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CONTROL SETS OF LINEAR CONTROL SYSTEMS ON $\mathbb{R}^2$. THE COMPLEX CASE

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Abstract. This paper explicitly computes the unique control set D with the non-empty interior of a linear control system on $\mathbb{R}^2$, when the associated matrix has complex eigenvalues. It turns out that the closure of D coincides with the region delimited by a computable periodic orbit $\mathcal{O}$ of the system.

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1. Introduction

Due to the intersection with several branches of mathematics, such as Lie groups, homogeneous spaces, and sub-Riemannian geometry [1], [5], [6], [9], [12]; and the number of relevant applications [2], [3], [7], [8], [10], linear and non-linear control systems have been developed for more than 70 years. In the center of all the applications, the classical problem of controllability keeps appearing since the possibility of connecting points without traveling back in time is the most desirable property. Even though controllability is important, it is actually a rare property for a control system to possess.

The second best option is the maximal regions of the state space where this important property holds, the so-called control sets. Not only controllability holds in the interior of such sets, but also contain fixed, periodic, and recurrent points, as also any bounded orbit of the system. Hence, a control set contains many relevant information about the system's dynamics. Despite their importance, control sets are quite difficult to compute explicitly. In some cases, it is possible to obtain their shape through numerical analysis (see [4] for examples and applications).

An important class of control systems for which several properties of control sets are known are the linear control systems (LCS) on Euclidean spaces defined by the family of ODEs

$$\dot{v}(t) = Av(t) + Bu(t), \qquad v(t) \in \mathbb{R}^n, \qquad u(t) \in \Omega, \qquad t \in \mathbb{R}, \tag{$\Sigma_{\mathbb{R}^n}$}$$

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where $\Omega \subset \mathbb{R}^m$ is a bounded subset, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are matrices. In general, it is well known that the Kalman rank condition

$$\operatorname{rank}[A|B] = \operatorname{rank}\left[A^{n-1}B|A^{n-2}B|\cdots|B\right] = n,$$

warrants the existence of a control set D with a non-empty interior of the LCS $\Sigma_{\mathbb{R}^n}$. Also, D is characterized by the positive and negative orbits of the system, which allows for determining some topological properties of D. Explicitly, if $0 \in \operatorname{int}\Omega$, the control set D is the unique control set with a nonempty interior of $\Sigma_{\mathbb{R}^n}$. This control set is bounded if and only if the matrix A has only eigenvalues with nonzero real parts and is closed (open) if and only if A has only eigenvalues with nonnegative (nonpositive) real parts (see, for instance, [4], Chap. 3).

In this article, we explicitly describe the control set D of an LCS in $\mathbb{R}^2$ associated with a matrix with a pair of complex eigenvalues. This task is done by building a periodic orbit $\mathcal{O}$ and proving that the closure of the region delimited by $\mathcal{O}$ coincides with the closure of the only control set with a non-empty interior. Basically, we compute two trajectories forming a periodic orbit of the system, which is the boundary of the control set we are looking for. The main contribution of our approach is that it permits us to recover all the known results about the controllability and control sets properties for this class of systems without the extra assumption that $0 \in \operatorname{int}\Omega$. Moreover, these computations provide an explicit way to obtain the control sets of LCSs under our setup, and they allow us to analyze the asymptotic behavior of such sets under perturbations of the control range and the eigenvalues of the involved matrix.

It is worth mentioning that our results are not limited to dimension two. In fact, for a general LCS on $\mathbb{R}^n$, if the matrix A is semisimple, *i.e.*, its complexification is diagonalizable, the system can be decomposed in blocks of dimensions 1 and 2 determined by the eigenvalues of A. Furthermore, if A has a couple of complex eigenvalues, an invariant plane exists for $\Sigma_{\mathbb{R}^n}$, and the restriction of the system to such a plane is exactly the one considered in the present paper.

The paper is divided as follows: After some preliminaries, Section 2 focuses on constructing the control sets of $\Sigma_{\mathbb{R}^2}$. If $\operatorname{tr} A = 0$, the system is controllable, and we show, for completeness' sake, trajectories connecting two arbitrary points of $\mathbb{R}^2$. If $\operatorname{tr} A \neq 0$, the system admits a control set with a non-empty interior. As commented previously, the most crucial issue with computing the control set D explicitly is to calculate a periodic orbit $\mathcal{O}$, which is obtained asymptotically by considering a solution whose control function switches between the boundary points in Ω. With that, it holds that $\operatorname{tr} A < 0$ and D is closed, and its boundary is $\mathcal{O}$, or $\operatorname{tr} A > 0$ and D is open and coincides with the region bounded by $\mathcal{O}$. Moreover, when $\operatorname{tr} A > 0$, the orbit $\mathcal{O}$ is also a control set of $\Sigma_{\mathbb{R}^2}$ with an empty interior. Section 3 concludes the analysis of the possible control sets of the LCS $\Sigma_{\mathbb{R}^2}$ by showing that such systems do not admit any other control sets different from the one obtained in the previous section. In Section 4, we make some remarks concerning the asymptotic analysis through the parameters determining the system's dynamics, *i.e.*, the eigenvalues and the range size determined by the controls u^- and u^+. In particular, controllability properties are recovered in some cases. We also mention that our method does not consider 0 in the range's interior as usual. The paper concludes with an appendix concerning some properties of spirals in $\mathbb{R}^2$. The results in this appendix are related to the geometrical approach we used to prove the results in the paper.

Notations: For any vector $v \in \mathbb{R}^2$ we denote by $\mathbb{R}\cdot v$ the line passing by the origin and parallel to v. We consider the natural order on $\mathbb{R}\cdot v$ as

$$v_1, v_2 \in \mathbb{R}\cdot v, \quad v_1 \leq v_2 \quad \iff \quad v_1 - v_2 = \alpha v, \;\; \alpha > 0.$$

For any $\tau \in \mathbb{R}$, we denote by R_τ the rotation of τ-degrees which is clockwise if $\tau < 0$ and counter-clockwise if $\tau > 0$. In particular, we define $\theta := R_{\pi/2}$.

2. Linear control systems on $\mathbb{R}^2$

A linear control system (LCS) on $\mathbb{R}^2$ is given by the family of ODEs

$$\dot{v}(t) = Av + u(t)\eta, \quad u(t) \in \Omega, \quad t \in \mathbb{R}, \tag{$\Sigma_{\mathbb{R}^2}$}$$

where $\Omega := [u^-, u^+]$ with $u^- < u^+$ and $\eta \in \mathbb{R}^2$ is a nonzero vector.

The set Ω is called the *control range* of the system $\Sigma_{\mathbb{R}^2}$. The family of the *control functions* $\mathcal{U}$ is, by definition, the set of all *piecewise constant functions* with image in Ω. The *solution* of $\Sigma_{\mathbb{R}^2}$ starting at $v \in \mathbb{R}^2$ and associated control $\mathbf{u} \in \mathcal{U}$ is the unique piecewise differentiable curve $s \in \mathbb{R} \mapsto \varphi(s, v, \mathbf{u})$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}s}\varphi(s, v, \mathbf{u}) = A\varphi(s, v, \mathbf{u}) + \mathbf{u}(s)\eta.$$

It is not hard to see that the solutions of $\Sigma_{\mathbb{R}^2}$ are given by concatenations of the curves associated with constant control functions.

