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OPTIMAL CONTROL FOR THE PANEITZ OBSTACLE PROBLEM

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Abstract. In this paper, we study a natural optimal control problem associated to the Paneitz obstacle problem on closed 4-dimensional Riemannian manifolds. We show the existence of an optimal control which is an optimal state and induces also a conformal metric with prescribed Q-curvature. We show also C^∞-regularity of optimal controls and some compactness results for the optimal controls. In the case of the 4-dimensional standard sphere, we characterize all optimal controls.

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1. Introduction and statement of the results

One of the most important problem in conformal geometry is the problem of finding conformal metrics with a prescribed curvature quantity. An example of curvature quantity which has received a lot of attention in the last decades is the Branson's Q-curvature. It is a Riemannian scalar invariant introduced by Branson–Oersted [4] (see also Branson [3]) for closed four-dimensional Riemannian manifolds.

Given (M, g) a four-dimensional closed Riemannian manifold with Ricci tensor Ric_g, scalar curvature R_g, and Laplace-Beltrami operator Δ_g, the Q-curvature of (M, g) is defined by

$$Q_g = -\frac{1}{12}(\Delta_g R_g - R_g^2 + 3|Ric_g|^2). \tag{1.1}$$

Under the conformal change of metric $g_u = e^{2u} g$ with u a smooth function on M, the Q-curvature transforms in the following way

$$P_g u + 2Q_g = 2Q_{g_u} e^{4u}, \tag{1.2}$$

where P_g is the Paneitz operator introduced by Paneitz [22] and is defined by the following formula

$$P_g \varphi = \Delta_g^2 \varphi + div_g \left((\frac{2}{3} R_g g - 2Ric_g) \nabla_g \varphi \right), \tag{1.3}$$

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where φ is any smooth function on M, div_g is the divergence of with respect to g, and ∇_g denotes the covariant derivative with respect to g. When, one changes conformally g as before, namely by $g_u = e^{2u}g$ with u a smooth function on M, P_g obeys the following simple transformation law

$$P_{g_u} = e^{-4u}P_g. \tag{1.4}$$

The equation (1.2) and the formula (1.4) are analogous to classical ones which hold on closed Riemannian surfaces. Indeed, given a closed Riemannian surface (Σ, g) and $g_u = e^{2u}g$ a conformal change of g with u a smooth function on Σ, it is well know that

$$\Delta_{g_u} = e^{-2u}\Delta_g, \qquad -\Delta_g u + K_g = K_{g_u}e^{2u}, \tag{1.5}$$

where for a background metric $\tilde{g}$ on Σ, $\Delta_{\tilde{g}}$ and $K_{\tilde{g}}$ are respectively the Laplace-Beltrami operator and the Gauss curvature of $(\Sigma, \tilde{g})$. In addition to these, we have an analogy with the classical Gauss-Bonnet formula

$$\int_\Sigma K_g dV_g = 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ and dV_g is the volume form of Σ with respect to g. In fact, we have the Chern—Gauss–Bonnet formula

$$\int_M (Q_g + \frac{|W_g|^2}{8})dV_g = 4\pi^2\chi(M),$$

where W_g denotes the Weyl tensor of (M, g) and $\chi(M)$ is the Euler characteristic of M. Hence, from the pointwise conformal invariance of $|W_g|^2 dV_g$, it follows that $\int_M Q_g dV_g$ is also conformally invariant and will be denoted by κ_g, namely

$$\kappa_g := \int_M Q_g dV_g. \tag{1.6}$$

When $(M, g) = (\mathbb{S}^4, g_{\mathbb{S}^4})$ is the 4-dimensional standard sphere, we have

$$\kappa_g = \kappa_{g_{\mathbb{S}^4}} = 8\pi^2. \tag{1.7}$$

Well-known results of Gursky [13] implies that if the Yamabe invariant $\mathcal{Y}(M, [g]) \geq 0$ and $\kappa_g \geq 0$, then

$$\ker P_g \simeq \mathbb{R} \quad \text{and} \quad P_g \geq 0,$$

where

$$\mathcal{Y}(M, [g]) := \inf_{\tilde{g}\in[g], vol_{\tilde{g}}=1} \int_M R_{\tilde{g}} dV_{\tilde{g}}, \;\; vol_{\tilde{g}} = \int_M dV_{\tilde{g}}, \;\; [g] = \{\tilde{g} = e^{2u}g, \quad u \in C^\infty(M)\}.$$

Moreover, the work of Gursky [13] implies that: if $\mathcal{Y}(M, [g]) \geq 0$, then $\kappa_g \leq 8\pi^2$ with equality if and only if (M, g) is conformally equivalent to $(\mathbb{S}^4, g_{\mathbb{S}^4})$. This latter Aubin's type inequality $\kappa_g \leq 8\pi^2$ was previously derived by Gursky [12] under additional assumption of smallness of the L^2 norm of the Weyl tensor.

Of particular importance in Conformal Geometry is the following Kazdan–Warner type problem. Given a smooth positive function K defined on a closed 4-dimensional Riemannian manifold (M, g), under which conditions on K there exists a Riemannian metric conformal to g with Q-curvature equal to K. Thanks to

(1.2), the problem is equivalent to finding a smooth solution of the fourth-order nonlinear partial differential equation

$$P_g u + 2Q_g = 2K\mathrm{e}^{4u} \qquad in \;\; M. \tag{1.8}$$

Equation (1.8) is usually refereed to as the prescribed Q-curvatre equation and has been studied in the framework of Calculus of Variations, Critical Points Theory, Morse Theory and Dynamical Systems, see [5], [9], [15], [16], [18], [19], [20], and the references therein.

In this paper, we investigate equation (1.8) in the context of Optimal Control Theory. Precisely, we study the following optimal control problem for the Paneitz obstacle problem

$$\text{Finding} \quad u_{min} \in H^2_Q(M) \;\text{ such that }\; I(u_{min}) = \min_{u \in H^2_Q(M)} I(u), \tag{1.9}$$

where

$$I(u) = \langle u, u \rangle_g - \kappa_g \log\left(\int_M K\mathrm{e}^{4T_g(u)} \mathrm{d}V_g \right), \quad u \in H^2_Q(M)$$

with

$$\langle u, u \rangle_g = \int_M \Delta_g u \Delta_g v \mathrm{d}V_g + \frac{2}{3}\int_M R_g \nabla_g u \cdot \nabla_g v \mathrm{d}V_g - \int_M 2Ric_g(\nabla_g u, \nabla_g v)\mathrm{d}V_g$$

$$T_g(u) = \arg \min_{v \in H^2_Q(M),\;\; v \geq u} \langle v, v \rangle_g$$

and

$$H^2_Q(M) := \{u \in H^2(M) : \;\; \int_M Q_g u \mathrm{d}V_g = 0\}$$

with $H^2(M)$ denoting the space of functions on M which are of class L^2, together with their first and second derivatives. Moreover, the symbol

$$\arg \min_{v \in H^2_Q(M),\;\; v \geq u} \langle v, v \rangle_g$$

denotes the unique solution to the minimization problem

$$\min_{v \in H^2_Q(M),\;\; v \geq u} \langle v, v \rangle_g \,,$$

see Lemma 3.1. We remark that for u smooth,

$$\langle u, u \rangle_g = \langle P_g u, u \rangle_{L^2(M)} \,,$$

where $\langle \cdot, \cdot \rangle_{L^2(M)}$ denotes the L^2 scalar product.

