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UNIFORMLY EXPONENTIAL STABILITY OF SEMI-DISCRETE SCHEME FOR A VIBRATION CABLE WITH A TIP MASS UNDER OBSERVER-BASED FEEDBACK CONTROL

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Abstract. In this paper, we consider uniformly exponential approximation for a vibrating cable with tip mass under a non-collocated output stabilizing feedback control. By designing an observer-based output feedback control, the closed-loop system is composed of the coupled same type of PDEs and ODEs. By order reduction method, we find a global Lyapunov functional for the closed-loop system. The closed-loop system is then semi-discretized by the finite difference method. For the discrete systems, we also construct the Lyapunov functions. The uniform exponential stability of the semi-discretized systems is then established analogously as the proof for the continuous counterpart via an indirect Lyapunov functional approach.

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1. Introduction

Flexible structures are widely used in many areas, such as aerospace, civil and mechanical engineering, and vibrating cable is one of the most important flexible structures in applications. Mathematically, a vibrating cable with a tip mass can be described by a partial differential equation (PDE) which represents the dynamics of the cable and an ordinary differential equation (ODE) which represents the dynamics of the tip mass. The vibration control is the most concerned problem from standpoint of the distributed parameter systems. In the past several decades, boundary pointwise control and observation is the major strategy for the vibration control of flexible structures. Examples can be found in [4, 6, 8, 18–20] and the references therein, among many others. Usually, for the collocated cases where the actuator and sensor are located at the same boundary, one can use most often the direct output feedback control based on passivity principle. However, for the non-collocated cases where the control and sensor are located in different boundaries, the control design and analysis is much challenging. As early as 1980s, the non-collocated control was used in robot control [3], see also [17] and [9]. For a non-collocated control system, the most often used control design is the observer-based feedback control which makes the feedback control intrinsically infinite-dimensional. A single PDE system under an observer-based feedback control becomes a coupled PDE system. In order to apply such designed feedback control, the

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discretization is thereby necessary. For a PDE system, one would rather discretize the system into an ODE system for which most of engineers are familiar with by keeping the time continuous and the spatial variable to be discretized. This process is called the semi-discretization scheme. However, there is a major concern from PDE to its semi-discrete counterpart. If the discretized ODE system preserves as much as possible the important control properties like uniform controllability, observability, and exponential stability, one can use the finite-dimensional semi-discretized system as a physical model itself instead of the original infinite-dimensional PDE model. In the past a few decades since from 1990s, there are many studies on semi-discretization of PDEs, for which a basic problem is the preservation of the uniform exponential stability. This is unfortunately not an easy question to be answered.

In paper [1], it was first pointed out that an exponentially stable PDE system might not be uniformly exponentially stable with respect to the mesh size by classical finite difference or finite element schemes although the semi-discrete systems are still exponentially stable for every step size. The non-uniformity problem was later studied in [23]. Gradually, researchers realize that the non-uniformity is caused by spurious high frequencies during the semi-discretization process. Many methods have thereafter been proposed to circumvent this problem, such as the Tychonoff regularization method [11], two-grid algorithms [12, 20], filtering techniques [2, 23]. Among them the numerical viscosity method proposed in [22] and mixed finite element method proposed in [5, 10] are most successful methods. However, the mixed finite element method is much complicated then the finite-difference method and the numerical viscosity method adds an artificial term in the classical semi-discrete scheme where the coefficients of the numerical viscosity term vary from PDE to PDE and the stability analysis relies on the special boundary conditions of the PDEs. A natural semi-discrete scheme was proposed in paper [14] by an order reduction finite-difference method, which demonstrates very different nature compared with other approaches. First, the scheme preserve uniformly the exponential stability and observability for many types of PDEs. Second, its mathematical proofs for uniform exponential stability follows the exponential stability of the PDE counterpart. These advantages have been explained in papers [7, 15, 16].

The papers aforementioned by order reduction finite-difference method, however, dealt with only single PDEs. In a recent paper [21], the method was used to deal with a wave PDE under an observer-based feedback control, which is a coupled PDE. Very surprisedly, a global Lyapunov functional was constructed in [21]. This motivates us to discuss the uniform exponential stability of the semi-discrete scheme for a vibrating cable with tip mass, for which a non-collocated control and observation was discussed in a recent paper [19]. The idea is still constructing a global Lyapunov functional by which we not only give the uniform exponential stability of the semi-discrete scheme but also simplifies significantly the exponential stability of the original PDE system discussed in [19]. A big difference with [21] is that the Lyapunov functional is not directly applied instead it is used by combining the C_0-semigroup property.

The system that we consider in this paper is described by the following wave equation with a tip mass:

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), x \in (0,1), t > 0, \\ w(0,t) = 0, t \geq 0, \\ w_x(1,t) + m w_{tt}(1,t) = U(t), t \geq 0, \\ y(t) = w_x(0,t), t \geq 0, \end{cases} \tag{1.1}$$

where $w(x,t)$ is the displacement at position x and time t, $m > 0$ is the tip mass at the actuator end, $U(t)$ is the boundary control (input), $y(t)$ is the observation (output). System (1.1) can be used to describe the vibration of a flexible cable with tip mass first discussed in [18]. We consider system (1.1) in the energy state space $\mathcal{H}_1 = H^1_E(0,1) \times L^2(0,1) \times \mathbb{C}, H^1_E(0,1) = \{f | f \in H^1(0,1), f(0) = 0\}$, with the system energy

$$E_0(t) = \frac{1}{2}\int_0^1 [w_t^2(x,t) + w_x^2(x,t)]\mathrm{d}x + \frac{1}{2}\frac{\eta^2(t)}{m+\alpha a},\ \eta(t) = a w_x(1,t) + m w_t(1,t). \tag{1.2}$$