For any $v \in \mathbb{R}^2$, the *positive* and the *negative orbits* of $\Sigma_{\mathbb{R}^2}$ are given, respectively, by the sets

$$\mathcal{O}^+(v) := \{\varphi(s, v, \mathbf{u}),\ s \geq 0, \mathbf{u} \in \mathcal{U}\} \qquad \text{and} \qquad \mathcal{O}^-(v) := \{\varphi(s, v, \mathbf{u}),\ s \leq 0, \mathbf{u} \in \mathcal{U}\}.$$

Definition 2.1. A *control set* of $\Sigma_{\mathbb{R}^2}$ is a subset D of $\mathbb{R}^2$ satisfying

(a) For any $v \in D$ there exists $\mathbf{u} \in \mathcal{U}$ such that $\varphi(\mathbb{R}^+, v, \mathbf{u}) \subset D$;
(b) For any $v \in D$ it holds that $D \subset \overline{\mathcal{O}^+(v)}$;
(c) D is maximal w.r.t. set inclusion satisfying (a) and (b).

If a control set D of $\Sigma_{\mathbb{R}^2}$ satisfies $D = \mathbb{R}^2$ we say the $\Sigma_{\mathbb{R}^2}$ is controllable.

For the matrix $A \in \mathfrak{gl}(2, \mathbb{R})$ let us denote by σ_A the number

$$\sigma_A := (\operatorname{tr} A)^2 - 4 \det A.$$

The number σ_A is related to the eigenvalues of A, and it is straightforward to see that A has a pair of complex eigenvalues if and only if $\sigma_A < 0$. Therefore, from here on we assume that the matrix $A \in \mathfrak{gl}(\mathbb{R}, 2)$ which defines our control system $\Sigma_{\mathbb{R}^2}$ satisfies $\sigma_A < 0$, and fix an orthonormal basis of $\mathbb{R}^2$ such that

$$A = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}.$$

In particular, the system $\Sigma_{\mathbb{R}^2}$ satisfies the Kalman rank condition if and only if $\eta \neq 0$. Moreover, since $\det A \neq 0$ it holds that

$$\varphi(s, v, u) = \mathrm{e}^{sA}(v - v(u)) + v(u), \quad \text{where} \quad v(u) := -uA^{-1}\eta,$$

are the equilibria of the system and $u \in \Omega$. In particular, the solutions of $\Sigma_{\mathbb{R}^2}$ for constant control functions coincide with the spirals $\varphi_A(s, v, v(u))$ if $\operatorname{tr} A \neq 0$ (see Appendix A) and lie on circumferences if $\operatorname{tr} A = 0$.

In what follows, we analyze the dynamics of the solutions of $\Sigma_{\mathbb{R}^2}$ to obtain a full characterization of the control sets of the system. Moreover, all the results that follow do not need the assumption that $0 \in \operatorname{int} \Omega$.

2.1. The control set with nonempty interior

In this section, we construct explicitly the control set of $\Sigma_{\mathbb{R}^2}$ with a non-empty interior by considering the possibilities for the trace of the matrix A.

2.1.1. *The case* $\operatorname{tr} A = 0$

In this case, the solutions of $\Sigma_{\mathbb{R}^2}$ for constant controls have the form

$$\varphi(s, v, u) = R_{s\mu}(v - v(u)) + v(u),$$

and they lie on the circumferences $C_{u,v}$ with center $v(u)$ and radius $|v - v(u)|$.

Theorem 2.2. *If the associated matrix A of $\Sigma_{\mathbb{R}^2}$ is such that $\operatorname{tr} A = 0$ and $\det A > 0$, then $\Sigma_{\mathbb{R}^2}$ is controllable.*

Proof. In order to show the result, it is enough to construct a periodic orbit between an arbitrary point $v \in \mathbb{R}^2$ and some fixed $v(u_0) \in v(\Omega)$, which we do as follows:

(a) $v(\Omega) = v([u^-, u^+])$ is a compact interval on the line $\mathbb{R} \cdot \theta\eta$;
(b) The circumference $C_{u^+,v}$ intersects the line $\mathbb{R} \cdot \theta\eta$ in two points. Denote by v_1 the point in this intersection close to $v(u^-)$. In particular, $v_1 = \varphi(s_1, v, u^+)$ for some $s_1 > 0$;
(c) If $v_1 \notin v(\Omega)$, we repeat the process in the previous item for the circumference C_{u^-,v_1}, obtaining a point v_2.
(d) Repeating the previous process, if $v_n \notin v(\Omega)$, we obtain in the same way, a point v_{n+1} belonging to the intersection of the circumference $C_{\bar{u},v_n}$, and the line $\mathbb{R} \cdot \theta\eta$, where $\bar{u} = u^+$ if n is even and $\bar{u} = u^-$ if n is odd. By induction, we quickly see that the radius R_n of $C_{\bar{u},v_n}$ satisfies

$$R_n = |v_n - v(\bar{u})| = |v - v(u^-)| - n|v(u^+) - v(u^-)|.$$

Therefore, there exists $N \in \mathbb{N}$ such that $v_N \in v(\Omega)$.
(e) Now, since $v_N \in v(\Omega)$ there exists, by continuity, $u_N \in \Omega$ satisfying $|v(u_N) - v(u_0)| = |v_N - v(u_N)|$. The circumference C_{u_N,v_N} passes through v_N and by the point $v(u_0)$. Therefore, there exists $s_N > 0$ such that $\varphi(s_N, v_N, u_N) = v(u_0)$ and by concatenation we get a trajectory from v to $v(u_0)$ (blue path in Fig. 1).
(f) By choosing the complementary path (red path in Fig. 1) on the circumferences constructed on the previous items, we obtain a trajectory from $v(u_0)$ to v, which gives us a periodic orbit as desired (Fig. 1).

□

2.1.2. *The case* $\operatorname{tr} A \neq 0$

Next, we construct a periodic orbit for $\Sigma_{\mathbb{R}^2}$ (see Fig. 2 The main result in this section will show that such orbit is the boundary of the unique control set of $\Sigma_{\mathbb{R}^2}$.

As commented in Appendix A, we will assume w.l.o.g. that the eigenvalues of A are $\lambda \pm \mu i$ with $\lambda < 0$ and $\mu > 0$. Define recurrently

$$P_0 = v(u^+), \quad P_{2n+1} := \varphi\left(\frac{\pi}{\mu}, P_{2n}, u^-\right) \quad \text{and} \quad P_{2n+2} := \varphi\left(\frac{\pi}{\mu}, P_{2n+1}, u^+\right), \quad n \geq 0.$$

A simple inductive process allows us to obtain

$$P_{2n} = -e^{\pi\frac{\lambda}{\mu}} \left[\sum_{j=0}^{2n-1} e^{j\pi\frac{\lambda}{\mu}}\right] v(u^-) + \left[\sum_{j=0}^{2n} e^{j\pi\frac{\lambda}{\mu}}\right] v(u^+), \qquad n \geq 1$$

