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INDEFINITE BACKWARD STOCHASTIC LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEMS

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Abstract. This paper is concerned with a backward stochastic linear-quadratic (LQ, for short) optimal control problem with deterministic coefficients. The weighting matrices are allowed to be indefinite, and cross-product terms in the control and state processes are presented in the cost functional. Based on a Hilbert space method, necessary and sufficient conditions are derived for the solvability of the problem, and a general approach for constructing optimal controls is developed. The crucial step in this construction is to establish the solvability of a Riccati-type equation, which is accomplished under a fairly weak condition by investigating the connection with forward stochastic LQ optimal control problems.

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1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a standard one-dimensional Brownian motion $W = \{W(t); t \geqslant 0\}$ is defined, and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geqslant 0}$ be the usual augmentation of the natural filtration generated by W. For a random variable ξ, we write $\xi \in \mathcal{F}_t$ if ξ is $\mathcal{F}_t$-measurable; and for a stochastic process φ, we write $\varphi \in \mathbb{F}$ if it is progressively measurable with respect to the filtration $\mathbb{F}$.

Consider the following controlled linear backward stochastic differential equation (BSDE, for short) over a finite horizon $[0, T]$:

$$\begin{cases} \mathrm{d}Y(t) = [A(t)Y(t) + B(t)u(t) + C(t)Z(t)]\mathrm{d}t + Z(t)\mathrm{d}W(t), \\ Y(T) = \xi, \end{cases} \tag{1.1}$$

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where the *coefficients* $A, C : [0,T] \to \mathbb{R}^{n\times n}$ and $B : [0,T] \to \mathbb{R}^{n\times m}$ of the *state equation* (1.1) are given bounded deterministic functions; the *terminal value* ξ is in $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, the space of $\mathbb{R}^n$-valued, $\mathcal{F}_T$-measurable, square-integrable random variables; and u, valued in $\mathbb{R}^m$, is the *control process*. The class of *admissible controls* for (1.1) is

$$\mathscr{U} := \Big\{ u : [0,T]\times\Omega \to \mathbb{R}^m \;\Big|\; u \in \mathbb{F} \text{ and } \mathbb{E}\textstyle\int_0^T |u(t)|^2 \mathrm{d}t < \infty \Big\},$$

and the associated *cost* is given by the following quadratic functional:

$$J(\xi;u) := \mathbb{E}\Bigg[\langle GY(0), Y(0)\rangle + \int_0^T \Bigg\langle \begin{pmatrix} Q(t) & S_1^\top(t) & S_2^\top(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t) \end{pmatrix} \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix} \Bigg\rangle \mathrm{d}t \Bigg], \tag{1.2}$$

where the superscript $\top$ denotes the transpose of a matrix, G is a symmetric $n \times n$ constant matrix, and

$$Q, \quad S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

are bounded deterministic matrix-valued functions of proper dimensions over $[0,T]$ such that the blocked matrix in the cost functional is symmetric. The optimal control problem of interest in the paper can be stated as follows.

Problem (BSLQ). For a given terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, find a control $u^* \in \mathscr{U}$ such that

$$J(\xi;u^*) = \inf_{u\in\mathscr{U}} J(\xi;u) \equiv V(\xi). \tag{1.3}$$

The above problem is usually referred to as a *backward stochastic linear-quadratic (LQ, for short) optimal control problem* (BSLQ problem, for short), due to the linearity of the backward state equation (1.1) and the quadratic form of the cost (1.2). A process $u^* \in \mathscr{U}$ satisfying (1.3) is called an *optimal control* for the terminal state ξ; the adapted solution (Y^*, Z^*) of the state equation (1.1) corresponding to $u = u^*$ is called an *optimal state process*; and the function V is called the *value function* of Problem (BSLQ).

The LQ optimal control problem of forward stochastic differential equations (FSDEs, for short) was first studied by Wonham [30] and has since then been investigated by many other researchers; see, for example, [1, 6] and the references therein. In 1998, Chen, Li, and Zhou [3] found that a forward stochastic LQ problem might still be solvable when the weighting matrices are not positive definite. Since then the so-called *indefinite* forward stochastic LQ problem has attracted considerable attention, and its theory has been well developed in recent years; see, for example, [4, 5, 7, 11, 14, 18, 22] as well as the recent books [24, 25] by Sun and Yong.

The study of optimal control for BSDEs is important and appealing, not only at the theoretical level, but also in financial applications. It is well known that BSDEs play a central role in stochastic control theory and have had fruitful applications in finance (see, *e.g.*, [15–17, 33, 35]). An optimal control problem for backward stochastic differential equations arises naturally when we look at a two-person zero-sum differential game from a leader-follower point of view (see, *e.g.*, [31] and [32], Chap. 6). In the case of stochastic differential games, the state equation involved in the backward optimal control problem becomes a BSDE, and it turns out that we need to handle a backward stochastic optimal control problem. On the other hand, in mathematical finance one frequently encounters financial investment problems with future conditions specified, for example, the mean-variance problem for a given level of expected return. Such kind of problems usually can be modeled as an optimal control problem with terminal constraints, and under certain conditions, it can be converted into a backward stochastic optimal control problem; see the recent work of Bi, Sun, and Xiong [2].

The LQ optimal control problem for BSDEs was initially investigated by Lim and Zhou [13], where *no cross terms* in (Y, Z, u) appear in the quadratic cost functional and all the weighting matrices are *positive semidefinite*.

They obtained a complete solution for such a BSLQ problem, using a forward formulation and a limiting procedure, together with the completion-of-squares technique. Along this line, a couple of following-up works appeared afterward. For example, Wang, Wu, and Xiong [26] studied a one-dimensional BSLQ problem under partial information; Li, Sun, and Xiong [12] generalized the results of Lim and Zhou [13] to the case of mean-field backward LQ optimal control problems; Sun and Wang [19] further carried out an investigation on the backward stochastic LQ problem with random coefficients; Huang, Wang, and Wu [10] studied a backward mean-field linear-quadratic-Gaussian game with full/partial information; Wang, Xiao, and Xiong [29] analyzed a kind of LQ nonzero-sum differential game with asymmetric information for BSDEs; Du, Huang, and Wu [8] considered a dynamic game of N weakly-coupled linear BSDE systems involving mean-field interactions; and based on [12, 13], Bi, Sun, and Xiong [2] developed a theory of optimal control for controllable stochastic linear systems. It is worthy to point out that the above mentioned works depend crucially on the positive/nonnegative definiteness assumption imposed on the weighting matrices, and most of them do not allow cross terms in (Y, Z, u) in cost functionals. However, one often encounters situations where the positive/nonnegative definiteness assumption is not fulfilled. For example, let us look at the following maximin control problem (this kind of problems arise in the study of zero-sum differential games):

$$\max_{v\in L^2_{\mathbb{F}}(0,1;\mathbb{R})}\ \min_{u\in L^2_{\mathbb{F}}(0,1;\mathbb{R})} \mathbb{E}\bigg\{|X(1)|^2+2\xi X(1)+\int_0^1\Big[|u(t)|^2-(a^2+1)|v(t)|^2\Big]\mathrm{d}t\bigg\}$$

subject to

$$\begin{cases}\mathrm{d}X(t)=u(t)\mathrm{d}t+[X(t)+v(t)]\mathrm{d}W(t), \quad t\in[0,1],\\ X(0)=0,\end{cases}$$

where $L^2_{\mathbb{F}}(0,1;\mathbb{R})$ is the space of $\mathbb{F}$-progressively measurable processes $\varphi:[0,1]\times\Omega\to\mathbb{R}$ with $\mathbb{E}\int_0^1|\varphi(t)|^2\mathrm{d}t<\infty$, ξ is an $\mathcal{F}_1$-measurable, bounded random variable, and $a>0$ is a constant. For a given $v\in L^2_{\mathbb{F}}(0,1;\mathbb{R})$, the minimization problem is a standard forward stochastic LQ optimal control problem. Applying the theory developed in [18] (see also [24], Chap. 2), we can easily obtain the minimum $V(\xi;v)$ (depending on ξ and v):

$$V(\xi;v)=\mathbb{E}\int_0^1\Big[-|\eta(t)|^2+2\zeta(t)v(t)-a^2|v(t)|^2\Big]\mathrm{d}t,$$

where (η,ζ) is the adapted solution to the BSDE

$$\begin{cases}\mathrm{d}\eta(t)=[\eta(t)-\zeta(t)-v(t)]\mathrm{d}t+\zeta(t)\mathrm{d}W(t), \quad t\in[0,1],\\ \eta(1)=\xi.\end{cases}$$

Using the transformations

$$Y(t)=\eta(t), \quad Z(t)=\zeta(t), \quad u(t)=v(t)-\frac{1}{a^2}\zeta(t),$$

we see the maximization problem is equivalent to the BSLQ problem with the state equation

$$\begin{cases}\mathrm{d}Y(t)=\Big[Y(t)-\dfrac{a^2+1}{a^2}Z(t)-u(t)\Big]\mathrm{d}t+Z(t)\mathrm{d}W(t), \quad t\in[0,1],\\ Y(1)=\xi\end{cases}$$

and the cost functional

$$J(\xi;u) := -V(\xi;v) = \mathbb{E}\int_0^1 \Big[|Y(t)|^2 - \frac{1}{a^2}|Z(t)|^2 + a^2|u(t)|^2\Big]\mathrm{d}t.$$

Clearly, the positive/nonnegative definiteness condition is not satisfied in this BSLQ problem since the coefficient of $|Z(t)|^2$ is negative.

The *indefinite* (by which we mean the weighting matrices in the cost functional are not necessarily positive semidefinite) backward stochastic LQ optimal control problem remains open. It is of great practical importance and more difficult than the definite case mentioned previously. The reason for this is that without the positive definiteness assumption one does not even know whether the problem admits a solution. On the other hand, even if optimal controls exist, it is by no means trivial to construct one by using the forward formulation and limiting procedure developed in [13], because the forward formulation is also indefinite and it is not clear whether the solution to its associated Riccati equation is invertible or not (and hence the limiting procedure cannot proceed). Moreover, the presence of the cross terms in (Y,Z,u) in the cost functional, especially the cross term in Y and Z, brings extra difficulty to the problem. As we shall see in Section 5, for the indefinite problem one cannot eliminate all cross terms simultaneously by transformations. We point out that taking cross terms into consideration is not just to make the framework more general, but also to prepare for solving LQ differential games. As mentioned earlier, a BSLQ problem arises when we look at the game in a leader-follower manner, whose cost functional is exactly of the form introduced in this paper.

The purpose of this paper is to carry out a thorough study of the indefinite BSLQ problem. We explore the abstract structure of Problem (BSLQ) from a Hilbert space point of view and derive necessary and sufficient conditions for the existence of optimal controls. The backward problem turns out to possess a similar structure as the forward stochastic LQ problem (Problem (FSLQ), for short), which suggests that the uniform convexity of the cost functional is the essential condition for solving Problem (BSLQ). We also establish a characterization of the optimal control by means of forward-backward stochastic differential equations (FBSDEs, for short). With this characterization the verification of the optimality of the control constructed in Section 5 becomes straightforward and no longer needs the completion-of-squares technique. The crucial step in constructing the optimal control is to establish the solvability of a Riccati-type equation. We accomplish this by examining the connection between Problem (BSLQ) and Problem (FSLQ) and discovering some nice properties of the solution to the Riccati equation associated with Problem (FSLQ). It is worth pointing out that a transformation, which converts the original backward problem to an equivalent one with $G = Q = 0$, plays a central role in our analysis. This transformation simplifies Problem (BSLQ), and more importantly, enables us to easily obtain the positivity of the solution to the Riccati equation associated with Problem (FSLQ). As we shall see, this positivity property is the key to employ the limiting procedure.

To conclude this section, we would like to mention that there are also many studies on LQ control of FBSDEs. For example, Huang, Li, and Shi [9] considered a forward-backward LQ stochastic optimal control problem with delay; Wang, Wu, and Xiong [27] carried out an analysis on the LQ optimal control problem of FBSDEs with partial information; Wang, Xiao, and Xing [28] studied an LQ optimal control problem for mean-field FBSDEs with noisy observation; Yu [34] investigated the LQ optimal control and nonzero-sum differential game of forward-backward stochastic system. Like the works on LQ optimal control problem for BSDEs mentioned earlier, in those works the positive-definiteness assumption on the weighting matrices was also taken for granted. To our best knowledge, the corresponding indefinite problems are still not completely solved so far. We hope to report some relevant results along this line in our future publications. We would also like to mention [20], which generalized our framework to the nonhomogeneous case. An old version of our paper can be found in [21]. Compared with [21], we have reorganized the structure slightly and corrected a few typos in the current paper.

The remainder of this paper is structured as follows. In Section 2 we give the preliminaries and collect some recently developed results on forward stochastic LQ optimal control problems. In Section 3 we study Problem (BSLQ) from a Hilbert space point of view and derive necessary and sufficient conditions for the existence of an optimal control. By means of forward-backward stochastic differential equations, a characterization of the optimal control is also present. The connection between Problem (BSLQ) and Problem (FSLQ) is discussed in Section 4. In Section 5 we simplify Problem (BSLQ) and construct the optimal control in the case that the cost functional is uniformly convex. Section 6 concludes the paper.