It was proved in [18] that system (1.1) under the collocated feedback control:

$$U(t) = -\alpha w_t(1,t) - a w_{xt}(1,t), \alpha, a > 0, \tag{1.3}$$

is exponentially stable in $\mathcal{H}_1$, that is,

$$E_0(t) \leq M_0 e^{-\omega_0 t} E_0(0), \quad \forall t \geq 0, \tag{1.4}$$

for some constants $M_0, \omega_0 > 0$ independent of t. Here it is noted that both signals $w_t(1,\cdot)$ and $w_{xt}(1,\cdot)$ belong to $L^2_{loc}(0,\infty)$. For instance, the transfer function from $U \to w_{xt}(1,\cdot)$ is computed as

$$\frac{s^2(e^{2s}+1)}{s(e^{2s}+1) + ms2(e^{2s}-1)}$$

which is bounded on open left complex plane, which means that $w_{xt}(1,\cdot) \in L^2_{loc}(0,\infty)$, and so is for $w_t(1,\cdot)$. In other words, the feedback control (1.3) makes sense that $U \in L^2_{loc}(0,\infty)$. The collocated control (1.3), however, has some disadvantages in measurement because when the cable is moving, measurement of the high order signal (angular velocity) $w_{xt}(1,t)$ in control (1.3) is not always feasible. The non-collocated measurement (vertical force) in system (1.1) is much easy. The control problem (1.1) was first proposed in [19] in which the following observer-based feedback control

$$U(t) = -\alpha \widehat{w}_t(1,t) - a\widehat{w}_{xt}(1,t), \tag{1.5}$$

was proposed with the following observer:

$$\begin{cases} \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t), x \in (0,1), t > 0, \\ \widehat{w}_x(0,t) = r\widehat{w}_t(0,t) + \beta\widehat{w}(0,t) + w_x(0,t), t \geq 0, \\ \widehat{w}_x(1,t) + m\widehat{w}_{tt}(1,t) = U(t), t \geq 0, \end{cases} \tag{1.6}$$

where the tuning parameters β and r are two strictly positive numbers. The closed-loop of the system (1.1) under output feedback (1.5) is then described by

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t), x \in (0,1), t > 0, \\ w(0,t) = 0, t \geq 0, \\ w_x(1,t) + mw_{tt}(1,t) = -\alpha\widehat{w}_t(1,t) - a\widehat{w}_{xt}(1,t), t \geq 0, \\ \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t), x \in (0,1), t > 0, \\ \widehat{w}_x(0,t) = r\widehat{w}_t(0,t) + \beta\widehat{w}(0,t) + w_x(0,t), t \geq 0, \\ \widehat{w}_x(1,t) + m\widehat{w}_{tt}(1,t) = -\alpha\widehat{w}_t(1,t) - a\widehat{w}_{xt}(1,t), t \geq 0. \end{cases} \tag{1.7}$$

Denote the observer error as

$$e(x,t) = \hat{w}(x,t) - w(x,t). \tag{1.8}$$

Then, the system (1.7) is equivalent to

$$\begin{cases} \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t),\ x \in (0,1),\ t > 0, \\ \widehat{w}(0,t) = e(0,t),\ t \geq 0, \\ \widehat{w}_x(1,t) + m\widehat{w}_{tt}(1,t) = -\alpha\widehat{w}_t(1,t) - a\widehat{w}_{xt}(1,t),\ t \geq 0, \\ e_{tt}(x,t) = e_{xx}(x,t),\ x \in (0,1),\ t > 0, \\ e_x(0,t) = re_t(0,t) + \beta e(0,t),\ t \geq 0, \\ e_x(1,t) + me_{tt}(1,t) = 0,\ t \geq 0. \end{cases} \tag{1.9}$$

Remark 1.1. In paper [19], the following system which is equivalent to (1.9)

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t),\ x \in (0,1),\ t > 0, \\ w(0,t) = 0,\ t \geq 0, \\ w_x(1,t) + mw_{tt}(1,t) = -\alpha w_t(1,t) - aw_{xt}(1,t) - \alpha e_t(1,t) - ae_{xt}(1,t),\ t \geq 0, \\ e_{tt}(x,t) = e_{xx}(x,t),\ x \in (0,1), t > 0, \\ e_x(0,t) = re_t(0,t) + \beta e(0,t),\ t \geq 0, \\ e_x(1,t) + me_{tt}(1,t) = 0,\ t \geq 0, \end{cases} \tag{1.10}$$

was shown to be exponentially stable by means of the Riesz basis approach and the author declaimed that finding a Lyapunov functional for the system (1.10) is difficult because at first glance, the system (1.10) is even not dissipative. In this paper, we turn to study the equivalent form system (1.9) for which we do find a Lyapunov functional. This not only simplifies significantly the exponential stability of system (1.10) or (1.9) but also makes the uniformly convergence analysis of the semi-discrete scheme for (1.9) possible.

We point out that the system (1.9) is a coupled PDE+ODE system. Actually, let

$$\eta(t) = m\widehat{w}_t(1,t) + a\widehat{w}_x(1,t),\ \eta_e(t) = e_t(1,t). \tag{1.11}$$

Then, (1.9) can be written into

$$\begin{cases} \widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t),\ x \in (0,1),\ t > 0, \\ \widehat{w}(0,t) = e(0,t),\ t \geq 0, \\ \widehat{w}_x(1,t) = -\frac{m}{a}\widehat{w}_t(1,t) + \frac{1}{a}\eta(t),\ t \geq 0, \\ \dot{\eta}(t) = -\dfrac{1}{a}\eta(t) - \dfrac{a\alpha - m}{a}\widehat{w}_t(1,t),\ t \geq 0, \\ e_{tt}(x,t) = e_{xx}(x,t),\ x \in (0,1),\ t > 0, \\ e_x(0,t) = re_t(0,t) + \beta e(0,t),\ t \geq 0, \\ e_t(1,t) = \eta_e(t), t \geq 0, \\ \dot{\eta}_e(t) = -\dfrac{1}{m}e_x(1,t),\ t \geq 0. \end{cases} \tag{1.12}$$

We consider the system (1.12) in the state space:

$$\mathcal{H} = \{(f_1, g_1, \eta_1, f_2, g_2, \eta_2) \in \mathcal{H}_2^2, f_1(0) = f_2(0)\}. \tag{1.13}$$

