

RATES OF CONVERGENCE IN PERIODIC HOMOGENIZATION OF NONLOCAL HAMILTON–JACOBI–BELLMAN EQUATIONS*,**

ANDREI RODRÍGUEZ-PAREDES¹ AND ERWIN TOPP^{2,***}

Abstract. In this paper we provide a rate of convergence for periodic homogenization of Hamilton–Jacobi–Bellman equations with nonlocal diffusion. The result is based on the regularity of the associated effective problem, where convexity plays a crucial role. The necessary regularity estimates are made possible by a representation formula we obtain for the effective Hamiltonian, a result that has an independent interest.

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1. INTRODUCTION

In this paper, we are interested in the periodic homogenization of fractional Hamilton–Jacobi–Bellman equations with the form

$$u^\epsilon + H\left(x, \frac{x}{\epsilon}, Du^\epsilon(x), u^\epsilon\right) = 0 \quad \text{for } x \in \mathbb{R}^N, \quad (1.1)$$

where $\epsilon \in (0, 1)$ is the perturbation parameter, and the Hamiltonian H is defined as the supremum of linear, uniformly elliptic integro-differential operators with periodic dependence on the fast variable x/ϵ .

More precisely, let Θ be a compact metric space. Denote \mathbb{T}^N the N -dimensional flat torus and \mathbb{S}^{N-1} the unit sphere in \mathbb{R}^N . Let $a \in C(\mathbb{T}^N \times \mathbb{S}^{N-1} \times \Theta)$ satisfying certain symmetry/ellipticity assumptions (cf. (1.8)). For $\sigma \in (1, 2)$ fixed, $\varphi : \mathbb{R}^N \rightarrow \mathbb{R}$ measurable, and $x \in \mathbb{R}^N$, we denote

$$L_y^\theta \varphi(x) = \text{P.V.} \int_{\mathbb{R}^N} [\varphi(x+z) - \varphi(x)] \frac{a^\theta(y, \hat{z})}{|z|^{N+\sigma}} dz, \quad (1.2)$$

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¹ Departamento de Matemática, Universidad de Concepción, Avenida Esteban Iturra s/n, Barrio Universitario, Casilla 160 C, Concepción, Chile.

² Departamento de Matemática y C.C., Universidad de Santiago de Chile, Casilla 307, Santiago, Chile, and Instituto de Matemáticas, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ 21941-909, Brazil.

*** Corresponding author: erwin.topp@usach.cl

whenever the integral makes sense. Here $\hat{z} = z/|z|$ for $z \neq 0$, and *P.V.* stands for the Cauchy Principal Value. Here and in what follows, we adopt the notation

$$a^\theta(\cdot, \cdot) = a(\cdot, \cdot, \theta), \quad (1.3)$$

and use it similarly for other functions.

If $a \equiv C_{N,\sigma} > 0$ for some suitable positive, normalizing constant, (1.2) is the well-known fractional Laplacian of order σ , given by

$$-(-\Delta)^{\sigma/2}\varphi(x) = C_{N,\sigma} \text{P.V.} \int_{\mathbb{R}^N} \frac{\varphi(x+z) - \varphi(x)}{|z|^{N+\sigma}} dz, \quad (1.4)$$

see [16] for a complete survey about this operator. In particular, it is nowadays well-known that the integral in (1.2) makes sense for functions φ which are $C^{\sigma+\alpha}$ around x for some $\alpha > 0$, and belonging to the class $L^1_\sigma(\mathbb{R}^N)$ of functions in $L^1_{loc}(\mathbb{R}^N)$ which are integrable against the kernel $(1+|x|)^{-N-\sigma}$, see Caffarelli and Silvestre [9].

We also consider functions $f \in C_b(\mathbb{R}^N \times \mathbb{T}^N \times \Theta; \mathbb{R})$, $l \in C_b(\mathbb{R}^N \times \mathbb{T}^N \times \Theta; \mathbb{R})$, and with this we define the Hamiltonian

$$H(x, y, p, \varphi) = \sup_{\theta \in \Theta} \{-L_y^\theta \varphi(x) - f^\theta(x, y) \cdot p - l^\theta(x, y)\}, \quad (1.5)$$

where we have adopted notation (1.3) for f and l .

Fractional PDEs have attracted the attention of the community in the last decade because of its connection with other branches in mathematics and applications. We are particularly interested in the connection between problem (1.1) and infinite horizon optimal control of jump diffusion processes. For instance, denoting by $S(N)$ the set of symmetric $N \times N$ matrices, and considering $A \in C(\mathbb{T}^N \times \Theta; S(N))$ with $|\det A(y, \theta)| = 1$, nonlocal operators in Lévy-Ito form

$$L_y^\theta \varphi(x) = \text{P.V.} \int_{\mathbb{R}^N} [\varphi(x + A(y, \theta)z) - \varphi(x)] \frac{dz}{|z|^{N+\sigma}}.$$

can be readily written in the form (1.2) after a change of variables. If $A(\cdot, \theta)$, $f(\cdot, \cdot, \theta)$ are Lipschitz continuous, for each $x \in \mathbb{R}^N$ there exists a unique solution $(X_t)_t$ for the SDE

$$dX_t = f(X_t, \epsilon^{-1}X_t, \theta_t)dt + \int_{\mathbb{R}^N} A(\epsilon^{-1}X_s, \theta_s)z \tilde{N}(dt, dz), \quad X_0 = x,$$

where for a given filtered probability space, \tilde{N} is the compensated Poisson random measure of the isotropic σ -stable Lévy process (whose infinitesimal generator is the fractional Laplacian (1.4)). In this context, the value function

$$u^\epsilon(x) = \inf_{\theta(\cdot)} \mathbb{E} \left\{ \int_0^\infty e^{-s} l(X_s, \epsilon^{-1}X_s, \theta_s) ds \right\},$$

is a viscosity solution to (1.1), see *e.g.* [24, 28]. Here, the infimum is taken over the set of adapted process with values in Θ . This optimal control perspective of the problem has applications to financial markets driven by jump processes, see Cont and Tankov [14]. Roughly speaking, the homogenization result for problem (1.1) is related to an ‘‘average behavior’’ of the value function of the optimization problem as $\epsilon \rightarrow 0$.

Periodic homogenization for nonlocal equations has been previously addressed in different frameworks, see for instance [2, 3, 13, 22, 29, 30] and references therein. In particular, using the viscosity solution’s approach for

nonlocal problems (c.f. [7]), together with the procedure introduced in the seminal paper of Lions, Papanicolaou and Varadhan [27], it is possible to prove that u^ϵ solving (1.1) converges uniformly as $\epsilon \rightarrow 0$ to a function u , solution to a limiting problem with the form

$$u + \bar{H}(x, Du(x), u) = 0 \quad \text{for } x \in \mathbb{R}^N, \quad (1.6)$$

see for instance [3, 13] for similar results posed in the evolution case. This convergence result is what we understood as *homogenization*; in this article, rather than addressing the convergence directly, we will provide an estimate on the difference $\|u^\epsilon - u\|_\infty$ in terms of ϵ , from which we conclude homogenization as a byproduct.

In (1.6), \bar{H} is the *effective Hamiltonian*, defined as follows: for $x, p \in \mathbb{R}^N$ and $\phi \in C^{\sigma+\alpha}(\mathbb{R}^N) \cap L^1_\sigma(\mathbb{R}^N)$, $\bar{H}(x, p, \phi)$ is defined as the unique constant $c \in \mathbb{R}$ for which the *cell problem*

$$\sup_{\theta} \{-L_y^\theta w(y) - L_y^\theta \phi(x) - f^\theta(x, y) \cdot p - l^\theta(x, y)\} = c \quad \text{for } y \in \mathbb{T}^N, \quad (1.7)$$

has a viscosity solution $w \in C(\mathbb{T}^N)$. As soon as well-posedness for (1.6) holds, homogenization is a consequence of the so-called perturbed test function method by Evans [17, 18]. The main problem here is that \bar{H} is not explicit in general, and it is not clear when comparison principles for (1.6) are available without further assumptions.

Now we introduce the assumptions on (1.1) and present the main results. Concerning the nonlocal operator, we assume that

$$\begin{cases} a \in C(\mathbb{T}^N \times \mathbb{S}^{N-1} \times \Theta); \quad \sup_{\theta \in \Theta} [a(\cdot, \cdot, \theta)]_{C^\alpha(\mathbb{T}^N \times \mathbb{S}^{N-1})} \leq C, \\ a(\cdot, w, \cdot) = a(\cdot, -w, \cdot), \quad \text{for all } w \in S^{N-1}, \\ \gamma \leq a \leq \Gamma, \end{cases} \quad (1.8)$$

for some constants $C > 0$, $\alpha \in (0, 1)$, and $0 < \gamma \leq \Gamma$ (these last two constants are the ellipticity constants in the sense of [9]).

We assume $f \in C_b(\mathbb{R}^N \times \mathbb{T}^N \times \Theta; \mathbb{R}^N)$, $l \in C_b(\mathbb{R}^N \times \mathbb{T}^N \times \Theta; \mathbb{R})$, and Lipschitz in x, y , that is there exists $C > 0$ such that

$$|D_x f^\theta(x, y)|, |D_y f^\theta(x, y)|, |D_x l^\theta(x, y)|, |D_y l^\theta(x, y)| \leq C \quad \text{for all } x, y, \theta, \quad (1.9)$$

where D_x, D_y represent the Lipschitz constant of the function with respect to x, y , respectively.

The main result of this paper is the following

Theorem 1.1. *Let a satisfying (1.8), f, l satisfying (1.9), and assume H has the form (1.5). Then, \bar{H} in (1.7) is well-defined, and problem (1.6) has a unique bounded, viscosity solution $u \in C^{\sigma+\iota}(\mathbb{R}^N)$ for some $\iota \in (0, 1)$.*

For each $\epsilon \in (0, 1)$, there exists a unique bounded, viscosity solution u^ϵ to (1.1), and there exist $\bar{\alpha} \in (0, \alpha)$ and $C > 0$ depending on the data, but not on ϵ , such that

$$\|u^\epsilon - u\|_\infty \leq C\epsilon^{\bar{\alpha}}.$$

The notion of viscosity solution is given in Definition 2.1 below. We eventually prove a representation formula for \bar{H} , and this notion of solution fits into (1.6) too (see Rem. 2.3). Existence, uniqueness and regularity estimates for u^ϵ are provided in Proposition 2.5.

