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A FREE BOUNDARY PROBLEM IN THERMAL INSULATION WITH A PRESCRIBED HEAT SOURCE

Paolo Acampora[1,*], Emanuele Cristoforoni[2], Carlo Nitsch[1] and Cristina Trombetti[1]

Abstract. We study the thermal insulation of a bounded body $\Omega \subset \mathbb{R}^n$, under a prescribed heat source $f > 0$, *via* a bulk layer of insulating material. We consider a model of heat transfer between the insulated body and the environment determined by convection; this corresponds to Robin boundary conditions on the free boundary of the layer. We show that a minimal configuration exists and that it satisfies uniform density estimates.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set with smooth boundary, let $f \in \mathrm{L}^2(\Omega)$ be a positive function and let β, C_0 be positive constants. We consider the following energy functional

$$\mathcal{F}(A, v) = \int_A |\nabla v|^2 \,\mathrm{d}\mathcal{L}^n + \beta \int_{\partial A} v^2 \,\mathrm{d}\mathcal{H}^{n-1} - 2\int_\Omega fv \,\mathrm{d}\mathcal{L}^n + C_0 \mathcal{L}^n(A \setminus \Omega), \tag{1.1}$$

and the variational problem

$$\inf \left\{ \mathcal{F}(A, v) \;\middle|\; \begin{array}{l} A \supseteq \Omega \text{ open, bounded and Lipschitz} \\ v \in \mathrm{W}^{1,2}(A),\ v \geq 0 \text{ in } A \end{array} \right\}. \tag{1.2}$$

This problem is related to the following thermal insulation problem: for a given heat source f distributed in a conductor Ω, find the best possible configuration of insulating material surrounding Ω. A similar problem has been studied in [2, 10] for a thin insulating layer, and in [4, 5, 7] for a prescribed temperature in Ω.

For a fixed open set A with Lipschitz boundary, we have, *via* the direct methods of the calculus of variations, that there exists $u_A \in \mathrm{W}^{1,2}(A)$ such that for all $v \in \mathrm{W}^{1,2}(A)$, with $v \geq 0$ in A. Furthermore u_A solves the

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[1] Dipartimento di Matematica e Applicazioni "R. Caccioppoli", Università degli studi di Napoli Federico II, Via Cintia, Complesso Universitario Monte S. Angelo, 80126 Napoli, Italy.
[2] Mathematical and Physical Sciences for Advanced Materials and Technologies, Scuola Superiore Meridionale, Largo San Marcellino 10, 80126 Napoli, Italy.

[*] Corresponding author: p.acampora98@gmail.com

following stationary problem, with Robin boundary condition on ∂A. Precisely

$$\begin{cases} -\Delta u_A = f & \text{in } \Omega, \\ \dfrac{\partial u_A^+}{\partial \nu} = \dfrac{\partial u_A^-}{\partial \nu} & \text{on } \partial\Omega, \\ \Delta u_A = 0 & \text{in } A \setminus \Omega, \\ \dfrac{\partial u_A}{\partial \nu} + \beta u_A = 0 & \text{on } \partial A, \end{cases}$$

where u_A^- and u_A^+ denote the traces of u_A on $\partial\Omega$ in Ω and in $A \setminus \Omega$ respectively. That is

$$\int_A \nabla u_A \cdot \nabla \varphi \, \mathrm{d}\mathcal{L}^n + \beta \int_{\partial A} u_A \varphi \, \mathrm{d}\mathcal{H}^{n-1} = \int_\Omega f\varphi \, \mathrm{d}\mathcal{L}^n, \tag{1.3}$$

for all $\varphi \in \mathrm{W}^{1,2}(A)$. The Robin boundary condition represents the case when the heat transfer with the environment is conveyed by convection.

If for any couple (A, v) with A an open bounded set with Lipschitz boundary containing Ω and $v \in \mathrm{W}^{1,2}(A)$, $v \geq 0$ in A, we identify v with $v\chi_A$, where χ_A is the characteristic function of A, and the set A with the support of v, then the energy functional (1.1) becomes

$$\mathcal{F}(v) = \int_{\mathbb{R}^n} |\nabla v|^2 \, \mathrm{d}\mathcal{L}^n + \beta \int_{J_v} \left(\overline{v}^2 + \underline{v}^2\right) \, \mathrm{d}\mathcal{H}^{n-1} - 2\int_\Omega fv \, \mathrm{d}\mathcal{L}^n + C_0\mathcal{L}^n(\{\, v > 0 \,\} \setminus \Omega), \tag{1.4}$$

and the minimization problem (1.2) becomes

$$\inf \left\{ \mathcal{F}(v) \;\middle|\; v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap \mathrm{W}^{1,2}(\Omega) \right\}, \tag{1.5}$$

where $\overline{v}$ and $\underline{v}$ are respectively the approximate upper and lower limits of v, J_v is the jump set and ∇v is the absolutely continuous part of the derivative of v. See Section 2 for the definitions.

We state the main results of this paper in the two following theorems.

Theorem 1.1. *Let $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $\mathrm{C}^{1,1}$ boundary, let $f \in \mathrm{L}^2(\Omega)$, with $f > 0$ almost everywhere in Ω. Assume in addition that, if $n = 2$,*

$$\|f\|_{2,\Omega}^2 < C_0 \lambda_\beta(B) \mathcal{L}^2(\Omega), \tag{1.6}$$

where B is a ball having the same measure of Ω. Then problem (1.5) admits a solution. Moreover, if $p > n$ and $f \in \mathrm{L}^p(\Omega)$, then there exists a positive constant $C = C(\Omega, f, p, \beta, C_0)$ such that if u is a minimizer to problem (1.5) then

$$\|u\|_\infty \leq C.$$

Theorem 1.2. *Let $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $\mathrm{C}^{1,1}$ boundary, let $p > n$ and let $f \in \mathrm{L}^p(\Omega)$, with $f > 0$ almost everywhere in Ω. Assume in addition that, if $n = 2$ condition (1.6) holds true. Then there exist positive constants $\delta_0 = \delta_0(\Omega, f, p, \beta, C_0)$, $c = c(\Omega, f, p, \beta, C_0)$, $C = C(\Omega, f, p, \beta, C_0)$ such that if u is a minimizer to problem (1.5) then*

$$u \geq \delta_0 \qquad \mathcal{L}^n\text{-a.e. in } \{\, u > 0 \,\},$$

and the jump set J_u *satisfies the density estimates*

$$cr^{n-1} \leq \mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq Cr^{n-1},$$

with $x \in \overline{J_u}$*, and* $0 < r < d(x, \partial\Omega)$*. In particular, we have*

$$\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0.$$

We refer to Section 2 for the definitions of $\lambda_\beta(B)$ in (1.6), and the distance $d(x, \partial\Omega)$ in Theorem 1.2. Section 3 is devoted to the proof of Theorem 1.1, while Section 4 is devoted to the proof of Theorem 1.2.

We notice that the assumptions on the function f do not seem to be sharp. Indeed, it is well known that (see for instance [12], Thm. 8.15), in the more regular case, the assumption $f \in \mathrm{L}^p(\Omega)$ with $p > n/2$ ensures the boundedness of solutions to equation (1.3).

2. Notation and tools

In this section we recall some definitions and proprieties of the spaces BV, SBV, and $\mathrm{SBV}^{\frac{1}{2}}$. We refer to [1, 3, 11] for a deep study of the properties of these functions.

In the following, given an open set $\Omega \subseteq \mathbb{R}^n$ and $1 \leq p \leq \infty$, we will denote the $\mathrm{L}^p(\Omega)$ norm of a function $v \in \mathrm{L}^p(\Omega)$ as $\|v\|_{p,\Omega}$, in particular when $\Omega = \mathbb{R}^n$ we will simply write $\|v\|_p = \|v\|_{p,\mathbb{R}^n}$.

Definition 2.1 (BV)**.** Let $u \in L^1(\mathbb{R}^n)$. We say that u is a function of *bounded variation* in $\mathbb{R}^n$ and we write $u \in \mathrm{BV}(\mathbb{R}^n)$ if its distributional derivative is a Radon measure, namely

$$\int_\Omega u \, \frac{\partial \varphi}{\partial x_i} = \int_\Omega \varphi \; \mathrm{dD}_i u \qquad \forall \varphi \in \mathrm{C}_c^\infty(\mathbb{R}^n),$$

with $\mathrm{D}u$ a $\mathbb{R}^n$-valued measure in $\mathbb{R}^n$. We denote with $|\mathrm{D}u|$ the total variation of the measure $\mathrm{D}u$. The space $\mathrm{BV}(\mathbb{R}^n)$ is a Banach space equipped with the norm

$$\|u\|_{\mathrm{BV}(\mathbb{R}^n)} = \|u\|_1 + |\mathrm{D}u|(\mathbb{R}^n).$$

Definition 2.2. Let $E \subseteq \mathbb{R}^n$ be a measurable set. We define the *set of points of density 1 for E* as

$$E^{(1)} = \left\{ x \in \mathbb{R}^n \;\middle|\; \lim_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} = 1 \right\},$$

and the *set of points of density 0 for E* as

$$E^{(0)} = \left\{ x \in \mathbb{R}^n \;\middle|\; \lim_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} = 0 \right\}.$$

Moreover, we define the *essential boundary* of E as

$$\partial^* E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}).$$

Definition 2.3 (Approximate upper and lower limits)**.** Let $u \colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We define the *approximate upper and lower limits* of u, respectively, as

$$\overline{u}(x) = \inf \left\{ t \in \mathbb{R} \;\middle|\; \limsup_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap \{ u > t \})}{\mathcal{L}^n(B_r(x))} = 0 \right\},$$

and

$$\underline{u}(x) = \sup \left\{ t \in \mathbb{R} \;\middle|\; \limsup_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap \{ u < t \})}{\mathcal{L}^n(B_r(x))} = 0 \right\}.$$

We define the *jump set* of u as

$$J_u = \{ x \in \mathbb{R}^n \mid \underline{u}(x) < \overline{u}(x) \}.$$

We denote by K_u the closure of J_u.

If $\overline{u}(x) = \underline{u}(x) = l$, we say that l is the approximate limit of u as y tends to x, and we have that, for any $\varepsilon > 0$,

$$\limsup_{r \to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap \{ |u - l| \geq \varepsilon \})}{\mathcal{L}^n(B_r(x))} = 0.$$

If $u \in \mathrm{BV}(\mathbb{R}^n)$, the jump set J_u is a $(n-1)$-rectifiable set, *i.e.* $J_u \subseteq \bigcup_{i \in \mathbb{N}} M_i$, up to a $\mathcal{H}^{n-1}$-negligible set, with M_i a C^1-hypersurface in $\mathbb{R}^n$ for every i. We can then define $\mathcal{H}^{n-1}$-almost everywhere on J_u a normal ν_u coinciding with the normal to the hypersurfaces M_i. Futhermore, the direction of $\nu_u(x)$ is chosen in such a way that the approximate upper and lower limits of u coincide with the approximate limit of u on the half-planes

$$H^+_{\nu_u} = \{ y \in \mathbb{R}^n \mid \nu_u(x) \cdot (y - x) \geq 0 \}$$

and

$$H^-_{\nu_u} = \{ y \in \mathbb{R}^n \mid \nu_u(x) \cdot (y - x) \leq 0 \}$$

respectively.