Since one of our main our goal is to develop an approach to solve (1.8), then recallling $K > 0$ it is necessary to assume $\kappa_g > 0$ and we will focus on this situation in the paper.

In the positive subcritical case, namely $0 < \kappa_g < 8\pi^2$, we prove the following result.

Theorem 1.1. *Assuming that* $P_g \geq 0$, $\ker P_g \simeq \mathbb{R}$, *and* $0 < \kappa_g < 8\pi^2$, *then there exists*

$$u_{min} \in C^{\infty}(M) \cap H^2_Q(M)$$

such that

$$I(u_{min}) = \min_{v \in H^2_Q(M)} I(v) \quad \textit{and} \quad u_{min} = T_g(u_{min}).$$

Moreover, setting

$$u_c = u_{min} - \frac{1}{4}\log\int_M K\mathrm{e}^{4u_{min}} + \frac{1}{4}\log\kappa_g \quad \textit{and} \quad g_c = \mathrm{e}^{2u_c}g,$$

we have

$$Q_{g_c} = K.$$

Remark 1.2. We would like to recall that the work of Gursky implies that $\mathcal{Y}(M,[g]) \geq 0$ and $\kappa_g \geq 0$ imply $P_g \geq 0$, $\ker P_g \simeq \mathbb{R}$, and $\kappa_g \leq 8\pi^2$ with equality if and only if (M,g) is conformally equivalent to $(\mathbb{S}^4, g_{\mathbb{S}^4})$. Hence, if $\mathcal{Y}(M,[g]) \geq 0$, $\kappa_g \geq 0$, and (M,g) is **not** conformally equivalent to $(\mathbb{S}^4, g_{\mathbb{S}^4})$, then the assumptions of Theorem 1.1 hold.

To state our existence result in the critical case, i.e $\kappa_g = 8\pi^2$, we first set some notations. We define $\mathcal{F}_K : M \longrightarrow \mathbb{R}$ as follows

$$\mathcal{F}_K(a) := 2\left(H(a,a) + \frac{1}{2}\log(K(a))\right), \quad a \in M \tag{1.10}$$

where H is the regular part of the Green's function G of $P_g(\cdot) + 2Q_g$ satisfying the normalization $\int_M Q_g(x)G(\cdot,x)\mathrm{d}V_g(x) = 0$, see Section 2. Furthermore, we define

$$Crit(\mathcal{F}_K) := \{a \in M : \ a \text{ is critical point of } \mathcal{F}_K\}. \tag{1.11}$$

Moreover, for $a \in M$ we set

$$\mathcal{F}^a(x) := \mathrm{e}^{(H(a,x)+\frac{1}{4}\log(K(x))}, \quad x \in M \tag{1.12}$$

and define

$$\mathcal{L}_K(a) := -\mathcal{F}^a(a)L_g(\mathcal{F}^a)(a), \tag{1.13}$$

where

$$L_g := -\Delta_g + \frac{1}{6}R_g$$

is the conformal Laplacian associated to g. We set also

$$\mathcal{F}^+_\infty := \{a \in Crit(\mathcal{F}_K) : \ \mathcal{L}_K(a) > 0\}. \tag{1.14}$$

With this notation, our existence result in the critical case reads as follows:

Theorem 1.3. *Assuming that* $P_g \geq 0$, $\ker P_g \simeq \mathbb{R}$, $\kappa_g = 8\pi^2$, *and* $\mathcal{F}_\infty^+ = Crit(\mathcal{F}_K)$, *then there exists*

$$u_{min} \in C^\infty(M) \cap H_Q^2(M)$$

such that

$$I(u_{min}) = \min_{v \in H_Q^2(M)} I(v) \quad \textit{and} \quad u_{min} = T_g(u_{min}).$$

Moreover, setting

$$u_c = u_{min} - \frac{1}{4}\log\int_M K\mathrm{e}^{4u_{min}} + \frac{1}{4}\log\kappa_g \quad \textit{and} \quad g_c = \mathrm{e}^{2u_c}g,$$

we have

$$Q_{g_c} = K.$$

Remark 1.4. We want to make some comments on the assumption $\mathcal{F}_\infty^+ = Crit(\mathcal{F}_K)$ in Theorem 1.3. In order to do that, we first recall that in our joint work with Ahmedou [1] (see also [20]), the critical points at infinity of J defined by

$$J(u) := < u, u >_g - \kappa_g \log\left(\int_M K\mathrm{e}^u \mathrm{d}V_g\right), \quad u \in H_Q^2(M)$$

has been characterized, see [1](or [2]) for a precise definition of critical point at infinity of J. What is important to us here is that under the non-degeneracy assumption $\mathcal{L}_K(a) \neq 0$ for all $a \in Crit(\mathcal{F}_K)$, the critical points at infinity of J are in bijection with the critical points of $\mathcal{F}_K$. They are divided into two categories according to the sign of $\mathcal{L}_K$ at critical points of $\mathcal{F}_K$. One type is called the true ones and correspond to critical point a of $\mathcal{F}_K$ with $\mathcal{L}_K(a) < 0$ and the others are called false ones and correspond to $\mathcal{L}_K(a) > 0$. With this, the assumption $\mathcal{F}_\infty^+ = Crit(\mathcal{F}_K)$ means just the non-degeneracy assumption $\mathcal{L}_K(a) \neq 0$ for all $a \in Crit(\mathcal{F}_K)$ holds and that all the critical points at infinity of J are false which in turn means that the variational problem corresponding J is compact in the sense that there is a pseudogradient of J for which all the flow lines with small speed are pre-compact. On the other hand, the critical points at infinity of J can also be described sequentially, see the works [19] and [21], or the works of Chen–Lin [7], [8] in a related context. In fact, under the non-degeneracy assumption $\mathcal{L}_K(a) \neq 0$ for all $a \in Crit(\mathcal{F}_K)$, the true critical points at infinity of J are in bijection with blowing up sequence $u_l \in H_Q^2(M)$ of critical points of J_{t_l} with $t_l < 1$ and $t_l \longrightarrow 1$ as $l \to \infty$, while false critical points at infinity are in bijection with blowing up sequence $u_l \in H_Q^2(M)$ of critical points of J_{t_l} with $t_l > 1$ and $t_l \longrightarrow 1$ as $l \to \infty$, where (for $0 < t < 2$)

$$J_t(u) := < u, u >_g - \kappa_g t \log\left(\int_M K\mathrm{e}^{4u} \mathrm{d}V_g\right), \quad u \in H_Q^2(M).$$

Hence, the assumption $\mathcal{F}_\infty^+ = Crit(\mathcal{F}_K)$ can be interpreted as the pre-compactness of all sequence u_l of critical points of J_{t_l} with $t_l < 1$ and $t_l \longrightarrow 1$ as $l \to \infty$.

Remark 1.5.