We are admittedly in debt with an exhaustive list of references concerning homogenization results, both for local and nonlocal problems, in divergence and non-divergence form. We refer to the reader to the articles mentioned above and references therein. However, to the best of our knowledge, the rate of convergence for the homogenization of nonlocal equations has not yet been addressed.

Concerning available results for local equations, rates of convergence for homogenization of first-order Hamilton–Jacobi equations was first established by Capuzzo–Dolcetta and Ishii in [11], obtaining the exponent $\bar{\alpha} = 1/3$ when the Hamiltonian is coercive. This method was adapted by Camilli and Marchi in [10] for second-order equations, where the exponent $\bar{\alpha}$ is not explicit in general and depends upon the various data parameters involved in $C^{2,\alpha}$ estimates for convex, uniformly elliptic equations. We follow this approach, studying the difference $u^\epsilon - u$ through the classical doubling variables method in the theory of viscosity solutions, adapted to the nonlocal setting, and obtain in the most general case an analogous non-explicit, polynomial rate, depending on the regularity of solutions to the effective problem and on the order of the nonlocal operator. As in the case of [10], there is no direct way of comparing the exponent found to the one in the first-order setting, $\bar{\alpha} = 1/3$. In this direction, $C^{\sigma+\alpha}$ estimates for (1.6) are crucial in the construction of appropriate correctors, and therefore we require a closer look into the structure of the effective Hamiltonian \bar{H} .

It is rather standard that some structural properties of \bar{H} are inherited from H . For instance, in [18] it is proven that properties such as uniform ellipticity, continuity, coercivity, and/or convexity hold for the effective Hamiltonian as soon as the original Hamiltonian has these properties. In the nonlocal case, Schwab [30] proved that the effective Hamiltonian associated to uniformly elliptic nonlocal problems fulfills a *nonlocal degenerate ellipticity*, which is summarized as follows: given $x, p \in \mathbb{R}^N$ and $\varphi_1, \varphi_2 \in C^{\sigma+\alpha}(\mathbb{R}^N) \cap L^1_\sigma(\mathbb{R}^N)$ such that $\varphi_1(x) = \varphi_2(x)$ and $\varphi_1 \geq \varphi_2$ in \mathbb{R}^N , then

$$\bar{H}(x, p, \varphi_1) \leq \bar{H}(x, p, \varphi_2). \quad (1.10)$$

This property, together with appropriate continuity of \bar{H} ensures comparison for classical solutions. However, the degenerate ellipticity property (1.10) alone does not guarantee uniqueness for (1.6) nor regularity of its solutions in general, see [7, 19, 26, 31]. This is mostly due to the x -dependence of \bar{H} , which is not explicit in the most general setting. Nevertheless, condition (1.10) by itself is sufficient to compare classical solutions of (1.6), and we search for classical solutions to the effective problem by means of representation formulas for the ergodic constant in (1.7) that have an interest in their own right. It dates back to Courrège [15] that linear operators satisfying (1.10) (or satisfying the *global comparison principle*) can be characterized of being of *Lévy type*. Some of these formulae are at hand, see for instance [20, 30], but they are not sufficient to conclude comparison principles and/or regularity estimates for equation (1.6) in the generality presented here. Following the lines of Ishii, Mitake and Tran [21], we arrive at an “average” formula for \bar{H} (see Thm. 2.8 below) that allows us to employ available elliptic regularity estimates for nonlocal equations.

For Hamiltonians with a simpler structure, it is possible to provide a more explicit expression of the exponent $\bar{\alpha}$ in Theorem 1.1.

Theorem 1.2. *Assume hypotheses of Theorem 1.1.*

- (i) *if $f^\theta \equiv 0$ and l^θ depends only on $y \in \mathbb{T}^N$ (i.e., $l^\theta(x, y) = l^\theta(y)$ for all $\theta \in \Theta$), there exists a constant $C > 0$ such that*

$$\|u - u^\epsilon\|_\infty \leq C\epsilon^\sigma, \quad \epsilon \in (0, 1).$$

- (ii) *if f^θ and l^θ depend only on $y \in \mathbb{T}^N$ (i.e., $f^\theta(x, y) = f^\theta(y)$, $l^\theta(x, y) = l^\theta(y)$ for all $\theta \in \Theta$), there exists a constant $C > 0$ such that*

$$\|u - u^\epsilon\|_\infty \leq C\epsilon^{\sigma-1}, \quad \epsilon \in (0, 1).$$

Aside from case (i) in Theorem 1.2, we think the convergence rates presented here are not sharp. Explicit estimates for periodic homogenization of first-order Hamilton–Jacobi equations are provided in the already mentioned paper [11], see also [1, 25] for evolution equations. Based on the regularizing effect of the diffusion, the methods presented here do not cover the case of time-dependent equations, nor the case $\sigma < 1$ (the case

$\sigma = 1$ is neither considered here, but from rather different reasons, see Sect. 2.3). Finally, we also mention that in the second-order setting (closer to our setting) higher order expansions of u^ϵ in terms of ϵ are obtained by Kim and Lee in [23] for second-order problems with higher-order regularity assumptions on the data. The methods presented therein do not seem to have a direct analog in the nonlocal setting, and further ideas must be brought into the analysis.

The paper is organized as follows. In Section 2, we provide the basic notation, the notion of solution and prove the representation formula for the effective Hamiltonian. In Section 3, we provide relevant estimates of the discount problem, the approximating equation that defines the effective Hamiltonian. Finally, in Section 4 we provide the rate of convergence for our homogenization problem.

2. REPRESENTATION FORMULA FOR \bar{H}

2.1. Preliminaries

Here and in what follows, for $x \in \mathbb{R}^N$ and $r > 0$, we denote by $B_r(x)$ the ball of radius r centered at x . If $x = 0$, we write B_r .

We provide here some definitions and basic properties for viscosity solutions in the nonlocal framework. Despite several properties can be extended for more general problems, and weaker assumptions on the data, our presentation is focused on the current problem.

For a as in (1.8), and for each $\theta \in \Theta$, we denote

$$K^\theta(x, z) = \frac{a^\theta(x, \hat{z})}{|z|^{N+\sigma}} \quad x \in \mathbb{R}^N, \quad z \in \mathbb{R}^N \setminus \{0\}. \quad (2.1)$$

We assume the existence of a modulus of continuity m such that

$$\sup_{\theta \in \Theta, \omega \in \mathbb{S}^{N-1}} |a^\theta(x, \omega) - a^\theta(y, \omega)| \leq m(|x - y|), \quad \text{for all } x, y \in \mathbb{R}^N. \quad (2.2)$$

We consider the problem

$$u + H(x, Du, u) = 0 \quad \text{in } \mathbb{R}^N, \quad (2.3)$$

where, for $x, p \in \mathbb{R}^N$, $\phi \in L^1_\sigma(\mathbb{R}^N)$ and smooth around x , and recalling the definition of the nonlocal operator L in (1.2), the Hamiltonian takes the form

$$H(x, p, \phi) = \sup_{\theta \in \Theta} \{-L^\theta_x \phi(x) - f^\theta(x) \cdot p - l^\theta(x)\}. \quad (2.4)$$

The data is intended to satisfy standard regularity and/or growth conditions at infinity in the viscosity theory, in such a way all the definitions and results can be applied to our problem. However, no periodicity assumptions need to be considered in this subsection.

Following Barles and Imbert [7], we provide the notion of viscosity solution for our problem.

Definition 2.1. An *u.s.c.* (resp. *l.s.c.*) function $u \in L^1_\sigma(\mathbb{R}^N)$ is a viscosity subsolution (resp. supersolution) to (2.3) at $x_0 \in \mathbb{R}^N$ if for all $\phi \in C^2$ such that ϕ touches u at x_0 from above (resp. from below) in $B_\delta(x_0)$ for some $\delta > 0$, then

$$u(x_0) + \sup_{\theta \in \Theta} \{-L^\theta_{x_0}[B_\delta]\phi(x_0) - L^\theta_{x_0}[B_\delta^c]u(x_0) - f^\theta(x_0) \cdot D\phi(x_0) - l^\theta(x_0)\} \leq (\text{resp. } \geq) 0,$$

where, for each $x, \xi \in \mathbb{R}^N$, for adequate function φ , and $A \subseteq \mathbb{R}^N$ measurable, we have adopted the notation

$$L_\xi^\theta[A]\varphi(x) = \text{P.V.} \int_A [\varphi(x+z) - \varphi(x)] K^\theta(\xi, z) dz.$$

We say that u is sub/supersolution in $A \subseteq \mathbb{R}^N$ if it is a sub/supersolution at every point in A .

A continuous function $u \in L_\sigma^1(\mathbb{R}^N)$ is a viscosity solution to (2.3) if it is a viscosity subsolution and supersolution simultaneously.

Remark 2.2. The above definition can be readily extended in the usual way to parabolic problem with the form

$$u_t + H(x, Du, u) = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty),$$

see [7].

Remark 2.3. For the sake of completeness, we mention that the notion of viscosity solution may be defined for nonlocal operators satisfying (1.10) even if the structure of the operator is not explicitly given by (2.4) (again, see [7]). In this case, an *u.s.c.* (resp. *l.s.c.*) function $u \in L_\sigma^1(\mathbb{R}^N)$ is a viscosity subsolution (resp. supersolution) to (2.3) at $x_0 \in \mathbb{R}^N$ if for all $\phi \in C^2$ such that ϕ touches u at x_0 from above (resp. from below) in $B_\delta(x_0)$ for some $\delta > 0$, then

$$H(x_0, D\phi(x_0), \mathbf{1}_{B_\delta(x_0)}\phi(\cdot) + \mathbf{1}_{B_\delta^c(x_0)}u(\cdot)) \leq (\text{resp. } \geq) 0.$$

This notion reduces to Definition 2.1 when a formula like (2.4) is available, of course, and this will be the case for (1.6) once Theorem 2.8 is proved.

