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GLOBAL CONTROL ASPECTS FOR LONG WAVES IN NONLINEAR DISPERSIVE MEDIA

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Abstract. A class of models of long waves in dispersive media with coupled quadratic nonlinearities on a periodic domain $\mathbb{T}$ are studied. We used two distributed controls, supported in $\omega \subset \mathbb{T}$ and assumed to be generated by a linear feedback law conserving the *"mass" (or "volume")*, to prove global control results. The first result, using spectral analysis, guarantees that the system in consideration is locally controllable in $H^s(\mathbb{T})$, for $s \geq 0$. After that, by certain properties of Bourgain spaces, we show a property of global exponential stability. This property together with the local exact controllability ensures for the first time in the literature that long waves in nonlinear dispersive media are globally exactly controllable in large time. Precisely, our analysis relies strongly on the *bilinear estimates* using the Fourier restriction spaces in two different dispersions that will guarantee a global control result for coupled systems of the Korteweg–de Vries type. This result, of independent interest in the area of control of coupled dispersive systems, provides a necessary first step for the study of global control properties to the coupled dispersive systems in periodic domains.

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1. Introduction

Nonlinear dispersive wave equations arise in a number of important application areas. Because of this, and because their mathematical properties are interesting and subtle, they have seen enormous development since the 1960s when they first came to the fore.[1] The theory for a single nonlinear dispersive wave equation is well developed by now, though there are still interesting open issues, however, the theory for coupled systems of such equations is much less developed, though they, too, arise as models of a range of physical phenomena.

Considered here is a class of such systems, namely coupled Korteweg–de Vries (KdV) equations. The systems we have in mind take the form

$$\begin{cases}\partial_t u + \partial_x^3 u + \partial_x P(u,v) = 0, & x \in \mathbb{T},\ t \in \mathbb{R},\\ \partial_t v + \alpha\partial_x^3 v + \partial_x Q(u,v) = 0, & x \in \mathbb{T},\ t \in \mathbb{R},\end{cases} \tag{1.1}$$

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[1] See [29] for a sketch of the early history of the subject.

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which comprises two linear Korteweg–de Vries equations coupled through their nonlinearity. Here, $u = u(x,t)$ and $v = v(x,t)$ are real-valued functions of variables $(x,t) \in \mathbb{T} \times \mathbb{R}$ and the nonlinearities P and Q are taken to be homogeneous quadratic polynomials.

As far as we know, there are no studies of the *global control properties* of this kind of coupled system in a periodic domain. Thus, in this article, the goal is to fill this gap by focusing on the global exact controllability and global asymptotic behavior to the solutions of the coupled system of KdV equations (1.1) when we add two control inputs in each equation and considering initial conditions $(u(x,0), v(x,0)) = (u_0(x), v_0(x))$ belonging in $H^s(\mathbb{T}) \times H^s(\mathbb{T})$, for any $s \geq 0$.

1.1. Models in the literature

Such systems arise as models for wave propagation in physical systems where both nonlinear and dispersive effects are important. Moreover, their close relatives arise as models for waves in a number of situations. Before presenting details about the problem that we will study, let us start to list a few specializations of systems (1.1) that appeared in the literature.

1.1.1. The coupled KdV system

The classical Boussinesq systems were first derived by Boussinesq in [6], to describe the two-way propagation of small amplitude, long wavelength gravity waves on the surface of the water in a canal. These systems and their higher-order generalizations also arise when modeling the propagation of long-crested waves on large lakes or on the ocean and in other contexts. Recently Bona *et al.* [1], derived a four-parameter family of Boussinesq systems to describe the motion of small amplitude long waves on the surface of an ideal fluid under the gravity force and in situations where the motion is sensibly two dimensional. More precisely, they studied a family of systems of the form

$$\begin{cases} \eta_t + w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} = 0, \\ w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0. \end{cases} \tag{1.2}$$

In (1.2), η is the elevation from the equilibrium position, and $w = w_\theta$ is the horizontal velocity in the flow at height θh, where h is the undisturbed depth of the liquid. The parameters a, b, c, d, that one might choose in a given modeling situation, are required to fulfill the relations

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad c + d = \frac{1}{2}(1 - \theta^2) \geq 0, \quad \theta \in [0,1],$$

where $\theta \in [0,1]$ specifies which horizontal velocity the variable w represents (*cf.* [1]). Consequently,

$$a + b + c + d = \frac{1}{3}.$$

As it has been proved in [1], the initial value problem for the linear system associated with (1.2) is well-posed on $\mathbb{R}$ if either C_1 or C_2 is satisfied, where

$$(C_1) \;\; b, d \geq 0, \; a \leq 0, \; c \leq 0;$$
$$(C_2) \;\; b, d \geq 0, \; a = c > 0.$$

When $b = d = 0$ and (C_2) is satisfied, then necessarily $a = c = 1/6$. Nevertheless, the scaling $x \to x/\sqrt{6}$, $t \to t/\sqrt{6}$ gives an system equivalent to (1.2) for which $a = c = 1$, namely

$$\begin{cases} \eta_t + w_x + w_{xxx} + (\eta w)_x = 0, \\ w_t + \eta_x + \eta_{xxx} + ww_x = 0, \end{cases} \tag{1.3}$$

which is the so-called *Boussinesq system of KdV-KdV type*.

1.1.2. *Gear-Grimshaw system*

In [15] a complex system of equations was derived by Gear and Grimshaw as a model to describe the strong interaction of two-dimensional, weakly nonlinear, long, internal gravity waves propagating on neighboring pycnoclines in a stratified fluid, where the two waves correspond to different modes. It has the structure of a pair of KdV equations with both linear and nonlinear coupling terms and has been the object of intensive research in recent years. The system can be read as follows

$$\begin{cases} u_t + uu_x + u_{xxx} + av_{xxx} + a_1 vv_x + a_2(uv)_x = 0, \\ cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + a_2 buu_x + a_1 b(uv)_x = 0, \end{cases} \tag{1.4}$$

where $a_1, a_2, a, b, c, r \in \mathbb{R}$ are physical constants and we may assume that

$$1 - a^2 b > 0 \quad \text{and} \quad b, c > 0.$$

1.1.3. *Majda-Biello system*

The following coupled system

$$\begin{cases} u_t + u_{xxx} = -vv_x, \\ v_t + \alpha v_{xxx} = -(uv)_x, \end{cases}$$

when $\alpha \in (0, 1)$,[2] was proposed by Majda and Biello in [27] as a reduced asymptotic model to study the nonlinear resonant interactions of long wavelength equatorial Rossby waves and barotropic Rossby waves.

1.1.4. *Hirota-Satsuma system*

In the eighties, Hirota and Satsuma introduced in [17] the set of two coupled KdV equations, namely

$$\begin{cases} u_t + au_{xxx} = 6auu_x + bvv_x, \\ v_t + v_{xxx} = -3uv_x, \end{cases}$$

with $a \neq 0$, where $a, b \in \mathbb{R}$ are constants that appear in the model deduction. This model describes the interaction of two long waves with different dispersion relations.

We caution that this is only a small sample of the extant equations with a similar structure to the system (1.1). For an extensive review of the physical meanings of these equations, as well as local and global well-posedness results, the authors suggest the following two nice references [2, 40].

[2] The parameter $\alpha > 0$ depends upon the Rossby wave in question and it typically has a value near 1.

1.2. Setting of the problem

Since any solution (u,v) of system (1.1) has its components with invariant mean value, we can introduce the numbers $[u]:=\beta$ and $[v]:=\gamma$. Setting $\widetilde{u}=u-\beta$ and $\widetilde{v}=v-\gamma$, we obtain $[\widetilde{u}]=[\widetilde{v}]=0$ and $(\widetilde{u},\widetilde{v})$ solves

$$\begin{cases}\partial_t\widetilde{u}+\partial_x^3\widetilde{u}+(2\beta A+\gamma B)\partial_x\widetilde{u}+(\beta B+\gamma C)\partial_x\widetilde{v}+\partial_x P(\widetilde{u},\widetilde{v})=0, & x\in\mathbb{T},\ t\in\mathbb{R},\\ \partial_t\widetilde{v}+\alpha\partial_x^3\widetilde{v}+(\beta B+\gamma C)\partial_x\widetilde{u}+(2\gamma D+\beta C)\partial_x\widetilde{v}+\partial_x Q(\widetilde{u},\widetilde{v})=0, & x\in\mathbb{T},\ t\in\mathbb{R}.\end{cases}\tag{1.5}$$

Throughout the paper, we will denote $\mu:=2\beta A+\gamma B$, $\eta:=\beta B+\gamma C$, $\zeta:=2\gamma D+\beta C$ which are real constants. Thus, as mentioned before, this article presents for the first time the global control results for a class of models of long waves with coupled quadratic nonlinearities. Precisely, thanks to (1.5) we will study the following system

$$\begin{cases}\partial_t u+\partial_x^3 u+\mu\partial_x u+\eta\partial_x v+\partial_x P(u,v)=p(x,t), & x\in\mathbb{T},\ t\in\mathbb{R},\\ \partial_t v+\alpha\partial_x^3 v+\zeta\partial_x v+\eta\partial_x u+\partial_x Q(u,v)=q(x,t), & x\in\mathbb{T},\ t\in\mathbb{R},\\ (u(x,0),v(x,0))=(u_0(x),v_0(x)), & x\in\mathbb{T},\end{cases}\tag{1.6}$$

with quadratic nonlinearities

$$\begin{cases}P(u,v)=Au^2+Buv+\frac{C}{2}v^2,\\ Q(u,v)=Dv^2+Cvu+\frac{B}{2}u^2,\end{cases}\tag{1.7}$$

where α, A, B, C and D real constants, from a control point of view with forcing terms $p=p(x,t)$ and $q=q(x,t)$ added to the equation as two control inputs on the periodic domain. Therefore, the following classical issues related to control theory are considered in this work.

Problem 1.1 (Exact controllability)**.** Given an initial state (u_0,v_0) and a terminal state (u_1,v_1) in a certain space, can one find two appropriate control inputs p and q so that the equation (1.6) admits a solution (u,v) which satisfies $(u(\cdot,0),v(\cdot,0)=(u_0,v_0)$ and $(u(\cdot,T),v(\cdot,T))=(u_1,v_1)$?

Problem 1.2 (Stabilizability)**.** Can one find some (linear) feedback controls $p=K_1(u,v)$ and $q=K_2(u,v)$ such that the resulting closed-loop system (1.6) is stabilized, *i.e.*, its solution (u,v) tends to zero in an appropriate space as $t\to\infty$?

Note that system (1.6) has the *mass (or volume)* and the *energy* conserved, which are

$$M_1(u,v)=\int_{\mathbb{T}}u(x,t)\mathrm{d}x,\qquad M_2(u,v)=\int_{\mathbb{T}}v(x,t)\mathrm{d}x,\quad E(u,v)=\frac{1}{2}\int_{\mathbb{T}}(u^2(x,t)+v^2(x,t))\mathrm{d}x,$$

respectively. In order to keep the mass M_1 and M_2 conserved, the two control inputs $p(x,t)$ and $q(x,t)$ will are chosen to be of the form $Gf(x,t)$ and $Gh(x,t)$, respectively, where this operator is defined by

$$(G\ell)(x,t):=g(x)\left(\ell(x,t)-\int_{\mathbb{T}}g(y)\ell(y,t)\mathrm{d}y\right),\tag{1.8}$$

where f and h are considered the new control inputs, and $g(x)$ is a given nonnegative smooth function such that $\{g>0\}=\omega\subset\mathbb{T}$ and

$$2\pi[g]=\int_{\mathbb{T}}g(x)\mathrm{d}x=1.$$

Due to such a choice of g, it is easy to see that for any solution (u, v) of (1.6) with $p = Gf$ and $q = Gh$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}M_1(u,v) = \int_{\mathbb{T}} Gf(x,t)\mathrm{d}x = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}M_2(u,v) = \int_{\mathbb{T}} Gh(x,t)\mathrm{d}x = 0,$$

that is, the *mass* of the system is indeed conserved.

To stabilize system (1.6) we want to employ two feedback control laws that help make the energy of the system decrease, that is, $E'(u, v) \leq 0$. We will see that this is possible, and so makes sense to show global answers to the Problems 1.1 and 1.2, mentioned before. Before it, let us give a state of the art of control theory for KdV-type systems.

1.3. State of the art

The study of the controllability and stabilization to the KdV equation started with the works of Russell and Zhang [36, 37] for the system

$$u_t + uu_x + u_{xxx} = f, \tag{1.9}$$

with periodic boundary conditions and an internal control f. Since then, both controllability and stabilization problems have been intensively studied [7, 9, 12, 34, 42].

Equation (1.9) is known to possess an infinite set of conserved integral quantities, of which the first three are

$$I_1(t) = \int_{\mathbb{T}} u(x,t)\mathrm{d}x, \qquad I_2(t) = \int_{\mathbb{T}} u^2(x,t)\mathrm{d}x \quad \text{and} \quad I_3(t) = \int_{\mathbb{T}} \left(u_x^2(x,t) - \frac{1}{3}u^3(x,t)\right)\mathrm{d}x.$$

From the historical origins of the KdV equation involving the behavior of water waves in a shallow channel [5, 11, 21, 29], it is natural to think of I_1 and I_2 as expressing conservation of *volume (or mass)* and *energy*, respectively. The Cauchy problem for equation (1.9) has been intensively studied for many years (see [4, 18, 19] and the references therein).

The first work of Russell and Zhang [36] is purely linear. They had to wait for several years to extend their results to the nonlinear systems [37] until Bourgain [4] discovered a subtle smoothing property of solutions of the KdV equation posed on a periodic domain, thanks to which Bourgain was able to show that the Cauchy problem (1.9) is well-posed in the space $H^s(\mathbb{T})$, for any $s \geq 0$. This novelty discovered the smoothing property of the KdV equation has played a crucial role in the proof of the results in [37].

Question 1: *Can one still guide the system by choosing appropriate control input h from a given initial state u_0 to a given terminal state u_1 when u_0 or u_1 have large amplitude?*

Question 2: *Do the large amplitude solutions of the closed-loop system (1.9) decay exponentially as $t \to \infty$?*

Laurent *et al.* [24] gave positive answers to these questions. These answers are established with the aid of certain properties of propagation of compactness and regularity in Bourgain spaces for the solutions of the associated linear system of (1.9).

We have to mention that other works in the literature deal with the models having similar structure as the system (1.1) in periodic domains. Micu *et al.* [28] gave a rather complete picture of the control properties of (1.2) on a periodic domain with a locally supported forcing term. According to the values of the four parameters a, b, c, and d, the linearized system may be controllable in any positive time, or only in large time, or it may not be controllable at all.

Recently, Capistrano-Filho *et al.* [10] considered the problem of controlling pointwise, by means of a time-dependent Dirac measure supported by a given point, a coupled system of two Korteweg–de Vries equations (1.4) on the unit circle. More precisely, using spectral analysis and Fourier expansion they proved, under general assumptions on the physical parameters of the system, a pointwise observability inequality that leads to the pointwise controllability by using two control functions. In addition, with a uniqueness property proved for the linearized system without control, they are able to show pointwise controllability when only one control function acts internally.

There are two important points to say about the results shown in [10, 28]. The first one is that the results presented in [28] are purely local (controllability and stability), the authors did not use propagation of singularities, provided by the Bourgain spaces, to obtain more general results. In fact, one of the problems left in [28] is to prove global results for systems like (1.6). With respect to the results proved in [10], the results are purely linear, and extensions to the non-linear system are only possible in regular spaces.

1.4. Notation and Main results

Let us introduce some notation and present the main results of the manuscript.

We denote $\mathcal{D}(\mathbb{T})$ the space of periodic distributions whose dual space is $C^\infty(\mathbb{T})$. The Fourier series of periodic distributions is given by

$$\mathcal{F}f(k) = \widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\mathrm{d}x, \quad k \in \mathbb{Z}$$

and the inverse Fourier series by

$$\mathcal{F}^{-1}f(x) = \sum_{k\in\mathbb{Z}} e^{ikx} f(x).$$

For $s > 0$, we use the operator $D^s = (-\Delta)^{\frac{1}{2}}$ given on the Fourier side as

$$\widehat{D^s f}(k) = |k|^s \widehat{f}(\xi).$$

Similarly, we have the operators J^s given on the Fourier side as

$$\widehat{J^s f}(k) = \langle k\rangle^s \, \widehat{f}(k)$$

where $\langle k\rangle := (1+|k|) \sim (1+|k|^2)^{\frac{1}{2}}$. Here we define the $H^s(\mathbb{R})$ Sobolev spaces, for $s \in \mathbb{R}$

$$H^s(\mathbb{T}) = \{f \in \mathcal{D}(\mathbb{T}) \ : \ \|f\|_s := \|J^s f\| < \infty\}$$

For the Cartesian spaces $H^s(\mathbb{T}) \times H^s(\mathbb{T})$ we define $\|(u,v)\|_s := \|(u,v)\|_{H^s(\mathbb{T})\times H^s(\mathbb{T})} = \|u\|_s + \|v\|_s$. Throughout this paper we will denote the norm $\|(\cdot,\cdot)\|_{L^2(\mathbb{T})\times L^2(\mathbb{T})}$ simply by $\|(\cdot,\cdot)\|$. Let X be one of the previously defined spaces, we will denote X_0 the function space belongs in X with mean-value null, *i.e.*, $X_0 := \{u \in X \ : \ [u] = 0\}$.

This manuscript aims to address the control and stabilization (global) issues. Precisely, we want to give answers for both questions (see Probs. 1.1 and 1.2) presented at the beginning of this introduction. As a first result, we will analyze the exact controllability for the following linear system

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x u + \eta\partial_x v = Gf, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ \partial_t v + \alpha\partial_x^3 v + \zeta\partial_x v + \eta\partial_x u = Gh, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & x \in \mathbb{T}. \end{cases} \tag{1.10}$$

Here, f and g are defined as two control inputs and the operator G is given by (1.8). We have established the following.

Theorem 1.3. *Let $T>0$ and $s\geq 0$ be given. Then for any (u_0,v_0), $(u_1,v_1)\in H_0^s(\mathbb{T})\times H_0^s(\mathbb{T})$, there exists a pair of control functions $(f,h)\in L_0^2(\mathbb{T})\times L_0^2(\mathbb{T})$, such that system (1.10) has a solution in the class*

$$(u,v)\in C([0,T];H_0^s(\mathbb{T}))\times C([0,T];H_0^s(\mathbb{T}))$$

satisfying

$$(u(x,0),v(x,0))=(u_0,v_0) \quad \text{and} \quad (u(x,T),v(x,T))=(u_1,v_1).$$

Taking advantage of the results obtained by Bourgain [4], we are able to extend the previous local result to the nonlinear system, which is represented by,

$$\begin{cases}\partial_t u+\partial_x^3 u+\mu\partial_x u+\eta\partial_x v+\partial_x P(u,v)=Gf, & x\in\mathbb{T},\ t\in\mathbb{R},\\ \partial_t v+\alpha\partial_x^3 v+\zeta\partial_x v+\eta\partial_x u+\partial_x Q(u,v)=Gh, & x\in\mathbb{T},\ t\in\mathbb{R},\\ (u(x,0),v(x,0))=(u_0(x),v_0(x)), & x\in\mathbb{T},\end{cases} \tag{1.11}$$

where $P(u,v)$, $Q(u,v)$ are defined by (1.7), G is represented by (1.8), with f and g are control inputs. Thus, our second result deals with the asymptotic behavior of the solutions of (1.6). In order to stabilize system (1.11), choose the two feedback controls

$$f=-G^*L^{-1}_{1,\mu,\lambda}u \quad \text{and} \quad h=-G^*L^{-1}_{\alpha,\zeta,\lambda}v,$$

in (1.11), to transform it in a resulting closed-loop system reads as follows

$$\begin{cases}\partial_t u+\partial_x^3 u+\mu\partial_x u+\eta\partial_x v+\partial_x P(u,v)=-K_{1,\mu,\lambda}u, & x\in\mathbb{T},\ t\in\mathbb{R},\\ \partial_t v+\alpha\partial_x^3 v+\zeta\partial_x u+\eta\partial_x u+\partial_x Q(u,v)=-K_{\alpha,\zeta,\lambda}v, & x\in\mathbb{T},\ t\in\mathbb{R},\\ (u(x,0),v(x,0))=(u_0(x),v_0(x)), & x\in\mathbb{T},\end{cases} \tag{1.12}$$

with the damping mechanism defined by

$$K_{\beta,\gamma,\lambda}:=GG^*L^{-1}_{\beta,\gamma,\lambda}.$$

Here, $L_{\beta,\gamma,\lambda}$ is a bounded linear operator from $H^s(\mathbb{T})$ to $H^s(\mathbb{T})$, $s\geq 0$, for details see Section 3. So, as for Problem 1.2, we have the following affirmative answer.

Theorem 1.4. *Let $s\geq 0$ and $\gamma\in\mathbb{R}$ be given. There exists a constant $\kappa>0$ such that for any $u_0,v_0\in H_0^s(\mathrm{T})$ the corresponding solution (u,v) of the system (1.12) satisfies*

$$\|(u,v)\|_s\leq a_{s,\gamma}\left(\|(u_0,v_0)\|_0\right)e^{-\kappa t}\|(u_0,v_0)\|_s,$$

for all $t\geq 0$. Here $a_{s,\gamma}:\mathbb{R}^+\to\mathbb{R}^+$ is a nondecreasing continuous function depending on s and γ.

To finalize, observe that Theorem 1.3 is purely linear. Thanks to Theorem 1.4 we guarantee global controllability for long waves, thus responding to Problem 1.1. The result can be read as follows.