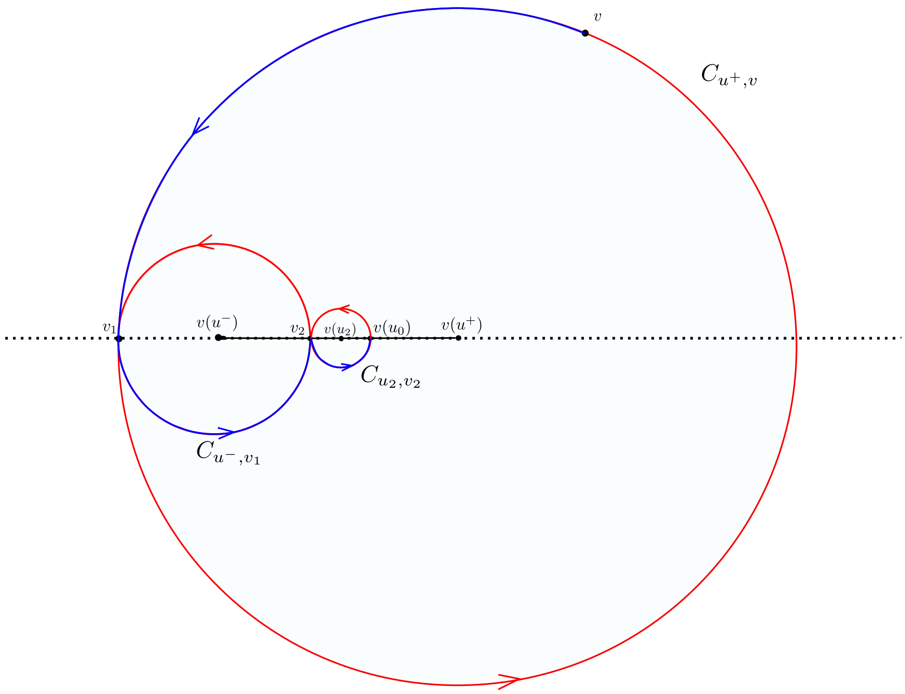

Figure 1. Periodic Orbit through $v(u_0)$ and v.

and

$$P_{2n+1} = \left[\sum_{j=0}^{2n-1} \mathrm{e}^{j\pi\frac{\lambda}{\mu}}\right] v(u^-) - \mathrm{e}^{\pi\frac{\lambda}{\mu}} \left[\sum_{j=0}^{2n} \mathrm{e}^{j\pi\frac{\lambda}{\mu}}\right] v(u^+), \qquad n \geq 0.$$

On the other hand,

$$\frac{\lambda}{\mu} < 0 \quad \Longrightarrow \quad \mathrm{e}^{2\pi\frac{\lambda}{\mu}} < 1 \quad \Longrightarrow \quad \sum_{j=0}^{m} \mathrm{e}^{j\pi\frac{\lambda}{\mu}} = \sum_{j=0}^{m} \left(\mathrm{e}^{\pi\frac{\lambda}{\mu}}\right)^j \to \frac{1}{1-\mathrm{e}^{\pi\frac{\lambda}{\mu}}} \quad \text{as} \quad m \to +\infty.$$

Consequently,

$$P_{2n} \to P^+ := \left(\frac{-u^+ + \mathrm{e}^{\pi\frac{\lambda}{\mu}}u^-}{1-\mathrm{e}^{\pi\frac{\lambda}{\mu}}}\right) A^{-1}\eta \quad \text{and} \quad P_{2n+1} \to P^- := \left(\frac{-u^- + \mathrm{e}^{\pi\frac{\lambda}{\mu}}u^+}{1-\mathrm{e}^{\pi\frac{\lambda}{\mu}}}\right) A^{-1}\eta. \tag{2.1}$$

Note that both of the points P^-, P^+ belong to the line $\mathbb{R}A^{-1}\eta$ and satisfy

$$P^+ - v(u^+) = \frac{(u^+ - u^-)\mathrm{e}^{\pi\frac{\lambda}{\mu}}}{u^+(1-\mathrm{e}^{\pi\frac{\lambda}{\mu}})} v(u^+) \quad \text{and} \quad P^- - v(u^-) = -\frac{(u^+ - u^-)\mathrm{e}^{\pi\frac{\lambda}{\mu}}}{u^-(1-\mathrm{e}^{\pi\frac{\lambda}{\mu}})} v(u^-),$$

implying that

$$P^- < v(u^-) < v(u^+) < P^+ \quad \text{on the line} \quad \mathbb{R}\cdot(-A^{-1}\eta).$$

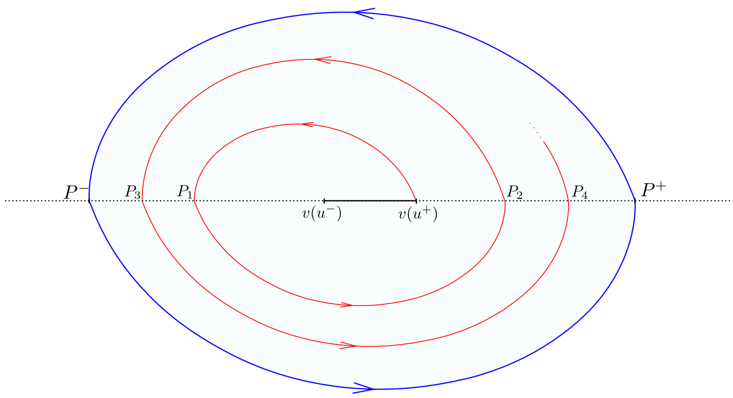

FIGURE 2. Periodic Orbit.

Moreover, it holds that

$$\varphi\left(\frac{\pi}{\mu}, P^+, u_-\right) = -\mathrm{e}^{\pi\frac{\lambda}{\mu}}P^+ + (1+\mathrm{e}^{\mu\frac{\lambda}{\mu}})v(u^-) = \left[-\mathrm{e}^{\pi\frac{\lambda}{\mu}}\left(\frac{-u^+ + \mathrm{e}^{\pi\frac{\lambda}{\mu}}u^-}{1-\mathrm{e}^{\pi\frac{\lambda}{\mu}}}\right) - (1+\mathrm{e}^{\mu\frac{\lambda}{\mu}})u^-\right]A^{-1}\eta$$

$$\left(\frac{\mathrm{e}^{\pi\frac{\lambda}{\mu}}u^+ - \mathrm{e}^{2\pi\frac{\lambda}{\mu}}u^- - (1-\mathrm{e}^{2\mu\frac{\lambda}{\mu}})u^-}{1-\mathrm{e}^{\pi\frac{\lambda}{\mu}}}\right)A^{-1}\eta = \left(\frac{-u^- + \mathrm{e}^{\pi\frac{\lambda}{\mu}}u^+}{1-\mathrm{e}^{\pi\frac{\lambda}{\mu}}}\right)A^{-1}\eta = P^-,$$

and analogously,

$$\varphi\left(\frac{\pi}{\mu}, P^-, u^+\right) = P^+,$$

showing the following:

Proposition 2.3. *The subset of* $\mathbb{R}^2$ *given by*

$$\mathcal{O} := \left\{\varphi(s, P^+, u^-), s \in \left[0, \frac{\pi}{\mu}\right]\right\} \cup \left\{\varphi(s, P^-, u^+), s \in \left[0, \frac{\pi}{\mu}\right]\right\},$$

is a periodic orbit of $\Sigma_{\mathbb{R}^2}$*.*

Let us denote by $\mathcal{C}$ the closure of the region delimited by the periodic orbit $\mathcal{O}$. The next result shows that $\mathcal{C}$, or its interior, is a control set of the system.

