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A UNIQUENESS RESULT FOR A NON-STRICTLY CONVEX PROBLEM IN THE CALCULUS OF VARIATIONS.

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Abstract. We prove the uniqueness of the solution for a non-strictly convex problem in the Calculus of Variations of the form $\int \varphi(\nabla v) - \lambda v$. Here, φ is a convex function not differentiable at the origin and λ is a Lipschitz function. To prove this result, we show that under fairly general assumptions the minimizers are globally Lipschitz continuous.

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1. Introduction

1.1. The Kohn and Strang's example

The aim of this article is the study of non-strictly convex problems in the Calculus of Variations in dimension $N \geq 2$. A first example is given by the convexification of a non-convex functional introduced by Kohn and Strang in [15–17]:

$$\text{Minimize } : u \to \int_\Omega F(\nabla u(x)) \mathrm{d}x, \tag{1.1}$$

on $W^{1,2}_\psi(\Omega)$ with

$$F(z) := \begin{cases} |z| & \text{if } |z| \leq 1, \\ \frac{1}{2}(|z|^2 + 1) & \text{if } 1 < |z|, \end{cases} \tag{1.2}$$

which is the convexification of

$$F_0(z) := \begin{cases} 0 & \text{if } z = 0, \\ \frac{1}{2}(|z|^2 + 1) & \text{if } 0 < |z|. \end{cases}$$

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In this paper, $|z|$ represents the Euclidean norm of z in $\mathbb{R}^N$. For this problem, the admissible functions belong to the Sobolev space $W^{1,2}(\Omega)$ and have a prescribed trace $\psi : \mathbb{R}^2 \to \mathbb{R}$ on the boundary $\partial\Omega$ of a bounded open set Ω of $\mathbb{R}^N$. It is important to notice that F is not strictly convex, hence the uniqueness is not guaranteed. Moreover, we observe that F is singular at the origin and strictly convex outside the unit ball. This article focuses on problems that share these three features.

1.2. The general problem

More precisely, our aim is to study functionals of the following form:

$$\mathcal{I}_\lambda : u \mapsto \int_\Omega \varphi(\nabla u) - \lambda(x) u \mathrm{d}x,$$

where Ω is a bounded open set in $\mathbb{R}^N$ and $\lambda \in L^\infty(\Omega)$. Throughout this paper, $\varphi := g(|\cdot|)$ with $g : \mathbb{R} \to \mathbb{R}_+$ a continuous even convex function $\mathcal{C}^1$ on $(0, +\infty)$ and $\mathcal{C}^2$ on $(1, +\infty)$. We assume that:

$$g(x) = x \text{ if } 0 \leq x \leq 1. \tag{A1}$$

By convexity of g, we have $g(x) \geq x$ and $g'(x) \geq 1$ for every $x \in \mathbb{R}_+^*$. We assume that g is strongly convex at $+\infty$ as follows:

$$\liminf_{x \to +\infty} \frac{x g''(x)}{g'(x)} > 0. \tag{A2}$$

Moreover, we assume that

$$\limsup_{x \to 1^+} g''(x) < +\infty \text{ and } g'' > 0 \text{ on } (1, +\infty). \tag{A3}$$

Remark 1.1. The assumption (A2) provides the existence of $x_0 > 1$, $p = \frac{\alpha}{2} + 1 > 1$ with $\alpha := \liminf_{x \to +\infty} \frac{x g''(x)}{g'(x)}$ and $C_p > 0$ such that $g'(x) \geq C_p x^{p-1}$ and $g(x) \geq C_p x^p$ for every $x \geq x_0$. Moreover, with the assumption (A1) and a possible smaller $C_p > 0$, we have that $g(x) \geq C_p |x|^p$ for every $x \in \mathbb{R}$.

In this setting, the admissible functions for $\mathcal{I}_\lambda$ belong to the Sobolev space $W^{1,p}_\psi(\Omega)$, which is the subset of functions in $W^{1,p}(\Omega)$ that have a prescribed trace $\psi : \mathbb{R}^N \to \mathbb{R}$ on the boundary $\partial\Omega$. Our goal is to prove the uniqueness of solutions for the following minimization problem:

$$\mathcal{P}_\lambda : \min_{u \in W^{1,p}_\psi(\Omega)} \mathcal{I}_\lambda(u).$$

Thanks to Remark 1.1, the problem $\mathcal{P}_\lambda$ has at least one solution. As in the preliminary example (1.1), φ is not strictly convex. Hence, there is no obvious reason for $\mathcal{P}_\lambda$ to have a unique solution. When λ is constant, $\partial\Omega$ is connected and φ as (1.2) it was first proven in [3] that $\mathcal{P}_\lambda$ has a unique solution. When we only assume that λ is constant, we have by [20] that if $\mathcal{P}_\lambda$ admits a uniformly continuous solution (which may not exist) then is the only one. It remains to be seen whether this last theorem can be extended to cases where λ is non-constant.

In Section 5, we prove that this result cannot be generalized to every function λ:

Proposition 1.2. *When Ω is a ball and $\psi \equiv 0$, there exists $\lambda \in \mathcal{C}^1(\overline{\Omega})$ such that the problem $\mathcal{P}_\lambda$ with φ as in (1.1) has more than one solution.*

Thus, if we want to have uniqueness, λ cannot be any function in $L^\infty(\Omega)$. The aim of this paper is to find some assumptions on λ to ensure the uniqueness of minimizers, while also examining its regularity properties.

1.3. The two main results

We assume that Ω is an open simply connected bounded set of $\mathbb{R}^N$ with a Lipschitz continuous boundary $\partial\Omega$ and ψ is a Lipschitz function on $\partial\Omega$.

Since we do not have uniqueness for any $\lambda \in L^\infty(\Omega)$, we restrict our study to a specific class of functions, namely the functions with *small oscillations* in a sense to be specified subsequently. In general, every solution u of $\mathcal{P}_\lambda$ is globally Lipschitz continuous and the $W^{1,\infty}$ norm of u does not depend on $\nabla\lambda$. By choosing $|\nabla\lambda|$ sufficiently small, we can prove the two main theorems of the article.

The first one is the following:

Theorem 1.3. *Let Ω be a connected bounded open set of $\mathbb{R}^N$ with $N \geq 2$. We assume that Ω has a $\mathcal{C}^{1,1}$ connected boundary, $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^N)$ and λ is Lipschitz continuous on $\overline{\Omega}$ with $\min_{\overline{\Omega}} \lambda > 0$. If φ satisfies the assumptions $(A1)$, $(A2)$ and $(A3)$, then there exists a positive constant*

$$C(p, C_p, N, |\Omega|, diam(\Omega), \max_{\overline{\Omega}} \lambda, \min_{\overline{\Omega}} \lambda, ||\psi||_{\mathcal{C}^{1,1}(\Omega)}, \kappa)$$

such that if $\|\nabla\lambda\|_{L^\infty(\Omega)} \leq C$ then $\mathcal{P}_\lambda$ admits a unique solution on $W^{1,p}_\psi(\Omega)$. Here, κ is the essential infimum of the signed principal curvatures of $\partial\Omega$, p and C_p are the constants introduced in Remark 1.1.

Part of the proof relies on a global Lipschitz estimate, which is proven in Section 4:

Theorem 1.4 (Global Lipschitz continuity for a general degenerate functional)**.** *Let Ω be a connected bounded open set of $\mathbb{R}^N$ with $N \geq 2$ and $\lambda \in L^\infty(\Omega)$. We assume that Ω has a $\mathcal{C}^{1,1}$ connected boundary, $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^N)$. If φ satisfies the assumption $(A2)$, then any minimizer u of $\mathcal{P}_\lambda$ is globally Lipschitz-continuous on Ω. Moreover,*

$$||\nabla u||_{L^\infty(\Omega)} \leq C(p, C_p, ||\lambda||_{L^\infty(\Omega)}, N, |\Omega|, diam(\Omega), ||\psi||_{\mathcal{C}^{1,1}(\Omega)}, \kappa)$$

where κ is the essential infimum of the signed principal curvatures of $\partial\Omega$.

Observe that in this result we do not assume any upper bound on φ. Moreover, the above statement improves several global regularity results in the literature. For instance, in comparison with [6] we do not require that $\varphi'(0) = 0$, $\varphi \in \mathcal{C}^2(\mathbb{R}_+)$ and $\lambda \equiv 0$.

The second main theorem requires less regularity on Ω and ψ. However, in that case, Ω has to be convex. In order to state it, we need the notion of bounded slope condition:

Definition 1.5. Let Ω be a bounded open set of $\mathbb{R}^N$. We say that a map $\psi : \partial\Omega \to \mathbb{R}$ satisfies the bounded slope condition of rank R if for every $y \in \partial\Omega$, there exist ζ_y^+, $\zeta_y^- \in \mathbb{R}^N$ such that $|\zeta_y^+|$, $|\zeta_y^-| \leq R$ and

$$\psi(y) + \langle \zeta_y^-, x - y\rangle \leq \psi(x) \leq \psi(y) + \langle \zeta_y^+, x - y\rangle \tag{1.3}$$

for every $x \in \partial\Omega$.

If Ω is convex and ψ satisfies the bounded slope condition, then [4], Main Theorem asserts that every minimizer of $\mathcal{P}_\lambda$ is globally Lipschitz-continuous. Therefore, we can prove the following result:

Theorem 1.6. *Let Ω be a convex set, λ a Lipschitz continuous function on $\overline{\Omega}$, $\min_{\overline{\Omega}} \lambda > 0$ and $\psi : \partial\Omega \to \mathbb{R}$ satisfies the bounded slope condition of rank R. If φ satisfies the assumptions $(A1)$, $(A2)$ and $(A3)$, then there* *exists a constant* $C := C(p, C_p, N, diam(\Omega), \max_{\overline{\Omega}} \lambda, \min_{\overline{\Omega}} \lambda, R) > 0$ *such that if* $\|\nabla\lambda\|_{L^\infty(\Omega)} \leq C$ *then* $\mathcal{P}_\lambda$ *admits a unique solution on* $W^{1,p}_\psi(\Omega)$*. Here,* p *and* C_p *are the constants introduced in Remark 1.1.*

Links with the existent literature

The question of uniqueness in the calculus of variations has aroused much interest over the last decades. The first example (1.1) of this article is drawn from [15–17] and is also studied in an article of Alibert, Bouchitté, Fragalà and Lucardesi [1]. A first proof of uniqueness for this particular problem with a general Ω can be found in a paper of Bouchitté and Bousquet [3]. In [18], Kawohl, Stara and Wittum study the uniqueness of the solutions when the integrand is the convexification of the minimum of two parabolas in dimension two. In this case, the proof requires different ideas as the region where the integrand is affine is no longer the unit ball, but an annulus. A part of our proof has been inspired by a seminal paper by Marcellini [22] on the uniqueness when the integrand is convex and depends solely on ∇u. In [21], Lussardi and Mascolo proved the result of Marcellini with fewer assumptions but only in dimension two. When λ is constant, a shorter proof of Theorem 1.3 can be found in [20]. Many of these articles make use of the regularity of the level sets. In this regard, we particularly rely on an article by Massari [23]. The *pseudo-Cheeger problem* introduced in the proof exploits ideas coming from papers by Buttazzo, Carlier and Comte [7, 8].

1.4. Structure of the proof

We want to prove the uniqueness of the solution for a variational problem. We rely on a method introduced by Marcellini in [22]. In his work, he considers the minimization of

$$v \to \int_\Omega g(|\nabla v(x)|)\mathrm{d}x,$$

with $g : [0, +\infty] \to [0, +\infty)$ an increasing convex function and Ω a bounded convex $\mathcal{C}^1$ set. The main result is the following: if there exists a solution $u \in \mathcal{C}^1(\overline{\Omega})$ such that ∇u does not vanish on $\overline{\Omega}$, then u is the unique solution in the class of Lipschitz continuous functions agreeing with u on $\partial\Omega$. The proof is based on two parts:

Part a Each level set of a solution u intersects the boundary $\partial\Omega$ of the domain.

Part b If v is another solution, then v is constant on the level sets of u.

Hence, if u and v are two solutions, then $u = v$ on each level set of u. Since each level set of u intersects $\partial\Omega$ where $u = v$, we have $u = v$ on $\overline{\Omega}$.

In our case, we have a lower order term $\int_\Omega \lambda u \mathrm{d}x$ which alters the geometry of the level sets. These level sets may no longer intersect the boundary. For instance, if Ω is the unit ball of $\mathbb{R}^N$, then for $\lambda \equiv \lambda_0 \in \mathbb{R}_+$ where $\lambda_0 > h_\Omega$ (see Def. 1.8), then $u \equiv 0$ is not a solution of $\mathcal{P}_\lambda$ on $W^{1,p}_0(\Omega)$ (see Lem. 3.10). Hence, only one level set of the solution intersects $\partial\Omega$.

Therefore, we must adapt the aforementioned strategy as in [3]. Our proof is based on the five following parts:

Part 1 There exists an open set U independent of the choice of the minimizer u such that $|\nabla u| > 1$ on U. This is a generalization of [3], Proposition 1.2 which is a consequence of [10], Theorem 1.1.

Proposition 1.7. *Let* u *be a Lipschitz continuous solution. When* $\lambda \in L^\infty(\Omega)$*, there exists an open subset* $U \subset \Omega$ *such that* $u \in \mathcal{C}^{1,\alpha}$ *on* U *with* $0 < \alpha < 1$*,* $|\nabla u(x)| > 1$ *for every* $x \in U$ *and* $|\nabla u(x)| \leq 1$ *for a.e.* $x \in \Omega \setminus U$*. Moreover, the restriction of* $|\nabla u|$ *to* U *is uniformly continuous on* $U \cap \Omega'$ *for every* $\Omega' \Subset \Omega$*. Hence, the restriction of* $|\nabla u|$ *to* U *can be extended as a continuous function on* $\overline{U} \cap \Omega$ *which is equal to* 1 *on* $\partial U \cap \Omega$*.*

Part 2 We study the minimizing properties of the super-level sets $E_t(u) = [u \geq t]$ for a given solution u. To do this, we introduce the concepts of *Cheeger problem* and *weighted Cheeger problem*.

Definition 1.8. The Cheeger constant of Ω is defined as:

$$h_\Omega = \inf_{D \subset \overline{\Omega}} \frac{\mathrm{Per}(D, \mathbb{R}^N)}{|D|}.$$

A set $D \subset \overline{\Omega}$ of finite perimeter with $|D| > 0$ is said to be a *Cheeger set* if $\mathrm{Per}(D, \mathbb{R}^N) = h_\Omega |D|$.

Here, $\mathrm{Per}(D, \mathbb{R}^N)$ is the perimeter of D in the sense of Caccioppoli sets, see Section 3 below for details. Additionally, we also use a *pseudo-perimeter*, $\widetilde{Per}$ which is a weighted perimeter defined in Section 6. In this case, we replace $|D|$ by $\int_D \lambda \mathrm{d}x$ in order to demonstrate that the super-level sets satisfy a variational problem:

Theorem 1.9. *Let u be a Lipschitz continuous solution of $\mathcal{P}_\lambda$ and $\lambda \in L^\infty(\Omega)$. Then for a.e. $s \in \mathbb{R}$ and for every set $F \subset \Omega$ with finite perimeter in Ω such that $F \Delta E_s \Subset \Omega$, we have*

$$\widetilde{Per}(E_s, \Omega) - \int_{E_s} \lambda \mathrm{d}x \leq \widetilde{Per}(F, \Omega) - \int_F \lambda \mathrm{d}x.$$

In the above statement $F \Delta E_s := (E_s \backslash F) \cup (F \backslash E_s)$. This result is crucial in order to prove the regularity of the super-level sets.

Part 3 We establish the following result on the super-level sets inspired by [3], Proposition 1.3 where it is formulated in the setting of the Kohn and Strang's problem. It states that generically, the super-level sets $E_t(u) = [u \geq t]$ of a solution u have a regular boundary outside $\overline{U}$.

