\phi^*)(\phi-\phi^*).
\end{aligned}$$

Since F is a cubic polynomial, we deduce that $f^*_{\mathsf{c}} = 6\beta_0(\phi^*(\phi-\phi^*)^2 + (\phi-\phi^*)^3)$. We proceed with the same duality argument as in the proof of Lemma 4.7. First, we deduce the following estimates:

$$\begin{aligned}
&\int_0^T \langle f^*_{\mathsf{o}}, \varphi\rangle_{W^{-1,2}(\Omega),W^{1,2}_0(\Omega)}\,\mathrm{d}t \le c\|\phi-\phi^*\|_{L^\infty(I;L^4(\Omega))}\|\boldsymbol{u}-\boldsymbol{u}^*\|_{L^2(I;\boldsymbol{L}^2(\Omega))}\|\varphi\|_{L^2(I;W^{1,4}_0(\Omega))}\\
&\int_0^T \langle f^*_{\mathsf{h}}, \vartheta\rangle_{W^{-1,2}(\Omega),W^{1,2}_0(\Omega)}\,\mathrm{d}t \le c\{\|\phi-\phi^*\|_{L^4(I;L^4(\Omega))} + \|\theta-\theta^*\|_{L^4(I;L^4(\Omega))}\}\|\boldsymbol{u}-\boldsymbol{u}^*\|_{L^2(I;\boldsymbol{L}^2(\Omega))}\|\vartheta\|_{L^4(I;W^{1,4}_0(\Omega))}\\
&\int_0^T \langle \boldsymbol{f}^*_{\mathsf{v}}, \boldsymbol{v}\rangle_{\boldsymbol{W}^{-1,2}(\Omega),\boldsymbol{W}^{1,2}_0(\Omega)}\,\mathrm{d}t \le c\|\boldsymbol{u}-\boldsymbol{u}^*\|_{L^4(I;\boldsymbol{L}^4(\Omega))}\|\boldsymbol{u}-\boldsymbol{u}^*\|_{L^2(I;\boldsymbol{L}^2(\Omega))}\|\boldsymbol{v}\|_{L^4(I;\boldsymbol{W}^{1,4}_0(\Omega))}\\
&\qquad + c\{\|\mu-\mu^*\|_{L^2(I;L^2(\Omega))} + \|\theta-\theta^*\|_{L^2(I;L^2(\Omega))}\}\|\phi-\phi^*\|_{L^2(I;W^{1,4}_0(\Omega))}\|\boldsymbol{v}\|_{L^\infty(I;\boldsymbol{L}^4(\Omega))}\\
&\int_0^T (f^*_{\mathsf{c}}, \eta)_{L^2(\Omega)}\,\mathrm{d}t \le c\{\|\phi^*\|_{L^\infty(I;L^6(\Omega))} + \|\phi-\phi^*\|_{L^\infty(I;L^6(\Omega))}\}\|\phi-\phi^*\|^2_{L^4(I;L^6(\Omega))}\|\eta\|_{L^2(I;L^2(\Omega))}.
\end{aligned}$$

As before, these estimates and the one that can be obtained from Lemma 4.8 give us

$$\begin{aligned}
\|(\psi,\zeta,\boldsymbol{w},\xi)\|_{\boldsymbol{\mathcal{T}}^2_2(Q)} &\le c_{F_\gamma}\|\mathbf{F}(\boldsymbol{s}) - \mathbf{F}(\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)}\|\mathbf{F}(\boldsymbol{s}) - \mathbf{F}(\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{T}}^2_2(Q)}\\
&\le c_{F_\gamma}\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)}\{\|(\psi,\zeta,\boldsymbol{w},\xi)\|_{\boldsymbol{\mathcal{T}}^2_2(Q)} + \|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)(\boldsymbol{s}-\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{T}}^2_2(Q)}\}.
\end{aligned}$$

The last inequality is due to Lemma 4.6, (4.35), and the triangle inequality. Choosing $\varepsilon_0 > 0$ such that $c_{F_\gamma}\varepsilon_0/(1-c_{F_\gamma}\varepsilon_0) \leq 1$ proves (4.34). $\square$

The next lemma deals with additional integrability for the weak solutions of the state system. We refer the reader to Section 3.1 for the definition of H.

Lemma 4.10. *Let (3.11) with $r > 8$ holds and suppose that $\boldsymbol{f} = (f_{\mathsf{o}}, f_{\mathsf{h}}, \boldsymbol{f}_{\mathsf{v}}, f_{\mathsf{c}}) \in \boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)$ and $(\phi_0, \theta_0, \boldsymbol{u}_0) \in \boldsymbol{\mathcal{D}}^{r,r/2}_{q,s,p}(\Omega)$. Then,*

$$\mathsf{H} \in C^\infty(\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q), \boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)).$$

In particular, $\mathsf{F} \in C^\infty(\boldsymbol{\mathcal{M}}^r, \boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q))$.

Proof. The map $\mathsf{H} : \boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q) \to \boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)$ is well-defined according to [43], Theorem 6.4. Define the operator

$$\mathsf{E} : \boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q) \times \boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q) \to \boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q) \times \boldsymbol{\mathcal{D}}^{r,r/2}_{q,s,p}(\Omega)$$

according to

$$\begin{aligned}
&\mathsf{E}((\phi, \theta, \boldsymbol{u}, \mu), (\widetilde{f}_{\mathsf{o}}, \widetilde{f}_{\mathsf{h}}, \widetilde{\boldsymbol{f}}_{\mathsf{v}}, \widetilde{f}_{\mathsf{c}})) = (\boldsymbol{F}, \boldsymbol{d})\\
&\boldsymbol{F} := \begin{pmatrix} \partial_t \phi + mA\mu + \operatorname{div}(\phi \boldsymbol{u}) - \widetilde{f}_{\mathsf{o}} \\ \partial_t(\theta - l_{\mathsf{h}}\phi) + \operatorname{div}((\theta - l_{\mathsf{h}}\phi)\boldsymbol{u}) + \kappa A\theta - \alpha \boldsymbol{g}\cdot \boldsymbol{u} - \widetilde{f}_{\mathsf{h}} \\ \partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nu \boldsymbol{A}\boldsymbol{u} - \ell(\phi, \theta)\boldsymbol{g} - \mathcal{K}(\mu - l_{\mathsf{c}}\theta)\nabla\phi - \widetilde{\boldsymbol{f}}_{\mathsf{v}} \\ \mu - \tau\partial_t\phi - \epsilon A\phi - l_{\mathsf{c}}\theta - F(\phi) - \widetilde{f}_{\mathsf{c}} \end{pmatrix}\\
&\boldsymbol{d} := \begin{pmatrix} \phi(0) - \phi_0 \\ \theta(0) - \theta_0 \\ \boldsymbol{u}(0) - \boldsymbol{u}_0 \end{pmatrix}.
\end{aligned}$$

Here, A and $\boldsymbol{A}$ are extensions of the Dirichlet Laplacian and Stokes operators associated with the weak formulation for the state system, see [43], Section 5 for the details. It can be checked that E is of class C^∞. Moreover, we have $\mathsf{E}(\mathsf{H}(\widetilde{\boldsymbol{f}}), \widetilde{\boldsymbol{f}}) = 0$ for $\widetilde{\boldsymbol{f}} \in \boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)$.

In virtue of the implicit function theorem [51], Section 4.7, to establish that $\mathsf{H} \in C^\infty(\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q), \boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q))$, it is enough to show that $\mathrm{D}\mathsf{H}(\widetilde{\boldsymbol{f}}) \in \mathcal{L}(\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q), {}_0\boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q))$ is an isomorphism for each $\widetilde{\boldsymbol{f}} \in \boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)$. We refer to (3.6) for the definition of the function space ${}_0\boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)$. Now, thanks to $\mathsf{F} = \mathsf{H} \circ \mathsf{I}$ and $\mathsf{I} \in C^\infty(\boldsymbol{\mathcal{M}}^r, \boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q))$, it will follow that $\mathsf{F} \in C^\infty(\boldsymbol{\mathcal{M}}^r, \boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q))$.

To this end, let $(\phi, \theta, \boldsymbol{u}, \mu) = \mathsf{H}(\widetilde{\boldsymbol{f}}) \in \boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)$ and $\mathbf{g} = (g_{\mathsf{o}}, g_{\mathsf{h}}, \boldsymbol{g}_{\mathsf{v}}, g_{\mathsf{c}}) \in \boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)$. Then, $(\psi, \zeta, \boldsymbol{w}, \xi) = \mathrm{D}\mathsf{H}(\widetilde{\boldsymbol{f}})\mathbf{g}$ satisfies the following linear system:

$$\left[\begin{array}{ll}
\partial_t\psi - m\Delta\xi = e_{\mathsf{o}} + g_{\mathsf{o}} & \text{in } Q,\\
\xi = \tau\partial_t\psi - \epsilon\Delta\psi - \beta_1\psi + l_{\mathsf{c}}\zeta + e_{\mathsf{c}} + g_{\mathsf{c}} & \text{in } Q,\\
\partial_t\zeta - l_{\mathsf{h}}\partial_t\psi - \kappa\Delta\zeta = \alpha\boldsymbol{g}\cdot\boldsymbol{w} + e_{\mathsf{h}} + g_{\mathsf{h}} & \text{in } Q,\\
\partial_t\boldsymbol{w} - \nu\Delta\boldsymbol{w} + \nabla\varpi = (\alpha_1\psi + \alpha_2\zeta)\boldsymbol{g} + \mathbf{e}_{\mathsf{v}} + \boldsymbol{g}_{\mathsf{v}} & \text{in } Q,\\
\operatorname{div}\boldsymbol{w} = 0 & \text{in } Q,\\
\psi = \Delta\psi = 0, \quad \zeta = 0, \quad \boldsymbol{w} = \mathbf{0} & \text{on } \Sigma,\\
\psi(0) = 0, \quad \zeta(0) = 0, \quad \boldsymbol{w}(0) = \mathbf{0} & \text{in } \Omega,
\end{array}\right. \tag{4.36}$$

where $\mathbf{e} = (e_{\mathsf{o}}, e_{\mathsf{h}}, \mathbf{e}_{\mathsf{v}}, e_{\mathsf{c}})$ has the components

$$\begin{aligned}
e_{\mathsf{o}} &:= -\operatorname{div}(\psi \boldsymbol{u}) - \operatorname{div}(\phi \boldsymbol{w})\\
e_{\mathsf{h}} &:= -\operatorname{div}((\zeta - l_{\mathsf{h}}\psi)\boldsymbol{u}) - \operatorname{div}((\theta - l_{\mathsf{h}}\phi)\boldsymbol{w})\\
\mathbf{e}_{\mathsf{v}} &:= -\operatorname{div}(\boldsymbol{w} \otimes \boldsymbol{u}) - \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{w}) + \mathcal{K}(\xi - l_{\mathsf{c}}\zeta)\nabla\phi + \mathcal{K}(\mu - l_{\mathsf{c}}\theta)\nabla\psi\\
e_{\mathsf{c}} &:= 3\beta_0\phi^2\psi.
\end{aligned}$$