- If $\kappa_g > 8\pi^2$, then the well-known fact

$$\inf_{u \in H^2(M)} J(u) = \inf_{u \in H_Q^2(M)} J(u) = -\infty$$

and Lemma 4.1 imply

$$\inf_{u\in H^2_Q(M)} I(u) = -\infty.$$

- The relation $u_{min} = T_g(u_{min})$ in the above theorems is an additional information with respect to the existence results based on Calculus of Variations, Critical Points Theory, Morse Theory, and Dynamical Systems. It provides the inequality

$$\langle u_{min}, u_{min}\rangle_g \leq \langle u, u\rangle_g\,, \qquad \forall\, u_{min} \leq u \in H^2_Q(M). \tag{1.15}$$

- We remark that the nonlocal character of $e^{4T_g(u)}$ in the definition of I with respect to e^{4u} appearing in the definition of J which is used in the existence approaches of (1.8) *via* Calculus of Variations, Critical Points Theory, Morse Theory, and Dynamical Systems. The trade of the local character to non-local is in contrast with the traditional approach in the study of Differential Equations, but have the advantage of providing automatically the variational inequality (1.15).
- The Q-curvature functional J is invariant by translation by constants, while the Q-optimal control functional I is not. The functional J is weakly lower semicontinuous, but the functional I is not. This makes it difficult to apply the Direct Methods in Calculus of Variations to study (1.9).
- We expect formula (1.15) to be useful to deal with the case $\kappa_g = 8\pi^2$ by helping to track down the loss of coercivity in the Variational Analysis of equation (1.8).

As a byproduct of our existence argument, we have the following regularity result for solutions of the optimal control problem (1.9).

Theorem 1.6. *Assuming that* $P_g \geq 0$, $\ker P_g \simeq \mathbb{R}$, $0 < \kappa_g \leq 8\pi^2$, *and* $u \in H^2_Q(M)$ *is a minimizer of* I *on* $H^2_Q(M)$*, then*

$$u \in C^\infty(M).$$

Another consequence of our existence argument is the following compactness theorems for the set of minimizers of I on $H^2_Q(M)$. We start with the subcritical case.

Theorem 1.7. *Assuming that* $P_g \geq 0$, $\ker P_g \simeq \mathbb{R}$, *and* $0 < \kappa_g < 8\pi^2$*, then* $\forall m \in \mathbb{N}$ *there exists* $C_m > 0$ *such that* $\forall u \in C^\infty(M) \cap H^2_Q(M)$ *minimizer of* I *on* $H^2_Q(M)$*, we have*

$$||u||_{C^m(M)} \leq C_m.$$

For the critical case, setting

$$\mathcal{F}^0_\infty := \{a \in Crit(\mathcal{F}_K) : \;\; \mathcal{L}_K(a) \neq 0\}, \tag{1.16}$$

we have:

Theorem 1.8. *Assuming that* $P_g \geq 0$, $\ker P_g \simeq \mathbb{R}$, $\kappa_g = 8\pi^2$, *and* $\mathcal{F}^0_\infty = Crit(\mathcal{F}_K)$*, then* $\forall m \in \mathbb{N}$ *there exists* $C_m > 0$ *such that* $\forall u \in C^\infty(M) \cap H^2_Q(M)$ *minimizer of* I *on* $H^2_Q(M)$*, we have*

$$||u||_{C^m(M)} \leq C_m.$$

Remark 1.9. Th assumption $\mathcal{F}^0_\infty = Crit(\mathcal{F}_K)$ in Theorem 1.8 means that all sequence of critical points of J on $H^2_Q(M)$ are pre-compact.

We derive also some Moser–Trudinger type inequalities and prove some results in the particular case of the 4-dimensional standard sphere, see Proposition 6.1, Theorem 6.2, and Corollary 6.3 in Section 6.

Remark 1.10. We would like to make some comments about our approach in this paper. First of all, to the best of our knowledge, this is the first work on optimal control in the study of the Q-curvature problem.

- The approach in this paper works also for the prescribed Gauss curvature problem in dimension 2, the prescribed Q-curvature problem in critical dimensions greater that 2, and Louiville type equations in general.
- The method of this paper can also be adapted to the Yamabe problem, the prescribed scalar curvature problem, and elliptic equations with critical Sobolev nonlinearity in general. We will pursue this line of study in a forthcoming paper.
- In this paper, we focus in the $\kappa_g > 0$, but an optimal control approach can also be developed when $\kappa_g \leq 0$. We will study this case in a forthcoming paper.

We feel some comments are in order about the motivation to study optimal control problems in this geometric context and the place of this work in the general control theory framework. We want also to give more motivations on the geometric obstacle problem studied in this paper.

Motivation to study optimal control problems in geometric context
One of the main motivation to study optimal control problems in a geometric context in this paper is in the search of *best* metrics. In Geometric Analysis, one of the main theme of research is the problem of finding conformal metrics with some prescribed curvature quantity verifying some variational inequalities. This is usually achieved *via* standard Calculus of Variations and leads to a solution which verifies **one** variational inequality by minimality in the involved Euler–Lagrange functional. However, in Optimal Control Theory, solutions usually not only verify a variational inequality by minimality in the involved Lagrangian but they do verify an **additional** variational inequality because of being **optimal states** too. Because of this additional property, we decide in this paper to investigate the 4-dimensional prescribed Q-curvature problem *via* optimal control theory. Even if we admit that it would have been more natural to start with the 2-dimensional case regarding Gaussian curvature, we have decided to start with the 4-dimensional one, because the geometry of 4-manifolds is less understood compared to the 2-dimensional case, and also there are a lot more works of variational nature in the 2-dimensional case compared to the 4-dimensional one.

Place in general control theory framework.
The optimal control problems that we are studying in this paper falls in the general control theory framework. In fact, it belongs to the class of optimal control problem of the type

$$\min_{u\in X} F(u, T(u)), \tag{1.17}$$

where $(X, ||\cdot||)$ is some subset of a Sobolev–Hilbert space with associated L^2-scalar product $\langle\cdot,\cdot\rangle$, $T(u)$ is the unique solution of the obstacle problem

$$\min_{v\in X,\ v\geq u} \langle Lu, u\rangle \tag{1.18}$$

with

$$F : X \times X \longrightarrow \mathbb{R} \quad \text{some continuous functional,}$$

and $L : X \longrightarrow L^2$ a continuous linear operator verifying

$$L \geq 0, \ \ \ker L = \{0\},$$

and

$$\sqrt{< Lu, u >} \text{ defines a norm equivalent to } \ ||\cdot||.$$

In fact, the problem under study falls in the particular case where

$$F(u, v) = \langle Lu, u\rangle - \kappa N(v), \tag{1.19}$$

with $N : X \longrightarrow \mathbb{R}$ increasing and continuous, and κ is a positive constant. We will investigate (in a paper to come) the abstract optimal control problems (1.17) with F as in (1.19) and

$$\min_{u\in X^*} F(u, T(u)), \tag{1.20}$$

where T is defined by (1.18) with X replaced by X^*,

$$F(u, v) = \frac{\langle Lu, u\rangle}{N(v)}, \quad u, v \in X^* := X \setminus \{0\},$$

and $N : X^* \longrightarrow \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ increasing and continuous and their applications in the study of a class of differential equations arising from Conformal Geometry.

More motivations on geometric obstacle problems

One of the main motivation to introduce the obstacle problem for the Paneitz operator and the associated functional I is to have a Euler–Lagrange functional for (1.8) with a better description of the bubbling phenomena involved in its variational study. Since in the study of many Geometric Variational Problems, the standard bubbles -here the solutions of the corresponding problem on the standard sphere with $K = 3$-are solely responsible of the involved bubbling phenomena, then one can think that a Euler–Lagrange functional with a good hope of an optimal description of the involved bubbling phenomena in its study should be in such a way that the variational properties of the standard bubbles are inherited by the blowing sequences involved in its bubbling phenomena. Since the standard bubbles (up to a constant) of (1.8) are solutions of their Paneitz obstacle problem (see Cor. 6.3), then it is natural to think that such a phenomena should be inherited by bubbling up sequence of a *good* Euler–Lagrange functional for (1.8). This is one of the main motivation of our introduction of the Paneitz obstacle problem and of the functional I.