System (1.12) can be written as an evolution equation in $\mathcal{H}$:

$$\frac{d}{dt}(\widehat{w}(\cdot,t),\widehat{w}_t(\cdot,t),\eta(t),e(\cdot,t),e_t(\cdot,t),\eta_e(t)) = A(\widehat{w}(\cdot,t),\widehat{w}_t(\cdot,t),\eta(t),e(\cdot,t),e_t(\cdot,t),\eta_e(t)), \tag{1.14}$$

where the operator A is defined by

$$\begin{cases} A(f_1,g_2,\eta_1,f_2,g_2,\eta_2) = \left(g_1, f_2'', -\frac{1}{a}\eta_1 - \frac{a\alpha - m}{a}g_1(1), g_2, f_2'', -\frac{1}{m}f_2'(1)\right), \\ D(A) = \Big\{(f_1,g_2,\eta_1,f_2,g_2,\eta_2), A(f_1,g_2,\eta_1,f_2,g_2,\eta_2) \in \mathcal{H} | \\ \qquad f_1'(1) = -\frac{m}{a}g_1(1) + \frac{1}{a}\eta_1, f_1'(0) = rg_2(0) + \beta f_2(0), g_2(1) = \eta_2\Big\}. \end{cases} \tag{1.15}$$

For notational simplicity, we omit the obvious spatial and time domains in all equations hereafter unless it is necessary. It is seen from (1.12) that the e-subsystem is independent of $(\widehat{w},\eta)$-subsystem. We consider the e-subsystem in the energy state space $\mathcal{H}_2 = H^1(0,1) \times L^2(0,1) \times \mathbb{C}$ with the system energy

$$F_e(t) = \frac{1}{2}\int_0^1 [e_t^2(x,t) + e_x^2(x,t)]\mathrm{d}x + \frac{\beta}{2}e^2(0,t) + \frac{m}{2}\eta_e^2(t). \tag{1.16}$$

It is well known that the e-subsystem associates with a C_0-semigroup solution in $\mathcal{H}_2$ and is exponentially stable:

$$F_e(t) \le M_e \mathrm{e}^{-\omega_e t}F_e(0), \forall t \ge 0,$$

for some constants $M_e, \omega_e > 0$ independent of t. The total energy $F(t)$ of the system (1.9) is given by

$$F(t) = F_{\widehat{w}}(t) + F_e(t), \tag{1.17}$$

where $F_e(t)$ is defined by (1.16) and

$$F_{\widehat{w}}(t) = \frac{1}{2}\int_0^1 [\widehat{w}_t^2(x,t) + \widehat{w}_x^2(x,t)]\mathrm{d}x + \frac{1}{2}\frac{\eta^2(t)}{m+\alpha a}. \tag{1.18}$$

Different to paper [19], we show by an (indirect) Lyapunov method that the system (1.12) is also exponentially stable, that is, there are constants M_1 and ω_1 independent of the initial value such that

$$F(t) \le M_1 \mathrm{e}^{-\omega_1 t}F(0),\ \forall t > 0. \tag{1.19}$$

Throughout the paper, we use $\dot{E}(t)$ for the derivative of the energy $E(t)$ with respect to time and $f'(t)$ for derivative of function $f(t)$ with respect to time, which are clear from the context. The rest of this paper is organized as follows. In Section 2, we transform the original coupled system (1.9) into an equivalent system by order reduction method and prove the exponential stability of the order reduced couple system by the Lyapunov function method. In Section 3, we construct a semi-discretized finite difference scheme for the order reduced coupled system. Equivalently, we get a semi-discretized finite difference scheme for the original coupled system (1.9). Section 4 is devoted to the proof of the uniform exponential stability of the semi-discretized system of the order reduced PDE, which leads to the uniform exponential stability of the semi-discretized system of the original system (1.9), followed up by concluding remarks in Section 5.

2. Exponential stability of continuous system

Since (1.9) and (1.12) are equivalent, we would rather use (1.9) in this section for the sake of notational simplicity. First, we introduce the following intermediate variables

$$z(x,t) = \widehat{w}_x(x,t), u(x,t) = e_x(x,t). \tag{2.1}$$

Then, the system (1.9) can be reformulated as

$$\begin{cases} \widehat{w}_{tt}(x,t) = z_x(x,t), \\ z(x,t) = \widehat{w}_x(x,t), \\ \widehat{w}(0,t) = e(0,t), \\ z(1,t) + m\widehat{w}_{tt}(1,t) = -\alpha\widehat{w}_t(1,t) - az_t(1,t), \\ e_{tt}(x,t) = u_x(x,t), \\ u(x,t) = e_x(x,t), \\ u(0,t) = re_t(0,t) + \beta e(0,t), \\ u(1,t) + me_{tt}(1,t) = 0. \end{cases} \tag{2.2}$$

It is noted that in the system (1.9), it involves the second derivative with respect to the spatial variable x, but in (2.2), the derivative with respect to x is the first order. This is what the appellation of order reduction means. System (2.2) is a differential algebraic system which is called singular system in literature. Its well-posedness and regularity are guaranteed by that of (1.12), which have been proved in [19] for its equivalent system (1.10). Having constructed the Lyapunov functional, this problem will become a trivial fact in present paper. The energy $E(t)$ of the system (2.2) now reads

$$E(t) = E_{\widehat{w}}(t) + E_e(t), \tag{2.3}$$

where

$$\begin{cases} E_{\widehat{w}}(t) = \dfrac{1}{2}\displaystyle\int_0^1 [\widehat{w}_t^2(x,t) + z^2(x,t)]\mathrm{d}x + \dfrac{1}{2}\dfrac{\eta^2(t)}{m+\alpha a}, \ \eta(t) = az(1,t) + m\widehat{w}_t(1,t), \\ E_e(t) = \dfrac{1}{2}\displaystyle\int_0^1 [e_t^2(x,t) + u^2(x,t)]\mathrm{d}x + \dfrac{m}{2}e_t^2(1,t) + \dfrac{\beta}{2}e^2(0,t). \end{cases} \tag{2.4}$$

The derivatives of $E_{\widehat{w}}(t)$ and $E_e(t)$ formally satisfy

$$\begin{cases} \dot{E}_{\widehat{w}}(t) = -\frac{1}{m+\alpha a}[az^2(1,t) + \alpha m\widehat{w}_t^2(1,t)] - z(0,t)e_t(0,t), \\ \dot{E}_e(t) = -re_t^2(0,t), \end{cases} \tag{2.5}$$

and hence

$$\dot{E}(t) = -\frac{1}{m+\alpha a}[az^2(1,t) + \alpha m\widehat{w}_t^2(1,t)] - z(0,t)e_t(0,t) - re_t^2(0,t). \tag{2.6}$$

This is the reason that the author of [19] considered the C_0-semigroup generation to be difficult because $\dot{E}(t)$ is not dissipative at a first glance.