Definition 2.4. Let $E \subseteq \mathbb{R}^n$ be a measurable set and let $\Omega \subseteq \mathbb{R}^n$ be an open set. We define the *relative perimeter* of E inside Ω as

$$P(E; \Omega) = \sup \left\{ \int_E \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n \;\middle|\; \begin{array}{c} \varphi \in \mathrm{C}^1_c(\Omega, \mathbb{R}^n) \\ |\varphi| \leq 1 \end{array} \right\}.$$

If $P(E; \mathbb{R}^n) < +\infty$ we say that E is a *set of finite perimeter.*

Theorem 2.5 (Relative Isoperimetric Inequality)**.** *Let Ω be an open, bounded, connected set with Lipschitz boundary. Then there exists a positive constants $C = C(\Omega)$ such that*

$$\min \{ \mathcal{L}^n(\Omega \cap E), \mathcal{L}^n(\Omega \setminus E) \}^{\frac{n-1}{n}} \leq C P(E; \Omega),$$

for every set E of finite perimeter.

See for instance [14] for the proof of this theorem.

Theorem 2.6. *Let Ω be an open, bounded, connected set with Lipschitz boundary. Then there exists a constant $C = C(\Omega) > 0$ such that*

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial\Omega) \leq C \mathcal{H}^{n-1}(\partial^* E \cap \Omega)$$

for every set of finite perimeter $E \subset \Omega$ with $0 < \mathcal{L}^n(E) \leq \mathcal{L}^n(\Omega)/2$.

We refer to Theorem 2.3 of [8] for the proof of the previous theorem, observing that if Ω is a Lipschitz set, then it is an admissible set in the sense defined in [8] (see [16], Rem. 5.10.2).

Theorem 2.7 (Decomposition of BV functions)**.** *Let $u \in \mathrm{BV}(\mathbb{R}^n)$. Then we have*

$$\mathrm{dD}u = \nabla u \; \mathrm{d}\mathcal{L}^n + |\overline{u} - \underline{u}|\nu_u \; \mathrm{d}\mathcal{H}^{n-1}\lfloor_{J_u} + \mathrm{dD}^c u,$$

where ∇u is the density of $\mathrm{D}u$ with respect to the Lebesgue measure, ν_u is the normal to the jump set J_u and $\mathrm{D}^c u$ is the Cantor part *of the measure $\mathrm{D}u$. The measure $\mathrm{D}^c u$ is singular with respect to the Lebesgue measure and concentrated out of J_u.*

Definition 2.8. Let $v \in \mathrm{BV}(\mathbb{R}^n)$, let $\Gamma \subseteq \mathbb{R}^n$ be a $\mathcal{H}^{n-1}$-rectifiable set and let $\nu(x)$ be the generalized normal to Γ defined for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. For $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ we define the traces $\gamma_\Gamma^\pm(v)(x)$ of v on Γ by the following Lebesgue-type limit quotient relation

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B_r^\pm(x)} |v(y) - \gamma_\Gamma^\pm(v)(x)| \; \mathrm{d}\mathcal{L}^n(y) = 0,$$

where

$$B_r^+(x) = \{\, y \in B_r(x) \mid \nu(x) \cdot (y - x) > 0 \,\},$$

$$B_r^-(x) = \{\, y \in B_r(x) \mid \nu(x) \cdot (y - x) < 0 \,\}.$$

Remark 2.9. Notice that, by Remark 3.79 of [1], for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$, $(\gamma_\Gamma^+(v)(x), \gamma_\Gamma^-(v)(x))$ coincides with either $(\overline{v}(x), \underline{v}(x))$ or $(\underline{v}(x), \overline{v}(x))$, while, for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus J_v$, we have that $\gamma_\Gamma^+(v)(x) = \gamma_\Gamma^-(v)(x)$ and they coincide with the approximate limit of v in x. In particular, if $\Gamma = J_v$, we have

$$\gamma_{J_v}^+(v)(x) = \overline{v}(x) \qquad \gamma_{J_v}^-(v)(x) = \underline{v}(x)$$

for $\mathcal{H}^{n-1}$-a.e. $x \in J_v$.

We now focus our attention on the BV functions whose Cantor parts vanish.

Definition 2.10 (SBV)**.** Let $u \in \mathrm{BV}(\mathbb{R}^n)$. We say that u is a *special function of bounded variation* and we write $u \in \mathrm{SBV}(\mathbb{R}^n)$ if $\mathrm{D}^c u = 0$.

For SBV functions we have the following.

Theorem 2.11 (Chain rule)**.** *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then if $u \in \mathrm{SBV}(\mathbb{R}^n)$, we have*

$$\nabla g(u) = g'(u)\nabla u.$$

Furthermore, if g is increasing,

$$\overline{g(u)} = g(\overline{u}), \quad \underline{g(u)} = g(\underline{u})$$

while, if g is decreasing,

$$\overline{g(u)} = g(\underline{u}), \quad \underline{g(u)} = g(\overline{u}).$$

We now give the definition of the following class of functions.

Definition 2.12 ($\mathrm{SBV}^{1/2}$)**.** Let $u \in \mathrm{L}^2(\mathbb{R}^n)$ be a non-negative function. We say that $u \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ if $u^2 \in \mathrm{SBV}(\mathbb{R}^n)$. In addition, we define

$$J_u := J_{u^2} \qquad \overline{u} := \sqrt{\overline{u^2}} \qquad \underline{u} := \sqrt{\underline{u^2}}$$
$$\nabla u := \frac{1}{2u}\nabla(u^2)\chi_{\{\, u>0\,\}}$$

Notice that this definition extends the validity of the Chain Rule to the functions in $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$. We refer to Lemma 3.2 of [3] for the coherence of this definition.

Theorem 2.13 (Compactness in $\mathrm{SBV}^{1/2}$)**.** *Let u_k be a sequence in $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ and let $C>0$ be such that for every $k \in \mathbb{N}$*

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 \,\mathrm{d}\mathcal{L}^n + \int_{J_{u_k}} \left(\overline{u}_k^2 + \underline{u}_k^2\right) \,\mathrm{d}\mathcal{H}^{n-1} + \int_{\mathbb{R}^n} u_k^2 \,\mathrm{d}\mathcal{L}^n < C$$

Then there exists $u \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ and a subsequence u_{k_j} such that

– Compactness:

$$u_{k_j} \xrightarrow{\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^n)} u$$

– Lower semicontinuity: *for every open set Ω we have*

$$\int_\Omega |\nabla u|^2 \,\mathrm{d}\mathcal{L}^n \le \liminf_{j\to+\infty} \int_\Omega |\nabla u_{k_j}|^2 \,\mathrm{d}\mathcal{L}^n$$

and

$$\int_{J_u\cap\Omega} \left(\overline{u}^2 + \underline{u}^2\right) \,\mathrm{d}\mathcal{H}^{n-1} \le \liminf_{j\to+\infty} \int_{J_{u_{k_j}}\cap\Omega} \left(\overline{u}_{k_j}^2 + \underline{u}_{k_j}^2\right) \,\mathrm{d}\mathcal{H}^{n-1}$$

Definition 2.14 (Robin Eigenvalue)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set with Lipschitz boundary, let $\beta > 0$. We define $\lambda_\beta(\Omega)$ as

$$\lambda_\beta(\Omega) = \inf\left\{ \frac{\displaystyle\int_\Omega |\nabla v|^2 \,\mathrm{d}\mathcal{L}^n + \beta \int_{\partial\Omega} v^2 \,\mathrm{d}\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2 \,\mathrm{d}\mathcal{L}^n} \;\middle|\; v \in \mathrm{W}^{1,2}(\Omega) \setminus \{\,0\,\} \right\}. \tag{2.1}$$

Remark 2.15. Standard tools of calculus of variation ensures that the infimum in (2.1) is achieved, see for instance.

Lemma 2.16. *For every $0<r<R$, the following inequality holds*

$$\lambda_\beta(B_r) \le \left(\frac{\mathcal{L}^n(B_R)}{\mathcal{L}^n(B_r)}\right)^{\frac{2}{n}} \lambda_\beta(B_R),$$

where B_R and B_r are balls with radii R and r respectively.

Proof. Let φ be a minimum of (2.1) for $\Omega = B_R$ and with $\|\varphi\|_{2,B_R} = 1$. We define

$$w(x) = \varphi\left(\frac{R}{r}x\right) \qquad \forall x \in B_r.$$

Therefore,

$$\begin{aligned}\lambda_\beta(B_r) &\le \frac{\displaystyle\int_{B_r} |\nabla w(x)|^2 \, \mathrm{d}\mathcal{L}^n(x) + \int_{\partial B_r} w(x)^2 \, \mathrm{d}\mathcal{H}^{n-1}(x)}{\displaystyle\int_{B_r} w(x)^2 \, \mathrm{d}\mathcal{L}^n(x)} \\ &= \frac{\displaystyle\left(\frac{r}{R}\right)^{n-2} \int_{B_R} |\nabla \varphi(y)|^2 \, \mathrm{d}\mathcal{L}^n(y) + \left(\frac{r}{R}\right)^{n-1} \int_{\partial B_R} \varphi(y)^2 \, \mathrm{d}\mathcal{H}^{n-1}(y)}{\displaystyle\left(\frac{r}{R}\right)^n}.\end{aligned}$$

Since $r/R < 1$, by minimality of φ, we get

$$\lambda_\beta(B_r) \le \frac{\left(\frac{r}{R}\right)^{n-2}}{\left(\frac{r}{R}\right)^n} \lambda_\beta(B_R) = \left(\frac{\mathcal{L}^n(B_r)}{\mathcal{L}^n(B_R)}\right)^{-\frac{2}{n}} \lambda_\beta(B_R).$$

□

Let $\beta, m > 0$, and let us denote by

$$\Lambda_{\beta,m} = \inf\left\{ \frac{\displaystyle\int_{\mathbb{R}^n} |\nabla v|^2 \, \mathrm{d}\mathcal{L}^n + \beta \int_{J_v} \left(\underline{v}^2 + \overline{v}^2\right) \, \mathrm{d}\mathcal{H}^{n-1}}{\displaystyle\int_{\mathbb{R}^n} v^2 \, \mathrm{d}\mathcal{L}^n} \;\middle|\; \begin{array}{l} v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \setminus \{0\} \\ \mathcal{L}^n(\{\, v > 0 \,\}) \le m \end{array} \right\}.$$

Here we state a theorem, referring to Theorem 5 of [3] for the proof.

Theorem 2.17. *Let $B \subseteq \mathbb{R}^n$ be a ball of volume m. Then*

$$\Lambda_{\beta,m} = \lambda_\beta(B).$$

We will denote by $d(x, \partial\Omega)$ the distance between $x \in \mathbb{R}^n$ and the boundary $\partial\Omega$, and for every $\varepsilon > 0$ we define

$$\Omega_\varepsilon = \{\, x \in \Omega \mid d(x, \partial\Omega) > \varepsilon \,\}.$$

We will use the following result.