To prove Theorem 1.1-Theorem 1.8, we first use the variational characterization of the solution of Paneitz obstacle problem $T_g(u)$ (see Lem. 3.1) to show that the Paneitz obstacle solution map T_g is idempotent, i.e $T_g^2 = T_g$, see Proposition 3.2. Next, using the idempotent property of T_g, we establish some monotonicity formulas, see Lemma 3.4, Lemma 4.1, and Lemma 4.3. Using the later monotonicity formulas, we show that any minimizer of J or any solution of the optimal control problem (1.9) is a fixed point of T_g, see Corollary 3.8 and Corollary 4.5. This allows us to show that the Q-curvature functional J and the Q-optimal control functional have the same minimizers on $H^2_Q(M)$, see Proposition 4.8 . With this at hand, Theorem 1.1 follows from the work of Chang–Yang [5] in the subcritical case, while Theorem 1.3 follows from our work in the critical case in [19]. Moreover, Theorem 1.6 follows from the regularity result of Uhlenbeck–Viaclosky [23]. Furthermore, Theorem 1.7 follows from the compactness result of Malchiodi [14] and Druet–Robert-[10], while Theorem 1.8 follows our compactness theorem in [19].

The structure of the paper is as follows. In Section 2, we collect some preliminaries and fix some notations. In Section 3, we discuss the Paneitz obstacle problem and some monotonicity formulas involving the Q-curvature functional J. We also present some consequences of the latter monotonicity formulas. In Section 4, we establish some monotonicity formulas for the Q-optimal control functional I and their consequences as well. In Section 5, we present the proof of Theorem 1.1-Theorem 1.8. Finally, in Section 6, we derive a Moser–Trudinger type inequality involving the obstacle operator T_g and discuss the particular case of the 4-dimensional standard sphere.

2. Notations and Preliminaries

In this brief section, we fix our notations and give some preliminaries. First of all, from now until the end of the paper, (M,g) and $K : M \longrightarrow \mathbb{R}_+$ are respectively the given underlying closed four-dimensional Riemannian manifold and the smooth positive function to prescribe.

We recall the function J used in other approaches to study (1.8).

$$J(u) := \langle u,u\rangle_g + 4\int_M Q_g u \mathrm{d}V_g - \kappa_g \log\left(\int_M K\mathrm{e}^{4u}\mathrm{d}V_g\right), \quad u \in H^2(M). \tag{2.1}$$

Moreover, we recall the perturbed functional J_t $(0 < t \leq 1)$ which plays also an important role in the study of minimizers of J.

$$J_t(u) := \langle u,u\rangle_g + 4t\int_M Q_g u \mathrm{d}V_g - t\kappa_g \log\left(\int_M K\mathrm{e}^{4u}\mathrm{d}V_g\right), \quad u \in H^2(M). \tag{2.2}$$

We observe that

$$J = J_1.$$

Moreover, we define

$$\overline{u}_Q = \frac{1}{\kappa_g}\int_M Q_g u \mathrm{d}V_g, \quad u \in H^2(M),$$

so that

$$H^2_Q(M) = \{u \in H^2(M) : \quad \overline{u}_Q = 0\}.$$

For $a \in M$, we let $G(a,\cdot)$ be the unique solution of the following system

$$\begin{cases} P_g G(a,\cdot) + 2Q_g(\cdot) = 16\pi^2\delta_a(\cdot) \text{ in } \quad M \\ \int_M Q_g(x)G(a,x)\mathrm{d}V_g(x) = 0. \end{cases} \tag{2.3}$$

It is a well know fact that $G(\cdot,\cdot)$ has a logarithmic singularity. In fact $G(\cdot,\cdot)$ decomposes as follows

$$G(a,x) = \log\left(\frac{1}{\chi^2(d_g(a,x))}\right) + H(a,x), \quad x \neq a \in M. \tag{2.4}$$

where $H(\cdot,\cdot)$ is the regular par of $G(\cdot,\cdot)$ and χ is some smooth cut-off function, see for example [24].

The decomposition of the Green's function G and the arguments of the proof of the Moser–Trudinger's inequality of Chang–Yang [5] imply the following Moser–Trudinger type inequality.

Proposition 2.1. *Assuming that* $P_g \geq 0$, $\ker P_g = \mathbb{R}$, *then there exists* $C = C(M,g) > 0$ *such that*

$$\log \int_M \mathrm{e}^{4u} \mathrm{d}V_g \leq C + \frac{1}{8\pi^2} \langle u, u \rangle_g , \quad \forall u \in H^2_Q(M).$$

Proof. As already said the proof of this Proposition 2.1 goes along the same line of the proof of Lemma 1.6 in [5] (whose proof is at the end of Sect. 1) as in Proposition 2.2 of [17] or Proposition 2.1 in [14]. Because of this, we will sketch only the main steps of the argument. First of all, without loss of generality we can assume $u \neq 0$. Now, using the Green's function G and $\overline{u}_Q = 0$, we have

$$u(x) = \int_M G(x,y) P_g u(y) \mathrm{d}V_g(y), \quad x \in M. \tag{2.5}$$

On the other hand as in the proof of Lemma 1.6 in [5], we have $\sqrt{P_g}$ is well-defined and (2.5) implies

$$u(x) = \int_M \sqrt{P_g} G(x,y) \sqrt{P}_g u(y) \mathrm{d}V_g(y), \;\; x \in M \tag{2.6}$$

From here, using the integral representation (2.6) and arguing as in Proposition 2.1 in [17] or Proposition 2.1 in [14] following the method of the proof of Lemma 1.6 in [5], we arrive to

$$\int_M \mathrm{e}^{32\pi^2 \frac{u^2}{\langle u,u \rangle_g}} \mathrm{d}V_g \leq \hat{C}.$$

for some $\hat{C} = \hat{C}(M,g) > 0$. Finally, taking exponential on both sides of the inequality

$$4u \leq 32\pi^2 \frac{u^2}{\langle u,u \rangle_g} + \frac{1}{8\pi^2} \langle u, u \rangle_g ,$$

integrating over M and taking logarithm on both sides of the resulting inequalities, we obtain

$$\log \int_M \mathrm{e}^{4u} \mathrm{d}V_g \leq C + \frac{1}{8\pi^2} \langle u, u \rangle_g ,$$

for some $C = C(M,g) > 0$ as desired. □

Remark 2.2. When $Q = \textit{constant}$, Proposition 2.1 is exactly Lemma 1.6 in [5].

When $(M,g) = (\mathbb{S}^4, g_{\mathbb{S}^4})$, we say v is a standard bubble if

$$P_{g_{\mathbb{S}^4}} v + 6 = 6\mathrm{e}^{4v} \;\; \text{on} \;\; \mathbb{S}^4. \tag{2.7}$$

By the result of Chang–Yang [6], v satisfies

$$\mathrm{e}^{2v} g_{\mathbb{S}^4} = \varphi^*(g_{\mathbb{S}^4}), .$$

for some φ conformal transformation of $\mathbb{S}^4$. It is well-known that the standard bubbles are related to the classical Moser–Trudinger–Onofri inequality. Indeed, we have:

Proposition 2.3. *Assuming that* $(M,g)=(\mathbb{S}^4, g_{\mathbb{S}^4})$ *and* $K=1$*, then*

$$J(u)\geq 0, \quad \forall u\in H^2(M). \tag{2.8}$$

Moreover, equality in (2.8) *holds if and only if*

$$v:=u-\frac{1}{4}\log\int_M e^{4u}+\frac{1}{4}\log\frac{\kappa_g}{3}$$

is a standard bubble.