In this section, we use an (indirect) Lyapunov function method to prove the exponential stability of the system (2.2), which is a preparation for the uniform exponential stability of its semi discrete counterpart in the next section. To this end, we construct the Lyapunov functional $L(t)$ and in particular $V(t)$:

$$L(t) = E_1(t) + \varepsilon P_1(t),\ V(t) = tL(t) + P(t), \tag{2.7}$$

where the auxiliary functions are given by

$$\begin{cases} E_1(t) = E_{\widehat{w}}(t) + \delta_1 E_e(t), \\ P_1(t) = 2\displaystyle\int_0^1 \widehat{w}_t(x,t)z(x,t)\mathrm{d}x, \\ \psi_1(t) = 2\displaystyle\int_0^1 [x-(m+1)]u(x,t)e_t(x,t)\mathrm{d}x, \\ \psi_2(t) = \displaystyle\int_0^1 e(0,t)e_t(x,t)\mathrm{d}x + me(0,t)e_t(1,t), \\ P(t) = NE_e(t) + \delta_1(\psi_1(t) + N_e\psi_2(t)) + 3\displaystyle\int_0^1 xz(x,t)\widehat{w}_t(x,t)\mathrm{d}x, \end{cases} \tag{2.8}$$

with constants $\varepsilon, \delta_1, N_e, N > 0$ satisfying

$$\begin{cases} 0 < \varepsilon < \min\left\{\dfrac{a}{m+\alpha a}, \dfrac{\alpha m}{m+\alpha a}, \dfrac{1}{2}\right\}, \\ 0 < \delta < 2\varepsilon, \\ \delta_1 > \dfrac{1}{2\delta r}, \\ N_e > \max\left\{(m+1)\beta + \dfrac{1}{2}, 2(m+1)\beta\right\}, \\ N > \delta_1(m+1)\dfrac{2(1+r^2)+N_e^2}{2r}. \end{cases} \tag{2.9}$$

In order to prove the uniform exponential stability of system (2.2), we need the following Lemmas 2.1–2.5. Doing so, we need an equivalent inner product on $\mathcal{H}$, which enables us to establish the C_0-semigroup generation of system (1.12) under this new product. Actually, notice that

$$\begin{cases} L(t) = E_{\widehat{w}}(t) + \delta_1 E_e(t) + 2\varepsilon\displaystyle\int_0^1 \widehat{w}_t(x,t)\widehat{w}_x(x,t)\mathrm{d}x, \\ E_{\widehat{w}}(t) = \dfrac{1}{2}\displaystyle\int_0^1 [\widehat{w}_t^2(x,t) + \widehat{w}_x^2(x,t)(x,t)]\mathrm{d}x + \dfrac{1}{2}\dfrac{1}{m+\alpha a}\eta^2(t), \\ E_e(t) = \dfrac{1}{2}\displaystyle\int_0^1 [e_t^2(x,t) + e_x^2(x,t)]\mathrm{d}x + \dfrac{m}{2}\eta_e^2(t) + \dfrac{\beta}{2}e^2(0,t). \end{cases} \tag{2.10}$$

The new product can be defined according to (2.10) as follows:

$$\begin{aligned}
&\langle (f_1, g_1, \eta_1, f_2, g_2, \eta_2, \hat{f}_1, \hat{g}_1, \hat{\eta}_1, \hat{f}_2, \hat{g}_2, \hat{\eta}_2 \rangle_{\mathcal{H}} \\
&= \int_0^1 [f_1'(x)\overline{\hat{f}_1'(x)} + g_1(x)\overline{\hat{g}_1(x)}]\mathrm{d}x + \frac{1}{m+\alpha a}\eta_1\overline{\eta_1} + 2\varepsilon \int_0^1 f_1'(x)\overline{g_1(x)}\mathrm{d}x \\
&+\delta_1 \left[\int_0^1 [f_2'(x)\overline{\hat{f}_2'(x)} + g_2(x)\overline{\hat{g}_2(x)}]\mathrm{d}x + \beta f_2(0)\overline{\hat{f}_2(0)} + m\eta_2\overline{\hat{\eta}_2} \right],
\end{aligned} \tag{2.11}$$

where the ε and δ_1 are those defined by (2.9). The following Lemma 2.1 is a trivial fact.

Lemma 2.1. *The function $E_1(t)$ defined by (2.8) is equivalent to the energy $E(t)$ of the system (2.2), i.e., there exist two positive constants C_1 and C_2 such that*

$$C_1 E(t) \le E_1(t) \le C_2 E(t), \tag{2.12}$$

where $C_1 = \min\{1, \delta_1\}$, $C_2 = \max\{1, \delta_1\}$ and δ_1 is selected by (2.9).

Lemma 2.2. *The function $L(t)$ defined by (2.7) is equivalent to the energy $E_1(t)$ defined by (2.8), i.e.*

$$(1-2\varepsilon)E_1(t) \le L(t) \le (1+2\varepsilon)E_1(t), \tag{2.13}$$

for any $0 < \varepsilon < \frac{1}{2}$.