2. Preliminaries

We begin by introducing some notations. Let $\mathbb{R}^{n\times m}$ be the Euclidean space of $n\times m$ real matrices, equipped with the Frobenius inner product

$$\langle M,N\rangle=\operatorname{tr}(M^\top N),\quad M,N\in\mathbb{R}^{n\times m},$$

where $\operatorname{tr}(M^\top N)$ is the trace of $M^\top N$. The norm induced by the Frobenius inner product is denoted by $|\cdot|$. The identity matrix of size n is denoted by I_n. When no confusion arises, we often suppress the index n and write I instead of I_n. Let $\mathbb{S}^n$ be the subspace of $\mathbb{R}^{n\times n}$ consisting of symmetric matrices. For $\mathbb{S}^n$-valued functions M and N, we write $M\geqslant N$ (respectively, $M>N$) if $M-N$ is positive semidefinite (respectively, positive definite) almost everywhere (with respect to the Lebesgue measure), and write $M\gg 0$ if there exists a constant $\delta>0$ such that $M\geqslant\delta I_n$. For a subset $\mathbb{H}$ of $\mathbb{R}^{n\times m}$, we denote by $C([0,T];\mathbb{H})$ the space of continuous functions from $[0,T]$ into $\mathbb{H}$, and by $L^\infty(0,T;\mathbb{H})$ the space of Lebesgue measurable, essentially bounded functions from $[0,T]$ into $\mathbb{H}$. Besides the space $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ introduced previously, the following spaces of stochastic processes will also be frequently used in the sequel:

$$\begin{aligned}
L^2_{\mathbb{F}}(0,T;\mathbb{H})&:=\Big\{\varphi:[0,T]\times\Omega\to\mathbb{H}\mid\varphi\in\mathbb{F}\text{ and }\mathbb{E}\textstyle\int_0^T|\varphi(t)|^2\mathrm{d}t<\infty\Big\},\\
L^2_{\mathbb{F}}(\Omega;C([0,T];\mathbb{H}))&:=\Big\{\varphi:[0,T]\times\Omega\to\mathbb{H}\mid\varphi\text{ has continuous paths, }\varphi\in\mathbb{F},\\
&\qquad\text{and }\mathbb{E}\Big[\textstyle\sup_{0\leqslant t\leqslant T}|\varphi(t)|^2\Big]<\infty\Big\}.
\end{aligned}$$

We impose the following conditions on the coefficients of the state equation (1.1) and the weighting matrices of the cost functional (1.2).

(A1). The coefficients of the state equation (1.1) satisfy

$$A\in L^\infty(0,T;\mathbb{R}^{n\times n}),\ B\in L^\infty(0,T;\mathbb{R}^{n\times m}),\ C\in L^\infty(0,T;\mathbb{R}^{n\times n}).$$

(A2). The weighting matrices in the cost functional (1.2) satisfy

$$G\in\mathbb{S}^n,\ Q\in L^\infty(0,T;\mathbb{S}^n),\ S\in L^\infty(0,T;\mathbb{R}^{(n+m)\times n}),\ R\in L^\infty(0,T;\mathbb{S}^{n+m}).$$

For the well-posedness of the state equation (1.1) we present the following lemma, which is a direct consequence of the theory of linear BSDEs; see [33, Chapter 7].

Lemma 2.1. *Let (A1) hold. Then for any $(\xi,u)\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)\times\mathscr{U}$, the state equation (1.1) admits a unique adapted solution*

$$(Y,Z)\in L^2_{\mathbb{F}}(\Omega;C([0,T];\mathbb{R}^n))\times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n).$$

Moreover, there exists a constant $K>0$, independent of ξ and u, such that

$$\mathbb{E}\left[\sup_{0\leqslant t\leqslant T}|Y(t)|^2+\int_0^T|Z(t)|^2\mathrm{d}t\right]\leqslant K\mathbb{E}\left[|\xi|^2+\int_0^T|u(t)|^2\mathrm{d}t\right].\tag{2.1}$$

We next collect some results from forward stochastic LQ optimal control theory. These results will be needed in Section 4 and Section 5. Consider the forward linear stochastic differential equation

$$\begin{cases} d\mathcal{X}(t) = [\mathcal{A}(t)\mathcal{X}(t) + \mathcal{B}(t)v(t)]dt + [\mathcal{C}(t)\mathcal{X}(t) + \mathcal{D}(t)v(t)]dW(t), \quad t \in [0,T], \\ \mathcal{X}(0) = x, \end{cases} \tag{2.2}$$

and the cost functional

$$\mathcal{J}(x;v) := \mathbb{E}\left[\langle \mathcal{G}\mathcal{X}(T), \mathcal{X}(T)\rangle + \int_0^T \left\langle \begin{pmatrix} \mathcal{Q}(t) & \mathcal{S}^\top(t) \\ \mathcal{S}(t) & \mathcal{R}(t) \end{pmatrix} \begin{pmatrix} \mathcal{X}(t) \\ v(t) \end{pmatrix}, \begin{pmatrix} \mathcal{X}(t) \\ v(t) \end{pmatrix} \right\rangle dt\right], \tag{2.3}$$

where in (2.2) and (2.3),

$$\begin{gathered} \mathcal{A}, \mathcal{C} \in L^\infty(0,T;\mathbb{R}^{n\times n}), \quad \mathcal{B}, \mathcal{D} \in L^\infty(0,T;\mathbb{R}^{n\times m}), \\ \mathcal{G} \in \mathbb{S}^n, \quad \mathcal{Q} \in L^\infty(0,T;\mathbb{S}^n), \quad \mathcal{S} \in L^\infty(0,T;\mathbb{R}^{m\times n}), \quad \mathcal{R} \in L^\infty(0,T;\mathbb{S}^m). \end{gathered}$$

The forward stochastic LQ optimal control problem is as follows.

Problem (FSLQ). For a given initial state $x \in \mathbb{R}^n$, find a control $v^* \in \mathscr{U} = L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ such that

$$\mathcal{J}(x;v^*) = \inf_{v\in\mathscr{U}} \mathcal{J}(x;v) \equiv \mathcal{V}(x). \tag{2.4}$$

The control v^* (if it exists) in (2.4) is called an *open-loop optimal control* for the initial state x, and $\mathcal{V}(x)$ is called the *value* of Problem (FSLQ) at x. Note that Problem (FSLQ) is an indefinite LQ optimal control problem, since we do not require the weighting matrices to be positive semidefinite. The following lemma establishes the solvability of Problem (FSLQ) under a condition that is nearly necessary for the existence of open-loop optimal controls. We refer the reader to Sun, Li, and Yong [18] and the recent book [24] by Sun and Yong for proofs and further information.

Lemma 2.2. *Suppose that there exists a constant $\alpha > 0$ such that*

$$\mathcal{J}(0;v) \geqslant \alpha\, \mathbb{E}\int_0^T |v(t)|^2 dt, \quad \forall v \in \mathscr{U}. \tag{2.5}$$

Then the Riccati differential equation

$$\begin{cases} \dot{\mathcal{P}} + \mathcal{P}\mathcal{A} + \mathcal{A}^\top\mathcal{P} + \mathcal{C}^\top\mathcal{P}\mathcal{C} + \mathcal{Q} \\ \quad - (\mathcal{P}\mathcal{B} + \mathcal{C}^\top\mathcal{P}\mathcal{D} + \mathcal{S}^\top)(\mathcal{R} + \mathcal{D}^\top\mathcal{P}\mathcal{D})^{-1}(\mathcal{B}^\top\mathcal{P} + \mathcal{D}^\top\mathcal{P}\mathcal{C} + \mathcal{S}) = 0, \\ \mathcal{P}(T) = \mathcal{G} \end{cases} \tag{2.6}$$

admits a unique solution $\mathcal{P} \in C([0,T];\mathbb{S}^n)$ such that

$$\mathcal{R} + \mathcal{D}^\top\mathcal{P}\mathcal{D} \gg 0.$$

Moreover, for each initial state x, a unique open-loop optimal control exists and is given by the following closed-loop form:

$$v^* = -(\mathcal{R} + \mathcal{D}^\top\mathcal{P}\mathcal{D})^{-1}(\mathcal{B}^\top\mathcal{P} + \mathcal{D}^\top\mathcal{P}\mathcal{C} + \mathcal{S})X,$$

and the value at x is given by

$$\mathcal{V}(x)=\langle\mathcal{P}(0)x,x\rangle,\quad\forall x\in\mathbb{R}^n.$$

We have the following corollary to Lemma 2.2.

Corollary 2.3. *Suppose that*

$$\mathcal{G}\geqslant 0,\quad \mathcal{R}\gg 0,\quad \mathcal{Q}-\mathcal{S}^\top\mathcal{R}^{-1}\mathcal{S}\geqslant 0. \tag{2.7}$$

Then (2.5) *holds for some constant $\alpha>0$, and the solution of* (2.6) *satisfies*

$$\mathcal{P}(t)\geqslant 0,\quad \forall t\in[0,T].$$

If, in addition to (2.7), *$\mathcal{G}>0$, then $\mathcal{P}(t)>0$ for all $t\in[0,T]$.*

Proof. Take an arbitrary initial state x and an arbitrary control $v\in\mathscr{U}$, and let $\mathcal{X}_x^v$ denote the solution of (2.2) corresponding to x and v. By Assumption (2.7), we have

$$\begin{aligned}\mathcal{J}(x;v)&=\mathbb{E}\langle\mathcal{G}\mathcal{X}_x^v(T),\mathcal{X}_x^v(T)\rangle+\mathbb{E}\int_0^T\Big\{\big\langle[\mathcal{Q}(t)-\mathcal{S}^\top(t)\mathcal{R}^{-1}(t)\mathcal{S}(t)]\mathcal{X}_x^v(t),\mathcal{X}_x^v(t)\big\rangle\\&\qquad+\big|\mathcal{R}^{\frac12}(t)[v(t)+\mathcal{R}^{-1}(t)\mathcal{S}(t)\mathcal{X}_x^v(t)]\big|^2\Big\}\mathrm{d}t\\&\geqslant\mathbb{E}\langle\mathcal{G}\mathcal{X}_x^v(T),\mathcal{X}_x^v(T)\rangle+\mathbb{E}\int_0^T\big|\mathcal{R}^{\frac12}(t)[v(t)+\mathcal{R}^{-1}(t)\mathcal{S}(t)\mathcal{X}_x^v(t)]\big|^2\mathrm{d}t.\end{aligned}\tag{2.8}$$

Since $\mathcal{G}\geqslant 0$ and $\mathcal{R}(t)\geqslant\delta I$ a.e. $t\in[0,T]$ for some constant $\delta>0$, (2.8) implies

$$\mathcal{J}(x;v)\geqslant\delta\,\mathbb{E}\int_0^T\big|v(t)+\mathcal{R}^{-1}(t)\mathcal{S}(t)\mathcal{X}_x^v(t)\big|^2\mathrm{d}t\geqslant 0. \tag{2.9}$$

For $x=0$, we can define a linear operator $\mathfrak{L}:\mathscr{U}\to\mathscr{U}$ by

$$\mathfrak{L}v=v+\mathcal{R}^{-1}\mathcal{S}\mathcal{X}_0^v.$$

It is easy to see that $\mathfrak{L}$ is bounded and bijective, with inverse $\mathfrak{L}^{-1}$ given by

$$\mathfrak{L}^{-1}v=v-\mathcal{R}^{-1}\mathcal{S}\tilde{\mathcal{X}}_0^v,$$

where $\tilde{\mathcal{X}}_0^v$ is the solution of

$$\begin{cases}\mathrm{d}\tilde{\mathcal{X}}_0^v(t)=\big[(\mathcal{A}-\mathcal{B}\mathcal{R}^{-1}\mathcal{S})\tilde{\mathcal{X}}_0^v+\mathcal{B}v\big]\mathrm{d}t\\\qquad\qquad+\big[(\mathcal{C}-\mathcal{D}\mathcal{R}^{-1}\mathcal{S})\tilde{\mathcal{X}}_0^v+\mathcal{D}v\big]\mathrm{d}W(t),\quad t\in[0,T],\\\tilde{\mathcal{X}}_0^v(0)=0.\end{cases}$$

By the bounded inverse theorem, $\mathfrak{L}^{-1}$ is also bounded with $\|\mathfrak{L}^{-1}\|>0$. Thus,

$$\begin{aligned}\mathcal{J}(0;v)&\geqslant\delta\,\mathbb{E}\int_0^T\big|v(t)+\mathcal{R}^{-1}(t)\mathcal{S}(t)\mathcal{X}_0^v(t)\big|^2\mathrm{d}t=\delta\,\mathbb{E}\int_0^T|(\mathfrak{L}v)(t)|^2\mathrm{d}t\\&\geqslant\frac{\delta}{\|\mathfrak{L}^{-1}\|^2}\,\mathbb{E}\int_0^T|v(t)|^2\mathrm{d}t.\end{aligned}$$

This shows that (2.5) holds with $\alpha = \frac{\delta}{\|\mathfrak{L}^{-1}\|^2}$. From (2.9) and Lemma 2.2 we obtain

$$\langle \mathcal{P}(0)x, x\rangle = \inf_{v\in\mathscr{U}} \mathcal{J}(x;v) \geqslant 0, \quad \forall x \in \mathbb{R}^n,$$

which implies $\mathcal{P}(0) \geqslant 0$. To see that $\mathcal{P}(0) > 0$ when $\mathcal{G} > 0$, we assume to the contrary that $\langle \mathcal{P}(0)x, x\rangle = 0$ for some $x \neq 0$. Let $\bar{v}$ be the optimal control for x. Then we have from (2.8) that

$$0 = \mathcal{J}(x;\bar{v}) \geqslant \mathbb{E}\langle \mathcal{G}\mathcal{X}_x^{\bar{v}}(T), \mathcal{X}_x^{\bar{v}}(T)\rangle + \mathbb{E}\int_0^T \left|\mathcal{R}^{\frac{1}{2}}(t)[\bar{v}(t) + \mathcal{R}^{-1}(t)\mathcal{S}(t)\mathcal{X}_x^{\bar{v}}(t)]\right|^2 \mathrm{d}t.$$

Since $\mathcal{G} > 0$ and $\mathcal{R} \gg 0$, the above implies

$$\mathcal{X}_x^{\bar{v}}(T) = 0, \quad \bar{v}(t) = -\mathcal{R}^{-1}(t)\mathcal{S}(t)\mathcal{X}_x^{\bar{v}}(t).$$

Therefore, $\mathcal{X}_x^{\bar{v}}$ satisfies the following equation:

$$\begin{cases} \mathrm{d}\mathcal{X}(t) = [\mathcal{A}(t) - \mathcal{B}(t)\mathcal{R}^{-1}(t)\mathcal{S}(t)]\mathcal{X}(t)\mathrm{d}t + [\mathcal{C}(t) - \mathcal{D}(t)\mathcal{R}^{-1}(t)\mathcal{S}(t)]\mathcal{X}(t)\mathrm{d}W(t), \\ \mathcal{X}(0) = x, \quad \mathcal{X}(T) = 0, \end{cases}$$

which is impossible because $x \neq 0$. Finally, by considering Problem (FSLQ) over the interval $[t, T]$ and repeating the preceding argument, we obtain $\mathcal{P}(t) \geqslant 0$ (respectively, $\mathcal{P}(t) > 0$ if $\mathcal{G} > 0$). $\square$