To end this section, we say w is a Q-normalized standard bubble, if

$$w=v-\overline{v}_Q, \tag{2.9}$$

with v a standard bubble.

3. Obstacle problem for the Paneitz operator

In this section, we study the obstacle problem for the Paneitz operator. Indeed in analogy to the classical obstacle problem for the Laplacian, given $u\in H^2_Q(M)$, we look for a solution to the minimization problem

$$\min_{v\in H^2_Q(M),\ v\geq u}\langle v,v\rangle_g. \tag{3.1}$$

We start with the following lemma providing the existence and uniqueness of solution for the obstacle problem associated to the Paneitz operator (3.1).

Lemma 3.1. *Assuming that* $P_g\geq 0$ *and* $\ker P_g\simeq\mathbb{R}$*, then* $\forall u\in H^2_Q(M)$*, there exists a unique* $T_g(u)\in H^2_Q(M)$ *such that*

$$\langle T_g(u),T_g(u)\rangle_g=\min_{v\in H^2_Q(M),\ v\geq u}\langle v,v\rangle_g \tag{3.2}$$

Proof. Since P_g is self-adjoint, $P_g\geq 0$ and $\ker P_g\simeq\mathbb{R}$, then $<\cdot,\cdot>_g$ defines a scalar product on $H^2_Q(M)$ inducing a norm equivalent to the standard $H^2(M)$-norm on $H^2_Q(M)$. Hence, as in the classical obstacle problem for the Laplacian, the lemma follows from Direct Methods in the Calculus of Variations. □

We study now some properties of the obstacle solution map $T_g : H^2_Q(M)\longrightarrow H^2_Q(M)$. We start with the following algebraic one.

Proposition 3.2. *Assuming that* $P_g\geq 0$*,* $\ker P_g\simeq\mathbb{R}$*, then the obstacle solution map* $T_g : H^2_Q(M)\longrightarrow H^2_Q(M)$ *is idempotent, i.e*

$$T_g^2=T_g.$$

Proof. Let $v\in H^2_Q(M)$ such that $v\geq T_g(u)$. Then $T_g(u)\geq u$ implies $v\geq u$. Thus by minimality, we obtain

$$\langle v,v\rangle_g\geq\langle T_g(u),T_g(u)\rangle_g.$$

Hence, since $T_g(u) \geq T_g(u)$ then by uniqueness we have

$$T_g(T_g(u)) = T_g(u),$$

thereby ending the proof. $\square$

Remark 3.3. Lemma 3.1 and Proposition 3.2 require no assumption on κ_g. However, we recall that ([13]), $Y(M,[g]) \geq 0$ and $\kappa_g \geq 0$ imply the assumption of Lemma 3.1 and Proposition 3.2.

Next, we discuss some monotonicity formulas. We start with the following one.

Lemma 3.4. *Assuming that* $P_g \geq 0$, $\ker P_g \simeq \mathbb{R}$, $0 < t \leq 1$ *and* $\kappa_g > 0$, *then*

$$J_t(u) - J_t(T_g(u)) \geq \langle u, u\rangle_g - \langle T_g(u)), T_g(u)\rangle_g \geq 0, \quad \forall u \in H^2_Q(M).$$

Proof. Using the definition of J_t (see (2.2)), we have

$$J_t(u) - J_t(T_g(u)) = \langle u, u\rangle_g - \langle T_g(u), T_g(u)\rangle_g - t\kappa_g \left(\log \frac{\int_M K e^{4u} dV_g}{\int_M K e^{4T_g(u)} dV_g}\right). \tag{3.3}$$

Hence the result follows from $K > 0$, $T_g(u) \geq u$, $\kappa_g > 0$, and Lemma 3.1 . $\square$

Lemma 3.4 imply the following rigidity result.

Corollary 3.5. *Assuming that* $P_g \geq 0$, $\ker P_g \simeq \mathbb{R}$, $0 < t \leq 1$ *and* $\kappa_g > 0$, *then* $\forall u \in H^2_Q(M)$,

$$J_t(T_g(u)) \leq J_t(u) \tag{3.4}$$

and

$$J_t(u) = J_t(T_g(u)) \implies u = T_g(u). \tag{3.5}$$

Proof. Using Lemma 3.4, we have

$$J_t(u) - J_t(T_g(u)) \geq \langle u, u\rangle_g - \langle T_g(u), T_g(u)\rangle_g \geq 0. \tag{3.6}$$

Thus, (3.4) follows from (3.6). If $J_t(u) = J_t(T_g(u))$, then (3.6) implies

$$\langle u, u\rangle_g = \langle T_g(u), T_g(u)\rangle_g .$$

Hence, since $u \geq u$, then the uniqueness part in Lemma 3.1 implies

$$u = T_g(u),$$

thereby ending the proof of the corollary. $\square$

Remark 3.6. Besides the fact that the necessary condition for solvability of (1.8) force us to consider $\kappa_g > 0$, we have also analytically the proof of Lemma 3.4 and Corollary 3.5 under $K > 0$ requires only $\kappa_g \geq 0$. We focus on $0 < t \leq 1$, because our applications below deal just with that case.

Remark 3.7. Under the assumption of Corollary 3.5, we have Proposition 3.2 and Corollary 3.5 imply that we can assume without loss of generality that any minimizing sequence $(u_l)_{l\geq 1}$ of J_t on $H^2_Q(M)$ satisfies

$$u_l = T_g(u_l), \quad \forall l \geq 1.$$

Indeed, suppose u_l is a minimizing sequence for J_t on $H^2_Q(M)$. Then $u_l \in H^2_Q(M)$ and

$$J_t(u_l) \longrightarrow \inf_{H^2_Q(M)} J_t.$$

Thus by definition of infimum and Corollary 3.5, we have

$$\inf_{H^2_Q(M)} J_t \leq J_t(T_g(u_l)) \leq J_t(u_l).$$

This implies

$$J_t(T_g(u_l)) \longrightarrow \inf_{H^2_Q(M)} J_t.$$

Hence setting

$$\hat{u}_l = T_g(u_l),$$

and using Proposition 3.2, we get

$$J_t(\hat{u}_l) \longrightarrow \inf_{H^2_Q(M)} J_t \quad \text{and} \quad \hat{u}_l = T_g(\hat{u}_l)$$

as desired. We add that analytically the latter observation is also true for $\kappa_g \geq 0$ if $K > 0$.