Proof. The second function of (2.8) satisfies

$$|P_1(t)| \le 2\int_0^1 |\widehat{w}_t(x,t)z(x,t)|\mathrm{d}x \le \int_0^1 [\widehat{w}_t^2(x,t) + \widehat{w}_x^2(x,t)]\mathrm{d}x \le 2E_1(t),$$

and hence

$$(1-2\varepsilon)E_1(t) \le L(t) \le (1+2\varepsilon)E_1(t),$$

for any $0 < \varepsilon < \frac{1}{2}$. □

Lemma 2.3. *Let the parameters $\varepsilon, \delta, \delta_1$ be those chosen in (2.9). Then, the derivative of $L(t)$ defined by (2.7) satisfies*

$$\dot{L}(t) \le 0. \tag{2.14}$$

Proof. Let $E_1(t)$ be defined by (2.8) and $P_1(t)$ be defined by (2.8). A simple calculation shows that

$$\begin{cases} \dot{E}_1(t) &= -\dfrac{a}{m+\alpha a}z^2(1,t) - \dfrac{\alpha m}{m+\alpha a}\widehat{w}_t^2(1,t) - z(0)e_t(0,t) - \delta_1 r e_t^2(0,t), \\ \dot{P}_1(t) &= [z^2(1,t) + \widehat{w}_t^2(1,t)] - [z^2(0,t) + \widehat{w}_t^2(0,t)]. \end{cases} \tag{2.15}$$

Hence,

$$\begin{aligned}
\dot{L}(t) = &\ \dot{E}_1(t) + \varepsilon \dot{P}_1(t) \\
= &\ -\frac{a}{m+\alpha a}z^2(1,t) - \frac{\alpha m}{m+\alpha a}\widehat{w}_t^2(1,t) - z(0,t)e_t(0,t) - \delta_1 r e_t^2(0,t) + \varepsilon[z^2(1,t) + \widehat{w}_t^2(1,t)]
\end{aligned}$$

$$\begin{aligned}
&-\varepsilon[z^2(0,t)+\widehat{w}_t^2(0,t)]\\
\leq\ & -\frac{a}{m+\alpha a}z^2(1,t)-\frac{\alpha m}{m+\alpha a}\widehat{w}_t^2(1)+\frac{\delta}{2}z^2(0,t)+\frac{1}{2\delta}e_t^2(0,t)-\delta_1 re_t^2(0,t)\\
&+\varepsilon[z^2(1,t)+\widehat{w}_t^2(1,t)]-\varepsilon[z^2(0,t)+\widehat{w}_t^2(0,t)]\\
\leq\ & -\left(\frac{a}{m+\alpha a}-\varepsilon\right)z^2(1,t)-\left(\frac{\alpha m}{m+\alpha a}-\varepsilon\right)\widehat{w}_t^2(1,t)\\
&-\left(\varepsilon-\frac{\delta}{2}\right)z^2(0,t)-\left(\delta_1 r-\frac{1}{2\delta}\right)e_t^2(0,t)\leq 0,
\end{aligned}\tag{2.16}$$

where ε,δ and δ_1 are those chosen in (2.9). □

Lemma 2.3 really means that the system (1.12) is dissipative under the equivalent inner product (2.11). The operator A defined by (1.15) is dissipative in $\mathcal{H}$ under the inner product (2.11). Since the boundedness of A^{-1} is a trivial fact, by the Lumer-Phillips theorem, A generates a C_0-semigroup on $\mathcal{H}$. The well-posedness of (1.12) and hence the differential algebraic system (2.2) is guaranteed. This simplifies significantly the argument by the Riesz basis approach presented in [19].

Lemma 2.4. *The function $V(t)$ is equivalent to the auxiliary function $L(t)$ defined by (2.7), i.e., there exists constant $C>0$ such that*

$$(t-C)L(t)\leq V(t)\leq (t+C)L(t),\forall t\geq 0,\tag{2.17}$$

where

$$C_3=\max\left\{3,\frac{N+2\delta_1(m+1)+\delta_1 N_e\max\left\{1,\frac{m+1}{\beta}\right\}}{\delta_1}\right\},C=\frac{C_3}{1-2\varepsilon}.$$

Proof. According to the auxiliary function $P(t)$ defined by (2.8), we have

$$P(t)=NE_e(t)+\delta_1(\psi_1(t)+N_e\psi_2(t))+3\int_0^1 xz(x,t)\widehat{w}_t(x,t)\mathrm{d}x=I_1(t)+I_2(t),\tag{2.18}$$

where

$$I_1(t)=NE_e(t)+\delta_1(\psi_1(t)+N_e\psi_2(t)),\ I_2(t)=3\int_0^1 xz(x,t)\widehat{w}_t(x,t)\mathrm{d}x.\tag{2.19}$$

The functions $\psi_1(t)$ and $\psi_2(t)$ defined by (2.8) satisfy

$$\begin{aligned}
|\psi_1(t)| &=2\left|\int_0^1[x-(m+1)]u(x,t)e_t(x,t)\mathrm{d}x\right|\\
&\leq (m+1)\left[\int_0^1 u^2(x,t)\mathrm{d}x+\int_0^1 e_t^2(x,t)\mathrm{d}x\right]\\
&\leq 2(m+1)E_e(t),
\end{aligned}\tag{2.20}$$

and

$$\begin{aligned} |\psi_2(t)| = & \left|\int_0^1 e(0,t)e_t(x,t)\mathrm{d}x + me(0,t)e_t(1,t)\right| \\ & \le \frac{1}{2}\int_0^1 e_t^2(x,t)\mathrm{d}x + \frac{1}{2}e^2(0,t) + \frac{m}{2}e_t^2(1,t) + \frac{m}{2}e^2(0,t) \\ & = \frac{1}{2}\int_0^1 e_t^2(x,t)\mathrm{d}x + \frac{m}{2}e_t^2(1,t) + \left(\frac{1}{2}+\frac{m}{2}\right)e^2(0,t) \\ & \le \max\left\{1, \frac{m+1}{\beta}\right\} E_e(t). \end{aligned} \tag{2.21}$$