3. A study from the Hilbert space point of view

Observe that $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, $\mathscr{U}$, and $L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$ are Hilbert spaces equipped with their usual L^2-inner products, and that $L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n)) \subseteq L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$. For a terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$ and a control $u \in \mathscr{U}$, we denote by (Y_ξ^u, Z_ξ^u) the adapted solution to the state equation (1.1). By the linearity of the state equation,

$$Y_\xi^u = Y_\xi^0 + Y_0^u, \quad Z_\xi^u = Z_\xi^0 + Z_0^u.$$

Note that (Y_ξ^0, Z_ξ^0) linearly depends on ξ and (Y_0^u, Z_0^u) linearly depends on u. Thus, Lemma 2.1 implies that the following are all bounded linear operators:

$$\begin{aligned}
&\mathcal{L}_0 : L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n) \to \mathbb{R}^n; && \mathcal{L}_0\xi := Y_\xi^0(0), \\
&\mathcal{L}_1 : L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n) \to L^2_{\mathbb{F}}(0,T;\mathbb{R}^n); && \mathcal{L}_1\xi := Y_\xi^0, \\
&\mathcal{L}_2 : L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n) \to L^2_{\mathbb{F}}(0,T;\mathbb{R}^n); && \mathcal{L}_2\xi := Z_\xi^0, \\
&\mathcal{K}_0 : \mathscr{U} \to \mathbb{R}^n; && \mathcal{K}_0 u := Y_0^u(0), \\
&\mathcal{K}_1 : \mathscr{U} \to L^2_{\mathbb{F}}(0,T;\mathbb{R}^n); && \mathcal{K}_1 u := Y_0^u, \\
&\mathcal{K}_2 : \mathscr{U} \to L^2_{\mathbb{F}}(0,T;\mathbb{R}^n); && \mathcal{K}_2 u := Z_0^u.
\end{aligned}$$

Let us denote by $\mathcal{T}^*$ the adjoint operator of a bounded linear operator $\mathcal{T}$ between Banach spaces, and denote the inner product of two elements ϕ and φ in an L^2 space by $[\![\phi, \varphi]\!]$. Set

$$M := \begin{pmatrix} Q & S_1^\top & S_2^\top \\ S_1 & R_{11} & R_{12} \\ S_2 & R_{21} & R_{22} \end{pmatrix}, \quad \mathcal{L} := \begin{pmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ 0 \end{pmatrix}, \quad \mathcal{K} := \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{I} \end{pmatrix},$$

where $\mathcal{I}$ is the identity operator. In terms of the above notation, the cost functional (1.2) can be written as follows:

$$\begin{aligned} J(\xi;u) &= [\![G(\mathcal{L}_0\xi+\mathcal{K}_0u),\mathcal{L}_0\xi+\mathcal{K}_0u]\!] + [\![M(\mathcal{L}\xi+\mathcal{K}u),\mathcal{L}\xi+\mathcal{K}u]\!] \\ &= [\![(\mathcal{K}_0^*G\mathcal{K}_0+\mathcal{K}^*M\mathcal{K})u,u]\!] + 2[\![(\mathcal{K}_0^*G\mathcal{L}_0+\mathcal{K}^*M\mathcal{L})\xi,u]\!] \\ &\quad + [\![(\mathcal{L}_0^*G\mathcal{L}_0+\mathcal{L}^*M\mathcal{L})\xi,\xi]\!] \\ &\equiv [\![\widehat{\mathcal{A}}u,u]\!] + 2[\![\widehat{\mathcal{B}}\xi,u]\!] + [\![\widehat{\mathcal{C}}\xi,\xi]\!]. \end{aligned} \tag{3.1}$$

Clearly,

$$\begin{aligned} &\widehat{\mathcal{A}} = \mathcal{K}_0^*G\mathcal{K}_0+\mathcal{K}^*M\mathcal{K}: \ \mathscr{U}\to\mathscr{U}, \\ &\widehat{\mathcal{C}} = \mathcal{L}_0^*G\mathcal{L}_0+\mathcal{L}^*M\mathcal{L} \ : \ L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)\to L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n) \end{aligned}$$

are bounded linear self-adjoint operators, and

$$\widehat{\mathcal{B}} = \mathcal{K}_0^*G\mathcal{L}_0+\mathcal{K}^*M\mathcal{L}: \ L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)\to\mathscr{U}$$

is a bounded linear operator.

From the representation (3.1), we obtain the following characterization of optimal controls.

Theorem 3.1. *Let (A1)–(A2) hold. For a given terminal state $\xi\in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$, a control $u^*\in\mathscr{U}$ is optimal if and only if*

(i) $\widehat{\mathcal{A}}\geqslant 0$ (i.e., $\widehat{\mathcal{A}}$ is a positive operator), and
(ii) $\widehat{\mathcal{A}}u^*+\widehat{\mathcal{B}}\xi=0$.

Proof. For every $u\in\mathscr{U}$ and $\lambda\in\mathbb{R}$, we have

$$\begin{aligned} J(\xi;u^*+\lambda u) &= [\![\widehat{\mathcal{A}}(u^*+\lambda u),u^*+\lambda u]\!] + 2[\![\widehat{\mathcal{B}}\xi,u^*+\lambda u]\!] + [\![\widehat{\mathcal{C}}\xi,\xi]\!] \\ &= J(\xi;u^*) + \lambda^2[\![\widehat{\mathcal{A}}u,u]\!] + 2\lambda[\![\widehat{\mathcal{A}}u^*+\widehat{\mathcal{B}}\xi,u]\!], \end{aligned}$$

from which it follows that u^* is optimal for ξ if and only if

$$\lambda^2[\![\widehat{\mathcal{A}}u,u]\!] + 2\lambda[\![\widehat{\mathcal{A}}u^*+\widehat{\mathcal{B}}\xi,u]\!]\geqslant 0, \quad \forall\lambda\in\mathbb{R}, \ \forall u\in\mathscr{U}. \tag{3.2}$$

Clearly, (i) and (ii) imply (3.2). Conversely, if (3.2) holds, taking $\lambda=\pm1$ in (3.2) and then adding the resulting inequalities, we obtain $\widehat{\mathcal{A}}\geqslant 0$. Dividing both sides of the inequality in (3.2) by $\lambda>0$ and then sending $\lambda\to 0$ gives

$$[\![\widehat{\mathcal{A}}u^*+\widehat{\mathcal{B}}\xi,u]\!]\geqslant 0, \quad \forall u\in\mathscr{U}.$$

Dividing both sides of the inequality in (3.2) by $\lambda<0$ and then sending $\lambda\to 0$ gives

$$[\![\widehat{\mathcal{A}}u^*+\widehat{\mathcal{B}}\xi,u]\!]\leqslant 0, \quad \forall u\in\mathscr{U}.$$

Therefore, $[\![\widehat{\mathcal{A}}u^*+\widehat{\mathcal{B}}\xi,u]\!]=0$ for all $u\in\mathscr{U}$ and thereby (ii) holds. ∎

Remark 3.2. The condition $\widehat{\mathcal{A}} \geqslant 0$ is compatible with indefinite weighting matrices. To see this, let us take $S_i = 0$; $i = 1, 2$, for simplicity. Then $\widehat{\mathcal{A}}$ can be rewritten as follows:

$$\widehat{\mathcal{A}} = \mathcal{K}_0^* G \mathcal{K}_0 + \mathcal{K}_1^* Q \mathcal{K}_1 + \mathcal{K}_2^* R_{11} \mathcal{K}_2 + R_{22}. \tag{3.3}$$

We will show later in Remark 4.3 that different from the forward stochastic LQ optimal control problem, for the operator $\widehat{\mathcal{A}}$ to be positive, the weighting matrix R_{22} must be positive semidefinite. Thus, "indefinite" means that the weighting matrices G, Q, and R_{11} could be indefinite. This is possible for $\widehat{\mathcal{A}} \geqslant 0$, as we can see from (3.3) that no matter how negative G, Q, and R_{11} are, $\widehat{\mathcal{A}}$ will be positive as long as R_{22} is positive enough. The following is such an example.

Example 3.3. Consider the one-dimensional state equation

$$\begin{cases} \mathrm{d}Y(t) = u(t)\mathrm{d}t + Z(t)\mathrm{d}W(t), \quad t \in [0,1], \\ Y(1) = \xi \end{cases}$$

and the cost functional

$$J(\xi; u) := \mathbb{E}\bigg\{ -|Y(0)|^2 + \int_0^1 \Big[-|Y(t)|^2 - |Z(t)|^2 + a|u(t)|^2 \Big]\mathrm{d}t \bigg\}.$$

From (3.1) we see that $[\![\widehat{\mathcal{A}}u, u]\!] = J(0; u)$. For the terminal state $\xi = 0$,

$$Y(t) = -\int_t^1 u(s)\mathrm{d}s - \int_t^1 Z(s)\mathrm{d}W(s).$$

Taking conditional expectations with respect to $\mathcal{F}_t$ yields

$$Y(t) = -\mathbb{E}\left[\int_t^1 u(s)\mathrm{d}s \,\bigg|\, \mathcal{F}_t \right].$$

Jensen's inequality implies that

$$\mathbb{E}|Y(t)|^2 \leqslant \mathbb{E}\left[\int_t^1 u(s)\mathrm{d}s \right]^2 \leqslant \mathbb{E}\int_0^1 |u(s)|^2 \mathrm{d}s, \quad \forall t \in [0,1].$$

Consequently,

$$\mathbb{E}\int_0^1 |Z(s)|^2 ds = \mathbb{E}\left[\int_0^1 Z(s)\mathrm{d}W(s) \right]^2 = \mathbb{E}\left[Y(0) + \int_0^1 u(s)\mathrm{d}s \right]^2 \leqslant 4\,\mathbb{E}\int_0^1 |u(t)|^2 \mathrm{d}t.$$

Now it is easy to see that $\widehat{\mathcal{A}} \geqslant 0$ if $a \geqslant 6$.

We see from Theorem 3.1 that the positivity condition $\widehat{\mathcal{A}} \geqslant 0$, which by (3.1) is equivalent to

$$J(0; u) \geqslant 0, \quad \forall u \in \mathscr{U},$$

is necessary for the existence of an optimal control. On the other hand, if $\widehat{\mathcal{A}}$ is uniformly positive, *i.e.*, there exists a constant $\delta > 0$ such that

$$[\![\widehat{\mathcal{A}}u, u]\!] = J(0; u) \geqslant \delta\, \mathbb{E}\int_0^T |u(t)|^2 \mathrm{d}t, \quad \forall u \in \mathscr{U},$$

then by Theorem 3.1, for each $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, an optimal control u^* uniquely exists and is given by

$$u^* = -\widehat{\mathcal{A}}^{-1}\widehat{\mathcal{B}}\xi.$$

When the necessity condition $\widehat{\mathcal{A}} \geqslant 0$ holds but it is not clear if $\widehat{\mathcal{A}}$ is uniformly positive, we can define, for each $\varepsilon > 0$, a new cost functional $J_\varepsilon(\xi; u)$ by

$$\begin{aligned} J_\varepsilon(\xi; u) &:= J(\xi; u) + \varepsilon\, \mathbb{E}\int_0^T |u(t)|^2 \mathrm{d}t \\ &= [\![(\widehat{\mathcal{A}} + \varepsilon\mathcal{I})u, u]\!] + 2[\![\widehat{\mathcal{B}}\xi, u]\!] + [\![\widehat{\mathcal{C}}\xi, \xi]\!]. \end{aligned}$$

For the new cost functional, the (unique) optimal control u^*_ε for ξ exists and is given by

$$u^*_\varepsilon = -(\widehat{\mathcal{A}} + \varepsilon\mathcal{I})^{-1}\widehat{\mathcal{B}}\xi.$$

In terms of the family $\{u^*_\varepsilon\}_{\varepsilon>0}$, we now provide a sufficient and necessary condition for the existence of optimal controls.

Theorem 3.4. *Let (A1)–(A2) hold and assume that the necessity condition $\widehat{\mathcal{A}} \geqslant 0$ holds. Then an optimal control exists for a given terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ if and only if one of the following conditions holds:*

(i) the family $\{u^*_\varepsilon\}_{\varepsilon>0}$ is bounded in the Hilbert space $\mathscr{U}$;
(ii) u^*_ε converges weakly in $\mathscr{U}$ as $\varepsilon \to 0$;
(iii) u^*_ε converges strongly in $\mathscr{U}$ as $\varepsilon \to 0$.

Whenever (i), (ii), or (iii) is satisfied, the strong (weak) limit $u^* = \lim_{\varepsilon\to 0} u^*_\varepsilon$ is an optimal control for ξ.