With regard to the components of the source vector $\mathbf{e}$, one can utilize Hölder's inequality and the Sobolev embedding to deduce the following estimates:

$$\|e_{\mathsf{o}}\|_{L^{r/2}(I;W^{-1,2}(\Omega))} \le c\{\|\psi\|_{L^r(I;L^4(\Omega))}\|\boldsymbol{u}\|_{L^r(I;\boldsymbol{L}^4(\Omega))} + \|\phi\|_{L^r(I;L^4(\Omega))}\|\boldsymbol{w}\|_{L^r(I;\boldsymbol{L}^4(\Omega))}\} \tag{4.37}$$

$$\begin{aligned}
\|e_{\mathsf{h}}\|_{L^{r/2}(I;W^{-1,2}(\Omega))} &\le c\{\|\zeta\|_{L^r(I;L^4(\Omega))} + \|\psi\|_{L^r(I;L^4(\Omega))}\}\|\boldsymbol{u}\|_{L^r(I;\boldsymbol{L}^4(\Omega))}\\
&\quad + c\{\|\theta\|_{L^r(I;L^4(\Omega))} + \|\phi\|_{L^r(I;L^4(\Omega))}\}\|\boldsymbol{w}\|_{L^r(I;\boldsymbol{L}^4(\Omega))}
\end{aligned} \tag{4.38}$$

$$\begin{aligned}
\|\mathbf{e}_{\mathsf{v}}\|_{L^{r/2}(I;\boldsymbol{W}^{-1,2}(\Omega))} &\le c\|\boldsymbol{u}\|_{L^r(I;\boldsymbol{L}^4(\Omega))}\|\boldsymbol{w}\|_{L^r(I;\boldsymbol{L}^4(\Omega))}\\
&\quad + c\{\|\xi\|_{L^{r/2}(I;L^4(\Omega))} + \|\zeta\|_{L^{r/2}(I;L^4(\Omega))}\}\|\phi\|_{L^\infty(I;W^{1,2}(\Omega))}\\
&\quad + c\{\|\mu\|_{L^{r/2}(I;L^4(\Omega))} + \|\theta\|_{L^{r/2}(I;L^4(\Omega))}\}\|\psi\|_{L^\infty(I;W^{1,2}(\Omega))}
\end{aligned} \tag{4.39}$$

$$\|e_{\mathsf{c}}\|_{L^{r/2}(I;W_0^{1,2}(\Omega))} \le c\|\phi\|^2_{L^r(I;W^{1,4}(\Omega))}\|\psi\|_{L^r(I;W^{2,4}(\Omega))}. \tag{4.40}$$

Let $N := \|(\phi, \theta, \boldsymbol{u}, \mu)\|_{\boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)}$ and c_N be a generic positive constant that depends on N. Replacing r by 4 in the estimates (4.37)–(4.40) and using (3.5), we have

$$\|\mathbf{e}\|_{\boldsymbol{\mathcal{N}}^2_{2,2,2}(Q)\times L^2(I;W_0^{1,2}(\Omega))} \le c_N\|(\psi, \zeta, \boldsymbol{w}, \xi)\|_{\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)} \tag{4.41}$$

$$\|(\psi, \zeta, \boldsymbol{w}, \xi)\|_{\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)} \le c\|\mathbf{g}\|_{\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)}. \tag{4.42}$$

We split the weak solution of (4.36) into two parts. Denote by $(\psi_{\mathbf{e}}, \zeta_{\mathbf{e}}, \boldsymbol{w}_{\mathbf{e}}, \xi_{\mathbf{e}}, \varpi_{\mathbf{e}})$ and $(\psi_{\mathbf{g}}, \zeta_{\mathbf{g}}, \boldsymbol{w}_{\mathbf{g}}, \xi_{\mathbf{g}}, \varpi_{\mathbf{g}})$ the weak solutions of (4.36) with $\mathbf{g}$ set to zero and $\mathbf{e}$ set to zero, respectively, so that $(\psi, \zeta, \boldsymbol{w}, \xi, \varpi) = (\psi_{\mathbf{e}}, \zeta_{\mathbf{e}}, \boldsymbol{w}_{\mathbf{e}}, \xi_{\mathbf{e}}, \varpi_{\mathbf{e}}) + (\psi_{\mathbf{g}}, \zeta_{\mathbf{g}}, \boldsymbol{w}_{\mathbf{g}}, \xi_{\mathbf{g}}, \varpi_{\mathbf{g}})$. Applying the extended maximal parabolic regularity for the linearized system in [43], Theorem 3.18, we obtain the following inequalities

$$\|(\psi_{\mathbf{g}}, \zeta_{\mathbf{g}}, \boldsymbol{w}_{\mathbf{g}}, \xi_{\mathbf{g}})\|_{\boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)} \le c\|\mathbf{g}\|_{\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)} \tag{4.43}$$

$$\|(\psi_{\mathbf{e}}, \zeta_{\mathbf{e}}, \boldsymbol{w}_{\mathbf{e}}, \xi_{\mathbf{e}})\|_{\mathcal{Z}^3_{2,2}(Q)\times\mathcal{Z}^1_{2,2}(Q)\times\boldsymbol{\mathcal{V}}^1_{2,2}(Q)\times L^2(I;W_0^{1,2}(\Omega))} \le c_N\|\mathbf{g}\|_{\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)}. \tag{4.44}$$

Here, we used (4.41) and (4.42) for the second inequality.

From the proof of [43], Theorem 6.2, the following compact embeddings hold

$$\mathcal{Z}^3_{2,r/2}(Q) \hookrightarrow\hookrightarrow L^r(I;W^{2,4}(\Omega)),\ \mathcal{Z}^1_{2,r/2}(Q) \hookrightarrow\hookrightarrow L^r(I;L^4(\Omega)),\ \boldsymbol{\mathcal{V}}^1_{2,r/2}(Q) \hookrightarrow\hookrightarrow L^r(I;\boldsymbol{L}^4(\Omega)).$$

In particular, this implies the continuous embedding

$$\boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q) \hookrightarrow L^r(I;W^{2,4}(\Omega)) \times L^r(I;L^4(\Omega)) \times L^r(I;\boldsymbol{L}^4(\Omega)) \times L^{r/2}(I;L^4(\Omega)). \tag{4.45}$$

The next step is to derive estimates for each component of $(\psi_{\mathbf{e}}, \zeta_{\mathbf{e}}, \boldsymbol{w}_{\mathbf{e}}, \xi_{\mathbf{e}})$ by applying maximal parabolic regularity results for the heat, viscous biharmonic heat, and Stokes equations. First, we start with an *enthalpy* *transformation* via $\beta_{\mathsf{e}} = \zeta_{\mathsf{e}} - l_{\mathsf{h}}\psi_{\mathsf{e}}$. Then, the linear system satisfied by $(\psi_{\mathsf{e}}, \zeta_{\mathsf{e}}, \boldsymbol{w}_{\mathsf{e}}, \xi_{\mathsf{e}})$ is equivalent to

$$\left[\begin{array}{ll} \partial_t\psi_{\mathsf{e}} - m\Delta\xi_{\mathsf{e}} = e_{\mathsf{o}} & \text{in } Q,\\ \xi_{\mathsf{e}} = \tau\partial_t\psi_{\mathsf{e}} - \epsilon\Delta\psi_{\mathsf{e}} - (\beta_1 - l_{\mathsf{c}}l_{\mathsf{h}})\psi_{\mathsf{e}} + l_{\mathsf{c}}\beta_{\mathsf{e}} + e_{\mathsf{c}} & \text{in } Q,\\ \partial_t\beta_{\mathsf{e}} - \kappa\Delta\beta_{\mathsf{e}} = \kappa l_{\mathsf{h}}\Delta\psi_{\mathsf{e}} + \alpha\boldsymbol{g}\cdot\boldsymbol{w}_{\mathsf{e}} + e_{\mathsf{h}} & \text{in } Q,\\ \partial_t\boldsymbol{w}_{\mathsf{e}} - \nu\Delta\boldsymbol{w}_{\mathsf{e}} + \nabla\varpi_{\mathsf{e}} = ((\alpha_1 + \alpha_2 l_{\mathsf{h}})\psi_{\mathsf{e}} + \alpha_2\beta_{\mathsf{e}})\boldsymbol{g} + \mathbf{e}_{\mathsf{v}} & \text{in } Q,\\ \operatorname{div}\boldsymbol{w}_{\mathsf{e}} = 0 & \text{in } Q,\\ \psi_{\mathsf{e}} = \Delta\psi_{\mathsf{e}} = 0, \quad \beta_{\mathsf{e}} = 0, \quad \boldsymbol{w}_{\mathsf{e}} = \mathbf{0} & \text{on } \Sigma,\\ \psi_{\mathsf{e}}(0) = 0, \quad \beta_{\mathsf{e}}(0) = 0, \quad \boldsymbol{w}_{\mathsf{e}}(0) = \mathbf{0} & \text{in } \Omega, \end{array}\right. \tag{4.46}$$

Applying the maximal parabolic regularity for the heat equation, see [43], Theorem 3.6 for example, we get

$$\|\beta_{\mathsf{e}}\|_{\mathcal{Z}^1_{2,r/2}(Q)} \le c\{\|\psi_{\mathsf{e}}\|_{L^{r/2}(I;W^{1,2}_0(\Omega))} + \|\boldsymbol{w}_{\mathsf{e}}\|_{L^{r/2}(I;\boldsymbol{L}^2(\Omega))} + \|e_{\mathsf{h}}\|_{L^{r/2}(I;W^{-1,2}(\Omega))}\}.$$

Using Lions Lemma [35], Theorem 16.4 to the function spaces $\mathcal{Z}^1_{2,r/2}(Q) \overset{c}{\hookrightarrow} L^r(I;L^4(\Omega)) \hookrightarrow L^2(I;L^2(\Omega))$ and $\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q) \overset{c}{\hookrightarrow} L^r(I;\boldsymbol{L}^4(\Omega)) \hookrightarrow L^2(I;\boldsymbol{L}^2(\Omega))$ in (4.38), one has

$$\begin{aligned}\|e_{\mathsf{h}}\|_{L^{r/2}(I;W^{-1,2}(\Omega))} &\le c\{c_\varepsilon\|\zeta\|_{L^2(I;L^2(\Omega))} + \varepsilon\|\zeta\|_{\mathcal{Z}^1_{2,r/2}(Q)} + \|\psi\|_{L^r(I;L^4(\Omega))}\}\|\boldsymbol{u}\|_{L^r(I;\boldsymbol{L}^4(\Omega))}\\ &\quad + c\{\|\theta\|_{L^r(I;L^4(\Omega))} + \|\phi\|_{L^r(I;L^4(\Omega))}\}\{c_\varepsilon\|\boldsymbol{w}\|_{L^2(I;\boldsymbol{L}^2(\Omega))} + \varepsilon\|\boldsymbol{w}\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)}\}.\end{aligned}$$

The last two inequalities along with (4.41), (4.42), and (4.45) give us

$$\|\beta_{\mathsf{e}}\|_{\mathcal{Z}^1_{2,r/2}(Q)} \le c_N\{c_\varepsilon\|\mathbf{g}\|_{\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)} + \varepsilon\|\zeta\|_{\mathcal{Z}^1_{2,r/2}(Q)} + \varepsilon\|\boldsymbol{w}\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)}\}. \tag{4.47}$$

From the maximal parabolic regularity for the viscous biharmonic heat equation, see [43], Theorem 3.11 for instance, we have

$$\|\psi_{\mathsf{e}}\|_{\mathcal{Z}^3_{2,r/2}(Q)} + \|\xi_{\mathsf{e}}\|_{L^{r/2}(I;W^{1,2}_0(\Omega))} \le c\{\|e_{\mathsf{o}}\|_{L^{r/2}(I;W^{-1,2}(\Omega))} + \|e_{\mathsf{c}}\|_{L^{r/2}(I;W^{1,2}_0(\Omega))} + \|\beta_{\mathsf{e}}\|_{L^{r/2}(I;W^{1,2}_0(\Omega))}\}.$$

Applying Lions Lemma to the function spaces $\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q) \overset{c}{\hookrightarrow} L^r(I;\boldsymbol{L}^4(\Omega)) \hookrightarrow L^2(I;\boldsymbol{L}^2(\Omega))$ and $\mathcal{Z}^3_{2,r/2}(Q) \overset{c}{\hookrightarrow} L^r(I;W^{2,4}(\Omega)) \hookrightarrow L^2(I;L^2(\Omega))$ in (4.37) and (4.40), respectively, we obtain

$$\begin{aligned}\|e_{\mathsf{o}}\|_{L^{r/2}(I;W^{-1,2}(\Omega))} &\le c\{\|\psi\|_{L^r(I;L^4(\Omega))}\|\boldsymbol{u}\|_{L^r(I;\boldsymbol{L}^4(\Omega))}\\ &\qquad\qquad + \|\phi\|_{L^r(I;L^4(\Omega))}(c_\varepsilon\|\boldsymbol{w}\|_{L^2(I;\boldsymbol{L}^2(\Omega))} + \varepsilon\|\boldsymbol{w}\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)})\}\\ \|e_{\mathsf{c}}\|_{L^{r/2}(I;W^{1,2}_0(\Omega))} &\le c\|\phi\|^2_{L^r(I;W^{1,4}(\Omega))}\{c_\varepsilon\|\psi\|_{L^2(I;L^2(\Omega))} + \varepsilon\|\psi\|_{\mathcal{Z}^3_{2,r/2}(Q)}\}.\end{aligned}$$