Theorem 2.4. *For the LCS* $\Sigma_{\mathbb{R}^2}$ *with* $\sigma_A < 0$ *and* $\operatorname{tr} A \neq 0$ *there holds that*

1. $\operatorname{tr} A < 0$ *and* $D = \mathcal{C}$ *is a control set;*
2. $\operatorname{tr} A > 0$ *and* $D = \operatorname{int} \mathcal{C}$ *is a control set.*

Proof. Let us start by showing, in the next steps, that

$$\forall v \in \operatorname{int} \mathcal{C}, \quad \operatorname{int} \mathcal{C} = \mathcal{O}^+(v) \quad \text{if} \quad \lambda < 0 \quad \text{and} \quad \operatorname{int} \mathcal{C} = \mathcal{O}^-(v) \quad \text{if} \quad \lambda > 0.$$

Since both cases are analogous, we will assume w.l.o.g. that $\lambda < 0$ and $\mu > 0$.

Step 1: $\mathcal{C}$ is positively invariant;
For any $u \in \Omega$, it turns out

$$\varphi(s, v, u) = \varphi_A(s, v, v(u)).$$

As a consequence, the region $\mathcal{C}$ can be decomposed in two regions

$$\mathcal{C}_A(P^+, v(u^-)) \quad \text{and} \quad \mathcal{C}_A(P^-, v(u^+)),$$

which are delimited, respectively, by the line passing through $v(u^+)$ and $v(u^-)$ and the curves

$$\left\{\varphi(s, P^+, u^-), s \in \left[0, \frac{\pi}{\mu}\right]\right\} \qquad \text{and} \qquad \left\{\varphi(s, P^-, u^+), s \in \left[0, \frac{\pi}{\mu}\right]\right\},$$

where P^+ and P^- are defined in equation (2.1).
Moreover, on the line $\mathbb{R} \cdot (-A^{-1}\eta)$, for any $u \in \Omega$ and $w \in \mathcal{C}_A(P^+, v(u^-))$, it holds that $v(u) \in [v(u^-), P^+]$. Therefore, by Proposition A.3 it holds that

$$\varphi(s, w, u) = \varphi_A(s, w, v(u)) \in \mathcal{C}_A(P^+, v(u^-)),$$

for any $s \in \left[0, \frac{\pi-\sigma}{\mu}\right]$. Here, σ is the angle between $v(u^+) - v(u^-)$ and $w - v(u)$. In particular,

$$P_1 := \varphi\left(\frac{\pi - \sigma}{\mu}, w, u\right) \in [P^-, P^+] \subset \mathcal{C}_A(P^-, v(u^+)).$$

Since $P_1 \in \mathcal{C}_A(P^-, v(u^+))$ and $v(u) \in [v(u^+), P^-]$, Proposition A.3 implies that

$$\varphi(s, P_1, u) = \varphi_A(s, P_1, v(u)) \in \mathcal{C}_A(P^-, v(u^+)),$$

for any $s \in \left[0, \frac{\pi}{\mu}\right]$. Again,

$$P_2 := \varphi\left(\frac{\pi}{\mu}, P_1, u\right) \in [P^-, P^+] \subset \mathcal{C}_A(P^+, v(u^-)).$$

Since we can repeat the process, we already prove the invariance of $\mathcal{C}$ in positive time.

Step 2: Controllability holds on $\operatorname{int}\mathcal{C}$;
The result certainly follows if we show the relationships

$$\forall v \in \operatorname{int}\mathcal{C}, \quad v \in \mathcal{O}^+(v(u^-)) \quad \text{and} \quad v(u^-) \in \mathcal{O}^+(v).$$

The assumption $\lambda < 0$ implies that

$$\forall u \in \Omega, \quad |\varphi(s, v, u)| \to +\infty \quad \text{as} \quad s \to -\infty.$$

Consequently, the compactness of $\mathcal{C}$ shows the existence of $s_0 > 0$ such that $\varphi(-s_0, v, u) \in \partial\mathcal{C} = \mathcal{O}$. Moreover, there exists $n \in \mathbb{N}$ and $t_0 > 0$ such that

$$\varphi(-s_0, v, u) = \varphi(t_0, P_{2n}, u^-) \qquad \text{or} \qquad \varphi(-s_0, v, u) = \varphi(t_0, P_{2n+1}, u^+).$$

By construction, the points $P_m, m \in \mathbb{N}$ are attained from $v(u^-)$ in positive time. Therefore, the previous arguments show that v is attained from $v(u^-)$, or equivalently $v \in \mathcal{O}^+(v(u^-))$. Furthermore, $s \mapsto \varphi(s, v, u^-)$ is a curve that revolves around $v(u^-)$ and $s \mapsto \varphi(s, v(u^-), u)$ revolves around $v(u)$. Then, for any $u \neq u^-$, there exist $s_0, t_0 > 0$ such that

$$\varphi(s_0, v, u^-) = \varphi(-t_0, v(u^-), u) \quad \Longrightarrow \quad v(u^-) \in \mathcal{O}^+(v),$$

proving the claim.

Step 3: It holds that

$$\forall v \in \operatorname{int} \mathcal{C}, \quad \operatorname{int} \mathcal{C} = \mathcal{O}^+(v).$$

Since, for any $s \in \mathbb{R}$ and $u \in \Omega$, the map

$$v \in \mathbb{R}^2 \mapsto \varphi(s, v, u) \in \mathbb{R}^2,$$

is a diffeomorphism. Step 1 implies that

$$\varphi(s, \operatorname{int} \mathcal{C}, u) \subset \operatorname{int} \mathcal{C}, \quad \forall s > 0, u \in \Omega.$$

As a consequence,

$$\forall v \in \operatorname{int} \mathcal{C}, \quad \mathcal{O}^+(v) \subset \operatorname{int} \mathcal{C}.$$

On the other hand, by Step 2, controllability holds inside $\operatorname{int} \mathcal{C}$. Consequently, for any $v, w \in \operatorname{int} \mathcal{C}$ we obtain

$$w \in \mathcal{O}^+(v) \quad \Longrightarrow \quad \operatorname{int} \mathcal{C} \subset \mathcal{O}^+(v),$$

showing the desired.

By the previous, it is straightforward to see that $\operatorname{int} \mathcal{C}$ satisfies conditions (a) and (b) of Definition 2.1. Therefore, there exists a control set D such that $\operatorname{int} \mathcal{C} \subset D$ and we have that:

1. If $\lambda < 0$, the positively invariance on item (a) implies that $\mathcal{O}^+(v) \subset \mathcal{C}$ for all $v \in \mathcal{C}$. Since $v \in D$, condition (b) in Definition 2.1 implies that $D \subset \overline{\mathcal{O}^+(v)}$ and hence

$$\overline{\mathcal{O}^+(v)} \subset \overline{\mathcal{C}} = \mathcal{C} \subset D \subset \overline{\mathcal{O}^+(v)},$$

showing that $D = \mathcal{C}$ is in fact the control set of $\Sigma_{\mathbb{R}^2}$.