Proposition 1.10. *Let $\lambda \in L^\infty(\Omega)$, $\lambda \geq 0$ a.e. on Ω and let u be a locally Lipschitz continuous solution of $\mathcal{P}_\lambda$. For a.e. $t \in \mathbb{R}$, there exists an open set W_t in $\Omega \backslash \overline{U}$ such that $\partial^e E_t \cap W_t$ is a $\mathcal{C}^1$ hypersurface and $\mathcal{H}^s(\Omega \backslash (W_t \cup \overline{U})) = 0$ for every $s > N - 8$.*

Here, $\partial^e E_t$ is the essential boundary of E_t, namely, the set of those $x \in \mathbb{R}^N$ such that $\forall \rho > 0$,

$$0 < |E_t \cap B_\rho(x)| < |B_\rho(x)|. \tag{1.4}$$

With this proposition, we can prove that almost every level set of a solution u intersects the boundary of the domain $\partial\Omega$ or the closure of the set U.

Part 4 We use the regularity of the super-level sets of u to prove that if v is another solution, then v is constant $\mathcal{H}^{N-1}$ a.e. on the boundary of the super-level sets of u.

Part 5 Since $u - v$ is constant on each connected component of U, we try to obtain information on the nature of U in order to prove the uniqueness. If λ is a constant, then the connected components of U intersect the boundary of Ω (see [3], Lem. 3.3) and this allows us to conclude. In this paper, we prove an equivalent result when λ has *small oscillations* in Lemma 7.1. The open set U can be very ugly, it is even possible that $|\partial U| > 0$ or that $\overline{U}$ is not equal to the union of the closures of its connected components, as in Proposition 5.13. This is a huge difficulty to prove the uniqueness for a larger class of functions λ.

1.5. Plan of the paper

In the next section, we present some elementary results in order to have some general information about the solutions. In the third section, we introduce the *Cheeger problem* and some comparison principles. We prove the global Lipschitz estimates in Section 4. The fifth section is devoted to the Euler equation and to finding some cases where we do not have a unique solution. In Section 6, we study the *pseudo-Cheeger problem* in order to obtain information about the level sets of the solutions. The last section is devoted to the proof of the main theorems.

2. Preliminary results

In this section, we state some elementary results used in the proofs presented in the following parts.

Lemma 2.1. *If u is a solution of $\mathcal{P}_\lambda$ on $W^{1,p}_\psi(\Omega)$, then $u+c$ is a solution of $\mathcal{P}_\lambda$ on $W^{1,p}_{\psi+c}(\Omega)$ for each $c \in \mathbb{R}$.*

Proof. Consider $c \in \mathbb{R}$, u a solution of $\mathcal{P}_\lambda$ on $W^{1,p}_\psi(\Omega)$ and $v \in W^{1,p}_{\psi+c}(\Omega)$. We have

$$\mathcal{I}_\lambda(v) = \int_\Omega \varphi(\nabla v) - \lambda(v-c) - \lambda c = \int_\Omega \varphi(\nabla(v-c)) - \lambda(v-c) - \lambda c \geq \mathcal{I}_\lambda(u) - \int_\Omega \lambda c = \mathcal{I}_\lambda(u+c).$$

Thus, $u+c$ is a solution of $\mathcal{P}_\lambda$ on $W^{1,p}_{\psi+c}(\Omega)$. □

As in [3], Lemma 2.1 we have the following estimate on the L^∞ norm of u which can be established with a similar proof:

Lemma 2.2. *If $\lambda \in L^\infty(\Omega)$, $\lambda \geq 0$ a.e. on Ω, then there exists $C>0$ which depends only on N such that for every $x \in \Omega$,*

$$\min_{\partial\Omega} \psi \leq u(x) \leq \max_{\partial\Omega} \psi + C\|\lambda\|^N_{L^\infty(\Omega)} \|(u - \max_{\partial\Omega}\psi)_+\|_{L^1(\Omega)}.$$

Moreover, if $\sup_\Omega u > \max_{\partial\Omega} \psi$, then

$$|\{x \in \Omega : u(x) = \sup_\Omega u\}| \geq \frac{1}{C\|\lambda\|^N_{L^\infty(\Omega)}}.$$

A direct consequence of this lemma is the following remark:

Remark 2.3. If $\lambda \in L^\infty(\Omega)$ and $\lambda \geq 0$ a.e. on Ω, then the solutions on $W^{1,p}_0(\Omega)$ are nonnegative.

The definition of φ allows comparing the gradients of two solutions, as in [3], Lemma 3.1:

Lemma 2.4. *Let $\lambda \in L^\infty(\Omega)$, $\lambda \geq 0$ a.e. on Ω and let u, v be two solutions of $\mathcal{P}_\lambda$. Then for a.e. $x \in \Omega$, either*

$$\max(|\nabla u(x)|, |\nabla v(x)|) \leq 1 \quad \text{and} \quad \nabla u(x), \nabla v(x) \text{ are positively colinear}$$

or

$$\nabla u(x) = \nabla v(x).$$

We have the following observation:

Remark 2.5. If $\mathcal{P}_\lambda$ has a Lipschitz continuous minimizer, then every minimizer is Lipschitz continuous.

We conclude this section with some uniform convexity results at infinity for φ.

Proposition 2.6. *There exists $D(C_p, \alpha) > 0$ such that*

$$D|z|^{p-2}|\xi|^2 \leq \langle \nabla^2 \varphi(z)\xi, \xi\rangle$$

for every $|z| \geq x_0$ and every $\xi \in \mathbb{R}^N$ with x_0, α and C_p defined in Remark 1.1.

Proof. If $|z| > 1$, then

$$\langle \nabla^2 \varphi(z)\xi, \xi\rangle = \frac{g''(|z|)}{|z|^2}\langle z, \xi\rangle^2 + \frac{g'(|z|)}{|z|}|\xi|^2 - \frac{g'(|z|)}{|z|^3}\langle z, \xi\rangle^2$$

for every $\xi \in \mathbb{R}^N$. Hence, $\nabla^2\varphi(z)$ has two eigenvalues: $g''(|z|)$ and $\frac{g'(|z|)}{|z|}$.

By assumption $(A2)$ there exists $x_0 > 1$ such that for every $x \geq x_0$, $\frac{xg''(x)}{g'(x)} \geq \frac{\alpha}{2} > 0$. By Remark 1.1 and the fact that $g(0) = 0$, there exist $p > 1$ and $C_p > 0$ such that:

$$C_p x^p \leq g(x) \leq xg'(x)$$

for every $x \geq x_0$. Hence, $C_p x^{p-2} \leq \frac{g'(x)}{x} \leq \frac{2g''(x)}{\alpha}$ for every $x \geq x_0$. Thus, we have

$$D|z|^{p-2}|\xi|^2 \leq \langle \nabla^2\varphi(z)\xi, \xi\rangle$$

for every $|z| \geq x_0$ and every $\xi \in \mathbb{R}^N$ with $D := \min\{\frac{\alpha}{2}, 1\}C_p$. □

Definition 2.7. A convex function is said to be uniformly convex at infinity if there exist $M > 0$ and $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ a function such that $\lim_{x\to+\infty} x\Phi(x) = +\infty$ and

$$\varphi\big(t\xi + (1-t)\zeta\big) \leq t\varphi(\xi) + (1-t)\varphi(\zeta) - t(1-t)\Phi\big(|\xi| + |\zeta|\big)|\xi - \zeta|^2$$

for every $\xi, \zeta \in \mathbb{R}^N$ satisfying $[\xi, \zeta] \cap B_M(0) = \emptyset$.

Remark 2.8. The lower bound in Proposition 2.6 is equivalent to the existence of $\tilde{D} > 0$ such that φ is uniformly convex outside $B_{x_0}(0)$ with $\Phi(x) = \tilde{D}x^{p-2}$:

$$\varphi\big(t\xi + (1-t)\zeta\big) \leq t\varphi(\xi) + (1-t)\varphi(\zeta) - \tilde{D}t(1-t)\big(|\xi| + |\zeta|\big)^{p-2}|\xi - \zeta|^2$$

for every $\xi, \zeta \in \mathbb{R}^N$ satisfying $[\xi, \zeta] \cap \overline{B_{x_0}(0)} = \emptyset$ and every $t \in [0, 1]$. Moreover, when $p = 2$, we can take $\tilde{D} = \frac{D}{2}$.

3. The cheeger problem

3.1. BV functions and perimeter

In this section, we summarize some results on the functions of bounded variations and the perimeters of Caccioppoli sets. For additional information, readers may refer to [11], Chapter 5 which expounds on the general results and their proofs.

Definition 3.1. A function $f \in L^1(\Omega)$ has bounded variation in Ω if

$$\int_\Omega |Df| := \sup\left\{ \int_\Omega f \text{ div } g \;\middle|\; g \in \mathcal{C}^1_0(\Omega; \mathbb{R}^N), |g(x)| \leq 1\,,\ \forall x \in \Omega \right\} < \infty.$$

We denote by $BV(\Omega)$ the set of functions of bounded variation in Ω.

If $f \in BV(\Omega)$, then the distributional gradient of f is a vector valued Radon measure that we denote by Df and $|Df|$ is the total variation of Df.

Definition 3.2. Let E be a Borel set. We say that E has finite perimeter in Ω if the characteristic function χ_E of E belongs to $BV(\Omega)$ with

$$\chi_E(x) = \begin{cases} 1 \text{ if } x \in E \\ 0 \text{ if } x \notin E. \end{cases}$$

The perimeter $\mathrm{Per}(E, \Omega)$ is defined as:

$$\mathrm{Per}(E, \Omega) = \int_\Omega |D\chi_E| = \sup\left\{ \int_E \mathrm{div}\ g : g \in \mathcal{C}_0^1(\Omega; \mathbb{R}^N), |g(x)| \leq 1\,,\ \forall x \in \Omega \right\}.$$

Remark 3.3. When $\Omega = \mathbb{R}^N$, we just denote $\mathrm{Per}(E, \mathbb{R}^N)$ by $\mathrm{Per}(E)$.

Definition 3.4. A Borel set E is a Caccioppoli set if for every bounded subset Ω of $\mathbb{R}^N$, we have $\mathrm{Per}(E, \Omega) < \infty$.

Definition 3.5. The reduced boundary $\partial^* E$ of E in Ω is the set of those $x \in (\mathrm{supp}|D\chi_E|) \cap \Omega$ such that

$$\nu_E := \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} D\chi_E}{\int_{B_\rho(x)} |D\chi_E|}$$

is well defined and $|\nu_E| = 1$.

By [11], Lemma 5.8.1 the reduced boundary is a subset of the essential boundary $\partial^e E$. By [13], Theorem 4.4, we have for every Borel set $F \subset \Omega$,

$$\int_F |D\chi_E| = \mathcal{H}^{N-1}(F \cap \partial^* E). \tag{3.1}$$

If $u \in BV(\Omega)$, then for a.e. $s \in \mathbb{R}$ the super-level set $E_s = [u \geq s]$ has finite perimeter and by the co-area formula from [2], Theorem 3.40 we have for every Borel set $F \subset \Omega$ that

$$\int_F |Du| = \int_\mathbb{R} \mathrm{d}s \int_F |D\chi_{E_s}|. \tag{3.2}$$

The two last equations give

$$\int_F |Du| = \int_\mathbb{R} \mathcal{H}^{N-1}(F \cap \partial^* E_s) \mathrm{d}s. \tag{3.3}$$

If we write it with characteristic functions, we obtain

$$\int_\Omega \chi_F \, d|Du| = \int_\mathbb{R} \mathrm{d}s \int_{\partial^* E_s} \chi_F \, \mathrm{d}\mathcal{H}^{N-1}. \tag{3.4}$$

Thus, by linearity and monotone convergence theorem we can generalize the last equality to every nonnegative Borel function f,

$$\int_\Omega f \, d|Du| = \int_\mathbb{R} \mathrm{d}s \int_{\partial^* E_s} f \, \mathrm{d}\mathcal{H}^{N-1}. \tag{3.5}$$

The following standard lemma asserts that the gradient of a Sobolev function u is orthogonal to the level sets of u, see *e.g.* [3], Lemma 2.6 for a proof.

Lemma 3.6. *Let $u \in W^{1,1}(\Omega)$. Then for a.e. $s \in \mathbb{R}$ and $\mathcal{H}^{N-1}$ a.e. $x \in \partial^* E_s$, $\nabla u(x) \neq 0$. Moreover, we have*

$$\nu_{E_s} = \frac{D\chi_{E_s}}{|D\chi_{E_s}|} = \frac{\nabla u}{|\nabla u|} \qquad |D\chi_{E_s}| \text{ a.e.} \tag{3.6}$$

3.2. Comparison principles

When $\psi \equiv 0$, information about the solutions can be obtained through the geometry of Ω and a comparison principle.

Definition 3.7. The Cheeger constant of Ω is defined as:

$$h_\Omega = \inf_{D \subset \overline{\Omega}} \frac{\text{Per } D}{|D|}.$$

A set $D \subset \overline{\Omega}$ of finite perimeter is said to be a Cheeger set if Per $D = h_\Omega |D|$.

In this subsection, we also use an equivalent definition of h_Ω:

Proposition 3.8. *We have the following inequality for every $p \geq 1$:*

$$h_\Omega = \tilde{h}_\Omega := \inf \left(\int_\Omega |\nabla u| : u \in W_0^{1,p}(\Omega), \int_\Omega u = 1 \right).$$

Proof. By [25], Proposition 3.1, there exists a Cheeger set E of Ω. By [25], Proposition 3.3 and [24], Theorem 2, there exists a sequence $(E_k)_{k \in \mathbb{N}}$ of smooth sets such that $\chi_{E_k} \to \chi_E$ in $L^1_{loc}(\mathbb{R}^N)$, $\text{Per}(E_k) \to \text{Per}(E)$ and $E_k \Subset \Omega$.

By [11], Theorem 5.2.2, for each $k \in \mathbb{N}$ there exists a sequence $(f_k^n)_{n \in \mathbb{N}} \in BV(\Omega) \cap \mathcal{C}^\infty(\Omega)$ such that $f_k^n \to \chi_{E_k}$ in $L^1_{loc}(\mathbb{R}^N)$ and $\int_\Omega |Df_k^n| \to \int_\Omega |D\chi_{E_k}|$ when $n \to +\infty$. For every $k \in \mathbb{N}$, we introduce $\zeta_k \in \mathcal{C}_0^\infty(\Omega)$, $\chi_{E_k} \leq \zeta_k \leq 1$. The function $g_k^n := f_k^n \zeta_k \in \mathcal{C}_0^\infty(\Omega)$ is such that $g_k^n \to \chi_{E_k}$ in $L^1_{loc}(\mathbb{R}^N)$ and $\int_\Omega |Dg_k^n| \leq \int_\Omega |Df_k^n| + \int_\Omega |\nabla \zeta_k| f_k^n \to \int_\Omega |D\chi_{E_k}|$ when n goes to $+\infty$.