Proof. Suppose that $u^* \in \mathscr{U}$ is optimal for ξ. Then by Theorem 3.1,

$$\widehat{\mathcal{A}}u^* + \widehat{\mathcal{B}}\xi = 0.$$

Write u^* as $u^* = u + v$ with $u \in \ker\widehat{\mathcal{A}}$ and $v \in \overline{\operatorname{im}\widehat{\mathcal{A}}}$, where $\ker\widehat{\mathcal{A}}$ and $\overline{\operatorname{im}\widehat{\mathcal{A}}}$ are the kernel and the closure of the image of $\widehat{\mathcal{A}}$, respectively. Then

$$\widehat{\mathcal{A}}v + \widehat{\mathcal{B}}\xi = 0,$$

and thereby v is also an optimal control for ξ. For a fixed but arbitrary $\delta > 0$, since $v \in \overline{\operatorname{im}\widehat{\mathcal{A}}}$, we can find a $w \in \mathcal{U}$ such that $\|\widehat{\mathcal{A}}w - v\| \leqslant \delta$. Then using the fact

$$-\widehat{\mathcal{B}}\xi = \widehat{\mathcal{A}}v, \quad \|(\widehat{\mathcal{A}} + \varepsilon\mathcal{I})^{-1}\| \leqslant \varepsilon^{-1}, \quad \|(\widehat{\mathcal{A}} + \varepsilon\mathcal{I})^{-1}\widehat{\mathcal{A}}\| \leqslant 1,$$

we have for any $\varepsilon > 0$ that

$$\begin{aligned}\|u_\varepsilon^* - v\| &= \| - (\widehat{\mathcal{A}} + \varepsilon\mathcal{I})^{-1}\widehat{\mathcal{B}}\xi - v\| = \|(\widehat{\mathcal{A}} + \varepsilon\mathcal{I})^{-1}\widehat{\mathcal{A}}v - v\| = \varepsilon\|(\widehat{\mathcal{A}} + \varepsilon\mathcal{I})^{-1}v\| \\ &\leqslant \varepsilon\|(\widehat{\mathcal{A}} + \varepsilon\mathcal{I})^{-1}(v - \widehat{\mathcal{A}}w)\| + \varepsilon\|(\widehat{\mathcal{A}} + \varepsilon\mathcal{I})^{-1}\widehat{\mathcal{A}}w\| \\ &\leqslant \delta + \varepsilon\|w\|.\end{aligned}$$

Letting $\varepsilon \to 0$ and noting that $\delta > 0$ is arbitrary, we conclude that u_ε^* converges strongly to the optimal control v as $\varepsilon \to 0$.

It is clear that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). So it remains to show that (i) implies the existence of an optimal control for ξ. When (i) holds, we can select a sequence $\{\varepsilon_n\}_{n=1}^\infty$ with ε_n decreasing to 0 (as $n \to \infty$) such that $u_{\varepsilon_n}^*$ converges weakly to some $v^* \in \mathscr{U}$. By Mazur's lemma, for each integer $k \geqslant 1$, there exists finite many positive numbers $\alpha_{k1}, \ldots, \alpha_{kN_k}$ with $\alpha_{k1} + \cdots + \alpha_{kN_k} = 1$ such that $v_k \triangleq \sum_{j=1}^{N_k} \alpha_{kj} u_{\varepsilon_{k+j}}^*$ converges strongly to v^* as $k \to \infty$. Then

$$\begin{aligned}J(\xi; v^*) &= \lim_{k\to\infty} \Big([\![\widehat{\mathcal{A}}v_k, v_k]\!] + 2[\![\widehat{\mathcal{B}}\xi, v_k]\!] + [\![\widehat{\mathcal{C}}\xi, \xi]\!] \Big) \\ &\leqslant \liminf_{k\to\infty} \sum_{j=1}^{N_k} \alpha_{kj} \Big([\![\widehat{\mathcal{A}}u_{\varepsilon_{k+j}}^*, u_{\varepsilon_{k+j}}^*]\!] + 2[\![\widehat{\mathcal{B}}\xi, u_{\varepsilon_{k+j}}^*]\!] + [\![\widehat{\mathcal{C}}\xi, \xi]\!] \Big) \\ &\leqslant \liminf_{k\to\infty} \sum_{j=1}^{N_k} \alpha_{kj} \Big([\![(\widehat{\mathcal{A}} + \varepsilon_{k+j}\mathcal{I})u_{\varepsilon_{k+j}}^*, u_{\varepsilon_{k+j}}^*]\!] + 2[\![\widehat{\mathcal{B}}\xi, u_{\varepsilon_{k+j}}^*]\!] + [\![\widehat{\mathcal{C}}\xi, \xi]\!] \Big) \\ &= \liminf_{k\to\infty} \sum_{j=1}^{N_k} \alpha_{kj} J_{\varepsilon_{k+j}}(\xi; u_{\varepsilon_{k+j}}^*). \end{aligned} \tag{3.4}$$

Since $\lim_{k\to\infty} \sum_{j=1}^{N_k} \alpha_{kj}\varepsilon_{k+j} = 0$ and for any $u \in \mathscr{U}$,

$$J_{\varepsilon_{k+j}}(\xi; u_{\varepsilon_{k+j}}^*) \leqslant J_{\varepsilon_{k+j}}(\xi; u) = J(\xi; u) + \varepsilon_{k+j}\|u\|^2,$$

we conclude from (3.4) that

$$J(\xi; v^*) \leqslant J(\xi; u), \quad \forall u \in \mathscr{U}.$$

This shows that v^* is an optimal control for ξ and hence completes the proof. □

Theorem 3.1 provides a characterization of the optimal control using the operators in (3.1). We now present an alternative characterization in terms of forward-backward stochastic differential equations, which is more convenient for the verification of optimal controls. This result will be used in Section 5 to prove the control constructed there is optimal. The proof is standard and therefore omitted here.

Theorem 3.5. *Let (A1)–(A2) hold and let the terminal state $\xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ be given. A control $u^* \in \mathscr{U}$ is optimal for ξ if and only if the following conditions hold:*

(i) $J(0; u) \geqslant 0$ for all $u \in \mathscr{U}$.

(ii) The adapted solution (X^*, Y^*, Z^*) *to the decoupled FBSDE*

$$\begin{cases} \mathrm{d}X^*(t) = (-A^\top X^* + QY^* + S_1^\top Z^* + S_2^\top u^*)\mathrm{d}t \\ \qquad\qquad + (-C^\top X^* + S_1 Y^* + R_{11} Z^* + R_{12} u^*)\mathrm{d}W, \\ \mathrm{d}Y^*(t) = (AY^* + Bu^* + CZ^*)\mathrm{d}t + Z^*\mathrm{d}W, \\ X^*(0) = GY^*(0), \quad Y^*(T) = \xi, \end{cases} \tag{3.5}$$

satisfies

$$S_2 Y^* + R_{21} Z^* - B^\top X^* + R_{22} u^* = 0. \tag{3.6}$$

4. Connections with forward stochastic LQ optimal control problems

It is not easy to decide whether Problem (BSLQ) admits an optimal control (let alone constructing one) for a given terminal state when the operator $\widehat{\mathcal{A}}$ is merely positive. However, if the case that $\widehat{\mathcal{A}}$ is uniformly positive can be solved, then we are at least able to construct a minimizing sequence for Problem (BSLQ), by using Theorem 3.4. To solve the uniform positivity case, we investigate in this section the connection between Problem (BSLQ) and the forward stochastic LQ optimal control problem (FSLQ problem, for short) under the uniform positivity condition:

(A3). There exists a constant $\delta > 0$ such that

$$[\![\widehat{\mathcal{A}}u, u]\!] = J(0; u) \geqslant \delta\, \mathbb{E} \int_0^T |u(t)|^2 \mathrm{d}t, \quad \forall u \in \mathscr{U}. \tag{4.1}$$

In light of Theorem 3.5, to obtain the optimal control u^* of Problem (BSLQ) we need to solve the *optimality system* (3.5)–(3.6). The basic idea is to decouple the optimality system by making the ansatz

$$Y^*(t) = -\Sigma(t)X^*(t) + \varphi(t),$$

where Σ is a deterministic $\mathbb{S}^n$-valued function with $\Sigma(T) = 0$, and φ is a stochastic process with $\varphi(T) = \xi$. The key of the decoupling method is the existence of such a Σ. It is possible to derive the differential equation for Σ by comparing the drift and the diffusion terms in the equations for Y^* and $-\Sigma X^* + \varphi$, but the solvability of the derived equation is difficult to establish directly. Inspired by Lim and Zhou [13], we consider Problem (BSLQ) in another way: Suppose that u^* is the optimal control of Problem (BSLQ) for the terminal state ξ. Then the initial value $Y^*(0)$ is some point in $\mathbb{R}^n$, say y^*. Now regard (u, Z) as the control of the FSDE

$$\begin{cases} \mathrm{d}Y(t) = [A(t)Y(t) + B(t)u(t) + C(t)Z(t)]\mathrm{d}t + Z(t)\mathrm{d}W(t), \quad t \in [0, T], \\ Y(0) = y^*, \end{cases}$$

and consider, for $\lambda > 0$, the cost functional

$$\mathcal{J}_\lambda(y^*; u, Z) := \mathbb{E}\left\{ \lambda |Y(T) - \xi|^2 + \int_0^T \left\langle \begin{pmatrix} Q(t) & S_1^\top(t) & S_2^\top(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t) \end{pmatrix} \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix} \right\rangle \mathrm{d}t \right\}.$$

Let (u_λ, Z_λ) be the optimal control of the FSLQ problem and Y_λ the optimal state process. Intuitively,

$$Y_\lambda(T) \to \xi \quad \text{as } \lambda \to \infty,$$

because if not, the term $\mathbb{E}(\lambda|Y(T)-\xi|^2)$ will go to infinity as $\lambda \to \infty$. This suggests that

$$(u_\lambda, Z_\lambda) \to (u^*, Z^*) \quad \text{as } \lambda \to \infty.$$

To make the above analysis rigorous, we need to establish the solvability of the above FSLQ problem. The crucial step is to prove that the uniform convexity condition (A3) implies the uniform convexity of the FSLQ problem, which is the main connection between the FSLQ problem and the original problem.

To be more precise, let us consider the controlled linear FSDE

$$\begin{cases} \mathrm{d}X(t) = [A(t)X(t) + B(t)u(t) + C(t)v(t)]\mathrm{d}t + v(t)\mathrm{d}W(t), \quad t \in [0,T], \\ X(0) = x, \end{cases} \tag{4.2}$$

and, for $\lambda > 0$, the cost functional

$$\mathcal{J}_\lambda(x; u, v) := \mathbb{E}\left\{\lambda|X(T)|^2 + \int_0^T \left\langle \begin{pmatrix} Q(t) & S_1^\top(t) & S_2^\top(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t) \end{pmatrix} \begin{pmatrix} X(t) \\ v(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ v(t) \\ u(t) \end{pmatrix} \right\rangle \mathrm{d}t \right\}. \tag{4.3}$$

In the above, the control is the pair

$$(u, v) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathbb{F}}(0,T;\mathbb{R}^n) \equiv \mathscr{U} \times \mathscr{V}.$$

We impose the following FSLQ problem.

Problem (FSLQ)$_\lambda$. For a given initial state $x \in \mathbb{R}^n$, find a control $(u^*, v^*) \in \mathscr{U} \times \mathscr{V}$ such that

$$\mathcal{J}_\lambda(x; u^*, v^*) = \inf_{(u,v)\in\mathscr{U}\times\mathscr{V}} \mathcal{J}_\lambda(x; u, v) \equiv \mathcal{V}_\lambda(x).$$

We have the following result, which plays a basic role in the subsequent analysis.

Theorem 4.1. *Let (A1)–(A2) hold. If (A3) holds, then there exist constants $\rho > 0$ and $\lambda_0 > 0$ such that for $\lambda \geqslant \lambda_0$,*

$$\mathcal{J}_\lambda(0; u, v) \geqslant \rho\, \mathbb{E}\int_0^T \Big[|u(t)|^2 + |v(t)|^2\Big]\mathrm{d}t, \quad \forall (u,v) \in \mathscr{U} \times \mathscr{V}. \tag{4.4}$$

If, in addition, $G = 0$, then for $\lambda \geqslant \lambda_0$,

$$\mathcal{J}_\lambda(x; u, v) \geqslant \rho\, \mathbb{E}\int_0^T \Big[|u(t)|^2 + |v(t)|^2\Big]\mathrm{d}t, \quad \forall (u,v) \in \mathscr{U} \times \mathscr{V},\ \forall x \in \mathbb{R}^n. \tag{4.5}$$

Proof. Fix $x \in \mathbb{R}^n$ and $(u,v) \in \mathscr{U} \times \mathscr{V}$, and let X be the solution of (4.2). We see that the random variable $\xi \triangleq X(T)$ belongs to the space $L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n)$. Denote by (Y_0, Z_0) and (Y, Z) the adapted solutions to

$$\begin{cases} \mathrm{d}Y_0(t) = [A(t)Y_0(t) + C(t)Z_0(t)]\mathrm{d}t + Z_0(t)\mathrm{d}W(t), \\ Y_0(T) = \xi, \end{cases}$$

and

$$\begin{cases} \mathrm{d}Y(t) = [A(t)Y(t) + B(t)u(t) + C(t)Z(t)]\mathrm{d}t + Z(t)\mathrm{d}W(t), \\ Y(T) = 0, \end{cases}$$

respectively. By the linearity of the above equations, we see that

$$(\hat{Y}(t), \hat{Z}(t)) := (Y_0(t) + Y(t), Z_0(t) + Z(t)); \quad t \in [0, T]$$

solves the BSDE

$$\begin{cases} \mathrm{d}\hat{Y}(t) = [A(t)\hat{Y}(t) + B(t)u(t) + C(t)\hat{Z}(t)]\mathrm{d}t + \hat{Z}(t)\mathrm{d}W(t), \\ \hat{Y}(T) = \xi. \end{cases} \tag{4.6}$$

Observe that the pair $(X(t), v(t))$ satisfies (noting that $\xi = X(T)$)

$$\begin{cases} \mathrm{d}X(t) = [A(t)X(t) + B(t)u(t) + C(t)v(t)]\mathrm{d}t + v(t)\mathrm{d}W(t), \\ X(T) = \xi. \end{cases}$$

This means that (X, v) is also an adapted solution to the BSDE (4.6). Therefore, by the uniqueness of an adapted solution,

$$X(t) = Y(t) + Y_0(t), \quad v(t) = Z(t) + Z_0(t); \quad t \in [0, T]. \tag{4.7}$$

For simplicity, we introduce the following the notation:

$$M(t) := \begin{pmatrix} Q(t) & S_1^\top(t) & S_2^\top(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t) \end{pmatrix}, \quad \alpha(t) := \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix}, \quad \beta(t) := \begin{pmatrix} Y_0(t) \\ Z_0(t) \\ 0 \end{pmatrix}.$$

By the definition (1.2), we have

$$J(0; u) = \mathbb{E}\bigg\{\langle GY(0), Y(0)\rangle + \int_0^T \langle M(t)\alpha(t), \alpha(t)\rangle \mathrm{d}t\bigg\}. \tag{4.8}$$