2. If $\lambda > 0$ let $v \in \mathbb{R}^2$ and assume that

$$\overline{\mathcal{O}^+(v)} \cap \operatorname{int} \mathcal{C} \neq \emptyset.$$

In particular, there exists $s > 0$, $\mathbf{u} \in \mathcal{U}$ such that

$$\varphi(s, v, \mathbf{u}) \in \operatorname{int} \mathcal{C} \quad \Longrightarrow \quad v \in \varphi(-s, \operatorname{int} \mathcal{C}, \mathbf{u}') \subset \operatorname{int} \mathcal{C},$$

implying the maximality of $\operatorname{int} \mathcal{C}$ and hence $D = \operatorname{int} \mathcal{C}$, concluding the proof.

□

Remark 2.5. The previous result implies that if $\operatorname{tr} A \neq 0$, the LCS admits a bounded control set with nonempty interior, which is closed if $\operatorname{tr} A < 0$ and open when $\operatorname{tr} A > 0$. Moreover, from Step 1 in the proof of Theorem 2.4, it holds that

$$\forall v \in \mathcal{C}, \mathbf{u} \in \mathcal{U} \quad \varphi(s, v, \mathbf{u}) \in \mathcal{C} \quad \text{if} \quad s \cdot \operatorname{tr} A < 0 \tag{2.2}$$

3. The possible control sets of a LCS

As is well stated in the literature, if $0 \in \operatorname{int} \Omega$, the control set D previously obtained is the only control set of $\Sigma_{\mathbb{R}^2}$ with a non-empty interior. This section shows that D is in fact the only control set with a non-empty interior, even without the condition $0 \in \operatorname{int} \Omega$. Moreover, if the trace of the associated matrix A is positive, the periodic orbit $\mathcal{O} = \partial D$ is also a control set of $\Sigma_{\mathbb{R}^2}$ and those are, in fact, the only possible control sets of a LCS whose associated matrix has a pair of complex eigenvalues.

To show the previous claim, the following statement will be crucial.

Lemma 3.1. *For any $v \in \mathbb{R}^2 \setminus \mathcal{C}$, $u \in \Omega$ it holds that:*

(a) $|\varphi(s, v, u) - \mathcal{C}| \leq e^{s\lambda} |v - \mathcal{C}|$ *if* $s\lambda < 0$*;*
(b) $|\varphi(s, v, u) - \mathcal{C}| \geq e^{s\lambda} |v - \mathcal{C}|$ *if* $s\lambda > 0$*, where* $2\lambda = \operatorname{tr} A$.

Proof. (a) Since $\mathcal{C}$ is compact, for any $v \in \mathbb{R}^2$ there exists $v_0 \in \mathcal{C}$ such that $|v - \mathcal{C}| = |v - v_0|$. By equation (2.2), it holds that

$$\lambda s < 0 \quad \Longrightarrow \quad \varphi(s, v_0, u) \in \mathcal{C}.$$

Consequently,

$$|\varphi(s, v, u) - \mathcal{C}| \leq |\varphi(s, v, u) - \varphi(s, v_0, u)| = e^{s\lambda} |v - v_0| = e^{s\lambda} |v - \mathcal{C}|,$$

showing the assertion.
(b) Let us assume the existence of $v_0 \in \mathbb{R}^2 \setminus \mathcal{C}$, $u_0 \in \Omega$ and $s_0 \in \mathbb{R}$ such that

$$\lambda s_0 > 0 \quad \text{and} \quad |\varphi(s_0, v_0, u_0) - \mathcal{C}| < e^{s_0\lambda} |v_0 - \mathcal{C}|.$$

Since,

$$w_0 = \varphi(s_0, v_0, u_0) \quad \Longleftrightarrow \quad v_0 = \varphi(-s_0, w_0, u_0),$$

we have that

$$w_0 \in \mathcal{C} \quad \text{and} \quad -\lambda s_0 < 0 \quad \overset{(2.2)}{\Longrightarrow} \quad v_0 = \varphi(-s_0, w_0, u_0) \in \mathcal{C},$$

which cannot happen. Therefore, $w_0 \in \mathbb{R}^2 \setminus \mathcal{C}$ and by item (a) we obtain

$$|\varphi(s_0, v_0, u_0) - \mathcal{C}| < e^{s_0\lambda} |v_0 - \mathcal{C}| = e^{s_0\lambda} |\varphi(-s_0, w_0, u_0) - \mathcal{C}| \overset{(a)}{\leq} e^{s_0\lambda} e^{-s_0\lambda} |w_0 - \mathcal{C}| = |\varphi(s_0, v_0, u_0) - \mathcal{C}|,$$

which is absurd. Therefore, item (b) holds.

□

We can now prove the main result concerning the control sets of an LCS on $\mathbb{R}^2$.

Theorem 3.2. *Let $\Sigma_{\mathbb{R}^2}$ be a LCS satisfying $\sigma_A < 0$ and $\operatorname{tr} A \neq 0$. It holds:*

1. *If* $\operatorname{tr} A < 0$ *the only control set of* $\Sigma_{\mathbb{R}^2}$ *is* D*;*
2. *If* $\operatorname{tr} A > 0$ *then* $\operatorname{int} D$ *and* ∂D *are the only control sets of* $\Sigma_{\mathbb{R}^2}$*.*

Proof. Since $\partial D = \mathcal{O}$ is a periodic orbit, it satisfies conditions (a) and (b) of Definition 2.1 and is therefore contained in a control set of $\Sigma_{\mathbb{R}^2}$. By Theorem 2.4, we know that $D = \mathcal{C}$ is a control set if $\operatorname{tr} A < 0$ and $D = \operatorname{int} \mathcal{C}$ is a control set if $\operatorname{tr} A > 0$. Therefore, the result follows if we show that no control set of $\Sigma_{\mathbb{R}^2}$ intersects $\mathbb{R}^2 \setminus \mathcal{C}$.

Since the solutions of $\Sigma_{\mathbb{R}^2}$ are given by concatenations of the solutions for constant controls, it is not hard to show by induction that for all $\mathbf{u} \in \mathcal{U}$ and $v \in \mathbb{R}^2 \setminus \mathcal{C}$,

$$|\varphi(s, v, \mathbf{u}) - \mathcal{C}| \leq e^{s\lambda}|v - \mathcal{C}| \quad \text{if} \quad \lambda s < 0,$$

and

$$|\varphi(s, v, \mathbf{u}) - \mathcal{C}| \geq e^{s\lambda}|v - \mathcal{C}| \quad \text{if} \quad \lambda s > 0.$$

In particular, if $|v - \mathcal{C}| = \epsilon > 0$ we have that

$$\mathcal{O}^+(v) \subset N_\epsilon(\mathcal{C}) \quad \text{if} \quad \lambda < 0 \quad \text{and} \quad \mathcal{O}^+(v) \subset \mathbb{R}^2 \setminus N_\epsilon(\mathcal{C}) \quad \text{if} \quad \lambda > 0$$

Let us assume that $\Sigma_{\mathbb{R}^2}$ admits a second control set D' satisfying $|v - \mathcal{C}| = \epsilon > 0$ for some $v \in D'$.