As we previously mentioned, comparison principle for equations like (2.3), where the nonlocal operator depends upon the state variable is not known in general. One case is when sub and/or supersolution are classical (say, in $L_\sigma^1(\mathbb{R}^N)$ and $C^{\sigma+\alpha}$ around the point).

For the existence of a continuous solution to (2.3), we approximate the problem through nonlocal operators that satisfy the comparison principle. For $\eta > 0$, we consider the kernel

$$K_\eta^\theta(x, z) = K^\theta(x, z)(1 - \chi_{B_\eta}(z)) + |z|^{-(N+\sigma)}\chi_{B_\eta}(z),$$

where χ_A is the indicator function of $A \subset \mathbb{R}^N$. We denote by L_η the nonlocal operator L in (1.2) defined through this modified kernel, that is

$$L_{\xi, \eta}^\theta \varphi(x) = \text{P.V.} \int_{\mathbb{R}^N} [\varphi(x+z) - \varphi(x)] K_\eta^\theta(\xi, z) dz.$$

We have the following

Lemma 2.4. *Assume f, l are bounded, continuous in $\mathbb{R}^N \times \Theta$ such that (1.9) holds (without y dependence), and let L be a nonlocal operator with the form (1.2) with $a \in C(\mathbb{R}^N \times \mathbb{S}^{N-1} \times \Theta)$ satisfying the symmetry and ellipticity assumption in (1.8), and the uniform continuity in (2.2). Let H_η be a Hamiltonian defined in (2.4) with L replaced by L_η . Then, problem*

$$u + H_\eta(x, Du, u) = 0 \quad \text{in } \mathbb{R}^N, \tag{2.5}$$

admits comparison principle among bounded, viscosity sub and supersolutions.

Proof. Let u be a bounded, viscosity subsolution to (2.5), and v a bounded, viscosity supersolution to (2.5). By contradiction, assume that $u(x_0) > v(x_0)$ for some $x_0 \in \mathbb{R}^N$.

Let $\psi \in C_b^2(\mathbb{R}^N)$ be a nonnegative, radially nondecreasing function such that $\psi(x) = |x|^2$ in B_1 , $\psi = 2 + R$ in B_2^c for $R = \|u\|_\infty + \|v\|_\infty$. For $\beta \in (0, 1)$, we denote $\psi_\beta(x) = \psi(\beta x)$.

Then, for all $\beta > 0$ small enough depending on x_0 (say, β such that $\psi_\beta(x_0) \leq (u(x_0) - v(x_0))/2$), the supremum

$$\sup_{x \in \mathbb{R}^N} \{u(x) - v(x) - \psi_\beta(x)\}$$

is positive, and it is attained.

Then, we double variables and following the standard procedure in the viscosity theory, for all $\epsilon > 0$ the supremum

$$M := \sup_{x, y \in \mathbb{R}^N} \{u(x) - v(y) - \epsilon^{-2}|x - y|^2 - \psi_\beta(x)\}, \quad (2.6)$$

is positive, and attained at some point $\bar{x}, \bar{y} \in \mathbb{R}^N$. In fact, there exists $C_\beta > 0$ not depending on ϵ such that $|\bar{x}|, |\bar{y}| \leq C_\beta$ for all ϵ , and we have $\epsilon^{-2}|\bar{x} - \bar{y}|^2 \rightarrow 0$ as $\epsilon \rightarrow 0$, when β is fixed. We use the corresponding viscosity inequalities for u at \bar{x} , and for v at \bar{y} . Namely, we denote $\varphi(x, y) = \epsilon^{-2}|x - y|^2 + \psi_\beta(x)$, and for every β, ϵ small, and all $\delta > 0$, there exists $\theta \in \Theta$ such that

$$\begin{aligned} u(\bar{x}) - L_{\bar{x}, \eta}^\theta[B_\delta]\varphi(\cdot, \bar{y})(\bar{x}) - L_{\bar{x}, \eta}^\theta u(\bar{x}) - f^\theta(\bar{x}) \cdot D_x \varphi(\bar{x}, \bar{y}) - l^\theta(\bar{x}) &\leq 0, \\ v(\bar{y}) - L_{\bar{y}, \eta}^\theta[B_\delta](-\varphi)(\bar{x}, \cdot)(\bar{y}) - L_{\bar{y}, \eta}^\theta v(\bar{y}) - f^\theta(\bar{y}) \cdot D_y(-\varphi)(\bar{x}, \bar{y}) - l^\theta(\bar{y}) &\geq -M/2, \end{aligned}$$

where we have dropped the sup in the viscosity inequality for v at the expense of the term $-M/2$ in the right-hand side.

Using the ellipticity assumptions of the kernel, and the estimates of Lemma 2.3 in [8], we have the existence of a constant $C > 0$ such that

$$|L_{\bar{x}, \eta}^\theta[B_\delta]\varphi(\cdot, \bar{y})(\bar{x})| \leq C(\epsilon^{-2} + \beta^2)\delta^{2-\sigma}; \quad |L_{\bar{y}, \eta}^\theta[B_\delta](-\varphi)(\bar{x}, \cdot)(\bar{y})| \leq C\epsilon^{-2}\delta^{2-\sigma}.$$

By the Lipschitz assumption on f and l , uniform in θ , we have

$$|f^\theta(\bar{x}) \cdot D_x \varphi(\bar{x}, \bar{y}) - f^\theta(\bar{y}) \cdot D_y(-\varphi)(\bar{x}, \bar{y})| = o_{\epsilon, \beta}(1); \quad |l^\theta(\bar{x}) - l^\theta(\bar{y})| = o_{\epsilon, \beta}(1),$$

where $o_{\epsilon, \beta} \rightarrow 0$ as $\epsilon \rightarrow 0$ when β is fixed, independent of δ . Then, subtracting the viscosity inequalities, and using that $u(\bar{x}) - v(\bar{y}) \geq M$, we arrive at

$$M \leq L_{\bar{x}, \eta}^\theta u(\bar{x}) - L_{\bar{y}, \eta}^\theta v(\bar{y}) + M/2 + C\epsilon^{-2}\delta^{2-\sigma} + o_{\epsilon, \beta}(1). \quad (2.7)$$

At this point we estimate $I = L_{\bar{x}, \eta}^\theta u(\bar{x}) - L_{\bar{y}, \eta}^\theta v(\bar{y})$. By definition of the truncated kernels K_η , using that (\bar{x}, \bar{y}) attains the maximum in (2.6), we have

$$\begin{aligned} I &= \int_{B_\eta \setminus B_\delta} [u(\bar{x} + z) - v(\bar{y} + z) - (u(\bar{x}) - v(\bar{y}))]|z|^{-(N+\sigma)} dz \\ &\quad + L_{\bar{x}}^\theta[B_\eta^c]u(\bar{x}) - L_{\bar{y}}^\theta[B_\eta^c]v(\bar{y}) \\ &\leq \int_{B_\eta \setminus B_\delta} [\psi_\beta(\bar{x} + z) - \psi_\beta(\bar{x})]|z|^{-(N+\sigma)} dz \end{aligned}$$

$$\begin{aligned}
& + \int_{B_\eta^c} [\psi_\beta(\bar{x} + z) - \psi_\beta(\bar{x})] K^\theta(\bar{x}, z) dz \\
& + L_{\bar{x}, \eta}^\theta[B_\eta^c]v(\bar{y}) - L_{\bar{y}, \eta}^\theta[B_\eta^c]v(\bar{y}).
\end{aligned}$$

Using again the estimates of Lemma 2.3 in [8], we conclude that

$$I \leq o_\beta(1) + L_{\bar{x}, \eta}^\theta[B_\eta^c]v(\bar{y}) - L_{\bar{y}, \eta}^\theta[B_\eta^c]v(\bar{y}),$$

and using the uniform continuity of a , together with the boundedness of v , by Dominated Convergence Theorem we get that

$$|L_{\bar{x}, \eta}^\theta[B_\eta^c]v(\bar{y}) - L_{\bar{y}, \eta}^\theta[B_\eta^c]v(\bar{y})| = o_{\epsilon, \beta}(1),$$

where $o_{\epsilon, \beta}(1)$ is as before (depending on η , but not on δ). Thus, replacing the last estimates into (2.7), we let $\delta \rightarrow 0$ first, later $\epsilon \rightarrow 0$ and finally $\beta \rightarrow 0$ to conclude the contradiction. \square

As a first corollary of this result, we have the following

Proposition 2.5. *Under the assumptions of Theorem 1.1, there exists a unique solution $u^\epsilon \in C_b^{\sigma+\iota}(\mathbb{R}^N)$ to problem (1.1), for some $\iota \in (0, 1)$. Moreover, we have the family $\{u^\epsilon\}_\epsilon$ is equibounded, and there exists $\alpha' \in (0, 1)$ and $C > 0$ such that*

$$\sup_{\epsilon \in (0, 1)} [u^\epsilon]_{C^{\alpha'}} \leq C.$$

Proof. For $\epsilon \in (0, 1)$ fixed, we consider the approximation problem

$$u + H_\eta(x, \frac{x}{\epsilon}, Du, u) = 0 \quad \text{in } \mathbb{R}^N.$$

It is easy to see that there exists $M > 0$ depending on the data, but not on ϵ , such that the constant function equal to M is a viscosity supersolution to the problem, and $-M$ is a viscosity subsolution. By Perron's method, there exists a bounded discontinuous viscosity solution $u^{\epsilon, \eta}$ to this problem in the sense of Definition 2.1 (namely, the upper semicontinuous envelope of u is a viscosity subsolution, and analogously for supersolution) such that $\|u^{\epsilon, \eta}\|_\infty \leq M$, and since the approximating equation admits comparison principle, we have $u^{\epsilon, \eta}$ is continuous.