By the relation (4.7), we have

$$\begin{pmatrix} X(t) \\ v(t) \\ u(t) \end{pmatrix} = \begin{pmatrix} Y(t) \\ Z(t) \\ u(t) \end{pmatrix} + \begin{pmatrix} Y_0(t) \\ Z_0(t) \\ 0 \end{pmatrix} = \alpha(t) + \beta(t).$$

Substituting the above into

$$\mathcal{J}_\lambda(x; u, v) := \mathbb{E}\bigg\{\lambda|X(T)|^2 + \int_0^T \bigg\langle \begin{pmatrix} Q(t) & S_1^\top(t) & S_2^\top(t) \\ S_1(t) & R_{11}(t) & R_{12}(t) \\ S_2(t) & R_{21}(t) & R_{22}(t) \end{pmatrix} \begin{pmatrix} X(t) \\ v(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ v(t) \\ u(t) \end{pmatrix} \bigg\rangle \mathrm{d}t\bigg\}$$

and using (4.8), we obtain

$$
\begin{aligned}
\mathscr{J}_\lambda(x;u,v) &= \mathbb{E}\bigg\{\lambda|X(T)|^2 + \int_0^T \langle M(t)[\alpha(t)+\beta(t)], \alpha(t)+\beta(t)\rangle \mathrm{d}t\bigg\} \\
&= \mathbb{E}\bigg\{\lambda|X(T)|^2 + \int_0^T \Big[\langle M(t)\alpha(t),\alpha(t)\rangle + \langle M(t)\beta(t),\beta(t)\rangle \\
&\quad + 2\langle M(t)\alpha(t),\beta(t)\rangle\Big]\mathrm{d}t\bigg\} \\
&= J(0;u) + \mathbb{E}\bigg\{\lambda|X(T)|^2 - \langle GY(0),Y(0)\rangle \\
&\quad + \int_0^T \langle M(t)\beta(t),\beta(t)\rangle \mathrm{d}t + 2\int_0^T \langle M(t)\alpha(t),\beta(t)\rangle \mathrm{d}t\bigg\}. \qquad (4.9)
\end{aligned}
$$

Since the weighting matrices in the cost functional are bounded, we can chose a constant $K \geqslant 1$ such that $|M(t)| \leqslant K$ for a.e. $t \in [0,T]$. Thus, by the Cauchy–Schwarz inequality we have

$$
\begin{aligned}
&\bigg|\mathbb{E}\bigg\{\int_0^T \langle M(t)\beta(t),\beta(t)\rangle \mathrm{d}t + 2\int_0^T \langle M(t)\alpha(t),\beta(t)\rangle \mathrm{d}t\bigg\}\bigg| \\
&\quad \leqslant K\bigg\{\mathbb{E}\int_0^T |\beta(t)|^2 \mathrm{d}t + 2\mathbb{E}\int_0^T |\alpha(t)||\beta(t)| \mathrm{d}t\bigg\} \\
&\quad \leqslant K\bigg\{(\mu+1)\,\mathbb{E}\int_0^T |\beta(t)|^2 \mathrm{d}t + \frac{1}{\mu}\,\mathbb{E}\int_0^T |\alpha(t)|^2 \mathrm{d}t\bigg\}, \qquad (4.10)
\end{aligned}
$$

where $\mu > 0$ is a constant to be chosen later. If we choose $K \geqslant 1$ large enough (still independent of $X(T)$ and (u,v)), then according to Lemma 2.1,

$$
\mathbb{E}\int_0^T |\alpha(t)|^2 \mathrm{d}t \leqslant K\mathbb{E}\int_0^T |u(t)|^2 \mathrm{d}t, \quad \mathbb{E}\int_0^T |\beta(t)|^2 \mathrm{d}t \leqslant K\mathbb{E}|X(T)|^2, \qquad (4.11)
$$

and for the case $x = 0$, we have

$$
|\langle GY(0),Y(0)\rangle| = |\langle GY_0(0),Y_0(0)\rangle| \leqslant K\mathbb{E}|X(T)|^2. \qquad (4.12)
$$

Combining (4.10) and (4.11), we obtain from (4.9) that

$$
\begin{aligned}
\mathscr{J}_\lambda(x;u,v) &\geqslant J(0;u) - \langle GY(0),Y(0)\rangle + [\lambda - K^2(\mu+1)]\mathbb{E}|X(T)|^2 - \frac{K^2}{\mu}\,\mathbb{E}\int_0^T |u(t)|^2 \mathrm{d}t \\
&\geqslant \left(\delta - \frac{K^2}{\mu}\right)\mathbb{E}\int_0^T |u(t)|^2 \mathrm{d}t + [\lambda - K^2(\mu+1)]\mathbb{E}|X(T)|^2 - \langle GY(0),Y(0)\rangle. \qquad (4.13)
\end{aligned}
$$

From (4.11) we have

$$\begin{aligned}\mathbb{E}\int_0^T|v(t)|^2\mathrm{d}t&=\mathbb{E}\int_0^T|Z(t)+Z_0(t)|^2\mathrm{d}t\leqslant 2\left[\mathbb{E}\int_0^T|Z(t)|^2\mathrm{d}t+\mathbb{E}\int_0^T|Z_0(t)|^2\mathrm{d}t\right]\\&\leqslant 2\left[\mathbb{E}\int_0^T|\alpha(t)|^2\mathrm{d}t+\mathbb{E}\int_0^T|\beta(t)|^2\mathrm{d}t\right]\\&\leqslant 2K\left[\mathbb{E}|X(T)|^2+\mathbb{E}\int_0^T|u(t)|^2\mathrm{d}t\right],\end{aligned}$$

which implies that

$$\mathbb{E}|X(T)|^2\geqslant\frac{1}{2K}\,\mathbb{E}\int_0^T|v(t)|^2\mathrm{d}t-\mathbb{E}\int_0^T|u(t)|^2\mathrm{d}t. \tag{4.14}$$

Taking $\mu=\frac{2K^2}{\delta}$ and substituting (4.14) into (4.13), we see that when $\lambda\geqslant\lambda_0\triangleq\frac{\delta}{4}+K+K^2(\mu+1)$,

$$\begin{aligned}\mathcal{J}_\lambda(x;u,v)&\geqslant\frac{\delta}{2}\,\mathbb{E}\int_0^T|u(t)|^2\mathrm{d}t+\Big(\frac{\delta}{4}+K\Big)\,\mathbb{E}|X(T)|^2-\langle GY(0),Y(0)\rangle\\&\geqslant\frac{\delta}{4}\,\mathbb{E}\int_0^T|u(t)|^2\mathrm{d}t+\frac{\delta}{8K}\,\mathbb{E}\int_0^T|v(t)|^2\mathrm{d}t+\Big[K\mathbb{E}|X(T)|^2-\langle GY(0),Y(0)\rangle\Big]\\&\geqslant\frac{\delta}{8K}\,\mathbb{E}\int_0^T\Big[|u(t)|^2+|v(t)|^2\Big]\mathrm{d}t+\Big[K\mathbb{E}|X(T)|^2-\langle GY(0),Y(0)\rangle\Big].\end{aligned} \tag{4.15}$$

When $x=0$, by using (4.12) we further obtain

$$\mathcal{J}_\lambda(0;u,v)\geqslant\frac{\delta}{8K}\,\mathbb{E}\int_0^T\Big[|u(t)|^2+|v(t)|^2\Big]\mathrm{d}t.$$

If $G=0$, we obtain from (4.15) that

$$\mathcal{J}_\lambda(x;u,v)\geqslant\frac{\delta}{8K}\,\mathbb{E}\int_0^T\Big[|u(t)|^2+|v(t)|^2\Big]\mathrm{d}t$$

for all $(u,v)\in\mathscr{U}\times\mathscr{V}$ and all $x\in\mathbb{R}^n$. ☐

Combining Theorem 4.1 and Lemma 2.2, we get the following corollary.

Corollary 4.2. *Under the assumptions of Theorem 4.1, we have the following results:*

(i) For $\lambda\geqslant\lambda_0$, the Riccati equation

$$\begin{cases}\dot P_\lambda+P_\lambda A+A^\top P_\lambda+Q\\\qquad-\begin{pmatrix}C^\top P_\lambda+S_1\\B^\top P_\lambda+S_2\end{pmatrix}^\top\begin{pmatrix}R_{11}+P_\lambda & R_{12}\\R_{21} & R_{22}\end{pmatrix}^{-1}\begin{pmatrix}C^\top P_\lambda+S_1\\B^\top P_\lambda+S_2\end{pmatrix}=0,\\P_\lambda(T)=\lambda I\end{cases} \tag{4.16}$$

admits a unique solution $P_\lambda \in C([0,T];\mathbb{S}^n)$ such that

$$\begin{pmatrix} R_{11}+P_\lambda & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \gg 0. \tag{4.17}$$

(ii) Problem (FSLQ)$_\lambda$ is uniquely solvable for $\lambda \geqslant \lambda_0$, and

$$\mathcal{V}_\lambda(x) = \langle P_\lambda(0)x, x\rangle, \quad \forall x \in \mathbb{R}^n.$$

(iii) If, in addition, $G=0$, then for $\lambda \geqslant \lambda_0$ the value function $\mathcal{V}_\lambda$ satisfies

$$\mathcal{V}_\lambda(x) \geqslant 0, \quad \forall x \in \mathbb{R}^n.$$

Proof. The results follow from (4.4) and Lemma 2.2. Indeed, if we let

$$\mathcal{B} = (C, B), \quad \mathcal{D} = (I, 0), \quad \mathcal{S} = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}, \tag{4.18}$$

and use $\boldsymbol{v}$ to denote the control $\begin{pmatrix} v \\ u \end{pmatrix}$, then the state equation (4.2) can be written as

$$\begin{cases} \mathrm{d}X(t) = [A(t)X(t) + \mathcal{B}(t)\boldsymbol{v}(t)]\mathrm{d}t + \mathcal{D}(t)\boldsymbol{v}(t)\mathrm{d}W(t), \quad t \in [0,T], \\ X(0) = x, \end{cases}$$

which takes the form of (2.2) with $\mathcal{C}=0$, and the cost functional (4.3) can be written as

$$\mathcal{J}_\lambda(x;\boldsymbol{v}) = \mathbb{E}\left\{\lambda|X(T)|^2 + \int_0^T \left\langle \begin{pmatrix} Q(t) & \mathcal{S}^\top(t) \\ \mathcal{S}(t) & \mathcal{R}(t) \end{pmatrix}\begin{pmatrix} X(t) \\ \boldsymbol{v}(t) \end{pmatrix}, \begin{pmatrix} X(t) \\ \boldsymbol{v}(t) \end{pmatrix}\right\rangle \mathrm{d}t\right\},$$

which takes the form of (2.3) with $\mathcal{G} = \lambda I$. We know from Theorem 4.1 that (4.4) holds, *i.e.*,

$$\mathcal{J}_\lambda(0;\boldsymbol{v}) \geqslant \rho\,\mathbb{E}\int_0^T |\boldsymbol{v}(t)|^2 \mathrm{d}t, \quad \forall \boldsymbol{v}.$$

Thus, by Lemma 2.2, the Riccati equation

$$\begin{cases} \dot{P}_\lambda + P_\lambda A + A^\top P_\lambda + Q \\ \qquad - (P_\lambda \mathcal{B} + \mathcal{S}^\top)(\mathcal{R} + \mathcal{D}^\top P_\lambda \mathcal{D})^{-1}(\mathcal{B}^\top P_\lambda + \mathcal{S}) = 0, \\ P_\lambda(T) = \lambda I \end{cases} \tag{4.19}$$

admits a unique solution $P_\lambda \in C([0,T];\mathbb{S}^n)$ such that

$$\mathcal{R} + \mathcal{D}^\top P_\lambda \mathcal{D} \gg 0. \tag{4.20}$$

Substituting (4.18) into (4.19) and (4.20) gives (4.16) and (4.17), respectively. Again, by Lemma 2.2, Problem (FSLQ)$_\lambda$ is uniquely solvable for $\lambda \geqslant \lambda_0$, and

$$\mathcal{V}_\lambda(x) = \langle P_\lambda(0)x, x\rangle, \quad \forall x \in \mathbb{R}^n.$$

When $G=0$, (4.5) implies that $\mathcal{V}_\lambda(x)\geqslant 0$ for all $\lambda\geqslant\lambda_0$ and all $x\in\mathbb{R}^n$. □

We conclude this section with a remark on the weighting matrix R_{22} of the control process.

Remark 4.3. Clearly, (4.17) implies that $R_{22}\gg 0$. Thus, in order for the cost functional of Problem (BSLQ) to be uniformly positive (or equivalently, in order for (A3) to hold), the weighting matrix $R_{22}(t)$ must be positive definite uniformly in $t\in[0,T]$. This is quite different from the forward stochastic LQ optimal control problem, in which the uniform positivity of the weighting matrix for the control is neither sufficient nor necessary for the uniform positivity of the cost functional. To better understand the requirement $R_{22}\gg 0$, we observe that (A3) implies that $J(0;u)\geqslant\delta\int_0^T|u(t)|^2\mathrm{d}t$ for all deterministic nonzero controls u. When u is a deterministic function, the state equation (1.1) with terminal state $\xi=0$ becomes an ordinary differential equation since the coefficients are all deterministic. Therefore, for deterministic controls u, the adapted solution (Y,Z) of (1.1) with terminal state $\xi=0$ satisfies $Z=0$ and

$$\begin{cases}\mathrm{d}Y(t)=[A(t)Y(t)+B(t)u(t)]\mathrm{d}t,\\ Y(T)=0,\end{cases}$$

and the corresponding cost becomes

$$J(0;u)=\langle GY(0),Y(0)\rangle+\int_0^T\left\langle\begin{pmatrix}Q(t)&S_2^\top(t)\\ S_2(t)&R_{22}(t)\end{pmatrix}\begin{pmatrix}Y(t)\\ u(t)\end{pmatrix},\begin{pmatrix}Y(t)\\ u(t)\end{pmatrix}\right\rangle\mathrm{d}t.$$

Then by reversing time, we can use the classical result from the deterministic forward LQ theory to conclude that if $\widehat{\mathcal{A}}$ is uniformly positive, then $R_{22}\gg 0$. In a similar manner, we can show that in order for the cost functional of Problem (BSLQ) to be positive, the weighting matrix $R_{22}(t)$ must be positive semidefinite for a.e. $t\in[0,T]$.