Proposition 2.18. *Let Ω be an open bounded set with $\mathrm{C}^{1,1}$ boundary, then there exist a constant $C = C(\Omega) > 0$ and a $\varepsilon_0 = \varepsilon_0(\Omega) > 0$ such that*

$$\mathcal{L}^n(\Omega \setminus \Omega_\varepsilon) \le C\varepsilon \qquad \forall \varepsilon < \varepsilon_0.$$

Proof. It is well known (see for instance Theorem 17.5 of [13]) that there exist a constant $C = C(\Omega)$ and $\varepsilon_0 = \varepsilon_0(\Omega) > 0$ such that

$$P(\Omega_\varepsilon) = P(\Omega) + C(\Omega)\,\varepsilon + O(\varepsilon^2),$$

for every $0 < \varepsilon < \varepsilon_0$. Let $r(x) = d(x, \partial\Omega)$ be the distance from the boundary of Ω. By coarea formula we have

$$\mathcal{L}^n(\Omega \setminus \Omega_\varepsilon) = \int_{\{\,0<r<\varepsilon\,\}} \mathrm{d}\mathcal{L}^n = \int_0^\varepsilon P(\Omega_t)\;\mathrm{d}t \leq C(\Omega)\varepsilon.$$

□

3. Existence of minimizers

In this section we prove Theorem 1.1: in Proposition 3.3 we prove the existence of a minimizer to problem (1.5); in Proposition 3.7 we prove the L^∞ estimate for a minimizer.

In this section, we will assume that $\Omega \subseteq \mathbb{R}^n$ is an open bounded set with $\mathrm{C}^{1,1}$ boundary, that $f \in \mathrm{L}^2(\Omega)$ is a positive function and that β, C_0 are positive constants. We consider the energy functional $\mathcal{F}$ defined in (1.4).

Lemma 3.1. *Let $n \geq 2$ and assume that, if $n = 2$, condition (1.6) holds true. Then there exist two positive constants $c = c(\Omega, f, \beta, C_0)$ and $C = C(\Omega, f, \beta, C_0)$ such that if $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap \mathrm{W}^{1,2}(\Omega)$, with $\mathcal{F}(v) \leq 0$ and $\Omega \subseteq \{\,v > 0\,\}$, then*

$$\mathcal{L}^n(\{\,v > 0\,\}) \leq c, \tag{3.1}$$

$$\|v\|_2 \leq C. \tag{3.2}$$

Proof. Let B' be a ball with the same measure as $\{\,v > 0\,\}$. By Theorem 2.17

$$\begin{aligned}0 \geq \mathcal{F}(v) \geq\ & \lambda_\beta\,(B') \int_{\mathbb{R}^n} v^2\;\mathrm{d}\mathcal{L}^n - 2\int_\Omega fv\;\mathrm{d}\mathcal{L}^n \\ &+ C_0\mathcal{L}^n\left(\{\,v > 0\,\} \setminus \Omega\right).\end{aligned}$$

By Lemma 2.16 and Hölder inequality

$$\begin{aligned}0 \geq &\lambda_\beta(B)\left(\frac{\mathcal{L}^n(\Omega)}{\mathcal{L}^n(\{\,v>0\,\})}\right)^{\frac{2}{n}}\|v\|_2^2 - 2\|f\|_{2,\Omega}\|v\|_2 \\ &+ C_0\mathcal{L}^n\left(\{\,v > 0\,\} \setminus \Omega\right)\end{aligned} \tag{3.3}$$

where B is a ball with the same measure as Ω. Obviously (3.3) implies that

$$\|f\|_{2,\Omega}^2 - \lambda_\beta(B)\left(\frac{\mathcal{L}^n(\Omega)}{\mathcal{L}^n(\{\,v>0\,\})}\right)^{\frac{2}{n}} C_0\mathcal{L}^n\left(\{\,v > 0\,\} \setminus \Omega\right) \geq 0.$$

Let $M = \mathcal{L}^n(\{\,v > 0\,\})$, and notice that, since $\Omega \subseteq \{\,v > 0\,\}$,

$$\mathcal{L}^n\left(\{\,v > 0\,\} \setminus \Omega\right) = M - \mathcal{L}^n(\Omega),$$

therefore

$$\|f\|_{2,\Omega}^2 \geq C_0\lambda_\beta(B)\left(\mathcal{L}^n(\Omega)\right)^{\frac{2}{n}}\left(M^{1-\frac{2}{n}} - M^{-\frac{2}{n}}\mathcal{L}^n(\Omega)\right).$$

This implies (taking into account (1.6) if $n = 2$) that there exists $c = c(\Omega, f, \beta, C_0) > 0$ such that

$$\mathcal{L}^n(\{\, v > 0 \,\}) < c.$$

Finally observe that by (3.3) it follows

$$\|v\|_2 \leq C(M), \tag{3.4}$$

where

$$\begin{aligned} C(M) &= \frac{M^{\frac{2}{n}}\left(\|f\|_{2,\Omega} + \sqrt{\|f\|_{2,\Omega}^2 - C_0\lambda_\beta(B)\left(\dfrac{\mathcal{L}^n(\Omega)}{M}\right)^{\frac{2}{n}}(M - \mathcal{L}^n(\Omega))}\right)}{\lambda_\beta(B)\mathcal{L}^n(\Omega)} \\ &\leq \frac{2c^{\frac{2}{n}}\|f\|_{2,\Omega}}{\lambda_\beta(B)\mathcal{L}^n(\Omega)} \end{aligned}$$

□

Remark 3.2. Let $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap \mathrm{W}^{1,2}(\Omega)$, it is always possible to choose a function v_0 such that $v_0 = v$ in $\mathbb{R}^n \setminus \Omega$, $\mathcal{F}(v_0) \leq \mathcal{F}(v)$, and $\Omega \subseteq \{\, v_0 > 0 \,\}$. Indeed the function $v_0 \in \mathrm{W}^{1,2}(\Omega)$, weak solution to the following boundary value problem

$$\begin{cases} -\Delta v_0 = f & \text{in } \Omega, \\ v_0 = \gamma^-_{\partial\Omega}(v) & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

satisfies

$$\int_\Omega \nabla v_0 \cdot \nabla\varphi \,\mathrm{d}\mathcal{L}^n = \int_\Omega f\varphi \,\mathrm{d}\mathcal{L}^n$$

for every $\varphi \in \mathrm{W}^{1,2}_0(\Omega)$ and $v_0 = \gamma^-_{\partial\Omega}(v)$ on $\partial\Omega$ in the sense of the trace. Then, extending v_0 to be equal to v outside of Ω, we have that $\Omega \subset \{\, v_0 > 0 \,\}$ and $\mathcal{F}(v_0) \leq \mathcal{F}(v)$.

Proposition 3.3 (Existence)**.** *Let $n \geq 2$ and, if $n = 2$, assume that condition* (1.6) *holds true. Then there exists a solution to problem* (1.5)*.*

Proof. Let $\{u_k\}$ be a minimizing sequence for problem (1.5). Without loss of generality we may always assume that, for all $k \in \mathbb{N}$, $\mathcal{F}(u_k) \leq \mathcal{F}(0) = 0$, and, by Remark 3.2, $\Omega \subseteq \{\, u_k > 0 \,\}$. Therefore we have

$$\begin{aligned} 0 \geq \mathcal{F}(u_k) &\geq \int_{\mathbb{R}^n} |\nabla u_k|^2 \,\mathrm{d}\mathcal{L}^n + \beta\int_{J_{u_k}} \left(\overline{u_k}^2 + \underline{u_k}^2\right) \,\mathrm{d}\mathcal{H}^{n-1} - 2\int_\Omega fv \,\mathrm{d}\mathcal{L}^n \\ &\geq \int_{\mathbb{R}^n} |\nabla u_k|^2 \,\mathrm{d}\mathcal{L}^n + \beta\int_{J_{u_k}} \left(\overline{u_k}^2 + \underline{u_k}^2\right) \,\mathrm{d}\mathcal{H}^{n-1} - 2\|f\|_{2,\Omega}\|u_k\|_{2,\Omega}\,, \end{aligned}$$

and by (3.2),

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 \,\mathrm{d}\mathcal{L}^n + \beta \int_{J_{u_k}} \left(\overline{u_k}^2 + \underline{u_k}^2\right) \,\mathrm{d}\mathcal{H}^{n-1} \leq C\|f\|_{2,\Omega}\,.$$

Then we have that there exists a positive constant still denoted by C, independent on the sequence $\{u_k\}$, such that

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 \,\mathrm{d}\mathcal{L}^n + \int_{J_{u_k}} \left(\overline{u}_k^2 + \underline{u}_k^2\right) \,\mathrm{d}\mathcal{H}^{n-1} + \int_{\mathbb{R}^n} u_k^2 \,\mathrm{d}\mathcal{L}^n < C. \tag{3.6}$$

The compactness theorem in $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ (Thm. 2.13), ensures that there exists a subsequence $\{u_{k_j}\}$ and a function $u \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap \mathrm{W}^{1,2}(\Omega)$, such that u_{k_j} converges to u strongly in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^n)$, weakly in $\mathrm{W}^{1,2}(\Omega)$, almost everywhere in $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} |\nabla u|^2 \,\mathrm{d}\mathcal{L}^n \leq \liminf_{j\to+\infty} \int_{\mathbb{R}^n} |\nabla u_{k_j}|^2 \,\mathrm{d}\mathcal{L}^n,$$

$$\int_{J_u} \left(\overline{u}^2 + \underline{u}^2\right) \,\mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{j\to+\infty} \int_{J_{u_{k_j}}} \left(\overline{u}_{k_j}^2 + \underline{u}_{k_j}^2\right) \,\mathrm{d}\mathcal{H}^{n-1},$$

$$\mathcal{L}^n(\{\, u > 0 \,\} \setminus \Omega) \leq \liminf_{j\to+\infty} \mathcal{L}^n(\{\, u_{k_j} > 0 \,\} \setminus \Omega).$$

Finally we have

$$\mathcal{F}(u) \leq \liminf_{j\to+\infty} \mathcal{F}(u_{k_j}) = \inf\left\{\, \mathcal{F}(v) \;\middle|\; v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap \mathrm{W}^{1,2}(\Omega) \,\right\},$$

Therefore u is a minimizer to problem (1.5). □

Theorem 3.4 (Euler-Lagrange equation)**.** *Let u be a minimizer to problem* (1.5)*, and let $v \in \mathrm{SBV}^{1/2}(\mathbb{R}^n)$ such that $J_v \subseteq J_u$. Assume that there exists $t > 0$ such that $\{\, v > 0 \,\} \subseteq \{\, u > t \,\}$ $\mathcal{L}^n$-a.e., and that*

$$\int_{J_u \setminus J_v} v^2 \,\mathrm{d}\mathcal{H}^{n-1} < +\infty.$$

Then

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \,\mathrm{d}\mathcal{L}^n + \beta \int_{J_u} \left(\overline{u}\gamma^+(v) + \underline{u}\gamma^-(v)\right) \,\mathrm{d}\mathcal{H}^{n-1} = \int_\Omega fv \,\mathrm{d}\mathcal{L}^n, \tag{3.7}$$

where $\gamma^\pm = \gamma^\pm_{J_u}$.