By a diagonal process, we can extract a sequence $(g_k^{n_k})_{k \in \mathbb{N}}$ of smooth functions compactly supported in Ω such that

$$\liminf_{k \to +\infty} \frac{\int_\Omega |Dg_k^{n_k}|}{\int_\Omega g_k^{n_k}} \leq \frac{Per E}{|E|} = h_\Omega.$$

If we normalize the $(g_k^{n_k})_{k \in \mathbb{N}}$ we obtain that $\tilde{h}_\Omega \leq h_\Omega$. To prove the equality, we consider $(u_n)_{n \in \mathbb{N}}$ a minimizing sequence of $\int_\Omega |\nabla u|$ in $\{u \in W_0^{1,p}(\Omega), \int_\Omega u = 1\}$:

$$0 = \lim_{n \to +\infty} \int_\Omega |\nabla u_n| - \tilde{h}_\Omega.$$

We can assume that $u_n \geq 0$ a.e. for every $n \in \mathbb{N}$. By the co-area formula and Fubini's theorem, we have

$$\int_\Omega |\nabla u_n| - \tilde{h}_\Omega = \int_\Omega \big(|\nabla u_n| - \tilde{h}_\Omega u_n \big) = \int_\mathbb{R} Per(E_s^n) - \tilde{h}_\Omega |E_s^n|$$

where $E_s^n := [u_n \geq s]$. Since $\tilde{h}_\Omega \leq h_\Omega$ we get

$$\int_{\mathbb{R}} Per(E_s^n) - \tilde{h}_\Omega |E_s^n| \geq \int_{\mathbb{R}} Per(E_s^n) - h_\Omega |E_s^n| \geq 0.$$

Hence,

$$0 = \lim_{n\to+\infty} \int_\Omega |\nabla u_n| - \tilde{h}_\Omega \geq \lim_{n\to+\infty} \int_\Omega |\nabla u_n| - h_\Omega \geq 0.$$

Thus, $\tilde{h}_\Omega = h_\Omega$.

□

The first result of this part is the following:

Proposition 3.9. *Let $\lambda \in L^\infty(\Omega)$, $\lambda \geq 0$ a.e. on Ω.*

- *If $\inf_{x\in\Omega} \lambda(x) > h_\Omega$, then 0 is not a solution of $\mathcal{P}_\lambda$ on $W_0^{1,p}(\Omega)$.*
- *If $\sup_{x\in\Omega} \lambda(x) \leq h_\Omega$, then 0 is the unique solution of $\mathcal{P}_\lambda$ on $W_0^{1,p}(\Omega)$.*

The proof of this proposition is based on the two following lemmata, see [3], Lemmata 1.5 and 4.1 for the specific case of the Kohn and Strang's problem (1.2):

Lemma 3.10. *Let $\lambda \equiv \lambda_0 \in \mathbb{R}_+$.*

- *If $\lambda > h_\Omega$, then 0 is not the solution of $\mathcal{P}_\lambda$ on $W_0^{1,p}(\Omega)$.*
- *If $\lambda \leq h_\Omega$, then 0 is the unique solution of $\mathcal{P}_\lambda$ on $W_0^{1,p}(\Omega)$.*

Proof of Lemma 3.10. If we suppose that 0 is the solution, then for every $v \in \mathcal{C}_0^\infty(\Omega)$,

$$\int_\Omega \varphi(\nabla v) - \lambda \int_\Omega v \geq 0.$$

Thus, by assumption $(A1)$ on φ we have

$$\lambda \int_\Omega v \leq \int_\Omega \varphi(\nabla v) = \int_{[|\nabla v| \leq 1]} |\nabla v| + \int_{[|\nabla v| > 1]} \varphi(\nabla v).$$

By replacing v by sv for some $s > 0$ and dividing by s, one gets

$$\lambda \int_\Omega v \leq \int_{[|\nabla v| \leq s^{-1}]} |\nabla v| + \frac{1}{s} \int_{[|\nabla v| > s^{-1}]} \varphi(s\nabla v).$$

Then we let $s \to 0^+$, by convexity of φ and the fact that $\varphi(0) = 0$ we have:

$$\lambda \int_\Omega v \leq \lim_{s\to 0} \int_{[|\nabla v| \leq s^{-1}]} |\nabla v| + \lim_{s\to 0} \int_{[|\nabla v| > s^{-1}]} \varphi(\nabla v).$$

Since the limit in the last term is 0 we get

$$\lambda \int_\Omega v \leq \int_\Omega |\nabla v|.$$

By density of smooth functions in $W_0^{1,p}(\Omega)$ and the second definition of h_Ω we deduce that $\lambda \le h_\Omega$.

Assume now that $\lambda \le h_\Omega$. Then for any $v \in W_0^{1,p}(\Omega)$ and u solution, we have

$$\int_\Omega \varphi(\nabla v) - \lambda \int_\Omega v \ge \int_\Omega \varphi(\nabla u) - \lambda \int_\Omega u.$$

By Lemma 2.2 we have that $u \ge 0$. Then, by Proposition 3.8,

$$\int_\Omega \varphi(\nabla v) - \lambda \int_\Omega v \ge \int_\Omega |\nabla u| - h_\Omega \int_\Omega u \ge 0. \tag{3.7}$$

This proves that 0 is the solution of $\mathcal{P}_\lambda$. By [20], when $\lambda \equiv \lambda_0 \in \mathbb{R}_+$, if $\mathcal{P}_\lambda$ has a uniformly continuous solution in $W_0^{1,p}(\Omega)$ then it is the unique solution. Thus, 0 is the unique solution of $\mathcal{P}_\lambda$ on $W_0^{1,p}(\Omega)$. □

Lemma 3.11. *Consider λ_1, $\lambda_2 \in L^\infty(\Omega)$, $\lambda_2 \ge \lambda_1 > 0$ a.e. on Ω, u_1 a solution of $\mathcal{P}_{\lambda_1}$ on $u_1 + W_0^{1,p}(\Omega)$ and u_2 a solution of $\mathcal{P}_{\lambda_2}$ on $u_2 + W_0^{1,p}(\Omega)$. We also assume that $u_1|_{\partial\Omega} \le u_2|_{\partial\Omega}$. If $\lambda_2 > \lambda_1$ a.e. on Ω or if u_1 is the unique minimizer of $\mathcal{P}_{\lambda_1}$ on $u_1 + W_0^{1,p}(\Omega)$ or if u_2 is the unique minimizer of $\mathcal{P}_{\lambda_2}$ on $u_2 + W_0^{1,p}(\Omega)$, then $u_1 \le u_2$ a.e. on Ω.*

Proof of Lemma 3.11. Since $I_{\lambda_1}(u_1) \le I_{\lambda_1}(\min(u_1, u_2))$, we have

$$\int_{[u_1>u_2]} \varphi(\nabla u_1) - \lambda_1 u_1 \le \int_{[u_1>u_2]} \varphi(\nabla u_2) - \lambda_1 u_2. \tag{3.8}$$

Since $I_{\lambda_2}(u_2) \le I_{\lambda_2}(\max(u_1, u_2))$,

$$\int_{[u_1>u_2]} \varphi(\nabla u_2) - \lambda_2 u_2 \le \int_{[u_1>u_2]} \varphi(\nabla u_1) - \lambda_2 u_1. \tag{3.9}$$

The sum of (3.8) and (3.9) gives

$$0 \le \int_{[u_2<u_1]} (\lambda_2 - \lambda_1)(u_2 - u_1).$$

We first assume that for a.e. $x \in \Omega$, $\lambda_2(x) > \lambda_1(x)$ and $|[u_2 < u_1]| > 0$. Since $u_2 - u_1$ is negative on the set $[u_2 < u_1]$, it follows that this last integral is also negative. This is a contradiction, hence, if for a.e. $x \in \Omega$, $\lambda_2(x) > \lambda_1(x)$ then $u_1 \le u_2$ a.e. on Ω, which completes the proof in that case. Otherwise, if u_1 is the unique minimum of $\mathcal{P}_{\lambda_1}$ on $u_1 + W_0^{1,2}(\Omega)$ and $|[u_2 < u_1]| > 0$ the inequality (3.8) becomes strict and with the help of equality (3.9) we have

$$0 < \int_{[u_2<u_1]} (\lambda_2 - \lambda_1)(u_2 - u_1).$$

Since the integral is nonpositive, this implies that $|[u_2 < u_1]| = 0$. Hence, $u_1 \le u_2$ a.e. on Ω, which completes the proof in that case as well. If u_2 is the unique solution of $\mathcal{P}_\lambda$, then the proof is similar and we omit it. □

Proof of Proposition 3.9. Let $\lambda \in L^\infty(\Omega)$.

If $\inf_\Omega \lambda > h_\Omega$, then for ϵ small enough $\inf_\Omega \lambda \ge h_\Omega + \epsilon$. By Lemma 3.11 for any solution u of $\mathcal{P}_\lambda$, $u \ge v$ where v is a solution of $\mathcal{P}_{h_\Omega + \frac{\epsilon}{2}}$. By Lemma 3.10, $v \ne 0$, and thus $u \ne 0$.

If $\lambda \le h_\Omega$, by Lemma 3.10, 0 is the only solution of $\mathcal{P}_{h_\Omega}$. Then, by Lemma 3.11 we have $u \le 0$ and by Lemma 2.2, $u \equiv 0$. □

When $\lambda \leq h_\Omega$, we have the following L^∞ estimate on the solutions:

Proposition 3.12. *Let $\lambda \in L^\infty(\Omega)$, $0 \leq \lambda \leq h_\Omega$ a.e. on Ω and let u be a solution of $\mathcal{P}_\lambda$ on $W^{1,p}_\psi(\Omega)$. We have for a.e. $x \in \overline{\Omega}$,*

$$\min_{\partial\Omega} \psi \leq u(x) \leq \max_{\partial\Omega} \psi.$$

Proof. We proved in Lemma 2.2 that $u \geq \min_{\partial\Omega} \psi$. We denote by $b := \max_{\partial\Omega} \psi$. Since $0 \leq \lambda \leq h_\Omega$, by Lemma 3.10 and Lemma 2.1, b is the unique solution of $\mathcal{P}_\lambda$ on $W^{1,p}_b$. By lemma 3.11 we have $u \leq b$. □

4. Global lipschitz regularity

In this section, we study two situations where the minimizers of $\mathcal{P}_\lambda$ are globally Lipschitz continuous.

4.1. Ω is convex

In the following statement, we assume that ψ satisfies the bounded slope condition of rank R introduced in Definition 1.5. This assumption on ψ and Remark 2.8 allow us to apply [4], Main Theorem in our case:

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded convex set, φ a uniformly convex function at infinity, ψ a Lipschitz continuous function that satisfies the bounded slope condition of rank $R \geq 0$ and $\lambda \in L^\infty(\Omega)$. Then every minimizer u of the problem $\mathcal{P}_\lambda$ is Lipschitz-continuous. More precisely, we have*

$$||\nabla u||_{L^\infty(\Omega)} \leq \mathcal{L}(\Phi, N, R, ||\lambda||_{L^\infty(\Omega)}, diam(\Omega))$$

with Φ given in Definition 2.7.

4.2. Ω is smooth

If Ω is not convex or if the bounded slope condition is not satisfied, we assume more regularity on $\partial\Omega$ and on the boundary condition ψ.

We recall Theorem 1.4:

Theorem 4.2 (Global Lipschitz continuity for a general degenerate functional)**.** *Let Ω be a connected bounded open set of $\mathbb{R}^N$ with $N \geq 2$. We assume that Ω has a $\mathcal{C}^{1,1}$ connected boundary, $\psi \in \mathcal{C}^{1,1}(\mathbb{R}^N)$ and $\lambda \in L^\infty(\Omega)$. If φ satisfies the assumption $(A2)$, then any minimizer u of $\mathcal{P}_\lambda$ is globally Lipschitz-continuous on Ω. Moreover,*

$$||\nabla u||_{L^\infty(\Omega)} \leq C(p, C_p, ||\lambda||_{L^\infty(\Omega)}, N, |\Omega|, diam(\Omega), ||\psi||_{\mathcal{C}^{1,1}(\Omega)}, \kappa)$$

with κ being the essential infimum of the signed principal curvatures of $\partial\Omega$.

The proof of this result is divided into four parts. In ***Step 1***, we start by approximating φ by smooth uniformly convex functions $(\varphi_\epsilon)_{\epsilon>0}$ in order to work with smooth minimizers $(u_\epsilon)_{\epsilon>0}$. Then in ***Step 2***, we construct two Lipschitz continuous functions l^+ and l^- such that $l^- \leq u_\epsilon \leq l^+$ for every $\epsilon > 0$. In the subsequent ***Step 3*** we prove that u_ϵ is globally Lipschitz continuous uniformly in ϵ. We pass to the limit when ϵ goes to 0 to conclude in ***Step 4***.

Step 1 By assumption $(A2)$ on g, there exist $x_0 > 1$ and $\alpha > 0$ such that for every $x \geq x_0$,

$$xg''(x) \geq \frac{\alpha}{2} g'(x).$$

Moreover, we take $\alpha \leq 2$ for the rest of the proof which is possible if we only assume $\alpha \leq \liminf_{x\to+\infty} \frac{xg''(x)}{g'(x)}$.

We introduce the following notations from [4], Lemma A5: $Q > x_0$, $J_Q(x) = (|x| - Q)_+^2$ and

$$\nu = \min\{1, \min_{x\in[2x_0,4Q]} \tilde{D}|x|^{p-2}\} \tag{4.1}$$

with $\tilde{D}$ introduced in Remark 2.8.

We define an approximation of g quadratic at $+\infty$:

$$\tilde{g}(x) := \begin{cases} g(x) & \text{if } |x| \le 3Q, \\ \frac{1}{2}g''(3Q)(x-3Q)^2 + g'(3Q)(x-3Q) + g(3Q) & \text{if } |x| > 3Q. \end{cases} \tag{4.2}$$

By Remark 1.1 and since we can take $\alpha \le 2$, we have that $x\tilde{g}''(x) \ge \frac{\alpha}{2}\tilde{g}'(x)$ for every $x \ge x_0$.

Let us consider $\eta \in \mathcal{C}_0^\infty((-1,1))$ an even function such that $\eta \ge 0$, $\int_{\mathbb{R}} \eta = 1$. We define for every $0 < \epsilon < \frac{1}{x_0}$,

$$\eta_\epsilon := \tfrac{1}{\epsilon}\eta(\tfrac{\cdot}{\epsilon}),\ g_\epsilon(x) := \big(\tilde{g} + \nu J_Q\big) * \eta_\epsilon(x) + \epsilon x^2,\ \varphi_\epsilon(\cdot) := g_\epsilon(|\cdot|) \text{ and } \lambda_\epsilon(z) := \lambda * \widetilde{\eta}_\epsilon(z)$$

where $(\widetilde{\eta}_\epsilon)_{\epsilon>0}$ is a mollifying sequence in $\mathbb{R}^N$. We prove the following regularity result on φ_ϵ:

Proposition 4.3. *The function φ_ϵ is smooth on $\mathbb{R}^N$.*

Proof. If $f : \mathbb{R} \to \mathbb{R}$ is a smooth even function, then $f(|\cdot|)$ is $\mathcal{C}^1$. Hence, φ_ϵ is $\mathcal{C}^1$ on $\mathbb{R}^N$. For every integer $1 \le i \le N$, we have $\partial_i\varphi_\epsilon(z) = \frac{g'_\epsilon(|z|)z_i}{|z|}$ if $z \ne 0$ and $\partial_i\varphi_\epsilon(0) = 0$. By a Taylor expansion we can see that

$$f(x) := \begin{cases} \frac{g'_\epsilon(x)}{x} & \text{if } x \ne 0, \\ g''_\epsilon(0) & \text{if } x = 0 \end{cases}$$

is a smooth even function. Thus, $\partial_i\varphi_\epsilon(z) = f(|z|)z_i$ is $\mathcal{C}^1$. Hence, φ_ϵ is $\mathcal{C}^2$. With the same arguments we can prove that this is also the case for $f(|\cdot|)$. By induction, φ_ϵ is $\mathcal{C}^\infty$. □

By the proof of [4], Lemma A5, we have the following uniform convexity result for every $0 < \epsilon < \frac{1}{2}x_0$:

$$g_\epsilon\big(tx + (1-t)y\big) \le tg_\epsilon(x) + (1-t)g_\epsilon(y) - t(1-t)\frac{\nu}{2}|x-y|^2$$

for every $t \in [0,1]$ and every $x, y \in \mathbb{R}$ such that $[x,y] \cap B_{\frac{3}{2}x_0}(0) = \emptyset$.