We then obtain

$$|\delta_1(\psi_1(t) + N_e\psi_2(t))| \le \delta_1\left(2(m+1) + N_e \max\left\{1, \frac{m+1}{\beta}\right\}\right) E_e(t), \tag{2.22}$$

and hence

$$|I_1(t)| \le \left(N + 2\delta_1(m+1) + \delta_1 N_e \max\left\{1, \frac{m+1}{\beta}\right\}\right) E_e(t). \tag{2.23}$$

Similarly, for $I_2(t)$ defined by (2.19), one has

$$|I_2(t)| = \left|3\int_0^1 xz(x,t)\widehat{w}_t(x,t)\mathrm{d}x\right| \le \frac{3}{2}\int_0^1 z^2(x,t) + \widehat{w}_t^2(x,t)\mathrm{d}x \le 3E_{\widehat{w}}(t). \tag{2.24}$$

By (2.23) and (2.24) with (2.18), we can easily obtain that

$$\begin{aligned} |P(t)| & = |I_1(t) + I_2(t)| \le \left(N + 2\delta_1(m+1) + \delta_1 N_e \max\left\{1, \frac{m+1}{\beta}\right\}\right) E_e(t) + 3E_{\widehat{w}}(t) \\ & \le C_3 E_1(t). \end{aligned} \tag{2.25}$$

By Lemma 2.2, we have further that

$$|P(t)| \le \frac{C_3}{1-2\varepsilon} L(t). \tag{2.26}$$

The function $V(t)$ then satisfies

$$(t-C)L(t) \le V(t) \le (t+C)L(t), \tag{2.27}$$

where

$$C_3 = \max\left\{\frac{N + 2\delta_1(m+1) + \delta_1 N_e \max\left\{1, \frac{m+1}{\beta}\right\}}{\delta_1}, 3\right\}, \ C = \frac{C_3}{1-2\varepsilon}.$$

$\square$

Lemma 2.5. *Let $V(t)$ be defined by (2.7). Then, there exists a constant $T>0$ such that*

$$\dot{V}(t) \leq 0,\ \forall t \geq T. \tag{2.28}$$

Proof. Finding the derivative of $\psi_1(t)$ defined by (2.8) with respect to t gives

$$\begin{aligned}
\dot{\psi}_1(t) =& 2\int_0^1 [x-(m+1)]u_t(x,t)e_t(x,t)\mathrm{d}x + 2\int_0^1 [x-(m+1)]u(x,t)e_{tt}(x,t)\mathrm{d}x \\
=& [x-(m+1)]e_t^2(x,t)|_0^1 + [x-(m+1)]u^2(x,t)|_0^1 - \int_0^1 \left[u^2(x,t)+e_t^2(x,t)\right]\mathrm{d}x \\
=& -me_t^2(1,t) + (m+1)e_t^2(0,t) - mu^2(1,t) + (m+1)u^2(0,t) - \int_0^1 \left[u^2(x,t)+e_t^2(x,t)\right]\mathrm{d}x \\
\leq& -\int_0^1 \left[u^2(x,t)+e_t^2(x,t)\right]\mathrm{d}x - me_t^2(1,t) + (m+1)e_t^2(0,t) + (m+1)u^2(0,t) \\
=& -\int_0^1 \left[u^2(x,t)+e_t^2(x,t)\right]\mathrm{d}x - me_t^2(1,t) + (m+1)(1+r^2)e_t^2(0,t) \\
&+ (m+1)\beta^2 e^2(0,t) + 2(m+1)r\beta e(0,t)e_t(0,t).
\end{aligned} \tag{2.29}$$

Finding the derivative of $\psi_2(t)$ defined by (2.8) with respect to t gives

$$\begin{aligned}
N_e\dot{\psi}_2(t) =& N_e e_t(0,t)\int_0^1 e_t(x,t)\mathrm{d}x + N_e e(0,t)\int_0^1 e_{tt}(x,t)\mathrm{d}x + N_e m e_t(0,t)e_t(1,t) + N_e m e(0,t)e_{tt}(1,t) \\
=& N_e\int_0^1 e_t(0,t)e_t(x,t)\mathrm{d}x + N_e m e_t(0,t)e_t(1,t) - N_e e(0,t)u(0,t) \\
\leq& \frac{1}{2}\int_0^1 e_t^2(x,t)\mathrm{d}x + \frac{N_e^2}{2}e_t^2(0,t) + \frac{m}{2}e_t^2(1,t) + \frac{mN_e^2}{2}e_t^2(0,t) - N_e e(0,t)u(0,t) \\
=& \frac{1}{2}\int_0^1 e_t^2(x,t)\mathrm{d}x + \frac{m}{2}e_t^2(1,t) + \frac{N_e^2}{2}(1+m)e_t^2(0,t) - N_e r e(0,t)e_t(0,t) - N_e\beta e^2(0,t).
\end{aligned} \tag{2.30}$$

By (2.29), (2.30) and the second equation of (2.5), the derivative of $I_1(t)$ defined by (2.19) satisfies

$$\begin{aligned}
\dot{I}_1(t) =& N\dot{E}_e(t) + \delta_1(\dot{\psi}_1(t) + N_e\dot{\psi}_2(t)) \\
\leq& -Nre_t^2(0,t) - \delta_1\int_0^1 u^2(x,t)\mathrm{d}x - \delta_1\frac{1}{2}\int_0^1 e_t^2(x,t)\mathrm{d}x - \delta_1[N_e\beta-(m+1)\beta^2]e^2(0,t) - \delta_1\frac{m}{2}e_t^2(1,t) \\
&- \delta_1\left[N_e r - 2(m+1)r\beta\right]e(0,t)e_t(0,t) + \delta_1\left[(m+1)(1+r^2) + \frac{N_e^2}{2}(m+1)\right]e_t^2(0,t) \\
=& -\delta_1\int_0^1 u^2(x,t)\mathrm{d}x - \delta_1\frac{1}{2}\int_0^1 e_t^2(x,t)\mathrm{d}x - \delta_1[N_e\beta-(m+1)\beta^2]e^2(0,t) - \delta_1\frac{m}{2}e_t^2(1,t) \\
&- \delta_1\left[N_e r - 2(m+1)r\beta\right]e(0,t)e_t(0,t) - \left[Nr - \delta_1(m+1)(1+r^2) - \delta_1\frac{N_e^2}{2}(m+1)\right]e_t^2(0,t) \\
\leq& -\delta_1 E_e(t) - \delta_1\left[N_e\beta - (m+1)\beta^2 - \frac{\beta}{2}\right]e^2(0,t) - \Big[Nr - \delta_1(m+1)(1+r^2) \\
&-\delta_1\frac{N_e^2}{2}(m+1)\Big]e_t^2(0,t) - \delta_1\left[N_e r - 2(m+1)r\beta\right]e(0,t)e_t(0,t),
\end{aligned}$$