Theorem 1.5. *Let $s\geq 0$ and $R_0>0$ be given. Consider $\alpha<0$ and $|\mu|+|\zeta|$ being sufficiently small, so there exists a time $T>0$ such that if (u_0,v_0), $(u_1,v_1)\in H_0^s(\mathbb{T})\times H_0^s(\mathbb{T})$ are such that*

$$\|(u_0,v_0)\|_s\leq R_0, \quad \|(u_1,v_1)\|_s\leq R_0,$$

then one can find two controls input $f, g \in L^2(0,T;H_0^s(\mathbb{T}))$ such that system (1.11) admits a solution

$$(u,v) \in C([0,T];H_0^s(\mathbb{T})) \times C([0,T];H_0^s(\mathbb{T}))$$

satisfying

$$(u(x,0),v(x,0)) = (u_0(x),v_0(x)) \quad \text{and} \quad (u(x,T),v(x,T)) = (u_1(x),v_1(x)).$$

It is important to point out that Theorems 1.4 and 1.5 are valid for the case when we consider in the systems above mentioned $\alpha < 0$ and $|\mu| + |\zeta|$ is small enough. This restriction is necessary because we need estimates for non-linear terms (see Lems. 3.7 and 3.8 in the Sect. 3) which needs to be verified when $|\mu| + |\zeta| << 1$, $\alpha < \frac{1}{4}$ and $\frac{1}{\alpha} < \frac{1}{4}$, simultaneously. However, if $B = C = 0$, *i.e.* $\eta = 0$ (see system (1.6)), we have two KdV-type systems coupled only in the nonlinear terms. Thus, $\eta = 0$ ensures that all the results presented in this manuscript remain valid without any restriction in the constants α, μ, and ζ.

1.5. Structure of the article

Section 2 is devoted to showing the spectral analysis necessary to prove the exact controllability result for the linear system associated with (1.6). Next, in Section 3, we present the Bourgain spaces and its property. Precisely, thanks to linear and nonlinear estimates we are able to prove the global well-posedness results for the system (1.12). In Section 4, the reader will find the proofs of the main theorems of the article. Section 5 is devoted to presenting the conclusion of the work and some open issues. Finally, in Appendix A, we collect results associated with the system (1.6), which were used throughout the paper.

2. Spectral problem

In this section, we study the spectral properties of the linear system associated with (1.10). Precisely, using Ingham's type theorem, we prove that the exact controllability for (1.6)–(1.7) holds. Consider the following operator

$$L = \begin{pmatrix} -\partial_x^3 - \mu\partial_x & -\eta\partial_x \\ -\eta\partial_x & -\alpha\partial_x^3 - \zeta\partial_x \end{pmatrix} \tag{2.1}$$

with domain $\mathcal{D}(L) = H^3(\mathbb{T}) \times H^3(\mathbb{T})$. This operator has the following properties.

Proposition 2.1. *Consider the operator L defined as in (2.1). If $\alpha < 0$ and $\zeta - \mu > 0$ then L generates a strongly continuous group $S(t)$ in $L^2(\mathbb{T}) \times L^2(\mathbb{T})$. Moreover, the eigenfunctions are defined by $e^{-ikx}Z_k^{\pm}$, with $k \in \mathbb{Z}$ and form an orthogonal basis in $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ satisfying*

$$Z_k^{\pm} \longrightarrow Z^{\pm}, \quad \text{as } k \to \pm\infty,$$

where $Z^+ := (0,0)$ and $Z^- := (0, 2(1-\alpha))$.

Proof. A simple calculation shows that $L^* = -L$ and $\langle Lu, u \rangle = -\langle u, Lu \rangle = 0$. Thus L and L^* are dissipative. Since $\mathcal{D}(L)$ is dense on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ follows from Corollary 4.4 of [32] that L is an infinitesimal generator of a strongly continuous group of contractions on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$.

We claim that, for each fixed $k \in \mathbb{Z}$, $e^{-ikx}(\sigma_k, \tau_k)$ is an eigenvector of L with eigenvalue $i\omega_k$ if and only if

$$\begin{cases} (k^3 - \mu k - \omega_k)\sigma_k - \eta k \tau_k = 0, \\ -\eta k \sigma_k + (\alpha k^3 - \zeta k - \omega_k)\tau_k = 0. \end{cases} \tag{2.2}$$

That is, there exist non-trivial solutions if and only if

$$\omega_k^2+\omega_k(\zeta+\mu-(1+\alpha)k^2)k+\alpha k^6-k^4(\zeta+\alpha\mu)-k^2(\eta^2-\mu\zeta)=0.$$

Hence, we have two possible exponents, given by the formula

$$\begin{aligned}2\omega_k^{\pm}&=k((1+\alpha)k^2-(\mu+\zeta))\pm\sqrt{k^2(\mu+\zeta-(1+\alpha)k^2)^2-4k^2(\alpha k^4-k^2(\alpha\mu+\zeta)-(\eta^2-\mu\zeta))}\\&=k^3(1+\alpha-(\mu+\zeta)k^{-2})\pm k^3\sqrt{[(1-\alpha)+k^{-2}(\zeta-\mu)]^2+4k^{-4}\eta^2},\end{aligned}$$

that is,

$$2\omega_k^{\pm}=k^3\left[(1+\alpha)-(\zeta+\mu)k^{-2}\pm\sqrt{[(1-\alpha)+k^{-2}(\zeta-\mu)]^2+4k^{-4}\eta^2)}\right]. \tag{2.3}$$

If $k\neq 0$, with $\eta\neq 0$, then $\omega_k^-\neq\omega_k^+$ and two corresponding non-zero eigenvectors are given by the formula

$$Z_k^{\pm}=(\sigma_k,\tau_k)=\ 2k^{-3}\left(\eta k,k^3-\mu k-\omega_k^{\pm}\right). \tag{2.4}$$

If $k=0$, then both eigenvalues are equal to zero and two linearly independent eigenvectors are given for example by

$$Z_0^{\pm}=(\sigma_0,\tau_0)=\left(2\eta,(1-\alpha)\mp\sqrt{(1-\alpha)^2+4\eta^2}\right) \tag{2.5}$$

A direct calculation show that $Z_k^+\cdot Z_k^-=0$, for all $k\in\mathbb{Z}$ and $Z_k^{\pm}\longrightarrow Z^{\pm}$ as $k\longrightarrow\pm\infty$. Thus, $(\phi_k^{\pm},\psi_k^{\pm})=e^{-ikx}\cdot Z_k^{\pm}$, where $Z_k^{\pm}=(\sigma_k^{\pm},\tau_k^{\pm})$ is defined as in (2.4)–(2.5), form an orthogonal basis in $L^2(\mathbb{T})\times L^2(\mathbb{T})$ with the eigenvalues given by (2.3), showing the proposition. □

We recall the definition of the *upper density* $D^+=D^+(\Omega)$ of Ω. For each $\ell>0$ we denote by $n^+(\ell)$ the largest number of exponents ω_k that we may find in an interval of length ℓ, and then we set

$$D^+:=\inf_{\ell>0}\frac{n^+(\ell)}{\ell}\in[0,\infty].$$

It can be shown (see, *e.g.*, [20], p. 174) that

$$D^+=\lim_{\ell\to\infty}\frac{n^+(\ell)}{\ell}.$$

It follows from the definition that D^+ is subadditive:

$$D^+(\Omega_1\cup\Omega_2)\leq D^+(\Omega_1)+D^+(\Omega_2) \tag{2.6}$$

for any families Ω_1 and Ω_2. If Ω is *uniformly separated*, *i.e.*, if

$$\gamma=\gamma(\Omega)=\inf\{|\omega_k-\omega_n|\ :\ k\neq n\}>0,$$

then $D^+\leq 1/\gamma$, and hence $D^+<\infty$. With this, we will prove the following.

Lemma 2.2. *Let $\omega_k^{\pm}$ be as in (2.3). We have*

$$\lim_{k\to\pm\infty}(\omega_{k+1}^{+}-\omega_k^{+})=+\infty \quad \textit{and} \quad \lim_{k\to\pm\infty}(\omega_{k+1}^{-}-\omega_k^{-})=-\infty$$

Consequently, we have that

$$D^{+}(\{\omega_k^{\pm}\})=0.$$

Proof. Since $\omega_{-k}^{+}=-\omega_k^{+}$, it suffices to consider the case $k\to+\infty$. One denotes

$$T^{\pm}(k)=(1+\alpha)-(\zeta+\mu)k^{-2}\pm\sqrt{[(1-\alpha)+(\zeta-\mu)k^{-2}]^2+4k^{-4}\eta^2}$$

Thus,

$$T^{+}(k)=2+O(k^{-2}) \ \text{ as } \ k\to\infty$$

and

$$\omega_k^{+}=\frac{1}{2}k^3T^{+}(k).$$

Hence

$$\omega_{k+1}^{+}-\omega_k^{+}=(k+1)^3-k^3+O(k)=3k^2+3k+1+O(k)\to+\infty, \text{ as } k\to+\infty.$$

In the similar way

$$\omega_{k+1}^{-}-\omega_k^{-}=\alpha[(k+1)^3-k^3]+O(k)\to-\infty, \text{ as } k\to+\infty,$$

where the last convergence is due to the fact that $\alpha<0$. Now, as a consequence of these converges and by definition of $D^{+}\leq 1/\gamma$, where $\gamma=\gamma(\Omega)=\inf\{|\omega_k-\omega_n| \ : \ k\neq n\}>0$, we have that $D^{+}(\{\omega_k^{\pm}\})=0$. □

We now need to order our orthonormal basis, let us do it as follows. Consider $(\phi_k,\psi_k)=e^{-ikx}(\sigma_k,\tau_k)$, so

$$(\phi_k,\psi_k):=\begin{cases}(\phi_k^{+},\psi_k^{+})=e^{-ikx}(\sigma_k^{+},\tau_k^{+})=e^{-ikx}Z_k^{+}, & \text{if} \quad k=2k' \text{ for all } k'\in\mathbb{Z},\\ (\phi_k^{-},\psi_k^{-})=e^{-ikx}(\sigma_k^{-},\tau_k^{-})=e^{-ikx}Z_k^{-}, & \text{if} \quad k=2k'+1 \text{ for all } k'\in\mathbb{Z}.\end{cases} \tag{2.7}$$

Therefore, any vector $(u,v)\in H^s(\mathbb{T})\times H^s(\mathbb{T})$ can be represented by

$$(u,v)=\left(\sum_{k\in\mathbb{Z}}a_k\phi_k,\sum_{k\in\mathbb{Z}}b_k\psi_k\right),$$

with the coefficients a_k and b_k are defined by

$$a_k=\langle u,\phi_k\rangle \ \text{ and } \ b_k=\langle v,\psi_k\rangle,$$

where $\langle\cdot,\cdot\rangle$ denoting the inner product in $L^2(\mathbb{T})$. Consider, also, the following

$$\omega_k=\begin{cases}\omega_k^{+}, & \text{if} \quad k=2k' \text{ for all } k'\in\mathbb{Z},\\ \omega_k^{-}, & \text{if} \quad k=2k'+1 \text{ for all } k'\in\mathbb{Z}.\end{cases} \tag{2.8}$$

With these notions in hand, the following lemma gives the behavior of $\omega_k^{\pm}$ and concludes that the upper density of the set $\{\omega_k^{\pm}\}$ is zero. It is important to notice that to use ([20], Thm. 4.6) we need the following uniform gap condition $\gamma = \inf_{k\neq n} |\omega_k - \omega_n| > 0$, where ω_k is defined by (2.8). The next proposition will give us such information.

Proposition 2.3 (Gap condition)**.** *Let ω_k be as in (2.8). Thus,*

$$\lim_{|k|,|r|\to+\infty} |\omega_k - \omega_r| = +\infty.$$

Proof. Start noting that Lemma 2.2 ensures the result for k and n both odd or both even. Now, we need to guarantee that the same is true for the other cases of k and n. Consider without loss of generality $r = 2k'$ and $k = 2(k' + k'') + 1$ for any $k' \in \mathbb{Z}$ and k'' is a fixed positive integer. Using the notation of Lemma 2.2, follows that

$$\omega_{2(k'+k'')+1} - \omega_{2k'} = 8(\alpha - 1)k'^3 + \alpha[12k'^2(2k'' + 1) + 6k'(2k'' + 1)^2 + (2k'' + 1)^3] + O(k').$$

Thus,

$$\lim_{|k'|\to+\infty} |\omega_{2(k'+k'')+1} - \omega_{2k'}| = +\infty,$$

and then the proposition is proved. □

Remark 2.4. Thanks to Theorem 4.6 of [20] and Proposition 2.3 there exists a subset $\mathbb{K} \subset \mathbb{Z}$ such that $\overline{span\{e^{-i\omega_k t}\}_{k\in\mathbb{K}}}^{L^2(0,T)}$ has a unique biorthogonal Riesz basis $\{q_k\} \subset L^2(0, T)$, where

$$\mathbb{K} = \{k \in \mathbb{Z}\ ;\ \omega_k \neq \omega_r \text{ for all } k \neq r\}. \tag{2.9}$$

2.1. Exact controllability: linear result

With this previous information that concerns the spectral properties of the operator L, in this section, we will analyze the exact controllability for the linear system (1.10).

Before presenting the main result of this section, let us first consider some properties of the homogeneous initial value problem (HIVP) associated with (1.10). It is well know, thanks to Proposition 2.1, that (1.10), with $f = g = 0$, has solution on the Sobolev space $H^s(\mathbb{T})$, for $s \in [0, 3]$, which is given by

$$(u(t), v(t)) = (S(t)u_0, S(t)v_0) := \left(\sum_k e^{-i(\omega_k t + kx)}\widehat{u}_0, \sum_k e^{-i(\omega_k t + kx)}\widehat{v}_0\right). \tag{2.10}$$

Additionally, using Semigroup Theory, see for instance Theorems 1.1 and 1.4 of [32], we have that the open loop control system has a unique solution in

$$C([0, T]; H^3(\mathbb{T})) \cap C^1([0, T]; L^2(\mathbb{T})) \times C([0, T]; H^3(\mathbb{T})) \cap C^1([0, T]; L^2(\mathbb{T})).$$

Remark 2.5. Operator G defined as in (1.8) from $L^2(\mathbb{T})$ to $L^2(\mathbb{T})$ is linear, bounded and self-adjoint. Actually, was proved in Remark 2.1 of [25] (see also [28], Lem. 2.20) that operator G is a linear bounded operator from $L^2(0, T; H^s(\mathbb{T}))$ into $L^2(0, T; H^s(\mathbb{T}))$, for any $s \geq 0$.

From now on, we are in a position to prove the exact controllability result.

Proof of Theorem 1.3. Since the functions (ϕ_k, ψ_k), defined by (2.7), form an orthonormal basis on $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ and the space $L^2_0(\mathbb{T}) \times L^2_0(\mathbb{T})$ is a closed space, we can represent the initial and terminal states like expansions, which are convergent in $H^s_0(\mathbb{T}) \times H^s_0(\mathbb{T})$, as follows

$$\begin{aligned} u_j &= \sum_{k\in\mathbb{Z}} u_{k,j}\phi_k, \quad u_{k,j} = \int_{\mathbb{T}} u_j(x)\overline{\phi_k(x)}\mathrm{d}x, \qquad \text{for } j = 0,1, \\ v_j &= \sum_{k\in\mathbb{Z}} v_{k,j}\psi_k, \quad v_{k,j} = \int_{\mathbb{T}} v_j(x)\overline{\psi_k(x)}\mathrm{d}x, \qquad \text{for } j = 0,1. \end{aligned} \tag{2.11}$$

The solution of the homogeneous (adjoint) system can be expressed by $(u_k(x,t), v_k(x,t)) = (e^{-i\omega_k t}\phi_k(x), e^{-i\omega_k t}\psi_k(x))$, where ω_k are the eigenvalues defined in (2.8). Pick smooth functions (f,h) on $\mathbb{T}\times\mathbb{T}$. Multiplying (1.10) by $(\overline{u_k(x,t)}, \overline{v_k(x,t)})^T$ and using integration by parts on $\mathbb{T}\times(0,T)$, we obtain

$$\begin{aligned} \int_{\mathbb{T}} u(x,T)\overline{u_k(x,T)}\mathrm{d}x - \int_{\mathbb{T}} u(x,0)\overline{u_k(x,0)}\mathrm{d}x &= \int_0^T \int_{\mathbb{T}} Gf(x,t)\overline{u_k(x,t)}\mathrm{d}x\mathrm{d}t, \\ \int_{\mathbb{T}} v(x,T)\overline{v_k(x,T)}\mathrm{d}x - \int_{\mathbb{T}} v(x,0)\overline{v_k(x,0)}\mathrm{d}x &= \int_0^T \int_{\mathbb{T}} Gh(x,t)\overline{v_k(x,t)}\mathrm{d}x\mathrm{d}t, \end{aligned} \tag{2.12}$$

with the previous equality valid for $f,\, h \in L^2([0,T]; H^s_0(\mathbb{T}))$, for any $s \geq 0$, where (u,v) satisfies (1.10). Observe that $(\overline{u_k}, \overline{v_k}) = (e^{i\omega_k t}\overline{\phi_k(x)}), e^{i\omega_k t}\overline{\psi_k(x)})$. Moreover, thanks to (2.12), we get that

$$\int_{\mathbb{T}} u(x,T)e^{i\omega_k T}\overline{\phi_k(x)}\mathrm{d}x - \int_{\mathbb{T}} u_0(x)\overline{\phi_k(x)}\mathrm{d}x = \int_0^T \int_{\mathbb{T}} Gf(x,t)e^{i\omega_k t}\overline{\phi_k(x)}\mathrm{d}x$$

and

$$\int_{\mathbb{T}} v(x,T)e^{i\omega_k T}\overline{\psi_k(x)}\mathrm{d}x - \int_0^T v_0(x)\overline{\psi_k(x)}\mathrm{d}x = \int_0^T \int_{\mathbb{T}} Gh(x,t)e^{i\omega_k t}\overline{\psi_k(x)}\mathrm{d}x.$$

Evaluation of the integrals in (2.12) with

$$w_k = \int_{\mathbb{T}} u(x,T)\overline{\phi_k(x)}\mathrm{d}x \quad \text{and} \quad z_k = \int_{\mathbb{T}} v(x,T)\overline{\psi_k(x)}\mathrm{d}x \tag{2.13}$$

gives that

$$\begin{aligned} w_k - u_{k,0}e^{-i\omega_k T} &= \int_0^T e^{-i\omega_k(T-t)} \int_{\mathbb{T}} Gf(x,t)\overline{\phi_k(x)}\mathrm{d}x\mathrm{d}t, \qquad \forall k \in \mathbb{Z}, \\ z_k - v_{k,0}e^{-i\omega_k T} &= \int_0^T e^{-i\omega_k(T-t)} \int_{\mathbb{T}} Gh(x,t)\overline{\psi_k(x)}\mathrm{d}x\mathrm{d}t, \qquad \forall k \in \mathbb{Z}. \end{aligned} \tag{2.14}$$

Let us take our control functions f and h in the following way

$$f(x,t) = \sum_{j\in\mathbb{Z}} f_j q_j(t) G\phi_j(x), \quad \text{and} \quad h(x,t) = \sum_{j\in\mathbb{Z}} h_j q_j(t) G\psi_j(x). \tag{2.15}$$

Here the coefficients f_j and h_j must be determined so that, among other things, the series (2.15) is appropriately convergent. Substituting (2.15) into (2.14) yields,

$$w_k - u_{k,0}e^{-i\omega_k T} = e^{-i\omega_k T}\sum_{j\in\mathbb{Z}} f_j \int_0^T e^{i\omega_k t} q_j(t)\mathrm{d}t \int_{\mathbb{T}} GG\phi_j(x)\overline{\phi_k(x)}\mathrm{d}x \tag{2.16}$$

and

$$z_k - v_{k,0}e^{-i\omega_k T} = e^{-i\omega_k T}\sum_{j\in\mathbb{Z}} h_j \int_0^T e^{i\omega_k t} q_j(t)\mathrm{d}t \int_{\mathbb{T}} GG\psi_j(x)\overline{\psi_k(x)}\mathrm{d}x. \tag{2.17}$$

Thanks to the fact that $\{q_k\}_{k\in\mathbb{K}}$ is a biorthogonal Riesz basis to $\{e^{-i\omega_k t}\}_{k\in\mathbb{K}}$ in $L^2_0(0,T)$, for $\mathbb{K}$ defined by (2.9), and due to the Remark 2.5 we can get that

$$\begin{aligned} w_k - u_{k,0}e^{-i\omega_k T} &= e^{-i\omega_k T} f_k \int_{\mathbb{T}} G\phi_k(x)\overline{G\phi_k(x)}\mathrm{d}x = e^{-i\omega_k T} f_k \|G\phi_k\|^2,\\ z_k - v_{k,0}e^{-i\omega_k T} &= e^{-i\omega_k T} h_k \int_{\mathbb{T}} G\psi_k(x), \overline{G\psi_k(x)}\mathrm{d}x = e^{-i\omega_k T} h_k \|G\psi_k\|^2, \end{aligned} \tag{2.18}$$

for all $k_j \in \mathbb{Z}\setminus \cup_{j=1}^{\ell}\mathbb{K}_j$, where $\mathbb{K}_j := \{k\in\mathbb{Z}\ ;\ \omega_k = \omega_{k_j} \text{ and } k\neq k_j\}$. By the definition of G, see (1.8), yield that

$$\|G\phi_k\|^2 = \int_{\mathbb{T}} \left| g(x)\left(\phi_k(x) - \int_{\mathbb{T}} g(y)\phi_k(y)\mathrm{d}y\right)\right|^2 \mathrm{d}x = |\sigma_k|^2\beta_k \tag{2.19}$$

and

$$\|G\psi_k\|^2 = \int_{\mathbb{T}} \left| g(x)\left(\psi_k(x) - \int_{\mathbb{T}} g(y)\psi_k(y)\mathrm{d}y\right)\right|^2 \mathrm{d}x = |\tau_k|^2\beta_k, \tag{2.20}$$

where

$$\beta_k := \left\| G\left(\frac{e^{-ikx}}{\sqrt{2\pi}}\right)\right\|^2.$$

Since $[g] = \frac{1}{2\pi}$ it is easy to see that $\beta_0 = 0$. The fact that $g(x)$ is real valued shows that $g(x)\frac{e^{-ikx}}{\sqrt{2\pi}}$ cannot be a constant multiple of $g(x)$ on any interval. Thus, follows that $\beta_k \neq 0$, $k>0$ and

$$\lim_{k\to\infty}\beta_k = \int_{\mathbb{T}} g(x)^2\mathrm{d}x \neq 0.$$

Its implies that there is a $\delta>0$ such that

$$|\beta_k| > \delta, \qquad \text{for } k\neq 0. \tag{2.21}$$

Due to the fact that $\sigma_k\neq 0$ and $\tau_k \neq 0$, for all k, we can putting $f_0 = h_0 = 0$ and

$$f_k = \frac{u_{k,1}e^{i\omega_k T} - u_{k,0}}{|\sigma_k|^2\beta_k} \quad \text{and} \quad h_k = \frac{v_{k,1}e^{i\omega_k T} - v_{k,0}}{|\tau_k|^2\beta_k}, \tag{2.22}$$

for all $k \in \mathbb{Z}^* \setminus \cup_{j=1}^{n} \mathbb{K}_j$. So we get, from (2.18), that $w_k = u_{k,1}$ and $z_k = v_{k,1}$, where $u_{k,1}$ and $v_{k,1}$ are given by (2.11)[3]. Since ω_k is given by a polynomial of degree 3, each set $\mathbb{K}_j$ has at most three elements. So, we can consider $k_{j,i} \in \cup_{j=1}^{\ell} \mathbb{K}_j$ for $i = 0, 1, 2$. In this case, from (2.16)–(2.17) follows that

$$\begin{aligned} w_{k_{j,i}} - u_{k_{j,i},0} e^{-i\omega_{k_{j,0}} T} &= \sigma_{k_{j,i}} e^{-i\omega_{k_{j,0}} T} \sum_{\ell=0}^{2} f_{k_{j,\ell}} \sigma_{k_{j,\ell}} M_{k_{j,\ell} k_{j,i}}, \\ z_{k_{j,i}} - v_{k_{j,i},0} e^{-i\omega_{k_{j,0}} T} &= \tau_{k_{j,i}} e^{-i\omega_{k_{j,0}} T} \sum_{\ell=0}^{2} h_{k_{j,\ell}} \tau_{k_{j,\ell}} M_{k_{j,\ell} k_{j,i}}, \end{aligned} \tag{2.23}$$

where

$$M_{k_{j,\ell} k_{j,i}} := \frac{1}{2\pi} \int_{\mathbb{T}} GG\left(e^{-ik_{j,\ell} x}\right) \overline{e^{-ik_{j,i} x}} \mathrm{d}x.$$