Proof. Notice that since $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ with $J_v \subseteq J_u$ we have that $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap \mathrm{W}^{1,2}(\Omega)$. Assume $v \in \mathrm{SBV}^{1/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. If $s \in \mathbb{R}$, recalling that $\{\, v > 0 \,\} \subseteq \{\, u > t \,\}$ $\mathcal{L}^n$-a.e.,

$$u(x) + sv(x) = u(x) \geq 0 \qquad \mathcal{L}^n\text{-a.e. } \forall x \in \{\, u \leq t \,\},$$

while, for $|s|$ small enough,

$$u(x) + sv(x) \geq t - |s|\,\|v\|_\infty > 0 \qquad \forall x \in \{\, u > t \,\}.$$

Therefore we still have

$$u + sv \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n, \mathbb{R}^+).$$

Moreover by minimality of u we have, for every $|s| \leq s_0$

$$\begin{aligned}\mathcal{F}(u) &\leq \mathcal{F}(u+sv)\\ &= \int_{\mathbb{R}^n} |\nabla u + s\nabla v|^2 \, \mathrm{d}\mathcal{L}^n\\ &\quad + \int_{J_{u+sv}} \left[\left(\gamma^+(u) + s\gamma^+(v)\right)^2 + \left(\gamma^-(u) + s\gamma^-(v)\right)^2 \right] \, \mathrm{d}\mathcal{H}^{n-1}\\ &\quad - 2\int_{\mathbb{R}^n} f(u+sv) \, \mathrm{d}\mathcal{L}^n + C_0\mathcal{L}^n(\{\, u > 0 \,\}).\end{aligned}$$

Claim: The set

$$S := \left\{ \, s \in [-s_0, s_0] \;\middle|\; \mathcal{H}^{n-1}(J_u \setminus J_{u+sv}) \neq 0 \, \right\}$$

is at most countable.

Let us define

$$\begin{aligned} D_0 &= \left\{ \, x \in J_u \;\middle|\; \gamma^+(u)(x) \neq \gamma^-(u)(x) \, \right\},\\ D_s &= \left\{ \, x \in J_u \;\middle|\; \gamma^+(u+sv)(x) \neq \gamma^-(u+sv)(x) \, \right\},\end{aligned}$$

and notice that

$$\mathcal{H}^{n-1}(J_u \setminus D_0) = 0, \qquad \mathcal{H}^{n-1}(J_{u+sv} \setminus D_s) = 0.$$

Then we have to prove that

$$\left\{ \, s \in [-s_0, s_0] \;\middle|\; \mathcal{H}^{n-1}(D_0 \setminus D_s) \neq 0 \, \right\}$$

is at most countable. Observe that if $t \neq s$,

$$(D \setminus D_t) \cap (D \setminus D_s) = \emptyset.$$

Indeed if $x \in D \setminus D_s$

$$\begin{gathered}\gamma^+(u)(x) \neq \gamma^-(u)(x),\\ \gamma^+(u) + s\gamma^+(v)(x) = \gamma^-(u) + s\gamma^-(v)(x),\end{gathered}$$

then

$$\gamma^+(v)(x) \neq \gamma^-(v)(x),$$

and so

$$s=\frac{\gamma^-(u)(x)-\gamma^+(u)(x)}{\gamma^+(v)(x)-\gamma^-(v)(x)}.$$

If $\mathcal{H}^0$ denotes the counting measure in $\mathbb{R}$, we can write

$$\int_{-s_0}^{s_0}\mathcal{H}^{n-1}(D_0\setminus D_s)\,\mathrm{d}\mathcal{H}^0=\mathcal{H}^{n-1}\left(\bigcup_{(-s_0,s_0)}D_0\setminus D_s\right)\le\mathcal{H}^{n-1}(J_u)<+\infty,$$

then the claim is proved.

We are now able to differentiate in $s=0$ the function $\mathcal{F}(u+sv)$, and observing that $0\notin S$ is a minimum for $\mathcal{F}(u+sv)$, we get

$$\frac{1}{2}\delta\mathcal{F}(u,v)=\int_{\mathbb{R}^n}\nabla u\cdot\nabla v\,\mathrm{d}\mathcal{L}^n+\beta\int_{J_u}\left[\overline{u}\gamma^+(v)+\underline{u}\gamma^-(v)\right]\,\mathrm{d}\mathcal{H}^{n-1}-\int_\Omega fv\,\mathrm{d}\mathcal{L}^n=0.$$

If $v\notin L^\infty(\mathbb{R}^n)$, we consider $v_h=\min\{\,v,h\,\}$. Then

$$\delta\mathcal{F}(u,v_h)=0\qquad\forall h>0.$$

Observe that, since $\gamma^\pm(v_h)=\min\{\,\gamma^\pm(v),h\,\}$,

$$\gamma^\pm(v_h)\to\gamma^\pm(v)\qquad\mathcal{H}^{n-1}\text{-a.e. in }J_u.$$

Therefore, passing to limit for $h\to+\infty$, by dominated convergence on the term

$$\int_{\mathbb{R}^n}\nabla u\cdot\nabla v_h\,\mathrm{d}\mathcal{L}^n,$$

and by monotone convergence on the terms

$$\beta\int_{J_u}\left[\overline{u}\gamma^+(v_h)+\underline{u}\gamma^-(v_h)\right]\,\mathrm{d}\mathcal{H}^{n-1},\qquad\int_\Omega fv_h\,\mathrm{d}\mathcal{L}^n,$$

we get

$$0=\lim_h\delta\mathcal{F}(u,v_h)=\delta\mathcal{F}(u,v).$$

□

We now want to use the Euler-Lagrange equation (3.7) to prove that if f belongs to $\mathrm{L}^p(\Omega)$ with $p>n$, and if u is a minimizer to problem (1.5) then u belongs to $\mathrm{L}^\infty(\mathbb{R}^n)$. In order to prove this we need the following

Lemma 3.5. *Let m be a positive real number. There exists a positive constant $C=C(m,\beta,n)$ such that, for every function $v\in\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ with $\mathcal{L}^n(\{\,v>0\,\})\le m$,*

$$\left(\int_{\mathbb{R}^n}v^{2\cdot 1^*}\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{1^*}}\le C\left[\int_{\mathbb{R}^n}|\nabla v|^2\,\mathrm{d}\mathcal{L}^n+\beta\int_{J_v}\left(\overline{v}^2+\underline{v}^2\right)\,\mathrm{d}\mathcal{H}^{n-1}\right],$$

where $1^* = \dfrac{n}{n-1}$ *is the Sobolev conjugate of* 1.

Proof. Classical Embedding of $BV(\mathbb{R}^n)$ in $\mathrm{L}^{1^*}(\mathbb{R}^n)$ ensures that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*}\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{1^*}} \leq & C(n)\left|\mathrm{D}v^2\right|(\mathbb{R}^n)\\ =& C(n)\left[\int_{\mathbb{R}^n} 2v|\nabla v|\,\mathrm{d}\mathcal{L}^n + \int_{J_v}\left(\overline{v}^2+\underline{v}^2\right)\,\mathrm{d}\mathcal{H}^{n-1}\right].\end{aligned}$$

For every $\varepsilon>0$, using Young's and Hölder's inequalities, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*}\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{1^*}} \leq & \frac{C(n)}{\varepsilon}\int_{\mathbb{R}^n} v^2\,\mathrm{d}\mathcal{L}^n\\ &+C(n)\left[\varepsilon\int_{\mathbb{R}^n}|\nabla v|^2\,\mathrm{d}\mathcal{L}^n+\int_{J_v}\left(\overline{v}^2+\underline{v}^2\right)\,\mathrm{d}\mathcal{H}^{n-1}\right]\\ \leq & \frac{C(n)\,m^{\frac{1}{n}}}{\varepsilon}\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*}\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{1^*}}\\ &+C(n)\left[\varepsilon\int_{\mathbb{R}^n}|\nabla v|^2\,\mathrm{d}\mathcal{L}^n+\int_{J_v}\left(\overline{v}^2+\underline{v}^2\right)\,\mathrm{d}\mathcal{H}^{n-1}\right].\end{aligned}$$

Setting $\varepsilon = 2C(n)m^{\frac{1}{n}}$, we can find two constants $C(m,n), C(m,\beta,n)>0$ such that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*}\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{1^*}} &\leq C(m,n)\left[\int_{\mathbb{R}^n}|\nabla v|^2\,\mathrm{d}\mathcal{L}^n+\int_{J_v}\left(\overline{v}^2+\underline{v}^2\right)\,\mathrm{d}\mathcal{H}^{n-1}\right]\\ &\leq C(m,\beta,n)\left[\int_{\mathbb{R}^n}|\nabla v|^2\,\mathrm{d}\mathcal{L}^n+\beta\int_{J_v}\left(\overline{v}^2+\underline{v}^2\right)\,\mathrm{d}\mathcal{H}^{n-1}\right].\end{aligned}$$

□

We refer to [15] for the following lemma.