where N_e and $N > 0$ satisfy (2.9). Therefore,

$$\dot{I}_1(t) \leq -\delta_1 E_e(t). \tag{2.31}$$

Finding the derivative of $I_2(t)$ defined by (2.19) with respect to t gives

$$\dot{I}_2(t) = \frac{3}{2}z^2(1,t) + \frac{3}{2}\widehat{w}_t^2(1,t) - \frac{3}{2}\int_0^1 [\widehat{w}_t^2(x,t) + z^2(x,t)]\mathrm{d}x. \tag{2.32}$$

Combining (2.31), (2.32) and (2.18) leads to

$$\dot{P}(t) = \dot{I}_1(t) + \dot{I}_2(t) \leq -\delta_1 E_e(t) + \frac{3}{2}z^2(1,t) + \frac{3}{2}\widehat{w}_t^2(1,t) - \frac{3}{2}\int_0^1 [\widehat{w}_t^2(x,t) + z^2(x,t)]\mathrm{d}x. \tag{2.33}$$

Now, differentiating (2.7), together with (2.8), (2.16) and (2.33), yields

$$\begin{aligned}
\dot{V}(t) = &\ t\dot{L}(t) + L(t) + \dot{P}(t) \\
= &\ -t\left(\frac{a}{m+\alpha a} - \varepsilon\right) z^2(1,t) - t\left(\frac{\alpha m}{m+\alpha a} - \varepsilon\right)\widehat{w}_t^2(1,t) - tz(0)e_t(0,t) - t\delta_1 r e_t^2(0,t) \\
&-t\varepsilon[z^2(0,t) + \widehat{w}_t^2(0,t)] + \frac{1}{2}\int_0^1 [\widehat{w}_t^2(x,t) + z^2(x,t)]\mathrm{d}x + \frac{1}{2}\frac{\eta^2}{m+\alpha a} + \delta_1 E_e(t) \\
&+2\varepsilon\int_0^1 \widehat{w}_t(x,t)z(x,t)\mathrm{d}x + \dot{P}(t) \\
\leq &\ -t\left(\frac{a}{m+\alpha a} - \varepsilon\right) z^2(1,t) - t\left(\frac{\alpha m}{m+\alpha a} - \varepsilon\right)\widehat{w}_t^2(1,t) + t\frac{\delta}{2}z^2(0,t) + t\frac{1}{2\delta}e_t^2(0,t) - t\delta_1 r e_t^2(0,t) \\
&-t\varepsilon z^2(0,t) + \frac{1}{2}\int_0^1 [\widehat{w}_t^2(x,t) + z^2(x,t)]\mathrm{d}x + \frac{a^2}{m+\alpha a}z^2(1,t) + \frac{m^2}{m+\alpha a}\widehat{w}_t^2(1,t) + \delta_1 E_e(t) \\
&+\int_0^1 [\widehat{w}_t^2(x,t) + z^2(x,t)]\mathrm{d}x + \dot{P}(t) \\
\leq &\ -\left[t\left(\frac{a}{m+\alpha a} - \varepsilon\right) - \frac{a^2}{m+\alpha a} - \frac{3}{2}\right] z^2(1,t) - \left[t\left(\frac{\alpha m}{m+\alpha a} - \varepsilon\right) - \frac{m^2}{m+\alpha a} - \frac{3}{2}\right]\widehat{w}_t^2(1,t) \\
&-t\left(\varepsilon - \frac{\delta}{2}\right) z^2(0,t) - t\left(\delta_1 r - \frac{1}{2\delta}\right) e_t^2(0,t)
\end{aligned} \tag{2.34}$$

Since ε, δ and δ_1 are selected by (2.9), there exists a constant $T > 0$ such that

$$\dot{V}(t) \leq 0, \ \forall t \geq T. \tag{2.35}$$

□

Theorem 2.6. *There exist strictly positive constants M_1 and ω_1, independent of the initial value, such that the energy $E(t)$ of the continuous system (2.2) satisfies*

$$E(t) \leq M_1 \mathrm{e}^{-\omega_1 t} E(0), \ \forall t > 0. \tag{2.36}$$

Proof. From Lemmas 2.4 and 2.5, it follows that

$$L(t) \leq \frac{T+C}{t-C} L(0), \forall t > \max\{T, C\}. \tag{2.37}$$

Hence, there exist constants $t_0 > 0, 0 < \gamma_0 < 1$, such that

$$L(t_0) \leq \gamma_0 L(0). \tag{2.38}$$

Since from Lemmas 2.2 and 2.1, $L(t)$, $E_1(t)$ and $E(t)$ are mutually equivalent, (2.38) means that the C_0-semigroup e^{At} generated by operator A define by (1.15) is exponentially stable on $\mathcal{H}$. This implies in turn that $E(t)$ decays uniformly exponentially: There exist positive constants M_1 and ω_1, independent of the initial value, such that

$$E(t) \leq M_1 \mathrm{e}^{-\omega_1 t} E(0), \ \forall t > 0. \tag{2.39}$$

□

It is seen that the exponential decay for our Lyapunov functional $L(t)$ uses an indirect argument (2.38) by the semigroup property.