Corollary 3.5 implies that minimizers of J_t on $H^2_Q(M)$ are fixed points of the obstacle solution map T_g. Indeed, we have:

Corollary 3.8. *Assuming that* $P_g \geq 0$, $\ker P_g \simeq \mathbb{R}$, $0 < t \leq 1$ *and* $0 < \kappa_g \leq 8\pi^2$, *then*

$$u \in H^2_Q(M) \ \text{ is a minimizer of } \ J_t \implies u = T_g(u).$$

Proof. $u \in H^2_Q(M)$ is a minimizer of J_t on $H^2_Q(M)$ implies

$$J_t(u) \leq J_t(T_g(u)). \tag{3.7}$$

Thus combining (3.4) and (3.7), we get

$$J_t(u) = J_t(T_g(u)). \tag{3.8}$$

Hence, combining (3.5) and (3.8), we obtain

$$u = T_g(u).$$

□

Remark 3.9. We put the the assumption $0 < \kappa_g \leq 8\pi^2$ in Corollary 3.8 for the statement to be sound, since if not J_t is unbounded for t close to 1 which is the case of interest in our applications below and that a necessary condition for for the existence of minimizer of J_t is $\kappa_g > 0$. However, as in Lemma 3.4 and Corollary 3.5, analytically the proof of Corollary 3.8 under $K > 0$ requires only $\kappa_g \geq 0$

Remark 3.10. We would like to make some comments on the dependence of $u = u_t$ minimizer of J_t with respect to t in Corollary 3.8. Beside u_t being a fixed point of T_g, we have

$$w_t := u_t - \frac{1}{4}\log \int_M K\mathrm{e}^{4u_t}\mathrm{d}V_g$$

satisfies

$$P_g w_t + 2tQ_g = 2t\kappa_g K\mathrm{e}^{4w_t}. \tag{3.9}$$

The asymptotic behavior of sequence of solutions $w_l = w_{t_l}$ of (3.9) with $t_l \to 1$ as $l \to \infty$ has been investigated in several works, see for example [10], [14], [19], [24].

4. Optimal control for the Paneitz operator

In this section, we study a natural optimal control problem associated to the obstacle problem for the Paneitz operator. Indeed, we look for solutions of

$$\min_{u \in H^2_Q(M)} I(u),$$

where I is the Q-optimal control functional defined by

$$I(u) := \langle u, u\rangle_g - \kappa_g \log\left(\int_M K\mathrm{e}^{4T_g(u)}\mathrm{d}V_g\right), \quad u \in H^2_Q(M). \tag{4.1}$$

Similarly to the Q-curvature functional J, for $0 < t \leq 1$ we define I_t by

$$I_t(u) := \langle u, u\rangle_g - t\kappa_g \log\left(\int_M K\mathrm{e}^{4T_g(u)}\mathrm{d}V_g\right), \quad u \in H^2_Q(M). \tag{4.2}$$

We start with the following comparison result.

Lemma 4.1. *Assuming that* $P_g \geq 0$, $\ker P_g \simeq \mathbb{R}$, $0 < t \leq 1$ *and* $\kappa_g > 0$, *then*

$$I_t \leq J_t \quad \text{on} \quad H^2_Q(M) \quad \text{and} \quad J_t \circ T_g = I_t \circ T_g \quad \text{on} \quad H^2_Q(M).$$

Proof. By definition of J_t and I_t (see (2.2) and (4.2)), we have

$$J_t(u) - I_t(u) = t\kappa_g \log\left(\frac{\int_M K\mathrm{e}^{4T_g(u)}}{\int_M K\mathrm{e}^{4u}}\right).$$

Thus $I_t(u) \leq J_t(u)$ follows from $T_g(u) \geq u$, $\kappa_g > 0$ and $K > 0$. Moreover, we have

$$J_t(T_g(u)) - I_t(T_g(u)) = t\kappa_g \log\left(\frac{\int_M K\mathrm{e}^{4T^2_g(u)}}{\int_M K\mathrm{e}^{4T_g(u)}}\right).$$

Hence, $T_g^2 = T_g$ (see Lem. 3.2) implies

$$J_t(T_g(u)) = I_t(T_g(u)).$$

□

Remark 4.2. In Lemma 4.1, we have $\kappa_g > 0$ is used in the conclusion $I_t \leq J_t$, but analytically only $\kappa_g \geq 0$ is required under the assumption $K > 0$. Moreover analytically, the conclusion $I_t \circ T_g = J_t \circ T_g$ require no assumption on κ_g.

We have the following monotonicity formula for the Q-optimal control functional I_t.

Lemma 4.3. *Assuming that* $P_g \geq 0$, $\ker P_g \simeq \mathbb{R}$, $0 < t \leq 1$ *and* $\kappa_g > 0$, *then* $\forall u \in H^2_Q(M)$,

$$I_t(u) - I_t(T_g(u)) = \langle u, u\rangle_g - \langle T_g(u), T_g(u)\rangle_g \geq 0.$$

Proof. By definition of I_t (see (4.2)), we have

$$I_t(u) - I_t(T_g(u)) = \langle u, u\rangle_g - \langle T_g(u), T_g(u)\rangle_g - t\kappa_g \log\left(\frac{\int_M K\mathrm{e}^{4T_g(u)}}{\int_M K\mathrm{e}^{4T_g^2(u)}}\right).$$

Using $T_g^2(u) = T_g(u)$ and the definition of T_g (see Lem. 3.1), we get

$$I_t(u) - I_t(T_g(u)) = \langle u, u\rangle_g - \langle T_g(u), T_g(u)\rangle_g \geq 0.$$

□

Remark 4.4. Analytically no assumption on κ_g is need in Lemma 4.3. As in the previous section and for the same reasons, here also we focus on the case $0 < t \leq 1$.

Lemma 3.1 and Lemma 4.3 imply that minimizers of I_t are fixed points of T_g.

Corollary 4.5. *Assuming that* $P_g \geq 0$, $\ker P_g = \mathbb{R}$, $0 < t \leq 1$ *and* $0 < \kappa_g \leq 8\pi^2$, *then*

$$u \in H^2_Q(M) \quad \textit{is a minimizer of} \quad I_t \implies u = T_g(u).$$

Proof. $u \in H^2_Q(M)$ is a minimizer of I_t implies

$$I_t(u) \leq I_t(T_g(u)).$$

Thus Lemma 4.3 gives

$$\langle u,u\rangle_g = \langle T_g(u), T_g(u)\rangle_g .$$

Hence, by uniqueness we have

$$u = T_g(u).$$

□

Remark 4.6. Analytically no assumption on κ_g is needed in Lemma 4.5. However, to have a sound statement it is necessary to have $\kappa_g \leq 8\pi^2$, since if not I_t is unbounded for t close to 1 which is the case of interest in our applications below.

Remark 4.7.
- Under the assumptions of Lemma 4.3 and using the same argument as in Remark 3.7, we have that Proposition 3.2 and Lemma 4.3 imply that for a minimizing sequence $(u_l)_{l\geq 1}$ of I_t on $H^2_Q(M)$, we can assume without loss of generality that

$$u_l = T_g(u_l), \quad \forall l \geq 1.$$

- As in Remark 4.6, analytically the first part of this remark holds with no assumption on κ_g.

We have the following proposition showing that I_t and J_t have the same minimizers on $H^2_Q(M)$.