Lemma 3.6. *Let* $g:[0,+\infty)\to[0,+\infty)$ *be a decreasing function and assume that there exist* $C,\alpha>0$ *and* $\theta>1$ *constants such that for every* $h>k\geq 0$,

$$g(h)\leq C(h-k)^{-\alpha}g(k)^{\theta}.$$

Then there exists a constant $h_0>0$ *such that*

$$g(h)=0 \qquad \forall h\geq h_0.$$

In particular we have

$$h_0 = C^{\frac{1}{\alpha}}g(0)^{\frac{\theta-1}{\alpha}}2^{\theta(\theta-1)}.$$

Proposition 3.7 (L^∞ bound)**.** *Let $n \geq 2$ and assume that, if $n = 2$, condition (1.6) holds true. Let $f \in \mathrm{L}^p(\Omega)$, with $p > n$. Then there exists a constant $C = C(\Omega, f, p, \beta, C_0) > 0$ such that if u is a minimizer to problem (1.5), then*

$$\|u\|_\infty \leq C.$$

Proof. Let $\gamma^\pm = \gamma^\pm_{J_u}$. For every $\varphi, \psi \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ satisfying $J_\varphi, J_\psi \subseteq J_u$, define

$$a(\varphi, \psi) = \int_{\mathbb{R}^n} \nabla\varphi \cdot \nabla\psi \; \mathrm{d}\mathcal{L}^n + \beta \int_{J_u} \left[\gamma^+(\varphi)\gamma^+(\psi) + \gamma^-(\varphi)\gamma^-(\psi)\right] \; \mathrm{d}\mathcal{H}^{n-1}.$$

For every v satisfying the assumptions of Theorem 3.4, it holds that

$$a(u, v) = \int_\Omega f v \; \mathrm{d}\mathcal{L}^n.$$

In particular, let us fix $k \in \mathbb{R}^+$ and define

$$\varphi_k(x) = \begin{cases} u(x) - k & \text{if } u(x) \geq k, \\ 0 & \text{if } u(x) < k, \end{cases}$$

then

$$\gamma^+(\varphi_k)(x) = \begin{cases} \overline{u}(x) - k & \text{if } \overline{u}(x) \geq k, \\ 0 & \text{if } \overline{u}(x) < k, \end{cases}$$

and analogously for $\gamma^-(\varphi_k)$. Furthermore, let us define

$$\mu(k) = \mathcal{L}^n(\{\, u > k \,\}).$$

We want to prove that $\mu(k) = 0$ for sufficiently large k. From Theorem 3.4, we have

$$a(u, \varphi_k) = \int_\Omega f\varphi_k \; \mathrm{d}\mathcal{L}^n, \tag{3.8}$$

and we can observe that

$$\begin{aligned} a(u, \varphi_k) &= \int_{\{\, u>k \,\}} |\nabla u|^2 \; \mathrm{d}\mathcal{L}^n + \beta \int_{J_u \cap \{\, u>k \,\}} \left[\overline{u}(\overline{u} - k) + \underline{u}(\underline{u} - k)\right] \; \mathrm{d}\mathcal{H}^{n-1} \\ &\geq \int_{\{\, u>k \,\}} |\nabla u|^2 \; \mathrm{d}\mathcal{L}^n + \beta \int_{J_u \cap \{\, u>k \,\}} \left[(\overline{u} - k)^2 + (\underline{u} - k)^2\right] \; \mathrm{d}\mathcal{H}^{n-1} \\ &= a(\varphi_k, \varphi_k). \end{aligned}$$

Moreover, by minimality, $\mathcal{F}(u) \leq \mathcal{F}(0) = 0$ and by Remark 3.2, $\Omega \subseteq \{\, u > 0 \,\}$. Therefore, (3.1) holds true and we can apply Lemma 3.5, having that there exists $C = C(\Omega, f, \beta, C_0) > 0$ such that

$$\int_\Omega f\varphi_k \; \mathrm{d}\mathcal{L}^n = a(u, \varphi_k) \geq a(\varphi_k, \varphi_k) \geq C\|\varphi_k\|^2_{2\cdot 1^*}. \tag{3.9}$$

On the other hand

$$\begin{aligned}\int_{\Omega} f\varphi_k \,\mathrm{d}\mathcal{L}^n = \int_{\Omega\cap\{\,u>k\,\}} f(u-k)\,\mathrm{d}\mathcal{L}^n &\leq \left(\int_{\Omega\cap\{\,u>k\,\}} f^{\frac{2n}{n+1}}\,\mathrm{d}\mathcal{L}^n\right)^{\frac{n+1}{2n}} \|\varphi_k\|_{2\cdot 1^*} \\ &\leq \|f\|_{p,\Omega}\,\|\varphi_k\|_{2\cdot 1^*}\,\mu(k)^{\frac{n+1}{2n\sigma'}},\end{aligned} \tag{3.10}$$

where

$$\sigma = \frac{p(n+1)}{2n} > 1,$$

since $p > n$. Joining (3.9) and (3.10), we have

$$\|\varphi_k\|_{2\cdot 1^*} \leq C\|f\|_{p,\Omega}\,\mu(k)^{\frac{n+1}{2n\sigma'}}. \tag{3.11}$$

Let $h > k$, then

$$\begin{aligned}(h-k)^{2\cdot 1^*}\mu(h) &= \int_{\{\,u>h\,\}} (h-k)^{2\cdot 1^*}\,\mathrm{d}\mathcal{L}^n \\ &\leq \int_{\{\,u>h\,\}} (u-k)^{2\cdot 1^*}\,\mathrm{d}\mathcal{L}^n \\ &\leq \int_{\{\,u>k\,\}} (u-k)^{2\cdot 1^*}\,\mathrm{d}\mathcal{L}^n = \|\varphi_k\|_{2\cdot 1^*}^{2\cdot 1^*}.\end{aligned}$$

Using (3.11) and the previous inequality, we have

$$\mu(h) \leq C(h-k)^{-2\cdot 1^*}\mu(k)^{\frac{n+1}{(n-1)\sigma'}}.$$

Since $p > n$, then $\sigma' < (n+1)/(n-1)$. By Lemma 3.6, we have that $\mu(h) = 0$ for all $h \geq h_0$ with $h_0 = h_0(\Omega, f, \beta, C_0) > 0$, which implies

$$\|u\|_\infty \leq h_0.$$

□

Proof of Theorem 1.1. The result is obtained by joining Proposition 3.3 and Proposition 3.7. □

4. Density estimates for the jump set

In this section we prove Theorem 1.2: in Proposition 4.6 we prove the lower bound for minimizers to problem (1.5); in Proposition 4.8 and Proposition 4.9 we prove the density estimates for the jump set of a minimizer to problem (1.5).

In this section, we will assume that $\Omega \subseteq \mathbb{R}^n$ is an open bounded set with $\mathrm{C}^{1,1}$ boundary, that $f \in \mathrm{L}^p(\Omega)$, with $p > n$, is a positive function, and that β, C_0 are positive constants. We consider the energy functional $\mathcal{F}$ defined in (1.4).

In order to show that if u is a minimizer to problem (1.5) then u is bounded away from 0, we will first prove that there exists a positive constant δ such that $u > \delta$ almost everywhere in Ω, and then we will show that this implies the existence of a positive constant δ_0 such that $u > \delta_0$ almost everywhere in the set $\{\,u > 0\,\}$. In the following we will denote by $U_t := \{\,u < t\,\} \cap \Omega$.

Remark 4.1. Let u be a minimizer to (1.5), by Remark 3.2, u is a solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega \end{cases}$$

Let $u_0 \in \mathrm{W}_0^{1,2}(\Omega)$ be the solution to the following boundary value problem

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Then, by maximum principle,

$$u \geq u_0 \quad \text{in } \Omega \subseteq \{\, u > 0 \,\} \quad \text{and} \quad \{\, u < t \,\} \cap \Omega = U_t \subseteq \{\, u_0 < t \,\} \cap \Omega.$$

Lemma 4.2. *There exist two positive constants $t_0 = t_0(\Omega, f)$ and $C = C(\Omega, f)$ such that if u is a minimizer to (1.5) then for every $t \in [0, t_0]$ it results*

$$\mathcal{L}^n(U_t) \leq C\, t. \tag{4.2}$$

Proof. Let u_0 be the solution to (4.1), fix $\varepsilon > 0$ such that the set

$$\Omega_\varepsilon = \{\, x \in \Omega \mid d(x, \partial\Omega) > \varepsilon \,\}$$

is not empty. Since u_0 is superharmonic and non-negative in Ω, by maximum principle we have that

$$\alpha = \inf_{\Omega_\varepsilon} u_0 > 0.$$

then u_0 solves

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \\ u_0 \geq \alpha & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Therefore, if we consider the solution v to the following boundary value problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \setminus \bar{\Omega}_\varepsilon, \\ v = 0 & \text{on } \partial\Omega, \\ v = \alpha & \text{in } \bar{\Omega}_\varepsilon, \end{cases}$$

we have that $u \geq u_0 \geq v$ almost everywhere in Ω and

$$\{\, u < t \,\} \cap \Omega = U_t \subseteq \{\, u_0 < t \,\} \cap \Omega \subseteq \{\, v < t \,\} \cap \Omega.$$

Hopf Lemma implies that
there exists a constant $\tau = \tau(\Omega) > 0$ such that

$$\frac{\partial v}{\partial \nu} < -\tau \qquad \text{on } \partial\Omega.$$

Let $x \in \bar{\Omega}$, and let x_0 be a projection of x onto the boundary $\partial\Omega$, then

$$|x - x_0| = d(x, \partial\Omega), \qquad \frac{x - x_0}{|x - x_0|} = -\nu_\Omega(x_0),$$

where ν_Ω denotes the exterior normal to $\partial\Omega$. We can write

$$\begin{aligned} v(x) &= \underbrace{v(x_0)}_{=0} + \nabla v(x_0) \cdot (x - x_0) + o\left(|x - x_0|\right) \\ &= -\frac{\partial v}{\partial \nu}(x_0)|x - x_0| + o\left(|x - x_0|\right) \\ &\geq \tau|x - x_0| + o\left(|x - x_0|\right) \\ &> \frac{\tau}{2}|x - x_0| = \frac{\tau}{2}d(x, \partial\Omega) \end{aligned} \tag{4.3}$$

for every x such that $d(x, \partial\Omega) < \sigma_0$ for a suitable $\sigma_0 = \sigma_0(\Omega, f) > 0$. Notice that if $\bar{x} \in \overline{\Omega}$ and $\lim_{x \to \bar{x}} v(x) = 0$ then necessarily $\bar{x} \in \partial\Omega$. Therefore, there exists a $t_0 = t_0(\Omega, f) > 0$ such that $v(x) < t_0$ implies $d(x, \partial\Omega) < \sigma_0$. Consequently, if $t < t_0$, we have that

$$\{ v < t \} \subseteq \{ d(x, \partial\Omega) < \sigma_0 \},$$

and by (4.3), we get

$$\mathcal{L}^n\left(U_t\right) \leq \mathcal{L}^n\left(\{ v < t \}\right) \leq \mathcal{L}^n\left(\left\{ x \in \Omega \,\middle|\, d(x, \partial\Omega) \leq \frac{2}{\tau} t \right\}\right).$$

Since Ω is $C^{1,1}$, by Proposition 2.18, we conclude the proof.