In other words, $f_{k_{j,\ell}}$ and $h_{k_{j,\ell}}$, for each $j = 1, 2, \cdots, n$ and $\ell = 0, 1, 2$, must be satisfy the following matrix identities

$$\begin{pmatrix} \sigma_{k_{j,0}} M_{k_{j,0},k_{j,0}} & \sigma_{k_{j,0}} M_{k_{j,1},k_{j,0}} & \sigma_{k_{j,0}} M_{k_{j,2},k_{j,0}} \\ \sigma_{k_{j,1}} M_{k_{j,0},k_{j,1}} & \sigma_{k_{j,1}} M_{k_{j,1},k_{j,1}} & \sigma_{k_{j,1}} M_{k_{j,2},k_{j,1}} \\ \sigma_{k_{j,2}} M_{k_{j,0},k_{j,2}} & \sigma_{k_{j,2}} M_{k_{j,1},k_{j,2}} & \sigma_{k_{j,2}} M_{k_{j,2},k_{j,2}} \end{pmatrix} \cdot \begin{pmatrix} \sigma_{k_{j,0}} f_{k_{j,0}} \\ \sigma_{k_{j,1}} f_{k_{j,1}} \\ \sigma_{k_{j,2}} f_{k_{j,2}} \end{pmatrix} = \begin{pmatrix} w_{k_{j,0}} e^{i\omega_{k_{j,0}} T} - u_{k_{j,0},0} \\ w_{k_{j,1}} e^{i\omega_{k_{j,0}} T} - u_{k_{j,1},0} \\ w_{k_{j,2}} e^{i\omega_{k_{j,0}} T} - u_{k_{j,2},0} \end{pmatrix}$$

and

$$\begin{pmatrix} \tau_{k_{j,0}} M_{k_{j,0},k_{j,0}} & \tau_{k_{j,0}} M_{k_{j,1},k_{j,0}} & \tau_{k_{j,0}} M_{k_{j,2},k_{j,0}} \\ \tau_{k_{j,1}} M_{k_{j,0},k_{j,1}} & \tau_{k_{j,1}} M_{k_{j,1},k_{j,1}} & \tau_{k_{j,1}} M_{k_{j,2},k_{j,1}} \\ \tau_{k_{j,2}} M_{k_{j,0},k_{j,2}} & \tau_{k_{j,2}} M_{k_{j,1},k_{j,2}} & \tau_{k_{j,2}} M_{k_{j,2},k_{j,2}} \end{pmatrix} \cdot \begin{pmatrix} \tau_{k_{j,0}} h_{k_{j,0}} \\ \tau_{k_{j,1}} h_{k_{j,1}} \\ \tau_{k_{j,2}} h_{k_{j,2}} \end{pmatrix} = \begin{pmatrix} z_{k_{j,0}} e^{i\omega_{k_{j,0}} T} - v_{k_{j,0},0} \\ z_{k_{j,1}} e^{i\omega_{k_{j,0}} T} - v_{k_{j,1},0} \\ z_{k_{j,2}} e^{i\omega_{k_{j,0}} T} - v_{k_{j,2},0} \end{pmatrix}.$$

In order to achieve the result, we will need to prove the following two claims.

Claim 1. The previous systems have a unique solution $(f_{k_{j,0}}, f_{k_{j,1}}, f_{k_{j,2}})$ and $(h_{k_{j,0}}, h_{k_{j,1}}, h_{k_{j,2}})$, for each $j = 1, 2, \cdots, n$.

Indeed, note that the determinant of the above matrices are given by $\sigma_{k_{j,0}} \times \sigma_{k_{j,1}} \times \sigma_{k_{j,2}} \times \det M_j$ and $\tau_{k_{j,0}} \times \tau_{k_{j,1}} \times \tau_{k_{j,2}} \times \det M_j$, respectively, with M_j defined by

$$M_j := \begin{pmatrix} m_{k_{j,0},k_{j,0}} & m_{k_{j,0},k_{j,1}} & m_{k_{j,0},k_{j,2}} \\ m_{k_{j,1},k_{j,0}} & m_{k_{j,1},k_{j,1}} & m_{k_{j,1},k_{j,2}} \\ m_{k_{j,2},k_{j,0}} & m_{k_{j,2},k_{j,1}} & m_{k_{j,2},k_{j,2}} \end{pmatrix}.$$

Since $\sigma_{k_{j,0}} \times \sigma_{k_{j,1}} \times \sigma_{k_{j,2}} \neq 0$ and $\tau_{k_{j,0}} \times \tau_{k_{j,1}} \times \tau_{k_{j,2}} \neq 0$, we only have show that the hermitian matrices M_j are invertible for all $j = 1, \cdots, \ell$. For fixed j, let us consider Σ_2 the space spanned by $\Upsilon_{k_{j,i}} = e^{-ik_{j,i}}$, $i = 0, 1, 2$. Let $\rho_{k_{j,\ell}}$ be the projection of $GG(\Upsilon_{k_{j,\ell}})$ onto the space Σ_2, that is,

$$\rho_{k_{j,\ell}} = \sum_{i=0}^{2} M_{k_{j,\ell} k_{j,i}} \Upsilon_{k_{j,i}}.$$

[3]Note that clearly w_0 and z_0 must be zero.

Now, it suffices to show that $\rho_{k_{j,\ell}}$, $\ell = 0, 1, 2$, is a linearly independent subset of Σ_2. Assume that there exist scalars λ_ℓ, $\ell = 0, 1, 2$, such that

$$\sum_{\ell=0}^{2} \lambda_\ell \rho_{k_{j,\ell}}(x) = 0 \iff \sum_{\ell,i=0}^{2} \lambda_\ell M_{k_{j,\ell},k_{j,i}} \Upsilon_{k_{j,i}}(x) = 0$$

Then, it yields that

$$\sum_{i=0}^{2}\sum_{\ell=0}^{2} \left\langle \lambda_\ell G\Upsilon_{k_{j,\ell}}, G\Upsilon_{k_{j,i}} \right\rangle \Upsilon_{k_{j,i}} = \sum_{i=0}^{2} \left\langle GG\left(\sum_{\ell=0}^{2} \lambda_\ell \Upsilon_{k_{j,\ell}}\right), \Upsilon_{k_{j,i}} \right\rangle \Upsilon_{k_{j,i}} = 0$$

Since $\Upsilon_{k_{j,i}}$ is a basis of Σ_2, follows that

$$\left\langle GG\left(\sum_{\ell=0}^{2} \lambda_\ell \Gamma_{k_{j,\ell}}\right), \Gamma_{k_{j,i}} \right\rangle = 0,$$

for each $i = 0, 1, 2$. As consequence of the last equality, we get

$$0 = \left\langle GG\left(\sum_{\ell=0}^{2} \lambda_\ell \Upsilon_{k_{j,\ell}}\right), \sum_{i=0}^{2} \lambda_i \Upsilon_{k_{j,i}} \right\rangle \iff \sum_{\ell=0}^{2} \lambda_\ell \Upsilon_{k_{j,\ell}} = 0 \iff \lambda_\ell = 0,$$

for $\ell = 0, 1, 2$, showing the Claim 1.

Claim 2. The functions f and h defined by (2.15) and (2.22) belongs to $L^2([0,T]; H_0^s(\mathbb{T}))$ provided that $(u_0, v_0), (u_1, v_1) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$.

In fact, let us write $G\phi_j(x)$ and $G\psi_j(x)$ as follows

$$G\phi_j(x) = \sum_{k\in\mathbb{Z}} a_{jk}\phi_k \quad \text{and} \quad G\psi_j(x) = \sum_{k\in\mathbb{Z}} b_{jk}\psi_k, \tag{2.24}$$

where

$$a_{jk} = \int_{\mathbb{T}} G\phi_j \overline{\phi_k(x)} \mathrm{d}x \quad \text{and} \quad b_{jk} = \int_{\mathbb{T}} G\psi_j \overline{\psi_k(x)} \mathrm{d}x, \; k \in \mathbb{Z}.$$

Therefore, we can see that

$$f(x,t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} f_j a_{jk} q_j(t)\phi_k(x) \quad \text{and} \quad h(x,t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} h_j b_{jk} q_j(t)\psi_k(x).$$

Consequently, this yields that

$$\|f\|^2_{L^2([0,T];H_0^s(\mathbb{T}))} = \int_0^T \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} \left|\sum_{j\in\mathbb{Z}} a_{jk} f_j q_j(t)\right|^2 \mathrm{d}t = \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} \int_0^T \left|\sum_{j\in\mathbb{Z}} a_{jk} f_j q_j(t)\right|^2 \mathrm{d}t.$$

As $\{q_k\}_{k\in\mathbb{K}}$ is a Bessel sequence and $\mathbb{Z}\setminus\mathbb{K}$ is a finite set, from the previous identity holds that

$$\|f\|_{L^2([0,T];H_0^s(\mathbb{T}))} \leq c\sum_{j\in\mathbb{Z}}|f_j|^2\sum_{k\in\mathbb{Z}}(1+|k|)^{2s}|a_{jk}|^2. \tag{2.25}$$

Analogously, we can obtain the following estimate for h, that is,

$$\|h\|_{L^2([0,T];H_0^s(\mathbb{T}))} \leq c\sum_{j\in\mathbb{Z}}|h_j|^2\sum_{k\in\mathbb{Z}}(1+|k|)^{2s}|b_{jk}|^2. \tag{2.26}$$

To finish the proof of Claim 2, let us prove that the right hand side of (2.25) and (2.26) are bounded. For this, note that

$$|a_{jk}| = |\langle G\phi_j(x),\phi_k(x)\rangle| \leq \frac{1}{\sqrt{2\pi}}|\sigma_{k-j}||\sigma_j||\sigma_k||\langle g,\phi_{k-j}\rangle| + |\sigma_k||\sigma_j||\langle g,\phi_k\rangle||\langle g,\phi_j\rangle|$$

and, in a similar way,

$$|b_{jk}| \leq \frac{1}{\sqrt{2\pi}}|\tau_{k-j}||\tau_j||\tau_k||\langle g,\psi_{k-j}\rangle| + |\tau_k||\tau_j||\langle g,\psi_k\rangle||\langle g,\psi_j\rangle|.$$

Hence,

$$|a_{jk}|^2 \leq 2|\sigma_j|^2(|\sigma_{k-j}|^2|\sigma_k|^2|\langle g,\phi_{k-j}\rangle|^2 + |\sigma_k|^2|\langle g,\phi_k\rangle|^2|\langle g,\phi_j\rangle|^2)$$

and

$$|b_{jk}|^2 \leq 2|\tau_j|^2(|\tau_{k-j}|^2|\tau_k|^2|\langle g,\psi_{k-j}\rangle|^2 + |\tau_k|^2|\langle g,\psi_k\rangle|^2|\langle g,\psi_j\rangle|^2).$$

Using the last inequalities we can estimate

$$\begin{aligned}\sum_{k\in\mathbb{Z}}(1+|k|)^{2s}|a_{jk}|^2 \leq& 2|\sigma_j|^2\left[(1+|j|)^{2s}\sum_{k\in\mathbb{Z}}(1+|k|)^{2s}|\langle g,\phi_k\rangle|^2 + |\langle g.\phi_j\rangle|^2\sum_{k\in\mathbb{Z}}(1+|k|)^{2s}|\langle g,\phi_k\rangle|^2\right]\\ \leq& 2|\sigma_j|^2\left[(1+|j|)^{2s} + |\langle g,\phi_j\rangle|^2\right]\|g\|_s^2,\end{aligned}$$

and analogously, we have

$$\sum_{k\in\mathbb{Z}}(1+|k|)^{2s}|b_{jk}|^2 \leq 2|\tau_j|^2\left[(1+|j|)^{2s} + |\langle g,\psi_j\rangle|^2\right]\|g\|_s^2.$$

Therefore, (2.25) and (2.26) together the previous inequality results

$$\|f\|^2_{L^2([0,T];H_0^s(\mathbb{T}))} \leq 2C_0\sum_{j\in\mathbb{Z}}\frac{|u_{j,1}e^{i\omega_jT}-u_{j,0}|^2}{|\sigma_j|^2|\beta_j|^2}[(1+|j|)^{2s} + |\langle g,\phi_j\rangle|^2]\|g\|_s^2,$$

where $C_0 = \max_{j=1,\cdots,n}\{1,\|M_j^{-1}\|^2\}$ and $\|M_j^{-1}\|$ denote the Euclidean norms of the Matrices M_j^{-1}. An analogous inequality is obtained for $\|h\|^2_{L^2([0,T];H_0^s(\mathbb{T}))}$. Putting all these inequalities together and using the relation

(2.21), we get

$$\|(f,h)\|^2_{L^2([0,T];H^s_0(\mathbb{T}))\times L^2([0,T];H^s_0(\mathbb{T}))} \le C_0\delta^{-2}\|g\|^2_{H^s_0(\mathbb{T})}\sum_{j\in\mathbb{Z}}(1+|j|)^{2s}\frac{|\sigma_j|^2(|\widetilde{u_{j,1}}|^2+|\widetilde{u_{j,0}}|^2)}{|\sigma_j|^2} + C_0\delta^{-2}\|g\|^2_{H^s_0(\mathbb{T})}\sum_{j\in\mathbb{Z}}(1+|j|)^{2s}\frac{|\tau_j|^2(|\widetilde{v_{j,1}}|^2+|\widetilde{v_{j,0}}|^2)}{|\tau_j|^2},$$

where $\widetilde{u_{j,i}}$ and $\widetilde{v_{j,i}}$ denote the Fourier coefficients with respect to the orthonormal base $\left\{\frac{e^{-ijx}}{\sqrt{2\pi}}\right\}_{j\in\mathbb{Z}}$. So,

$$\|(f,h)\|^2_{L^2([0,T];H^s_0(\mathbb{T}))\times L^2([0,T];H^s_0(\mathbb{T}))} \le K_0\delta^{-2}\|g\|^2_{H^s_0(\mathbb{T})}(\|(u_0,v_0)\|^2_{H^s_0(\mathbb{T})\times H^s_0(\mathbb{T})}+\|(u_1,v_1)\|^2_{H^s_0(\mathbb{T})\times H^s_0(\mathbb{T})})$$

completing the proof of Claim 2 and showing Theorem 1.3. □

As a consequence of Theorem 1.3 we have the next result, which will be important to extend the result to the nonlinear system.

Corollary 2.6. *Equations* (2.15), (2.22) *and* (2.24) *define, for* $s \ge 0$, *two bounded operators* $\Phi(u_0, v_0) = f$ *and* $\Psi(u_1, v_1) = h$ *from* $H^s_0(\mathbb{T}) \times H^s_0(\mathbb{T})$ *to* $L^2([0,T]; H^s_0(\mathbb{T}))$ *such that*

$$S(T)(u_0,v_0)+\int_0^T S(T-\tau)(G\Phi(u_0,u_1),G\Psi(v_0,v_1))(\cdot,\tau)\mathrm{d}\tau=(u_1,v_1),$$

for any (u_0, u_1), $(v_0, v_1) \in H^s_0(\mathbb{T}) \times H^s_0(\mathbb{T})$. *Moreover, there exists a constant* $C_{T,g} := C(T, g)$ *such that the following inequality is verified*

$$\|(\Phi(u_0,u_1),\Psi(v_0,v_1))\|_{[L^2([0,T];H^s_0(\mathbb{T}))]^2} \le C_{T,g}\left(\|(u_0,v_0)\|_s+\|(u_1,v_1)\|_s\right).$$

3. Well-posedness theory in Bourgain spaces

It is well known that Bourgain in [4] discovered a subtle smoothing property of solutions of the KdV equation posed on Torus, thanks to which he was able to show that the KdV equation is well-posed in the space $H^s(\mathbb{T})$, for any $s \ge 0$. In this section we will present the smoothing properties of the IVP (1.10), considering $f = g = 0$, which are the key to proving the global control results in this manuscript, using the techniques introduced by Bourgain.

3.1. Fourier restriction space

Observe that the IVP (1.10), with $f = g = 0$, can be rewrite as

$$\begin{cases} \begin{pmatrix} u_t \\ v_t \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} u_{xxx} \\ v_{xxx} \end{pmatrix} + \begin{pmatrix} \mu & \eta \\ \eta & \zeta \end{pmatrix}\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & x\in\mathrm{T}, t\in\mathbb{R}, \\ \begin{pmatrix} u \\ v \end{pmatrix}\Big|_{t=0} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \in H^s(\mathbb{T})\times H^s(\mathbb{T}). \end{cases}$$

To find an appropriate way to define the $X_{s,b}$ for the targeted system (1.10), taking into account that $f = g = 0$, consider the following equivalent system

$$\begin{cases}\partial_t w+\beta\partial_x^3 w+\gamma\partial_x w=0, & x\in\mathbb{T},\ t\in\mathbb{R},\\ w(0)=w_0, & \in H^s(\mathbb{T}).\end{cases} \tag{3.1}$$

The solution to (3.1) is given explicitly by

$$w(x,t)=\sum_{k\in\mathbb{Z}} e^{ikx}e^{i\phi^{\beta,\gamma}(k)t}\widehat{w_0}(k):=S^{\beta,\gamma}(t)w_0 \tag{3.2}$$

with

$$\phi^{\beta,\gamma}(k):=\beta k^3-\gamma k.$$

For convenience, $\phi^{1,0}$ will be written as ϕ.

Remark 3.1. With the notation (3.2), note that when we put $\eta=0$ in (2.1), the operator L remains an infinitesimal generator of a strongly continuous group of contraction on $L^2(\mathbb{T})\times L^2(\mathbb{T})$ which is given by

$$S(t)=e^{-tL}=\begin{pmatrix} e^{-t(\partial_x^3+\mu\partial_x)} & 0\\ 0 & e^{-t(\alpha\partial_x^3+\zeta\partial_x)}\end{pmatrix}$$

Hence, $S(t)(u_0,v_0)=(S^{1,\mu}(t)u_0,S^{\alpha,\zeta}(t)v_0)$. In this way, the Corollary 2.6 also is obtained for $\eta=0$.

Definition 3.2. [4] For any $\beta,\gamma,s,b\in\mathbb{R}$, the Fourier restriction space $X_{s,b}^{\beta,\gamma}$ is defined to be the completion of the Schwartz space $\mathcal{S}(\mathbb{T}\times\mathbb{R})$ with respect to the norm

$$\|w\|_{X_{s,b}^{\beta,\gamma}}:=\left\|\langle k\rangle^s\left\langle\tau-\phi^{\beta,\gamma}(k)\right\rangle^b\widetilde{w}(k,\tau)\right\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})},$$

where $\widetilde{v}$ refers to the space-time Fourier transform of v. In addition, for any $T>0$,

$$X_{s,b}^{\beta}([0,T]):=X_{s,b}^{\beta,\gamma,T}$$

denotes the restriction of $X_{s,b}^{\beta,\gamma}$ on the domain $\mathbb{T}\times[0,T]$ which is a Banach space when equipped with the usual quotient norm.

As well known (see *e.g.* [19]), for the periodic KdV equation, one needs to take $b=\frac{1}{2}$. But, this space barely fails to be in $C(\mathbb{R}_t;H_x^s)$. To ensure the continuity of the time flow of the solution will be used the norm $Y_{s,b}^{\beta,\gamma}$ given by

$$\|w\|_{Y_{s,b}^{\beta,\gamma}}=\left\|\langle k\rangle^s\langle\tau-\phi_{\beta,\gamma}\rangle^b\widetilde{w}(k,\tau)\right\|_{\ell^2(\mathbb{Z})L^1(\mathbb{R})}$$

and the companion spaces will be defined as

$$Z_{s,b}^{\beta,\gamma}=X_{s,b}^{\beta,\gamma}\cap Y_{s,b-\frac{1}{2}}^{\beta,\gamma},\quad b,s\in\mathbb{R},$$

endowed with the norm

$$\|w\|_{Z_{s,b}^{\beta,\gamma}}=\|w\|_{X_{s,b}^{\beta,\gamma}}+\|w\|_{Y_{s,b-\frac{1}{2}}^{\beta,\gamma}}.$$

[4] We infer for more details the two references [4, 19].