5. Construction of optimal controls

In this section, we construct the optimal control of Problem (BSLQ) under the uniform positivity condition (A3). As mentioned in Section 4, once the case of (A3) is solved, we can develop an ε-approximation scheme that is asymptotically optimal for the general case, thanks to Theorem 3.4.

First, we observe that the uniform positivity condition (A3) implies $R_{22}\gg 0$ (Rem. 4.3). This enables us to simplify Problem (BSLQ) by assuming

$$G=0,\quad Q(t)=0,\quad R_{12}(t)=R_{21}^\top(t)=0;\quad\forall t\in[0,T].\tag{5.1}$$

In fact, using the transformations

$$\begin{aligned}\mathscr{S}_1&:=S_1-R_{12}R_{22}^{-1}S_2, &\qquad \mathscr{R}_{11}&:=R_{11}-R_{12}R_{22}^{-1}R_{21},\\ \mathscr{C}&:=C-BR_{22}^{-1}R_{21}, & v&:=u+R_{22}^{-1}R_{21}Z,\end{aligned}\tag{5.2}$$

the original Problem (BSLQ) is equivalent to the backward stochastic LQ optimal control problem with state equation

$$\begin{cases}\mathrm{d}Y(t)=[A(t)Y(t)+B(t)v(t)+\mathscr{C}(t)Z(t)]\mathrm{d}t+Z(t)\mathrm{d}W(t),\\ Y(T)=\xi,\end{cases}\tag{5.3}$$

and cost functional

$$\mathscr{J}(\xi;v):=\mathbb{E}\bigg\{\langle GY(0),Y(0)\rangle+\int_0^T\bigg\langle\begin{pmatrix}Q(t)&\mathscr{S}_1^\top(t)&S_2^\top(t)\\ \mathscr{S}_1(t)&\mathscr{R}_{11}(t)&0\\ S_2(t)&0&R_{22}(t)\end{pmatrix}\begin{pmatrix}Y(t)\\ Z(t)\\ v(t)\end{pmatrix},\begin{pmatrix}Y(t)\\ Z(t)\\ v(t)\end{pmatrix}\bigg\rangle\mathrm{d}t\bigg\}. \tag{5.4}$$

Furthermore, letting $H\in C([0,T];\mathbb{S}^n)$ be the unique solution to the linear ordinary differential equation (ODE, for short)

$$\begin{cases}\dot H(t)+H(t)A(t)+A(t)^\top H(t)+Q(t)=0, \quad t\in[0,T],\\ H(0)=-G,\end{cases} \tag{5.5}$$

and then applying the integration by parts formula to $t\mapsto\langle H(t)Y(t),Y(t)\rangle$, where Y is the state process determined by (5.3), we obtain

$$\begin{aligned}&\mathbb{E}\langle H(T)\xi,\xi\rangle+\mathbb{E}\langle GY(0),Y(0)\rangle\\ &\quad=\mathbb{E}\int_0^T\Big[\langle(\dot H+HA+A^\top H)Y,Y\rangle+2\langle B^\top HY,v\rangle+2\langle\mathscr{C}^\top HY,Z\rangle+\langle HZ,Z\rangle\Big]\mathrm{d}t\\ &\quad=\mathbb{E}\int_0^T\Big[-\langle QY,Y\rangle+2\langle B^\top HY,v\rangle+2\langle\mathscr{C}^\top HY,Z\rangle+\langle HZ,Z\rangle\Big]\mathrm{d}t\\ &\quad=\mathbb{E}\int_0^T\bigg\langle\begin{pmatrix}-Q&H\mathscr{C}&HB\\ \mathscr{C}^\top H&H&0\\ B^\top H&0&0\end{pmatrix}\begin{pmatrix}Y\\ Z\\ v\end{pmatrix},\begin{pmatrix}Y\\ Z\\ v\end{pmatrix}\bigg\rangle\mathrm{d}t.\end{aligned}$$

Substituting for $\mathbb{E}\langle GY(0),Y(0)\rangle$ in the cost functional (5.4) yields

$$\mathscr{J}(\xi;v)=\mathbb{E}\int_0^T\bigg\langle\begin{pmatrix}0&(S_1^H)^\top&(S_2^H)^\top\\ S_1^H&R_{11}^H&0\\ S_2^H&0&R_{22}\end{pmatrix}\begin{pmatrix}Y\\ Z\\ v\end{pmatrix},\begin{pmatrix}Y\\ Z\\ v\end{pmatrix}\bigg\rangle\mathrm{d}t-\mathbb{E}\langle H(T)\xi,\xi\rangle,$$

where

$$S_1^H:=\mathscr{S}_1+\mathscr{C}^\top H,\quad S_2^H:=S_2+B^\top H,\quad R_{11}^H:=\mathscr{R}_{11}+H. \tag{5.6}$$

Thus, for a given terminal state ξ, minimizing $J(\xi;u)$ subject to (1.1) is equivalent to minimizing the cost functional

$$J^H(\xi;v):=\mathbb{E}\int_0^T\bigg\langle\begin{pmatrix}0&(S_1^H)^\top&(S_2^H)^\top\\ S_1^H&R_{11}^H&0\\ S_2^H&0&R_{22}\end{pmatrix}\begin{pmatrix}Y\\ Z\\ v\end{pmatrix},\begin{pmatrix}Y\\ Z\\ v\end{pmatrix}\bigg\rangle\mathrm{d}t, \tag{5.7}$$

subject to the state equation (5.3). Therefore, in the rest of this section we may assume without loss of generality that (5.1) holds. The general case will be discussed in Section 6.

Observe that in the case of (5.1), the Riccati equation (4.16) becomes

$$\begin{cases}\dot P_\lambda+P_\lambda A+A^\top P_\lambda-\begin{pmatrix}C^\top P_\lambda+S_1\\ B^\top P_\lambda+S_2\end{pmatrix}^\top\begin{pmatrix}R_{11}+P_\lambda&0\\ 0&R_{22}\end{pmatrix}^{-1}\begin{pmatrix}C^\top P_\lambda+S_1\\ B^\top P_\lambda+S_2\end{pmatrix}=0,\\ P_\lambda(T)=\lambda I.\end{cases} \tag{5.8}$$

Proposition 5.1. *Let (A1)–(A3) and* (5.1) *hold. Let $\lambda_0 > 0$ be the constant in Theorem 4.1. Then for $\lambda \geqslant \lambda_0$, the solution of* (5.8) *satisfies*

$$P_\lambda(t) \geqslant 0, \quad \forall t \in [0, T]. \tag{5.9}$$

Moreover, for every $\lambda_2 > \lambda_1 \geqslant \lambda_0$, we have

$$P_{\lambda_2}(t) > P_{\lambda_1}(t), \quad \forall t \in [0, T]. \tag{5.10}$$

Proof. Consider Problem (FSLQ)$_\lambda$ for $\lambda \geqslant \lambda_0$. Since $G = 0$, we see from Corollary 4.2 that

$$\langle P_\lambda(0)x, x\rangle = \mathcal{V}_\lambda(x) \geqslant 0, \quad \forall x \in \mathbb{R}^n,$$

and hence $P_\lambda(0) \geqslant 0$. With the notation

$$\mathcal{Q}_\lambda := \begin{pmatrix} C^\top P_\lambda + S_1 \\ B^\top P_\lambda + S_2 \end{pmatrix}^\top \begin{pmatrix} R_{11} + P_\lambda & 0 \\ 0 & R_{22} \end{pmatrix}^{-1} \begin{pmatrix} C^\top P_\lambda + S_1 \\ B^\top P_\lambda + S_2 \end{pmatrix}$$

and with Φ denoting the solution to the matrix ODE

$$\begin{cases} \dot{\Phi}(t) = A(t)\Phi(t), \quad t \in [0, T], \\ \Phi(0) = I_n, \end{cases}$$

we can rewrite (5.8) in the integral form

$$P_\lambda(t) = \left[\Phi^{-1}(t)\right]^\top \left[P_\lambda(0) + \int_0^t \Phi(s)^\top \mathcal{Q}_\lambda(s)\Phi(s)\mathrm{d}s\right] \Phi^{-1}(t), \quad t \in [0, T].$$

This implies (5.9) because $P_\lambda(0) \geqslant 0$ and $\mathcal{Q}_\lambda(t) \geqslant 0$ a.e. by (4.17). To prove (5.10), let us consider $\mathcal{P}(t) := P_{\lambda_2}(t) - P_{\lambda_1}(t)$, which satisfies the following equation:

$$\begin{cases} \dot{\mathcal{P}} + \mathcal{P}A + A^\top \mathcal{P} + \mathcal{Q} - (\mathcal{P}\mathcal{B} + \mathcal{S}^\top)(\mathcal{R} + \mathcal{D}^\top \mathcal{P}\mathcal{D})^{-1}(\mathcal{B}^\top \mathcal{P} + \mathcal{S}) = 0, \\ \mathcal{P}(T) = (\lambda_2 - \lambda_1)I, \end{cases} \tag{5.11}$$

where we have employed the notation

$$\mathcal{B} := (C, B), \quad \mathcal{D} := (I, 0), \quad \mathcal{R} := \begin{pmatrix} R_{11} + P_{\lambda_1} & 0 \\ 0 & R_{22} \end{pmatrix}, \quad \mathcal{S} := \begin{pmatrix} C^\top P_{\lambda_1} + S_1 \\ B^\top P_{\lambda_1} + S_2 \end{pmatrix},$$
$$\mathcal{Q} := \begin{pmatrix} C^\top P_{\lambda_1} + S_1 \\ B^\top P_{\lambda_1} + S_2 \end{pmatrix}^\top \begin{pmatrix} R_{11} + P_{\lambda_1} & 0 \\ 0 & R_{22} \end{pmatrix}^{-1} \begin{pmatrix} C^\top P_{\lambda_1} + S_1 \\ B^\top P_{\lambda_1} + S_2 \end{pmatrix} = \mathcal{S}^\top \mathcal{R}^{-1}\mathcal{S}.$$

Clearly, the matrices $\mathcal{G} := (\lambda_2 - \lambda_1)I > 0$, $\mathcal{Q}$, $\mathcal{S}$, and $\mathcal{R}$ satisfies the condition (2.7), so Corollary 2.3 implies (5.10). □

For notational convenience we write for an $\mathbb{S}^n$-valued function $\Sigma : [0, T] \to \mathbb{S}^n$,

$$\begin{aligned} \mathcal{B}(t, \Sigma(t)) &:= B(t) + \Sigma(t)S_2(t)^\top, \\ \mathcal{C}(t, \Sigma(t)) &:= C(t) + \Sigma(t)S_1(t)^\top, \\ \mathcal{R}(t, \Sigma(t)) &:= I + \Sigma(t)R_{11}(t). \end{aligned}$$

When there is no risk for confusion we will frequently suppress the argument t from our notation and write $\mathcal{B}(t,\Sigma(t))$, $\mathcal{C}(t,\Sigma(t))$, and $\mathcal{R}(t,\Sigma(t))$ as $\mathcal{B}(\Sigma)$, $\mathcal{C}(\Sigma)$, and $\mathcal{R}(\Sigma)$, respectively. In order to construct the optimal control of Problem (BSLQ), we now introduce the following Riccati equation:

$$\begin{cases}\dot\Sigma(t)-A(t)\Sigma(t)-\Sigma(t)A(t)^\top+\mathcal{B}(t,\Sigma(t))[R_{22}(t)]^{-1}\mathcal{B}(t,\Sigma(t))^\top \\ \qquad +\,\mathcal{C}(t,\Sigma(t))[\mathcal{R}(t,\Sigma(t))]^{-1}\Sigma(t)\mathcal{C}(t,\Sigma(t))^\top=0, \quad t\in[0,T], \\ \Sigma(T)=0. \end{cases} \tag{5.12}$$

Theorem 5.2. *Let (A1)–(A3) and* (5.1) *hold. Then the Riccati equation* (5.12) *admits a unique positive semidefinite solution $\Sigma\in C([0,T];\mathbb{S}^n)$ such that $\mathcal{R}(\Sigma)$ is invertible a.e. on $[0,T]$ and $\mathcal{R}(\Sigma)^{-1}\in L^\infty(0,T;\mathbb{R}^n)$.*

Proof. Uniqueness. Suppose that Σ and Π are two solutions of (5.12) satisfying the properties stated in the theorem. Then $\Delta:=\Sigma-\Pi$ satisfies $\Delta(T)=0$ and

$$\begin{aligned}\dot\Delta &= A\Delta+\Delta A^\top-\Delta S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top-\mathcal{B}(\Pi)R_{22}^{-1}S_2\Delta \\ &\quad -\Delta S_1\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top-\mathcal{C}(\Pi)\Big[\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top-\mathcal{R}(\Pi)^{-1}\Pi\mathcal{C}(\Pi)^\top\Big].\end{aligned}$$

Note that

$$\begin{aligned}&\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top-\mathcal{R}(\Pi)^{-1}\Pi\mathcal{C}(\Pi)^\top \\ &\quad=-\mathcal{R}(\Sigma)^{-1}\Delta R_{11}\mathcal{R}(\Pi)^{-1}\Sigma\mathcal{C}(\Sigma)^\top+\mathcal{R}(\Pi)^{-1}\Big[\Sigma\mathcal{C}(\Sigma)^\top-\Pi\mathcal{C}(\Pi)^\top\Big] \\ &\quad=-\mathcal{R}(\Sigma)^{-1}\Delta R_{11}\mathcal{R}(\Pi)^{-1}\Sigma\mathcal{C}(\Sigma)^\top+\mathcal{R}(\Pi)^{-1}\Big[\Delta\mathcal{C}(\Sigma)^\top+\Pi S_1\Delta\Big].\end{aligned}$$

It follows that

$$\begin{aligned}\dot\Delta(t) &= A\Delta+\Delta A^\top-\Delta S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top-\mathcal{B}(\Pi)R_{22}^{-1}S_2\Delta-\Delta S_1\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top \\ &\quad+\mathcal{C}(\Pi)\mathcal{R}(\Sigma)^{-1}\Delta R_{11}\mathcal{R}(\Pi)^{-1}\Sigma\mathcal{C}(\Sigma)^\top-\mathcal{C}(\Pi)\mathcal{R}(\Pi)^{-1}\Big[\Delta\mathcal{C}(\Sigma)^\top+\Pi S_1\Delta\Big] \\ &\equiv f(t,\Delta(t)).\end{aligned}$$

Noting that $\Delta(T)=0$ and $f(t,x)$ is Lipschitz-continuous in x, we conclude by Gronwall's inequality that $\Delta(t)=0$ for all $t\in[0,T]$.