□

Lemma 4.3. *Let $g : [0, t_1] \to [0, +\infty)$ be an increasing, absolutely continuous function such that*

$$g(t) \leq Ct^\alpha \left(g'(t)\right)^\sigma \qquad \forall t \in [0, t_1], \tag{4.4}$$

with $C > 0$ and $\alpha > \sigma > 1$. Then there exists $t_0 > 0$ such that

$$g(t) = 0 \qquad \forall t \leq t_0.$$

Precisely,

$$t_0 = \left(\frac{C(\alpha - \sigma)}{\sigma - 1} g(t_1)^{\frac{\sigma - 1}{\sigma}} + t_1^{\frac{\sigma - \alpha}{\sigma}} \right)^{\frac{\sigma}{\sigma - \alpha}}.$$

Proof. Assume by contradiction that $g(t) > 0$ for every $t > 0$. Inequality (4.4) implies

$$\frac{g'}{g^{\frac{1}{\sigma}}} \geq \frac{1}{C} t^{-\frac{\alpha}{\sigma}}.$$

Integrating between t_0 and t_1, we have

$$\frac{\sigma}{\sigma-1}\left(g(t_1)^{\frac{\sigma-1}{\sigma}}-g(t_0)^{\frac{\sigma-1}{\sigma}}\right)\geq\frac{1}{C}\frac{\sigma}{\sigma-\alpha}\left(t_1^{\frac{\sigma-\alpha}{\sigma}}-t_0^{\frac{\sigma-\alpha}{\sigma}}\right).$$

Since $\alpha>\sigma>1$, we have

$$0\leq g(t_0)^{\frac{\sigma-1}{\sigma}}\leq\frac{\sigma-1}{C\,(\alpha-\sigma)}\left(t_1^{\frac{\sigma-\alpha}{\sigma}}-t_0^{\frac{\sigma-\alpha}{\sigma}}\right)+g(t_1)^{\frac{\sigma-1}{\sigma}},$$

which is a contradiction if

$$t_0\leq\left(\frac{C(\alpha-\sigma)}{\sigma-1}g(t_1)^{\frac{\sigma-1}{\sigma}}+t_1^{\frac{\sigma-\alpha}{\sigma}}\right)^{\frac{\sigma}{\sigma-\alpha}}.$$

□

Remark 4.4. Let g be as in Lemma 4.3 and assume that $g(t_1)\leq K$, then $g(t)=0$ for all $0<t<\tilde{t}$ where

$$\tilde{t}=\left(\frac{C(\alpha-\sigma)}{\sigma-1}K^{\frac{\sigma-1}{\sigma}}+t_1^{\frac{\sigma-\alpha}{\sigma}}\right)^{\frac{\sigma}{\sigma-\alpha}}.$$

We now have the tools to prove the lower bound inside Ω.

Proposition 4.5. *There exists a positive constant $\delta=\delta(\Omega,f,p,\beta,C_0)>0$ such that if u is a minimizer to problem (1.5) then*

$$u\geq\delta$$

almost everywhere in Ω.

Proof. Assume that Ω is connected and define the function

$$u_t(x)=\begin{cases}\max\{\,u,t\,\} & \text{in } \Omega,\\ u & \text{in } \mathbb{R}^n\setminus\Omega\end{cases}$$

Recalling that $U_t=\{\,u<t\,\}\cap\Omega$, we have

$$J_{u_t}\setminus\partial^*U_t=J_u\setminus\partial^*U_t,$$

and on this set $\underline{u_t}=\underline{u}$ and $\overline{u_t}=\overline{u}$.

Then we get by minimality of u, and using the fact that $J_{u_t} \cap \partial^* U_t \subseteq \partial\Omega$,

$$\begin{aligned} 0 &\geq \mathcal{F}(u) - \mathcal{F}(u_t) \\ &= \int_{U_t} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n - 2\int_{U_t} f(u-t) \, \mathrm{d}\mathcal{L}^n + \beta \int_{\partial^* U_t \cap J_u} \left(\underline{u}^2 + \overline{u}^2\right) \, \mathrm{d}\mathcal{H}^{n-1} + \\ &\quad - \beta \int_{J_{u_t} \cap \partial^* U_t \cap J_u} \left[t^2 + (\gamma^+_{\partial\Omega}(u))^2\right] \, \mathrm{d}\mathcal{H}^{n-1} - \beta \int_{\left(J_{u_t} \cap \partial^* U_t\right) \setminus J_u} \left(t^2 + u^2\right) \, \mathrm{d}\mathcal{H}^{n-1} \\ &\geq \int_{U_t} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n - 2\beta t^2 \mathcal{H}^{n-1}\left(\partial^* U_t \cap \partial\Omega\right) \end{aligned}$$

where we ignored all the non-negative terms except the integral of $|\nabla u|^2$, and we used that $u \leq t$ in $\partial^* U_t \setminus J_u$. By Lemma 4.2, we can choose t small enough to have $\mathcal{L}^n(U_t) \leq \mathcal{L}^n(\Omega)/2$, then applying the isoperimetric inequality in Theorem 2.6 to the set $E = U_t$, we get

$$\int_{U_t} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n \leq 2\beta \, C \, t^2 \, P(U_t; \Omega). \tag{4.5}$$

Let us define

$$p(t) = P(U_t; \Omega),$$

and consider the absolutely continuous function

$$g(t) = \int_{U_t} u \, |\nabla u| \, \mathrm{d}\mathcal{L}^n = \int_0^t s \, p(s) \, \mathrm{d}s.$$

By minimality of u we can apply the a priori estimates (3.6) to prove the equiboundedness of g, *i.e.* there exists $K = K(\Omega, f, \beta, C_0) > 0$ such that $g(t) \leq K$ for all $t > 0$. Using the Hölder inequality and the estimate (4.5) we have

$$g(t) \leq \left(\int_{U_t} u^2 \, \mathrm{d}\mathcal{L}^n\right)^{\frac{1}{2}} \left(\int_{U_t} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n\right)^{\frac{1}{2}} \leq \sqrt{2\beta C} \, t \, \mathcal{L}^n(U_t)^{\frac{1}{2}} (t^2 p(t))^{\frac{1}{2}}.$$

Fix $1 > \varepsilon > 0$. Then we can write $\mathcal{L}^n(U_t) = \mathcal{L}^n(U_t)^{\varepsilon} \mathcal{L}^n(U_t)^{1-\varepsilon}$, and by Lemma 4.2 there exists a constant $C = C(\Omega, f, \beta) > 0$ such that

$$g(t) \leq C \, t^{2 + \frac{1-\varepsilon}{2}} \, \mathcal{L}^n(U_t)^{\frac{\varepsilon}{2}} \, p(t)^{\frac{1}{2}}.$$

By the relative isoperimetric inequality in Theorem 2.5, we can estimate

$$\mathcal{L}^n(U_t)^{\frac{\varepsilon}{2}} \leq C(\Omega, n) p(t)^{\frac{\varepsilon n}{2(n-1)}},$$

and, noticing that $p(t) = g'(t)/t$, we get

$$g(t) \leq C t^{\alpha} (g'(t))^{\sigma},$$

where

$$\alpha = 2 - \frac{\varepsilon}{2}\left(1 + \frac{n}{n-1}\right), \qquad \sigma = \frac{1}{2} + \frac{\varepsilon}{2}\frac{n}{n-1}.$$

In particular, if we choose

$$\varepsilon \in \left(\frac{n-1}{n}, \frac{3n-3}{3n-1}\right),$$

we have that $\alpha > \sigma > 1$, and then, using Lemma 4.3 and Remark 4.4, there exists a $\delta = \delta(\Omega, f, p, \beta, C_0) > 0$ such that $g(t) = 0$ for every $t < \delta$. Then $\mathcal{L}^n(\{ u < t \} \cap \Omega) = 0$ for every $t < \delta$, hence

$$u \geq \delta$$

almost everywhere in Ω.

When Ω is not connected, then

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_N,$$

with Ω_i pairwise disjoint connected open sets. Using u_t as the function u truncated inside a single Ω_i, we find constants $\delta_i > 0$ such that

$$u(x) \geq \delta_i$$

almost everywhere in Ω_i. Therefore choosing $\delta = \min \{ \delta_1, \dots, \delta_N \}$ we have $u(x) > \delta$ almost everywhere in Ω. □

Finally, following the approach in [7], we have

Proposition 4.6 (Lower Bound)**.** *There exists a positive constant $\delta_0 = \delta_0(\Omega, f, p, \beta, C_0)$ such that if u is a minimizer to problem* (1.5) *then*

$$u \geq \delta_0$$

almost everywhere in $\{ u > 0 \}$.

Proof. Let δ be the constant in Proposition 4.5. For every $0 < t \leq \delta$ let us define the absolutely continuous function

$$h(t) = \int_{\{ u \leq t \} \setminus J_u} u|\nabla u| \, \mathrm{d}\mathcal{L}^n = \int_0^t sP(\{ u > s \}; \mathbb{R}^n \setminus J_u) \, \mathrm{d}s.$$

By minimality of u we can apply the a priori estimates (3.6) to prove the equiboundedness of h, *i.e.* there exists $K = K(\Omega, f, \beta, C_0) > 0$ such that $h(t) \leq K$ for all $t > 0$.

We will show that h satisfies a differential inequality. For any $0 < t < \delta$, let us consider $u^t = u\chi_{\{ u>t \}}$, where $\chi_{\{ u>t \}}$ is the characteristic funtion of the set $\{ u > t \}$, as a competitor for u. We observe that, by Proposition 4.5,

$\Omega \subseteq \{ u > t \}$, so we have that

$$\begin{aligned}0 \geq &\mathcal{F}(u) - \mathcal{F}(u^t) \\ = &\int_{\{ u \leq t \} \setminus J_u} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n + \beta \int_{J_u \cap \{ u > t \}^{(0)}} \left(\underline{u}^2 + \overline{u}^2 \right) \, \mathrm{d}\mathcal{H}^{n-1} \\ &+ \beta \int_{J_u \cap \partial^* \{ u > t \}} \underline{u}^2 \, \mathrm{d}\mathcal{H}^{n-1} - \beta \int_{\partial^* \{ u > t \} \setminus J_u} u^2 \, \mathrm{d}\mathcal{H}^{n-1} \\ &+ C_0 \mathcal{L}^n \left(\{ 0 < u \leq t \} \right).\end{aligned}$$

Rearranging the terms,

$$\begin{aligned}&\int_{\{ u \leq t \} \setminus J_u} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n + \beta \int_{J_u \cap \{ u > t \}^{(0)}} \left(\underline{u}^2 + \overline{u}^2 \right) \, \mathrm{d}\mathcal{H}^{n-1} \\ &+ \beta \int_{J_u \cap \partial^* \{ u > t \}} \underline{u}^2 \, \mathrm{d}\mathcal{H}^{n-1} + C_0 \mathcal{L}^n \left(\{ 0 < u \leq t \} \right) \\ &\leq \beta t^2 P(\{ u > t \} ; \mathbb{R}^n \setminus J_u) = \beta t h'(t).\end{aligned} \tag{4.6}$$

On the other hand using Hölder's inequality, we have

$$h(t) \leq \left(\int_{\{ u \leq t \}} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{2}} \mathcal{L}^n \left(\{ 0 < u \leq t \} \right)^{\frac{1}{2n}} \left(\int_{\{ u \leq t \}} u^{2 \cdot 1^*} \, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{2 \cdot 1^*}}. \tag{4.7}$$

Classical Embedding of BV in L^{1^*} applied to $u^2 \chi_{\{ u \leq t \}}$, ensures

$$\left(\int_{\{ u \leq t \}} u^{2 \cdot 1^*} \, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{1^*}} \leq C(n) \left| \mathrm{D}\left(u^2 \chi_{\{ u \leq t \}} \right) \right| (\mathbb{R}^n),$$

and, using (4.6),

$$\begin{aligned}\left| \mathrm{D}\left(u^2 \chi_{\{ u \leq t \}} \right) \right| (\mathbb{R}^n) = &2 \int_{\{ u \leq t \}} u |\nabla u| \, \mathrm{d}\mathcal{L}^n + \int_{J_u \cap \{ u > t \}^{(0)}} \left(\underline{u}^2 + \overline{u}^2 \right) \, \mathrm{d}\mathcal{H}^{n-1} \\ &+ \int_{J_u \cap \partial^* \{ u > t \}} \underline{u}^2 \, \mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^* \{ u > t \} \setminus J_u} u^2 \, \mathrm{d}\mathcal{H}^{n-1} \\ \leq &2t \left(\mathcal{L}^n \left(\{ 0 < u \leq t \} \right) \int_{\{ u \leq t \} \setminus J_u} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n \right)^{\frac{1}{2}} + 3 t h'(t) \\ \leq &\left(2 \frac{\delta \beta}{\sqrt{C_0}} + 3 \right) t h'(t).\end{aligned} \tag{4.8}$$