Now, notice that $u^{\epsilon, \eta}$ solves, in the viscosity sense, the inequalities

$$\begin{aligned}
u - \mathcal{M}^+ u - \|f\|_\infty |Du| &\leq \|l\|_\infty \quad \text{in } \mathbb{R}^N, \\
u - \mathcal{M}^- u + \|f\|_\infty |Du| &\geq -\|l\|_\infty \quad \text{in } \mathbb{R}^N,
\end{aligned}$$

where we have denoted by \mathcal{M}^\pm the nonlocal extremal Pucci operators associated to the class of kernels with ellipticity constants $\min\{1, \gamma\}$, $\max\{1, \Gamma\}$, see [9]. Then, by the Hölder estimates of Chang-Lara and Dávila [12], we have the family $\{u^{\epsilon, \eta}\}_{\epsilon, \eta}$ is equi-bounded with uniform C^α estimates.

Then, we pass to the limit as $\eta \rightarrow 0$, and by stability of viscosity solutions we arrive at a Hölder continuous solution u^ϵ to (1.1). By the estimates in Barles, Chasseigne, Imbert and Ciomaga [5], we conclude that u^ϵ is Lipschitz continuous. Then, by $C^{1, \alpha}$ estimates of Caffarelli and Silvestre [9] we conclude that u^ϵ is in $C^{1, \alpha}$, and then we use the result by Serra [31] to conclude that u^ϵ is a classical solution to (1.1).

Finally, uniqueness is a consequence of comparison principle among classical solutions. Indeed, if u, v are bounded, classical solutions of this problem and we assume that $M := \sup_{x \in \mathbb{R}^N} \{u - v\} = \max_{x \in \mathbb{R}^N} \{u - v\} > 0$, then we arrive at a contradiction by evaluating the equations at a maximum point and using the fact that the

equation is proper. If the supremum is not attained, we fix $\sigma' \in (0, \sigma)$ and use the fact that for all $\beta > 0$ small enough (in terms of the data and M), the function

$$x \mapsto u(x) - v(x) - \psi_\beta(x), \quad x \in \mathbb{R}^N,$$

attains its maximum at a point $x_\beta \in \mathbb{R}^N$. Here, ψ_β is as in Lemma 2.4. Since classical solutions are viscosity solutions, we can use $v + \psi_\beta$ as test function for u , and conclude as in Lemma 2.4, using the properness. \square

2.2. Representation formulas

We also consider a function $\ell \in C(\mathbb{T}^N \times \Theta)$ satisfying, for some $\alpha \in (0, 1)$ and $C > 0$, that

$$\sup_{\theta \in \Theta} [\ell^\theta]_{C^\alpha(\mathbb{T}^N)} \leq C. \quad (2.8)$$

For simplicity, we write

$$F(\psi, y) := \sup_{\theta} \{-L_y^\theta \psi(y) - \ell^\theta(y)\}.$$

We start with the solvability of the ergodic problem.

Proposition 2.6. *Assume (1.8), (2.8) hold. Then, there exists a unique constant $c \in \mathbb{R}$ for which the problem*

$$F(\psi, y) = c \quad \text{in } \mathbb{T}^N, \quad (2.9)$$

has a solution $\psi \in C^{\sigma+\iota}(\mathbb{T}^N)$ for some $\iota \in (0, \alpha)$. The solution ψ is unique up to additive constants.

Proof. The proof goes along the lines of Theorem 1 in [6], but the y -dependence of the kernel defining the nonlocal operator prevents the direct use of this result. For $\delta > 0$, we consider the approximation problem

$$\delta \psi^\delta + F(\psi^\delta, y) = 0 \quad \text{in } \mathbb{T}^N, \quad (2.10)$$

which is solvable by similar arguments as in Proposition 2.5. The solution ψ^δ ends up being in $C^{\sigma+\iota}(\mathbb{T}^N)$, and since it is a classical solution, it is unique. Once we have the existence of ψ^δ , we can prove that the function $\tilde{\psi}^\delta(y) = \psi^\delta(y) - \psi^\delta(0)$ is equibounded and equicontinuous just as in Proposition 1 in [6]. This function solves the problem

$$\delta \tilde{\psi}^\delta + F(\tilde{\psi}^\delta, y) = -\delta \psi^\delta(0) \quad \text{in } \mathbb{T}^N.$$

From (2.10), $\delta \psi^\delta(0)$ is bounded, hence there exists a sequence $\delta_k \rightarrow 0$ such that $\tilde{\psi}^{\delta_k} \rightarrow \psi$ uniformly in \mathbb{T}^N , $\delta_k \tilde{\psi}^{\delta_k}(0) \rightarrow -c$ as $k \rightarrow 0$, where the pair (ψ, c) solves (2.9). By the regularity results in [31] we have ψ is in $C^{\sigma+\iota}$. The uniqueness of c follows as in Theorem 1 in [6] by comparison among classical solutions, and the uniqueness up to additive constants for ψ is a consequence of Strong Maximum Principle. \square

Now we present the first main result of this section, which is a representation formula for c in the previous proposition. Given L as above, define the set

$$\mathcal{G}_0 = \{\phi \in C(\mathbb{T}^N \times \Theta) : \exists u \text{ s.t. } \sup_{\theta \in \Theta} \{-L_y^\theta u(y) - \phi^\theta(y)\} \leq 0 \text{ for } y \in \mathbb{T}^N\}, \quad (2.11)$$

where the equation satisfied by u is understood in the viscosity sense. Using similar arguments as in Lemma 2.8 in [21], \mathcal{G}_0 is a nonempty convex cone with vertex at the origin. We also denote \mathcal{P} the space of probability measures on $\mathbb{T}^N \times \Theta$, and consider the polar cone \mathcal{G}'_0 given by

$$\mathcal{G}'_0 := \left\{ \mu \in \mathcal{P} : \int \phi d\mu \geq 0 \text{ for all } \phi \in \mathcal{G}_0 \right\}. \quad (2.12)$$

Proposition 2.7. *Assume hypotheses of Proposition 2.6 hold, and let c be the ergodic constant associated to problem (2.9). Then, c can be characterized as follows*

$$\begin{aligned} c &= - \inf_{\mu \in \mathcal{P}} \sup_{\psi \in C^{\sigma+\iota}} \int_{\mathbb{T}^N \times \Theta} \{L_y^\theta \psi(y) + \ell^\theta(y)\} d\mu \\ &= - \inf_{\mu \in \mathcal{G}'_0} \int_{\mathbb{T}^N \times \Theta} \ell^\theta(y) d\mu, \end{aligned} \quad (2.13)$$

where $\iota \in (0, 1)$ in the first expression, and \mathcal{G}'_0 in the second expression is defined in (2.12).

Proof. We start proving the following characterization for the ergodic constant c in Proposition 2.6:

$$c = \inf \{ \bar{c} \in \mathbb{R} : \exists \psi \text{ s.t. } F(\psi, y) \leq \bar{c} \text{ in } \mathbb{T}^N \}, \quad (2.14)$$

where the inequality inside the inf is understood in the viscosity sense. This follows the lines of Proposition 3.2 in [33], but we provide the details for completeness. Denote c_1 the right-hand side in (2.14), and notice that $c \geq c_1$. If $c > c_1$, then we can choose $c_2 \in (c_1, c)$ such that there exists ψ_2 u.s.c. in \mathbb{T}^N such that $F(\psi, y) \leq c_2$ ($y \in \mathbb{T}^N$) in the viscosity sense. Then, it is possible to see that the function $w_2(y, t) = \psi_2(y) + (c - c_2)t$ is a viscosity subsolution to the parabolic problem

$$u_t + F(u, y) = c \quad \text{in } \mathbb{T}^N \times (0, +\infty).$$

Let $\psi \in C^{\sigma+\iota}$ be a solution to (2.9) such that $\psi \geq \psi_2$ in \mathbb{T}^N . Then, ψ is a smooth solution to the above problem, from which we get from comparison principle that $w_2 \leq \psi$ in $\mathbb{T}^N \times (0, +\infty)$, that is $\psi_2(y) + (c - c_2)t \leq \psi(y)$ for all y and all $t > 0$. Despite the nonlocal operator depends on the state variable, we have comparison principle since one of the functions to be compared is smooth. Dividing by t and letting $t \rightarrow +\infty$ we get $c \leq c_2$, which is a contradiction.

Now, we prove that (2.14) leads to a second characterization, namely

$$c = \inf_{\psi \in C^{\sigma+\iota}} \sup_{y \in \mathbb{T}^N} F(\psi, y).$$

Indeed, let us denote by \bar{c} the right-hand side in the last equality. Since there exists a smooth solution to (2.9), we clearly have $\bar{c} \leq c$. On the other hand, if $\bar{c} < c$, there exists $\epsilon > 0$ small and ψ smooth such that

$$F(\psi, y) \leq \bar{c} + \epsilon < c \quad \text{for all } y \in \mathbb{T}^N,$$

which contradicts the characterization (2.14).

Using this last formula for c and the continuity of the map

$$(y, \theta) \mapsto L_y^\theta \psi(y) + \ell^\theta(y)$$

when ψ is smooth, we conclude that

$$c = - \sup_{\psi \in C^{\sigma+\iota}} \inf_{\mu \in \mathcal{P}} \int_{\mathbb{T}^N \times \Theta} \{L_y^\theta \psi(y) + \ell^\theta(y)\} d\mu.$$

Note that for each ψ smooth, the map

$$\mu \mapsto \int_{\mathbb{T}^N \times \Theta} \{L_y^\theta \psi(y) + \ell^\theta(y)\} d\mu$$

is a bounded linear map when the $*$ -weak topology is considered on \mathcal{P} . On the other hand, given $\mu \in \mathcal{P}$, the map

$$\psi \mapsto \int_{\mathbb{T}^N \times \Theta} \{L_y^\theta \psi(y) + \ell^\theta(y)\} d\mu$$

is affine (hence, concave). Then, using Sion's Theorem [32], we get that c can be written as

$$-c = \inf_{\mu \in \mathcal{P}} \sup_{\psi \in C^{\sigma+\iota}} \int_{\mathbb{T}^N \times \Theta} \{L_y^\theta \psi(y) + \ell^\theta(y)\} d\mu, \quad (2.15)$$

which is the first characterization in (2.13).