By condition (a) in Definition 2.1, there exists $\mathbf{u} \in \mathcal{U}$ such that $\varphi(s, v, \mathbf{u}) \in D'$ for all $s > 0$. If $v \notin \mathcal{C}$, we have by invariance (see Eq. (2.2)), that $\varphi(s, v, \mathbf{u}) \notin \mathcal{C}$ for all $s > 0$. Moreover, by condition (b) in Definition 2.1 and the previous calculations, it holds that $D' \subset \overline{\mathcal{O}^+(\varphi(s, v, \mathbf{u}))}$ and by the previous

$$D' \subset N_{e^{s\lambda}\epsilon}(\mathcal{C}) \quad \text{if} \quad \lambda < 0 \quad \text{and} \quad D' \subset \mathbb{R}^2 \setminus N_\epsilon(\mathcal{C}) \quad \text{if} \quad \lambda > 0.$$

Consequently, for all $s > 0$

$$|v - \mathcal{C}| \leq e^{s\lambda}|v - \mathcal{C}| \quad \text{if} \quad \lambda < 0 \quad \text{and} \quad |v - \mathcal{C}| \geq e^{s\lambda}|v - \mathcal{C}| \quad \text{if} \quad \lambda > 0,$$

which is not possible if $|v - \mathcal{C}| \neq 0$.

Therefore, any control set D' of $\Sigma_{\mathbb{R}^2}$ satisfies $D' \subset \mathcal{C}$ concluding the proof.

□

4. Remarks on continuity and asymptotic behavior of control sets

Constructing the periodic orbit $\mathcal{O}$ allows us to analyze the asymptotic behavior of the control set D as the control range grows.

Fix a real number $\nu \in \mathbb{R}$ and define $\Omega_{\alpha,\rho} = [\alpha, \rho]$ where $\alpha < \nu < \rho$. Define the LCS

$$\dot{v} = Av + u\eta, \quad u \in \Omega_{\alpha,\rho}, \tag{$\Sigma_{\mathbb{R}^2}^{\alpha,\rho}$}$$

where $\eta \neq 0$ and the matrix A satisfies $\sigma_A < 0$ and $\operatorname{tr} A < 0$. By Theorem 2.4 the LCS $\Sigma_{\mathbb{R}^2}^{\alpha,\rho}$ admits a unique control set with nonempty interior $D^{\alpha,\rho}$ whose boundary is the periodic orbit

$$\mathcal{O}^{\alpha,\rho} := \left\{\varphi(s, P^+_{\alpha,\rho}, \alpha), s \in \left[0, \frac{\pi}{\mu}\right]\right\} \cup \left\{\varphi(s, P^-_{\alpha,\rho}, \rho), s \in \left[0, \frac{\pi}{\mu}\right]\right\},$$

with

$$P^+_{\alpha,\rho} := \left(\frac{-\rho + \mathrm{e}^{\pi\frac{\lambda}{\mu}}\alpha}{1-\mathrm{e}^{\pi\frac{\lambda}{\mu}}}\right) A^{-1}\eta \quad \text{and} \quad P^-_{\alpha,\rho} := \left(\frac{-\alpha + \mathrm{e}^{\pi\frac{\lambda}{\mu}}\rho}{1-\mathrm{e}^{\pi\frac{\lambda}{\mu}}}\right) A^{-1}\eta.$$

The maps

$$(\alpha,\rho) \mapsto P^+_{\alpha,\rho} \quad \text{and} \quad (\alpha,\rho) \mapsto P^-_{\alpha,\rho},$$

are continuous, and it holds that

$$\alpha \to -\infty \quad \text{or} \quad \rho \to +\infty \quad \Longrightarrow \quad P^+_{\alpha,\rho} \to +\infty \quad \text{and} \quad P^-_{\alpha,\rho} \to -\infty,$$

on the line $\mathbb{R}\cdot(-A^{-1}\eta)$. Therefore, one easily shows:

Proposition 4.1. *Any LCS on $\mathbb{R}^2$ whose control range Ω is unbounded is controllable, if the associated matrix A satisfies $\sigma_A < 0$.*

Remark 4.2. To obtain controllability, the previous result only requires that Ω is unbounded and not necessarily the whole real line (see [11]).

Also, by using the fact that

$$(s,\alpha,\rho) \mapsto \varphi(s, P^+_{\alpha,\rho}, \alpha) \quad \text{and} \quad (s,\alpha,\rho) \mapsto \varphi(s, P^-_{\alpha,\rho}, \rho),$$

are continuous maps, one can easily show that the map

$$(\alpha,\rho) \in (-\infty,\nu)\times(\nu,+\infty) \mapsto \overline{D_{\alpha,\rho}},$$

is continuous in the Hausdorff measure.

Appendix A. Geometric properties of spirals in $\mathbb{R}^2$

This section analyzes the dynamics of spirals in the Euclidean space $\mathbb{R}^2$. In particular, we show that spirals with the center in the same line have a particular kind of invariance.

Let $A \in \mathfrak{gl}(2,\mathbb{R})$ and denote by σ_A the number

$$\sigma_A := (\operatorname{tr} A)^2 - 4\det A.$$

The number σ_A is related to the eigenvalues of A, and it is straightforward to see that A has a pair of complex eigenvalues if and only if $\sigma_A < 0$.

Definition A.1. For any $A \in \mathfrak{gl}(2,\mathbb{R})$ with $\sigma_A < 0$ we define the spiral φ_A to be the function

$$(\tau, v_1, v_2) \in \mathbb{R}\times\left\{(\mathbb{R}^2\times\mathbb{R}^2)\setminus\Delta\right\} \mapsto \varphi_A(\tau,v_1,v_2) := \mathrm{e}^{\tau A}(v_1 - v_2) + v_2,$$

where $\Delta \subset \mathbb{R}^2\times\mathbb{R}^2$ is the diagonal.

Since $\sigma_A < 0$, there exists an orthonormal basis of $\mathbb{R}^2$ such that

$$A = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}, \quad \text{where} \quad 2\lambda = \operatorname{tr} A \quad \text{and} \quad \mu^2 = |\sigma_A|.$$

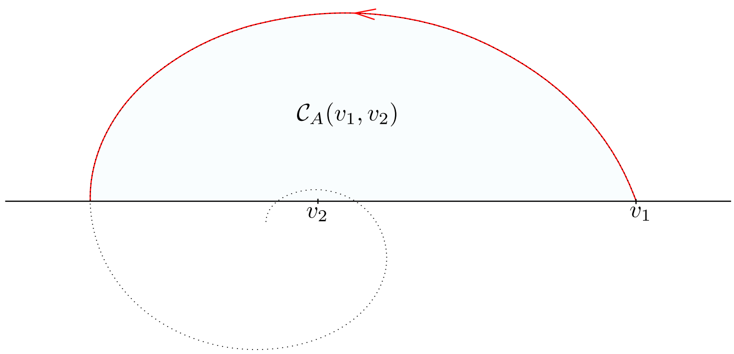

FIGURE A.1. The region $\mathcal{C}_A(v_1, v_2)$.

Consequently, the spiral φ_A can be written on such basis, as

$$\varphi_A(\tau, v_1, v_2) = \mathrm{e}^{\tau\lambda} R_{\tau\mu}(v_1 - v_2) + v_2,$$

where $R_{\mu\tau}$ is the rotation of $\mu\tau$-degrees with relation to the previous basis, which is clockwise if $\mu\tau < 0$ and counter-clockwise if $\mu\tau > 0$.