Therefore, joining (4.7), (4.6), and (4.8), we have

$$h(t) \leq C_3 \left(t h'(t) \right)^{1 + \frac{1}{2n}}, \tag{4.9}$$

where

$$C_3 = \beta^{\frac{1}{2}} \left(\frac{\beta}{C_0}\right)^{\frac{1}{2n}} C(n)^{\frac{1}{2}} \left(2\frac{\delta\beta}{\sqrt{C_0}} + 3\right)^{\frac{1}{2}}.$$

By (4.9) we now want to show that there exists $\delta_0 = \delta_0(\Omega, f, p, \beta, C_0) > 0$ such that $h(t) = 0$ for every $0 \leq t < \delta_0$. Indeed assume by contradiction that $h(t) > 0$ for every $0 < t \leq \delta$. We have

$$\frac{h'(t)}{h(t)^{\frac{2n}{2n+1}}} \geq \frac{C_3^{-\frac{2n}{2n+1}}}{t}.$$

Integrating from $t_0 > 0$ to δ, we get

$$\left(h(\delta)^{\frac{1}{2n+1}} - h(t_0)^{\frac{1}{2n+1}}\right) \geq C_4 \log\left(\frac{\delta}{t_0}\right),$$

where

$$C_4 = \frac{C_3^{-\frac{2n}{2n+1}}}{2n+1}.$$

Then

$$h(t_0)^{\frac{1}{2n+1}} \leq h(\delta)^{\frac{1}{2n+1}} + C_4 \log\left(\frac{t_0}{\delta}\right).$$

Finally, for any

$$0 < t_0 \leq \tilde{\delta} = \delta \exp\left(-h(\delta)^{\frac{1}{2n+1}}/C_4\right),$$

we have $h(t_0) < 0$, which is a contradiction. Then, setting $\delta_0 = \delta \exp\left(-K^{\frac{1}{2n+1}}/C_4\right) \leq \tilde{\delta}$, we conclude that $h(t) = 0$ for any $0 < t < \delta_0$, from which we have

$$u \geq \delta_0$$

almost everywhere in $\{ u > 0 \}$. □

Remark 4.7. From Proposition 4.6, if u is a minimizer to problem (1.5), we have that

$$\partial^* \{ u > 0 \} \subseteq J_u \subseteq K_u. \tag{4.10}$$

Indeed, on $\partial^* \{ u > 0 \}$ we have that, by definition, $\underline{u} = 0$ and that, since $u \geq \delta_0$ $\mathcal{L}^n$-a.e. in $\{ u > 0 \}$, $\overline{u} \geq \delta_0$.

Proposition 4.8 (Density Estimates)**.** *There exist positive constants $C = C(\Omega, f, p, \beta, C_0)$, $c = c(\Omega, f, p, \beta, C_0)$ and $\delta_1 = \delta_1(\Omega, f, p, \beta, C_0)$ such that if u is a minimizer to problem (1.5) then for every $B_r(x)$ such that $B_r(x) \cap \Omega = \emptyset$, we have:*

(a) For every $x \in \mathbb{R}^n \setminus \Omega$,

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq Cr^{n-1}; \tag{4.11}$$

(b) For every $x \in K_u$,

$$\mathcal{L}^n(B_r(x) \cap \{ u > 0 \}) \geq cr^n; \tag{4.12}$$

(c) The function u has bounded support, namely

$$\{ u > 0 \} \subseteq B_{1/\delta_1}.$$

Proof. This theorem is a consequence of Proposition 4.6, since we immediately have

$$\int_{J_u \cap B_r(x)} \left(\underline{u}^2 + \overline{u}^2\right) \, \mathrm{d}\mathcal{H}^{n-1} \geq \delta_0^2 \mathcal{H}^{n-1}(J_u \cap B_r(x)),$$

and by minimality of u we have *(a)*

$$\begin{aligned} 0 &\geq \mathcal{F}(u) - \mathcal{F}(u\chi_{\mathbb{R}^n \setminus B_r(x)}) \\ &\geq \int_{J_u \cap B_r(x)} \left(\underline{u}^2 + \overline{u}^2\right) \, \mathrm{d}\mathcal{H}^{n-1} - \int_{\partial B_r(x) \setminus J_u} u^2 \, \mathrm{d}\mathcal{H}^{n-1} \\ &\geq \int_{J_u \cap B_r(x)} \left(\underline{u}^2 + \overline{u}^2\right) \, \mathrm{d}\mathcal{H}^{n-1} - \int_{\partial B_r(x) \cap \{ u > 0 \}^{(1)}} \left(\underline{u}^2 + \overline{u}^2\right) \, \mathrm{d}\mathcal{H}^{n-1} \\ &\geq \int_{J_u \cap B_r(x)} \left(\underline{u}^2 + \overline{u}^2\right) \, \mathrm{d}\mathcal{H}^{n-1} - 2\|u\|_\infty^2 \mathcal{H}^{n-1}\left(\partial B_r(x) \cap \{ u > 0 \}^{(1)}\right), \end{aligned}$$

where, in the second inequality, we have used (4.10). Thus we have

$$\mathcal{H}^{n-1}\left(J_u \cap B_r(x)\right) \leq \frac{2\|u\|_\infty^2}{\delta_0^2} \mathcal{H}^{n-1}(\partial B_r(x) \cap \{ u > 0 \}^{(1)}) \leq Cr^{n-1}, \tag{4.13}$$

where $C = C(\Omega, f, p, \beta, C_0) > 0$.

(b) We now want to use the estimate (4.11) together with the relative isoperimetric inequality in order to get a differential inequality for the volume of $B_r(x) \cap \{ u > 0 \}^{(1)}$. Let $x \in K_u$, then for almost every r we have

$$\begin{aligned} 0 < V(r) := \mathcal{L}^n(B_r(x) \cap \{ u > 0 \}^{(1)}) &\leq k \, P(B_r(x) \cap \{ u > 0 \}^{(1)})^{\frac{n}{n-1}} \\ &\leq k \, \mathcal{H}^{n-1}(\partial B_r(x) \cap \{ u > 0 \}^{(1)})^{\frac{n}{n-1}}, \end{aligned}$$

where $k = k(\Omega, f, p, \beta, C_0) > 0$, and in the last inequality we used that (4.10) and (4.13) imply

$$\begin{aligned} P\left(B_r(x) \cap \{ u > 0 \}^{(1)}\right) &\leq \mathcal{H}^{n-1}\left(\partial B_r(x) \cap \{ u > 0 \}^{(1)}\right) + \mathcal{H}^{n-1}\left(J_u \cap B_r(x)\right) \\ &\leq \left(1 + \frac{2\|u\|_\infty^2}{\delta_0^2}\right) P(B_r(x); \{ u > 0 \}^{(1)}). \end{aligned}$$

Then we have

$$\frac{V'(r)}{V(r)^{\frac{n-1}{n}}} \geq \frac{1}{k},$$

which implies

$$\mathcal{L}^n(B_r(x) \cap \{ u > 0 \}^{(1)}) \geq c\, r^n.$$

(c) Finally, let $x \in K_u$ such that $d(x, \partial\Omega) \geq 1/\delta_1$. From (4.12), noticing that $\mathcal{F}(u) \leq \mathcal{F}(0) = 0$, we have that

$$c\, \delta_1^{-n} \leq \mathcal{L}^n(\{ u > 0 \} \setminus \Omega) \leq \frac{2\|u\|_\infty}{C_0} \int_\Omega f \, \mathrm{d}\mathcal{L}^n,$$

which is a contradiction if δ_1 is sufficiently small. Then the thesis is given by (4.10). □

Finally, we have

Proposition 4.9 (Lower Density Estimate)**.** *There exists a positive constant $c = c(\Omega, f, p, \beta, C_0)$ such that if u is a minimizer to problem* (1.5) *then*

1. *For any $x \in K_u$ and $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$,*

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \geq c r^{n-1};$$

2. *J_u is essentially closed, namely*

$$\mathcal{H}^{n-1}(K_u \setminus J_u) = 0;$$

The proof of Proposition 4.9 relies on classical techniques used in [9] to prove density estimates for the jump set of almost-quasi minimizers of the Mumford-Shah functional. We refer to Theorem 5.1 of [7] and Corollary 5.4 of [7] for the details of the proof.

Proof of Theorem 1.2. The result is obtained by joining Proposition 4.6, Proposition 4.8, and Proposition 4.9. □

Remark 4.10. Given the summability assumption on the function f and the lower bound given in Proposition 4.6, we have that minimizers to (1.2) are almost-quasi-minimizers of the functional $\mathcal{G}$, defined on $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap \mathrm{W}^{1,2}(\Omega)$ as

$$\mathcal{G}(v) = \int_{\mathbb{R}^n} |\nabla v|^2 \, \mathrm{d}\mathcal{L}^n + \lambda \mathcal{H}^{n-1}(J_v),$$

that is, there exists $C(\Omega, f, p, \beta, C_0) > 0$, $\Lambda(\Omega, f, p, \beta, C_0) \geq \lambda$ and $\alpha(n,p) > n-1$ such that, if $B_r(x)$ is a ball of radius $r \leq 1$, and $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap \mathrm{W}^{1,2}(\Omega)$, with $\{ u \neq v \} \subset B_r(x)$, then

$$\mathcal{G}_\lambda(u; B_r(x)) \leq \mathcal{G}_\Lambda(v; B_r(x)) + C r^\alpha,$$

where

$$\mathcal{G}_\lambda(v; B_r(x)) := \int_{B_r(x)} |\nabla v|^2 \, \mathrm{d}\mathcal{L}^n + \lambda \mathcal{H}^{n-1}(J_v \cap B_r(x)).$$

Indeed, let u be a minimizer to (1.2), let $B_r(x)$ be a ball of radius $r \leq 1$, and let $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap \mathrm{W}^{1,2}(\Omega)$, with $\{ u \neq v \} \subset B_r(x)$, and let

$$w = \min \{ \max \{ v, 0 \}, \|u\|_\infty \}.$$

By minimality of u we have that

$$\mathcal{G}_\lambda(u;B_r(x)) \leq \mathcal{G}_\Lambda(v;B_r(x)) + Cr^n,$$

where $\lambda = \beta\delta_0^2$ and $\Lambda = 2\beta\|u\|_\infty^2$. Moreover,

$$\int_{\Omega\cap B_r(x)} fu \ \mathrm{d}\mathcal{L}^n \leq \|f\|_{p,\Omega}\|u\|_\infty \mathcal{L}^n(B_r)^{1/p'} = C(\Omega,f,p,\beta,C_0)r^\alpha,$$

where

$$n > \alpha = \frac{n}{p'} > n-1.$$

Finally, we have

$$\mathcal{G}_\lambda(u;B_r(x)) \leq \mathcal{G}_\Lambda(v;B_r(x)) + Cr^\alpha.$$