For the second, we denote

$$\bar{c} = \inf_{\mu \in \mathcal{G}'_0} \int_{\mathbb{T}^N \times \Theta} \ell^\theta(y) d\mu.$$

Note that if $\psi \in C^{\sigma+\iota}(\mathbb{T}^N)$, the function $(y, \theta) \mapsto -L_y^\theta \psi(y)$ is in \mathcal{G}_0 . Then, by (2.15) we immediately see that $-c \leq \bar{c}$.

On the other hand, using that \mathcal{G}_0 is a cone with vertex at the origin, for each $\mu \notin \mathcal{G}'_0$ we have $\inf_{\phi \in \mathcal{G}_0} \int \phi d\mu = -\infty$; whereas, if $\mu \in \mathcal{G}'_0$ we have $\inf_{\phi \in \mathcal{G}_0} \int \phi d\mu = 0$. Thus, we can write

$$\begin{aligned} \bar{c} &= \inf_{\mu \in \mathcal{P}} \left\{ \int_{\mathbb{T}^N \times \Theta} \ell^\theta(y) d\mu - \inf_{\phi \in \mathcal{G}_0} \int_{\mathbb{T}^N \times \Theta} \phi^\theta(y) d\mu \right\} \\ &= \inf_{\mu \in \mathcal{P}} \sup_{\phi \in \mathcal{G}_0} \left\{ \int_{\mathbb{T}^N \times \Theta} (\ell^\theta - \phi^\theta) d\mu \right\}. \end{aligned}$$

Invoking again Sion's Theorem, we conclude that

$$\bar{c} = \sup_{\phi \in \mathcal{G}_0} \inf_{\mu \in \mathcal{P}} \left\{ \int_{\mathbb{T}^N \times \Theta} (\ell^\theta - \phi^\theta) d\mu \right\}.$$

Then, if by contradiction we assume that $-c < \bar{c}$, there exists $\epsilon > 0$ small and $\phi \in \mathcal{G}_0$ such that

$$-c + \epsilon < \int_{\mathbb{T}^N \times \Theta} (\ell^\theta - \phi^\theta) d\mu \quad \text{for all } \mu \in \mathcal{P}.$$

Since the Dirac delta is in \mathcal{P} , in particular we have

$$-c + \epsilon < \ell^\theta(y) - \phi^\theta(y) \quad \text{for all } (y, \theta) \in \mathbb{T}^N \times \Theta.$$

Thus, by definition of \mathcal{G}_0 , there exists u such that

$$-L_y^\theta u(y) - \ell^\theta(y) < c - \epsilon \quad \text{for all } (y, \theta) \in \mathbb{T}^N \times \Theta,$$

but this contradicts the characterization of c in (2.14). This concludes the second characterization in (2.13). \square

Theorem 2.8. *Under the assumptions of Theorem 1.1, for all $x, p \in \mathbb{R}^N$ and $\phi \in C^{\sigma+\nu}$, the function \bar{H} defined in (1.7) has the form*

$$\bar{H}(x, p, \phi) = \sup_{\mu \in \mathcal{P} \cap \mathcal{G}'_0} \left\{ -\bar{L}^\mu \phi(x) - \bar{f}^\mu(x) \cdot p - \bar{l}^\mu(x) \right\},$$

where \mathcal{G}'_0 is defined in (2.12), and for each $\mu \in \mathcal{P}$ we have denoted

$$\begin{aligned} \bar{L}^\mu \phi(x) &= \int_{\mathbb{R}^N} [\phi(x+z) - \phi(x)] \bar{K}^\mu(z) \, dz; & \bar{K}^\mu(z) &= \int_{\mathbb{T}^N \times \Theta} K^\theta(y, z) \, d\mu, \\ \bar{f}^\mu(x) &= \int_{\mathbb{T}^N \times \Theta} f(x, y, \theta) \, d\mu, \\ \bar{l}^\mu(x) &= \int_{\mathbb{T}^N \times \Theta} l(x, y, \theta) \, d\mu. \end{aligned}$$

Proof. As described in the introduction, \bar{H} is defined as the ergodic constant in (1.7). Therefore, for given $x, p \in \mathbb{R}^N$ and $\phi \in C_b^2(\mathbb{R}^N)$, we apply the second characterization in (2.13) with $\ell^\theta(y) = L_y^\theta \phi(x) + f^\theta(x, y) \cdot p + l^\theta(x, y)$, and conclude by integrating over $\mathbb{T}^N \times \Theta$. \square

In spite of \bar{H} not being translation-invariant, the representation formula obtained in Theorem 2.8 allows us to establish comparison –and thus, homogenization– through a superposition of available regularity results which allows us to ensure $C^{\sigma+\alpha}$ estimates for the effective problem, see Corollary 2.9.

As we mentioned in the Introduction, we have the following regularity result for the solution of the effective problem (1.6).

Corollary 2.9. *Assume hypotheses of Theorem (1.1) hold. Then, there exists a unique solution u of (1.6) and this solution is in $C^{\sigma+\nu}$, for some $\nu \in (0, \alpha)$.*

Proof. Notice that in view of the representation formula for \bar{H} we have the nonlocal operators \bar{L}^μ are autonomous, and comparison principle holds, see [7]. Thus, Perron's solution is continuous, and from here we get the higher order regularity by similar arguments as in Proposition 2.5. \square

2.3. A discussion on the dependency of \bar{H} on the slow variable

We notice here that the arguments of Proposition 2.7 and Theorem 2.8 apply equally to the more complicated case of x -dependent kernels, for instance, for kernels with the form

$$K^\theta(x, y, z) = \frac{a^\theta(x, y, \hat{z})}{|z|^{N+\sigma}} \quad y \in \mathbb{T}^N, \quad z \in \mathbb{R}^N \setminus \{0\}, \quad (2.16)$$

with certain continuity assumptions on a^θ with respect to the slow variable x . In this setting, we can still produce a representation formula, but with the new index set depending on x ,

$$\bar{H}(x, p, \phi) = \sup_{\mu \in \mathcal{G}'_0(x)} \left\{ -\bar{L}^\mu \phi(x) - \bar{f}^\mu(x) \cdot p - \bar{l}^\mu(x) \right\}. \quad (2.17)$$

The presence of $\mathcal{G}'_0(x)$ in (2.17) does not allow us to use the available regularity results to conclude an analogue of Corollary 2.9. As the method to obtain rates of convergence in the following section relies on such estimates, we are unable to reach a conclusion in the setting imposed by (2.16).

Nevertheless, we highlight that the structure of (2.17) is in accordance with phenomena arising, for instance, in first-order equations. For Hamiltonians of the form

$$H(x, y, p) = \sup_{\theta \in \Theta} \{-f^\theta(x, y) \cdot p - l^\theta(x, y)\},$$

under certain controllability assumptions on the family of fluxes f , the associated effective Hamiltonian takes the form

$$\bar{H}(x, p) = \sup_{\mu \in Z(x)} \{-f^\mu(x) \cdot p - l^\mu(x)\},$$

where for each $x \in \mathbb{R}^N$, $Z(x)$ is a subset of the space of probability measures on $\mathbb{T}^N \times \Theta$ – the set of “occupational measures”, as described in [34] and references therein. In particular, it is proven there that the set valued mapping $x \mapsto \{f^\mu(x)\}_{\mu \in Z(x)}$ is Lipschitz continuous with respect to the Hausdorff distance in \mathbb{R}^N . This allows us to employ the standard doubling-variables method to get comparison principles for viscosity solutions. This makes possible to identify the effective problem with a optimal control problem with average trajectories and costs, see Bardi and Terrone in [4].

The case of nonlocal terms of order $\sigma \in (0, 1]$ contains similar difficulties related to those of the previous point. First, we observe that the structure of the cell problem changes depending on the value of σ , as it is shown in [3]. For the critical value $\sigma = 1$ – the case which is studied in detail in [13] – the scaling of the problem makes the cell problem (1.7) depends also on the gradient of the corrector w . Consequently, the inequality defining \mathcal{G}_0 in (2.11) is now

$$\sup_{\theta \in \Theta} \{-L_y^\theta v(y) - f^\theta(x, y) \cdot Dv(y) - \phi_\theta(y)\} \leq 0 \text{ in } \mathbb{T}^N.$$

This implies that the sets \mathcal{G}_0 and \mathcal{G}'_0 both depend on the slow variable x , even if K^θ has the simpler form (2.1), and we are again in the situation described in the previous remark.

3. ESTIMATES FOR THE DISCOUNT PROBLEM

In the study of the cell problem (1.7) it is convenient to consider the approximation problem

$$\lambda w_\lambda + \sup_{\theta} \{-L_y^\theta w_\lambda(y) - L_y^\theta \phi(x) - f^\theta(x, y) \cdot p - l^\theta(x, y)\} = 0 \quad \text{in } \mathbb{T}^N, \quad (3.1)$$

for $\lambda > 0$, commonly referred to as the *vanishing discount problem*: by the same arguments provided in Proposition 2.5 this problem has a unique solution, and the solvability of the eigenvalue problem (1.7) is obtained in the passage to the limit as $\lambda \rightarrow 0^+$ in (3.1). This problem plays a key role in our main result Theorem 1.1.

To stress the dependence of w_λ on x, p and ϕ , we write $w_\lambda = w_\lambda(y; x, p, \phi)$ for the solution of (3.1) and, similarly, $w = w(y; x, p, \phi)$ for the solution of (1.7).