The last three inequalities, along with (4.42) and (4.45) provide us the estimate

$$\begin{aligned}&\|\psi_{\mathsf{e}}\|_{\mathcal{Z}^3_{2,r/2}(Q)} + \|\xi_{\mathsf{e}}\|_{L^{r/2}(I;W^{1,2}_0(\Omega))} - c_N\|\beta_{\mathsf{e}}\|_{\mathcal{Z}^1_{2,r/2}(Q)}\\ &\qquad \le c_N\{c_\varepsilon\|\mathbf{g}\|_{\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)} + \varepsilon\|\psi\|_{\mathcal{Z}^3_{2,r/2}(Q)} + \varepsilon\|\boldsymbol{w}\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)}\}.\end{aligned} \tag{4.48}$$

Finally, using the maximal parabolic regularity for the Stokes equation in [10], Theorem 2.4, we have

$$\|\boldsymbol{w}_{\mathsf{e}}\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)} \le c\{\|\psi_{\mathsf{e}}\|_{L^{r/2}(I;L^2(\Omega))} + \|\beta_{\mathsf{e}}\|_{L^{r/2}(I;L^2(\Omega))} + \|\mathbf{e}_{\mathsf{v}}\|_{L^{r/2}(I;\boldsymbol{W}^{-1,2}(\Omega))}\}.$$

From the continuous embedding $\mathcal{Z}^3_{q,r}(Q) + \mathcal{Z}^3_{2,2}(Q) \hookrightarrow L^\infty(I; W^{1,2}(\Omega))$ and another application of Lions Lemma to the function spaces $\mathcal{Z}^1_{2,r/2}(Q) \rightrightarrows L^{r/2}(I; L^4(\Omega)) \hookrightarrow L^2(I; L^2(\Omega))$ and $\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q) \rightrightarrows L^r(I; \boldsymbol{L}^4(\Omega)) \hookrightarrow L^2(I; \boldsymbol{L}^2(\Omega))$ in (4.39) leads to

$$\begin{aligned}\|\mathbf{e}_{\mathsf{v}}\|_{L^{r/2}(I;\boldsymbol{W}^{-1,2}(\Omega))} &\le c\|\boldsymbol{u}\|_{L^r(I;\boldsymbol{L}^4(\Omega))}\{c_\varepsilon\|\boldsymbol{w}\|_{L^2(I;\boldsymbol{L}^2(\Omega))} + \varepsilon\|\boldsymbol{w}\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)}\}\\ &\quad + c\{\|\xi\|_{L^{r/2}(I;L^4(\Omega))} + c_\varepsilon\|\zeta\|_{L^2(I;L^2(\Omega))} + \varepsilon\|\zeta\|_{\mathcal{Z}^1_{2,r/2}(Q)}\}\|\phi\|_{\mathcal{Z}^3_{q,r}(Q)+\mathcal{Z}^3_{2,2}(Q)}\\ &\quad + c\{\|\mu\|_{L^{r/2}(I;L^4(\Omega))} + \|\theta\|_{L^{r/2}(I;L^4(\Omega))}\}\|\psi\|_{\mathcal{Z}^3_{q,r}(Q)+\mathcal{Z}^3_{2,2}(Q)}.\end{aligned}$$

With regard to the second term on the right-hand side, note that

$$\|\xi\|_{L^{r/2}(I;L^4(\Omega))} \le c\{\|\xi_{\mathbf{e}}\|_{L^{r/2}(I;L^4(\Omega))} + \|\xi_{\mathbf{g}}\|_{L^{r/2}(I;L^4(\Omega))}\}.$$

These estimates together with (4.43), (4.44), and (4.45) give us

$$\|\boldsymbol{w}_{\mathbf{e}}\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)} - c_N\|\xi_{\mathbf{e}}\|_{L^{r/2}(I;W^{1,2}_0(\Omega))} \le c_N\{c_\varepsilon\|\mathbf{g}\|_{\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)} + \varepsilon\|\zeta\|_{\mathcal{Z}^1_{2,r/2}(Q)} + \varepsilon\|\boldsymbol{w}\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)}\}. \tag{4.49}$$

Let $\delta > 0$. Multiplying (4.48) and (4.49) by δ and δ^2, respectively, and then taking the sum of the resulting inequalities with (4.47), we obtain

$$\begin{aligned}&(1-\delta c_N)\|\beta_{\mathbf{e}}\|_{\mathcal{Z}^1_{2,r/2}(Q)} + \delta\|\psi_{\mathbf{e}}\|_{\mathcal{Z}^3_{2,r/2}(Q)} + \delta(1-\delta c_N)\|\xi_{\mathbf{e}}\|_{L^{r/2}(I;W^{1,2}_0(\Omega))} + \delta^2\|\boldsymbol{w}_{\mathbf{e}}\|_{\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)}\\ &\quad\le c_{N,\delta}\{c_\varepsilon\|\mathbf{g}\|_{\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)} + \varepsilon\|(\psi,\zeta,\boldsymbol{w},\xi)\|_{\boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)}\}.\end{aligned}$$

Choosing $\delta < 1/c_N$, recalling that $\beta_{\mathbf{e}} = \zeta_{\mathbf{e}} - l_{\mathsf{h}}\psi_{\mathbf{e}}$, and then using the continuous embedding $\mathcal{Z}^3_{2,r/2}(Q) \hookrightarrow \mathcal{Z}^1_{2,r/2}(Q)$, we arrive at the following estimate

$$\begin{aligned}&\|(\psi_{\mathbf{e}},\zeta_{\mathbf{e}},\boldsymbol{w}_{\mathbf{e}},\xi_{\mathbf{e}})\|_{\mathcal{Z}^3_{2,r/2}(Q)\times\mathcal{Z}^1_{2,r/2}(Q)\times\boldsymbol{\mathcal{V}}^1_{2,r/2}(Q)\times L^{r/2}(I;W^{1,2}_0(\Omega))}\\ &\quad\le c_N\{c_\varepsilon\|\mathbf{g}\|_{\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)} + \varepsilon\|(\psi,\zeta,\boldsymbol{w},\xi)\|_{\boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)}\}.\end{aligned} \tag{4.50}$$

By the triangle inequality and the definition of the norm for the sum of Banach spaces, we obtain from (4.43) and (4.50) that

$$\|(\psi,\zeta,\boldsymbol{w},\xi)\|_{\boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)} \le \frac{c_\varepsilon c_N + c}{1-\varepsilon c_N}\|\mathbf{g}\|_{\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)}.$$

Taking $0 < \varepsilon < 1/c_N$ establishes the boundedness of the linear operator $\mathrm{D}\mathbf{H}(\widetilde{\boldsymbol{f}})$ as a map from $\boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)$ onto ${}_0\boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)$. Finally, it is clear that $\mathrm{D}\mathbf{H}(\widetilde{\boldsymbol{f}})$ is injective. This completes the proof of the lemma. □

Now that we have higher integrability for the solutions of the state system, we shall utilize this in the succeeding lemma to establish an improved regularity for the solution of the adjoint system. For the definitions of $\boldsymbol{\mathcal{G}}^{r/2}_4(Q)$ and $\boldsymbol{\mathcal{Y}}^{r/2}_4(Q)$, we refer to (3.9) and (3.10), respectively.

Lemma 4.11. *Let (3.11) with $r > 8$ holds and assume that $(\phi,\theta,\boldsymbol{u},\mu) \in \boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)$ and $(g_{\mathsf{o}}, g_{\mathsf{h}}, \boldsymbol{g}_{\mathsf{v}}, g_{\mathsf{c}}) \in \boldsymbol{\mathcal{G}}^{r/2}_4(Q)$. The weak solution of the adjoint problem (3.8) satisfies $(\varphi,\vartheta,\boldsymbol{v},\eta) \in \boldsymbol{\mathcal{Y}}^{r/2}_4(Q)$ having the associated pressure $\pi \in L^{r/2}(I;\widehat{W}^{1,4}(\Omega))$. Furthermore, there exists a continuous function $\mathscr{C} : [0,\infty) \to [0,\infty)$ such that*

$$\|(\varphi,\vartheta,\boldsymbol{v},\eta)\|_{\boldsymbol{\mathcal{Y}}^{r/2}_4(Q)} + \|\pi\|_{L^{r/2}(I;\widehat{W}^{1,4}(\Omega))} \le \mathscr{C}(\|(\phi,\theta,\boldsymbol{u},\mu)\|_{\boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)})\|(g_{\mathsf{o}},g_{\mathsf{h}},\boldsymbol{g}_{\mathsf{v}},g_{\mathsf{c}})\|_{\boldsymbol{\mathcal{G}}^{r/2}_4(Q)}.$$

Proof. We shall eliminate η in (3.8) by substitution and proceed with an argument presented in [9], Lemma 4.8. Consider the subspace $\mathcal{Z}^3_{4,r/2,T}(Q) := \{\varphi \in \mathcal{Z}^3_{4,r/2}(Q) : \varphi(T) = 0\}$ of $\mathcal{Z}^3_{4,r/2}(Q)$ and follow similar definitions for $\mathcal{Z}^2_{4,r/2,T}(Q)$ and $\boldsymbol{\mathcal{V}}^2_{4,r/2,T}(Q)$. Let us introduce the function spaces

$$\begin{aligned}\boldsymbol{\mathcal{X}}_0(Q) &:= \mathcal{Z}^3_{4,r/2,T}(Q) \times \mathcal{Z}^2_{4,r/2,T}(Q) \times \boldsymbol{\mathcal{V}}^2_{4,r/2,T}(Q) \times L^{r/2}(I; \widehat{W}^{1,4}(\Omega))\\ \boldsymbol{\mathcal{X}}_1(Q) &:= L^{r/2}(I; W^{-1,4}(\Omega)) \times L^{r/2}(I; L^4(\Omega)) \times L^{r/2}(I; \boldsymbol{L}^4(\Omega)).\end{aligned}$$

For each $\varrho \in [0, 1]$, define the linear map $\mathbf{A}_\varrho : \boldsymbol{\mathcal{X}}_0(Q) \to \boldsymbol{\mathcal{X}}_1(Q)$ according to

$$\mathbf{A}_\varrho(\varphi, \vartheta, \boldsymbol{v}, \pi) := \begin{pmatrix} -\partial_t(\varphi - m\tau\Delta\varphi) + m\epsilon\Delta^2\varphi - \varrho e_{\mathsf{o}}(\varphi, \vartheta, \boldsymbol{v}) \\ -\partial_t\vartheta - \kappa\Delta\vartheta - \varrho e_{\mathsf{h}}(\varphi, \vartheta, \boldsymbol{v}) \\ -\partial_t\boldsymbol{v} - \nu\Delta\boldsymbol{v} + \nabla\pi - \varrho\mathbf{e}_{\mathsf{v}}(\varphi, \vartheta, \boldsymbol{v}), \end{pmatrix}$$

where the last terms in each component are given by

$$\begin{aligned} e_{\mathsf{o}}(\varphi, \vartheta, \boldsymbol{v}) &:= -l_{\mathsf{h}}\partial_t\vartheta - \tau\mathcal{K}\partial_t\boldsymbol{v} \cdot \nabla\phi - \tau\mathcal{K}\boldsymbol{v} \cdot \nabla\partial_t\phi + mF'(\phi)\Delta\varphi + m\mathcal{K}F'(\phi)\boldsymbol{v} \cdot \nabla\phi \\ &\qquad + \alpha_1\boldsymbol{g} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \nabla(\varphi - l_{\mathsf{h}}\vartheta) - \mathcal{K}\boldsymbol{v} \cdot \nabla(\mu - l_{\mathsf{c}}\theta) - \epsilon\mathcal{K}\Delta(\boldsymbol{v} \cdot \nabla\phi) \\ e_{\mathsf{h}}(\varphi, \vartheta, \boldsymbol{v}) &:= \boldsymbol{u} \cdot \nabla\vartheta + \alpha_2\boldsymbol{g} \cdot \boldsymbol{v} + ml_{\mathsf{c}}\Delta\varphi \\ \mathbf{e}_{\mathsf{v}}(\varphi, \vartheta, \boldsymbol{v}) &:= \boldsymbol{u} \cdot \nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\top\boldsymbol{u} + \alpha\vartheta\boldsymbol{g} + \phi\nabla\varphi + (\theta - l_{\mathsf{h}}\phi)\nabla\vartheta. \end{aligned}$$

Observe that the adjoint system (3.8) is equivalent to the equation

$$\mathbf{A}_1(\varphi, \vartheta, \boldsymbol{v}, \pi) = (g_{\mathsf{o}} - \tau\partial_t g_{\mathsf{c}} - \epsilon\Delta g_{\mathsf{c}} + F'(\phi)g_{\mathsf{c}}, g_{\mathsf{h}} + l_{\mathsf{c}}g_{\mathsf{c}}, \boldsymbol{g}_{\mathsf{v}}) \in \boldsymbol{\mathcal{X}}_1(Q). \tag{4.51}$$

Thus, it is enough to prove that $\mathbf{A}_1 : \boldsymbol{\mathcal{X}}_0(Q) \to \boldsymbol{\mathcal{X}}_1(Q)$ is an isomorphism.