Proposition 4.4. *We consider $0 < \epsilon < \frac{1}{2}x_0$. If we set $\tilde{\nu} := \min\{\nu, \frac{1}{2}C_p, \frac{1}{2}C_pQ^{p-2}\} > 0$, then*

$$\varphi_\epsilon\big(t\xi + (1-t)\zeta\big) \le t\varphi_\epsilon(\xi) + (1-t)\varphi_\epsilon(\zeta) - t(1-t)\frac{\tilde{\nu}}{2}|\xi-\zeta|^2$$

for every $t \in [0,1]$ and every $\xi, \zeta \in \mathbb{R}^N$ such that $[\xi,\zeta] \cap B_{\frac{3}{2}x_0}(0) = \emptyset$.

Proof. By Remark 2.8, we have to find $\tilde{\nu} > 0$ such that $\langle \nabla^2\varphi_\epsilon(z)\xi,\xi\rangle \ge \tilde{\nu}|\xi|^2$ for every $z \in \mathbb{R}^N \backslash B_{\frac{3}{2}x_0}(0)$ and every $\xi \in \mathbb{R}^N$. If we reproduce the beginning of the proof of Proposition 2.6, we obtain that $\langle \nabla^2\varphi_\epsilon(z)\xi,\xi\rangle \ge \min\{g''_\epsilon(|z|), \frac{g'_\epsilon(|z|)}{|z|}\}|\xi|^2$. Since g_ϵ is uniformly convex outside $B_{\frac{3}{2}x_0}(0)$, we have that $g''_\epsilon(|z|) \ge \nu$. If $x > 2Q + \epsilon$, by definition of J_Q, we obtain that $\frac{g'_\epsilon(x)}{|x|} \ge \nu$. Hence,

$$\langle \nabla^2\varphi_\epsilon(z)\xi,\xi\rangle \ge \min\big\{\nu, \min_{x\in[x_0,3Q]} \frac{g'_\epsilon(x)}{x}\big\}|\xi|^2$$

for every $z \in \mathbb{R}^N$ with $|z| \geq \frac{3}{2}x_0$ and every $\xi \in \mathbb{R}^N$.

By Remark 1.1, $\frac{g'(x)}{x} \geq C_p x^{p-2}$ on $[x_0, 3Q]$. Thus, for every $x \in [x_0, 3Q]$, $\frac{g'(x)}{x} \geq C_p \min\{1, Q^{p-2}\}$. Then, we can set $\tilde{\nu} := \min\{\nu, \frac{1}{2}C_p, \frac{1}{2}C_p Q^{p-2}\} > 0$ such that

$$\langle \nabla^2 \varphi_\epsilon(z)\xi, \xi\rangle \geq \tilde{\nu}|\xi|^2$$

for every z with $|z| \geq \frac{3}{2}x_0$ and every $\xi \in \mathbb{R}^N$. □

Step 2 We introduce the following problem:

$$\mathcal{P}^\epsilon_{\lambda_\epsilon} : \min_{u \in W^{1,2}_\psi(\Omega)} I^\epsilon_{\lambda_\epsilon}(u) \tag{4.3}$$

where the functional $I^\epsilon_{\lambda_\epsilon}$ is the following:

$$I^\epsilon_{\lambda_\epsilon} : u \in W^{1,2}_\psi(\Omega) \to \int_\Omega \varphi_\epsilon(\nabla u) - \lambda_\epsilon u \mathrm{d}x.$$

For each $\epsilon > 0$, the problem $\mathcal{P}^\epsilon_{\lambda_\epsilon}$ has a unique solution that we call u_ϵ. We first prove that the solutions $(u_\epsilon)_{\epsilon>0}$ are bounded uniformly in ϵ:

Proposition 4.5. *We have* $\sup_{\epsilon>0} ||u_\epsilon||_{L^\infty(\Omega)} < +\infty$. *Moreover, the supremum can be bonded uniformly in* $Q > x_0$.

Proof. We introduce $\Psi = \max_{x \in \partial\Omega} |\psi(x)|$, $\Lambda = \sup_{x \in \Omega, \epsilon>0} \lambda_\epsilon(x)$ and $B_\Omega = B_{\mathrm{diam}(\Omega)}(x_\Omega)$ the smallest ball containing Ω. By Lemma 2.2, $u^*_\epsilon \geq \Psi$ on $\partial\Omega$ where u^*_ϵ is the solution of $\mathcal{P}^\epsilon_\Lambda$ on $W^{1,2}_\Psi(B_\Omega)$. Hence, by Lemma 3.11, we have that $||u_\epsilon||_{L^\infty(\Omega)} \leq ||u^*_\epsilon||_{L^\infty(\Omega)} \leq ||u^*_\epsilon||_{L^\infty(B_\Omega)}$. By [9], Theorem 1, $||u^*_\epsilon||_{L^\infty(B_\Omega)} \leq \Psi + \frac{N}{\Lambda} g^*_\epsilon\big(\frac{\Lambda}{N}\mathrm{diam}(\Omega)\big)$ where $g^*_\epsilon(x) = \sup_{y\geq 0} xy - g_\epsilon(y)$. By strict convexity of g_ϵ, $g^*_\epsilon(x) = x {g'_\epsilon}^{-1}(x) - g_\epsilon({g'_\epsilon}^{-1}(x))$.

By (4.2), we have $\lim_{x\to+\infty} \tilde{g}'(x) = +\infty$. Hence, there exists $x > 0$ such that $g'(x) = \frac{\Lambda}{N}\mathrm{diam}(\Omega) + 1$. By (4.2), for every Q such that $3Q > x$, we have $\tilde{g}'(x) = g'(x) = \frac{\Lambda}{N}\mathrm{diam}(\Omega) + 1$. Since $\tilde{g}$ is convex, $\tilde{g}'(y) \geq \frac{\Lambda}{N}\mathrm{diam}(\Omega) + 1$ for every $y \geq x$. Thus, for every $\epsilon < 1$, $g'_\epsilon(t) \geq \frac{\Lambda}{N}\mathrm{diam}(\Omega) + 1$ for every $t \geq x + 1$. Hence, ${g'_\epsilon}^{-1}\big(\frac{\Lambda}{N}\mathrm{diam}(\Omega)\big)$ is bounded uniformly in ϵ and Q. Thus, $||u_\epsilon||_{L^\infty(\Omega)}$ is bounded uniformly in ϵ and Q. □

We introduce

$$M := \sup_{x \in \Omega, \epsilon > 0} |u_\epsilon(x) - \psi(x)|.$$

The minimizer of (4.3) is a solution of the following partial differential equation:

$$\mathrm{div}\ (\nabla \varphi_\epsilon(\nabla u)) + \lambda_\epsilon = 0.$$

Hence,

$$L_\epsilon(u_\epsilon) := \sum_{i,j} \partial^2_{ij}\varphi_\epsilon(\nabla u_\epsilon)\partial^2_{ij}u_\epsilon + \lambda_\epsilon = 0. \tag{4.4}$$

This is a quasi-linear elliptic partial differential equation with smooth coefficients. Thus, $u_\epsilon \in \mathcal{C}^\infty(\Omega)$, by [19], Theorem 1 we have $u_\epsilon \in \mathcal{C}^1(\overline{\Omega})$ and [12], Lemma 9.15 shows that $u_\epsilon \in W^{2,2}_\psi(\Omega)$.

We establish the following estimate:

Proposition 4.6. *There exists* $\mu := \mu(x_0, \alpha, N, ||\lambda||_{L^\infty(\Omega)}) > 0$ *such that*

$$|z| \sum_{i,j} |\partial^2_{ij}\varphi_\epsilon(z)| + ||\lambda_\epsilon||_{L^\infty(\Omega)} \leq \mu \sum_{i,j} \partial^2_{ij}\varphi_\epsilon(z) z_i z_j$$

for every $|z| \geq \mu$ *and every* $0 < \epsilon < \frac{1}{2}x_0$.

Proof. By construction of g_ϵ, we have for every $x \geq \frac{3}{2}x_0$ that

$$x g''_\epsilon(x) = x \int_{\mathbb{R}} \big(\tilde{g}''(x-t) + 2\nu\chi_{\{y>Q\}}(x-t) + 2\epsilon\big)\eta_\epsilon(t)dt.$$

Since $x\tilde{g}''(x) \geq \frac{\alpha}{2}\tilde{g}'(x)$ for every $x \geq \frac{3}{2}x_0$, we get

$$\begin{aligned} x g''_\epsilon(x) &\geq \int_{\mathbb{R}} \frac{x}{x-t}\bigg(\frac{\alpha}{2}\tilde{g}'(x-t) + 2\nu(x-t)\chi_{\{y>Q\}}(x-t) + 2\epsilon(x-t)\bigg)\eta_\epsilon(t)dt \\ &\geq \frac{1}{2} \times \frac{\alpha}{2}\int_{\mathbb{R}} \bigg(\tilde{g}'(x-t) + 2\nu(x-t)\chi_{\{y>Q\}}(x-t) + 2\epsilon(x-t)\bigg)\eta_\epsilon(t)dt \geq \frac{1}{2} \times \frac{\alpha}{2} g'_\epsilon(x). \end{aligned}$$

Moreover,

$$\partial^2_{ij}\varphi_\epsilon(z) = \frac{g''_\epsilon(|z|)}{|z|^2} z_i z_j + \frac{g'_\epsilon(|z|)}{|z|}\delta_{ij} - \frac{g'_\epsilon(|z|)}{|z|^3} z_i z_j$$

for every $z \in \mathbb{R}^N$. By the previous estimate for every $|z| \geq \frac{3}{2}x_0$ we have:

$$|z| \sum_{i,j} |\partial^2_{ij}\varphi_\epsilon(z)| + ||\lambda||_{L^\infty(\Omega)} \leq C(\alpha, N) g''_\epsilon(|z|)|z| + ||\lambda||_{L^\infty(\Omega)}.$$

We also have $\sum_{i,j} \partial^2_{ij}\varphi_\epsilon(z) z_i z_j = g''_\epsilon(|z|)|z|^2$ and $\inf_{x > \frac{3}{2}x_0, 0<\epsilon<\frac{1}{2}x_0} g''_\epsilon(x)x > 0$. Hence, there exists $\mu > \frac{3}{2}x_0$ independent of ϵ, $\tilde{\nu}$ and Q such that

$$|z| \sum_{i,j} |\partial^2_{ij}\varphi(z)| + ||\lambda||_{L^\infty(\Omega)} \leq \mu \sum_{i,j} \partial^2_{ij}\varphi(z) z_i z_j$$

for every $|z| \geq \mu$.

□

We introduce $R := \dfrac{1}{-\kappa} > 0$ if κ the essential infimum of the principal curvatures of $\partial\Omega$ is negative. In the case where $\kappa \geq 0$ we can take R as large as we want. By [12], Appendix 14.6 and the regularity assumption on Ω, for every $y \in \partial\Omega$ there exists y_0 such that $y \in \overline{B_R(y_0)} \cap \overline{\Omega} = \overline{B_R(y_0)} \cap \partial\Omega \neq \emptyset$. We define the distance function $d_y(x) := dist(x, \partial B_R(y_0))$ and $\mathcal{N}_y := \{x \in \Omega \mid d_y(x) < A\}$ with A a constant to be specified later.

We estimate the norm of ∇u_ϵ at the boundary of Ω:

Lemma 4.7. *For every* $y \in \partial\Omega$, *there exist two* L_1*-Lipschitz continuous functions* l^+_y, l^-_y *such that* $l^-_y(y) = u_\epsilon(y) = l^+_y(y)$ *and* $l^-_y \leq u_\epsilon \leq l^+_y$ *on* Ω *for every* $0 < \epsilon < \frac{1}{2}x_0$ *with*

$$L_1 := L_1(||\lambda||_{L^\infty(\Omega)}, M, N, \kappa, \alpha, x_0, ||\nabla\psi||_{L^\infty(\mathbb{R}^N)}, ||\nabla^2\psi||_{L^\infty(\mathbb{R}^N)})$$

independent of ϵ, $\tilde{\nu}$ *introduced in Proposition 4.4 and* Q.

Proof. We follow the discussion of [12], Chapter 14, Section 1 to construct barriers for u_ϵ.

We introduce the following constants:

$$\tilde{\mu} = 4\big(\mu(1 + ||\nabla\psi||^2_{L^\infty(\mathbb{R}^N)}) + ||\nabla^2\psi||_{L^\infty(\mathbb{R}^N)}\big),\ \omega = (1 + \tfrac{N-1}{R})\tilde{\mu},\ k = \tilde{\mu}\omega e^{\omega M} \text{ and } A = \tfrac{1-e^{-\omega M}}{\tilde{\mu}\omega}.$$

Hence, we can define the following smooth function:

$$F_y(x) = \tfrac{1}{\omega}\log(1 + kd_y(x)).$$

The constant A is chosen such that $F_y(x) = M$ on $\partial\mathcal{N}_y \cap \Omega$. Thus, $\psi(x) \pm F_y(x)$ are two Lipschitz continuous functions, with $\psi(x) - F_y(x) \le u_\epsilon(x) \le \psi(x) + F_y(x)$ on $\partial\mathcal{N}_y$.

Moreover, using Proposition 4.6 we can prove as in [12], Chapter 14, Section 1 that $L_\epsilon(\psi(x) + F_y(x)) \le L_\epsilon(u_\epsilon) \le L_\epsilon(\psi(x) - F_y(x))$. By [12], Theorem 10.1, we have that $\psi(x) - F_y(x) \le u_\epsilon(x) \le \psi(x) + F_y(x)$ for every $x \in \Omega \cap \mathcal{N}_y$. Since $F_y \ge M$ on $\Omega\backslash\mathcal{N}_y$, we have that $\psi - F_y \le u_\epsilon \le \psi + F_y$ on Ω.

If we set $L_1 := \frac{k}{\omega} + ||\nabla\psi||_{L^\infty(\mathbb{R}^N)}$, then $l_y^\pm(x) := \psi(x) \pm F_y(x)$ are L_1-Lipschitz continuous on $\mathcal{N}_y$ for every $y \in \partial\Omega$.

□

Step 3 We prove the main part of Theorem 1.4:

Lemma 4.8. *There exists a constant $\mathcal{L} := \mathcal{L}(||\lambda||_{L^\infty(\Omega)}, M, N, \kappa, \alpha, x_0, ||\psi||_{\mathcal{C}^{1,1}(\mathbb{R}^N)})$ independent of ϵ, $\tilde{\nu}$ and Q such that*

$$||\nabla u_\epsilon||_{L^\infty(\overline{\Omega})} \le \frac{\mathcal{L}}{\tilde{\nu}}.$$

Proof. We define the two following functions:

$$l^+(x) = \begin{cases} \inf\limits_{y\in\partial\Omega} l_y^+(x) & \text{if } x \in \Omega, \\ \psi(x) & \text{if } x \notin \Omega \end{cases} \tag{4.5}$$

and

$$l^-(x) = \begin{cases} \sup\limits_{y\in\partial\Omega} l_y^-(x) & \text{if } x \in \Omega, \\ \psi(x) & \text{if } x \notin \Omega \end{cases} \tag{4.6}$$

with $M := \sup\limits_{x\in\Omega, \epsilon>0} |u_\epsilon(x) - \psi(x)|$.

We prove that l^+ and l^- are continuous on $\mathbb{R}^N$. Since l^+ is the infimum on Ω of L_1-Lipschitz continuous functions, l^+ is L_1-Lipschitz continuous in Ω. Let us consider $y \in \partial\Omega$ and $(x_n)_{n\in\mathbb{N}}$ a sequence in Ω converging to y. For n large enough, $x_n \in \mathcal{N}_y$. By definition of l^+ and the fact that $l_y^+(x) := \psi(x) + F_y(x)$ on $\mathcal{N}_y$, we have $\psi(x_n) \le l^+(x_n) \le \psi(x_n) + F_y(x_n)$. Since $F_y(z)$ decreases to 0 when $d_y(z)$ goes to 0, we have that $l^+(x_n) \to \psi(y)$ when $n \to +\infty$. Hence, l^+ is continuous on $\mathbb{R}^N$.