Existence. According to Proposition 5.1, for $\lambda>\lambda_0$, the solution P_λ of (5.8) is positive definite on $[0,T]$. Thus we may define

$$\Sigma_\lambda(t):=P_\lambda^{-1}(t), \quad t\in[0,T].$$

Again, by Proposition 5.1, for each fixed $t\in[0,T]$, $\Sigma_\lambda(t)$ is decreasing in λ and bounded below by zero, so the family $\{\Sigma_\lambda(t)\}_{\lambda>\lambda_0}$ is bounded uniformly in $t\in[0,T]$ and converges pointwise to some positive semidefinite function $\Sigma:[0,T]\to\mathbb{S}^n$. Next we shall prove the following:

(a) $\mathcal{R}(t,\Sigma(t))=I+\Sigma(t)R_{11}(t)$ is invertible for a.e. $t\in[0,T]$;
(b) $\mathcal{R}(\Sigma)^{-1}\in L^\infty(0,T;\mathbb{R}^n)$; and
(c) Σ solves the equation (5.12).

For (a) and (b), we observe first that for $\lambda>\lambda_0$, $I+\Sigma_\lambda R_{11}$ is invertible a.e. on $[0,T]$ since by (4.17),

$$P_\lambda(I+\Sigma_\lambda R_{11})=P_\lambda+R_{11}\gg 0.$$

Define $K := R_{11} + P_{\lambda_0}$ and $L_\lambda := P_\lambda - P_{\lambda_0}$. For every $\lambda > \lambda_0$,

$$0 \leqslant (K + L_\lambda)^{-1} = (R_{11} + P_\lambda)^{-1} \leqslant (R_{11} + P_{\lambda_0})^{-1},$$

from which we obtain

$$|(K + L_\lambda)^{-1}| \leqslant |(R_{11} + P_{\lambda_0})^{-1}|, \quad \forall \lambda > \lambda_0,$$

and hence for every $x \in \mathbb{R}^n$,

$$\begin{aligned}
\langle P_\lambda (R_{11} + P_\lambda)^{-2} P_\lambda x, x\rangle &= |(K + L_\lambda)^{-1}(L_\lambda + P_{\lambda_0})x|^2 \\
&\leqslant 2|(K + L_\lambda)^{-1} L_\lambda x|^2 + 2|(K + L_\lambda)^{-1} P_{\lambda_0} x|^2 \\
&= 2|x - (K + L_\lambda)^{-1} K x|^2 + 2|(K + L_\lambda)^{-1} P_{\lambda_0} x|^2 \\
&\leqslant 4\Big[1 + \big|(K + L_\lambda)^{-1}\big|^2 \Big(|K|^2 + |P_{\lambda_0}|^2\Big)\Big]|x|^2 \\
&\leqslant 4\Big[1 + \big|(R_{11} + P_{\lambda_0})^{-1}\big|^2 \Big(|K|^2 + |P_{\lambda_0}|^2\Big)\Big]|x|^2.
\end{aligned}$$

It follows that for every $\lambda > \lambda_0$,

$$\begin{aligned}
(I + \Sigma_\lambda R_{11})(I + \Sigma_\lambda R_{11})^\top &= \Big[P_\lambda (R_{11} + P_\lambda)^{-2} P_\lambda\Big]^{-1} \\
&\geqslant \frac{1}{4}\Big[1 + \big|(R_{11} + P_{\lambda_0})^{-1}\big|^2 \Big(|K|^2 + |P_{\lambda_0}|^2\Big)\Big]^{-1} I.
\end{aligned}$$

Letting $\lambda \to \infty$ yields

$$(I + \Sigma R_{11})(I + \Sigma R_{11})^\top \geqslant \frac{1}{4}\Big[1 + \big|(R_{11} + P_{\lambda_0})^{-1}\big|^2 \Big(|K|^2 + |P_{\lambda_0}|^2\Big)\Big]^{-1} I.$$

This implies (a) and (b). For (c), we have from the identity

$$\dot{\Sigma}_\lambda(t) P_\lambda(t) + \Sigma_\lambda(t) \dot{P}_\lambda(t) = \frac{\mathrm{d}}{\mathrm{d}t}[\Sigma_\lambda(t) P_\lambda(t)] = 0$$

that

$$\begin{aligned}
\dot{\Sigma}_\lambda(t) &= -\Sigma_\lambda(t) \dot{P}_\lambda(t) \Sigma_\lambda(t) \\
&= A\Sigma_\lambda + \Sigma_\lambda A^\top - \begin{pmatrix} C^\top + S_1\Sigma_\lambda \\ B^\top + S_2\Sigma_\lambda \end{pmatrix}^\top \begin{pmatrix} R_{11} + P_\lambda & 0 \\ 0 & R_{22} \end{pmatrix}^{-1} \begin{pmatrix} C^\top + S_1\Sigma_\lambda \\ B^\top + S_2\Sigma_\lambda \end{pmatrix} \\
&= A\Sigma_\lambda + \Sigma_\lambda A^\top - \mathcal{B}(\Sigma_\lambda) R_{22}^{-1} \mathcal{B}(\Sigma_\lambda)^\top - \mathcal{C}(\Sigma_\lambda)(R_{11} + P_\lambda)^{-1} \mathcal{C}(\Sigma_\lambda)^\top \\
&= A\Sigma_\lambda + \Sigma_\lambda A^\top - \mathcal{B}(\Sigma_\lambda) R_{22}^{-1} \mathcal{B}(\Sigma_\lambda)^\top - \mathcal{C}(\Sigma_\lambda)\mathcal{R}(\Sigma_\lambda)^{-1} \Sigma_\lambda \mathcal{C}(\Sigma_\lambda)^\top.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\Sigma_\lambda(t) = \lambda^{-1} I - \int_t^T \Big[&A\Sigma_\lambda + \Sigma_\lambda A^\top - \mathcal{B}(\Sigma_\lambda) R_{22}^{-1} \mathcal{B}(\Sigma_\lambda)^\top \\
&- \mathcal{C}(\Sigma_\lambda)\mathcal{R}(\Sigma_\lambda)^{-1} \Sigma_\lambda \mathcal{C}(\Sigma_\lambda)^\top\Big] \mathrm{d}s. \qquad (5.13)
\end{aligned}$$

Letting $\lambda \to \infty$ in (5.13), we obtain by the bounded convergence theorem that

$$\Sigma(t) = -\int_t^T \Big[A\Sigma + \Sigma A^\top - \mathcal{B}(\Sigma)R_{22}^{-1}\mathcal{B}(\Sigma)^\top - \mathcal{C}(\Sigma)^\top\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)\Big]\mathrm{d}s,$$

which is the integral version of (5.12). □

With the solution Σ to the Riccati equation (5.12), we further introduce the following linear BSDE:

$$\begin{cases} \mathrm{d}\varphi(t) = \Big\{[A - \mathcal{B}(\Sigma)R_{22}^{-1}S_2 - \mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\Sigma S_1]\varphi \\ \qquad\qquad + \mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\beta\Big\}\mathrm{d}t + \beta\mathrm{d}W(t), \quad t \in [0,T], \\ \varphi(T) = \xi. \end{cases} \tag{5.14}$$

Since $R_{22} \gg 0$ and $\mathcal{R}(\Sigma)^{-1} \in L^\infty(0,T;\mathbb{R}^n)$, the BSDE (5.14) is clearly uniquely solvable.

In terms of the solution Σ to the Riccati equation (5.12) and the adapted solution (φ,β) to the BSDE (5.14), we can construct the optimal control of Problem (BSLQ) as follows.

Theorem 5.3. *Let (A1)–(A3) and (5.1) hold. Let (φ,β) be the adapted solution to the BSDE (5.14) and X the solution to the following SDE:*

$$\begin{cases} \mathrm{d}X(t) = \Big\{\big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top + S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top - A^\top\big]X \\ \qquad\qquad - \big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1}S_2\big]\varphi + S_1^\top\mathcal{R}(\Sigma)^{-1}\beta\Big\}\mathrm{d}t \\ \qquad\qquad - \big[\mathcal{R}(\Sigma)^{-1}\big]^\top\big[\mathcal{C}(\Sigma)^\top X - S_1\varphi - R_{11}\beta\big]\mathrm{d}W(t), \\ X(0) = 0. \end{cases} \tag{5.15}$$

Then the optimal control of Problem (BSLQ) for the terminal state ξ is given by

$$u(t) = [R_{22}(t)]^{-1}[\mathcal{B}(t,\Sigma(t))^\top X(t) - S_2(t)\varphi(t)], \quad t \in [0,T]. \tag{5.16}$$

Proof. We first point out that the SDE (5.15) is uniquely solvable. Indeed, (5.15) is a linear SDE. The coefficients of X are bounded, and the nonhomogenous terms are square-integrable processes since φ and β are square-integrable. Thus, by [23, Proposition 2.1], (5.15) is uniquely solvable. Now let us define for $t \in [0,T]$,

$$Y(t) := -\Sigma(t)X(t) + \varphi(t), \tag{5.17}$$

$$Z(t) := \mathcal{R}(t,\Sigma(t))^{-1}[\Sigma(t)\mathcal{C}(t,\Sigma(t))^\top X(t) - \Sigma(t)S_1(t)\varphi(t) + \beta(t)]. \tag{5.18}$$

We observe that

$$R_{22}u = \mathcal{B}(\Sigma)^\top X - S_2\varphi = B^\top X + S_2(\Sigma X - \varphi) = B^\top X - S_2 Y. \tag{5.19}$$

Furthermore, using (5.16) and (5.17) we obtain

$$\begin{aligned} S_1^\top Z + S_2^\top u &= S_1^\top\mathcal{R}(\Sigma)^{-1}[\Sigma\mathcal{C}(\Sigma)^\top X - \Sigma S_1\varphi + \beta] + S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top X - S_2^\top R_{22}^{-1}S_2\varphi \\ &= \big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top + S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top\big]X \\ &\quad - \big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1}S_2\big]\varphi + S_1^\top\mathcal{R}(\Sigma)^{-1}\beta, \end{aligned}$$

from which it follows that

$$\begin{aligned}-A^\top X+S_1^\top Z+S_2^\top u &= \big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top+S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top-A^\top\big]X\\ &\quad -\big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1+S_2^\top R_{22}^{-1}S_2\big]\varphi+S_1^\top\mathcal{R}(\Sigma)^{-1}\beta.\end{aligned}\tag{5.20}$$

Similarly, we can get

$$\begin{aligned}-C^\top X+S_1Y+R_{11}Z &= -(C^\top+S_1\Sigma)X+S_1\varphi+R_{11}Z\\ &= [R_{11}\mathcal{R}(\Sigma)^{-1}\Sigma-I]\mathcal{C}(\Sigma)^\top X+[I-R_{11}\mathcal{R}(\Sigma)^{-1}\Sigma]S_1\varphi\\ &\quad +R_{11}\mathcal{R}(\Sigma)^{-1}\beta.\end{aligned}$$

Noting that

$$\begin{aligned}&R_{11}\mathcal{R}(\Sigma)^{-1}=R_{11}(I+\Sigma R_{11})^{-1}=(I+R_{11}\Sigma)^{-1}R_{11},\\ &I-R_{11}\mathcal{R}(\Sigma)^{-1}\Sigma=(I+R_{11}\Sigma)^{-1}=\big[\mathcal{R}(\Sigma)^{-1}\big]^\top,\end{aligned}$$

we further obtain

$$-C^\top X+S_1Y+R_{11}Z=-\big[\mathcal{R}(\Sigma)^{-1}\big]^\top[\mathcal{C}(\Sigma)^\top X-S_1\varphi-R_{11}\beta].\tag{5.21}$$

This implies that the solution of (5.15) satisfies the equation

$$\begin{cases}\mathrm{d}X(t)=(-A^\top X+S_1^\top Z+S_2^\top u)\mathrm{d}t+(-C^\top X+S_1Y+R_{11}Z)\mathrm{d}W,\\ X(0)=0.\end{cases}\tag{5.22}$$

Next, for simplicity let us set

$$\alpha:=[A-\mathcal{B}(\Sigma)R_{22}^{-1}S_2-\mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\Sigma S_1]\varphi+\mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\beta.\tag{5.23}$$

By Itô's rule, we have

$$\begin{aligned}\mathrm{d}Y &= -\dot\Sigma X\mathrm{d}t-\Sigma\mathrm{d}X+\mathrm{d}\varphi\\ &= [\alpha-\dot\Sigma X-\Sigma(-A^\top X+S_1^\top Z+S_2^\top u)]\mathrm{d}t+[\beta-\Sigma(-C^\top X+S_1Y+R_{11}Z)]\mathrm{d}W.\end{aligned}$$

Using (5.20) and (5.23) we get

$$\begin{aligned}&\alpha-\dot\Sigma X-\Sigma(-A^\top X+S_1^\top Z+S_2^\top u)\\ &\quad=\alpha-\big[\dot\Sigma-\Sigma A^\top+\Sigma S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top+\Sigma S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top\big]X\\ &\qquad+\Sigma\big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1+S_2^\top R_{22}^{-1}S_2\big]\varphi-\Sigma S_1^\top\mathcal{R}(\Sigma)^{-1}\beta\\ &\quad=\alpha-\big[A\Sigma-C\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top-BR_{22}^{-1}\mathcal{B}(\Sigma)^\top\big]X\\ &\qquad+\Sigma\big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1+S_2^\top R_{22}^{-1}S_2\big]\varphi-\Sigma S_1^\top\mathcal{R}(\Sigma)^{-1}\beta\\ &\quad=AY+[C\mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top+BR_{22}^{-1}\mathcal{B}(\Sigma)^\top]X-BR_{22}^{-1}S_2\varphi\\ &\qquad-C\mathcal{R}(\Sigma)^{-1}(\Sigma S_1\varphi-\beta)\\ &\quad=AY+BR_{22}^{-1}[\mathcal{B}(\Sigma)^\top X-S_2\varphi]+C\mathcal{R}(\Sigma)^{-1}[\Sigma\mathcal{C}(\Sigma)^\top X-\Sigma S_1\varphi+\beta]\\ &\quad=AY+Bu+CZ.\end{aligned}$$