From the maximal parabolic regularity theory for the viscous biharmonic heat, Stokes, and heat equations, we deduce that $\mathbf{A}_0$ is an isomorphism. Next, we show that $\mathbf{A}_\varrho$ is well-defined and bounded for $\varrho > 0$. Indeed, we infer from Hölder inequality the following estimates:

$$\|e_{\mathsf{h}}(\varphi, \vartheta, \boldsymbol{v})\|_{L^{r/2}(I;L^4(\Omega))} \le c\{\|\boldsymbol{u}\|_{L^{r/2}(I;\boldsymbol{L}^4(\Omega))}\|\vartheta\|_{C(\bar{I};C^1(\Omega))} + \|\boldsymbol{v}\|_{L^{r/2}(I;\boldsymbol{L}^4(\Omega))} + \|\varphi\|_{L^{r/2}(I;W^{2,4}(\Omega))}\} \tag{4.52}$$

$$\begin{aligned}\|\mathbf{e}_{\mathsf{v}}(\varphi, \vartheta, \boldsymbol{v})\|_{L^{r/2}(I;\boldsymbol{L}^4(\Omega))} &\le c\{\|\boldsymbol{u}\|_{L^{r/2}(I;\boldsymbol{L}^4(\Omega))}\|\boldsymbol{v}\|_{C(\bar{I};\boldsymbol{C}^1(\Omega))} + \|\phi\|_{L^\infty(I;L^\infty(\Omega))}\|\varphi\|_{L^{r/2}(I;W^{1,4}(\Omega))} \\ &\quad + (\|\theta\|_{L^{r/2}(I;L^4(\Omega))} + \|\phi\|_{L^{r/2}(I;L^4(\Omega))})\|\vartheta\|_{C(\bar{I};C^1(\Omega))} + \|\vartheta\|_{L^{r/2}(I;L^4(\Omega))}\}\end{aligned} \tag{4.53}$$

$$\begin{aligned}\|e_{\mathsf{o}}(\varphi, \vartheta, \boldsymbol{v})\|_{L^{r/2}(I;W^{-1,4}(\Omega))} &\le c\{\|\partial_t\vartheta\|_{L^{r/2}(I;L^4(\Omega))} + \|\partial_t\boldsymbol{v}\|_{L^{r/2}(I;\boldsymbol{L}^4(\Omega))}\|\phi\|_{L^\infty(I;L^\infty(\Omega))} \\ &\quad + \|\boldsymbol{v}\|_{C(\bar{I};\boldsymbol{C}(\Omega))}\|\partial_t\phi\|_{L^{r/2}(I;L^4(\Omega))} + \|F'(\phi)\|_{L^\infty(I;L^\infty(\Omega))}\|\varphi\|_{L^{r/2}(I;W^{2,4}(\Omega))} \\ &\quad + \|F'(\phi)\|_{L^\infty(I;L^\infty(\Omega))}\|\boldsymbol{v}\|_{C(\bar{I};\boldsymbol{C}(\Omega))}\|\phi\|_{L^{r/2}(I;W^{1,4}(\Omega))} + \|\boldsymbol{u}\|_{L^{r/2}(I;\boldsymbol{L}^4(\Omega))}(\|\varphi\|_{C(\bar{I};C^1(\Omega))} + \|\vartheta\|_{C(\bar{I};C^1(\Omega))}) \\ &\quad + \|\boldsymbol{v}\|_{L^{r/2}(I;\boldsymbol{L}^4(\Omega))} + \|\boldsymbol{v}\|_{C(\bar{I};\boldsymbol{C}(\Omega))}(\|\mu\|_{L^{r/2}(I;L^4(\Omega))} + \|\theta\|_{L^{r/2}(I;L^4(\Omega))})\} \\ &\quad + \|\boldsymbol{v}\|_{C(\bar{I};\boldsymbol{C}^1(\Omega))}\|\phi\|_{L^{r/2}(I;W^{2,4}(\Omega))}.\end{aligned} \tag{4.54}$$

Using the following compact embeddings from the proof of [9], Lemma 4.8

$$\mathcal{Z}^2_{4,r/2}(Q) \hookrightarrow\hookrightarrow C(\bar{I}; C^1(\Omega)), \quad \boldsymbol{\mathcal{V}}^2_{4,r/2}(Q) \hookrightarrow\hookrightarrow C(\bar{I}; \boldsymbol{C}^1(\Omega)),$$

we have that $\mathbf{A}_\varrho(\varphi, \vartheta, \boldsymbol{v}, \pi) \in \boldsymbol{\mathcal{X}}_1(Q)$ for every $(\varphi, \vartheta, \boldsymbol{v}, \pi) \in \boldsymbol{\mathcal{X}}_0(Q)$.

Again, we set $N := \|(\phi,\theta,\boldsymbol{u},\mu)\|_{\boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)}$. Based on the above estimates and the previous embeddings, we can see that

$$\begin{aligned}\|(\mathbf{A}_\varrho-\mathbf{A}_\varsigma)(\varphi,\vartheta,\boldsymbol{v},\pi)\|_{\boldsymbol{\mathcal{X}}_1(Q)} &\le |\varrho-\varsigma|\|(e_{\mathsf{o}}(\varphi,\vartheta,\boldsymbol{v}),e_{\mathsf{h}}(\varphi,\vartheta,\boldsymbol{v}),\boldsymbol{e}_{\mathsf{v}}(\varphi,\vartheta,\boldsymbol{v}))\|_{\boldsymbol{\mathcal{X}}_1(Q)}\\ &\le c_N|\varrho-\varsigma|\|(\varphi,\vartheta,\boldsymbol{v})\|_{\mathcal{Z}^3_{4,r/2}(Q)\times\mathcal{Z}^2_{4,r/2}(Q),\times\boldsymbol{\mathcal{V}}^2_{4,r/2}(Q)}\end{aligned}$$

and thus,

$$\|\mathbf{A}_\varrho-\mathbf{A}_\varsigma\|_{\mathcal{L}(\boldsymbol{\mathcal{X}}_0(Q),\boldsymbol{\mathcal{X}}_1(Q))}\le c_N|\varrho-\varsigma|. \tag{4.55}$$

Denote by S the set of all $\varrho\in[0,1]$ such that $\mathbf{A}_\varrho$ is an isomorphism. Then, $0\in S$ and S is open relative to $[0,1]$ from (4.55) and the fact that the set of all isomorphisms from $\boldsymbol{\mathcal{X}}_0(Q)$ onto $\boldsymbol{\mathcal{X}}_1(Q)$ is open in $\mathcal{L}(\boldsymbol{\mathcal{X}}_0(Q),\boldsymbol{\mathcal{X}}_1(Q))$.

Let us show that S is also closed in $[0,1]$. Let $\varrho\in[0,1]$ and suppose that $\varrho_k\to\varrho$ where $\varrho_k\in S$ for each positive integer k. It is clear that $\mathbf{A}_\varrho$ is injective. Now we show that $\mathbf{A}_\varrho$ is surjective. Given $(h_{\mathsf{o}},h_{\mathsf{h}},\boldsymbol{h}_{\mathsf{v}})\in\boldsymbol{\mathcal{X}}_1(Q)$, since $\mathbf{A}_{\varrho_k}$ is surjective, we have $\mathbf{A}_{\varrho_k}(\varphi_k,\vartheta_k,\boldsymbol{v}_k,\pi_k)=(h_{\mathsf{o}},h_{\mathsf{h}},\boldsymbol{h}_{\mathsf{v}})$ for some $(\varphi_k,\vartheta_k,\boldsymbol{v}_k,\pi_k)\in\boldsymbol{\mathcal{X}}_0(Q)$. By adapting the proof of Theorem 3.7, we see that there exists a constant $c>0$ independent of k such that

$$\|(\varphi_k,\vartheta_k,\boldsymbol{v}_k,\pi_k)\|_{\boldsymbol{\mathcal{Y}}^2_2(Q)}\le c\|(h_{\mathsf{o}},h_{\mathsf{h}},\boldsymbol{h}_{\mathsf{v}})\|_{\boldsymbol{\mathcal{X}}_1(Q)}. \tag{4.56}$$

Applying separately the maximal parabolic regularity for the viscous biharmonic heat, heat, and Stokes equations on each component of the equation

$$\mathbf{A}_{\varrho_k}(\varphi_k,\vartheta_k,\boldsymbol{v}_k,\pi_k)=(h_{\mathsf{o}},h_{\mathsf{h}},\boldsymbol{h}_{\mathsf{v}}),$$

and using $0\le\varrho_k\le1$, we have

$$\|\varphi_k\|_{\mathcal{Z}^3_{4,r/2}(Q)}\le c\{\|h_{\mathsf{o}}\|_{L^{r/2}(I;W^{-1,4}(\Omega))}+\|e_{\mathsf{o}}(\varphi_k,\vartheta_k,\boldsymbol{v}_k)\|_{L^{r/2}(I;W^{-1,4}(\Omega))}\} \tag{4.57}$$

$$\|\vartheta_k\|_{\mathcal{Z}^2_{4,r/2}(Q)}\le c\{\|h_{\mathsf{h}}\|_{L^{r/2}(I;L^4(\Omega))}+\|e_{\mathsf{h}}(\varphi_k,\vartheta_k,\boldsymbol{v}_k)\|_{L^{r/2}(I;L^4(\Omega))}\} \tag{4.58}$$

$$\|\boldsymbol{v}_k\|_{\boldsymbol{\mathcal{V}}^2_{4,r/2}(Q)}+\|\pi_k\|_{L^{r/2}(I;\widehat{W}^{1,4}(\Omega))}\le c\{\|\boldsymbol{h}_{\mathsf{v}}\|_{L^{r/2}(I;\boldsymbol{L}^4(\Omega))}+\|\mathbf{e}_{\mathsf{v}}(\varphi_k,\vartheta_k,\boldsymbol{v}_k)\|_{L^{r/2}(I;\boldsymbol{L}^4(\Omega))}\}. \tag{4.59}$$

In what follows, we provide further estimates for the second terms on the right-hand sides of the last three inequalities.