The same reasoning leads to the same conclusion for l^-. Moreover, l^+ and l^- are L_1-Lipschitz continuous on $\mathbb{R}^N$ with L_1 independent of ϵ. Those two Lipschitz continuous functions are equal to ψ on $\partial\Omega$ and $l^-(x) \le u_\epsilon(x) \le l^+(x)$ on $\overline{\Omega}$, this is the definition of barriers in [4].

Then, by applying Proposition 4.4 and following the methods outlined in [4], Section 4.4, we can establish that

$$||\nabla u_\epsilon||_{L^\infty(\overline{\Omega})} \le \frac{\mathcal{L}}{\tilde{\nu}},$$

with $\mathcal{L} := \mathcal{L}(||\lambda||_\infty, M, N, \kappa, \alpha, x_0, ||\psi||_{\mathcal{C}^{1,1}(\mathbb{R}^N)})$ independent of ϵ, $\tilde{\nu}$ and Q. □

Step 4 We conclude by the proof of Theorem 1.4.

Proof of Theorem 1.4. By definition of ν in (4.1), definition of $\tilde{\nu}$ in Proposition 4.4 and growing assumptions stated in Remark 1.1, there exists $C > 0$ such that $\frac{\mathcal{L}}{\tilde{\nu}} \leq CQ^{2-p}$ when $Q \to +\infty$. Thus, we can choose Q such that $\dfrac{\mathcal{L}}{\tilde{\nu}} \leq Q - 1$. Hence, for $\epsilon < \frac{1}{2}x_0$, we have that $\varphi_\epsilon = g * \eta_\epsilon(|\cdot|) + \epsilon|\cdot|^2$ on $B_{\frac{\mathcal{L}}{\tilde{\nu}}}(0)$. Thus, if we consider the Euler–Lagrange equation (see Prop. 5.6) we can prove that u_ϵ is a minimizer on $W^{1,2}_\psi(\Omega)$ of

$$v \to \int_\Omega g * \eta_\epsilon(|\nabla v|) + \epsilon|\nabla v|^2 - \lambda_\epsilon v.$$

We proved that the family $(u_\epsilon)_{\epsilon>0}$ is equi-bounded in $W^{1,\infty}(\Omega)$. Hence, by the Arzelà-Ascoli Theorem, $u_\epsilon \overset{*}{\rightharpoonup} \tilde{u}$ up to a sub-sequence in $W^{1,\infty}(\Omega)$. Hence, $\tilde{u} \in W^{1,p}_\psi(\Omega)$ and we demonstrate that $\widetilde{u}$ is a minimizer of $\mathcal{P}_\lambda$.

Let $v \in W^{1,\infty}_\psi(\Omega)$, by minimality of u_ϵ we have that

$$\int_\Omega g * \eta_\epsilon(|\nabla u_\epsilon|) + \epsilon|u_\epsilon|^2 - \lambda_\epsilon u_\epsilon \leq \int_\Omega g * \eta_\epsilon(|\nabla v|) + \epsilon|\nabla v|^2 - \lambda_\epsilon v. \tag{4.7}$$

By Jensen's inequality, $g * \eta_\epsilon \geq \varphi$ on $B_{\frac{\mathcal{L}}{\tilde{\nu}}}(0)$. Hence,

$$\int_\Omega \varphi(\nabla u_\epsilon) \leq \int_\Omega g * \eta_\epsilon(|\nabla u_\epsilon|).$$

By weak lower semi-continuity, we obtain that

$$\int_\Omega \varphi(\nabla \tilde{u}) \leq \liminf_{\epsilon\to 0} \int_\Omega g * \eta_\epsilon(|\nabla u_\epsilon|).$$

Thus,

$$\int_\Omega \varphi(\nabla \tilde{u}) - \lambda\tilde{u} \leq \liminf_{\epsilon\to 0} \int_\Omega g * \eta_\epsilon(|\nabla u_\epsilon|) + \epsilon|\nabla u_\epsilon|^2 - \lambda_\epsilon u_\epsilon. \tag{4.8}$$

By Equations (4.7) and (4.8):

$$\int_\Omega \varphi(\nabla \tilde{u}) - \lambda\tilde{u} \leq \liminf_{\epsilon\to 0} \int_\Omega g * \eta_\epsilon(|\nabla v|) + \epsilon|\nabla v|^2 - \lambda_\epsilon v.$$

We apply the dominated convergence theorem to this last term to obtain that

$$\int_\Omega \varphi(\nabla \tilde{u}) - \lambda\tilde{u} \leq \int_\Omega \varphi(\nabla v) - \int_\Omega \lambda v.$$

By [5], Theorem 1.1 and the regularity of $\partial\Omega$, $\tilde{u}$ is a minimizer of $\mathcal{P}_\lambda$ on $W^{1,p}_\psi(\Omega)$. By Lemma 2.4 we have $||\nabla\widetilde{u} - \nabla u||_{L^\infty(\Omega)} \leq 1$. Thus, $u \in W^{1,\infty}(\Omega)$ and

$$||\nabla u||_{L^\infty(\Omega)} \leq C(p, C_p, ||\lambda||_{L^\infty(\Omega)}, N, |\Omega|, diam(\Omega), ||\psi||_{\mathcal{C}^{1,1}(\Omega)}, \kappa).$$

□

5. Non uniqueness cases

5.1. The Euler–Lagrange equation

We establish the weak Euler–Lagrange equation for Lipschitz continuous minimizer to obtain some regularity information on $|\nabla u|$. A proof of the following lemma can be found in [3]:

Lemma 5.1 (Euler–Lagrange equation)**.** *Let $\lambda \in L^\infty(\Omega)$ and u be a solution of $\mathcal{P}_\lambda$. If u is Lipschitz continuous, there exists $\sigma \in L^\infty(\Omega;\mathbb{R}^N)$ such that div $\sigma = -\lambda$ and*

$$\sigma \in \partial\varphi(\nabla u) \quad a.e..$$

Here $\partial\varphi$ is the convex subdifferential of φ.

We first exploit the Euler–Lagrange equation to show that every solution is $\mathcal{C}^1$ on the set where the norm of its gradient is larger than 1. In fact, we use [10], Theorem 1 and proceed as in [3]:

Lemma 5.2. *Let $\lambda \in L^\infty(\Omega)$ and u be a Lipschitz solution of $\mathcal{P}_\lambda$. Then there exists an open subset $U \subset \Omega$ such that $u \in \mathcal{C}^{1,\alpha}$ on U with $0 < \alpha < 1$, $|\nabla u(x)| > 1$ for every $x \in U$ and $|\nabla u(x)| \leq 1$ for a.e. $x \in \Omega \setminus U$. Moreover, the restriction of $|\nabla u|$ to U is uniformly continuous on $U \cap \Omega'$ for all $\Omega' \Subset \Omega$. Hence, the restriction of $|\nabla u|$ to U can be extended as a continuous function on $\overline{U} \cap \Omega$ which is equal to 1 on $\partial U \cap \Omega$.*

Proof. By [10], Theorem 1.1, for every continuous function $\mathcal{H} : \mathbb{R}^N \to \mathbb{R}$ such that $\mathcal{H} = 0$ on $B_1(0)$, the function $\mathcal{H}(\nabla u)$ has a continuous representative on Ω. If we apply this result to $\mathcal{H}(y) = (|y| - 1)_+$, we obtain that $(|\nabla u| - 1)_+$ has a continuous representative on Ω. Thus, the set $U := [\mathcal{H}(\nabla u) > 0]$ is open.

On the open set U, $|\nabla u| > 1$ a.e.. Hence, the function σ introduced in Lemma 5.1 satisfies $\sigma = \nabla\varphi(\nabla u)$ a.e. on U. Thus, u is a weak solution of the quasilinear elliptic equation:

$$\operatorname{div}\big(\nabla\varphi(\nabla u)\big) = -\lambda. \tag{5.1}$$

By [14], Theorem 8.8, $u \in \mathcal{C}^{1,\alpha}_{loc}$ on U for every $\alpha < 1$. Since $|\nabla u| = \mathcal{H}(\nabla u) + 1$ is uniformly continuous on $U \cap \Omega'$ for every $\Omega' \Subset \Omega$, it follows that $|\nabla u|$ can be extended as a continuous function on $\overline{U} \cap \Omega$ which is equal to 1 on $\partial U \cap \Omega$.

Finally, on $\Omega \setminus U$, $\mathcal{H}(\nabla u) = 0$ and thus $|\nabla u| \leq 1$ a.e. there.

□

Proposition 5.3. *The open set U does not depend on the choice of a Lipschitz continuous solution u of $\mathcal{P}_\lambda$ on $W^{1,p}_\psi(\Omega)$.*

Proof. Let u and v be two minimizers of the same problem. By Lemma 2.4 and the strict convexity of φ outside the unit ball, the functions $(|\nabla u(x)| - 1)_+$ and $(|\nabla v(x)| - 1)_+$ are equal almost everywhere and have a continuous representative. Hence, they have the same continuous representative. Thus, U is uniquely defined. □

Remark 5.4. Since φ is differentiable on $\mathbb{R}^N \backslash \{0\}$, $\sigma(x) = \nabla\varphi(\nabla u(x))$ a.e. on $[\nabla u \neq 0]$. In particular, if $V := \Omega \backslash \overline{U}$,

$$\sigma(x) = \nabla\varphi(\nabla u(x)) \text{ on } U, \tag{5.2}$$

$$\sigma(x) = \frac{\nabla u(x)}{|\nabla u(x)|} \text{ a.e. } x \in [\nabla u \neq 0] \cap V, \tag{5.3}$$

$$|\sigma(x)| \leq 1 \text{ a.e. on } [\nabla u = 0]. \tag{5.4}$$

Since div$\sigma = -\lambda$, one also has

$$\int_\Omega \lambda v \leq \int_{[\nabla u = 0]} |\nabla v| + \int_{[\nabla u \neq 0]} \langle \nabla\varphi(\nabla u), \nabla v \rangle, \quad \forall v \in W_0^{1,p}(\Omega). \tag{5.5}$$

Proposition 5.5. *On $W_0^{1,p}(\Omega)$, $u \equiv 0$ is a solution of $\mathcal{P}_\lambda$ if and only if $U = \emptyset$.*

Proof. If $u \equiv 0$ is a solution, then by lemma 2.4 for any other solution v we have $|\nabla v| \leq 1$, and thus $U = \emptyset$. If $U = \emptyset$ and v is a solution, we have

$$\int_\Omega \varphi(\nabla v) - \lambda v \mathrm{d}x = \int_\Omega \varphi(\nabla v) - \langle \sigma, \nabla v \rangle \mathrm{d}x = 0.$$

The last equality relies on the equation (5.3). Then $u \equiv 0$ is also a solution. □

5.2. Counter-examples

In this subsection, we restrict our attention to the case $p = 2$ with φ as in (1.1) to simplify the presentation. If λ is no longer a constant, there may exist multiple solutions. To illustrate this, we present the following reciprocal result of Lemma 5.1 in the case $p = 2$ as a tool to provide examples.

Proposition 5.6. *Let u be in $W^{1,2}(\Omega)$. If there exist $\sigma \in L^2(\Omega;\mathbb{R}^N)$ and $\lambda \in L^\infty(\Omega)$ such that div $\sigma = -\lambda$ and*

$$\sigma \in \partial\varphi(\nabla u) \quad a.e.,$$

then u is a minimizer of $\mathcal{P}_\lambda$ on $W_u^{1,2}(\Omega)$.

Proof. Since $\sigma \in \partial\varphi(\nabla u)$, for every $v \in W_u^{1,2}(\Omega)$:

$$\int_\Omega \varphi(\nabla v) \geq \int_\Omega \varphi(\nabla u) + \langle \sigma, \nabla v - \nabla u \rangle.$$

Since div $\sigma = -\lambda$ this gives:

$$\int_\Omega \langle \sigma, \nabla v - \nabla u \rangle = \int_\Omega \lambda(v-u) + \int_{\partial^*\Omega} \langle \sigma, \nu_\Omega \rangle (v-u)\mathcal{H}^{N-1} = \int_\Omega \lambda(v-u).$$

Hence, $\mathcal{I}_\lambda(v) \geq \mathcal{I}_\lambda(u)$ for every $v \in W_u^{1,2}(\Omega)$. Thus, u is a minimizer of $\mathcal{P}_\lambda$ on $W_u^{1,2}(\Omega)$. □

A first example of non uniqueness arises when $\Omega = (-1,1)$.

Proposition 5.7. *There exists λ a nonnegative Lipschitz continuous function on $[-1,1]$ such that $\mathcal{P}_\lambda$ has more than one solution on $W_0^{1,2}((-1,1))$.*

Proof. Let us consider the following functions:

$$u(x) = \begin{cases} \frac{1}{2} & \text{if } |x| < \frac{1}{2}, \\ 1 - |x| & \text{if } |x| \geq \frac{1}{2}, \end{cases}$$

$$\lambda(x) = \begin{cases} 8(\frac{1}{2} - |x|) & \text{if } |x| < \frac{1}{2}, \\ 0 & \text{if } |x| \geq \frac{1}{2}. \end{cases}$$

$$\sigma(x) = \begin{cases} 1 - \int_{-\frac{1}{2}}^{x} \lambda & \text{if } |x| < \frac{1}{2}, \\ -\frac{x}{|x|} & \text{if } |x| \geq \frac{1}{2}. \end{cases}$$

Direct computations show that $\sigma \in \partial\varphi(u')$, $\lambda = -\sigma'$ and

$$\int_{-1}^{1} \varphi(u') - \lambda u = 2\int_{0}^{1} \varphi(u') - \lambda u = 1 - \int_{0}^{\frac{1}{2}} \lambda = 0. \tag{5.6}$$

Then u and 0 are solutions of $\mathcal{P}_\lambda$ on $W_0^{1,2}((-1,1))$ thanks to Proposition 5.6. □

Remark 5.8. In dimension one, if $\lambda > 0$ then we have uniqueness. Indeed, since $\sigma' = -\lambda$ there is at most two points where $|\sigma| = 1$. If u and v are two solutions, by definition of σ, $u'(x) \in (0,1]$ for at most two points $x \in \Omega$ and $v'(y) \in (0,1]$ for at most two points $y \in \Omega$. Therefore, for a.e. $x \in \Omega$ we either have $u'(x) = v'(x) = 0$ or $u'(x) = v'(x)$ with $x \in U$. Hence, $u \equiv v$.