Since the second term $\|\langle k\rangle^s \widehat{w}(k,\tau)\|_{\ell^2(\mathbb{Z})L^1(\mathbb{R})}$ has already dominated the $L_t^\infty H_x^s$ norm of v, it follows that $Z^{\beta,\gamma}_{s,\frac{1}{2}} \subset C(\mathbb{R}_t; H_x^s)$ continuously. Lastly, the spaces

$$Z^{\beta,\gamma}_{s,b}([0,T]) := Z^{\beta,\gamma,T}_{s,b}$$

denotes the restriction of $Z^{\beta,\gamma}_{s,b}$ on the domain $\mathbb{T}\times[0,T]$ which is a Banach space when equipped with the usual quotient norm.

Remark 3.3. When $b=-\frac{1}{2}$, the companion spaces $Z^{\beta,\gamma}_{s,-\frac{1}{2}}$ *via* the norm previously defined is so introduced to control the $Z^{\beta,\gamma}_{s,\frac{1}{2}}$-norm of the integral term from the Duhamel principle (see Lem. 3.4)

$$\|w\|_{Z^{\beta,\gamma}_{s,-\frac{1}{2}}} = \|w\|_{X^{\beta,\gamma}_{s,-\frac{1}{2}}} + \left\|\frac{\langle k\rangle^s \widehat{w}(k,\tau)}{\langle \tau-\phi^{\beta,\gamma}(k)\rangle}\right\|_{\ell^2(\mathbb{Z})L^1(\mathbb{R})}.$$

3.2. Linear and nonlinear estimates

To obtain a global well-posedness result for the system (1.6), with $p=q=0$, we will need some estimates related to linear and nonlinear IVP associated with this system. Let us first recall some classic results in the literature for dispersive systems.

Lemma 3.4.[5] *Let s, $b\in\mathbb{R}$ and $T>0$ be given. There exists a constant $C_0>0$ such that:*

(i) For any $w\in H^s(\mathbb{T})$,

$$\|S(t)^{\beta,\gamma}w\|_{X^{\beta,\gamma,T}_{s,b}} \le C_0\|w\|_s;$$
$$\|S(t)^{\beta,\gamma}w\|_{Z^{\beta,\gamma,T}_{s,b}} \le C_0\|w\|_s;$$

(ii) For any $f\in X^{\beta,\gamma}_{s,b-1}$,

$$\left\|\int_0^t S^{\beta,\gamma}(t-\tau)f(\tau)\mathrm{d}\tau\right\|_{X^{\beta,\gamma,T}_{s,b}} \le C_0\|f\|_{X^{\beta,\gamma,T}_{s,b-1}}$$

provided that $b>\frac{1}{2}$;

(iii) For any $f\in Z^{\beta,T}_{s,-\frac{1}{2}}$,

$$\left\|\int_0^t S^{\beta,\gamma}(t-\tau)f(\tau)\mathrm{d}\tau\right\|_{Z^{\beta,\gamma,T}_{s,\frac{1}{2}}} \le C_0\|f\|_{Z^{\beta,\gamma,T}_{s,-\frac{1}{2}}}.$$

Observe that the Bourgain spaces associated to (3.1) will be $X^{1,\mu}_{s,b}$ and $X^{\alpha,\zeta}_{s,b}$ ($Z^{1,\mu}_s$ and $Z^{\alpha,\zeta}_s$, respectively). In our case, it is important to see that $\sup_{k\in\mathbb{Z}}|\phi^\mu-\phi^{\alpha,\zeta}|=\infty$, which results that the norms $\|\cdot\|_{X^{1,\mu}_{s,b}}$ and $\|\cdot\|_{X^{\alpha,\zeta}_{s,b}}$ never will be equivalent (see for instance, [16], Rem. 1.1). To overcome this difficulty we need appropriate lemmas introduced first by T. Oh in [30] and, more recently, by Yang and Zhang in [40]. Consider $X^{\beta_i,\gamma_i}_{s,b}$ for β_i and γ_i, $i=1$ and 2. So, we present a lemma proved in Lemma 3.10 of [40] for the case $b=\frac{1}{2}$, here, we are able to extend the result for $b\in(\frac{1}{3},\frac{1}{2}]$.

[5]For details about this lemma, the authors suggest the following references [13, 39, 40].

Lemma 3.5. *Let $\beta_1 \neq \beta_2$, $s \in \mathbb{R}$, $\frac{1}{3} < b \leq \frac{1}{2}$ and $0 < T < 1$. There exist constants $\epsilon = \epsilon(\beta_1, \beta_2)$, $C_1 = C_1(\beta_1, \beta_2)$ and $\theta > 0$ such that for any γ_1, γ_2 with $|\gamma_1| + |\gamma_2| < \epsilon$*

$$\|\partial_x w\|_{Z^{\beta_2,\gamma_2,T}_{s,b-1}} \leq C_1 T^\theta \|w\|_{X^{\beta_1,\gamma_1,T}_{s,b}} \tag{3.3}$$

is verified for any $w \in X^{\beta_1,\gamma_1,T}_{s,b}$.

Proof. We must to prove that

$$\|\partial_x w\|_{X^{\beta_2,\gamma_2,T}_{s,b-1}} \leq C_1 T^\theta \|w\|^{\beta_1,\gamma_1,T}_{X_{s,b},} \quad \text{and} \quad \|\partial_x w\|_{Y^{\beta_2,\gamma_2,T}_{s,b-\frac{3}{2}}} \leq C_1 T^\theta \|w\|_{X^{\beta_1,\gamma_1,T}_{s,b}}. \tag{3.4}$$

Thus, it is sufficient to show the following estimates

$$\|\partial_x w\|_{X^{\beta_2,\gamma_2}_{s,b-1}} \leq C_1 \|w\|_{X^{\beta_1,\gamma_1}_{s,b^-}} \quad \text{and} \quad \|\partial_x w\|_{Y^{\beta_2,\gamma_2}_{s,b-\frac{3}{2}}} \leq C_1 \|w\|_{X^{\beta_1,\gamma_1,T}_{s,b^-}}, \tag{3.5}$$

here b^- denote $b - \widetilde{\epsilon}$ for $\widetilde{\epsilon} \ll 1$.

We will start by showing that the first inequality of (3.5) holds. Using the duality approach and Plancherel theorem, we get that

$$\begin{aligned}\|\partial_x w\|_{X^{\beta_2,\gamma_2}_{s,b-1}} &= \sup_{\|g\|_{X^{\beta_2,\gamma_2}_{-s,1-b}} \leq 1} \left| \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} ik \widetilde{w}(k,\tau) \widetilde{g}(k,\tau) \mathrm{d}\tau \right| \\ &= \sup_{\|g\|_{X^{\beta_2,\gamma_2}_{-s,1-b}} \leq 1} \left| \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} H(k,\tau) \widetilde{W}(k,\tau) \widetilde{G}(k,\tau) \mathrm{d}\tau \right|,\end{aligned}$$

where

$$H(k,\tau) = \frac{ik}{\left\langle \tau - \phi^{\beta_1,\gamma_1}(k) \right\rangle^{b^-} \left\langle \tau - \phi^{\beta_2,\gamma_2}(k) \right\rangle^{1-b}},$$

$$\widetilde{W}(k,\tau) = \langle k \rangle^s \left\langle \tau - \phi^{\beta_1,\gamma_1}(k) \right\rangle^{b^-} \widetilde{w}(k,\tau)$$

and

$$\widetilde{G}(k,\tau) = \langle k \rangle^{-s} \left\langle \tau - \phi^{\beta_2,\gamma_2}(k) \right\rangle^{1-b} \widetilde{g}(k,\tau).$$

The following claim shows that the function $H(k,\tau)$ is bounded.

Claim: For some constant $C_1 > 0$, which depends only on β_1, β_2, we have that

$$\sup_{(k,\tau) \in \mathbb{Z} \times \mathbb{R}} |H(k,\tau)| \leq C_1.$$

In fact, if $|k| \leq 1$ is immediate. If $|k| > 1$ note that

$$\left\langle \tau - \phi^{\beta_1,\gamma_1}(k) \right\rangle \left\langle \tau - \phi^{\beta_2,\gamma_2}(k) \right\rangle \geq \left| \phi^{\beta_1,\gamma_1}(k) - \phi^{\beta_2,\gamma_2}(k) \right| = |(\beta_1 - \beta_2)k^3 - (\gamma_1 - \gamma_2)k|.$$

Since $\beta_1 \neq \beta_2$ we can choose $\epsilon \ll 1$ such that $|\gamma_1| + |\gamma_2| \leq \epsilon$ and consequently

$$|\gamma_1 - \gamma_2||k| \leq \frac{1}{2}|\beta_1 - \beta_2||k|^3.$$

So, using this previous inequality, yields that

$$\left\langle \tau - \phi^{\beta_1,\gamma_1}(k)\right\rangle \left\langle \tau - \phi^{\beta_2,\gamma_2}(k)\right\rangle \geq |\beta_1 - \beta_2||k|^3 - |\gamma_1 - \gamma_2||k| \geq \frac{1}{2}|\beta_1 - \beta_2||k|^3.$$

Thus, we obtain

$$|H(k,\tau)| \leq \frac{C_1(\beta_1,\beta_2)|k|}{|k|^{3(b^-)} \langle \tau - \phi^{\beta_2,\gamma_2}(k)\rangle^{(1-2b)^+}} \leq \frac{C_1(\beta_1,\beta_2)}{|k|^{3(b^-)-1}} \leq C_1(\beta_1,\beta_2)$$

where we use the fact the $b \in (\frac{1}{3}, \frac{1}{2}]$ in the second and third inequality, respectively. This ends the proof of the claim.

With this in hand, we infer that

$$\begin{aligned}
\|\partial_x w\|_{X^{\beta_2,\gamma_2}_{s,b-1}} &\leq C_1 \sup_{\|g\|_{X^{\beta_2,\gamma_2}_{-s,1-b}} \leq 1} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \left|\widetilde{W}(k,\tau)\right| \left|\widetilde{G}(k,\tau)\right| \mathrm{d}\tau \\
&\leq C_1 \sup_{\|g\|_{X^{\beta_2,\gamma_2}_{-s,1-b}} \leq 1} \left\|\widetilde{W}\right\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})} \left\|\widetilde{G}\right\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})} \\
&\leq C_1 \|w\|_{X^{\beta_1,\gamma_1}_{s,b^-}}.
\end{aligned}$$

Consequently, for $\frac{1}{3} < b \leq \frac{1}{2}$, there exists $\theta > 0$ such that

$$\|\partial_x w\|_{X^{\beta_2,\gamma_2,T}_{s,b-1}} \leq C_1 \|w\|_{X^{\beta_1,\gamma_1,T}_{s,b^-}} \leq C_1 T^\theta \|w\|_{X^{\beta_1,\gamma_1,T}_{s,b}},$$

reaching estimate (3.4).

Now, to prove the second inequality in (3.5), note that by duality we have

$$\begin{aligned}
\|\partial_x w\|_{Y^{\beta_2,\gamma_2}_{s,b-\frac{3}{2}}} &= \sup_{\substack{\|a_k\|_{\ell^2(\mathbb{Z})} \\ a_k \neq 0}} \sum_{k\in\mathbb{Z}} a_k \int_{\mathbb{R}} \frac{ik\langle k\rangle^s |\widetilde{w}(k,\tau)|}{\langle \tau - \phi_{\beta_2,\gamma_2}(k)\rangle^{\frac{3}{2}-b}} \mathrm{d}\tau \\
&\leq \sup_{\substack{\|a_k\|_{\ell^2(\mathbb{Z})} \\ a_k \neq 0}} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} H(k,\tau) \frac{a_k}{\langle \tau - \phi^{\beta_2,\gamma_2}(k)\rangle^{\frac{1}{2}^+}} \widetilde{W}(k,\tau)\mathrm{d}\tau
\end{aligned}$$

where

$$H(k,\tau) = \frac{|k|}{\langle \tau - \phi^{\beta_1,\gamma_1}(k)\rangle^{b^-} \langle \tau - \phi^{\beta_2,\gamma_2}(k)\rangle^{(1-b)^-}}$$

and

$$\widetilde{W}(k,\tau) = \langle k \rangle^s \left\langle \tau - \phi^{\beta_1,\gamma_1}(k) \right\rangle^{b^-} |\widetilde{w}(k,\tau)|.$$

The claim proved before give us

$$\begin{aligned}\|\partial_x w\|_{Y^{\beta_2,\gamma_2}_{s,-b-\frac{1}{2}}} \leq & C_1 \sup_{\|a_k\|_{\ell^2(\mathbb{Z})}\leq 1} \|w\|_{X^{\beta_1,\gamma_1}_{s,b^-}} \left\| \frac{|a_k|}{\langle \tau - \phi^{\beta_2,\gamma_2}(k)\rangle^{\frac{1}{2}+}} \right\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})} \\ \leq & C_1 \sup_{\|a_k\|_{\ell^2(\mathbb{Z})}\leq 1} \|w\|_{X^{\beta_1,\gamma_1}_{s,b^-}} \|a_k\|_{\ell^2(\mathbb{Z})} \leq C_1 \|w\|_{X^{\beta_1,\gamma_1}_{s,b^-}},\end{aligned}$$

showing the second estimate of (3.5), and consequently, Lemma 3.5 is proved. □

Remark 3.6. It is important to point out that the following inequality

$$\|u\|_{X^{\alpha,\beta,T}_{s,b'}} \leq CT^{b''-b'} \|u\|_{X^{\alpha,\beta,T}_{s,b''}} \quad \forall u \in X^{\alpha,\beta,T}_{s,b''}, \tag{3.6}$$

holds for any $-\frac{1}{2} < b' < b'' < \frac{1}{2}$, $0 < T < 1$ and $s \in \mathbb{R}$, where $C > 0$ independent on T. For the previous Lemma 3.5 replacing b^-, in the proof, by any b_1 satisfying $\frac{1}{3} \leq b_1 < b$ and $b + b_1 \leq 1$, we can apply (3.6) with $b' = \frac{1}{3}$ and $b'' = b' + \theta$, since we can take $b'' < \frac{1}{2}$, to get $0 < \theta < \frac{1}{6}$.

The next lemma was borrowed from Lemmas 4.1 and 4.2 of [40] and concerns with the *bilinear estimates* in Bourgain spaces for the term $\partial_x(uv)$ when the functions u and v belong in $X^{\beta_i,\gamma_i}_{s,b}$ for β_i and γ_i, $i = 1$ and 2, distinct. In fact, the authors in [40] showed the result for general cases on a domain $\mathbb{T}_\lambda \times \mathbb{R}$, for $\lambda \geq 1$. Here, we will revisit the result proving in a simpler way the *bilinear estimates* on $\mathbb{T} \times [0,T]$, which will be used for obtaining our future results.

Lemma 3.7. *Let $s \geq 0$, $T \in (0,1)$ and $\beta_1, \beta_2 \in \mathbb{R}^*$, with $\beta_1 \neq \beta_2$[6]. Also consider that $\frac{\beta_1}{\beta_2} < \frac{1}{4}$. Let u and v functions such that with $[u] = [v] = 0$. There exist constants $\theta > 0$, $\epsilon = \epsilon(\beta_1,\beta_2) > 0$ and $C_2 = C_2(\beta_1,\beta_2) > 0$, independent of T, u and v, such that if $|\gamma_1| + |\gamma_2| < \epsilon$, we have:*

a) If $u \in X^{\beta_1,\gamma_1,T}_{s,\frac{1}{2}}$ and $v \in X^{\beta_2,\gamma_2,T}_{s,\frac{1}{2}}$, then

$$\|\partial_x(uv)\|_{Z^{\beta_2,\gamma_2,T}_{s,-\frac{1}{2}}} \leq C_2 T^\theta \|u\|_{X^{\beta_1,\gamma_1,T}_{s,\frac{1}{2}}} \|v\|_{X^{\beta_2,\gamma_2,T}_{s,\frac{1}{2}}}. \tag{3.7}$$

b) If $u, v \in X^{\beta_2,\gamma_2,T}_{s,\frac{1}{2}}$ then

$$\|\partial_x(uv)\|_{Z^{\beta_1,\gamma_1,T}_{s,-\frac{1}{2}}} \leq C_2 T^\theta \|u\|_{X^{\beta_2,\gamma_2,T}_{s,\frac{1}{2}}} \|v\|_{X^{\beta_2,\gamma_2,T}_{s,\frac{1}{2}}}. \tag{3.8}$$

Proof. We will prove the estimate (3.7). The proof of (3.8) is shown similarly and we omit its demonstration. Let $u \in X^{\beta_1,\gamma_1}_{s,\frac{1}{2}}$ and $v \in X^{\beta_2,\gamma_2}_{s,\frac{1}{2}}$, with $[u] = [v] = 0$. Necessarily, we must prove

$$\|\partial_x(uv)\|_{X^{\beta_2,\gamma_2}_{s,-\frac{1}{2}}} \leq C_2 T^\theta \|u\|_{X^{\beta_1,\gamma_1}_{s,\frac{1}{2}}} \|v\|_{X^{\beta_2,\gamma_2}_{s,\frac{1}{2}}} \tag{3.9}$$

[6]The cases $\beta_1 = \beta_2$ and $\gamma_1 = \gamma_2$ is known be true and can be seen in Proposition 5 of [13].

and

$$\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2}_{s,-1}} \leq C_2 T^{\theta}\|u\|_{X^{\beta_1,\gamma_1}_{s,\frac{1}{2}}}\|v\|_{X^{\beta_2,\gamma_2}_{s,\frac{1}{2}}}, \tag{3.10}$$

for some $\theta > 0$.

Firstly, we will show (3.9). To this end from the Plancherel theorem, the duality approach and convolution properties yields that

$$\|\partial_x(uv)\|_{X^{\beta_2,\gamma_2}_{s,-\frac{1}{2}}} = \sup_{\|g\|_{X^{\beta_2,\gamma_2}_{-s,\frac{1}{2}}}\leq 1} \sum_{\Gamma}\int_{\Lambda} \frac{|k_3|\langle k_3\rangle^s}{\langle k_1\rangle^s\langle k_2\rangle^s}\prod_{i=1}^{3}\frac{|f_i(k_i,\tau_i)|}{\langle L_i(k_i,\tau_i)\rangle^{\frac{1}{2}}}\mathrm{d}\Lambda. \tag{3.11}$$

Here $L_i(k_i,\tau_i)=\tau_i-\phi^{\beta_i,\gamma_i}(k_i)$, for $i=1,2$, $L_3(k_3,\tau_3)=\tau_3-\phi^{\beta_2,\gamma_2}(k_3)$, Γ and Λ given by

$$\Gamma := \left\{(k_1,k_2,k_3)\in\mathbb{Z}^3 \quad : \quad \sum_{i=1}^{3}k_i=0\right\} \text{ and } \Lambda := \left\{(k_1,k_2,k_3)\in\mathbb{R}^3 \quad : \quad \sum_{i=1}^{3}\tau_i=0\right\},$$

respectively, with

$$f_1(k_1,\tau_1) = \langle k_1\rangle^s\left\langle\tau_1-\phi^{\beta_1,\gamma_1}(k_1)\right\rangle^{\frac{1}{2}}\widetilde{u}(k_1,\tau_1),$$

$$f_2(k_2,\tau_2) = \langle k_2\rangle^s\left\langle\tau_2-\phi^{\beta_2,\gamma_2}(k_2)\right\rangle^{\frac{1}{2}}\widetilde{v}(k_2,\tau_2)$$

and

$$f_3(k_3,\tau_3) = \langle k_3\rangle^{-s}\left\langle\tau_3-\phi^{\beta_2,\gamma_2}(k_3)\right\rangle^{\frac{1}{2}}\widetilde{g}(k_3,\tau_3).$$

The condition $[u]=[v]=0$ together with the fact that $(k_1,k_2,k_3)\in\mathbb{Z}^3$ ensure us that we only need to consider the case $|k_i|\geq 1$ for $i=1,2,3$. In addition, as $(k_1,k_2,k_3)\in\Gamma$, we have $\frac{\langle k_3\rangle}{\langle k_1\rangle\langle k_2\rangle}\leq 1$. Thus,

$$\frac{|k_3|\langle k_3\rangle^s}{\langle k_1\rangle^s\langle k_2\rangle^s} \leq (|k_1||k_2||k_3|)^{\frac{1}{2}},$$

for all $s\geq 0$. Define

$$H(k_1,k_2,k_3) := \phi^{\beta_1,\gamma_1}(k_1)+\phi^{\beta_2,\gamma_2}(k_2)+\phi^{\beta_2,\gamma_2}(k_3). \tag{3.12}$$

Claim. Let $\frac{\beta_1}{\beta_2}<\frac{1}{4}$. There exist $\epsilon=\epsilon(\beta_1,\beta_2)$ and $\delta>0$ such that if $|\mu|+|\zeta|<\epsilon$ then the function H defined in (3.12) is δ-significant on $\mathbb{Z}$, *i.e.*,

$$\langle H(k_1,k_2,k_3)\rangle \geq \delta\prod_{i=1}^{3}|k_i| \text{ for any } (k_1,k_2,k_3)\in\Gamma.$$

Assume that the claim holds true. Thus,

$$\prod_{i=1}^{3}|k_i| \lesssim \langle H(k_1,k_2,k_3)\rangle = \left\langle \sum_{i=1}^{3} L_i(k_i,\tau_i)\right\rangle.$$

Using the last inequality we obtain

$$\begin{aligned}\sum_{\Gamma}\int_{\Lambda}\frac{|k_3|\langle k_3\rangle^s}{\langle k_1\rangle^s\langle k_2\rangle^s}\prod_{i=1}^{3}\frac{|f_i(k_i,\tau_i)|}{\langle L_i(k_i,\tau_i)\rangle^{\frac12}}\mathrm{d}\Lambda &\lesssim \sum_{\Gamma}\int_{\Lambda}\left\langle\sum_{j=1}^{3}L_i(k_i,\tau_i)\right\rangle^{\frac12}\prod_{i=1}^{3}\frac{f_i(k_i,\tau_i)}{\langle L_i(k_i,\tau_i)\rangle^{\frac12}}\mathrm{d}\Lambda \\ &\lesssim \sum_{j=1}^{3}\sum_{\Gamma}\int_{\Lambda}\frac{\prod_{i=1}^{3}\langle L_j(k_j,\tau_j)\rangle^{\frac12}\prod_{i=1}^{3}f_i(k_i,\tau_i)}{\langle L_i(k_i,\tau_i)\rangle^{\frac12}}\mathrm{d}\Lambda.\end{aligned} \tag{3.13}$$