Now, by utilizing Lions Lemma to the function spaces $\mathcal{Z}^2_{4,r/2}(Q)\overset{c}{\hookrightarrow}C(\bar I;C^1(\Omega))\hookrightarrow L^2(I;L^2(\Omega))$ and $\mathcal{Z}^3_{4,r/2}(Q)\overset{c}{\hookrightarrow}L^{r/2}(I;W^{2,4}(\Omega))\hookrightarrow L^2(I;L^2(\Omega))$, along with the embeddings $\boldsymbol{\mathcal{V}}^2_{2,2}(Q)\hookrightarrow L^{r/2}(I;\boldsymbol{L}^4(\Omega))$ and (4.45) in (4.52), one has

$$\begin{aligned}\|e_{\mathsf{h}}(\varphi_k,\vartheta_k,\boldsymbol{v}_k)\|_{L^{r/2}(I;L^4(\Omega))}&\le c_N\{c_\varepsilon\|\vartheta_k\|_{L^2(I;L^2(\Omega))}+\varepsilon\|\vartheta_k\|_{\mathcal{Z}^2_{4,r/2}(Q)}\\ &+\|\boldsymbol{v}_k\|_{\boldsymbol{\mathcal{V}}^2_{2,2}(Q)}+c_\varepsilon\|\varphi_k\|_{L^2(I;L^2(\Omega))}+\varepsilon\|\varphi_k\|_{\mathcal{Z}^3_{4,r/2}(Q)}\}.\end{aligned}$$

This inequality and (4.56) when applied to (4.58) give us:

$$(1-\varepsilon c_N)\|\vartheta_k\|_{\mathcal{Z}^2_{4,r/2}(Q)}\le c_N\{c_\varepsilon\|(h_{\mathsf{o}},h_{\mathsf{h}},\boldsymbol{h}_{\mathsf{v}})\|_{\boldsymbol{\mathcal{X}}_1(Q)}+\varepsilon\|\varphi_k\|_{\mathcal{Z}^3_{4,r/2}(Q)}\}. \tag{4.60}$$

A similar procedure as above along with $\boldsymbol{\mathcal{V}}^2_{4,r/2}(Q)\overset{c}{\hookrightarrow}C(\bar I;\boldsymbol{C}^1(\Omega))\hookrightarrow L^2(I;\boldsymbol{L}^2(\Omega))$ when applied to (4.53) yields

$$\|\mathbf{e}_{\mathsf{v}}(\varphi_k,\vartheta_k,\boldsymbol{v}_k)\|_{L^{r/2}(I;\boldsymbol{L}^4(\Omega))}\le c_N\{c_\varepsilon\|\boldsymbol{v}_k\|_{L^2(I;L^2(\Omega))}+\varepsilon\|\boldsymbol{v}_k\|_{\boldsymbol{\mathcal{V}}^2_{4,r/2}(Q)}$$

$$+ \|\varphi_k\|_{\mathcal{Z}^3_{2,2}(Q)} + c_\varepsilon \|\vartheta_k\|_{L^2(I;L^2(\Omega))} + \varepsilon \|\vartheta_k\|_{\mathcal{Z}^2_{4,r/2}(Q)} + \|\vartheta_k\|_{\mathcal{Z}^2_{2,2}(Q)}\}.$$

Substituting this inequality in (4.59) and then using the estimate (4.56), we have

$$(1-\varepsilon c_N)\|\boldsymbol{v}_k\|_{\boldsymbol{\mathcal{V}}^2_{4,r/2}(Q)} + \|\pi_k\|_{L^{r/2}(I;\widehat{W}^{1,4}(\Omega))} \le c_N\{c_\varepsilon \|(h_{\mathsf{o}}, h_{\mathsf{h}}, \boldsymbol{h}_{\mathsf{v}})\|_{\boldsymbol{\mathcal{X}}_1(Q)} + \varepsilon \|\vartheta_k\|_{\mathcal{Z}^2_{4,r/2}(Q)}\}. \tag{4.61}$$

Lastly with the same methods, we obtain from (4.54) the following inequality

$$\begin{aligned}\|e_{\mathsf{o}}(\varphi_k, \vartheta_k, \boldsymbol{v}_k)\|_{L^{r/2}(I;W^{-1,4}(\Omega))} &\le c_N\{\|\vartheta_k\|_{\mathcal{Z}^2_{4,r/2}(Q)} + \|\boldsymbol{v}_k\|_{\boldsymbol{\mathcal{V}}^2_{4,r/2}(Q)} + c_\varepsilon \|\boldsymbol{v}_k\|_{L^2(I;L^2(\Omega))} + \varepsilon \|\boldsymbol{v}_k\|_{\boldsymbol{\mathcal{V}}^2_{4,r/2}(Q)} \\ &+ c_\varepsilon \|\varphi_k\|_{L^2(I;L^2(\Omega))} + \varepsilon \|\varphi_k\|_{\mathcal{Z}^3_{4,r/2}(Q)} + \|\boldsymbol{v}_k\|_{\boldsymbol{\mathcal{V}}^2_{2,2}(Q)} + c_\varepsilon \|\vartheta_k\|_{L^2(I;L^2(\Omega))} + \varepsilon \|\vartheta_k\|_{\mathcal{Z}^2_{4,r/2}(Q)}\}.\end{aligned}$$

With this estimate and (4.56) in (4.57), we deduce

$$\begin{aligned}(1-\varepsilon c_N)\|\varphi_k\|_{\mathcal{Z}^3_{4,r/2}(Q)} &- c_N\|\vartheta_k\|_{\mathcal{Z}^2_{4,r/2}(Q)} - c_N\|\boldsymbol{v}_k\|_{\boldsymbol{\mathcal{V}}^2_{4,r/2}(Q)} \\ &\le c_N\{c_\varepsilon \|(h_{\mathsf{o}}, h_{\mathsf{h}}, \boldsymbol{h}_{\mathsf{v}})\|_{\boldsymbol{\mathcal{X}}_1(Q)} + \varepsilon \|\vartheta_k\|_{\mathcal{Z}^2_{4,r/2}(Q)} + \varepsilon \|\boldsymbol{v}_k\|_{\boldsymbol{\mathcal{V}}^2_{4,r/2}(Q)}\}.\end{aligned} \tag{4.62}$$

Multiplying (4.62) by $\delta > 0$ and then taking the sum of the resulting inequality with (4.60) and (4.61), we obtain

$$\begin{aligned}\delta(1-\varepsilon c_N)\|\varphi_k\|_{\mathcal{Z}^3_{4,r/2}(Q)} &+ (1-\delta c_N - \varepsilon c_N)\|\vartheta_k\|_{\mathcal{Z}^2_{4,r/2}(Q)} + (1-\delta c_N - \varepsilon c_N)\|\boldsymbol{v}_k\|_{\boldsymbol{\mathcal{V}}^2_{4,r/2}(Q)} \\ &+ \|\pi_k\|_{L^{r/2}(I;\widehat{W}^{1,4}(\Omega))} \le c_{N,\delta}\{c_\varepsilon \|(h_{\mathsf{o}}, h_{\mathsf{h}}, \boldsymbol{h}_{\mathsf{v}})\|_{\boldsymbol{\mathcal{X}}_1(Q)} + \varepsilon \|(\varphi_k, \vartheta_k, \boldsymbol{v}_k, \pi_k)\|_{\boldsymbol{\mathcal{X}}_0(Q)}\}.\end{aligned}$$

Taking $\delta = 1/(2c_N)$ and then $\varepsilon > 0$ small enough so that

$$\widetilde{c}_N := \min\left\{\frac{1-\varepsilon c_N}{2c_N}, \frac{1}{2} - \varepsilon c_N\right\} - c_{N,\delta}\varepsilon > 0,$$

we see from the above inequality that

$$\|(\varphi_k, \vartheta_k, \boldsymbol{v}_k, \pi_k)\|_{\boldsymbol{\mathcal{X}}_0(Q)} \le \frac{c_{N,\delta} c_\varepsilon}{\widetilde{c}_N} \|(h_{\mathsf{o}}, h_{\mathsf{h}}, \boldsymbol{h}_{\mathsf{v}})\|_{\boldsymbol{\mathcal{X}}_1(Q)}. \tag{4.63}$$

The previous inequality implies that the sequence $\{(\varphi_k, \vartheta_k, \boldsymbol{v}_k, \pi_k)\}_{k=1}^\infty$ is bounded in the reflexive space $\boldsymbol{\mathcal{X}}_0(Q)$. Thus, $(\varphi_k, \vartheta_k, \boldsymbol{v}_k, \pi_k) \rightharpoonup (\varphi, \vartheta, \boldsymbol{v}, \pi)$ in $\boldsymbol{\mathcal{X}}_0(Q)$ for some $(\varphi, \vartheta, \boldsymbol{v}, \pi) \in \boldsymbol{\mathcal{X}}_0(Q)$. Passing to the weak limit in the equation $\mathbf{A}_{\varrho_k}(\varphi_k, \vartheta_k, \boldsymbol{v}_k, \pi_k) = (h_{\mathsf{o}}, h_{\mathsf{h}}, \boldsymbol{h}_{\mathsf{v}})$ leads to $\mathbf{A}_\varrho(\varphi, \vartheta, \boldsymbol{v}, \pi) = (h_{\mathsf{o}}, h_{\mathsf{h}}, \boldsymbol{h}_{\mathsf{v}})$. This proves that $\mathbf{A}_\varrho$ is surjective. As a consequence, $\varrho \in S$ and so S is closed in $[0,1]$. Since the only non-void subset of the connected space $[0,1]$ that is both open and closed is itself, we have $S = [0,1]$. Therefore, $\mathbf{A}_1$ is an isomorphism. The estimate stated by the lemma follows from (4.51), the estimate

$$\|\eta\|_{\mathcal{Z}^1_{4,r/2}(Q)} \le c_N\{\|\varphi\|_{\mathcal{Z}^3_{4,r/2}(Q)} + \|\boldsymbol{v}\|_{\boldsymbol{\mathcal{V}}^2_{4,r/2}(Q)} + \|g_{\mathsf{c}}\|_{\mathcal{Z}^1_{4,r/2}(Q)}\},$$

and the one that can be obtained from (4.63) by passing to the limit inferior. □

The succeeding lemma deals with the stability of solutions for the adjoint system under the norm of $\boldsymbol{\mathcal{Y}}_4^{r/2}(Q)$.

Lemma 4.12. *Consider the assumptions of Lemma 4.11 and suppose that $\phi_d, \theta_d \in L^{r/2}(I; L^4(\Omega))$ and $\boldsymbol{\psi}_d, \boldsymbol{v}_d \in L^{r/2}(I; \boldsymbol{L}^4(\Omega))$. For every $\boldsymbol{s} \in \boldsymbol{\mathcal{M}}_{\mathsf{ad}}^\infty$, we have $\mathbf{D}(\boldsymbol{s}) \in \boldsymbol{\mathcal{Y}}_4^{r/2}(Q)$ and there exists $c_3 > 0$ independent on $\boldsymbol{s}$ such that*

$$\|\mathbf{D}(\boldsymbol{s}) - \mathbf{D}(\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{Y}}_4^{r/2}(Q)} \leq c_3 \|\mathbf{F}(\boldsymbol{s}) - \mathbf{F}(\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{W}}_{q,s,p}^{r,r/2}(Q)}.$$

Proof. The fact that $\mathbf{D}(\boldsymbol{s}) \in \boldsymbol{\mathcal{Y}}_4^{r/2}(Q)$ follows immediately from Lemmas 4.10 and 4.11. Moreover, from (4.14) we deduce that

$$D_\gamma := \sup_{\boldsymbol{s} \in \boldsymbol{\mathcal{M}}_{\mathsf{ad}}^\infty} \|\mathbf{D}(\boldsymbol{s})\|_{\boldsymbol{\mathcal{Y}}_4^{r/2}(Q)} < \infty. \tag{4.64}$$