In higher dimension, even with $\lambda > 0$ we can have two solutions. For example, if we consider $\Omega = B_1(0)$ the unit ball in dimension two:

Proposition 5.9. *When $N = 2$, there exists $\lambda \in \mathcal{C}^1(\overline{B_1(0)})$, $\lambda > 0$ such that the solutions of $\mathcal{P}_\lambda$ on $W_0^{1,2}(B_1(0))$ are not unique.*

Proof. Let us consider the following function:

$$u(x) = \begin{cases} \frac{1}{2} & \text{if } |x| < \frac{1}{2}, \\ 1 - |x| & \text{if } |x| \geq \frac{1}{2}. \end{cases}$$

By definition of u, $u(x) = f(|x|)$ with

$$f(r) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq r < \frac{1}{2}, \\ 1 - r & \text{if } r \geq \frac{1}{2}. \end{cases}$$

Our goal is to find $\sigma \in \mathcal{C}^0(\overline{B_1(0)})$ and $\lambda \in \mathcal{C}^1(\overline{B_1(0)})$ strictly positive and radial such that $\sigma \in \partial\varphi(\nabla u)$, $\lambda = -\text{div }\sigma$ and

$$\int_{B_1(0)} \varphi(\nabla u) - \lambda u = 0. \tag{5.7}$$

In this case, thanks to Proposition 5.6, u and 0 are solutions of $\mathcal{P}_\lambda$ on $W_0^{1,2}(B_1(0))$. By the definition of u and the Euler–Lagrange equation, we have necessarily that $\sigma = -\frac{x}{|x|}$ and $\lambda = \frac{1}{|x|}$ when $|x| \geq \frac{1}{2}$. It remains to find σ and λ on $B_{\frac{1}{2}}(0)$.
Since we want 0 to be a solution, in polar coordinates we must have that

$$\begin{aligned} 0 &= \int_0^1 (|f'(r)| - \lambda(r)f(r))r\mathrm{d}r = \int_0^{\frac{1}{2}} -\lambda(r)f(r)r\mathrm{d}r - \int_{\frac{1}{2}}^{1} rf'(r) + f(r)\mathrm{d}r \\ &= -\int_0^1 f'(r)r\mathrm{d}r + \int_0^{\frac{1}{2}} -\lambda(r)f(r)r\mathrm{d}r - \int_{\frac{1}{2}}^{1} f(r)\mathrm{d}r. \end{aligned}$$

With an integration by parts we have

$$0 = \int_0^{\frac{1}{2}} (1 - r\lambda(r))f(r)\mathrm{d}r = \frac{1}{2}\int_0^{\frac{1}{2}} (1 - r\lambda(r))\mathrm{d}r.$$

Then

$$\int_0^{\frac{1}{2}} \lambda(r)r\mathrm{d}r = \frac{1}{2}. \tag{5.8}$$

Conversely, if λ satisfies the equation (5.8) then (5.7) holds true. We are now looking for a σ such that $|\sigma| \leq 1$ on $B(0, \frac{1}{2})$ and $\lambda = -\operatorname{div} \sigma$ satisfies (5.8). In polar coordinates, we have $-\lambda = \operatorname{div} \sigma = \frac{1}{r}\partial_r(r\sigma_r) + \frac{1}{r}\partial_\theta\sigma_\theta$. Since $\sigma_r = -1$ and $\sigma_\theta = 0$ when $|x| \geq \frac{1}{2}$, we set $\sigma_\theta \equiv 0$ on $B_1(0)$. Thus, $-\lambda = \frac{1}{r}\partial_r(r\sigma_r)$ and (5.8) is satisfied for all smooth σ_r, $|\sigma_r| \leq 1$ such that $\sigma_r(\frac{1}{2}) = -1$ and $\sigma_r'(\frac{1}{2}) = 0$. Moreover, λ is Lipschitz on $B_1(0)$. Since λ has to be bounded and positive, we require that σ_r is negative, strictly decreasing and growth at 0 in such a way that $\lim\limits_{r\to 0} \frac{\sigma_r}{r}$ is finite. Since we want λ to be $\mathcal{C}^1$, we have to take σ_r such that $\lim\limits_{r\to \frac{1}{2}+} \sigma_r''(r) = 0$ and $\sigma_r = -r$ around 0. In this case $\nabla\lambda(0) = 0$ and $\nabla\lambda(x) = -\frac{x}{|x|^3}$ for every x such that $|x| = \frac{1}{2}$.

Since $|\sigma| \leq 1$ when $|x| \leq \frac{1}{2}$, we have $\sigma \in \partial\varphi(\nabla u)$. Since σ is continuous, we have div $\sigma = -\lambda$. Thus, 0 and u are two solutions of $\mathcal{P}_\lambda$ on $W_0^{1,2}(B_1(0))$. □

In the previous example $U = \emptyset$ but even when $U \neq \emptyset$ we can have two solutions.

Proposition 5.10. *There exists $\lambda \in \mathcal{C}^0(\overline{\Omega})$, $\lambda > 0$ such that the solutions of $\mathcal{P}_\lambda$ on $W_0^{1,2}(\Omega)$ are not unique and $U \neq \emptyset$.*

Proof. Let us consider the following functions:

$$u(x) = \begin{cases} \frac{5}{4} + \frac{1}{2}\ln(2) & \text{if } |x| < \frac{1}{2}, \\ \frac{7}{4} + \frac{1}{2}\ln(2) - |x| & \text{if } \frac{1}{2} \leq |x| \leq 1, \\ 1 + \frac{1}{2}\ln(2) - \frac{1}{2}\ln(|x|) - \frac{|x|^2}{4} & \text{if } |x| > 1 \end{cases}$$

and

$$v(x) = \begin{cases} \frac{3}{4} + \frac{1}{2}\ln(2) & \text{if } |x| \leq 1, \\ 1 + \frac{1}{2}\ln(2) - \frac{1}{2}\ln(|x|) - \frac{|x|^2}{4} & \text{if } |x| > 1, \end{cases}$$

defined on $\Omega := B_2(0) \subset \mathbb{R}^2$.

Here, the set U is $\{x \in B_2(0) \,/\, |x| > 1\}$ and we fix $\sigma = \nabla u = \nabla v$ on U. Hence, we define $\lambda(x) = -\operatorname{div} \sigma(x) = 1$ on this set. For $|x| \leq 1$, u and v are up to a constant the same functions as in the previous proof. Thus, since there is no continuity problem on $\partial B_1(0)$, we can take σ and λ as in the previous proof on $B_1(0)$.

Hence, u and v are solutions of the same problem $\mathcal{P}_\lambda$ on $W_0^{1,2}(B_2(0))$ with $\lambda(x) = \frac{1}{|x|}$ on $B_1(0)\backslash B_{\frac{1}{2}}(0)$, $\lambda(x) = 1$ on $B_2(0)\backslash B_1(0)$ and λ defined as in the previous proposition on $B_{\frac{1}{2}}(0)$. □

Definition 5.11. We say that a solution u of $\mathcal{P}_\lambda$ is special when $|\nabla u| \in \{0\} \cup (1, +\infty)$ *a.e.* in Ω.

Remark 5.12. Before proving the uniqueness when λ is constant in [3], it was proved in [1] that if there exists a special solution, then it is the unique solution. In the last proof v is a special solution, then when λ is not constant the existence of a special solution does not guarantee the uniqueness.

Proposition 5.13. *It is possible to have two solutions even if each level set intersects $\overline{U}$.*

Proof. For $N=2$ and $\Omega:=B_1(0)$, we introduce

$$A_n=\{x\in B_1(0),\ |\sin(n)|-\epsilon^{n+1}<|x|<|\sin(n)|+\epsilon^{n+1}\}\text{ for }n\in\mathbb{N}\text{ and }A:=\bigcup_{n\in\mathbb{N}}A_n.$$

The set A is dense in $B_1(0)$ but is not equal to $B_1(0)$ for ϵ small enough. We call $U_0:=B_{r_0}(0)$ the connected component of A that contains 0 and $U:=A\backslash U_0$. By definition of A and its density, we have $|\partial U|>0$. Since U is an open set we have $U=\bigcup_{i\in\mathbb{N}^*}U_i$ with $U_i:=\{x\in\Omega,\ r_i<|x|<R_i\}$.

We construct two radial functions $u,v\in W_0^{1,2}(B_1(0))$, regular on U. We introduce $\tilde{u},\tilde{v}:[0,1]\to\mathbb{R}$ such that $u(x)=\tilde{u}(|x|)$ and $v(x)=\tilde{v}(|x|)$. We start by defining $\tilde{u}'$ and $\tilde{v}'$:

- If $0\le r<r_0$, then $\tilde{u}'(r)=\tilde{v}'(r)=0$.
- If there exists $i\in\mathbb{N}^*$ such that $r_i<r<R_i$, then $\tilde{u}'(r)=\tilde{v}'(r)=f_i(r):=\gamma(R_i-r_i)^2f\big(\frac{r-r_i}{R_i-r_i}\big)-1$ with $f(t)=-t^2(t-1)^2$ and $\gamma>0$ independent of $i\in\mathbb{N}^*$ small enough such that $-\frac{f_i(r)}{r}-f_i'(r)>0$.
- If $r\ge r_0$ and $r\notin(r_i,R_i)$ for every $i\in\mathbb{N}^*$, then $\tilde{u}'(r)=-1$ and $\tilde{v}'(r)=0$.

Since $\tilde{u}'$ and $\tilde{v}'$ are bounded, we can define $\tilde{u}$ and $\tilde{v}$ on $[0,1]$ such that $\tilde{u}(1)=\tilde{v}(1)=0$. It remains to find λ and σ such that u and v are minimizers of $\mathcal{I}_\lambda$ on $W_0^{1,2}(B_1(0))$:

- u and v are constant on the disk U_0. Hence, we define λ and σ on U_0 as in the previous counterexamples by changing $\frac{1}{2}$ by r_0 and adjusting the constants.
- On every U_i, u satisfies $\partial_r u<-1$. We set $\lambda=-\Delta u=-\frac{\partial_r u}{r}-\partial_r^2u$. Hence, $\lambda(x)=\tilde{\lambda}(|x|)$ with $\tilde{\lambda}(r)=-\frac{\tilde{u}'(r)}{r}-\tilde{u}''(r)=-\frac{f_i(r)}{r}-f_i'(r)>0$ for $r_i<r<R_i$. On U_i, we set $\sigma(x)=\nabla u(x)=f_i(|x|)\frac{x}{|x|}$. Hence, div $\sigma=-\lambda$ on U.
- As in the previous example, $\nabla v=0$ and $\nabla u=-\frac{x}{|x|}$ on $\Omega\backslash(U_0\cup U)$. We set $\lambda=\frac{1}{|x|}$ and $\sigma=-\frac{x}{|x|}$ on $\Omega\backslash(U_0\cup U)$.

By definition of f, λ is continuous, strictly positive on Ω and σ is continuous on Ω. It remains to check that div $\sigma=-\lambda$ on $\Omega\backslash(U_0\cup U)$.

For every $i\in\mathbb{N}^*$, we have $|\tilde{\sigma}'(r)|=|f_i'(r)|\le4\gamma$ on (r_i,R_i) and $\int_{r_i}^{R_i}\tilde{\sigma}'(r)\mathrm{d}r=0$.

Moreover, $\tilde{\sigma}=-1$ on $[r_0,1]\backslash\big\{\bigcup_{i\in\mathbb{N}^*}(r_i,R_i)\big\}$. Therefore, by considering the different cases, we have that

$$\tilde{\sigma}(t)-\tilde{\sigma}(s)=\int_s^t\tilde{\sigma}'(r)\chi_{\cup_{i\in\mathbb{N}^*}(r_i,R_i)}\mathrm{d}r$$

for every $t,s\in[r_0,1]$. Hence, $\tilde{\sigma}$ is Lipschitz continuous and $\tilde{\sigma}'=\tilde{\sigma}'\chi_{\cup_{i\in\mathbb{N}^*}(r_i,R_i)}$ a.e. on $[r_0,1]$. Since σ is radial, we have div $\sigma(x)=\tilde{\sigma}'(|x|)+\frac{1}{|x|}\tilde{\sigma}(|x|)$. Thus, by definition of λ we obtain that div $\sigma=-\lambda$.

Hence, u and v are two different solutions of the same problem $\mathcal{P}_\lambda$ and $\forall s\in\mathbb{R}$ such that $\partial E_s\neq\emptyset$ we have $\partial E_s\cap\overline{U}\neq\emptyset$. □

6. Level sets

In this section, we consider only Lipschitz continuous minimizers. We adapt the definition of the perimeter to our problem in order to demonstrate that the level-sets of a Lipschitz continuous solution are generically $\mathcal{C}^1$ outside $\overline{U}$.

6.1. Pseudo-perimeter and the pseudo-Cheeger problem

In this subsection, we demonstrate that super-level sets of a Lipschitz continuous solution u are solutions of a weighted Cheeger problem. To do so, we use the function $\max(1,|\nabla\varphi(\nabla u)|)$ where u is a Lipschitz minimizer of $\mathcal{P}_\lambda$. By Lemma 5.2, the function $\max(1,|\nabla\varphi(\nabla u)|)$ is continuous on Ω. Moreover, it does not depend on the choice of the minimizer by Lemma 2.4. Therefore, $\max(1, |\nabla\varphi(\nabla u)|)$ is intrinsic to the problem $\mathcal{P}_\lambda$ and we can introduce the following quantity:

Definition 6.1. For every Caccioppoli set $F \subset \Omega$ and every measurable open set $V \subset \Omega$ we introduce:

$$\widetilde{Per}(F, V) = \int_{\partial^* F \cap V} \max(1, |\nabla\varphi(\nabla u)|) \mathrm{d}\mathcal{H}^{N-1},$$

where u is a Lipschitz continuous minimizer of $\mathcal{P}_\lambda$. We call $\widetilde{Per}(F, V)$ the *pseudo-perimeter* of F in V.

In the following proposition, we use the notation ν_F which is defined in Definition 3.5.

Proposition 6.2. *Let $V \Subset \Omega$ be an open set and F a Caccioppoli set such that $\widetilde{Per}(F, V) < \infty$. Then,*

$$\widetilde{Per}(F, V) = \sup_{g \in \mathcal{C}_0^1(V;\mathbb{R}^N), |g| \leq 1} \int_{\partial^* F \cap V} \max(1, |\nabla\varphi(\nabla u)|) \langle g, \nu_F \rangle \mathrm{d}\mathcal{H}^{N-1}.$$

Proof. We have that

$$\widetilde{Per}(F, V) \geq \sup_{g \in \mathcal{C}_0^1(V;\mathbb{R}^N), |g| \leq 1} \int_{\partial^* F \cap V} \max(1, |\nabla\varphi(\nabla u)|) \langle g, \nu_F \rangle \mathrm{d}\mathcal{H}^{N-1}.$$

By [8], Section 2, Lemma 1 there exists a sequence $(g_k)_{k\in\mathbb{N}} \in \mathcal{C}_0^1(V; \mathbb{R}^N)$, with $|g_k| \leq 1$ such that

$$\langle g_k, \nu_F \rangle \to 1 \text{ in } L^1_{\mathcal{H}^{N-1}}(\partial^* F \cap V).$$

Since $\max(1, |\nabla\varphi(\nabla u)|)$ is continuous by Lemma 5.2 and $\widetilde{Per}(F, V) < +\infty$, we can apply the dominated convergence theorem to obtain:

$$\int_{\partial^* F \cap V} \max(1, |\nabla\varphi(\nabla u)|) \langle g_k, \nu_F \rangle \mathrm{d}\mathcal{H}^{N-1} \to \int_{\partial^* F \cap V} \max(1, |\nabla\varphi(\nabla u)|) \mathrm{d}\mathcal{H}^{N-1} = \widetilde{Per}(F, V),$$

when $k \to +\infty$. Thus,

$$\widetilde{Per}(F, V) \leq \sup_{g \in \mathcal{C}_0^1(V;\mathbb{R}^N), |g| \leq 1} \int_{\partial^* F \cap V} \max(1, |\nabla\varphi(\nabla u)|) \langle g, \nu_F \rangle \mathrm{d}\mathcal{H}^{N-1}.$$

□

In the following proposition, we prove that the *pseudo-perimeter* is lower semi-continuous. Our proof draws inspiration from [8], Section 2, Lemma 2.