Let us estimate each term of the right-hand side of (3.13). For simplicity, we will present the estimate corresponding to $j=1$. The other terms will be estimated similarly. For this case, we have

$$\sum_{\Gamma}\int_{\Lambda}|f_1(k_1,\tau_1)|\frac{|f_2(k_2,\tau_2)|}{\langle L_2(k_2,\tau_2)\rangle^{\frac12}}\frac{|f_3(k_3,\tau_2)|}{\langle L_3(k_3,\tau_3)\rangle^{\frac12}} = \sum_{\Gamma}\int_{\Lambda}\widetilde{g_1}(k_1,\tau_1)\widetilde{g_2}(k_2,\tau_2)\widetilde{g_3}(k_3,\tau_3)$$

with

$$\widetilde{g_1} = |f_1(k_1,\tau_1)| \quad\text{and}\quad \widetilde{g_i} = \frac{|f_i(k_i,\tau_i)|}{\langle L_i(k_i,\tau_i)\rangle^{\frac12}} \quad\text{for}\quad i=2,3.$$

Thus,

$$\begin{aligned}\sum_{\Gamma}\int_{\Lambda}|f_1(k_1,\tau_1)|\frac{|f_2(k_2,\tau_2)|}{\langle L_2(k_2,\tau_2)\rangle^{\frac12}}\frac{|f_3(k_3,\tau_2)|}{\langle L_2(k_3,\tau_3)\rangle^{\frac12}} &\lesssim \sum_{k_1\in\mathbb{Z}}\int_{\mathbb{R}}\widetilde{g_1}(-k_1,-\tau_1)\widetilde{g_2 g_3}(k_1,\tau_1)\mathrm{d}\tau_1 \\ &\lesssim \|g_1\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})}\|g_2\|_{\ell^4(\mathbb{Z})L^4(\mathbb{R})}\|g_3\|_{\ell^4(\mathbb{Z})L^4(\mathbb{R})} \\ &\lesssim \|u\|_{X^{\beta_1,\gamma_1}_{s,\frac12}}\|v\|_{X^{\beta_2,\gamma_2}_{s,\frac13}}\|g\|_{X^{\beta_2,\gamma_2}_{-s,\frac13}},\end{aligned}$$

where we have used that $X^{\beta,\gamma}_{0,\frac13}$ is continuously imbedded in the space $\ell^4(\mathbb{Z})L^4(\mathbb{R})$[7]. Replacing the last inequality in (3.13) we conclude from (3.11) that

$$\|\partial_x(uv)\|_{X^{\beta_2,\gamma_2}_{s,-\frac12}} \lesssim \left(\|u\|_{X^{\beta_1,\gamma_1}_{s,\frac12}}\|v\|_{X^{\beta_2,\gamma_2}_{s,\frac13}} + \|u\|_{X^{\beta_1,\gamma_1}_{s,\frac13}}\|v\|_{X^{\beta_2,\gamma_2}_{s,\frac12}}\right),$$

which implies that for any $T\in(0,1)$, thanks to the Remark 3.6, there exists a positive constant C_2, independent of T, such that

$$\|\partial_x(uv)\|_{X^{\beta_2,\gamma_2,T}_{s,-\frac12}} \le C_2T^{\theta}\|u\|_{X^{\beta_1,\gamma_1,T}_{s,\frac12}}\|v\|_{X^{\beta_2,\gamma_2,T}_{s,\frac12}},$$

for $\theta\in\left(0,\frac16\right)$, showing (3.9).

[7]See Lemma 3.2 of [24] or Lemma 3.9 of [40].

Before presenting the proof of (3.10), let us prove the claim. Note that the function H defined by (3.12) can be rewritten as

$$H(k_1,k_2,k_3) = -3\beta_2 k_1^3 h\left(\frac{k_2}{k_1}\right) - (\gamma_1-\gamma_2)k_1,$$

where $h(x) = x^2 + x + \frac{1}{3}(1-\frac{\beta_1}{\beta_2})$. Since $\frac{\beta_1}{\beta_2} < \frac{1}{4}$, so h does not have real roots. Thus, there exists $\delta_1 > 0$ such that $h(x) \geq \delta_1(x^2+1)$ for all $x \in \mathbb{R}$. In addition, we can take ϵ sufficiently small, only depending on β_2, such that $|\gamma_1-\gamma_2||k_1| \leq \frac{1}{2} + \delta_1|\beta_2||k_1|^3$ for any $k_1 \in \mathbb{Z}^*$. Hence, for $k_1,k_2,k_3 \in \mathbb{Z}^*$ satisfying $\sum_{i=1}^3 k_i = 0$ we have

$$\begin{aligned}\langle H(k_1,k_2,k_3)\rangle \geq & 1 + 3|\beta_2||k_1|^3 h\left(\frac{k_2}{k_1}\right) - |\gamma_1-\gamma_2||k_1| \geq \frac{1}{2} + 3\delta_1|\beta_2||k_1||k_1|^2\left(\frac{k_2^2}{k_1^2}+1\right) - \delta_1|\beta_2||k_1|^3\\ \geq & \delta\left(1+|k_1|\sum_i^3 |k_i|^2\right) \geq \delta\prod_{i=1}^3 |k_i|,\end{aligned}$$

where δ is a positive constant which depends on β_1, β_2 and the claim is verified.

To prove (3.10) using Cauchy-Schwarz inequality and for arguments similar to the one used previously, follows that

$$\|\partial_x(uv)\|_{Y_{s,-1}^{\beta_2,\gamma_2}} \leq I \times \sup_{\substack{\|a_k\|_{\ell^2(\mathbb{Z})}\leq 1\\ a_k\neq 0}} \left\|\frac{\chi_{\Omega(k_3)}(L_3)a_{k_3}}{\langle L_3\rangle^{1-a}}\right\|_{\ell^2_{k_3}(\mathbb{Z})L^2_{\tau_3}(\mathbb{R})} \tag{3.14}$$

with

$$I = \sup_{\|f_3\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})}\leq 1} \sum_{\Gamma}\int_{\Lambda} \frac{|k_3|\langle k_3\rangle^s}{\langle k_2\rangle^s\langle k_1\rangle^s}\frac{|f_1(k_1,\tau_1)|}{\langle L_1(k_1,\tau_1)\rangle^{\frac{1}{2}}}\frac{|f_2(k_2,\tau_2)|}{\langle L_2(k_2,\tau_2)\rangle^{\frac{1}{2}}}\frac{|\widetilde{f}_3(k_3,\tau_3)|}{\langle L_3(k_3,\tau_3)\rangle^{a}}\mathrm{d}\Lambda$$

and f_i, L_i, for $i=1,2,3$, defined as in (3.11). Here the characteristic function $\chi_{\Omega(k_3)}(L_3)$ will be chosen so that

$$\left\|\frac{\chi_{\Omega(k_3)}(L_3)}{\langle L_3\rangle^{1-a}}\right\|_{L^2_{\tau_3}(\mathbb{R})} \lesssim 1$$

uniformly in the parameter k_3,$|k_i| \geq 1$, for $i=1,2,3$, and $a>0$ to be chosen conveniently.

Now, define $MAX := \max\{\langle L_1(k_1,\tau_1)\rangle, \langle L_2(k_2,\tau_2)\rangle, \langle L_3(k_3,\tau_3)\rangle\}$. Since H is δ-significant, we have

$$\prod_{i=1}^3 |k_i| \lesssim H(k_1,k_2,k_3) = \sum_{i=1}^3 L_i(k_i,\tau_i) \lesssim \sum_i^3 \langle L_i(k_i,\tau_i)\rangle \lesssim MAX.$$

The rest of the proof will be split in two cases.

Case 1: $MAX = \langle L_1(k_1,\tau_1)\rangle$ or $MAX = \langle L_2(k_2,\tau_2)\rangle$.

Assume without loss of generality $MAX = \langle L_1(k_1,\tau_1)\rangle$. Take $a = \frac{1}{2}^- = \frac{1}{2} - \epsilon'$ with $12\epsilon' = \frac{1}{100}$. In this case, we can put $\Omega(k_3) = \mathbb{R}$. Thus, from (3.14) we have

$$\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2}_{s,-1}} \lesssim \sup_{\|f_3\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})}\leq 1} \sum_{\Gamma}\int_{\Lambda} |f_1(k_1,\tau_1)| \frac{|f_2(k_2,\tau_2)|}{\langle L_2(k_2,\tau_2)\rangle^{\frac{1}{2}}} \frac{|\widetilde{f_3}(k_3,\tau_3)|}{\langle L_3(k_3,\tau_3)\rangle^{\frac{1}{2}^-}} \mathrm{d}\Lambda.$$

Similarly to $X^{\beta_2,\gamma_2}_{s,-\frac{1}{2}}$-norm, we obtain

$$\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2}_{s,-1}} \lesssim \sup_{\|f_3\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})}\leq 1} \lesssim \|u\|_{X^{\beta_1,\gamma_1}_{s,\frac{1}{2}}} \|v\|_{X^{\beta_2,\gamma_2}_{s,\frac{1}{3}}} \|f_3\|_{X^{\beta_2,\gamma_2}_{0,-\frac{1}{6}+}} \lesssim \|u\|_{X^{\beta_1,\gamma_1}_{s,\frac{1}{2}}} \|v\|_{X^{\beta_2,\gamma_2}_{s,\frac{1}{3}}}. \tag{3.15}$$

Case 2: $MAX = \langle L_3(k_3,\tau_3)\rangle$

This case will be divided in two parts.

Part I. $\langle L_3(k_3,\tau_3)\rangle^{\frac{1}{100}} \leq \delta \langle L_1(k_1,\tau_1)\rangle \langle L_2(k_2,\tau_2)\rangle$.

Observe that

$$\left(\prod_{i=1}^{3}|k_i|\right)^{\frac{1}{2}} \lesssim \langle L_3(k_3,\tau_3)\rangle^{\frac{1}{2}} \lesssim \delta^{\frac{1}{12}} \langle L_3(k_3,\tau_3)\rangle^{\frac{1}{2}-\epsilon'} \langle L_1(k_1,\tau_1)\rangle^{\frac{1}{12}} \langle L_2(k_2,\tau_2)\rangle^{\frac{1}{12}}. \tag{3.16}$$

Therefore, choosing a and $\Omega(k_3)$ as in the Case 1, from (3.14) results

$$\begin{aligned}\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2}_{s,-1}} &\lesssim \sup_{\|f_3\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})}\leq 1} \sum_{\Gamma}\int_{\Lambda} \frac{|f_1(k_1,\tau_1)|}{\langle L_1(k_1,\tau_1)\rangle^{\frac{5}{12}}} \frac{|f_2(k_2,\tau_2)|}{\langle L_2(k_2,\tau_2)\rangle^{\frac{5}{12}}} |\widetilde{f_3}(k_3,\tau_3)| \mathrm{d}\Lambda \\ &\lesssim \|u\|_{X^{\beta_1,\gamma_1}_{s,\frac{5}{12}}} \|v\|_{X^{\beta_2,\gamma_2}_{s,\frac{5}{12}}}\end{aligned}$$

So, for any $\theta \in (0,\frac{1}{6})$ we have

$$\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2,T}_{s,-1}} \lesssim T^{\theta} \|u\|_{X^{\beta_1,\gamma_1,T}_{s,\frac{5}{12}+\theta}} \|v\|_{X^{\beta_2,\gamma_2,T}_{s,\frac{5}{12}+\theta}} \lesssim T^{\theta} \|u\|_{X^{\beta_1,\gamma_1,T}_{s,\frac{1}{2}}} \|v\|_{X^{\beta_2,\gamma_2,T}_{s,\frac{1}{2}}}.$$

Part II. $\langle L_1(k_1,\tau_1)\rangle \langle L_2(k_2,\tau_2)\rangle \ll \delta \langle L_3(k_3,\tau_3)\rangle^{\frac{1}{100}}$

Note that

$$\langle L_3(k_3,\tau_3) + H(k_1,k_2,k_3)\rangle = \langle L_1 + L_2\rangle \ll \delta \langle L_3\rangle^{\frac{1}{100}}.$$

Thus, $|H(k_1,k_2,k_3)| \sim |L_3(k_3,\tau_3)|$ and

$$\langle L_3(k_3,\tau_3) + H(k_1,k_2,k_3)\rangle \ll \delta \langle H(k_1,k_2,k_3)\rangle^{\frac{1}{100}}. \tag{3.17}$$

Define for any $k_3 \in \mathbb{Z}^*$ the set

$$\Omega^\delta(k_3) := \Big\{ \tau \in \mathbb{R} \ : \ \exists\, k_1, k_2, \in \mathbb{Z} \ \text{ such that } \ \sum_{i=1}^{3} k_i = 0 \ \text{ and } \\ \langle L_3(k_3,\tau_3) + H(k_1,k_2,k_3)\rangle \ll \delta \, \langle H(k_1,k_2,k_3)\rangle^{\frac{1}{100}} \Big\}. \tag{3.18}$$

From (3.17) follows that $L_3(k_3,\tau_3) \in \Omega^\delta(k_3)$. Taking $a=\frac{1}{2}$ and $\Omega(k_3)=\Omega^\delta(k_3)$ defined by (3.18) we have[8]

$$\left\| \frac{\chi_{\Omega(k_3)}(L_3)}{\langle L_3\rangle^{\frac{1}{2}}} \right\|_{L^2_{\tau_3(\mathbb{R})}} \lesssim 1$$

uniformly in k_3 implying

$$\sup_{\substack{\|a_k\|_{\ell^2(\mathbb{Z})}\leq 1 \\ a_k \neq 0}} \left\| \frac{a_{k_3}}{\langle L_3\rangle^{\frac{1}{2}}} \right\|_{\ell^2_{k_3}(\mathbb{Z}) L^2_{\tau_3}(\mathbb{R})} \lesssim 1.$$

Hence, using the first inequality in (3.16) we obtain

$$\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2}_{s,-1}} \lesssim \sup_{\|f_3\|_{\ell^2(\mathbb{Z})L^2(\mathbb{R})}\leq 1} \sum_{\Gamma} \int_{\Lambda} \frac{|f_1(k_1,\tau_1)|}{\langle L_1(k_1,\tau_1)\rangle^{\frac{1}{2}}} \frac{|f_2(k_2,\tau_2)|}{\langle L_2(k_2,\tau_2)\rangle^{\frac{1}{2}}} |\widetilde{f}_3(k_3,\tau_3)| \lesssim \|u\|_{X^{\beta_1,\gamma_1}_{s,\frac{1}{3}}} \|v\|_{X^{\beta_2,\gamma_2}_{s,\frac{1}{3}}}.$$

Then, in both situation we obtain

$$\|\partial_x(uv)\|_{Y^{\beta_2,\gamma_2,T}_{s,-1}} \lesssim T^\theta \|u\|_{X^{\beta_1,\gamma_1,T}_{s,\frac{1}{2}}} \|v\|_{X^{\beta_2,\gamma_2,T}_{s,\frac{1}{2}}},$$

for any $\theta \in \left(0,\frac{1}{6}\right)$, showing (3.10) and, consequently, finishing the demonstration of the lemma. □

Finally, to finish this section we will prove nonlinear estimates associated with the solutions of (1.6) with $p=q=0$. To do it, we introduce the following notation

$$\mathcal{Z}_i := Z^{\beta_i,\gamma_i,T}_{s,-\frac{1}{2}},$$

for $i=1,2$ and

$$\mathcal{Z} := Z^{\beta_1,\gamma_1,T}_{s,-\frac{1}{2}} \times Z^{\beta_2,\gamma_2,T}_{s,-\frac{1}{2}}.$$

Lemma 3.8. *Let (u,v) and (w,z) belong to $\mathcal{Z}$ with $[u]=[v]=0$. Consider s,β_i,γ_i, $i=1,2$, as in Lemma 3.7 satisfying $\frac{\beta_2}{\beta_1}<0$, P and Q defined by (1.7). Then, there exist constants $\theta>0$, $\epsilon=\epsilon(\beta_1,\beta_2)>0$ and $C_3=C_3(\beta_1,\beta_2)>0$, independent of T, u and v, such that if $|\gamma_1|+|\gamma_2|<\epsilon$, the following estimates are satisfied*

$$\|\partial_x(P(u,v),Q(u,v))\|_{\mathcal{Z}} \leq C_3 T^\theta \|(u,v)\|^2_{X^{\beta_1,\gamma_1,T}_{s,\frac{1}{2}} \times X^{\beta_2,\gamma_2,T}_{s,\frac{1}{2}}}, \tag{3.19}$$

[8] This proof is extremely technical and can be found in Lemma 6.2 case 5.2.2. and Lemma 6.3 of [40] which in turn was inspired by Lemma 7.4 of [13].

$$\|\partial_x(P(u,v)-P(w,z))\|_{\mathcal{Z}_1} \leq C_3T^{\theta}\|(u,v)-(w,z)\|_{\mathcal{Z}}\left(\|(u,v)\|_{\mathcal{Z}}+\|(w,z)\|_{\mathcal{Z}}\right) \tag{3.20}$$

and

$$\|\partial_x(Q(u,v)-Q(w,z))\|_{\mathcal{Z}_2} \leq C_3T^{\theta}\|(u,v)-(w,z)\|_{\mathcal{Z}}\left(\|(u,v)\|_{\mathcal{Z}}+\|(w,z)\|_{\mathcal{Z}}\right). \tag{3.21}$$

Proof. First of all, (3.19) is a direct consequence of Lemma 3.7. Just applying (3.7) for P and Q provides us the existence of positive constants C_3, θ and ϵ such that

$$\begin{aligned}\|\partial_x(P(u,v),Q(u,v))\|_{\mathcal{Z}} \leq C_2T^{\theta}\Bigg(&|A|\|u\|^2_{\mathcal{Z}_1}+|B|\|u\|_{\mathcal{Z}_1}\|v\|_{\mathcal{Z}_2}+\frac{|C|}{2}\|v\|_{\mathcal{Z}_2}+|D|\|v\|^2_{\mathcal{Z}_2}\\ &+|C|\|u\|_{\mathcal{Z}_1}\|v\|_{\mathcal{Z}_2}+\frac{|B|}{2}\|u\|^2_{\mathcal{Z}_1}\Bigg)\\ \leq\ & C_3T^{\theta}\|(u,v)\|^2_{X^{\beta_1,\gamma_1,T}_{s,\frac12}\times X^{\beta_2,\gamma_2,T}_{s,\frac12}},\end{aligned}$$

whenever $|\gamma_1|+|\gamma_2|<\epsilon$, where $C_3=C_2\cdot 2\max\{|A|,|B|,|C|,|D|\}$.

Let us now prove (3.20). As the proof of (3.21) is analogous we will omit it. Note that we can write

$$P(u,v)-P(w,z)=A(u-w)(u+w)+B(u-w)v+B(v-z)w+\frac{C}{2}(v-z)(v+z)$$

Thus, again by (3.7), we get that

$$\begin{aligned}\|\partial_x(P(u,v)-P(w,z))\|_{\mathcal{Z}_1} \leq & C_3T^{\theta}\left(\|(u-w)\|_{\mathcal{Z}_1}\|u+w\|_{\mathcal{Z}_1}+\|u-w\|_{\mathcal{Z}_1}\|v\|_{\mathcal{Z}_2}\right.\\ &\left.+\|v-z\|_{\mathcal{Z}_2}\|w\|_{\mathcal{Z}_1}+\|v-z\|_{\mathcal{Z}_2}\|v+z\|_{\mathcal{Z}_2}\right),\end{aligned}$$

which implies

$$\|\partial_x(P(u,v)-P(w,z))\|_{\mathcal{Z}_1} \leq C_3T^{\theta}\|(u,v)-(w,z)\|_{\mathcal{Z}}\left(\|(u,v)\|_{\mathcal{Z}}+\|(w,z)\|_{\mathcal{Z}}\right).$$

Therefore, (3.20) and (3.21) is verified and the proof of the lemma is complete. □

3.3. Local well-posedness

Throughout the article, from now on, we will consider the following notations

$$\mathcal{Z}_{s,b}:=Z^{1,\mu,T}_{s,b}\times Z^{\alpha,\zeta,T}_{s,b},\quad \mathcal{Z}^1_{s,b}:=Z^{1,\mu,T}_{s,b},\quad \mathcal{Z}^{\alpha}_{s,b}:=Z^{\alpha,\zeta,T}_{s,b},$$

and

$$\mathcal{X}_{s,b}:=X^{1,\mu,T}_{s,b}\times X^{\alpha,\zeta,T}_{s,b},\quad \mathcal{X}^1_{s,b}:=X^{1,\mu,T}_{s,b},\quad \mathcal{X}^{\alpha}_{s,b}:=X^{\alpha,\zeta,T}_{s,b}.$$

Additionally, when $b=\frac12$, we will denote

$$\mathcal{Z}_{s}:=Z^{1,\mu,T}_{s,\frac12}\times Z^{\alpha,\zeta,T}_{s,\frac12},\quad \mathcal{Z}^1_{s}:=Z^{1,\mu,T}_{s,\frac12},\quad \mathcal{Z}^{\alpha}_{s}:=Z^{\alpha,\zeta,T}_{s,\frac12},$$

and

$$\mathcal{X}_s := X^{1,\mu,T}_{s,\frac{1}{2}} \times X^{\alpha,\zeta,T}_{s,\frac{1}{2}}, \quad \mathcal{X}^1_s := X^{1,\mu,T}_{s,\frac{1}{2}}, \quad \mathcal{X}^\alpha_s := X^{\alpha,\zeta,T}_{s,\frac{1}{2}}.$$

Let us now consider the IVP (1.11). For given $\lambda > 0$, let us define

$$L_{\beta,\gamma,\lambda}\phi = \int_0^1 e^{-2\lambda\tau} S^{\beta,\gamma}(-\tau)GG^* S^*(-\tau)\phi \mathrm{d}\tau,$$

for any $\phi \in H^s(\mathbb{T})$ and $s \geq 0$. Clearly, L_λ is a bounded linear operator from $H^s_0(\mathbb{T})$ to $H^s_0(\mathbb{T})$. Moreover, $L_{\beta,\gamma,\lambda}$ is a self-adjoint positive operator on $L^2_0(\mathbb{T})$, and so is its inverse $L^{-1}_{\beta,\gamma,\lambda}$. Therefore $L_{\beta,\gamma,\lambda}$ is an isomorphism from $L^2_0(\mathbb{T})$ onto itself, and the same is true on $H^s_0(\mathbb{T})$, with $s \geq 0$ (see, for instance, [24], Lem. 2.4).