The spiral φ_A intersects the line passing by v_1 and v_2 for any $\tau \in k\frac{\pi}{\mu}\mathbb{Z}$. Moreover,

$$|\varphi_A(\tau, v_1, v_2) - v_2| = \mathrm{e}^{\tau\lambda}|v_1 - v_2|,$$

showing that $\varphi_A(\tau, v_1, v_2)$ belongs to the circumference with center v_2 and radius $\mathrm{e}^{\tau\lambda}|v_1 - v_2|$. In particular,

$$\varphi_A(\tau, v_1, v_2) \to v_2 \quad \text{when} \quad \tau\lambda \to -\infty.$$

Remark A.2. Note that, by reverting the time, we can relate the spirals associated with λ and $-\lambda$. Also, if $B : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear map whose matrix on the previous basis is

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{then} \quad B^2 = I_{\mathbb{R}^2} \quad \text{and} \quad BAB = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix},$$

implying that the spirals associated with μ and $-\mu$ are related by conjugation.

By the previous Remark let us assume w.l.o.g. that $\lambda < 0$ and $\mu > 0$ and consider $v_1, v_2 \in (\mathbb{R}^2 \times \mathbb{R}^2) \setminus \Delta$. Denote by $\mathcal{C}_A(v_1, v_2)$, the region (see Fig. A.1) delimited by the line passing through v_1 and v_2, and the curve

$$\left\{ \varphi_A(\tau, v_1, v_2) \quad \tau \in \left[0, \frac{\pi}{\mu}\right] \right\}.$$

By our choices, such a region can be algebraically described as

$$\mathcal{C}_A(v_1, v_2) = \left\{ v \in \mathbb{R}^2; \;\; \langle v - v_2, \theta(v_1 - v_2)\rangle \geq 0 \quad \text{and} \quad \langle v - \varphi_A(\tau, v_1, v_2), \theta A \mathrm{e}^{\tau A}(v_1 - v_2)\rangle \geq 0, \; \forall \tau \in \left[0, \frac{\pi}{\mu}\right] \right\},$$

where θ is the counter-clockwise rotation of $\pi/2$-degrees. The following result analyzes a kind of invariance for the region $\mathcal{C}_A(v_1, v_2)$.

Proposition A.3. *For any $w_2 \in [v_1, v_2]$ and $w_1 \in \mathcal{C}_A(v_1, v_2)$, it holds that*

$$\varphi_A(s, w_1, w_2) \in \mathcal{C}_A(v_1, v_2), \quad s \in \left[0, \frac{\pi - \sigma}{\mu}\right],$$

where $\sigma \in [0, \pi]$ is the angle between $v_1 - v_2$ and $w_1 - w_2$. Here, we assume $\lambda < 0$ and $\mu > 0$.

Proof. Since,

$$\varphi_A(\tau, v_1, v_2) = \varphi_A(\tau, v_1 - v_2, 0) + v_2,$$

the region $\mathcal{C}_A(v_1, v_2)$ is obtained from $\mathcal{C}_A(v_1 - v_2, 0)$ through a translation by v_2. Therefore, it is enough to show the result assuming that $v_2 = 0$. For this case, we have

$$\mathcal{C}_A(v_1, 0) = \left\{v \in \mathbb{R}^2; \ \langle v, \theta v_1\rangle \geq 0 \quad \text{and} \quad \langle v - \mathrm{e}^{\tau\lambda} R_{\tau\mu} v_1, \theta A \mathrm{e}^{\tau A} v_1\rangle \geq 0, \ \forall \tau \in \left[0, \frac{\pi}{\mu}\right]\right\}.$$

Moreover, in this case $w_2 \in (0, v_1)$ and σ is the angle between v_1 and $w_1 - w_2$. Thus, we already have that,

$$\langle \varphi_A(s, w_1, w_2), \theta v_1\rangle = \langle \mathrm{e}^{sA}(w_1 - w_2) + w_2, \theta v_1\rangle \overset{w_2 \in (0, v_2)}{=} \mathrm{e}^{s\lambda}\langle R_{s\mu}(w_1 - w_2), \theta v_1\rangle$$

$$= \mathrm{e}^{s\lambda}\frac{|w_1 - w_2|}{|v_1|}\langle R_{\mu s + \sigma} v_1, \theta v_1\rangle = \mathrm{e}^{s\lambda}|w_1 - w_2||v_1|\sin(\mu s + \sigma) \geq 0, \quad \text{since} \quad \mu s \in [0, \pi - \sigma],$$

showing that

$$\langle w_1, \theta v_1\rangle \geq 0 \quad \Longrightarrow \quad \langle \varphi_A(s, w_1, w_2), \theta v_1\rangle \geq 0 \ \ \forall s \in \left[0, \frac{\pi - \sigma}{\mu}\right].$$

Define now the function

$$g : C_\sigma \to \mathbb{R}, \qquad g(s, \tau) := \langle \varphi_A(s, w_1, w_2) - \mathrm{e}^{\tau A} v_1, \theta A \mathrm{e}^{\tau A} v_1\rangle,$$

where $C_\sigma := \left[0, \frac{\pi - \sigma}{\mu}\right] \times \left[0, \frac{\pi}{\mu}\right]$. To conclude the result, it is enough to show that g is nonnegative, that is,

$$\forall (s, \tau) \in C_\sigma, \quad g(s, \tau) \geq 0,$$

which we will do in the next steps.

Step 1.: g is nonnegative on critical points in $\operatorname{int} C_\sigma$;
By simple calculations, we get that

$$\frac{\partial g}{\partial s}(s, \tau) = \langle A\mathrm{e}^{sA}(w_1 - w_2), \theta A \mathrm{e}^{\tau A} v_1\rangle = \det A\, \mathrm{e}^{(s+\tau)\lambda}\frac{|w_1 - w_2|}{|v_1|}\langle R_{\mu s + \sigma} v_1, \theta R_{\mu\tau} v_1\rangle$$

$$= \det A\, \mathrm{e}^{(s+\tau)\lambda}\frac{|w_1 - w_2|}{|v_1|}\sin(\mu(s - \tau) + \sigma) = 0 \quad \overset{(s,\tau) \in \operatorname{int} D_\sigma}{\Longleftrightarrow} \quad \mu s + \sigma = \mu\tau.$$

Also,

$$0=\frac{\partial g}{\partial \tau}(s,\tau)=\langle \varphi_A(s,w_1,w_2)-\mathrm{e}^{\tau A}v_1, \theta A^2\mathrm{e}^{\tau A}v_1\rangle,$$

if and only if there exists $\gamma\in\mathbb{R}$ such that,

$$\varphi_A(s,w_1,w_2)-\mathrm{e}^{\tau A}v_1=\gamma A^2\mathrm{e}^{\tau A}v_1.$$

The relation $\mu s+\sigma=\mu\tau$ gives us that

$$\varphi_A(s,w_1,w_2)=\mathrm{e}^{sA}(w_1-w_2)+w_2=\mathrm{e}^{\frac{-\sigma}{\mu}\lambda}\mathrm{e}^{\tau A}R_{-\sigma}(w_1-w_2)+w_2=\mathrm{e}^{\frac{-\sigma}{\mu}\lambda}\frac{|w_1-w_2|}{|v_1|}\mathrm{e}^{\tau A}v_1+w_2,$$

and hence

$$\langle \varphi_A(s,w_1,w_2)-\mathrm{e}^{\tau A}v_1, \theta\mathrm{e}^{\tau A}v_1\rangle=\langle w_2,\theta\mathrm{e}^{\tau A}v_1\rangle=-\mathrm{e}^{\tau\lambda}|w_2||v_1|\sin\mu\tau.$$