Proposition 6.3. *Let $V \Subset \Omega$ be an open set, $(F_k)_{k\in\mathbb{N}}$ a sequence of Caccioppoli sets and F a Caccioppoli set such that $\chi_{F_k} \to \chi_F$ when $k \to \infty$ in $L^1(\Omega)$. We have*

$$\widetilde{Per}(F, V) \leq \liminf_{k\to\infty} \widetilde{Per}(F_k, V).$$

Proof. Let us consider $f \in \mathcal{C}^1(V, \mathbb{R}_+)$, $\psi \in \mathcal{C}_0^1(V; \mathbb{R}^N)$, $|g| \leq 1$. Since

$$\int_{\partial^* F_k \cap V} f \langle \psi, \nu_{F_k} \rangle \mathrm{d}\mathcal{H}^{N-1} = -\int_V \chi_{F_k} \mathrm{div}(f\psi) \mathrm{d}x,$$

and $\chi_{F_k} \to \chi_F$ when $k \to \infty$ in $L^1(\Omega)$, the dominated convergence theorem gives that

$$\int_{\partial^* F \cap V} f\langle \psi, \nu_F\rangle \mathrm{d}\mathcal{H}^{N-1} = \lim_{k\to\infty} \int_{\partial^* F_k \cap V} f\langle \psi, \nu_{F_k}\rangle \mathrm{d}\mathcal{H}^{N-1}.$$

Thus,

$$\int_{\partial^* F \cap V} f\langle \psi, \nu_F\rangle \mathrm{d}\mathcal{H}^{N-1} \leq \liminf_{k\to\infty} \sup_{g\in\mathcal{C}_0^1(V;\mathbb{R}^N), |g|\leq 1} \int_{\partial^* F_k \cap V} f\langle g, \nu_{F_k}\rangle \mathrm{d}\mathcal{H}^{N-1}.$$

This inequality is true for every $\psi \in \mathcal{C}_0^1(V;\mathbb{R}^N)$, $|\psi| \leq 1$, then

$$\sup_{\psi\in\mathcal{C}_0^1(V;\mathbb{R}^N), |\psi|\leq 1} \int_{\partial^* F \cap V} f\langle \psi, \nu_F\rangle \mathrm{d}\mathcal{H}^{N-1} \leq \liminf_{k\to\infty} \sup_{g\in\mathcal{C}_0^1(V;\mathbb{R}^N), |g|\leq 1} \int_{\partial^* F_k \cap V} f\langle g, \nu_{F_k}\rangle \mathrm{d}\mathcal{H}^{N-1}.$$

Let us introduce

$$H_f(F) := \sup_{g\in\mathcal{C}_0^1(V;\mathbb{R}^N), |g|\leq 1} \int_{\partial^* F \cap V} f\langle g, \nu_F\rangle \mathrm{d}\mathcal{H}^{N-1}.$$

Then H_f is lower semi-continuous for every $f \in \mathcal{C}^1(V, \mathbb{R}_+)$. Since $\max(1, |\nabla\varphi(\nabla u)|)$ is continuous and nonnegative, there exists a sequence of nonnegative $\mathcal{C}^1$ functions $(f_n)_{n\in\mathbb{N}}$ converging uniformly to $\max(1, |\nabla\varphi(\nabla u)|)$ on $\overline{V}$ such that for every $n \in \mathbb{N}$, $f_n \leq \max(1, |\nabla\varphi(\nabla u)|)$. The proof of Proposition 6.2 applied to $(f_n)_{n\in\mathbb{N}}$ implies that

$$H_{\max(1,|\nabla\varphi(\nabla u)|)}(E) = \widetilde{Per}(E, V) \geq \sup_{n\in\mathbb{N}} \int_{\partial^* E \cap V} f_n \mathrm{d}\mathcal{H}^{N-1} = \sup_{n\in\mathbb{N}} H_{f_n}(E),$$

for every Caccioppoli set E. Hence, $H_{\max(1,|\nabla\varphi(\nabla u)|)}(E) \geq \sup_{n\in\mathbb{N}} H_{f_n}(E)$ for every Caccioppoli set E.

Let us consider $(g_n)_{n\in\mathbb{N}}$ a maximizing sequence for $H_{\max(1,|\nabla\varphi(\nabla u)|)}(E)$. Since $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $\max(1, |\nabla\varphi(\nabla u)|)$ we have

$$\lim_{n\to\infty} \int_{\partial^* E \cap V} f_n\langle g_n, \nu_E\rangle \mathrm{d}\mathcal{H}^{N-1} = H_{\max(1,|\nabla\varphi(\nabla u)|)}(E). \tag{6.1}$$

Thus, $H_{\max(1,|\nabla\varphi(\nabla u)|)}(E) \leq \sup_{n\in\mathbb{N}} H_{f_n}(E)$ for every Caccioppoli set E. We proved that

$$H_{\max(1,|\nabla\varphi(\nabla u)|)} = \sup_{n\in\mathbb{N}} H_{f_n}.$$

Hence, $H_{\max(1,|\nabla\varphi(\nabla u)|)}$ is lower semi-continuous as the supremum of lower semi-continuous functions. Thus, by Proposition 6.2 we have

$$\widetilde{Per}(F, V) \leq \liminf_{k\to\infty} \widetilde{Per}(F_k, V).$$

□

We generalize to Ω, [3], Proposition 2.7 that states that the super-level sets satisfy a minimization problem:

Theorem 6.4. *Let u be a Lipschitz continuous solution of $\mathcal{P}_\lambda$. Then for a.e. $s \in \mathbb{R}$, for every set $F \subset \Omega$ with finite perimeter in Ω such that $F \Delta E_s \Subset \Omega$, we have*

$$\widetilde{Per}(E_s, \Omega) - \int_{E_s} \lambda \mathrm{d}x \leq \widetilde{Per}(F, \Omega) - \int_F \lambda \mathrm{d}x.$$

Proof. We divide the proof into three steps.

Step 1 The first part is identical to [3], Proposition 2.7, **Step 1** with an approximation of σ by smooth functions.

We extend σ by 0 outside Ω and we define $\sigma_n := \sigma * \rho_n$, where $(\rho_n)_{n\geq 1} \subset \mathcal{C}_0^\infty(B_{1/n})$ is a sequence of mollifiers. Then, up to a sub-sequence, σ_n converges to σ a.e. on Ω and $\forall K \Subset \Omega$ compact, $\forall n \geq \dfrac{1}{\text{dist }(K, \partial\Omega)}$,

$$\text{div } \sigma_n = (\text{div } \sigma) * \rho_n = -\lambda * \rho_n =: -\lambda_n \quad \text{a.e. on } K.$$

Then we prove that there exists a sub-sequence that we do not relabel, $(\sigma_n)_{n\geq 1}$ such that for *a.e.* $s \in \mathbb{R}$, $\forall K \Subset \Omega$,

$$\lim_{n\to+\infty} \int_{K\cap\partial^* E_s} |\sigma_n - \sigma| = 0.$$

If $K \Subset \Omega$, then by the co-area formula (3.5), for every $n \geq 1$,

$$\int_{\mathbb{R}} \int_{K\cap\partial^* E_s} |\sigma_n - \sigma| \, \mathrm{d}\mathcal{H}^{N-1} \mathrm{d}s = \int_K |\nabla u||\sigma_n - \sigma|. \tag{6.2}$$

The integrand in the right-hand side is bounded from above by

$$\|\nabla u\|_{L^\infty(\Omega)}(\|\sigma_n\|_{L^\infty(\mathbb{R}^N)} + \|\sigma\|_{L^\infty(\mathbb{R}^N)}) \leq 2\|\nabla u\|_{L^\infty(\Omega)}\|\sigma\|_{L^\infty(\Omega)}.$$

Since $(\sigma_n)_{n\in\mathbb{N}}$ converges a.e. to σ on Ω, we can apply the dominated convergence theorem to obtain that

$$\lim_{n\to+\infty} \int_K |\nabla u||\sigma_n - \sigma| = 0.$$

By equation (6.2), there exists a sub-sequence that we do not relabel, $(\sigma_n)_{n\in\mathbb{N}^*}$ such that for a.e. $s \in \mathbb{R}$:

$$\lim_{n\to+\infty} \int_{K\cap\partial^* E_s} |\sigma_n - \sigma| \, \mathrm{d}\mathcal{H}^{N-1} = 0. \tag{6.3}$$

Let $(K_m)_{m\geq 1}$ be an increasing sequence of compact subsets of Ω such that $\bigcup_{m\geq 1} \text{int } K_m = \Omega$. We apply the previous reasoning to each K_m. Hence, with a diagonal process, we can extract a sub-sequence $(\sigma_n)_{n\geq 1}$ such that for a.e. $s \in \mathbb{R}$, (6.3) is true for every K_m. Since every compact subset $K \Subset \Omega$ is inside K_m for m large enough, (6.3) is valid for every compact subsets of Ω.

Step 2 For every $F \subset \Omega$ as in the statement of the proposition, for a.e. $s \in \mathbb{R}$ and for every $\theta \in \mathcal{C}_0^\infty(\Omega)$ such that $\theta \equiv 1$ on $F \Delta E_s$ and $0 \le \theta \le 1$, we claim that:

$$\int_{\mathbb{R}^N} \lambda(\chi_F - \chi_{E_s})\theta \le \int_{\partial^* F} \theta \max(1, |\nabla\varphi(\nabla u)|) \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial^* E_s} \theta \max(1, |\nabla\varphi(\nabla u)|) \mathrm{d}\mathcal{H}^{N-1}. \tag{6.4}$$

We prove this inequality in the remaining part of **Step 2**. By the co-area formula and Remark 5.4, we have that $\nabla u(x) \neq 0$ and $\langle \sigma(x), \dfrac{\nabla u(x)}{|\nabla u(x)|}\rangle = \max(1, |\nabla\varphi(\nabla u)|)$ for a.e. $s \in \mathbb{R}$ and for $\mathcal{H}^{N-1}$ a.e. $x \in \partial^* E_s$. We fix any s for which this property as well as (3.6) and (6.3) hold true.

Since $\nabla\theta = 0$ on $E_s \Delta F$, we get the following equality:

$$-\int_{\mathbb{R}^N} (\chi_F - \chi_{E_s})\theta \mathrm{div}\ \sigma_n = -\int_{\mathbb{R}^N} (\chi_F - \chi_{E_s}) \mathrm{div}\ (\theta\sigma_n). \tag{6.5}$$

But $\theta\sigma_n \in \mathcal{C}_0^\infty(\Omega)$ and for every $n > \frac{1}{\text{dist (supp } \theta, \partial\Omega)}$,

$$|\theta\sigma_n| \le \theta(|\sigma| * \rho_n) \le \theta(\max(1, |\nabla\varphi(\nabla u)|) * \rho_n).$$

Hence,

$$\left|\int_{\mathbb{R}^N} \chi_F \mathrm{div}\ (\theta\sigma_n)\right| = \left|\int_{\partial^* F} \theta\langle\sigma_n, \nu_F\rangle \mathrm{d}\mathcal{H}^{N-1}\right| \le \int_{\partial^* F \cap \Omega} \theta \max(1, |\nabla\varphi(\nabla u)|) * \rho_n \mathrm{d}\mathcal{H}^{N-1}. \tag{6.6}$$

Since $\max(1, |\nabla\varphi(\nabla u)|)$ is continuous by Lemma 5.2, when $n \to \infty$, we have $\max(1, |\nabla\varphi(\nabla u)|) * \rho_n \to \max(1, |\nabla\varphi(\nabla u)|)$ uniformly on supp θ. Thus, by the dominated convergence theorem we have

$$\limsup_{n\to\infty} \left|\int_{\mathbb{R}^N} \chi_F \mathrm{div}\ (\theta\sigma_n)\right| \le \int_{\partial^* F} \theta \max(1, |\nabla\varphi(\nabla u)|) \mathrm{d}\mathcal{H}^{N-1}. \tag{6.7}$$

Since we selected $s \in \mathbb{R}$ such that (3.6) is satisfied,

$$\int_{\mathbb{R}^N} \chi_{E_s} \mathrm{div}\ (\theta\sigma_n) = -\int_{\mathbb{R}^N} \theta\langle\sigma_n, \frac{D\chi_{E_s}}{|D\chi_{E_s}|}\rangle d|D\chi_{E_s}| = -\int_{\mathbb{R}^N} \theta\langle\sigma_n, \frac{\nabla u}{|\nabla u|}\rangle d|D\chi_{E_s}|.$$

With (3.1) we obtain that

$$\int_{\mathbb{R}^N} \chi_{E_s} \mathrm{div}\ (\theta\sigma_n) = -\int_{\partial^* E_s} \theta\langle\sigma_n, \frac{\nabla u}{|\nabla u|}\rangle \mathrm{d}\mathcal{H}^{N-1}. \tag{6.8}$$

Then, since $0 \le \theta \le 1$ we get that

$$\left|\int_{\partial^* E_s} \theta\langle\sigma_n, \frac{\nabla u}{|\nabla u|}\rangle \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial^* E_s} \theta\langle\sigma, \frac{\nabla u}{|\nabla u|}\rangle \mathrm{d}\mathcal{H}^{N-1}\right| \le \int_{\text{supp } \theta \cap \partial^* E_s} |\sigma_n - \sigma| \, \mathrm{d}\mathcal{H}^{N-1}.$$

Thanks to **Step 1**, the right-hand side goes to 0 when $n \to +\infty$. Thus,

$$\lim_{n\to+\infty} \int_{\partial^* E_s} \theta\langle\sigma_n, \frac{\nabla u}{|\nabla u|}\rangle \mathrm{d}\mathcal{H}^{N-1} = \int_{\partial^* E_s} \theta\langle\sigma, \frac{\nabla u}{|\nabla u|}\rangle \mathrm{d}\mathcal{H}^{N-1}. \tag{6.9}$$

But for $\mathcal{H}^{N-1}$ a.e. $x \in \partial^* E_s \cap \Omega$, $\langle \sigma(x), \nabla u(x)/|\nabla u(x)|\rangle = \max(1, |\nabla\varphi(\nabla u)|)$. Hence, it follows from (6.8) and (6.9) that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \chi_{E_s} \mathrm{div}\,(\theta\sigma_n) = -\int_{\partial^* E_s} \theta \max(1, |\nabla\varphi(\nabla u)|) \mathrm{d}\mathcal{H}^{N-1}. \tag{6.10}$$

Since $F\Delta E_s$ is compactly contained in Ω, we have for n large enough, div $\sigma_n = -\lambda_n \to -\lambda$ *a.e.* on $F\Delta E_s$ when $n \to \infty$. Thus,

$$\int_{\mathbb{R}^{\mathbb{N}}} (\chi_F - \chi_{E_s})\theta \text{ div } \sigma_n \mathrm{d}x \to \int_{\mathbb{R}^{\mathbb{N}}} (\chi_F - \chi_{E_s})\theta\lambda \mathrm{d}x. \tag{6.11}$$

The claim (6.4) is a consequence of (6.5), (6.6), (6.10) and (6.11).

Step 3 We conclude as in [3], Proposition 2.7. We replace the function θ introduced in the previous step by a sequence $(\theta_k)_{k\geq 1}$ such that each θ_k satisfies the same assumptions as θ, and $\theta_k \to 1$ a.e. on Ω. By letting $k \to +\infty$ in (6.4), we obtain

$$\int_{\mathbb{R}^N} \lambda(\chi_F - \chi_{E_s}) \leq \liminf_{k\to+\infty} \left(\int_{\partial^* F} \theta_k \max(1, |\nabla\varphi(\nabla u)|) \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial^* E_s} \theta_k \max(1, |\nabla\varphi(\nabla u)|) \mathrm{d}\mathcal{H}^{N-1} \right).$$

By the dominated convergence theorem we have

$$\int_{\mathbb{R}^N} \lambda(\chi_F - \chi_{E_s}) \leq \widetilde{\mathrm{Per}}(F, \Omega) - \widetilde{\mathrm{Per}}(E_s, \Omega).$$

Hence, we get

$$\widetilde{\mathrm{Per}}(E_s, \Omega) - \int_{E_s} \lambda \mathrm{d}x \leq \widetilde{\mathrm{Per}}(F, \Omega) - \int_F \lambda \mathrm{d}x.$$

□

Remark 6.5. For $s \in \mathbb{R}$ and F as in the statement of the previous proposition, we consider $V \subset \Omega$ such that $F\Delta E_s \Subset V$. Since F and E_s are identical outside V, we have

$$\widetilde{\mathrm{Per}}(E_s, V) - \int_{E_s\cap V} \lambda \mathrm{d}x \leq \widetilde{\mathrm{Per}}(F, V) - \int_{F\cap V} \lambda \mathrm{d}x.$$

Proposition 6.6. *If $F \Subset \Omega$ is a set of finite perimeter, then*

$$\int_F \lambda \mathrm{d}x \leq \widetilde{Per}(F, \Omega).$$

Moreover, for a.e. $s \in \mathbb{R}$, if $E_s \Subset \Omega$, then

$$\int_{E_s} \lambda \mathrm{d}x = \widetilde{Per}(E_s, \Omega).$$

Proof. The first part of the proposition is a consequence of the equation (6.7) with $\theta \equiv 1$ on F. The second part of the proposition can be proven using the same arguments as outlined in the preceding proof, specifically the equations (6.8) and (6.10), when applied to E_s.