With this information in hand, choose the two feedback controls $f = -G^* L^{-1}_{\beta,\gamma,\lambda} u$ and $h = -G^* L^{-1}_{\beta,\gamma,\lambda} v$, in (1.11), to transform this system in a resulting closed-loop system reads as follows

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu\partial_x u + \eta\partial_x v + \partial_x P(u,v) = -K_{1,\mu,\lambda}u, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ \partial_t v + \alpha\partial_x^3 v + \zeta\partial_x u + \eta\partial_x u + \partial_x Q(u,v) = -K_{\alpha,\zeta,\lambda}v, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)), & x \in \mathbb{T}, \end{cases} \tag{3.22}$$

with $K_{\beta,\gamma,\lambda} := GG^* L^{-1}_{\beta,\gamma,\lambda}$ for $\beta = 1$ and $\gamma = \mu$, and for $\beta = \alpha$ and $\gamma = \eta$. If $\lambda = 0$, we have $K_0 = GG^*$.

We will prove that the IVP (3.22) is well-posed in the spaces $H^s_0(\mathbb{T}) \times H^s_0(\mathbb{T})$, for $s \geq 0$. To prove it, we will borrow the following lemma shown in Lemma 4.2 of [24] for the case $\beta = 1$ and $\gamma = \mu > 0$. The proof in the general case, presented below, is similar to the one made there and will be omitted.

Lemma 3.9. *For any $\widetilde{\epsilon} > 0$ and $\phi \in Z^{\beta,\gamma,T}_{s,\frac{1}{2}}$ there exists a positive constant $C(\widetilde{\epsilon}) > 0$ such that*

$$\left\| \int_0^t S^{\beta,\gamma}(t-\tau)(K_\lambda \phi)(\tau)\mathrm{d}\tau \right\|_{Z^{\beta,\gamma,T}_{s,\frac{1}{2}}} \leq C(\widetilde{\epsilon}) T^{1-\widetilde{\epsilon}} \|\phi\|_{Z^{\beta,\gamma,T}_{s,\frac{1}{2}}}.$$

The next local well-posedness result is a consequence of Lemmas 3.4, 3.8 and 3.9, and its proof is classical, so we will omit it.

Theorem 3.10. *Let $\lambda \geq 0$ and $s \geq 0$ be given. Then, there exists $\epsilon = \epsilon(\alpha)$ with $|\mu| + |\zeta| < \epsilon$ such that for $T > 0$, small enough, and any $(u_0, v_0) \in H^s_0(\mathbb{T}) \times H^s_0(\mathbb{T})$ there exists a unique solution (u, v) of (3.22) in the class*

$$(u,v) \in \Xi_s := \mathcal{Z}^1_s \cap C([0,T]; L^2_0(\mathbb{T})) \times \mathcal{Z}^\alpha_s \cap C([0,T]; L^2_0(\mathbb{T})). \tag{3.23}$$

Furthermore, the following estimate holds

$$\|(u,v)\|_{\mathcal{Z}_s} \leq a_{T,s}(\|(u_0,v_0)\|)\|(u_0,v_0)\|_{H^s(\mathbb{T})\times H^s(\mathbb{T})}, \tag{3.24}$$

where $a_{s,T} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a nondecreasing continuous function depending only of T, s and constants α, μ, ζ.

In addition, for any $T_0 \in (0,T)$ there exists a neighborhood U_0 of (u_0, v_0) such that the mapping $(u_0, v_0) \longmapsto (u, v)$ from U_0 into Ξ_s is Lipschitz.

3.4. Global well-posedness

We check that the system (3.22) is globally well-posed in the space $H^s(\mathbb{T})$, for any $s \geq 0$. Precisely, the result can be read as follows.

Theorem 3.11. *Let $(u_0, v_0) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$, for any $s \geq 0$. Then the solution $(u, v) \in \Xi_s$ given in Theorem 3.10 can be extended for any $T > 0$ and still satisfies (3.24).*

Proof. Assume first $s = 0$. Multiplying the first equation of (3.22) by u and the second one by v, integrating on $\mathbb{T} \times (0, t)$, for $t \geq 0$, we have

$$\|(u(\cdot,t), v(\cdot,t))\|^2 \leq 2\|G\|^2 \| \left[L_{1,\mu,\lambda}^{-1}\| + \|L_{\alpha,\zeta,\lambda}^{-1}\| \right] \int_0^t \|(u,v)(\cdot,\tau)\|^2,$$

since G and $L_{\beta,\gamma,\lambda}^{-1}$ are continuous in $L^2(\mathbb{T})$ and

$$\int_{\mathbb{T}} \partial_x P(u,v)u + \partial_x Q(u,v)v = \frac{2}{3}\int_{\mathbb{T}} \frac{\mathrm{d}}{\mathrm{d}x}\left[(Au^3 + Dv^3) + Bu^2 v + Cuv^2 \right] = 0. \tag{3.25}$$

Using Grönwall's inequality holds that

$$\|(u(\cdot,t), v(\cdot,t))\|^2 \leq \|(u_0, v_0)\|^2 e^{C_5 t}, \tag{3.26}$$

with $C_5 = 2\|G\|^2 \left[\|L_{1,\mu,\lambda}^{-1}\| + \|L_{\alpha,\zeta,\lambda}^{-1}\| \right]$. In particular, for $\lambda = 0$, from the energy identity, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u(\cdot,t), v(\cdot,t))\|^2 = -\|(Gu, Gv)\|^2 \leq 0 \tag{3.27}$$

and

$$\|(u(\cdot,t), v(\cdot,t))\|^2 \leq \|(u_0, v_0)\|^2,$$

which ensures that (3.22) is globally well-posed in $L_0^2(\mathbb{T}) \times L_0^2(\mathbb{T})$.

Next, we show that (3.22) is globally well-posed in the space $H_0^3(\mathbb{T}) \times H_0^3(\mathbb{T})$. For a smooth solution (u, v) of (3.22), let $(\widetilde{u}, \widetilde{v}) = (\partial_t u, \partial_t v)$. Then

$$\begin{cases} \partial_t \widetilde{u} + \partial_x^3 \widetilde{u} + \mu \partial_x \widetilde{u} + \eta \widetilde{v} + \partial_x \widetilde{P}(\widetilde{u}, \widetilde{v}) = -K_\lambda \widetilde{u}, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ \partial_t \widetilde{v} + \alpha \partial_x^3 \widetilde{v} + \zeta \partial_x \widetilde{v} + \eta \widetilde{u} + \partial_x \widetilde{Q}(\widetilde{u}, \widetilde{v}) = -K_\lambda \widetilde{v}, & x \in \mathbb{T},\ t \in \mathbb{R}, \\ (\widetilde{u}(x,0), \widetilde{v}(x,0)) = (\widetilde{u}_0, \widetilde{v}_0), & x \in \mathbb{T}, \end{cases}$$

where

$$\begin{aligned} \widetilde{P}(\widetilde{u}, \widetilde{v}) &= 2Au\widetilde{u} + B\widetilde{u}v + Bu\widetilde{v} + Cv\widetilde{v}, \\ \widetilde{Q}(\widetilde{u}, \widetilde{v}) &= 2Dv\widetilde{v} + C\widetilde{v}u + Cv\widetilde{u} + Bu\widetilde{u}, \\ \widetilde{u}_0 &= -K_\lambda u_0 - u_0''' - \mu u_0' - \eta v_0' - P'(u_0, v_0), \\ \widetilde{v}_0 &= -K_\lambda v_0 - v_0''' - \zeta v_0' - \eta u_0' - Q'(u_0, v_0), \end{aligned}$$

with " $'$ " denoting here the derivative with respect to variable x. Observe that

$$\begin{aligned}\|(P(u_0,v_0))',(Q(u_0,v_0))'\| \leq& \frac{2C_3}{C_2}\|(u_0,v_0)\|\|(u_0',v_0')\|_{L^\infty(\mathbb{T})\times L^\infty(\mathbb{T})}\\ \leq& \frac{2C_3}{C_2}\|(u_0,v_0)\|C_5\|(u_0,v_0)\|^{\frac{1}{2}}\|(\partial_x^3u_0,\partial_x^3v_0)\|^{\frac{1}{2}}\\ \leq& \frac{2C_3C_5}{C_2}\|(u_0,v_0)\|^{\frac{3}{2}}\|(u_0,v_0)\|_3^{\frac{1}{2}},\end{aligned}$$

where C_5 is due to the Gagliardo-Nirenberg inequality. So, we have

$$\begin{aligned}\|(\widetilde{u}_0,\widetilde{v}_0)\| \leq& \|(K_\lambda u_0,K_\lambda v_0)\|+C_6\|(u_0,v_0)\|_3+2(|\mu|+|\eta|+|\zeta|)\|(u_0,v_0)\|_1\\ &+\|(P(u_0,v_0))',(Q(u_0,v_0))'\|\\ \leq& \|(u_0,v_0)\|+2(|\mu|+|\eta|+|\zeta|)\|(u_0,v_0)\|_1+(1+C_6)\|(u_0,v_0)\|_3\\ &+\left(\frac{C_3C_5}{C_2}\right)^2\|(u_0,v_0)\|^3.\end{aligned} \tag{3.28}$$

where $C_6=\max\{1,|\alpha|\}$. Note that (u,v) is a smooth solution of (3.22). Moreover, $\widetilde{P}(\tilde{u},\tilde{v})$ and $\widetilde{Q}(\tilde{u},\tilde{v})$ depends of (u,v). So, applying (3.7) and (3.8) for $u\tilde{u}, v\tilde{u}, u\tilde{v}$ and $v\tilde{v}$, we get

$$\|\partial_x(\widetilde{P}(\tilde{u},\tilde{v}),\widetilde{Q}(\tilde{u},\tilde{v}))\|_{\mathcal{Z}_s} \leq 2C_3T^\theta\|(u,v)\|_{\mathcal{Z}_s}\|(\tilde{u},\tilde{v})\|_{\mathcal{Z}_s},$$

with $|\mu|+|\zeta|<\epsilon$, for some ϵ that depends only α, we obtain from (3.28) that

$$\|(\tilde{u},\tilde{v})\|_{\mathcal{Z}_0} \leq C_0\|(\tilde{u}_0,\tilde{v}_0)\|+(C(\tilde{\epsilon})T^{1-\tilde{\epsilon}}+C_1T^\theta)\|(\tilde{u},\tilde{v})\|_{\mathcal{Z}_0}+2C_0C_3T^\theta\|(u,v)\|_{\mathcal{Z}_0}\|(\tilde{u},\tilde{v})\|_{\mathcal{Z}_0}.$$

Choosing T as

$$C(\epsilon)T^{1-\tilde{\epsilon}}+C_1T^\theta+2C_0C_3T^\theta d<\frac{1}{2},$$

we have

$$\|(u,v)\|_{\mathcal{Z}_0} \leq d=2C_0\|(u_0,v_0)\|$$

and consequently

$$\|(\tilde{u},\tilde{v})\|_{\mathcal{Z}_0} \leq C_0\|(\tilde{u}_0,\tilde{v}_0)\|+(C(\epsilon)T^{1-\epsilon}+C_1T^\theta+4C_0^2C_3T^\theta\|(u_0,v_0)\|)\|(\tilde{u},\tilde{v})\|_{\mathcal{Z}_0}.$$

Hence, for T_1 satisfying

$$C(\epsilon)T_1^{1-\epsilon}+C_1T^\theta+4C_0^2C_3T_1^\theta\|(u_0,v_0)\|<\frac{1}{2},$$

we obtain

$$\|(\tilde{u},\tilde{v})\|_{\mathcal{Z}_0} \leq 2C_0\|(\tilde{u}_0,\tilde{v}_0)\|.$$

Therefore, for $T_0 = \min\{T, T_1\}$, we see that

$$\|(\widetilde{u},\widetilde{v})\|_{L^\infty(0,T_0;L^2(\mathbb{T})\times L^\infty(0,T_0;L^2(\mathbb{T})} \leq C_7\|(\widetilde{u},\widetilde{v})\|_{\mathcal{Z}_0} \leq 2C_0C_7\|(\widetilde{u}_0,\widetilde{v}_0)\|.$$

From the following equations

$$\begin{cases}\partial_x^3 u = -\widetilde{u} - \mu\partial_x u - \eta\partial_x v - \partial_x P(u,v) - K_\lambda u,\\ \partial_x^3 v = -\widetilde{v} - \zeta\partial_x v - \eta\partial_x u - \partial_x Q(u,v) - K_\lambda v,\end{cases}$$

we infer that

$$\begin{aligned}\|(\partial_x^3 u,\partial_x^3 v)\| \leq&\|(\widetilde{u},\widetilde{v})\| + \|(u,v)\| + \left(C_7\sqrt{2\pi} + \frac{2C_3}{C_2}\|(u,v)\|\right)\|(\partial_x u,\partial_x v)\|_{L^\infty(\mathbb{T})\times L^\infty(\mathbb{T})}\\ \leq&\|(\widetilde{u},\widetilde{v})\| + \|(u,v)\| + C_6\left(C_7\sqrt{2\pi} + \frac{2C_3}{C_2}\|(u,v)\|\right)\|(u,v)\|^{\frac{1}{2}}\|(\partial_x^3 u,\partial_x^3 v)\|^{\frac{1}{2}}\\ \leq&\|(\widetilde{u},\widetilde{v})\| + \frac{1}{2}\|(\partial_x^3 u,\partial_x^3 v)\| + \left[1 + \frac{C_6^2}{2}\left(C_7\sqrt{2\pi} + \frac{2C_3}{C_2}\|(u,v)\|\right)^2\right]\|(u,v)\|,\end{aligned}$$

for $0 < t < T_0$ and $C_7 = 2(|\mu| + |\eta| + |\zeta|)$. Consequently, since $(\widetilde{u}_0,\widetilde{v}_0)$ satisfies (3.28) , we get that

$$\|(u,v)\|_{L^\infty(0,T;H_0^3(\mathbb{T}))\times L^\infty(0,T;H_0^3(\mathbb{T}))} \leq a_{T,3}(\|(u_0,v_0)\|)\|(u_0,v_0)\|_3.$$

Combining to (3.26), this shows that $(u,v) \in C(\mathbb{R}^+; H_0^3(\mathbb{T})) \times C(\mathbb{R}^+; H_0^3(\mathbb{T}))$ and (3.24) holds true for $s = 3$. A similar result can be obtained for any $s \in 3\mathbb{N}^*$. Note that for other values of s, the global well-posedness follows by nonlinear interpolation as done in [3]. This achieves the result. □

4. Control results: nonlinear problems

4.1. A local result: stabilization result

Our first result is local in the sense that the initial data need to be in a small ball in the energy space to ensure that the solution of the system goes to zero exponentially, for t sufficiently large. The result is the following one.

Theorem 4.1. *Let $0 < \lambda' < \lambda$ and $s \geq 0$ be given. There exists $\delta > 0$ such that for any $(u_0,v_0) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$ and $\|(u_0,v_0)\|_s \leq \delta$, the corresponding solution (u,v) of (3.22) satisfies*

$$\|(u(\cdot,t),v(\cdot,t))\|_s \leq Ce^{-\lambda' t}\|(u_0,v_0)\|_s, \quad \forall t \geq 0,$$

where $C > 0$ is a constant independent of (u_0,v_0).

Proof. Since K_λ is a bounded operator, the solution of (3.22) can be rewritten in its integral form

$$\begin{aligned}u(t) &= S_\lambda^{1,\mu}(t)u_0 - \eta\int_0^t S_\lambda^{1,\mu}(t-\tau)\partial_x v(\tau)\mathrm{d}\tau - \int_0^t S_\lambda^{1,\mu}(t-\tau)\partial_x P(u,v)(\tau)\mathrm{d}\tau\\ v(t) &= S_\lambda^{\alpha,\zeta}(t)v_0 - \eta\int_0^t S_\lambda^{\alpha,\zeta}(t-\tau)\partial_x u(\tau)\mathrm{d}\tau - \int_0^t S_\lambda^{\alpha,\zeta}(t-\tau)\partial_x Q(u,v)(\tau)\mathrm{d}\tau,\end{aligned} \tag{4.1}$$

where $S_\lambda^{\beta,\gamma}(t) = e^{-t(\beta\partial_x^3 + \gamma\partial_x + K_\lambda)}$ is the group to the linear system associated to (3.22).

Now, let us consider α, μ, ζ satisfying the hypothesis of the Lemma 3.8 and 3.9. Next, using Lemma 3.4 twice, Lemmas 3.8 and 3.9 and finally, the fact that $L_{\beta,\gamma,\lambda}$ is a bounded linear operator from $H_0^s(\mathbb{T})$ to $H_0^s(\mathbb{T})$, for all $s \geq 0$, we can guarantee the existence of a constant $c > 0$ such that the following inequalities are verified

$$\|S_\lambda^{\beta,\gamma}(t)\phi\|_{Z^{\beta,\gamma,T}_{s,\frac{1}{2}}} \leq c\|\phi\|_s, \tag{4.2}$$

for any $\phi \in H_0^s(\mathbb{T})$,

$$\left\|\int_0^t S_\lambda^{1,\mu}(t-\tau)\partial_x v(\tau)\mathrm{d}\tau\right\|_{\mathcal{Z}_s^1} \leq c\|v\|_{\mathcal{Z}_s^\alpha}, \tag{4.3}$$

$$\left\|\int_0^t S_\lambda^{\alpha,\zeta}(t-\tau)\partial_x u(\tau)(\tau)\mathrm{d}\tau\right\|_{\mathcal{Z}_s^\alpha} \leq c\|u\|_{\mathcal{Z}_s^1} \tag{4.4}$$

for any $(u, v) \in \mathcal{Z}_s$,

$$\left\|\int_0^t S_\lambda^{1,\mu}(t-\tau)\partial_x P(u,v)(\tau)\mathrm{d}\tau\right\|_{\mathcal{Z}_s^1} \leq c\|(u,v)\|^2_{\mathcal{Z}_s}, \tag{4.5}$$

and

$$\left\|\int_0^t S_\lambda(t-\tau)\partial_x Q(u,v)(\tau)\mathrm{d}\tau\right\|_{\mathcal{Z}_s^\alpha} \leq c\|(u,v)\|^2_{\mathcal{Z}_s}, \tag{4.6}$$

for any $(u, v) \in \mathcal{Z}_s$ with $[u] = [v] = 0$.

Now, thanks to the ideas introduced by Theorem 2.1 of [38] and Proposition 2.5 of [24], for given $s \geq 0$, there exists some positive constant M_s such that

$$\|(S_\lambda^{1,\mu}(t)u_0, S_\lambda^{\alpha,\zeta}(t)v_0)\|_s \leq M_s e^{-\lambda t}\|(u_0,v_0)\|_s, \ \forall\, t \geq 0,$$

where $S_\lambda^{\beta,\gamma}$ with $\beta = 1$ and $\gamma = \mu$, and for $\beta = \alpha$ and $\gamma = \eta$ are the groups associated to the linear system (3.22). Pick $T > 0$ such that $2M_s e^{-\lambda T} \leq e^{-\lambda' T}$. We seek a solution (u, v) to the integral equations (4.1), as a fixed point of the following map

$$\Gamma_\lambda(w,z) = (\Gamma_\lambda^1(w,z), \Gamma_\lambda^2(w,z)),$$

defined by

$$\Gamma_\lambda^1(w,z) = S_\lambda^{1,\mu}(t)u_0 - \int_0^t S_\lambda^{1,\mu}(t-\tau)\partial_x v(\tau)\mathrm{d}\tau - \int_0^t S_\lambda^{1,\mu}(t-\tau)(\partial_x P(w,z))(\tau)\mathrm{d}\tau$$

and

$$\Gamma_\lambda^2(w,z) = S_\lambda^{\alpha,\zeta}(t)v_0 - \int_0^t S_\lambda^{\alpha,\zeta}(t-\tau)\partial_x u(\tau)\mathrm{d}\tau - \int_0^t S_\lambda^{\alpha,\zeta}(t-\tau)(\partial_x Q(w,z))(\tau)\mathrm{d}\tau,$$

in some closed ball

$$B_R(0) \subset Z_s^1 \cap L^2(0,T;L_0^2(\mathbb{T})) \times Z_s^\alpha \cap L^2(0,T;L_0^2(\mathbb{T}))$$

for the $\|(w,z)\|_{\mathcal{Z}_s}$-norm. This will be done provided that $\|(u_0,v_0)\|_s \leq \delta$, where δ is a small number to be determined. Furthermore, to ensure the exponential stability with the claimed decay rate, the numbers δ and R will be chosen in such a way that

$$\|(u(T),v(T))\|_s \leq e^{-\lambda' T}\|(u_0,v_0)\|_s.$$

Since Γ_λ is a contraction in $B_R(0)$, by a fixed point argument, its unique fixed point $(u,v) \in B_R(0)$ fulfills

$$\|(u(T),v(T))\| = \|\Gamma_\lambda(u,v)\|_s \leq e^{-\lambda' T}\delta.$$

Finally, assume that $\|(u_0,v_0)\|_s < \delta$. Changing δ into $\delta' := \|(u_0,v_0)\|_s$ and R into $R' = \left(\frac{\delta'}{\delta}\right)^{\frac{1}{2}} R$, we infer that

$$\|(u(T),v(T))\|_s \leq e^{-\lambda' T}\|(u_0,v_0)\|_s$$

and by induction yields

$$\|(u(nT),v(nT))\|_s \leq e^{-\lambda' nT}\|(u_0,v_0)\|_s,$$

for any $n \geq 0$. As $\mathcal{Z}_s \cap L^2(0,T;L_0^2(\mathbb{T})) \subset C([0,T];H_0^s(\mathbb{T}))$, we infer by the semigroup property that there exists some constant $C' > 0$ such that

$$\|(u(t),v(t))\|_s \leq C' e^{-\lambda' t}\|(u_0,v_0)\|_s,$$

provided that $\|(u_0,v_0)\|_s \leq \delta$ and the proof of Theorem 4.1 is completed. □

4.2. A global result: stabilization result

As mentioned the stability result presented in Theorem 4.1 is purely local. Now we are in a position to extend it to global stability. It is well-known [8, 24, 41] in Control Theory that Theorem 1.4 is a direct consequence of the following observability inequality.