Such a minimality property can be used to prove that the lower density estimate in Proposition 4.9 still holds even when $B_r(x)\cap\Omega$ is non-empty. Indeed, the above density estimate is a consequence of the following decay lemma

Lemma 4.11 (Decay lemma). *Let $1 > \gamma > n-\alpha$. There exists $\tau_0 = \tau_0(n,\Omega,\gamma,\lambda) > 0$ such that for every $\tau_0 > \tau > 0$ there exist $r_0 = r_0(\tau,\Omega)$, $\varepsilon_0 = \varepsilon_0(\tau,\Omega) > 0$ such that, if $x_0 \in \partial\Omega$, $r_0 > r > 0$, and u is a almost-quasi minimizer on $B_r = B_r(x_0)$ for the functional $\mathcal{G}$ such that*

$$\mathcal{H}^{n-1}(J_u\cap B_r) \leq \varepsilon_0 r^{n-1},$$

then we have that either

$$\mathcal{G}_\lambda(u;B_r) \leq r^{n-\gamma},$$

or

$$\mathcal{G}_\lambda(u;B_{\tau r}) \leq \tau^{n-\gamma}\mathcal{G}_\lambda(u;B_r).$$

Proof. The proof of the decay lemma is similar to the one in Lemma 5.3 of [7], Section 4 of [6], Lemma 4.9 of [9]; the main difference is in the construction of the blow-up sequence of almost-quasi minimizers.

Let u_k be a sequence of almost-quasi minimizer on B_{r_k} contradicting the lemma, with $\lim_k r_k = 0$. To reach a contradiction one usually constructs a sequence of functions $\tilde{v}_k$ on the unit ball, related to the sequence u_k, that converges to an harmonic function v. In order to prove that v is harmonic we construct a sequence of admissible test functions ψ_k on B_{r_k} and use the minimality property of u_k. If $d(x_0,\Omega) > 0$, then the test function are only required to be in $\mathrm{SBV}(B_{r_k})$, while, if $x_0 \in \partial\Omega$ the additional constraint $\psi_k \in \mathrm{SBV}(B_{r_k})\cap \mathrm{W}^{1,2}(\Omega\cap B_{r_k})$ should be treated with more carefulness.

Without loss of generality let $x_0 = 0$ and let $E_k = r_k^{2-n}\mathcal{G}_\lambda(u_k;B_{r_k})$, and define

$$v_k(x) = \frac{1}{E_k^{1/2}}u_k(r_k x).$$

For any k, we extend $u_k \in \mathrm{W}^{1,2}(\Omega \cap B_{r_k})$ to $Lu_k \in \mathrm{W}^{1,2}(B_{r_k})$, which is a function such that $u_k - Lu_k \equiv 0$ in Ω. Let us define, with a slight abuse of notation,

$$Lv_k(x) = \frac{1}{E_k^{1/2}} Lu_k(r_k x), \qquad\qquad w_k = v_k - Lv_k,$$

so that, by construction, and by properties of the blow-up,

$$\liminf_k \mathcal{L}^n(\{w_k = 0\}) \geq \liminf_k r_k^{-n} \mathcal{L}^n(\Omega \cap B_{r_k}) > 0. \tag{4.14}$$

This is the key property: by Poincaré inequality in SBV, there exist $\tilde{w}_k$ truncated functions, such that

$$\lim_k \mathcal{L}^n(\{w_k \neq \tilde{w}_k\}) = 0, \tag{4.15}$$

and, up to subtracting medians, w_k converge in L^2 to some Sobolev function. By (4.14) and (4.15), and considering that $\tilde{w}_k$ is a truncation of w_k, then for big enough k, up to $\mathcal{L}^n$-negligible sets,

$$\{w_k = 0\} \subseteq \{w_k = \tilde{w}_k\}.$$

This means that if we define $\tilde{v}_k = \tilde{w}_k + Lv_k$, then the scaled back functions

$$\tilde{u}_k(x) := E_k^{1/2} \tilde{v}_k\left(\frac{x}{r_k}\right)$$

respect the property $\tilde{u}_k \equiv u_k$ in $\Omega \cap B_{r_k}$. Moreover, it is possible to choose an extension L (see Lem. 4.12) such that, combining the Poincaré inequality in SBV and the Poincaré inequality in $\mathrm{W}^{1,2}$, then there exist constants c_k such that $\tilde{v}_k - c_k$ converge in L^2 to a function $v \in \mathrm{W}^{1,2}(B_1)$. This ensures that, if we take $\rho < \rho'$ small enough, η cut-off functions between B_ρ and $B_{\rho'}$, and $\varphi \in \mathrm{W}^{1,2}(B_1)$, then the test functions $\psi_k = E_k^{1/2} \varphi_k(x/r_k)$, with

$$\varphi_k = \Big(\eta\big(\varphi + c_k\big) + (1-\eta)\tilde{v}_k\Big)\chi_{B_{\rho'}} + v_k \chi_{B_1 \setminus B_{\rho'}},$$

are admissible test functions for any $\varphi \in \mathrm{W}^{1,2}(B_1)$, leading to similar computations that can be found in the aforementioned papers. □

Lemma 4.12. *Let Ω an open set with Lipschitz boundary, and let $x_0 \in \partial\Omega$. There exist positive constants $\rho_0 = \rho_0(\Omega, x_0)$, $C = C(\Omega, x_0)$, $\delta = \delta(\Omega, x_0) > 1$, and an extension operator*

$$L : \mathrm{W}^{1,2}(\Omega) \to \mathrm{W}^{1,2}(B_{\rho_0}(x_0))$$

such that, for any $u \in \mathrm{W}^{1,2}(\Omega)$, and for any $r < \rho_0$, we have that $Lu \equiv u$ in $\Omega \cap B_{\rho_0}(x_0)$ and

$$\int_{B_r(x_0)} |\nabla Lu|^2 \, \mathrm{d}\mathcal{L}^n \leq C \int_{\Omega \cap B_{\delta r}(x_0)} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n. \tag{4.16}$$

Proof. We can assume without loss of generality that $x_0 = 0$, and, if s is small enough, we have that, up to rotations,

$$\Omega \cap B_s = \{\, (x', x_n) \in B_s \mid \gamma(x') < x_n \,\},$$

for a suitable Lipschitz function γ, with $\gamma(0) = 0$. We denote by Φ the diffeomorphism that flattens the boundary $\partial\Omega$, namely

$$\Phi(x', x_n) = (x', x_n - \gamma(x')), \qquad \Phi^{-1}(y', y_n) = (y', y_n + \gamma(y')).$$

Let $M = \|\nabla\gamma\|_\infty$, we claim that for any $r < (1+M)^{-2}s$ we have

$$\Phi(B_r) \subset B_{(1+M)r} \subset \Phi(B_{(1+M)^2 r}). \tag{4.17}$$

Indeed, let $x \in B_r$, then

$$|\Phi(x)|^2 \leq |x|^2 + 2|x_n\gamma(x)| + |\gamma(x)|^2,$$

so that, we have

$$|\gamma(x)| \leq |x| \|\nabla\gamma\|_\infty,$$

and then

$$|\Phi(x)| \leq (1+M)r.$$

In a similar way, we have that for any $x \in B_{(1+M)r}$,

$$|\Phi^{-1}(x)| \leq (1+M)^2 r,$$

thus the claim is proved.

Let us take a ball B_t such that $\Phi^{-1}(B_t) \subset B_s$, which we can find thanks to (4.17), and let us reflect the function $v(x) = u(\Phi(x))$ as follows: for any $x \in B_t$, we define

$$Lv(x) = \begin{cases} v(x) & \text{if } x_n < 0, \\ -3v(x', -x_n) + 4v\left(x', -\frac{x_n}{2}\right) & \text{if } x_n > 0, \end{cases}$$

which is still a Sobolev function in B_t. Moreover, we have

$$\int_{B_t} |\nabla Lv|^2 \, \mathrm{d}\mathcal{L}^n \leq C \int_{B_t \cap \{x_n < 0\}} |\nabla v|^2 \, \mathrm{d}\mathcal{L}^n,$$

where C is independent of Ω. We put $Lu(x) = Lv(\Phi^{-1}(x))$, and by change of variables, we get

$$\int_{\Phi^{-1}(B_t)} |\nabla Lu|^2 \, \mathrm{d}\mathcal{L}^n \leq C(\Omega) \int_{\Omega \cap \Phi^{-1}(B_t)} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n. \tag{4.18}$$

Finally, taking $\rho_0 = (1+M)^{-2}s$, and $t_0 = (1+M)^{-1}s$, we have $B_{\rho_0} \subset \Phi^{-1}(B_{t_0})$. Therefore, denoting by $\delta = (1+M)^2$, by (4.17) and (4.18), we get, for $r < \rho_0$,

$$\int_{B_r(x_0)} |\nabla Lu|^2 \, \mathrm{d}\mathcal{L}^n \leq \int_{\Phi^{-1}(B_{\sqrt{\delta} r})} |\nabla Lu|^2 \, \mathrm{d}\mathcal{L}^n \leq C \int_{\Omega \cap \Phi^{-1}(B_{\sqrt{\delta} r})} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n \leq C \int_{\Omega \cap B_{\delta r}} |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n.$$

□

Remark 4.13. Notice that if Ω is bounded, the constants in Lemma 4.12 can be chosen independent of the point x_0.

Remark 4.14. Let u be a minimizer to (1.5) and let $A = \{\, \overline{u} > 0 \,\} \setminus K_u$, then the boundary of A is equal to K_u: in first place, assume by contradiction that there exists an $x \in (\partial A) \setminus K_u$, then u is superharmonic in a small ball centered in x with radius r. Therefore, being

$$\{\, u > 0 \,\} \cap B_r(x) \neq \emptyset,$$

it is necessary that $u > 0$ in the entire ball, and then $x \notin \partial A$, which is a contradiction. In other words,

$$\partial A \subseteq K_u$$

By the same argument we also have that A is open, and moreover $J_u \subseteq \partial A$, then

$$K_u \subseteq \partial A.$$

In particular, the pair (A, u) is a minimizer for the functional

$$\mathcal{F}(E, v) = \int_E |\nabla v|^2 \, \mathrm{d}\mathcal{L}^n - 2 \int_\Omega f v \, \mathrm{d}\mathcal{L}^n + \int_{\partial E} \left(\underline{v}^2 + \overline{v}^2 \right) \, \mathrm{d}\mathcal{H}^{n-1} + C_0 \mathcal{L}^n(E \setminus \Omega)$$

over all pairs (E, v) with E open set of finite perimeter containing Ω and $v \in \mathrm{W}^{1,2}(E)$.

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