Let $(\phi_{\boldsymbol{s}}, \theta_{\boldsymbol{s}}, \boldsymbol{u}_{\boldsymbol{s}}, \mu_{\boldsymbol{s}}) = \mathbf{F}(\boldsymbol{s})$ and $(\varphi_{\boldsymbol{s}}, \vartheta_{\boldsymbol{s}}, \boldsymbol{v}_{\boldsymbol{s}}, \eta_{\boldsymbol{s}}) = \mathbf{D}(\boldsymbol{s})$. Also, let us recall that $(\phi^*, \theta^*, \boldsymbol{u}^*, \mu^*) = \mathbf{F}(\boldsymbol{s}^*)$ and $(\varphi^*, \vartheta^*, \boldsymbol{v}^*, \eta^*) = \mathbf{D}(\boldsymbol{s}^*)$. Then, the difference $(\varphi, \theta, \boldsymbol{v}, \eta) = (\varphi_{\boldsymbol{s}}, \vartheta_{\boldsymbol{s}}, \boldsymbol{v}_{\boldsymbol{s}}, \eta_{\boldsymbol{s}}) - (\varphi^*, \vartheta^*, \boldsymbol{v}^*, \eta^*)$ satisfies the linear system (3.8) with $(\phi, \theta, \boldsymbol{u}, \mu) = (\phi^*, \theta^*, \boldsymbol{u}^*, \mu^*)$ and

$$\begin{aligned}
g_{\mathsf{o}} &= \lambda_{\mathsf{o}1}(\phi_{\boldsymbol{s}} - \phi^*) + \lambda_{\mathsf{o}2}(\Delta\phi_{\boldsymbol{s}} - \Delta\phi^*) + (\boldsymbol{u}_{\boldsymbol{s}} - \boldsymbol{u}^*) \cdot \nabla(\varphi_{\boldsymbol{s}} - l_{\mathsf{h}}\vartheta_{\boldsymbol{s}}) \\
&\quad + (F'(\phi_{\boldsymbol{s}}) - F'(\phi^*))\eta_{\boldsymbol{s}} - \mathcal{K}\boldsymbol{v}_{\boldsymbol{s}} \cdot \nabla(\mu_{\boldsymbol{s}} - \mu^* - l_{\mathsf{c}}\theta_{\boldsymbol{s}} + l_{\mathsf{c}}\theta^*) \\
g_{\mathsf{c}} &= \mathcal{K}\boldsymbol{v}_{\boldsymbol{s}} \cdot \nabla(\phi_{\boldsymbol{s}} - \phi^*) \\
g_{\mathsf{h}} &= \lambda_{\mathsf{h}}(\theta_{\boldsymbol{s}} - \theta^*) + (\boldsymbol{u}_{\boldsymbol{s}} - \boldsymbol{u}^*) \cdot \nabla\vartheta_{\boldsymbol{s}} - \mathcal{K} l_{\mathsf{c}} \boldsymbol{v}_{\boldsymbol{s}} \cdot \nabla(\phi_{\boldsymbol{s}} - \phi^*) \\
\boldsymbol{g}_{\mathsf{v}} &= \lambda_{\mathsf{v}}(\boldsymbol{u}_{\boldsymbol{s}} - \boldsymbol{u}^*) + (\boldsymbol{u}_{\boldsymbol{s}} - \boldsymbol{u}^*) \cdot \nabla\boldsymbol{v}_{\boldsymbol{s}} + (\nabla\boldsymbol{v}_{\boldsymbol{s}})^\top(\boldsymbol{u}_{\boldsymbol{s}} - \boldsymbol{u}^*) \\
&\quad + (\phi_{\boldsymbol{s}} - \phi^*)\nabla\varphi_{\boldsymbol{s}} + (\theta_{\boldsymbol{s}} - \theta^* - l_{\mathsf{h}}\phi_{\boldsymbol{s}} + l_{\mathsf{h}}\phi^*)\nabla\vartheta_{\boldsymbol{s}}.
\end{aligned}$$

Then, we can adapt the same methods as in the previous lemma and use (4.64) so that

$$\|(\varphi, \vartheta, \boldsymbol{v}, \eta)\|_{\boldsymbol{\mathcal{Y}}_4^{r/2}(Q)} \leq c F_\gamma D_\gamma \|(\phi_{\boldsymbol{s}} - \phi^*, \theta_{\boldsymbol{s}} - \theta^*, \boldsymbol{u}_{\boldsymbol{s}} - \boldsymbol{u}^*, \mu_{\boldsymbol{s}} - \mu^*)\|_{\boldsymbol{\mathcal{W}}_{q,s,p}^{r,r/2}(Q)}$$

for some $c > 0$ independent of $\boldsymbol{s}$. Observe that this is precisely the estimate stated by the lemma with $c_3 = c F_\gamma D_\gamma$. ☐

Our next lemma provides an estimate for the action of the second-order derivatives of the cost functional.

Lemma 4.13. *Consider the framework of Lemma 4.12. Given $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that for every $\boldsymbol{s} \in \boldsymbol{\mathcal{M}}_{\mathsf{ad}}^\infty \cap \boldsymbol{\mathcal{B}}_{\varepsilon_\delta}^r(\boldsymbol{s}^*)$ and $\boldsymbol{r} \in \boldsymbol{\mathcal{M}}^r$, we have*

$$|[\mathrm{D}^2 J(\boldsymbol{s}) - \mathrm{D}^2 J(\boldsymbol{s}^*)](\boldsymbol{r}, \boldsymbol{r})| \leq \frac{\delta}{4} \|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}^2.$$

Proof. Let $(\psi_{\boldsymbol{r}}^*, \zeta_{\boldsymbol{r}}^*, \boldsymbol{w}_{\boldsymbol{r}}^*, \xi_{\boldsymbol{r}}^*) = \mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}$ and $(\psi_{\boldsymbol{r}}, \zeta_{\boldsymbol{r}}, \boldsymbol{w}_{\boldsymbol{r}}, \xi_{\boldsymbol{r}}) = \mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r}$. We shall estimate $\mathrm{D}^2 J(\boldsymbol{s})(\boldsymbol{r}, \boldsymbol{r}) - \mathrm{D}^2 J(\boldsymbol{s}^*)(\boldsymbol{r}, \boldsymbol{r})$ by using the representation of the second-order derivative of J provided in Theorem 4.1. First, applying the Cauchy–Schwarz inequality and the estimates in Lemma 4.7, we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega \lambda_{\mathsf{o}1}|\psi_{\boldsymbol{r}}+\psi_{\boldsymbol{r}}^*||\psi_{\boldsymbol{r}}-\psi_{\boldsymbol{r}}^*|+\lambda_{\mathsf{o}2}|\nabla\psi_{\boldsymbol{r}}+\nabla\psi_{\boldsymbol{r}}^*||\nabla\psi_{\boldsymbol{r}}-\nabla\psi_{\boldsymbol{r}}^*|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\int_\Omega \lambda_{\mathsf{h}}|\zeta_{\boldsymbol{r}}+\zeta_{\boldsymbol{r}}^*||\zeta_{\boldsymbol{r}}-\zeta_{\boldsymbol{r}}^*|+\lambda_{\mathsf{v}}|\boldsymbol{w}_{\boldsymbol{r}}+\boldsymbol{w}_{\boldsymbol{r}}^*||\boldsymbol{w}_{\boldsymbol{r}}-\boldsymbol{w}_{\boldsymbol{r}}^*|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le c\{\|\mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}+\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}\}\|\mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r}-\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}\\
&\quad\le cc_1(c_2+1)\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)}\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|^2_{\boldsymbol{\mathcal{T}}_2^2(Q)}.
\end{aligned}$$

Next, by the triangle inequality, we have

$$\begin{aligned}
&\int_0^T\int_\Omega |2(\boldsymbol{w}_{\boldsymbol{r}}\cdot\nabla\varphi)\psi_{\boldsymbol{r}}-2(\boldsymbol{w}_{\boldsymbol{r}}^*\cdot\nabla\varphi^*)\psi_{\boldsymbol{r}}^*|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le 2\int_0^T\int_\Omega |((\boldsymbol{w}_{\boldsymbol{r}}-\boldsymbol{w}_{\boldsymbol{r}}^*)\cdot\nabla\varphi)\psi_{\boldsymbol{r}}|\,\mathrm{d}x\,\mathrm{d}t+2\int_0^T\int_\Omega |(\boldsymbol{w}_{\boldsymbol{r}}^*\cdot\nabla(\varphi-\varphi^*))\psi_{\boldsymbol{r}}|\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+2\int_0^T\int_\Omega |(\boldsymbol{w}_{\boldsymbol{r}}^*\cdot\nabla\varphi^*)(\psi_{\boldsymbol{r}}-\psi_{\boldsymbol{r}}^*)|\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The terms on the right-hand side of this inequality can be estimated from above as follows

$$\begin{aligned}
&\int_0^T\int_\Omega |((\boldsymbol{w}_{\boldsymbol{r}}-\boldsymbol{w}_{\boldsymbol{r}}^*)\cdot\nabla\varphi)\psi_{\boldsymbol{r}}|\,\mathrm{d}x\,\mathrm{d}t\le c\|\boldsymbol{w}_{\boldsymbol{r}}-\boldsymbol{w}_{\boldsymbol{r}}^*\|_{L^2(I;L^2(\Omega))}\|\varphi\|_{C(\bar I;C^1(\Omega))}\|\psi_{\boldsymbol{r}}\|_{L^2(I;L^2(\Omega))}\\
&\int_0^T\int_\Omega |(\boldsymbol{w}_{\boldsymbol{r}}^*\cdot\nabla(\varphi-\varphi^*))\psi_{\boldsymbol{r}}|\,\mathrm{d}x\,\mathrm{d}t\le c\|\boldsymbol{w}_{\boldsymbol{r}}^*\|_{L^2(I;L^2(\Omega))}\|\varphi-\varphi^*\|_{C(\bar I;C^1(\Omega))}\|\psi_{\boldsymbol{r}}\|_{L^2(I;L^2(\Omega))}\\
&\int_0^T\int_\Omega |(\boldsymbol{w}_{\boldsymbol{r}}^*\cdot\nabla\varphi^*)(\psi_{\boldsymbol{r}}-\psi_{\boldsymbol{r}}^*)|\,\mathrm{d}x\,\mathrm{d}t\le c\|\boldsymbol{w}_{\boldsymbol{r}}^*\|_{L^2(I;L^2(\Omega))}\|\varphi^*\|_{C(\bar I;C^1(\Omega))}\|\psi_{\boldsymbol{r}}-\psi_{\boldsymbol{r}}^*\|_{L^2(I;L^2(\Omega))}.
\end{aligned}$$

Taking the sum of these inequalities and using the continuous embeddings presented in the proof of Lemma 4.11, we get

$$\begin{aligned}
&\int_0^T\int_\Omega |2(\boldsymbol{w}_{\boldsymbol{r}}\cdot\nabla\varphi)\psi_{\boldsymbol{r}}-2(\boldsymbol{w}_{\boldsymbol{r}}^*\cdot\nabla\varphi^*)\psi_{\boldsymbol{r}}^*|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\le c\|\mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r}-\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}\|\mathbf{D}(\boldsymbol{s})\|_{\boldsymbol{\mathcal{Y}}_4^{r/2}(Q)}\|\mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}\\
&\quad+c\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}\|\mathbf{D}(\boldsymbol{s})-\mathbf{D}(\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{Y}}_4^{r/2}(Q)}\|\mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}\\
&\quad+c\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}\|\mathbf{D}(\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{Y}}_4^{r/2}(Q)}\|\mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r}-\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}.
\end{aligned}$$

According to Lemma 4.10, we have $\|\mathbf{F}(\boldsymbol{s})-\mathbf{F}(\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{W}}^{r,r/2}_{q,s,p}(Q)}=z(\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)})$, with $z(\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)})\to 0$ as $\boldsymbol{s}\to\boldsymbol{s}^*$ in $\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)$. In what follows, the function z may be different at each line. Due to Lemma 4.12, it follows also that $\|\mathbf{D}(\boldsymbol{s})-\mathbf{D}(\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{Y}}_4^{r/2}(Q)}=z(\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)})$. Based on these and from Lemma 4.7, we deduce that

$$\int_0^T\int_\Omega |2(\boldsymbol{w}_{\boldsymbol{r}}\cdot\nabla\varphi)\psi_{\boldsymbol{r}}-2(\boldsymbol{w}_{\boldsymbol{r}}^*\cdot\nabla\varphi^*)\psi_{\boldsymbol{r}}^*|\,\mathrm{d}x\,\mathrm{d}t\le z(\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)})\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|^2_{\boldsymbol{\mathcal{T}}_2^2(Q)}.$$