Proposition 4.8. *Assuming that* $P_g \geq 0$, $\ker P_g \simeq \mathbb{R}$, $0 < t \leq 1$ *and* $0 < \kappa_g \leq 8\pi^2$, *then*

$$u \in H^2_Q(M) \text{ is a minimizer of } J_t \text{ is equivalent to } u \in H^2_Q(M) \text{ is a minimizer of } I_t.$$

Proof. Suppose $u \in H^2_Q(M)$ is a minimizer of J_t. Then Corollary 3.8 implies

$$u = T_g(u).$$

Thus using Lemma 4.1 we have

$$I_t(u) = J_t(u)$$

For $v \in H^2_Q(M)$, we have Lemma 4.1, Lemma 4.3, and $u \in H^2_Q(M)$ is a minimizer of J_t imply

$$I_t(v) \geq I_t(T_g(v)) = J_t(T_g(v)) \geq J_t(u) = I_t(u).$$

Hence $u \in H^2_Q(M)$ is a minimizer of I_t on $H^2_Q(M)$. Similarly, suppose $u \in H^2_Q(M)$ is a minimizer of I_t. Then Corollary 4.5 implies

$$u = T_g(u).$$

Thus using again Lemma 4.1 we have

$$I_t(u) = J_t(u).$$

For $v \in H^2_Q(M)$, we have Lemma 4.1, Lemma 3.4, and $u \in H^2_Q(M)$ is a minimizer of I_t imply

$$J_t(v) \geq J_t(T_g(v)) = I_t(T_g(v)) \geq I_t(u) = J_t(u).$$

Hence $u \in H^2_Q(M)$ is a minimizer of J_t on $H^2_Q(M)$. □

Remark 4.9. As in Remark 3.9 and for the same reasons, in Proposition 4.8, the assumption $0 < \kappa_g \leq 8\pi^2$ is needed for the statement to be sound. However, analytically the assumption $\kappa_g \geq 0$ is the only thing needed for the presented proof to work under the assumption $K > 0$.

Remark 4.10. Because of the equivalence in Proposition 4.8, we have the assumption $0 < \kappa_g \leq 8\pi^2$ is needed for the statement in Corollary 4.5 to be sound.

5. Proof of Theorem 1.1 -Theorem 1.8

In this section, we present the proof of Theorem 1.1 -Theorem 1.8. As already mentioned in the introduction, the proofs are based on Proposition 4.8 and some contributions of Chang–Yang [5], Druet–Robert [10], Malchiodi [14], the author [19] and Uhlenbeck–Viaclovsky [23] in the the study of the fourth-order nonlinear partial differential equation (1.8).

Proof of Theorem 1.1. Since $P_g \geq 0$, $\ker P_g = \mathbb{R}$, and $0 < \kappa_g < 8\pi^2$, then the works of Chang–Yang [5] and Uhlenbeck–Viaclosvky [23] imply the existence of $u_0 \in C^\infty(M)$ such that

$$J(u_0) = \min_{u \in H^2(M)} J(u).$$

Since J is translation invariant, then setting

$$u_{min} = u_0 - \overline{(u_0)}_Q,$$

we have

$$u_{min} \in C^\infty(M) \cap H^2_Q(M)$$

and

$$J(u_{min}) = \min_{u \in H^2_Q(M)} J(u).$$

Using Proposition 4.8, we get

$$I(u_{min}) = \min_{u \in H^2_Q(M)} I(u)$$

Thus Corollary 4.5 implies

$$u_{min} = T_g(u_{min}).$$

Recalling that

$$J(u_{min}) = J(u_0) = \min_{u \in H^2(M)} J(u),$$

we have

$$P_g u_{min} + 2Q_g = 2\kappa_g \frac{K e^{4u_{min}}}{\int_M K e^{4u_{min}}}.$$

Thus, setting

$$u_c = u_{min} - \frac{1}{4}\log\int_M K e^{4u_{min}} + \frac{1}{4}\log\kappa_g,$$

we have

$$P_g u_c + 2Q_g = 2K e^{4u_c}.$$

Hence, setting

$$g_{u_c} = e^{2u_c} g,$$

we obtain

$$Q_{g_{u_c}} = K.$$

thereby ending the proof. □

Proof of Theorem 1.3. Let $\varepsilon_l \in (0,1)$ with $\varepsilon_l \to 0$. For $l \geq 1$, we define

$$J_l := J_{1-\varepsilon_l} \quad \text{and} \quad I_l := I_{1-\varepsilon_l}$$

As in the proof of Theorem 1.1, for $l \geq 1$ the works of Chang–Yang [5] and Uhlenbeck–Viaclosvky [23] give the existence of

$$u^l_{min} \in C^\infty(M) \cap H^2_Q(M)$$

such that

$$J_l(u^l_{min}) = \min_{u\in H^2(M)} J_l(u). \tag{5.1}$$

Thus, using Proposition 4.8, we get

$$I_l(u^l_{min}) = \min_{u\in H^2_Q(M)} I_l(u). \tag{5.2}$$

Clearly (5.1) imply,

$$P_g u^l_{min} + 2Q_g(1-\varepsilon_l) = 2\kappa_g(1-\varepsilon_l)\frac{K e^{4u^l_{min}}}{\int_M K e^{4u^l_{min}}}. \tag{5.3}$$

Hence, setting

$$u^l_c = u^l_{min} - \frac{1}{4}\log\int_M K e^{4u^l_{min}} + \frac{1}{4}\log\kappa_g, \tag{5.4}$$

we obtain

$$P_g u^l_c + 2Q_g(1-\varepsilon_l) = 2K(1-\varepsilon_l)\, e^{4u^l_c}. \tag{5.5}$$

Thus our bubbling rate formula in Theorem 1.1 in [19] (applied to $w_l := u^l_{min} - \frac{1}{4}\log\int_M Ke^{4u^l_{min}}$) and the assumption $\mathcal{F}^+_\infty = Crit(\mathcal{F}_K)$ prevents the sequence u^l_c from bubbling. Hence we have

$$u^l_c \longrightarrow u_c \quad \text{smoothly, as} \quad l \longrightarrow \infty. \tag{5.6}$$

Thus (5.5) gives

$$P_g u_c + 2Q_g = 2K e^{4u_c}. \tag{5.7}$$

Recalling $u^l_{min} \in H^2_Q(M)$, we have (5.4) and (5.6) imply

$$u^l_{min} \longrightarrow u_{min} \quad \text{smoothly}. \tag{5.8}$$

and

$$u_c = u_{min} - \frac{1}{4}\log\int_M Ke^{4u_{min}} + \frac{1}{4}\log\kappa_g.$$

Clearly (5.8) and (5.2) imply

$$I(u_{min}) = \min_{u\in H^2_Q(M)} I(u).$$

Hence Corollary 4.5 and (5.7) imply

$$u_{min} = T_g(u_{min}).$$

and

$$Q_{g_{u_c}} = K.$$

□

Proof of Theorem 1.6. It follows directly from Proposition 4.8, the translation invariant property of J and the regularity result of Uhlenbeck–Viaclovsky [23]. □

Proof of Theorem 1.7. Let $u \in C^\infty(M) \cap H^2_Q(M)$ be a minimizer of I on $H^2_Q(M)$. Then the translation invariance property of J and Proposition 4.8 imply u is a minimizer of J on $H^2(M)$. Hence u satisfies

$$P_g u + 2Q_g = 2\kappa_g \frac{Ke^{4u}}{\int_M Ke^{4u}}.$$

Then, setting

$$v = u - \frac{1}{4}\log\int_M Ke^{4u} + \frac{1}{4}\log\kappa_g, \tag{5.9}$$

we get

$$P_g v + 2Q_g = 2Ke^{4v}$$

Thus, since $0 < \kappa_g < 8\pi^2$, then the compactness result of Malchiodi [14] and Druet–Robert [10] imply $\forall m \in \mathbb{N}$, there exists $\tilde{C}_m > 0$ such that

$$||v||_{C^m(M)} \leq \tilde{C}_m.$$

Hence, $u \in H^2_Q(M)$ and (5.9) give the existence of $C_m > 0$ such that

$$||u||_{C^m(M)} \leq C_m,$$

thereby ending the proof. □

Proof of Theorem 1.8. The proof is a small modification of the one of Theorem 1.7. For the sake of completeness, we repeat all the steps. Let $u \in C^\infty(M) \cap H^2_Q(M)$ be a minimizer of I on $H^2_Q(M)$. Then as in the proof of Theorem 1.7, u is a minimizer of J on $H^2(M)$. Hence u satisfies