Let $T > 0$ and $R_0 > 0$ be given. There exists a constant $\rho > 0$ such that for any $(u_0,v_0) \in L_0^2(\mathbb{T}) \times L_0^2(\mathbb{T})$ satisfying $\|(u_0,v_0)\| \leq R_0$, the corresponding solution (u,v) of (3.22) satisfies

$$\|(u_0,v_0)\|^2 \leq \rho \int_0^T \|(Gu,Gv)\|^2(t)\mathrm{d}t. \tag{4.7}$$

So, our next steps is to show the observability inequality.

Proof of (4.7). Suppose that (4.7) does not occur. Thus, for any $n \in \mathbb{N}$, there exists $(u_{n,0},v_{n,0}) := (u_n(0),v_n(0)) \in L_0^2(\mathbb{T}) \times L_0^2(\mathbb{T})$ such that the solution $(u_n,v_n) \in X$ of IVP (3.22), given by Theorem 3.11, satisfies

$$\|(u_{n,0},v_{n,0})\| \leq R_0 \tag{4.8}$$

and

$$\int_0^T \|(Gu_n, Gv_n)\|^2(t)\mathrm{d}t < \frac{1}{n}\|(u_{n,0}, v_{n,0})\|^2. \tag{4.9}$$

Since $a_n := \|(u_{n,0}, v_{n,0})\|$ is a bounded sequence in $\mathbb{R}$, we can choose a subsequence of $\{a_n\}$, still denoted by $\{a_n\}$, such that $\lim_{n\to\infty} a_n = a$. So, there are two possibilities for the limit, which will be divided in the following cases: $a > 0$ and $a = 0$.

i. Case $a > 0$.

Note that the sequence (u_n, v_n) is bounded in $L^\infty(0,T;L^2(\mathbb{T})) \times L^\infty(0,T;L^2(\mathbb{T}))$ and, also, in $\mathcal{X}_0$. Thus, applying Lemma 3.7 in each term of P and Q we have that $\partial_x P(u_n, v_n)$ and $\partial_x Q(u_n, v_n)$ are bounded in $\mathcal{Z}^1_{0,-\frac{1}{2}}$ and $\mathcal{Z}^\alpha_{0,-\frac{1}{2}}$, respectively, with $|\zeta| + |\mu| < \epsilon$, for some $\epsilon \ll 1$ and $\alpha < 0$. In particular, is bounded in $\mathcal{X}^1_{0,-\frac{1}{2}}$ and $\mathcal{X}^\alpha_{0,-\frac{1}{2}}$, respectively. Additionally, Bourgain spaces are reflexive and have the following compact embedding $\mathcal{X}_0 \hookrightarrow \mathcal{X}_{0,-1}$. Therefore, we can extract a subsequence of $\{(u_n, v_n)\}$, still denoted by $\{(u_n, v_n)\}$, such that

$$(u_n, v_n) \rightharpoonup (u, v) \text{ in } \mathcal{X}_0, \quad (u_n, v_n) \longrightarrow (u, v) \text{ in } \mathcal{X}_{0,-1}$$

and

$$(\partial_x P(u_n, v_n), \partial_x Q(u_n, v_n)) \rightharpoonup (f, g) \text{ in } \mathcal{X}_{0,-\frac{1}{2}},$$

where $(u, v) \in \mathcal{X}_0$ and $(f, g) \in \mathcal{X}_{0,-\frac{1}{2}}$. Moreover, since $\mathcal{X}_0$ is continuously embedded in $L^4((0,T) \times \mathbb{T})$, we have

$$\|u_n v_n\|_{L^2((0,T)\times\mathbb{T})} \leq \|u_n\|^2_{L^4((0,T)\times\mathbb{T})}\|v_n\|^2_{L^4((0,T)\times\mathbb{T})} \lesssim \|(u_n, v_n)\|^4_{\mathcal{X}_0},$$

which implies that $(P(u_n, v_n), Q(u_n, v_n))$ is bounded in $L^2((0,T) \times \mathbb{T}) \times L^2((0,T) \times \mathbb{T})$. Hence, it follows that $\partial_x(P(u_n, v_n), Q(u_n, v_n))$ is bounded in $L^2(0,T;H^{-1}(\mathbb{T})) \times L^2(0,T;H^{-1}(\mathbb{T})) = \mathcal{X}_{-1,0}$. Interpolating the spaces $\mathcal{X}_{0,-\frac{1}{2}}$ and $\mathcal{X}_{-1,0}$, we obtain that $\partial_x(P(u_n, v_n), Q(u_n, v_n))$ is bounded in $\mathcal{X}_{-\theta,-\frac{1}{2}(1-\theta)}$ for any $\theta \in [0,1]$. As $0 < \theta < 1$, it follows that $\mathcal{X}_{-\theta,-\frac{1}{2}(1-\theta)}$ is compactly embedded in $\mathcal{X}_{-1,-\frac{1}{2}}$. Thus, we can extract a subsequence of $\{(u_n, v_n)\}$, still denoted by $\{(u_n, v_n)\}$, such that

$$\partial_x(P(u_n, v_n), Q(u_n, v_n)) \longrightarrow (f, g) \text{ in } \mathcal{X}_{-1,-\frac{1}{2}}. \tag{4.10}$$

Thanks to (4.9) and the continuity of G, we ensures that

$$\int_0^T \|(Gu_n, Gv_n)\|^2\mathrm{d}t \longrightarrow \int_0^T \|(Gu, Gv)\|^2\mathrm{d}t = 0. \tag{4.11}$$

This convergence means that $(Gu, Gv) = (0, 0)$. Besides, since g is positive on $\omega \subset \mathbb{T}$, we have from definition (1.8) that

$$u(x,t) = \int_{\mathbb{T}} g(y)u(y,t)\mathrm{d}y = c_1(t) \quad \text{and} \quad v(x,t) = \int_{\mathbb{T}} g(y)v(y,t)\mathrm{d}y = c_2(t) \quad \text{on } \omega \times (0,T).$$

Thus, letting $n \longrightarrow \infty$, we obtain from (3.22) that

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu \partial_x u + \eta \partial_x v = f, & \text{on } \mathbb{T} \times (0,T), \\ \partial_t v + \alpha \partial_x^3 v + \zeta \partial_x v + \eta \partial_x u = g, & \text{on } \mathbb{T} \times (0,T), \\ u = c_1(t), \quad v = c_2(t), & \text{on } \omega \times (0,T). \end{cases}$$

Consider $w_n = u_n - u$, $z_n = v_n - v$, $P_n = -\partial_x P(u_n, v_n) + f - K_0 u_n$ and $Q_n = -\partial_x Q(u_n, v_n) + g - K_0 v_n$. Thus,

$$(w_n, z_n) \rightharpoonup (0,0) \quad \text{in} \quad \mathcal{X}_0$$

and satisfies

$$\begin{cases} \partial_t w_n + \partial_x^3 w_n + \mu \partial_x w_n + \eta \partial_x z_n = P_n, \\ \partial_t z_n + \alpha \partial_x^3 z_n + \zeta \partial_x z_n + \eta \partial_x w_n = Q_n. \end{cases}$$

Now, note that using the linearity of G we can rewrite

$$\begin{aligned} \int_0^T \|(Gw_n, Gz_n)\|^2 \mathrm{dt} = & \int_0^T \|(Gu_n, Gv_n)\|^2 \mathrm{dt} + \int_0^T \|(Gu, Gv)\|^2 \mathrm{dt} - 2\int_0^T \langle Gu_n, Gu \rangle \, \mathrm{d}t \\ & - 2\int_0^T \langle Gv_n, Gv \rangle \, \mathrm{d}t \end{aligned}$$

From (4.11), we obtain

$$\int_0^T \|(Gw_n, Gz_n)\|^2 \mathrm{dt} \longrightarrow 0. \tag{4.12}$$

On the other hand,

$$\int_0^T \|(Gw_n, Gz_n)\|^2 \mathrm{dt} = I + II + III, \tag{4.13}$$

where,

$$I = \int_0^T \int_{\mathbb{T}} g^2(x) \left[w_n^2(x,t) + z_n^2(x,t) \right] \mathrm{d}x\mathrm{d}t,$$

$$II = \int_0^T \left(\int_{\mathbb{T}} g^2(x) \mathrm{d}x \right) \left(\int_{\mathbb{T}} g(y) w_n(y,t) \mathrm{d}y \right)^2 \mathrm{d}t + \int_0^T \left(\int_{\mathbb{T}} g^2(x) \mathrm{d}x \right) \left(\int_{\mathbb{T}} g(y) z_n(y,t) \mathrm{d}y \right)^2 \mathrm{d}t$$

and

$$\begin{aligned} III = & -2\int_0^T \left(\int_{\mathbb{T}} g^2(x) w_n(x,t) \mathrm{d}x \right) \left(\int_{\mathbb{T}} g(y) w_n(y,t) \mathrm{d}y \right) \mathrm{d}t \\ & -2\int_0^T \left(\int_{\mathbb{T}} g^2(x) z_n(x,t) \mathrm{d}x \right) \left(\int_{\mathbb{T}} g(y) z_n(y,t) \mathrm{d}y \right) \mathrm{d}t. \end{aligned}$$

Let us prove that each previous term tends to zero as $n \longrightarrow \infty$. First, a direct application of Lemma A.1 implies that $II \longrightarrow 0$. Now, we can estimate III as follows

$$\begin{aligned}|III| &\leq \|b_n\|_{L^2(0,T)} \left\|\int_{\mathbb{T}} g^2 w_n(x,t)\mathrm{d}x\right\|_{L^2(0,T)} + \|c_n\|_{L^2(0,T)} \left\|\int_{\mathbb{T}} g^2 z_n(x,t)\mathrm{d}x\right\|_{L^2(0,T)} \\ &\leq \|g\|^2_{L^4(\mathbb{T})}\|(b_n,c_n)\|_{L^2(0,T)\times L^2(0,T)}\|(w_n,z_n)\|_{\mathcal{X}_0} \\ &\leq C\|(b_n,c_n)\|_{L^2(0,T)\times L^2(0,T)},\end{aligned}$$

where C is a positive constant. So, it follows that $III \longrightarrow 0$, when $n \longrightarrow \infty$, again thanks to the Lemma A.1. Lastly, combining the last two convergences with (4.12), we infer by (4.13) that $I \longrightarrow 0$.

We claim that

$$\|(w_n,z_n)\|_{L^2(0,T;L^2(\widetilde{\omega}))\times L^2(0,T;L^2(\widetilde{\omega}))} \longrightarrow 0, \quad \text{as } n \to \infty. \tag{4.14}$$

In fact, it is sufficient to observe that

$$\begin{aligned}\|(w_n,z_n)\|_{L^2(0,T;L^2(\widetilde{\omega}))\times L^2(0,T;L^2(\widetilde{\omega}))} &= \int_0^T \int_{\widetilde{\omega}} |g(x)|^{-2}\left[|g(x)|^2 w_n^2(x,t) + |g(x)|^2 z_n^2(x,t)\right]\mathrm{d}x\mathrm{d}t \\ &\leq \frac{4}{\|g\|^2_{L^\infty(\mathbb{T})}}\cdot\int_0^T\int_{\mathbb{T}} g^2(x)\left[w_n^2(x,t)+z_n^2(x,t)\right]\mathrm{d}x\mathrm{d}t,\end{aligned}$$

where $\widetilde{\omega} := \left\{g(x) > \frac{\|g\|_{L^\infty(\mathbb{T})}}{2}\right\}$ and using this previous inequality (4.14) follows. Additionally, note that

$$\begin{aligned}\|(P_n,Q_n)\|_{\mathcal{X}_{-1,-\frac{1}{2}}} \leq& \|GG^*(u_n,v_n)\|_{\mathcal{X}_{-1,-\frac{1}{2}}} + \|(\partial_x P(u_n,v_n) - f, \partial_x Q(u_n,v_n) - g)\|_{\mathcal{X}_{-1,-\frac{1}{2}}} \\ \leq& C\|(Gu_n,Gv_n)\|_{\mathcal{X}_{0,0}} + \|(\partial_x P(u_n,v_n) - f, \partial_x Q(u_n,v_n) - g)\|_{\mathcal{X}_{-1,-\frac{1}{2}}}.\end{aligned}$$

From (4.10), (4.11) and the previous inequality we obtain

$$\|(P_n,Q_n)\|_{\mathcal{X}_{-1,-\frac{1}{2}}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Applying Proposition A.2 with $b' = 0$ and $b = \frac{1}{2}$ yields that

$$(w_n,z_n) \longrightarrow 0 \text{ in } L^2_{loc}(0,T;L^2(\mathbb{T})) \times L^2_{loc}(0,T;L^2(\mathbb{T})). \tag{4.15}$$

Consequently,

$$(P(u_n,v_n),Q(u_n,v_n)) \longrightarrow (P(u,v),Q(u,v)) \text{ in } L^1_{loc}(0,T;L^1(\mathbb{T})) \times L^1_{loc}(0,T;L^1(\mathbb{T})) \tag{4.16}$$

and

$$(\partial_x P(u_n,v_n), \partial_x Q(u_n,v_n)) \longrightarrow (\partial_x P(u,v), \partial_x Q(u,v)),$$

in the distributional sense. Therefore, $(f,g) = (\partial_x P(u,v), \partial_x Q(u,v))$ and $(u,v) \in \mathcal{X}_0$ satisfies

$$\begin{cases} \partial_t u + \partial_x^3 u + \mu \partial_x u + \eta \partial_x v + \partial_x P(u,v) = 0, & \text{on } \mathbb{T} \times (0,T), \\ \partial_t v + \alpha \partial_x^3 v + \zeta \partial_x v + \eta \partial_x u + \partial_x Q(u,v) = 0, & \text{on } \mathbb{T} \times (0,T), \\ (u,v) = (c_1(t), c_2(t)), & \text{on } \omega \times (0,T). \end{cases}$$

From Corollary A.4, we infer that $(u,v) = (0,0)$ on $\mathbb{T} \times (0,T)$, which combined with (4.15) yields that $(u_n, v_n) \to (0,0)$ in $L^2_{loc}((0,T); L^2(\mathbb{T})) \times L^2_{loc}((0,T); L^2(\mathbb{T}))$. We can pick some time $t_0 \in [0,T]$ such that $(u_n(t_0), v_n(t_0)) \longrightarrow (0,0)$ in $L^2(\mathbb{T}) \times L^2(\mathbb{T})$. Since

$$\|(u_n(0), v_n(0))\|^2 = \|(u_n(t_0), v_n(t_0))\|^2 + \int_0^{t_0} \|(Gu_n, Gv_n)\|^2 \mathrm{d}t,$$

it is inferred that $a_n = \|(u_n(0), v_n(0))\| \longrightarrow 0$ which is a contradiction to the assumption $a > 0$.

ii. Case $a = 0$.

Note first that $a_n > 0$ for all n. Let $(w_n, z_n) = \left(\frac{u_n}{a_n}, \frac{v_n}{a_n}\right)$ for all $n \geq 1$. Then

$$\begin{cases} \partial_t w_n + \partial_x^3 w_n + \mu \partial_x w_n + \eta \partial_x z_n + K_0 w_n + a_n \partial_x P(w_n, z_n) = 0, \\ \partial_t z_n + \alpha \partial_x^3 z_n + \zeta \partial_x z_n + \eta \partial_x w_n + K_0 z_n + a_n \partial_x Q(w_n, z_n) = 0, \end{cases}$$

$$\int_0^T \|(Gw_n, Gz_n)\|^2 \mathrm{d}t < \frac{1}{n} \tag{4.17}$$

and

$$\|(w_n(0), z_n(0))\| = \frac{\|u_n(0)\|}{\|(u_{0,n}, v_{0,n})\|} + \frac{\|v_n(0)\|}{\|(u_{0,n}, v_{0,n})\|} = 1. \tag{4.18}$$

So, we obtain that the sequence $\{(w_n, z_n)\}$ which are bounded in both spaces $L^\infty(0,T; L^2(\mathbb{T})) \times L^\infty(0,T; L^2(\mathbb{T}))$ and $\mathcal{X}_0$. Indeed, $\|(w_n(t), z_n(t))\|$ is a nonincreasing function of t and since a_n is bounded, we have

$$\|(w_n, z_n)\|_{\mathcal{X}_0} \leq C_0 + \frac{(C(\widetilde{\epsilon}) T^{1-\widetilde{\epsilon}} + C_1 T^\theta)}{a_n} \|(u_n, v_n)\|_{\mathcal{X}_0} + \frac{C_0 C_3 T^\theta}{a_n^2} \|(u_n, v_n)\|_{\mathcal{X}_0}.$$

We can extract a subsequence of $\{(w_n, z_n)\}$, still denoted by $\{(w_n, z_n)\}$, such that

$$(w_n, z_n) \rightharpoonup (w,z) \text{ in } \mathcal{X}_0, \quad (w_n, z_n) \to (w,z) \text{ in } \mathcal{X}_{-1,-\frac{1}{2}} \quad \text{and} \quad (w_n, z_n) \to (w,z) \text{ in } \mathcal{X}_{-1,0},$$

as $n \to \infty$. Moreover, the sequence $\{(\partial_x P(w_n, z_n), \partial_x Q(w_n, z_n))\}$ is bounded in the space $\mathcal{X}_{0,-\frac{1}{2}}$, and therefore

$$a_n(\partial_x P(w_n, z_n), \partial_x Q(w_n, z_n)) \longrightarrow 0 \text{ in } \mathcal{X}_{0,-\frac{1}{2}},$$

when $n \to \infty$. Thus, (w, z) is solution of

$$\begin{cases} \partial_t w + \partial_x^3 w + \mu \partial_x w + \eta \partial_x z = 0, & \text{on } \mathbb{T} \times (0, T), \\ \partial_t z + \alpha \partial_x^3 z + \zeta \partial_x z + \eta \partial_x w = 0, & \text{on } \mathbb{T} \times (0, T), \\ w = c_1(t), \quad z = c_2(t), & \text{on } \omega \times (0, T). \end{cases}$$

Using Holmgren's uniqueness theorem, we can deduce that $w(x,t) = c_1(t) = c_1$ and $z(x,t) = c_2(t) = c_2$. However, as $[w] = [z] = 0$, we infer that $c_1 = c_2 = 0$.

According to (4.17)

$$\int_0^T \|(Gw_n, Gz_n)\|^2 \mathrm{d}t \longrightarrow 0, \qquad \text{as } n \to \infty,$$

and consequently $(K_0 w_n, K_0 z_n)$ converges strongly to $(0,0)$ in $\mathcal{X}_{-1,-\frac{1}{2}}$. Now, arguing as in the case $a > 0$, that is, applying again Proposition A.2, it follows that

$$(w_n, z_n) \longrightarrow (0,0) \text{ in } L^2_{loc}(0,T; L^2(\mathbb{T})) \times L^2_{loc}(0,T; L^2(\mathbb{T})),$$

and consequently $\|(w_n(0), z_n(0))\| \to 0$ which contradicts (4.18) and (4.7) is shown. □

4.3. Controllability result: nonlinear problem

We now consider the controllability properties of the nonlinear open loop control system (1.11). The following result is local and classical.

Theorem 4.2. *Let $T > 0$ and $s \geq 0$ be given. Then there exists a $\delta > 0$ such that for any (u_0, v_0), $(u_1, v_1) \in H_0^s(\mathbb{T}) \times H_0^s(\mathbb{T})$ and $\|(u_0, v_0)\|_s \leq \delta$, $\|(u_1, v_1)\|_s \leq \delta$, one can find two control inputs $(f, h) \in L^2([0,T]; H_0^s(\mathbb{T})) \times L^2([0,T]; H_0^s(\mathbb{T}))$ such that equation (1.11) has a solution*

$$(u, v) \in C([0,T]; H_0^s(\mathbb{T})) \times C([0,T]; H_0^s(\mathbb{T}))$$

satisfying $(u(x,0), v(x,0)) = (u_0(x), v_0(x))$ and $(u(x,T), v(x,T)) = (u_1(x), v_1(x))$.

Proof. The proof is analogous as done in [8, 24, 41], precisely the result is consequence of Corollary 2.6 and Remark 3.1 taking into account the following maps

$$\begin{aligned} u(t) = S^{1,\mu}(t) u_0 + \int_0^t S^{1,\mu}(t-\tau)(Gf)(\tau) - \eta \int_0^t S^{1,\mu}(t-\tau) \partial_x v(\tau) \mathrm{d}\tau \\ - \int_0^t S^{1,\mu}(t-\tau) \partial_x (P(u,v))(\tau) \mathrm{d}\tau \end{aligned}$$

and

$$\begin{aligned} v(t) = S^{\alpha,\zeta}(t) v_0 + \int_0^t S^{\alpha,\zeta}(t-\tau)(Gh)(\tau) - \eta \int_0^t S^{\alpha,\zeta}(t-\tau) \partial_x u(\tau) \mathrm{d}\tau \\ - \int_0^t S^{\alpha,\zeta}(t-\tau) \partial_x (Q(u,v))(\tau) \mathrm{d}\tau. \end{aligned}$$

□

5. Further comments and open problems

This article deals for the first time with the global aspect of control problems for the long waves in dispersive media, precisely systems like (1.1) with the coupled nonlinearity (1.7). From the perspective of considering the functions with null mean, the presence of the terms involving the constants B and C in the nonlinearities make the system studied coupled in the linear part (as well as in the nonlinear part) as can be seen in (1.5), that is, precisely the system with the terms $\eta\partial_x v$ and $\eta\partial_x u$, in the first and second equations of the system of (1.6). Unless that η can be considered 0, when $[u] = [v] = \beta$ and $B = -C$, the matrix of the operator L defined by (2.1) is not a diagonal matrix. For this reason, the arguments used in general for a singular dispersive equation (*e.g.* as the KdV case [24, 37]) in the study of the existence of solutions, controllability, and stabilization can not be directly applied here.