In a similar fashion, we can establish the following estimate for the other terms related to the convection and surface tension

$$\begin{aligned}&\int_0^T\int_\Omega 2|(\boldsymbol{w}_{\boldsymbol{r}}\cdot\nabla\vartheta)(\zeta_{\boldsymbol{r}}-l_{\mathsf{h}}\psi_{\boldsymbol{r}})-(\boldsymbol{w}^*_{\boldsymbol{r}}\cdot\nabla\vartheta^*)(\zeta^*_{\boldsymbol{r}}-l_{\mathsf{h}}\psi^*_{\boldsymbol{r}})|\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^T\int_\Omega 2\mathcal{K}|(\boldsymbol{v}\cdot\nabla\psi_{\boldsymbol{r}})(\xi_{\boldsymbol{r}}-l_{\mathsf{c}}\zeta_{\boldsymbol{r}})-(\boldsymbol{v}^*\cdot\nabla\psi^*_{\boldsymbol{r}})(\xi^*_{\boldsymbol{r}}-l_{\mathsf{c}}\zeta^*_{\boldsymbol{r}})|\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^T\int_\Omega 2|(\boldsymbol{w}_{\boldsymbol{r}}\cdot\nabla\boldsymbol{v})\cdot\nabla\boldsymbol{w}_{\boldsymbol{r}}-(\boldsymbol{w}^*_{\boldsymbol{r}}\cdot\nabla\boldsymbol{v}^*)\cdot\nabla\boldsymbol{w}^*_{\boldsymbol{r}}|\,\mathrm{d}x\,\mathrm{d}t\\&\quad\le z(\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)})\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|^2_{\boldsymbol{\mathcal{T}}^2_2(Q)}.\end{aligned}$$

Now for the remaining cubic terms for the second-order derivative, we shall estimate from above according to

$$\begin{aligned}&\int_0^T\int_\Omega 6\beta_0|\phi\psi^2_{\boldsymbol{r}}\eta_{\boldsymbol{r}}-\phi^*\psi^{*2}_{\boldsymbol{r}}\eta^*_{\boldsymbol{r}}|\,\mathrm{d}x\,\mathrm{d}t\\&\quad\le 6\beta_0\int_0^T\int_\Omega|[(\phi-\phi^*)\eta^*_{\boldsymbol{r}}+\phi(\eta_{\boldsymbol{r}}-\eta^*)]\psi^2_{\boldsymbol{r}}|\,\mathrm{d}x\,\mathrm{d}t+6\beta_0\int_0^T\int_\Omega|\phi^*\eta^*_{\boldsymbol{r}}(\psi_{\boldsymbol{r}}+\psi^*_{\boldsymbol{r}})(\psi_{\boldsymbol{r}}-\psi^*_{\boldsymbol{r}})|\,\mathrm{d}x\,\mathrm{d}t\\&\quad\le c\{\|\phi-\phi^*\|_{C(\bar I;L^\infty(\Omega))}\|\eta^*_{\boldsymbol{r}}\|_{C(\bar I;L^2(\Omega))}\|\psi_{\boldsymbol{r}}\|^2_{L^2(I;L^4(\Omega))}+\|\phi\|_{C(\bar I;L^\infty(\Omega))}\|\eta-\eta^*_{\boldsymbol{r}}\|_{C(\bar I;L^2(\Omega))}\|\psi_{\boldsymbol{r}}\|^2_{L^2(I;L^4(\Omega))}\\&\qquad+\|\phi^*\|_{C(\bar I;L^\infty(\Omega))}\|\eta^*_{\boldsymbol{r}}\|_{C(\bar I;L^2(\Omega))}(\|\psi_{\boldsymbol{r}}\|_{L^2(I;L^4(\Omega))}+\|\psi^*_{\boldsymbol{r}}\|_{L^2(I;L^4(\Omega))})\|\psi_{\boldsymbol{r}}-\psi^*_{\boldsymbol{r}}\|_{L^2(I;L^4(\Omega))}\}.\end{aligned}$$

With the same reasoning as above, one can obtain the estimate

$$\int_0^T\int_\Omega 6\beta_0|\phi\psi^2_{\boldsymbol{r}}\eta_{\boldsymbol{r}}-\phi\psi^{*2}_{\boldsymbol{r}}\eta^*_{\boldsymbol{r}}|\,\mathrm{d}x\,\mathrm{d}t\le z(\|\boldsymbol{s}-\boldsymbol{s}^*\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)})\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|^2_{\boldsymbol{\mathcal{T}}^2_2(Q)}.$$

Applying the triangle inequality and taking the sum of the above estimates give us the desired result. □

The proof of the following lemma is analogous to the previous one, and for this reason we shall omit the details.

Lemma 4.14. *Consider the framework of Lemma 4.12. Then, there exists $c_4>0$ such that for every $\boldsymbol{s}\in\boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$ and $\boldsymbol{r}\in\boldsymbol{\mathcal{M}}^r$ we have*

$$|[\mathrm{D}^2J(\boldsymbol{s})](\boldsymbol{r},\boldsymbol{r})|\le c_4\|\mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r}\|^2_{\boldsymbol{\mathcal{T}}^2_2(Q)}.$$

We are now in position to prove the main result of this section. In order to formulate the second-order sufficient conditions, we consider the cone of directions $\boldsymbol{\mathcal{C}}^r_\beta(\mathsf{s}^*)\subset\boldsymbol{\mathcal{M}}^{\mathsf{r}}$ with $\beta>0$, defined as follows:

$$\boldsymbol{\mathcal{C}}^r_\beta(\boldsymbol{s}^*):=\left\{\boldsymbol{r}\in\boldsymbol{\mathcal{M}}^r\;\middle|\;\begin{array}{l}\partial_{\boldsymbol{s}}L(\boldsymbol{s}^*,\boldsymbol{m}^*)\boldsymbol{r}\le\beta\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)},\\ \partial_{\boldsymbol{s}}\Lambda(\boldsymbol{s}^*,\boldsymbol{m}^*)\boldsymbol{r}\ge-\beta\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)},\\ \|\sigma^*_{\mathsf{k}}(t)\|_{M(\omega_{\mathsf{k}})}=\gamma_{\mathsf{k}}\text{ implies }\partial\|\sigma^*_{\mathsf{k}}(t)\|_{M(\omega_{\mathsf{k}})}\rho_{\mathsf{k}}(t)\le 0\\ \text{for a.a. }t\in I\text{ and for every }\mathsf{k}\in K\end{array}\right\}.$$

Theorem 4.15. *Let (3.11) with $r > 8$, $\boldsymbol{f} = (f_{\mathsf{o}}, f_{\mathsf{h}}, \boldsymbol{f}_{\mathsf{v}}, f_{\mathsf{c}}) \in \boldsymbol{\mathcal{F}}^{r,r/2}_{q,s,p}(Q)$, $(\phi_0, \theta_0, \boldsymbol{u}_0) \in \boldsymbol{\mathcal{D}}^{r,r/2}_{q,s,p}(\Omega)$, $\phi_d, \theta_d \in L^{r/2}(I; L^4(\Omega))$, $\boldsymbol{\psi}_d, \boldsymbol{v}_d \in L^{r/2}(I; \boldsymbol{L}^4(\Omega))$, and (4.5) be fulfilled. Assume that $\boldsymbol{s}^* \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$ satisfies*

$$\mathrm{D}^2\mathbf{F}(\boldsymbol{s}^*)(\boldsymbol{r}, \boldsymbol{r}) \geq \delta \|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|^2_{\boldsymbol{\mathcal{T}}^2_2(Q)} \qquad \forall\, \boldsymbol{r} \in \boldsymbol{\mathcal{C}}^r_\beta(\boldsymbol{s}^*) \tag{4.65}$$

for some $\delta > 0$ and $\beta > 0$. Then, there exist $\varepsilon = \varepsilon_{\beta,\delta} > 0$ and $\varrho = \varrho_{\beta,\delta} > 0$ such that

$$J(\boldsymbol{s}^*) + \varrho \|\mathbf{F}(\boldsymbol{s}) - \mathbf{F}(\boldsymbol{s}^*)\|^2_{\boldsymbol{\mathcal{T}}^2_2(Q)} \leq J(\boldsymbol{s}) \qquad \forall\, \boldsymbol{s} \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}} \cap \boldsymbol{\mathcal{B}}^r_\varepsilon(\boldsymbol{s}^*). \tag{4.66}$$

In particular, $\boldsymbol{s}^*$ is a strict local solution of (3.7) with respect to the topology of $\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)$, and hence to the topology of $\boldsymbol{\mathcal{M}}^\infty$ as well.

Proof. Suppose that $\boldsymbol{s} = (\sigma_{\mathsf{o}}, \sigma_{\mathsf{h}}, \boldsymbol{\sigma}_{\mathsf{v}}) \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}} \cap \boldsymbol{\mathcal{B}}^r_\varepsilon(\boldsymbol{s}^*)$, where $\varepsilon > 0$ will be chosen below. Let $\boldsymbol{r} = (\rho_{\mathsf{o}}, \rho_{\mathsf{h}}, \boldsymbol{\rho}_{\mathsf{v}}) = \boldsymbol{s} - \boldsymbol{s}^* \in \boldsymbol{\mathcal{M}}^r$. Given $\mathsf{k} \in K$, if $m^*_{\mathsf{k}}(t) = \|y^*_{\mathsf{k}}(t)\|_{C_0(\omega)} > 0$, then $\|\sigma^*_{\mathsf{k}}(t)\|_{M(\omega_{\mathsf{k}})} = \gamma_{\mathsf{k}}$ according to (4.5). Hence, (4.9) and $\|\sigma_{\mathsf{k}}(t)\|_{M(\omega_{\mathsf{k}})} \leq \gamma_{\mathsf{k}}$ lead to $\partial \|\sigma^*_{\mathsf{k}}(t)\|_{M(\omega_{\mathsf{k}})} \rho_{\mathsf{k}}(t) \leq 0$. Thus,

$$\partial_{\boldsymbol{s}} \Lambda(\boldsymbol{s}^*, \boldsymbol{m}^*)\boldsymbol{r} = \sum_{\mathsf{k} \in K} \int_{\{m^*_{\mathsf{k}} > 0\}} m^*_{\mathsf{k}} \partial \|\sigma^*_{\mathsf{k}}\|_{M(\omega_{\mathsf{k}})} \rho_{\mathsf{k}} \,\mathrm{d}t \leq 0. \tag{4.67}$$

Next, we derive some estimates based on the previous lemmas. From the continuous embedding $\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q) \hookrightarrow \boldsymbol{\mathcal{T}}^2_2(Q)$, $\|\boldsymbol{r}\|_{\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q)} \leq \varepsilon$, Theorem 3.3, Corollary 3.4, and setting

$$c_5 := \|\mathrm{D}\mathbf{H}(\mathbf{I}(\boldsymbol{s}^*))\|_{\mathcal{L}(\boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q), \boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q))} \|\mathrm{D}\mathbf{I}\|_{\mathcal{L}(\boldsymbol{\mathcal{N}}^r_{q,s,p}(Q), \boldsymbol{\mathcal{F}}^{r,2}_{q,s,p}(Q))} < \infty,$$

we have

$$\begin{aligned} \|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)} &\leq c \|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)} \\ &\leq c \|\mathrm{D}\mathbf{H}(\mathbf{I}(\boldsymbol{s}^*))\mathrm{D}\mathbf{I}\boldsymbol{r}\|_{\boldsymbol{\mathcal{W}}^{r,2}_{q,s,p}(Q)} \leq c c_5 \varepsilon. \end{aligned}$$

In particular, for $\varepsilon \leq 1/(cc_5)$ we have $\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)} \leq 1$. Moreover, according to Lemma 4.7, one has

$$\|\mathrm{D}\mathbf{F}(\boldsymbol{s})\boldsymbol{r}\|^2_{\boldsymbol{\mathcal{T}}^2_2(Q)} \leq c_2^2 \|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|^2_{\boldsymbol{\mathcal{T}}^2_2(Q)} \leq c c_2^2 c_5 \varepsilon \|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)}. \tag{4.68}$$

In light of Lemma 4.9, the triangle inequality, and upon taking $\varepsilon \leq \varepsilon_0$, the norm on the left-hand side can be bounded from below, thanks to

$$\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}^2_2(Q)} \geq \frac{1}{2} \|\mathbf{F}(\boldsymbol{s}) - \mathbf{F}(\boldsymbol{s}^*)\|_{\boldsymbol{\mathcal{T}}^2_2(Q)}. \tag{4.69}$$

Performing a second-order Taylor expansion to J around $\boldsymbol{s}^*$, we have

$$J(\boldsymbol{s}) = J(\boldsymbol{s}^*) + \mathrm{D}J(\boldsymbol{s}^*)\boldsymbol{r} + \frac{1}{2}\mathrm{D}^2 J(\boldsymbol{s}_\omega)(\boldsymbol{r}, \boldsymbol{r}), \tag{4.70}$$

where $\omega \in [0, 1]$ and we have set $\boldsymbol{s}_\omega := \omega \boldsymbol{s}^* + (1 - \omega)\boldsymbol{s} \in \boldsymbol{\mathcal{M}}^\infty_{\mathsf{ad}}$. As in [10], we shall proceed by considering two cases, namely, when $\boldsymbol{r} \in \boldsymbol{\mathcal{C}}^r_\beta(\boldsymbol{s}^*)$ or $\boldsymbol{r} \notin \boldsymbol{\mathcal{C}}^r_\beta(\boldsymbol{s}^*)$.