On the other hand,

$$\langle A^2\mathrm{e}^{\tau A}v_1,\theta\mathrm{e}^{\tau A}v_1\rangle=2\lambda\mu\mathrm{e}^{2\tau\lambda}|v_1|^2,$$

implying that

$$\gamma=-\frac{-\mathrm{e}^{-\tau\lambda}}{2\lambda\mu}\frac{|w_2|}{|v_1|}\sin\mu\tau\geq 0,\quad \text{ since }\ \lambda<0.$$

In particular, if g admits a critical point $(s,\tau)\in\operatorname{int}D_\sigma$, by the previous arguments, we get

$$g(s,\tau)=\langle \varphi_A(s,w_1,w_2)-\mathrm{e}^{\tau A}v_1, \theta A\mathrm{e}^{\tau A}v_1\rangle=\gamma\langle A^2\mathrm{e}^{\tau A}v_1,\theta A\mathrm{e}^{\tau A}v_1\rangle$$

$$=\gamma\det A\,\mathrm{e}^{2\tau\lambda}\langle Av_1,\theta v_1\rangle=\gamma\det A\,\mathrm{e}^{2\tau\lambda}\mu|v_1|^2\geq 0,$$

showing the assertion.

Step. 2: g is nonnegative on ∂C_σ.

Let us start by noticing the point $\varphi_A\left(\frac{\pi-\sigma}{\mu},w_1,w_2\right)$ belongs to the line $\mathbb{R}v_1$ and that

$$\varphi_A\left(\frac{\pi-\sigma}{\mu},w_1,w_2\right)=\mathrm{e}^{\frac{\pi-\sigma}{\mu}A}(w_1-w_2)+w_2=-\mathrm{e}^{\frac{\pi-\sigma}{\mu}\lambda}R_{-\sigma}(w_1-w_2)+w_2$$

$$=-\mathrm{e}^{\frac{\pi-\sigma}{\mu}\lambda}\frac{|w_1-w_2|}{|v_1|}v_1+w_2=\left(1-\mathrm{e}^{\frac{\pi-\sigma}{\mu}\lambda}\frac{|w_1-w_2|}{|w_2|}\right)w_2,$$

showing that $\varphi_A\left(\frac{\pi-\sigma}{\mu},w_1,w_2\right)\leq w_2\leq v_1$. On the other hand, if $\beta\in[0,\pi]$ is the angle between v_1 and w_1 we obtain that $\beta<\sigma$ (see Fig. A.2) and

$$\mathrm{e}^{\frac{\beta}{\mu}A}v_1=w_1\quad\Longrightarrow\quad \mathrm{e}^{\frac{\pi}{\mu}\lambda}|v_1|=\mathrm{e}^{\frac{\pi-\beta}{\mu}\lambda}\left|\mathrm{e}^{\frac{\beta}{\mu}A}v_1\right|=\mathrm{e}^{\frac{\pi-\beta}{\mu}\lambda}|w_1|.$$

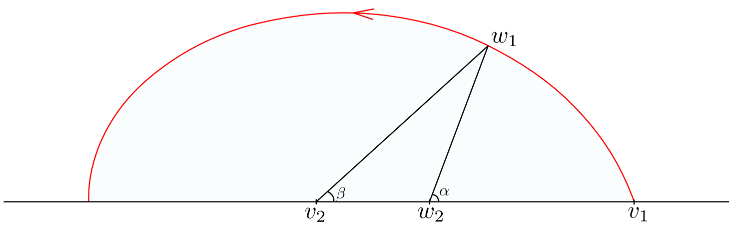

FIGURE A.2. The invariance of $\mathcal{C}_A(v_1, v_2)$.

Consequently,

$$|w_2| - \mathrm{e}^{\frac{\pi-\sigma}{\mu}\lambda}|w_1 - w_2| + \mathrm{e}^{\frac{\pi}{\mu}\lambda}|v_1| = |w_2| - \mathrm{e}^{\frac{\pi-\sigma}{\mu}\lambda}|w_1 - w_2| + \mathrm{e}^{\frac{\pi-\beta}{\mu}\lambda}|w_1|$$

$$= \mathrm{e}^{\frac{\pi-\beta}{\mu}\lambda}\left(|w_1| + \mathrm{e}^{-\frac{\pi-\beta}{\mu}\lambda}|w_2| - \mathrm{e}^{\frac{-(\sigma-\beta)}{\mu}\lambda}|w_1 - w_2|\right) \geq \mathrm{e}^{\frac{\pi-\beta}{\mu}\lambda}\left(|w_1| + |w_2| - |w_1 - w_2|\right) \geq 0,$$

implying that

$$\varphi_A\left(\frac{\pi-\sigma}{\mu}, w_1, w_2\right) - \mathrm{e}^{\frac{\pi}{\mu}A}v_1 = \left(|w_2| - \mathrm{e}^{\frac{\pi-\sigma}{\mu}\lambda}|w_1 - w_2| + \mathrm{e}^{\frac{\pi}{\mu}\lambda}|v_1|\right)\frac{v_1}{|v_1|} \geq 0,$$

and allowing us to conclude that

$$\mathrm{e}^{\frac{\pi}{\mu}A}v_1 \leq \varphi_A\left(\frac{\pi-\sigma}{\mu}, w_1, w_2\right) \leq v_1 \quad \Longrightarrow \quad \varphi_A\left(\frac{\pi-\sigma}{\mu}, w_1, w_2\right) \in \mathcal{C}_A(v_1, 0).$$

Therefore,

$$\forall \tau \in \left[0, \frac{\pi}{\mu}\right], \quad g(0, \tau) \geq 0 \quad \text{and} \quad g\left(\frac{\pi-\sigma}{\mu}, \tau\right) \geq 0.$$

By the previous calculations,

$$\frac{\partial g}{\partial s}(s, \tau) = \det A\, \mathrm{e}^{(s+\tau)\lambda}\frac{|w_1 - w_2|}{|v_1|}\sin(\mu(s-\tau)+\sigma),$$

implying that,

$$\forall s \in \left(0, \frac{\pi-\sigma}{\mu}\right), \quad \frac{\partial g}{\partial s}(s, 0) > 0 \quad \text{and} \quad \frac{\partial g}{\partial s}\left(s, \frac{\pi}{\mu}\right) < 0.$$

As a consequence, it follows that

$$\forall s \in \left(0, \frac{\pi-\sigma}{\mu}\right), \quad g(s, 0) \geq g(0, 0) \geq 0 \quad \text{and} \quad g\left(s, \frac{\pi}{\mu}\right) \geq g\left(\frac{\pi-\sigma}{\mu}, \frac{\pi}{\mu}\right) \geq 0.$$

Since C_σ is a compact subset and g is smooth, the Weierstrass Theorem assures the existence of a global minimum for g on C_σ. Since the possible candidates for such minimum were calculated in Steps 1. and 2. we conclude that g is nonnegative on C_σ, ending the proof.

$\square$

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