Lemma 3.1. *There exists a constant C_1 such that the solution of (3.1) satisfies the following: for all $x, p \in \mathbb{R}^N$ and $\phi \in C^{\sigma+\iota}(\mathbb{R}^N)$ with $\|\phi\|_{C^{\sigma+\iota}(\mathbb{R}^N)} < +\infty$, we have*

$$(a) \quad \|w_\lambda(\cdot; x, p, \phi)\|_\infty \leq \lambda^{-1} C_1 (1 + |p| + \|\phi\|_{C^{\sigma+\iota}});$$

(b) for some $\alpha \in (0, 1)$,

$$\|w_\lambda(\cdot; x, p, \phi) - w_\lambda(0; x, p, \phi)\|_{C^{\sigma+\alpha}(\mathbb{R}^N)} \leq C_1 (1 + |p| + \|\phi\|_{C^{\sigma+\iota}})$$

(c) $|D_p w_\lambda| \leq \lambda^{-1} C_1$, $|D_x w_\lambda| \leq \lambda^{-1} C_1 (1 + |p| + \|\phi\|_{C^{\sigma+\iota}})$ (in the viscosity sense); and if $\phi_i \in C^{\sigma+\iota}(\mathbb{R}^N)$ for $i = 1, 2$, then

$$\|w_\lambda(\cdot; x, p, \phi_1) - w_\lambda(\cdot; x, p, \phi_2)\|_\infty \leq \lambda^{-1} C_1 \|\phi_1 - \phi_2\|_{C^{\sigma+\iota}}$$

(d) for all $y \in \mathbb{T}^N$, $|w_\lambda(y; x, p, \phi) + \bar{H}(x, p, \phi)| \leq \lambda C_1 (1 + |p| + \|\phi\|_{C^{\sigma+\iota}})$.

Proof. (a) From the structure of H , we infer that $C = \pm \lambda^{-1} C_1 (1 + |p| + \|\phi\|_{C^{\sigma+\iota}(\mathbb{R}^N)})$ are respectively super- and subsolutions of (3.1). Indeed, it is immediate that

$$L_y^\theta C \equiv 0, \quad |f^\theta(x, y) \cdot p| \leq C_1 |p|, \quad |l^\theta(x, y)| \leq C_1,$$

for sufficiently large C_1 . For the principal part, we have

$$|L_y^\theta \phi(x)| \leq |L_y^\theta [B_1] \phi(x)| + |L_y^\theta [B_1^c] \phi(x)| =: I_1 + I_2.$$

Using the symmetry of K^θ , we have

$$\begin{aligned} I_1 &= \left| \int_{B_1} (\phi(x+z) - \phi(x) - D\phi(x) \cdot z) K^\theta(y, z) dz \right| \\ &= \left| \int_{B_1} \left(\int_0^1 D\phi(x+tz) \cdot z dt \right) - D\phi(x) \cdot z K^\theta(y, z) dz \right| \\ &\leq \int_{B_1} \left(\int_0^1 |D\phi(x+tz) - D\phi(x)| |z| dt \right) K^\theta(y, z) dz \\ &\leq \int_{B_1} [\phi]_{C^{1+\beta}} |z|^{1+\beta} K^\theta(y, z) dz \leq C\Gamma[\phi]_{C^{1+\beta}}. \end{aligned}$$

On the other hand, it is easy to see that $I_2 \leq C\Gamma\|\phi\|_\infty$, and by combining both estimates we conclude.

(b) We must first establish that for all λ, x, p and ϕ as above,

$$\|w_\lambda(\cdot; x, p, \phi) - w_\lambda(0; x, p, \phi)\|_\infty \leq C_1 (1 + |p| + \|\phi\|_{C^{\sigma+\iota}}). \quad (3.2)$$

Assume, on the contrary, that there exist sequences $(\lambda_k)_k$ and $((x_k, p_k, \phi_k))_k$ such that $\lambda_k \rightarrow 0$ and $w_k = w_{\lambda_k}(\cdot; x_k, p_k, \phi_k)$ satisfies

$$\|w_k - w_k(0)\|_\infty \geq k (1 + |p_k| + \|\phi_k\|_{C^{\sigma+\iota}}).$$

For $\eta_k = \|w_k - w_k(0)\|_\infty^{-1}$, we define $\tilde{w}_k = \eta_k (w_k - w_k(0))$ and note that \tilde{w}_k satisfies $\tilde{w}_k(0) = 0$, $\|\tilde{w}_k\|_\infty = 1$ and

$$\lambda_k \tilde{w}_k + \lambda_k \eta_k w_k(0) + \sup_\theta \{-L_y^\theta \tilde{w}_k(y) - \eta_k \tilde{l}_k^\theta(y)\} = 0, \quad (3.3)$$

where $\tilde{l}_k^\theta(y) = L_y^\theta \phi_k(x_k) + f^\theta(x_k, y) \cdot p_k + l^\theta(x_k, y)$.

By part (a) – again using (1.9) – we have that

$$\eta_k \lambda_k |\tilde{w}_k(0)| + \eta_k |\tilde{l}_k^\theta|_\infty \leq \frac{C}{k}.$$

for some $C > 0$. We may thus apply the regularity results of [9] to (3.3) to find that (\tilde{w}_k) is bounded in $C^{\bar{\nu}}$ for some $\bar{\nu} > 0$, uniformly with respect to k , and thus converges up to a subsequence to some $\tilde{w} \in C^{\bar{\nu}}(\mathbb{T}^N)$. Passing to the limit in (3.3) in the viscosity sense, we find that \tilde{w} is a solution of

$$\sup_{\theta} \{-L_y^\theta \tilde{w}(y)\} = 0 \quad \text{in } \mathbb{T}^N.$$

Since \tilde{w} is periodic, it achieves its maximum at some point; hence, it is constant by the strong maximum principle. This contradicts the fact that, as \tilde{w} is a limit of \tilde{w}_k , we have $\tilde{w}(0) = 0$ and $\|\tilde{w}\|_\infty = 1$, thus establishing (3.2).

We now note that $v_\lambda = w_\lambda - w_\lambda(0)$ satisfies

$$\lambda v_\lambda + \lambda w_\lambda(0) + \sup_{\theta} \{-L_y^\theta v_\lambda(y) - \tilde{l}^\theta(y)\} = 0 \quad (3.4)$$

where $\tilde{l}^\theta(y) = L_y^\theta \phi(x) + f^\theta(x, y) \cdot p + l^\theta(x, y)$, and using (3.2) and the results of [9] we conclude that $\|v_\lambda\|_{C^\alpha(\mathbb{T}^N)} \leq C_1(1 + |p| + \|\phi\|_{C^{\sigma+\iota}})$ for some $\alpha > 0$, taking a larger C_1 if necessary.

We wish to conclude by applying the results of [31] to (3.4). To this end, we now show that \tilde{l}^θ as defined above is uniformly bounded in C^α .

Let $y_i \in \mathbb{T}^N$, $i = 1, 2$, $z \in \mathbb{R}^N \setminus \{0\}$. Given the assumptions (1.9), it is immediate that

$$\|f^\theta(x, \cdot) \cdot p + l^\theta(x, \cdot)\|_{C^\alpha(\mathbb{T}^N)} \leq C_1(1 + |p|),$$

in fact for any $\alpha \in (0, 1]$. To bound the nonlocal term appearing in \tilde{l}^θ we compute, using (1.8),

$$\begin{aligned} |L_{y_1}^\theta \phi(x) - L_{y_2}^\theta \phi(x)| &\leq \int_{\mathbb{R}^N} |\phi(x+z) - \phi(x)| |K^\theta(y_1, z) - K^\theta(y_2, z)| dz \\ &= \int_{\mathbb{R}^N} |\phi(x+z) - \phi(x)| |a^\theta(y_1, \hat{z}) - a^\theta(y_2, \hat{z})| |z|^{-(N+\sigma)} dz \\ &\leq C |y_1 - y_2|^\alpha \int_{\mathbb{R}^N} |\phi(x+z) - \phi(x)| |z|^{-(N+\sigma)} dz \\ &\leq C |y_1 - y_2|^\alpha \|\phi\|_{C^{\sigma+\iota}}. \end{aligned}$$

Here we have removed the principal value from each of the integrals defining $L_{y_1}^\theta \phi$ and $L_{y_2}^\theta \phi$ by using that $\phi \in C^{\sigma+\iota}$ and computing as in part (a) of the lemma. The last inequality is similarly obtained.

Again using (1.8), we apply the results of [31] to equation (3.4), and with this we conclude.

(c) Let $w_i = w_i(\cdot; x_i, p_i, \phi_i)$ for $x_i, p_i \in \mathbb{R}^N$, $\phi_i \in C^{\sigma+\iota}(\mathbb{R}^N)$, $i = 1, 2$. Using the assumptions on the structure of H , it is easy to see that $w_\pm := w_1 \pm \lambda^{-1} [(|x_1 - x_2| + |p_2| + \|\phi_1\|_{\sigma+\iota}) + |p_1 - p_2| + \|\phi_1 - \phi_2\|_{C^{\sigma+\iota}}]$ are respectively a super- and a subsolution of (3.1) centered in (x_2, p_2, ϕ_2) . The claim follows by comparison.

(d) Adapting the arguments of [10], [21], we consider fixed (x, p, ϕ) and define

$$\Gamma_\lambda = \lambda \sup_y w_\lambda(y; x, p, \phi).$$

We first claim that

$$\Gamma_\lambda \geq -\bar{H}(x, p, \phi). \quad (3.5)$$

Indeed, by noting that w_λ satisfies

$$\Gamma_\lambda + \sup_\theta \{-L_y^\theta w_\lambda(y) - L_y^\theta \phi(x) - f^\theta(x, y) \cdot p - l^\theta(x, y)\} \geq 0,$$

the claim follows from applying the following characterization of the ergodic constant: using the notation of Section 2,

$$c = \sup\{\tilde{c} \in \mathbb{R} : \exists \psi \text{ s.t. } F(\psi, y) \geq \tilde{c} \text{ in } \mathbb{T}^N\}.$$

This is in a sense dual to (2.14) and can be proved in exactly the same way.