Suppose that $\boldsymbol{r} \in \boldsymbol{\mathcal{C}}_\beta^r(\boldsymbol{s}^*)$. Since $\mathrm{D}J(\boldsymbol{s}^*)\boldsymbol{r} \geq 0$ and $\mathrm{D}^2 J(\boldsymbol{s}^*)(\boldsymbol{r},\boldsymbol{r}) \geq \delta \|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|^2_{\boldsymbol{\mathcal{T}}_2^2(Q)}$ from (4.65), we obtain from Lemma 4.13, (4.70), and (4.69) that

$$\begin{aligned} J(\boldsymbol{s}) &\geq J(\boldsymbol{s}^*) + \frac{1}{2}\mathrm{D}^2 J(\boldsymbol{s}^*)(\boldsymbol{r},\boldsymbol{r}) + \frac{1}{2}[\mathrm{D}^2 J(\boldsymbol{s}_\omega) - \mathrm{D}^2 J(\boldsymbol{s}^*)](\boldsymbol{r},\boldsymbol{r}) \\ &\geq J(\boldsymbol{s}^*) + \frac{\delta}{4}\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|^2_{\boldsymbol{\mathcal{T}}_2^2(Q)} \\ &\geq J(\boldsymbol{s}^*) + \frac{\delta}{16}\|\mathbf{F}(\boldsymbol{s}) - \mathbf{F}(\boldsymbol{s}^*)\|^2_{\boldsymbol{\mathcal{T}}_2^2(Q)} \end{aligned}$$

provided that $\varepsilon \leq \varepsilon_\delta$.

Now assume that $\boldsymbol{r} \notin \boldsymbol{\mathcal{C}}_\beta^r(\boldsymbol{s}^*)$. From the definition of $\boldsymbol{\mathcal{C}}_\beta^r(\boldsymbol{s}^*)$ and the statement at the beginning of the proof, either we have $\partial_{\boldsymbol{s}} L(\boldsymbol{s}^*,\boldsymbol{m}^*)\boldsymbol{r} \geq \beta\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}$ or $\partial_{\boldsymbol{s}} \varLambda(\boldsymbol{s}^*,\boldsymbol{m}^*)\boldsymbol{r} \leq -\beta\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}$. In any case, we claim that

$$\mathrm{D}J(\boldsymbol{s}^*)\boldsymbol{r} \geq \beta\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}.$$

Indeed, if the first inequality holds, then due to (4.67), one has

$$\begin{aligned} \mathrm{D}J(\boldsymbol{s}^*)\boldsymbol{r} &= \partial_{\boldsymbol{s}} L(\boldsymbol{s}^*,\boldsymbol{m}^*)\boldsymbol{r} - \partial_{\boldsymbol{s}} \varLambda(\boldsymbol{s}^*,\boldsymbol{m}^*)\boldsymbol{r} \\ &\geq \partial_{\boldsymbol{s}} L(\boldsymbol{s}^*,\boldsymbol{m}^*)\boldsymbol{r} \geq \beta\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)} \end{aligned}$$

If the second inequality is satisfied then from (4.13), we have

$$\mathrm{D}J(\boldsymbol{s}^*)\boldsymbol{r} \geq -\partial_{\boldsymbol{s}} \varLambda(\boldsymbol{s}^*,\boldsymbol{m}^*)\boldsymbol{r} \geq \beta\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}.$$

Utilizing $\mathrm{D}J(\boldsymbol{s}^*)\boldsymbol{r} \geq \beta\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)}$, $\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)} \leq 1$, (4.68), (4.69), (4.70), and Lemma 4.14, we get

$$\begin{aligned} J(\boldsymbol{s}) &\geq J(\boldsymbol{s}^*) + \beta\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|_{\boldsymbol{\mathcal{T}}_2^2(Q)} + \frac{1}{2}\mathrm{D}^2 J(\boldsymbol{s}_\omega)(\boldsymbol{r},\boldsymbol{r}) \\ &\geq J(\boldsymbol{s}^*) + \beta\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|^2_{\boldsymbol{\mathcal{T}}_2^2(Q)} - \frac{1}{2}cc_2^2c_4c_5\varepsilon\|\mathrm{D}\mathbf{F}(\boldsymbol{s}^*)\boldsymbol{r}\|^2_{\boldsymbol{\mathcal{T}}_2^2(Q)} \\ &\geq J(\boldsymbol{s}^*) + \frac{1}{8}(2\beta - cc_2^2c_4c_5\varepsilon)\|\mathbf{F}(\boldsymbol{s}) - \mathbf{F}(\boldsymbol{s}^*)\|^2_{\boldsymbol{\mathcal{T}}_2^2(Q)}. \end{aligned}$$

Based on the above discussions, choosing $\varepsilon = \min\{1/(cc_5), \varepsilon_0, \varepsilon_\delta, 2\beta/(cc_2^2c_4c_5)\}$ and $\varrho = \min\{\delta/16, (2\beta - cc_2^2c_4c_5\varepsilon)/8\}$ yield the desired estimate (4.66). This completes the proof of the theorem. □

Appendix

For the sake of the reader, we provide the computations leading to (4.19) that was used in the proof of Lemma 4.7. Starting from the adjoint system (3.8), we integrate by parts, use the vanishing initial conditions for the linearized system, the vanishing terminal conditions for the adjoint system, and (4.17). For the adjoint equation involving the linearized Cahn–Hilliard system, we obtain

$$\begin{aligned} \int_0^T \langle g_{\mathsf{o}}, \psi\rangle_{W^{-1,2}(\Omega),W_0^{1,2}(\Omega)}\,\mathrm{d}t &= \int_0^T \langle -\partial_t\varphi + l_{\mathsf{h}}\partial_t\vartheta + \tau\partial_t\eta - \boldsymbol{u}\cdot\nabla(\varphi - l_{\mathsf{h}}\vartheta) + \epsilon\Delta\eta, \psi\rangle_{W^{-1,2}(\Omega),W_0^{1,2}(\Omega)}\,\mathrm{d}t \\ &\quad - \int_0^T \langle F'(\phi)\eta + \alpha_1\boldsymbol{g}\cdot\boldsymbol{v} - \mathcal{K}\boldsymbol{v}\cdot\nabla(\mu - l_{\mathsf{c}}\theta), \psi\rangle_{W^{-1,2}(\Omega),W_0^{1,2}(\Omega)}\,\mathrm{d}t \end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \langle \partial_t \psi + \operatorname{div}(\psi \boldsymbol{u}), \varphi \rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)} \,\mathrm{d}t \\
&\quad + \int_0^T (-\tau \partial_t \psi + \epsilon \Delta \psi - F'(\phi)\psi, \eta)_{L^2(\Omega)} \,\mathrm{d}t \\
&\quad + \int_0^T \langle -\alpha_1 \psi \boldsymbol{g} - \mathcal{K}(\mu - l_{\mathsf{c}}\theta)\nabla \psi, \boldsymbol{v} \rangle_{\boldsymbol{W}^{-1,2}(\Omega), \boldsymbol{W}_0^{1,2}(\Omega)} \,\mathrm{d}t \\
&\quad + \int \langle -l_{\mathsf{h}} \partial_t \psi - l_{\mathsf{h}} \operatorname{div}(\psi \boldsymbol{u}), \vartheta \rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)} \,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
\int_0^T (g_{\mathsf{c}}, \xi)_{L^2(\Omega)} \,\mathrm{d}t &= \int_0^T (\eta - m\Delta\varphi - \mathcal{K}\boldsymbol{v} \cdot \nabla \phi, \xi)_{L^2(\Omega)} \,\mathrm{d}t \\
&= \int_0^T (\xi, \eta)_{L^2(\Omega)} \,\mathrm{d}t + \int_0^T \langle -m\Delta \xi, \varphi \rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)} \,\mathrm{d}t \\
&\quad + \int_0^T \langle -\mathcal{K}\xi \nabla \phi, \boldsymbol{v} \rangle_{\boldsymbol{W}^{-1,2}(\Omega), \boldsymbol{W}_0^{1,2}(\Omega)} \,\mathrm{d}t.
\end{aligned}
$$

Also, for the adjoint equation associated with the linearized convection-diffusion equation, we have

$$
\begin{aligned}
\int_0^T (g_{\mathsf{h}}, \zeta)_{L^2(\Omega)} \,\mathrm{d}t &= \int_0^T (-\partial_t \vartheta - \boldsymbol{u} \cdot \nabla \vartheta + \mathcal{K} l_{\mathsf{c}} \boldsymbol{v} \cdot \nabla \phi - \kappa \Delta \vartheta - \alpha_2 \boldsymbol{g} \cdot \boldsymbol{v} - l_{\mathsf{c}} \eta, \zeta)_{L^2(\Omega)} \,\mathrm{d}t \\
&= \int_0^T (-l_{\mathsf{c}} \zeta, \eta)_{L^2(\Omega)} \,\mathrm{d}t + \int_0^T \langle \partial_t \zeta + \operatorname{div}(\zeta \boldsymbol{u}) - \kappa \Delta \zeta, \vartheta \rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)} \,\mathrm{d}t \\
&\quad + \int_0^T \langle \mathcal{K} l_{\mathsf{c}} \zeta \nabla \phi - \alpha_2 \zeta \boldsymbol{g}, \boldsymbol{v} \rangle_{\boldsymbol{W}^{-1,2}(\Omega), \boldsymbol{W}_0^{1,2}(\Omega)} \,\mathrm{d}t
\end{aligned}
$$

and for the adjoint equation corresponding to the linearized Navier–Stokes equation, one has

$$
\begin{aligned}
\int_0^T (\boldsymbol{g}_{\mathsf{v}}, \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t &= \int_0^T (-\partial_t \boldsymbol{v} - \boldsymbol{u} \cdot \nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^\top \boldsymbol{u} - \nu \Delta \boldsymbol{v} + \nabla \pi, \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \\
&\quad - \int_0^T (\alpha \vartheta \boldsymbol{g} + \phi \nabla \varphi + (\theta - l_{\mathsf{h}} \phi) \nabla \vartheta, \boldsymbol{w})_{\boldsymbol{L}^2(\Omega)} \,\mathrm{d}t \\
&= \int_0^T \langle \operatorname{div}(\phi \boldsymbol{w}), \varphi \rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)} \,\mathrm{d}t \\
&\quad + \int_0^T \langle -\alpha \boldsymbol{g} \cdot \boldsymbol{w} + \operatorname{div}((\theta - l_{\mathsf{h}} \phi) \boldsymbol{w}), \vartheta \rangle_{W^{-1,2}(\Omega), W_0^{1,2}(\Omega)} \,\mathrm{d}t \\
&\quad + \int_0^T \langle \partial_t \boldsymbol{w} + \operatorname{div}(\boldsymbol{w} \otimes \boldsymbol{u}) + \operatorname{div}(\boldsymbol{w} \otimes \boldsymbol{u}) - \nu \Delta \boldsymbol{w} + \nabla \varpi, \boldsymbol{v} \rangle_{\boldsymbol{W}^{-1,2}(\Omega), \boldsymbol{W}_0^{1,2}(\Omega)} \,\mathrm{d}t.
\end{aligned}
$$

Taking the sum of these equations and utilizing the equations for the linearized system $(\psi, \zeta, \boldsymbol{w}, \xi) = \mathrm{D}\mathbf{H}(\mathbf{I}(\boldsymbol{s}))\boldsymbol{f}_{\boldsymbol{s}}^*$, see Theorem 3.3, we can easily obtain (4.19).

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