Note that the previous works are concentrated in a single KdV equation (or, as in the case of [10, 28], linear results or local results for coupled KdV systems). It is important to point out that in [28] the authors left some open problems concerning global controllability. In this spirit, our work is dedicated to covering this lack of results, that is, when presenting global results the article intends to give the first step to understanding global control problems in periodic domains for systems like (1.6) with quadratic nonlinearities.

Remark 5.1. In what concerns our main results the following remarks are worth mentioning:

- The system (1.1), differently from what happens for the KdV or for a pair of KdV which is decoupled only by the linear part, admits two families of eigenvalues associated with two different families of eigenfunctions. In this way, we proceed carefully to guarantee some spectral properties essential for obtaining a gap condition that satisfies an Ingham type theorem (see [20], Thm. 4.6).
- Another new and important fact, and itself interesting, is that the orthonormal basis for the space $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ formed by the eigenfunctions is not a pair compound for two identical copies of the basis $\{\frac{1}{\sqrt{2\pi}}e^{ikx}\}_{k\in\mathbb{Z}}$, as usual in this kind of problem, which makes the controls obtained for each equation different but comparable to each other.
- The global results presented in Theorems 1.4 and 1.5 are truly nonlinear and, which is more important, are global properties, which means that the initial and final data are controlled in a ball with no size restrictions.
- The novelty is that for the first time *Fourier restriction spaces* introduced by Bourgain [4] are used in two different dispersions to ensure the global control results. In fact, T. Oh [30] noticed, for the first time, the possibility of the occurrence of some resonance when dealing with systems of KdV-type equations in the periodic setting. This fact might obstruct the gain of extra regularity. Also, more recently, Yang and Zhang [40] proved new estimates related to this kind of system (see Sect. 3).
- It is important to mention that the propagation results have been successfully applied in control theory in several systems represented by single equations, such as wave equation [14], after that for the Schrödinger equation [22], for the Benjamin-Ono equation [26], KdV equation [24], the Kawahara equation [41], biharmonic Schrödinger equation [8], for the Benney-Luke equation [33] and, finally, for the Benjamin equation [31].

So, this work opens a series of situations that can be studied to understand the well-posedness theory and global controllability problems for long waves in dispersive media. We will now detail below the novelties of this work and open issues that seem interesting from a mathematical point of view.

5.1. Control problems

The problems in this work were solved by requiring some conditions over the constants of the system, namely α, μ, and ζ. Under the conditions $\alpha < 0$ and $\zeta - \mu > 0$, we find eigenfunctions associated with the operator L that define an orthonormal basis in $L^2(\mathbb{T}) \times L^2(\mathbb{T})$, so through a spectral analysis we were able to show, using the *moment method* [35], that linear system (1.10) is exactly controllable.

In addition, the global control problems are also verified thanks to the smoothing properties of the Bourgain spaces. This is the main novelty in this work since it is not to our knowledge that systems like (1.6) have global control properties (see [10, 28] for local results). In fact, the main tool used here is Bourgain spaces in different dispersions. With the smoothing properties of the Bourgain spaces in hand, the propagation of singularities showed in [24], for the single KdV equation, can be extended to the coupled KdV system defined by the operator L and together with the *bilinear estimates* and a *unique continuation property* we can achieve the global control results.

It is important to point out that even Bourgain spaces are important to establish the results for data in $L^2_0(\mathbb{T}) \times L^2_0(\mathbb{T})$, the propagation of compactness property is fundamental to obtain our global results. Indeed, in [37] Bourgain spaces were used however the results were local and the arguments there were "basically" small perturbations of linear results.

Finally, there is a drawback in our method, we are able to solve these global control problems only with two controls input. The use of control in one of the equations is still an open issue.

5.2. Well-posedness theory

As mentioned before, the Bourgain spaces are the key point in showing global results in this article. These spaces have been applied with success in the literature for global control results in less regular spaces (see, for instance, [8, 22–24, 31]).

The Bourgain spaces related to the linear system associated to (1.10), with $f = g = 0$, can be defined *via* the norm

$$\|W(-t)u\|_{H^s_x H^b_t} =: \|u\|_{X_{s,b}}$$

where $W(t) := e^{-tL}$ is the strongly continuous group generated by the operator L. However, in the aspect of the nonlinear problem, it is necessary to make bilinear estimates and there are no studies on such estimates with Bourgain spaces of this nature. To get around this situation, we use two Bourgain spaces associated with each dispersion present in (1.10). This strategy was used in [30, 40] for the study of the well-posedness of KdV-KdV type systems. Thus, the terms $-\eta\partial_x v$ and $-\eta\partial_x u$ are treated as part of nonlinearity in both studies: the well-posedness theory and in the control problem that concerns the asymptotic behavior of solutions to the problem (3.22).

Since such spaces do not have equivalent norms, a natural question that arises is whether we can estimate the components of the first equation with Bourgain's norm regarding the dispersion of the second and the converse. In this sense, lemmas 3.5 and 3.7 provide answers to this problem and are essential for this work.

Lemma 3.5 is an extension of Lemma 3.10 presented in [40]. In this work we verify that the result is still valid in the range $\frac{1}{3} < b < \frac{1}{2}$. Since $\beta_1, \beta_2, \gamma_1$ and γ_2 are fixed constants, the lemma is still true if we change the hypothesis $|\gamma_1| + |\gamma_2| < \epsilon$ for $\nu := |\beta_2 - \beta_1| - |\gamma_1 - \gamma_2| \geq \widetilde{\delta} > 0$, but the condition $\beta_2 \neq \beta_1$ is still required.

In turn, the key point of Lemma 3.7 is that the function H defined by (3.12) is δ-significant, and for it the condition $|\eta| + |\mu| < \epsilon$ is required. Furthermore, due to the nature of the nonlinearity given in (1.7) it is also essential that α is strictly negative. To extend the results presented here to a larger class of constants $\alpha, \mu, \zeta, \eta \in \mathbb{R}^*$ it is necessary to find conditions for which the function H is δ-significant. So, the well-posedness theory, global controllability and global stabilization properties for the system (1.5) where ζ and μ are not small enough is still an open problem, as well as the case when $\alpha > 0$.

Appendix A. Auxiliary results

In this appendix, we will give some results which were used throughout the paper. The first result gives us the properties of the two integral quantities.

Lemma A.1. *Let* $(w_n, z_n) \in \mathcal{X}_s$ *be bounded sequences. Define*

$$b_n := \int_{\mathbb{T}} g(y) w_n(y,t)\mathrm{d}y \quad \text{and} \quad c_n := \int_{\mathbb{T}} g(y) z_n(y,t)\mathrm{d}y.$$

If $(w_n, z_n) \rightharpoonup (0,0)$ *in* $\mathcal{X}_s$ *then*

$$b_n, c_n \longrightarrow 0 \text{ in } L^2(0,T). \tag{A.1}$$

Proof. By Cauchy-Schwarz inequality we have

$$\begin{aligned}\|(b_n, c_n)\|^2_{L^2_T \times L^2_T} &\leq \int_0^T \|g\|^2_{L^2(\mathbb{T})} \|(w_n(\cdot,t), z_n(\cdot,t))\|^2_{L^2(\mathbb{T})}\mathrm{d}t \\ &\leq C\|g\|^2_{L^2(\mathbb{T})}\|(w_n, z_n)\|_{\mathcal{X}_{0,0}},\end{aligned}$$

for some constant $C > 0$. By hypothesis

$$(w_n, z_n) \rightharpoonup (0,0) \text{ in } \mathcal{X}_0,$$

since $\mathcal{X}_0$ is compactly embedded in $\mathcal{X}_{0,0}$ the result is proved. □

The next result of this appendix shows that we can propagate, due to the smoothing effect of the Bourgain spaces, the compactness of a $\omega \subset \mathbb{T}$ for the entire space $\mathbb{T}$.

Proposition A.2 (Propagation of compactness)**.** *Let* $T > 0$ *and* $0 \leq b' < b \leq 1$ *be given. Suppose that* $(u_n, v_n) \in \mathcal{X}_{0,b}$ *and* $(f_n, g_n) \in \mathcal{X}_{-2+2b,-b}$ *satisfies*

$$\begin{cases} \partial_t u_n + \partial_x^3 u_n + \mu \partial_x u_n + \eta \partial_x v_n = f_n, & \text{on } \mathbb{T} \times (0,T), \\ \partial_t v_n + \alpha \partial_x^3 v_n + \zeta \partial_x v_n + \eta \partial_x u_n \ = g_n, & \text{on } \mathbb{T} \times (0,T), \end{cases}$$

for $n \in \mathbb{N}$*. Assume that there exists a constant* $C > 0$ *such that*

$$\|(u_n, v_n)\|_{\mathcal{X}_{0,b}} \leq C, \quad \forall n \geq 1, \tag{A.2}$$

and that

$$\begin{gathered}\|(u_n, v_n)\|_{\mathcal{X}_{-2+2b,-b}} \longrightarrow 0, \ \text{ as } n \longrightarrow \infty, \qquad \|(f_n, g_n)\|_{\mathcal{X}_{-2+2b,-b}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \\ \|(u_n, v_n)\|_{\mathcal{X}_{-1+2b,-b}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{gathered} \tag{A.3}$$

In addition, assume that for some nonempty open set $\omega \subset \mathbb{T}$ *it holds*

$$(u_n, v_n) \longrightarrow (0,0) \text{ in } L^2(0,T; L^2(\omega)) \times L^2(0,T; L^2(\omega)).$$

Then,

$$(u_n, v_n) \longrightarrow (0,0) \text{ in } L^2_{loc}(0,T; L^2(\mathbb{T})) \times L^2_{loc}(0,T; L^2(\mathbb{T})).$$

Proof. Pick $\phi \in C^\infty(\mathbb{T})$ and $\psi \in C_0^\infty((0,T))$ real valued and set

$$\Phi = \phi(x) D^{-2} \quad \text{and} \quad \Psi = \psi(t) B,$$

where D is defined by

$$\widehat{D^r u}(k) = \begin{cases} |k|^r \, \hat{u}(k) & \text{if } k \neq 0, \\ \hat{u}(0) & \text{if } k = 0. \end{cases} \tag{A.4}$$

Since

$$\int_0^T \int_{\mathbb{T}} \Psi u(x,t) v(x,t) \mathrm{d}x\mathrm{d}t = \int_0^T \int_{\mathbb{T}} u(x,t)\phi(t) D^{-2}(\phi(x)v(x,t))\mathrm{d}x\mathrm{d}t.$$

we have $\Psi^* = \psi(t)D^{-2}\phi(x)$. For any $\epsilon > 0$, let $\Psi_\epsilon = \Phi e^{\epsilon\partial_x^2} = \psi(t)\Phi_\epsilon$ be regularization of Ψ. We define

$$\alpha_\epsilon^1(u_n, v_n) := \langle [\Psi_\epsilon, \mathcal{L}_1]u_n, u_n \rangle_{L^2(\mathbb{T}\times(0,T))}, \qquad \alpha_\epsilon^2(v_n, v_n) := \langle [\Psi_\epsilon, \mathcal{L}_2]v_n, v_n \rangle_{L^2(\mathbb{T}\times(0,T))},$$
$$\alpha_\epsilon^3(u_n, v_n) := \langle [\Psi_\epsilon, \mathcal{L}_3]u_n, v_n \rangle_{L^2(\mathbb{T}\times(0,T))},$$

where

$$\mathcal{L}_1 = \partial_t + \partial_x^3 + \mu\partial_x, \quad \mathcal{L}_2 = \partial_t + \alpha\partial_x^3 + \zeta\partial_x \quad \text{and} \quad \mathcal{L}_3 = \eta\partial_x.$$

Note that

$$[\Psi_\epsilon, \partial_t]w_n(x,t) = -\psi'(t)\phi(x)D^{-2}w_n(x,t).$$

Denoting $\alpha_{n,\epsilon} := \alpha_\epsilon^1(u_n,u_n) + \alpha_\epsilon^3(v_n,u_n) + \alpha_\epsilon^2(v_n,v_n) + \alpha_\epsilon^3(u_n,v_n)$, we have

$$\begin{aligned} \alpha_{n,\epsilon} = &\left\langle [\Psi_\epsilon, \partial_x^3 + \mu\partial_x]u_n, u_n \right\rangle - \langle \psi'(t)\Phi_\epsilon u_n, u_n\rangle + \left\langle [\Psi_\epsilon, \alpha\partial_x^3 + \zeta\partial_x]v_n, v_n \right\rangle \\ &- \langle \psi'(t)\Phi_\epsilon v_n, v_n\rangle + \langle [\Psi_\epsilon, \eta\partial_x]u_n, v_n\rangle + \langle [\Psi_\epsilon, \eta\partial_x]v_n, u_n\rangle . \end{aligned}$$

On the other hand,

$$\begin{aligned} \alpha_\epsilon^1(u_n,u_n) + \alpha_\epsilon^3(v_n,u_n) &= \langle \Psi_\epsilon(\mathcal{L}_1 u_n), u_n)\rangle + \langle \Psi_\epsilon(\mathcal{L}_3 v_n), u_n\rangle - \langle \mathcal{L}_1(\Psi_\epsilon u_n), u_n\rangle - \langle \mathcal{L}_3(\Psi_\epsilon v_n), u_n\rangle \\ &= \langle f_n, \Psi_\epsilon^* u_n\rangle + \langle \Psi_\epsilon u_n, \mathcal{L}_1 u_n\rangle + \langle \Psi_\epsilon v_n, \mathcal{L}_3 u_n\rangle , \end{aligned}$$

since $\mathcal{L}_1 u_n + \mathcal{L}_3 v_n = f_n$, $\mathcal{L}_1^* = -\mathcal{L}_1$ and $\mathcal{L}_3^* = -\mathcal{L}_3$. Similarly

$$\alpha_\epsilon^2(v_n,v_n) + \alpha_\epsilon^3(u_n,v_n) = \langle g_n, \Psi_\epsilon^* v_n\rangle + \langle \Psi_\epsilon v_n, \mathcal{L}_2 v_n\rangle + \langle \Psi_\epsilon u_n, \mathcal{L}_3 v_n\rangle .$$

Thus,

$$\alpha_{n,\epsilon} = \langle f_n, \Psi_\epsilon^* u_n\rangle + \langle g_n, \Psi_\epsilon^* v_n\rangle + \langle \Psi_\epsilon u_n, f_n\rangle + \langle \Psi_\epsilon v_n, g_n\rangle .$$

Now, following the ideas almost as done in Proposition 3.5 of [24] the result is achieved. □

The following result concerns the propagation of regularity. Precisely, the result guarantees that if we have a gain of derivatives in the spatial space in a subset ω of $\mathbb{T}$, then this is also valid in the whole space $\mathbb{T}$.

Proposition A.3 (Propagation of regularity)**.** *Let $T>0$, $0\le b<1$, $r\in\mathbb{R}$ and $(p,q)\in\mathcal{X}_{r,-b}$. Let $(u,v)\in\mathcal{X}_{r,b}$ be a solution of*

$$\begin{cases}\partial_t u+\partial_x^3 u+\mu\partial_x u+\eta\partial_x v=p, & \text{on } \mathbb{T}\times(0,T),\\ \partial_t v+\alpha\partial_x^3 v+\zeta\partial_x v+\eta\partial_x u=q, & \text{on } \mathbb{T}\times(0,T).\end{cases}$$

If there exists a nonempty open set ω of $\mathbb{T}$ such that

$$(u,v)\in L^2_{loc}(0,T;H^{r+\rho}(\omega))\times L^2_{loc}(0,T;H^{r+\rho}(\omega)),$$

for some ρ satisfying

$$0<\rho\le\min\left\{1-b,\frac{1}{2}\right\},$$

then $(u,v)\in L^2_{loc}(0,T;H^{r+\rho}(\mathbb{T}))\times L^2_{loc}(0,T;H^{r+\rho}(\mathbb{T}))$.

Proof. Set $s=r+\rho$ and for $n\in\mathbb{N}$ consider

$$\begin{cases}u_n=e^{\frac{1}{n}\partial_x^2}u=:\Xi_n u, & p_n=\Xi_n p=\mathcal{L}_1u_n+\mathcal{L}_3v_n,\\ v_n=e^{\frac{1}{n}\partial_x^2}v=:\Xi_n v, & q_n=\Xi_n q=\mathcal{L}_2v_n+\mathcal{L}_3u_n.\end{cases}$$

There exists a constant $C>0$ such that

$$\|(u_n,v_n)\|_{\mathcal{X}_{r,b}}\le C \quad \text{and} \quad \|(p_n,q_n)\|_{\mathcal{X}_{r,-b}}\le C, \quad \forall\ n\in\mathbb{N}.$$

Pick $\phi\in C^\infty(\mathbb{T})$ and $\psi\in C_0^\infty((0,T))$ as in the proof of Proposition A.2, and set $\Phi=D^{2s-2}\phi(x)$ and $\Psi=\psi(t)B$, where D is defined by (A.4). We have

$$\begin{aligned}&\langle\mathcal{L}_1u_n\ +\mathcal{L}_3v_n,\Psi^*u_n,\rangle+\langle\Psi u_n,\mathcal{L}_1u_n+\mathcal{L}_3v_n\rangle\\&\quad=\left\langle[\Psi,\partial_x^3+\mu\partial_x]u_n,u_n\right\rangle+\langle[\Psi,\eta\partial_x]v_n,u_n\rangle-\langle\psi'(t)\Phi u_n,u_n\rangle,\end{aligned}$$

and, similarly

$$\begin{aligned}&\langle\mathcal{L}_2v_n\ +\mathcal{L}_3u_n,\Psi^*v_n,\rangle+\langle\Psi v_n,\mathcal{L}_2v_n+\mathcal{L}_3u_n\rangle\\&\quad=\left\langle[\Psi,\alpha\partial_x^3+\zeta\partial_x]v_n,v_n\right\rangle+\langle[\Psi,\eta\partial_x]u_n,v_n\rangle-\langle\psi'(t)\Phi v_n,v_n\rangle,\end{aligned}$$

where $\mathcal{L}_i$, for $i=1,2,3$, were defined on Proposition A.2. With this in hand, the result follows in a similar way as proved in Proposition 3.6 of [24]. □

We are in a position to prove the unique continuation property for our dispersive operator.

Corollary A.4. *Let ω be a nonempty set in $\mathbb{T}$ and*

$$(u,v)\in\mathcal{X}_0^1\cap C([0,T];L_0^2(\mathbb{T}))\times\mathcal{X}_0^\alpha\cap C([0,T];L_0^2(\mathbb{T}))$$

be solution of (3.22)*. Suppose that* (u,v) *satisfies*

$$\begin{cases}\partial_t u+\partial_x^3 u+\mu\partial_x u+\eta\partial_x v+\partial_x P(u,v)=0, & \text{on } \mathbb{T}\times(0,T),\\ \partial_t v+\partial_x^3 v+\zeta\partial_x v+\eta\partial_x u+\partial_x Q(u,v)=0, & \text{on } \mathbb{T}\times(0,T),\\ (u(x,t),v(x,t))=(c_1(t),c_2(t)), & \text{for a.e. } (x,t)\in\omega\times(0,T),\end{cases}$$

where $c_1,\ c_2\in L^2(0,T)\times L^2(0,T)$*. Then* $(u(x,t),v(x,t))\equiv(0,0)$ *for a.e.* $(x,t)\in\mathbb{T}\times(0,T)$*.*

Proof. Since $(u(x,t),v(x,t))=(c_1(t),c_2(t))$ for a.e. $(x,t)\in\omega\times(0,T)$, we have that

$$\partial_t u=c_1'(t)=0 \quad\text{and}\quad \partial_t v=c_2'(t)=0. \tag{A.5}$$

Pick a time $t\in(0,T)$ as above and define $(p,q):=(\partial_x^3 u(\cdot,t),\partial_x^3 v(\cdot,t))$. Thus, it holds that $(p,q)\in H^{-3}(\mathbb{T})\times H^{-3}(\mathbb{T})$ with $(p,q)=(0,0)$. Decompose p and q as

$$p(x)=\sum_{k\in\mathbb{Z}}\widehat{p_k}e^{ikx} \text{ and } q(x)=\sum_{k\in\mathbb{Z}}\widehat{q_k}e^{ikx}.$$

the convergence of the Fourier series being in $H^{-3}(\mathbb{T})$. Since p and q are real-valued functions, we also have that $\widehat{p_{-k}}=\widehat{p_k}$ for all k and the same is true for q. Then,

$$0=p(x)=\sum_{k>0}\widehat{p_k}e^{ikx} \quad\text{and}\quad 0=q(x)=\sum_{k>0}\widehat{q_k}e^{ikx},$$

for each $x\in\omega$. Applying Lemma 2.9 of [26] to p and q we obtain $(p,q)\equiv(0,0)$ on $\mathbb{T}$. It follows, $\partial_x^3 u=\partial_x^3 v=\partial_x u=\partial_x v=0$ on $\mathbb{T}$ for a.e $t\in(0,T)$. Hence,

$$(u(x,t),v(x,t))=(c_1(t),c_2(t)) \text{ for a.e. } (x,t)\in\mathbb{T}\times(0,T).$$

As in (A.5) we deduce that for a.e. $(x,t)\in\mathbb{T}\times(0,T)$ we have $(c_1'(t),c_2'(t))=(0,0)$ for $c_1,\ c_2\in\mathbb{R}$, which, combined with the fact that $[u_0]=[v_0]=0$, gives that $c_1=c_2=0$. The proof of Corollary A.4 is complete. □

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