We then note that part (b) of the lemma implies in particular that

$$\lambda |w_\lambda(y_1; x, p, \phi) - w_\lambda(y_2; x, p, \phi)| \leq \lambda C_1 \sqrt{N} (1 + |p| + \|\phi\|_{C^{\sigma+\iota}}),$$

given that \sqrt{N} is the diameter of the unit hypercube in \mathbb{R}^N . Plugging (3.5) into the previous inequality, we find

$$\lambda w_\lambda(y; x, p, \phi) \geq -\bar{H}(x, p, \phi) - \lambda C_1 \sqrt{N} (1 + |p| + \|\phi\|_{C^{\sigma+\iota}}) \quad \text{for all } y \in \mathbb{T}^N.$$

The proof of the corresponding upper bound is similarly obtained. \square

In the proof of Theorem 1.1 we will make use of the following estimate.

Proposition 3.2. *Let $\phi \in C^{\sigma+\alpha}(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$. Then, for each $\alpha' \in (0, \alpha)$, there exists $C_1 > 0$ just depending on α' but not on ϕ , such that*

$$|L_y^\theta \phi(x_1) - L_y^\theta \phi(x_2)| \leq C |x_1 - x_2|^{\alpha'} \|\phi\|_{C^{\sigma+\alpha}} \quad (3.6)$$

for all $x_1, x_2, z \in \mathbb{R}^N$, $y \in \mathbb{T}^N$.

Proof. We split each integral term as

$$L_y^\theta \phi(x) = L_y^\theta [B_1] \phi(x) + L_y^\theta [B_1^c] \phi(x).$$

By the regularity of ϕ we have that

$$|L_y^\theta [B_1^c] \phi(x_1) - L_y^\theta [B_1^c] \phi(x_2)| \leq 2\Gamma \|D\phi\|_\infty |x_1 - x_2| \int_{B_1^c} |z|^{-N-\sigma} dz \leq C \|\phi\|_{C^1} |x_1 - x_2|.$$

Now we deal with the integral on the ball B . Denoting

$$\delta_{x_1, x_2}(z) = \phi(x_1 + z) - \phi(x_1) - D\phi(x_1) \cdot z - (\phi(x_2 + z) - \phi(x_2) - D\phi(x_2) \cdot z),$$

we have

$$|L_y^\theta [B_1] \phi(x_1) - L_y^\theta [B_1] \phi(x_2)| \leq \Gamma \int_{B_1} |\delta_{x_1, x_2}(z)| |z|^{-(N+\sigma)} dz.$$

At one hand, we can choose $\beta > 0$ such that $\sigma < 1 + \beta < \min\{2, \sigma + \alpha\}$ and from here we get

$$\begin{aligned} |\delta_{x_1, x_2}(z)| &\leq |\phi(x_1 + z) - \phi(x_1) - D\phi(x_1) \cdot z| + |\phi(x_2 + z) - \phi(x_2) - D\phi(x_2) \cdot z| \\ &\leq 2[\phi]_{C^{1+\beta}} |z|^{1+\beta}. \end{aligned}$$

On the other hand, we have

$$\delta_{x_1, x_2}(z) = \int_0^1 (D\phi(x_1 + tz) - D\phi(x_2 + tz)) \cdot z dt - (D\phi(x_1) - D\phi(x_2)) \cdot z,$$

from which we get

$$|\delta_{x_1, x_2}(z)| \leq 2[\phi]_{C^{1+\beta}} |x_1 - x_2|^\beta |z|.$$

Hence, denoting $\delta_{x_1, x_2}(z)$ as δ for simplicity, we use the above estimates to write

$$\delta = \delta^s \delta^{1-s} \leq 2[\phi]_{C^{1+\beta}} |x_1 - x_2|^{s\beta} |z|^{s+(1+\beta)(1-s)},$$

for each $s \in (0, 1)$.

Since $1 + \beta > \sigma$, we can choose s small enough such that $s + (1 + \beta)(1 - s) > \sigma$. We fix s in that way, which allows us to obtain

$$\begin{aligned} &|L_y^\theta[B_1]\phi(x_1) - L_y^\theta[B_1]\phi(x_2)| \\ &\leq 2\Gamma[\phi]_{C^{1+\beta}} |x_1 - x_2|^{s\beta} \int_{B_1} |z|^{s+(1+\beta)(1-s)} |z|^{-(N+\sigma)} dz \\ &\leq C\Gamma[\phi]_{C^{1+\beta}} |x_1 - x_2|^{s\beta} \end{aligned}$$

where C depends on the data and on the difference $s + (1 + \beta)(1 - s) - \sigma > 0$. Joining the estimates on B and B_1^c we conclude the result. \square

4. RATE OF CONVERGENCE: PROOF OF THEOREM 1.1

This section is entirely devoted to the

Proof of Theorem 1.1: The proof follows that of Theorem 2.1 in [10]. Crucially, we will employ the following bound, consequence of Corollary 2.9:

$$\|u\|_{C^{\sigma+\iota}(\mathbb{R}^N)} \leq M, \tag{4.1}$$

where $M > 0$ and $\iota \in (0, \alpha)$ depends on the data. From now on, we consider $\alpha' \in (0, \iota)$, and assume without loss of generality that C_1 in Lemma 3.1 and Proposition 3.2 are the same.

For $\beta \in (0, 1)$, let ψ, ψ_β as in the proof of Lemma 2.4, with $R = \sup_\epsilon \|u^\epsilon\|_\infty + \|u\|_\infty + 1$, which is a constant just depending on the data.

We immediately note that there exists $M_1 > 0$ such that, for each $\theta \in \Theta$ and $x, \xi \in \mathbb{T}^N$ we have

$$|D\psi(x)|, |L_\xi^\theta \psi(x)| \leq M_1; \quad |D\psi_\beta(x)| \leq M_1\beta, \quad |L_\xi^\theta \psi_\beta(x)| \leq M_1\beta^\sigma. \tag{4.2}$$

For each $x, z \in \mathbb{R}^N$ we write $w_\lambda(z; [u](x)) = w_\lambda(z; x, Du(x), u)$ for short. In view of (4.1) and Lemma 3.1, we also have the existence of $C > 0$ such that

$$\sup_{x, z} |w_\lambda(z, [u](x))| \leq C\lambda^{-1}M.$$

We would like to estimate $u^\epsilon - u$. To this end, we define

$$\varphi(x) = u^\epsilon(x) - u(x) - \epsilon^\sigma w_\lambda(x/\epsilon; [u](x)) - \psi_\beta(x),$$

where we have assumed that λ depends on ϵ in such a way that $\epsilon^\sigma \lambda^{-1} \rightarrow 0$ as $\epsilon \rightarrow 0$ (and which is going to be made explicit later). Then, for sufficiently small ϵ and β , φ attains a maximum at some point $\hat{x} \in \mathbb{R}^N$.

For $c > 0$ to be fixed later, define

$$\tilde{\varphi}(x) = u^\epsilon(x) - u(x) - \epsilon^\sigma w_\lambda(x/\epsilon; [u](\hat{x})) - \psi_\beta(x) - c\psi(x - \hat{x}).$$

Given $\tau \in (0, 1)$, we claim that c can be chosen so that $\tilde{\varphi}$ attains a global maximum $\tilde{x} \in B_\tau(\hat{x})$. Indeed, set

$$c = \frac{\epsilon^\sigma}{\lambda\tau^{2-\alpha'}} 2C(1 + 2M)$$

for some $C > C_1$.

Let $x \in \mathbb{R}^N \setminus B_\tau(\hat{x})$. By construction we have $\tilde{\varphi}(\hat{x}) = \varphi(\hat{x}) \geq \varphi(x)$, hence

$$\tilde{\varphi}(\hat{x}) - \tilde{\varphi}(x) \geq \varphi(x) - \tilde{\varphi}(x) = -\epsilon^\sigma [w_\lambda(x/\epsilon; [u](x)) - w_\lambda(x/\epsilon; [u](\hat{x}))] + c\psi(x - \hat{x}).$$

If in addition $\tau \leq |x - \hat{x}| < 1$, we have

$$c\psi(x - \hat{x}) = c|x - \hat{x}|^2 = \frac{\epsilon^\sigma}{\lambda\tau^{2-\alpha'}} 2C(1 + 2M)|x - \hat{x}|^2 > \frac{\epsilon^\sigma}{\lambda} 2C_1(1 + 2M)|x - \hat{x}|^{\alpha'},$$

while Lemma 3.1(c) and (3.6) gives that

$$[w_\lambda(x/\epsilon; [u](x)) - w_\lambda(x/\epsilon; [u](\hat{x}))] \leq \lambda^{-1}C_1(1 + 2M)|x - \hat{x}|^{\alpha'}.$$

Combining these inequalities we obtain that $\tilde{\varphi}(\hat{x}) - \tilde{\varphi}(x) > 0$.

On the other hand, if $|x - \hat{x}| \geq 1$, then $c\psi(x - \hat{x}) \geq c$, and from Lemma 3.1 (a), again using $\tau < 1$, we have

$$\epsilon^\sigma [w_\lambda(x/\epsilon; [u](x)) - w_\lambda(x/\epsilon; [u](\hat{x}))] \leq \epsilon^\sigma 2\lambda^{-1}C_1(1 + 2M) < c.$$

This again implies that $\tilde{\varphi}(\hat{x}) - \tilde{\varphi}(x) > 0$. Thus $\tilde{\varphi}$ attains a global maximum in $B_\tau(\hat{x})$, which we denote by \tilde{x} .

In particular, we have the following behavior at \tilde{x} :

$$Du^\epsilon(\tilde{x}) = Du(\tilde{x}) + \epsilon^{\sigma-1} D_y w_\lambda(\tilde{x}/\epsilon; [u](\hat{x})) + D\psi_\beta(\tilde{x}) + 2c(\tilde{x} - \hat{x}), \quad (4.3)$$

since $|\tilde{x} - \hat{x}| < \tau < 1$ implies that $\psi(x - \hat{x}) = c|x - \hat{x}|^2$ in a neighborhood of \tilde{x} .