□

Remark 6.7. The minimization problem satisfied by the super-level sets is a kind of analogue of the problem $\mathcal{P}_\lambda$ for sets. When $\psi \equiv 0$, we have for *a.e.* $s > 0$,

$$0 = \widetilde{Per}(E_s, \Omega) - \int_{E_s} \lambda \mathrm{d}x = \inf_{F \Subset \Omega} \widetilde{Per}(F, \Omega) - \int_F \lambda \mathrm{d}x.$$

If we call *pseudo-Cheeger set* a set $F \subset \Omega$ such that

$$\frac{\widetilde{Per}(F, \Omega)}{\int_F \lambda \mathrm{d}x} = \inf_{D \Subset \Omega} \frac{\widetilde{Per}(D, \Omega)}{\int_D \lambda \mathrm{d}x},$$

then for a.e. $s > 0$, E_s is a *pseudo-Cheeger set*.

6.2. Regularity of the level sets

The minimizing property of the super-level sets of a solution allows us to show Proposition 1.10. This result states that the super-level sets of a solution are $\mathcal{C}^1$, up to $\overline{U}$ and a negligible set:

Proposition 6.8. *Given $\lambda \in L^\infty(\Omega)$, $\lambda \geq 0$ a.e. on Ω, let u be a locally Lipschitz continuous solution of $\mathcal{P}_\lambda$ and U be the open set in Ω defined by $[|\nabla u| > 1]$. Then for a.e. $t \in \mathbb{R}$, there exists an open set W_t in $\Omega \backslash \overline{U}$ such that $W_t \cap \partial^e E_t$ is a $\mathcal{C}^1$ hypersurface and $\mathcal{H}^s(\Omega \setminus (W_t \cup \overline{U})) = 0$ for every $s > N - 8$ with $W_t \cap \partial^e E_t \subset \partial^* E_t$.*

Proof. By Proposition 6.4 and Remark 6.5, we have for a.e $t \in \mathbb{R}$, for every $x \in \Omega$, $\rho > 0$ in order that $B_\rho(x) \subset \Omega$ and every Caccioppoli set F such that $F \Delta E_t \Subset B_\rho(x)$:

$$\widetilde{Per}(E_t, B_\rho(x)) - \int_{E_t \cap B_\rho(x)} \lambda \mathrm{d}x \leq \widetilde{Per}(F, B_\rho(x)) - \int_{F \cap B_\rho(x)} \lambda \mathrm{d}x.$$

But if $B_\rho(x) \subset \Omega \backslash \overline{U}$, then

$$\widetilde{Per}(\cdot, B_\rho(x)) = Per(\cdot, B_\rho(x)).$$

Hence, for a.e $t \in \mathbb{R}$, for every $x \in \Omega \backslash \overline{U}$, $\rho > 0$ such that $B_\rho(x) \subset \Omega \backslash \overline{U}$ and every Caccioppoli set F that satisfies $F \Delta E_t \Subset B_\rho(x)$:

$$Per(E_t, B_\rho(x)) - \int_{E_t \cap B_\rho(x)} \lambda \mathrm{d}x \leq Per(F, B_\rho(x)) - \int_{F \cap B_\rho(x)} \lambda \mathrm{d}x.$$

Then, by [23], Theorems 5.1 and 5.2, there exists an open set W_t in $\Omega \backslash \overline{U}$ such that $\partial^e E_t \cap W_t$ is a $N - 1$ dimensional manifold of class $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha < 1$ and $\mathcal{H}^s(\Omega \setminus (W_t \cup \overline{U})) = 0$ for every $s > N - 8$. Moreover, $W_t \cap \partial^e E_t \subset \partial^* E_t$. □

6.3. Relation between U and the super-level sets

The minimizing property of the super-level sets allows us to show that the level sets intersect the set $\overline{U} = \overline{[|\nabla u| > 1]}$ for some classes of functions λ.

We consider λ Lipschitz continuous on $\overline{\Omega}$, $\lambda > 0$. We assume that there exist $x_\lambda \in \mathbb{R}^N$ and $l > 0$ such that $\forall x \in \Omega$,

$$\lambda(x) - ||\nabla\lambda||_{L^\infty(\Omega)}|x - x_\lambda| \geq l. \tag{6.12}$$

Then we have:

Proposition 6.9. *If $E \Subset \Omega$ is a Caccioppoli set with $|E| > 0$ and*

$$\int_E \lambda = \widetilde{Per}(E, \Omega),$$

then $\partial E \cap \overline{U} \neq \emptyset$.

Proof. If $\partial E \cap \overline{U} = \emptyset$, then

$$\widetilde{Per}(E, \Omega) = Per(E, \Omega),$$

and for $t \geq 1$ close to 1, we have $\partial^e E^t \cap (\partial\Omega \cup \overline{U}) = \emptyset$ with $E^t = t(E - x_\lambda) + x_\lambda$. Thus, we get

$$\int_{E^t} \lambda \leq \widetilde{Per}(E^t, \Omega) = Per(E_t, \Omega) = t^{N-1} Per(E, \Omega) = t^{N-1} \int_E \lambda.$$

Hence, we have

$$\int_E t\lambda(t(x - x_\lambda) + x_\lambda) - \lambda(x)\mathrm{d}x \leq 0.$$

By (6.12) we get a contradiction when $t \to 1$. Thus, $\partial E \cap (\partial\Omega \cup \overline{U}) \neq \emptyset$.

□

Remark 6.10. If $||\nabla\lambda||_{L^\infty(\Omega)} < K \inf_\Omega \lambda$ with $K := \frac{1}{\mathrm{diam}(\Omega)}$, then λ satisfies (6.12) with any $x_\lambda \in \Omega$.

7. Proof of the main theorems

In this section, we consider λ with small oscillations. We assume there exists a constant $\mathcal{L} > 0$ such that

$$||\nabla u||_{L^\infty(\Omega)} \leq \mathcal{L},$$

for every minimizer u of $\mathcal{P}_\lambda$. The following lemma shows that when $\nabla\lambda$ is small, then $U \cup \partial\Omega$ is connected.

Lemma 7.1. *Assume that λ is Lipschitz continuous on $\overline{\Omega}$, $\min_{\overline{\Omega}} \lambda > 0$ and $\|\nabla\lambda\|_{L^\infty(\Omega)} < \frac{1}{G\mathcal{L}} \min_{\overline{\Omega}} \lambda^2$ with $G = \sup_{x\in(1,\mathcal{L})} g''(x) + (N-1)\frac{g'(x)}{x}$. Then, each connected component U_0 of the open set U satisfies*

$$\partial U_0 \cap \partial\Omega \neq \emptyset.$$

The proof of this lemma is inspired by the ones of [1], Proposition 7.3 and [3], Lemma 3.3.

Proof. Assume by contradiction that $\overline{U_0} \subset \Omega$. The function $|\nabla u||_{U_0}$ can be extended as a uniformly continuous function on $\overline{U_0}$ and $|\nabla u| \equiv 1$ on ∂U_0, see Lemma 5.2. We introduce $\widetilde{U_0} \Subset U_0$ a smooth set. By continuity of ∇u on U, there exists $\delta > 0$ such that $|\nabla u(x)| \geq 1 + 2\delta$ for every $x \in \widetilde{U_0}$. We regularize φ and λ by convolution to obtain a sequence $(u_\epsilon)_{\epsilon>0}$ of smooth solutions on Ω to approximated problems of $\mathcal{P}_\lambda$. By equations (100) and (102) in [10], $(|\nabla u_\epsilon| - 1 - \delta)_+ \to (|\nabla u_0| - 1 - \delta)_+$ uniformly on $\overline{\widetilde{U_0}}$ with u_0 a minimizer of $\mathcal{P}_\lambda$ on $W^{1,p}_\psi(\Omega)$. By Proposition 2.4 and the fact that $|\nabla u(x)| \geq 1 + 2\delta$ on $\widetilde{U}_0$, we have that $|\nabla u_\epsilon| \to |\nabla u|$ uniformly on $\overline{\widetilde{U}_0}$. If $||\nabla\lambda||_{L^\infty(\Omega)} < \frac{\min_\Omega \lambda^2}{G\mathcal{L}}$, then (15.11) in [12] is satisfied for u_ϵ with ϵ small enough. Hence, we can apply [12], Theorem 15.1 to obtain $\sup_{\widetilde{U}_0} |\nabla u_\epsilon| = \sup_{\partial\widetilde{U}_0} |\nabla u_\epsilon|$. Hence, $\sup_{\widetilde{U}_0} |\nabla u| = \sup_{\partial\widetilde{U}_0} |\nabla u|$. Since $\widetilde{U}_0$ can be any smooth subset of U_0, we have $\sup_{U_0} |\nabla u| = \sup_{\partial U_0} |\nabla u| = 1$. Since $|\nabla u| > 1$ on U_0 that is a contradiction and $\partial U_0 \cap \partial\Omega \neq \emptyset$ as desired. □

Remark 7.2. When λ is constant, $\nabla\lambda = 0$. Hence, every λ strictly positive satisfies the assumptions of the previous lemma. Thus, there is no need for the assumption that the minimizers are globally Lipschitz continuous in this case.

The following lemma coming from [3], Lemma 3.8 is useful to prove that generically, other solutions are constant on the boundary of the connected components of E_s where $E_s := [u \geq s]$.

Lemma 7.3. *Let V be a bounded open subset of $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus V$ is connected. Let M be a $\mathcal{C}^1$ orientable hypersurface compactly contained in V. If $\mathcal{H}^{N-1}(M) < \infty$ and $\mathcal{H}^{N-2}(\overline{M} \setminus M) = 0$, then there exists a non-empty connected open set $E \subset V$ such that $\partial E \subset \overline{M}$.*

The following proposition demonstrates that the level-sets of two solutions coincide generically.

Proposition 7.4. *Let u and v be two Lipschitz continuous solutions of $\mathcal{P}_\lambda$ on $W^{1,p}_\psi(\Omega)$ with λ satisfying (6.12) and the assumptions of Lemma 7.1. For a.e. $t \in \mathbb{R}$, for every connected component M of $\partial^e E_t(u) \cap W_t(u)$ we have $\overline{M} \cap (\partial\Omega \cup \overline{U}) \neq \emptyset$ where $W_t(u)$ is defined in Proposition 6.8. Moreover, $u - v$ is constant on $\overline{M}$.*

Proof. Let S be the set of those $x \in \Omega$ such that u or v is not differentiable at x or $\nabla u(x) = 0$.

By the co-area formula for Lipschitz continuous functions, see [11], Theorem 1, Section 3.4.2 we have

$$0 = \int_S |\nabla u| \mathrm{d}x = \int_{\mathbb{R}} \mathcal{H}^{N-1}(u^{-1}(t) \cap S) dt.$$

Hence, for *a.e.* $t \in \mathbb{R}$, for $\mathcal{H}^{N-1}$ *a.e.* $x \in u^{-1}(t)$ we have that $\nabla u(x) \neq 0$. By proposition 6.8, for a.e. $t \in \mathbb{R}$, there exists an open set $W_t(u)$ such that $\partial^e E_t(u) \cap W_t(u)$ is an orientable $\mathcal{C}^1$ hypersurface with $\mathcal{H}^s(\Omega \setminus (W_t(u) \cup \overline{U})) = 0$ for every $s > N - 8$. Thus, a.e $t \in \mathbb{R}$ satisfy all these conditions.

For such a t, we consider M a connected component of $\partial^e E_t(u) \cap W_t(u)$. Since M is a $\mathcal{C}^1$ hypersurface, $\nabla v(x)$ is orthogonal to M at x. Hence, u and v are constant on M and on $\overline{M}$.

If we assume that $\overline{M} \cap (\partial\Omega \cup \overline{U}) = \emptyset$, we can show as in [3], Theorem 1.1, p. 18 that $\mathcal{H}^{N-2}(\overline{M} \setminus M) = 0$. Then we can apply Lemma 7.3 with $V = \Omega$. Thus, there exists a non-empty connected open set $E \subset \Omega$ such that $\partial E \subset \overline{M}$. Since $u \equiv t$ on ∂E by Lemma 2.2, $u \geq t$ on E. For every $s > t$, we consider $F_s := [u \geq s] \cap E$. We have that $F_s \subset E$, hence, $F_s \cap \partial\Omega = \emptyset$. If $F_s \cap \overline{U} \neq \emptyset$, then there exists a connected component U_i of U such that $U_i \cap E \neq \emptyset$. In this case, by Lemma 7.1 we have that $\partial E \cap U_i \neq \emptyset$ which is a contradiction with $\partial E \subset M$ and $\overline{M} \cap (\partial\Omega \cup \overline{U}) = \emptyset$. Thus, $F_s \cap \overline{U} = \emptyset$. By Proposition 6.6 applied to F_s with E instead of Ω, we obtain that $\int_{F_s} \lambda = \widetilde{Per}(F_s, E) = \widetilde{Per}(F_s, \Omega)$. By Proposition 6.9, $F_s = \emptyset$ for a.e. $s > t$. Hence, $u = t$ on E and $\nabla u = 0$ a.e. on E. Then $E \subset u^{-1}(t) \cap S$. Thus, E is a non-empty open set with $|E| \leq |u^{-1}(t) \cap S| = 0$. That is a contradiction. Hence, $\overline{M} \cap (\partial\Omega \cup \overline{U}) \neq \emptyset$. □

Finally, we can prove the two main theorems:

Proof of Theorem 1.3 and Theorem 1.6. We consider two minimizers u and v. We assume that λ satisfies the assumptions of Lemma 7.1 and Remark 6.10. Since $\nabla u = \nabla v$ on U we have that $u = v$ on $\partial\Omega \cap \overline{U}$. By Proposition 7.4 and the continuity of u and v that is also the case on $\partial^e E_s \cap W_s$ for a.e. $s \in \mathbb{R}$ with $E_s = [u \geq s]$. By (3.3) we have

$$\int_{[u\neq v]} |\nabla u| = \int_{\mathbb{R}} \mathcal{H}^{N-1}(\partial^* E_s \cap [u \neq v]).$$

We have for a.e. $s \in \mathbb{R}$, $\partial^* E_s \cap [u \neq v] \subset \partial^e E_s \backslash (\partial^e E_s \cap (W_s \cup \overline{U}))$. Since $\mathcal{H}^{N-1}(\partial^e E_s \backslash (\partial^e E_s \cap (W_s \cup \overline{U}))) = 0$, we obtain that $\mathcal{H}^{N-1}(\partial^* E_s \cap [u \neq v]) = 0$. Hence, $\nabla u = 0$ a.e. on $[u \neq v]$. With the same arguments, we can prove that this is also the case for ∇v. Thus, $\nabla(u - v) = 0$ a.e. on Ω. Since $u - v$ is Lipschitz continuous on Ω and $u - v = 0$ on $\partial\Omega$, we have $u = v$ on $\overline{\Omega}$. □

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