Using (5.21) and the relations

$$\Sigma\big[\mathcal{R}(\Sigma)^{-1}\big]^\top = \Sigma(I+R_{11}\Sigma)^{-1} = (I+\Sigma R_{11})^{-1}\Sigma = \mathcal{R}(\Sigma)^{-1}\Sigma,$$
$$I-\mathcal{R}(\Sigma)^{-1}\Sigma R_{11} = I-(I+\Sigma R_{11})^{-1}\Sigma R_{11} = (I+\Sigma R_{11})^{-1} = \mathcal{R}(\Sigma)^{-1},$$

we get

$$\begin{aligned}&\beta-\Sigma(-C^\top X+S_1Y+R_{11}Z)\\&\quad=\beta+\Sigma\big[\mathcal{R}(\Sigma)^{-1}\big]^\top[\mathcal{C}(\Sigma)^\top X-S_1\varphi-R_{11}\beta]\\&\quad=\mathcal{R}(\Sigma)^{-1}[\Sigma\mathcal{C}(\Sigma)^\top X-\Sigma S_1\varphi]+[I-\mathcal{R}(\Sigma)^{-1}\Sigma R_{11}]\beta\\&\quad=Z.\end{aligned}$$

Therefore, the pair (Y,Z) defined by (5.17)–(5.18) satisfies the backward equation

$$\begin{cases}\mathrm{d}Y(t)=(AY+Bu+CZ)\mathrm{d}t+Z\mathrm{d}W,\\ Y(T)=\xi.\end{cases}\tag{5.24}$$

Combining (5.19), (5.22) and (5.24), we see that the solution X of (5.15), the pair (Y,Z) defined by (5.17)–(5.18), and the control u defined by (5.16) satisfy the FBSDE

$$\begin{cases}\mathrm{d}X(t)=(-A^\top X+S_1^\top Z+S_2^\top u)\mathrm{d}t+(-C^\top X+S_1Y+R_{11}Z)\mathrm{d}W,\\ \mathrm{d}Y(t)=(AY+Bu+CZ)\mathrm{d}t+Z\mathrm{d}W,\\ X(0)=0,\quad Y(T)=\xi,\end{cases}\tag{5.25}$$

and the condition

$$S_2Y-B^\top X+R_{22}u=0.\tag{5.26}$$

Therefore, by Theorem 3.5, u is the (unique) optimal control for the terminal state ξ. □

We conclude this section with a representation of the value function $V(\xi)$.

Theorem 5.4. *Let (A1)–(A3) and (5.1) hold. Then the value function of Problem (BSLQ) is given by*

$$\begin{aligned}V(\xi)=\mathbb{E}\int_0^T\Big\{&\langle R_{11}\mathcal{R}(\Sigma)^{-1}\beta,\beta\rangle+2\langle S_1^\top\mathcal{R}(\Sigma)^{-1}\beta,\varphi\rangle\\&-\langle[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1+S_2^\top R_{22}^{-1}S_2]\varphi,\varphi\rangle\Big\}\mathrm{d}t,\end{aligned}\tag{5.27}$$

where (φ,β) is the adapted solution to the BSDE (5.14).

Proof. Let u be the optimal control for the terminal state ξ. Then, by Theorem 3.5, the adapted solution (X,Y,Z) of (5.25) satisfies (5.26). Observe that

$$\begin{aligned}V(\xi)=J(\xi;u)&=\mathbb{E}\int_0^T\Big[2\langle S_1Y,Z\rangle+2\langle S_2Y,u\rangle+\langle R_{11}Z,Z\rangle+\langle R_{22}u,u\rangle\Big]\mathrm{d}t\\&=\mathbb{E}\int_0^T\Big[\langle S_1^\top Z+S_2^\top u,Y\rangle+\langle S_1Y+R_{11}Z,Z\rangle+\langle S_2Y+R_{22}u,u\rangle\Big]\mathrm{d}t\\&=\mathbb{E}\int_0^T\Big[\langle S_1^\top Z+S_2^\top u,Y\rangle+\langle S_1Y+R_{11}Z,Z\rangle+\langle B^\top X,u\rangle\Big]\mathrm{d}t.\end{aligned}$$

Integration by parts yields

$$\begin{aligned}\mathbb{E}\langle X(T),Y(T)\rangle &= \mathbb{E}\int_0^T \Big[\langle X, AY+Bu+CZ\rangle + \langle -A^\top X + S_1^\top Z + S_2^\top u, Y\rangle \\ &\quad + \langle -C^\top X + S_1 Y + R_{11}Z, Z\rangle\Big]\mathrm{d}t \\ &= \mathbb{E}\int_0^T \Big[\langle X, Bu\rangle + \langle S_1^\top Z + S_2^\top u, Y\rangle + \langle S_1 Y + R_{11}Z, Z\rangle\Big]\mathrm{d}t \\ &= V(\xi).\end{aligned}$$

From the proof of Theorem 5.3, we see that X also satisfies the equation (5.15). Using (5.15) and integration by parts again, we obtain

$$\begin{aligned}\mathbb{E}\langle X(T),\varphi(T)\rangle &= \mathbb{E}\int_0^T \Big\{\big\langle\big[S_1^\top \mathcal{R}(\Sigma)^{-1}\Sigma\mathcal{C}(\Sigma)^\top + S_2^\top R_{22}^{-1}\mathcal{B}(\Sigma)^\top - A^\top\big]X,\varphi\big\rangle \\ &\quad - \big\langle\big[S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1}S_2\big]\varphi,\varphi\big\rangle + \big\langle S_1^\top\mathcal{R}(\Sigma)^{-1}\beta,\varphi\big\rangle \\ &\quad + \big\langle X, [A - \mathcal{B}(\Sigma)R_{22}^{-1}S_2 - \mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\Sigma S_1]\varphi\big\rangle + \big\langle X, \mathcal{C}(\Sigma)\mathcal{R}(\Sigma)^{-1}\beta\big\rangle \\ &\quad - \big\langle\big[\mathcal{R}(\Sigma)^{-1}\big]^\top[\mathcal{C}(\Sigma)^\top X - S_1\varphi - R_{11}\beta],\beta\big\rangle\Big\}\mathrm{d}t \\ &= \mathbb{E}\int_0^T \Big\{\langle R_{11}\mathcal{R}(\Sigma)^{-1}\beta,\beta\rangle + 2\langle S_1^\top\mathcal{R}(\Sigma)^{-1}\beta,\varphi\rangle \\ &\quad - \langle [S_1^\top\mathcal{R}(\Sigma)^{-1}\Sigma S_1 + S_2^\top R_{22}^{-1}S_2]\varphi,\varphi\rangle\Big\}\mathrm{d}t.\end{aligned}$$

The representation (5.27) then follows from the fact

$$V(\xi) = \mathbb{E}\langle X(T),Y(T)\rangle = \mathbb{E}\langle X(T),\xi\rangle = \mathbb{E}\langle X(T),\varphi(T)\rangle.$$

The proof is complete. □

6. Conclusion

For the reader's convenience, we conclude the paper by generalizing the results obtained in Section 5 to the case without the assumption (5.1). We shall only present the result, as the proof can be easily given using the argument at the beginning of Section 5 and the results established there for the case (5.1).

Recall the notation

$$\begin{aligned}\mathscr{C}(t) &:= C(t) - B(t)[R_{22}(t)]^{-1}R_{21}(t), \\ \mathscr{S}_1(t) &:= S_1(t) - R_{12}(t)[R_{22}(t)]^{-1}S_2(t), \\ \mathscr{R}_{11}(t) &:= R_{11}(t) - R_{12}(t)[R_{22}(t)]^{-1}R_{21}(t).\end{aligned}$$

Let $H \in C([0,T];\mathbb{S}^n)$ be the unique solution to the linear ODE

$$\begin{cases}\dot H(t) + H(t)A(t) + A(t)^\top H(t) + Q(t) = 0, \quad t\in[0,T], \\ H(0) = -G,\end{cases}$$

and let

$$
\begin{aligned}
S_1^H(t) &:= \mathscr{S}_1(t) + \mathscr{C}(t)^\top H(t), & \mathcal{B}^H(t,\Sigma(t)) &:= B(t) + \Sigma(t)[S_2^H(t)]^\top, \\
S_2^H(t) &:= S_2(t) + B(t)^\top H(t), & \mathcal{C}^H(t,\Sigma(t)) &:= \mathscr{C}(t) + \Sigma(t)[S_1^H(t)]^\top, \\
R_{11}^H(t) &:= \mathscr{R}_{11}(t) + H(t), & \mathcal{R}^H(t,\Sigma(t)) &:= I + \Sigma(t)R_{11}^H(t).
\end{aligned}
$$

Theorem 6.1. *Let (A1)–(A3) hold. We have the following results.*

(i) The Riccati equation

$$
\begin{cases}
\dot{\Sigma}(t) - A(t)\Sigma(t) - \Sigma(t)A(t)^\top + \mathcal{B}^H(t,\Sigma(t))[R_{22}(t)]^{-1}[\mathcal{B}^H(t,\Sigma(t))]^\top \\
\qquad + \mathcal{C}^H(t,\Sigma(t))[\mathcal{R}^H(t,\Sigma(t))]^{-1}\Sigma(t)[\mathcal{C}^H(t,\Sigma(t))]^\top = 0, \\
\Sigma(T) = 0
\end{cases}
\tag{6.1}
$$

admits a unique positive semidefinite solution $\Sigma \in C([0,T];\mathbb{S}^n)$ such that $\mathcal{R}^H(\Sigma)$ is invertible a.e. on $[0,T]$ and $[\mathcal{R}^H(\Sigma)]^{-1} \in L^\infty(0,T;\mathbb{R}^n)$.

(ii) Let (φ,β) be the adapted solution to the BSDE

$$
\begin{cases}
\mathrm{d}\varphi(t) = \Big\{[A - \mathcal{B}^H(\Sigma)R_{22}^{-1}S_2^H - \mathcal{C}^H(\Sigma)[\mathcal{R}^H(\Sigma)]^{-1}\Sigma S_1^H]\varphi \\
\qquad\qquad + \mathcal{C}^H(\Sigma)[\mathcal{R}^H(\Sigma)]^{-1}\beta\Big\}\mathrm{d}t + \beta \mathrm{d}W(t), \quad t \in [0,T], \\
\varphi(T) = \xi,
\end{cases}
\tag{6.2}
$$

and let X be the solution to the following SDE:

$$
\begin{cases}
\mathrm{d}X(t) = \Big\{\Big[(S_1^H)^\top[\mathcal{R}^H(\Sigma)]^{-1}\Sigma[\mathcal{C}^H(\Sigma)]^\top + (S_2^H)^\top R_{22}^{-1}[\mathcal{B}^H(\Sigma)]^\top - A^\top\Big]X \\
\qquad\qquad - \Big[(S_1^H)^\top[\mathcal{R}^H(\Sigma)]^{-1}\Sigma S_1^H + (S_2^H)^\top R_{22}^{-1}S_2^H\Big]\varphi + (S_1^H)^\top[\mathcal{R}^H(\Sigma)]^{-1}\beta\Big\}\mathrm{d}t \\
\qquad\qquad - \Big[\mathcal{R}^H(\Sigma)^{-1}\Big]^\top[\mathcal{C}^H(\Sigma)^\top X - S_1^H\varphi - R_{11}^H\beta]\mathrm{d}W, \\
X(0) = 0.
\end{cases}
$$

Then the optimal control of Problem (BSLQ) for the terminal state ξ is given by

$$
\begin{aligned}
u = R_{22}^{-1}\Big\{&[\mathcal{B}^H(\Sigma)^\top - R_{21}\mathcal{R}^H(\Sigma)^{-1}\Sigma\mathcal{C}^H(\Sigma)^\top]X \\
&+ [R_{21}\mathcal{R}^H(\Sigma)^{-1}\Sigma S_1^H - S_2^H]\varphi - R_{21}\mathcal{R}^H(\Sigma)^{-1}\beta\Big\}.
\end{aligned}
\tag{6.3}
$$

(iii) The value function of Problem (BSLQ) is given by

$$
\begin{aligned}
V(\xi) = -\mathbb{E}\langle H(T)\xi,\xi\rangle + \mathbb{E}\int_0^T &\Big\{\big\langle R_{11}^H\mathcal{R}^H(\Sigma)^{-1}\beta,\beta\big\rangle + 2\big\langle (S_1^H)^\top\mathcal{R}^H(\Sigma)^{-1}\beta,\varphi\big\rangle \\
&- \big\langle[(S_1^H)^\top\mathcal{R}^H(\Sigma)^{-1}\Sigma S_1^H + (S_2^H)^\top R_{22}^{-1}S_2^H]\varphi,\varphi\big\rangle\Big\}\mathrm{d}t,
\end{aligned}
\tag{6.4}
$$

where (φ,β) is the adapted solution to the BSDE (6.2).

To summarize, we have investigated an indefinite backward stochastic LQ optimal control problem with deterministic coefficients and have developed a general procedure for constructing optimal controls. The crucial idea is to establish the connection between backward stochastic LQ optimal control problems and forward stochastic LQ optimal control problems (see Sect. 4) and to convert the backward stochastic LQ optimal control problem into an equivalent one for which the limiting procedure applies (see Sect. 5). The results obtained in the paper provide insight into some related topics, especially into the study of zero-sum stochastic differential games (as mentioned in the introduction). We hope to report some relevant results along this line in our future publications.

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