Moreover, we have

$$\begin{aligned} & L_{\tilde{x}/\epsilon}^\theta u^\epsilon(\tilde{x}) \\ & \leq L_{\tilde{x}/\epsilon}^\theta u(\tilde{x}) + \epsilon^\sigma L_{\tilde{x}/\epsilon}^\theta w_\lambda(\cdot/\epsilon; [u](\hat{x}))(\tilde{x}) + L_{\tilde{x}/\epsilon}^\theta \psi_\beta(\tilde{x}) + cL_{\tilde{x}/\epsilon}^\theta \psi(\cdot - \hat{x})(\tilde{x}) \end{aligned}$$

$$= L_{\tilde{x}/\epsilon}^\theta u(\tilde{x}) + L_{\tilde{x}/\epsilon}^\theta w_\lambda(\cdot; [u](\hat{x}))(\tilde{x}/\epsilon) + L_{\tilde{x}/\epsilon}^\theta \psi_\beta(\tilde{x}) + cL_{\tilde{x}/\epsilon}^\theta \psi(\tilde{x} - \hat{x}), \quad (4.4)$$

using the homogeneity and translation invariance of L_x^θ .

For the following computation we write $w_\lambda = w_\lambda(\cdot; [u](\hat{x}))$ to ease notation. Evaluating (1.1) at \tilde{x} , we have

$$u^\epsilon(\tilde{x}) + \sup_\theta \left\{ -L_{\tilde{x}/\epsilon}^\theta u^\epsilon(\tilde{x}) - f^\theta(\tilde{x}, \tilde{x}/\epsilon) \cdot Du^\epsilon(\tilde{x}) - l^\theta(\tilde{x}, \tilde{x}/\epsilon) \right\} = 0, \quad (4.5)$$

and continue estimating:

$$\begin{aligned} & \sup_\theta \left\{ -L_{\tilde{x}/\epsilon}^\theta u^\epsilon(\tilde{x}) - f^\theta(\tilde{x}, \tilde{x}/\epsilon) \cdot Du^\epsilon(\tilde{x}) - l^\theta(\tilde{x}, \tilde{x}/\epsilon) \right\} \\ & \geq \sup_\theta \left\{ -L_{\tilde{x}/\epsilon}^\theta u(\tilde{x}) - L_{\tilde{x}/\epsilon}^\theta w_\lambda(\tilde{x}/\epsilon) - L_{\tilde{x}/\epsilon}^\theta \psi_\beta(\tilde{x}) - cL_{\tilde{x}/\epsilon}^\theta \psi(\tilde{x} - \hat{x}) \right. \\ & \quad \left. - f^\theta(\tilde{x}, \tilde{x}/\epsilon) \cdot (Du(\tilde{x}) + \epsilon^{\sigma-1} D_y w_\lambda(\tilde{x}/\epsilon) + D\psi_\beta(\tilde{x}) + 2c(\tilde{x} - \hat{x})) - l^\theta(\tilde{x}, \tilde{x}/\epsilon) \right\} \\ & \geq \sup_\theta \left\{ -L_{\tilde{x}/\epsilon}^\theta u(\tilde{x}) - L_{\tilde{x}/\epsilon}^\theta w_\lambda(\tilde{x}/\epsilon) - f^\theta(\tilde{x}, \tilde{x}/\epsilon) \cdot Du(\tilde{x}) - l^\theta(\tilde{x}, \tilde{x}/\epsilon) \right\} \\ & \quad - M_2(\epsilon^{\sigma-1} + \beta + c) \\ & \geq \sup_\theta \left\{ -L_{\tilde{x}/\epsilon}^\theta u(\hat{x}) - L_{\tilde{x}/\epsilon}^\theta w_\lambda(\tilde{x}/\epsilon) - f^\theta(\hat{x}, \tilde{x}/\epsilon) \cdot Du(\hat{x}) - l^\theta(\hat{x}, \tilde{x}/\epsilon) \right\} \\ & \quad - M_2(\tau^{\alpha'} + \tau) - M_2(\epsilon^{\sigma-1} + \beta + c). \end{aligned}$$

Here we have used (4.3) and (4.4) for the first inequality. For the second inequality, we have used (4.2) and the estimate (b) in Lemma 3.1. The constant M_2 just depends on M_1 and C_1 . For the third inequality, we have used estimate (3.6) and the Lipschitz continuity of the data, and we have relabeled the constant M_2 , which depends on the data but not on $\epsilon, \lambda, \tau, \beta$ and c .

We now use that $w_\lambda(\cdot; [u](\hat{x}))$ and u are solutions of (3.1) and (1.6), respectively, together with Lemma 3.1 (d) to obtain

$$\begin{aligned} & \sup_\theta \left\{ -L_{\tilde{x}/\epsilon}^\theta u(\hat{x}) - L_{\tilde{x}/\epsilon}^\theta w_\lambda(\tilde{x}/\epsilon) - f^\theta(\hat{x}, \tilde{x}/\epsilon) \cdot Du(\hat{x}) - l^\theta(\hat{x}, \tilde{x}/\epsilon) \right\} \\ & = -\lambda w_\lambda(\tilde{x}/\epsilon; [u](\hat{x})) \geq \bar{H}(\hat{x}, Du(\hat{x}), u) - \lambda C_1(1 + 2M) \\ & \geq -u(\hat{x}) - \lambda C_1(1 + 2M). \end{aligned}$$

Combining this with the previous computation and replacing into (4.5), we arrive at

$$u^\epsilon(\tilde{x}) - u(\hat{x}) \leq M_2(\lambda + \epsilon^{\sigma-1} + \beta + c + \tau^{\alpha'}),$$

for some $M_2 > 0$ just depending on the data.

By construction, for all $x \in \mathbb{R}^N$ we have $\varphi(x) \leq \varphi(\hat{x}) = \tilde{\varphi}(\hat{x}) \leq \tilde{\varphi}(\tilde{x})$, hence

$$\begin{aligned} u^\epsilon(x) - u(x) & \leq (u^\epsilon(\tilde{x}) - u(\hat{x})) + (u(\hat{x}) - u(\tilde{x})) \\ & \quad + \epsilon^\sigma [w_\lambda(x/\epsilon; [u](x)) - w_\lambda(\tilde{x}/\epsilon; [u](\hat{x}))] + [\psi_\beta(x) - \psi_\beta(\tilde{x})] \\ & \leq M_2(\lambda + \epsilon^{\sigma-1} + \beta + c + \tau^{\alpha'}) + M\tau + \frac{\epsilon^\sigma}{\lambda} 2C_1(1 + 2M) + M_1\beta, \end{aligned}$$

by Lemma 3.1 (a), (4.2) and (4.1).

Sending $\beta \rightarrow 0$ and recalling the definition of c , this gives

$$u^\epsilon(x) - u(x) \leq M_2 \left(\lambda + \epsilon^{\sigma-1} + \frac{\epsilon^\sigma}{\lambda\tau^{2-\alpha'}} + \tau^{\alpha'} \right),$$

again by taking a larger value of M_2 if necessary. Thus, we set

$$\lambda = \epsilon^{\frac{\sigma\alpha'}{2+\alpha'}}, \quad \tau = \epsilon^{\frac{\sigma}{2+\alpha'}},$$

and obtain

$$u^\epsilon(x) - u(x) \leq 4M_2\epsilon^{\frac{\sigma\alpha'}{2+\alpha'}}.$$

□

We proceed with the proof of the more explicit rates of convergence, in which some of the dependencies of H are dropped. We note that these results are independent of Theorems 2.8 and 1.1, see [10, 11].

Proof of Theorem 1.2. (i) From the assumptions, (1.1) may be written as

$$u^\epsilon + H\left(x, \frac{x}{\epsilon}, u^\epsilon\right) = 0 \quad \text{in } \mathbb{R}^N, \quad (4.6)$$

where

$$H(x, y, \varphi) = \sup_{\theta \in \Theta} \{-L_y^\theta \varphi(x) - l^\theta(y)\}.$$

For a given $x \in \mathbb{R}^N$, $u \in C_b^2(\mathbb{R}^N)$, the associated cell problem

$$\sup_{\theta \in \Theta} \{-L_y^\theta w(y) - L_y^\theta u(x) - l^\theta(y)\} = \bar{H}(x, u) \quad \text{in } \mathbb{T}^N, \quad (4.7)$$

has a unique (up to additive constants) \mathbb{T}^N -periodic solution $w = w(y)$, by the results of Lemma 3.1. As the right-hand side of (4.7) only depends on x through evaluating u in the nonlinear term $L_y^\theta u(x)$, a solution is given by $u, \bar{H}(\cdot, 0) \equiv \text{const.}$; *i.e.*, the effective problem

$$u + \bar{H}(x, u) = 0 \quad \text{in } \mathbb{R}^N$$

has the constant solution $u \equiv -\bar{H}(0, 0)$.

Thus $v^\pm(x) = -\bar{H}(0, 0) + \epsilon^\sigma w(x/\epsilon) \pm \epsilon^\sigma \|w\|_\infty$ are respectively a sub- and supersolution of (4.6), and the desired bound follows by comparison.

(ii) Arguing as in the first part of the proof we have that u^ϵ and $u \equiv -\bar{H}(0, 0, 0)$ are respectively solutions of

$$u^\epsilon + H(x, x/\epsilon, Du^\epsilon, u^\epsilon) = 0, \quad \text{in } \mathbb{R}^N, \quad (4.8)$$

where $H(x, y, p, \varphi) = \sup_{\theta \in \Theta} \{-L_y^\theta \varphi(x) - f^\theta(y) \cdot p - l^\theta(y)\}$, and

$$u + H(x, Du, u) = 0 \quad \text{in } \mathbb{R}^N.$$

Define $v^\pm(x) = -\bar{H}(0, 0, 0) + \epsilon^\sigma w(x/\epsilon) \pm C\epsilon^{\sigma-1}\|w\|_{C^1}$, where C is given by (1.9); these are respectively a sub- and a supersolution of (4.8). By using the continuity assumption (1.9) and comparison once more, we conclude. \square

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