$$P_g u + 2Q_g = 2\kappa_g \frac{K\mathrm{e}^{4u}}{\int_M K\mathrm{e}^{4u}}.$$

Then, setting

$$v = u - \frac{1}{4}\log \int_M K\mathrm{e}^{4u} + \frac{1}{4}\log \kappa_g,$$

we get

$$P_g v + 2Q_g = 2K\mathrm{e}^{4v}.$$

Thus since $\mathcal{F}^0_\infty = Crit(\mathcal{F}_K)$, then our compactness result in Corollary 1.5 of [19] (applied to $w := u - \frac{1}{4}\log\int_M K\mathrm{e}^{4u}$) imply that $\forall m \in \mathbb{N}$ there exists $\tilde{C}_m > 0$ such that

$$||v||_{C^m(M)} \leq \tilde{C}_m.$$

Hence recalling that $u \in H^2_Q(M)$, then there exists $C_m > 0$ such that

$$||u||_{C^m(M)} \leq C_m.$$

□

6. Obstacle problem and Moser–Trudinger type inequality

In this section, we discuss some Moser–Trudinger type inequalities related to the Paneitz obstacle problem. In particular, we specialize to the case of the 4-dimensional standard sphere $(\mathbb{S}^4, g_{\mathbb{S}^4})$.

We have the following obstacle Moser–Trudinger type inequality.

Proposition 6.1. *Assuming that* $P_g \geq 0$, $\ker P_g = \mathbb{R}$, *then there exists* $C = C(M,g) > 0$ *such that*

$$\log \int_M \mathrm{e}^{4u} dV_g \leq \log \int_M \mathrm{e}^{4T_g(u)} dV_g \leq C + \frac{1}{8\pi^2}\langle T_g(u), T_g(u)\rangle_g \leq C + \frac{1}{8\pi^2}\langle u, u\rangle_g, \quad \forall u \in H^2_Q(M).$$

Proof. Clearly $u \le T_g(u)$ gives

$$\log \int_M e^{4u} dV_g \le \log \int_M e^{4T_g(u)} dV_g. \tag{6.1}$$

Since $P_g \ge 0$ and $\ker P_g = \mathbb{R}$, then the classical Moser–Trudinger inequality in Proposition 2.1 implies the existence of $C = C(M, g) > 0$ such that

$$\log \int_M e^{4T_g(u)} dV_g \le C + \frac{1}{8\pi^2} \langle T_g(u), T_g(u) \rangle_g . \tag{6.2}$$

Using the definition of T_g, we get

$$\langle T_g(u), T_g(u) \rangle_g \le \langle u, u \rangle_g . \tag{6.3}$$

Hence combining (6.1)-(6.3), we get

$$\log \int_M e^{4u} dV_g \le \log \int_M e^{4T_g(u)} dV_g \le C + \frac{1}{8\pi^2} \langle T_g(u), T_g(u) \rangle_g \le C + \frac{1}{8\pi^2} \langle u, u \rangle_g .$$

□

When $(M, g) = (\mathbb{S}^4, g_{\mathbb{S}^4})$ and $K = 1$, we have the following sharp obstacle Moser–Trudinger type inequality.

Theorem 6.2. *Assuming that $(M, g) = (\mathbb{S}^4, g_{\mathbb{S}^4})$ and $K = 1$, then*

$$I \ge 0 \quad on \quad H^2_Q(M),$$

i.e

$$\log \int_M e^{4T_g(u)} dV_g \le \frac{1}{8\pi^2} \langle P_g u, u \rangle , \quad \forall u \in H^2_Q(M). \tag{6.4}$$

Moreover equality in (6.4) holds if and only if

$$v := u - \frac{1}{4} \log \int_M e^{4u} + \frac{1}{4} \log \frac{\kappa_g}{3}$$

is a standard bubble, see (2.7) for its definition.

Proof. Since $(M, g) = (\mathbb{S}^4, g_{\mathbb{S}^4})$ and $K = 1$, then by the classical Moser–Trudinger–Onofiri ineqality in Proposition 2.3, we have

$$J \ge 0 \quad \text{on} \quad H^2(M) \tag{6.5}$$

and

$$J(u) = 0 \quad \text{is equivalent to} \quad v := u - \frac{1}{4} \log \int_M e^{4u} + \frac{1}{4} \log \frac{\kappa_g}{3} \quad \text{is a standard bubble.} \tag{6.6}$$

Using Lemma 4.3, we get

$$I \geq I \circ T_g \quad \text{on} \quad H^2_Q(M). \tag{6.7}$$

Thus, using Lemma 4.1 and (6.7), we have

$$I \geq J \circ T_g \quad \text{on} \quad H^2_Q(M). \tag{6.8}$$

So, combining (6.5) and (6.8), we get

$$I \geq 0 \quad \text{on} \quad H^2_Q(M). \tag{6.9}$$

Hence, recalling the definition of I (see (4.1)) and (1.7), we have (6.9) is equivalent to

$$\log \int_M e^{4T_g(u)} dV_g \leq \frac{1}{8\pi^2} \langle u, u \rangle_g, \quad \forall u \in H^2_Q(M).$$

Suppose

$$v := u - \frac{1}{4} \log \int_M e^{4u} + \frac{1}{4} \log \frac{\kappa_g}{3}$$

is a standard bubble with $u \in H^2_Q(M)$. Then (6.6) implies

$$J(u) = 0 \tag{6.10}$$

Thus (6.10), Lemma 4.1 and the first part (namely (6.9)) imply

$$I(u) = 0.$$

Hence we have the equality case in (6.4). Suppose we have the equality case in (6.4) with $u \in H^2_Q(M)$. Then

$$I(u) = 0. \tag{6.11}$$

Thus, using (6.9) and (6.11) we get

$$I(u) = \min_{v \in H^2_Q(M)} I(v). \tag{6.12}$$

Using (6.12) and Corollary 4.5, we obtain

$$u = T_g(u). \tag{6.13}$$

So Lemma 4.1 , (6.11) and (6.13) imply

$$J(u) = 0. \tag{6.14}$$

Hence using (6.6) and (6.14), we have $v := u - \frac{1}{4} \log \int_M e^{4u} + \frac{1}{4} \log \frac{\kappa_g}{3}$ is a standard bubble. $\square$

Theorem 6.2 implies the following corollary stating that Q-normalized standard bubbles (see (2.9) for their definitions) are fixed points of the obstacle solution map T_g.

Corollary 6.3. *Assuming that* $(M, g) = (\mathbb{S}^4, g_{\mathbb{S}^4})$ *and* w *is a* Q*-normalized standard bubble (see* (2.9) *for its definition), then*

$$T_g(w) = w.$$

Proof. Since w is a Q-normalized standard bubble, then

$$w := v - \overline{(v)}_Q$$

with v is a standard bubble. Thus, Lemma 4.1, Theorem 6.2, and the translation invariant property of J imply

$$0 \leq I(w) \leq J(w) = J(v) = 0.$$

Using again Theorem 6.2, we obtain

$$I(w) = \min_{v \in H^2_Q(M)} I(v)$$

Hence using Corollary 4.5, we get

$$w = T_g(w).$$

□

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