3. Semi-discrete finite difference schemes

In this section, we first construct a semi-discrete finite difference scheme for the order reduced system (2.2). Let N be a non-negative integer and the mesh size be $h = \frac{1}{N+1}$. An equidistance partition of the spatial interval $[0, 1]$ is given by

$$0 = x_0 < \ldots < x_j = jh < \ldots < x_{N+1} = 1, \ j = 0, 1, \ldots, N+1.$$

To simplify the notation, we introduce the following notation. Let the sequence $\{z_j\}_0^{N+1}$ be denoted by $\{z_j\}_j$ and $\{u_j\}_0^{N+1}$ by $\{u_j\}_j$. Introduce the average operators $z_{j+\frac{1}{2}}$ and $u_{j+\frac{1}{2}}$, and the difference operators $\delta_x z_{j+\frac{1}{2}}$, $\delta_x^2 z_{j+\frac{1}{2}}$, $\delta_x u_{j+\frac{1}{2}}$, and $\delta_x^2 u_{j+\frac{1}{2}}$ as follows:

$$z_{j+\frac{1}{2}} = \frac{z_j + z_{j+1}}{2}, \ \delta_x z_{j+\frac{1}{2}} = \frac{z_{j+1} - z_j}{h},$$

$$\delta_x^2 z_{j+\frac{1}{2}} = \frac{\delta_x z_{j+\frac{1}{2}} - \delta_x z_{j-\frac{1}{2}}}{h} = \frac{z_{j+1} - 2z_j + z_{j-1}}{h^2},$$

$$u_{j+\frac{1}{2}} = \frac{u_j + u_{j+1}}{2}, \ \delta_x u_{j+\frac{1}{2}} = \frac{u_{j+1} - u_j}{2},$$

$$\delta_x^2 u_{j+\frac{1}{2}} = \frac{\delta_x u_{j+\frac{1}{2}} - \delta_x u_{j-\frac{1}{2}}}{h} = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}.$$

Denote

$$\widehat{w}_j = \widehat{w}(x_j, t), z_j = z(x_j, t), e_j = e(x_j, t), u_j = u(x_j, t), j = 0, 1, \cdots, N+1.$$

Denote by the prime $'$ the derivative with respect to time t. The first two equations of system (2.2) holds at $(x_{j+\frac{1}{2}}$,t), *i.e.*,

$$\widehat{w}''(x_{j+\frac{1}{2}},t) = z_x(x_{j+\frac{1}{2}},t), z(x_{j+\frac{1}{2}},t) = \widehat{w}_x(x_{j+\frac{1}{2}},t),$$

where $x_{j+\frac{1}{2}} = (j+\frac{1}{2})h$. Replace the differential operator ∂x with difference operator δx to get

$$\widehat{w}''_{j+\frac{1}{2}} = \delta_x z_{j+\frac{1}{2}} + \mathcal{O}(h^2), \quad z_{j+\frac{1}{2}} = \delta_x \widehat{w}_{j+\frac{1}{2}} + \mathcal{O}(h^2). \tag{3.1}$$

Similarly, for the fifth and sixth equations of system (2.2), we have

$$e''_{j+\frac{1}{2}} = \delta_x u_{j+\frac{1}{2}} + \mathcal{O}(h^2), \quad u_{j+\frac{1}{2}} = \delta_x e_{j+\frac{1}{2}} + \mathcal{O}(h^2). \tag{3.2}$$

It is noted in (3.1) and (3.2) that $\mathcal{O}(h^2)$ is formal, which is for the functions with respect to x not for the sequences. By omitting the infinitesimal terms in (3.1) and (3.2), and still denoting by $\widehat{w}_j$, $\widehat{w}_j$, e_j and u_j all related sequences without the infinitesimal terms, we can get the following semi-discretized finite difference scheme for (2.2):

$$\begin{cases} \widehat{w}''_{j+\frac{1}{2}} = \delta_x z_{j+\frac{1}{2}}, & 0 \le j \le N, \\ z_{j+\frac{1}{2}} = \delta_x \widehat{w}_{j+\frac{1}{2}}, & 0 \le j \le N, \\ \widehat{w}_0 = e_0, & \\ z_{N+1} + m\widehat{w}''_{N+1} = -\alpha \widehat{w}'_{N+1} - a z'_{N+1}, & \\ e''_{j+\frac{1}{2}} = \delta_x u_{j+\frac{1}{2}}, & 0 \le j \le N, \\ u_{j+\frac{1}{2}} = \delta_x e_{j+\frac{1}{2}}, & 0 \le j \le N, \\ u_0 = r e'_0 + \beta e_0, & \\ u_{N+1} + m e''_{N+1} = 0. & \end{cases} \tag{3.3}$$

which is a second-order accuracy scheme at each $x_{j+\frac{1}{2}}$. Since$\{z_j\}_j$ and $\{u_j\}_j$ in (3.3) are additionally introduced, we can cancel the intermediate variables $\{z_j\}_j$ and $\{u_j\}_j$ to get the semi-discretized finite difference scheme for the original system(1.9). For this purpose, we first eliminate the variable $\{z_j\}_j$ related terms from (3.3). To this end, multiplying the first equation of (3.3) by $\frac{h}{2}$ and adding to the second equation of (3.3) yields

$$z_{j+1} = \delta_x \widehat{w}_{j+\frac{1}{2}} + \frac{h}{2}\widehat{w}''_{j+\frac{1}{2}}, \quad 0 \le j \le N,$$

and hence

$$z_j = \delta_x \widehat{w}_{j-\frac{1}{2}} + \frac{h}{2}\widehat{w}''_{j-\frac{1}{2}}, \quad 1 \le j \le N. \tag{3.4}$$

Similarly, multiplying the first equation of (3.3) by $\frac{h}{2}$ and subtracting from the second equation of (3.3) gives

$$-z_j = -\delta_x \widehat{w}_{j+\frac{1}{2}} + \frac{h}{2}\widehat{w}''_{j+\frac{1}{2}}, \quad 1 \le j \le N. \tag{3.5}$$

Adding up (3.4) and (3.5), for $1 \le j \le N$, we derive

$$\frac{1}{2}(\widehat{w}''_{j+\frac{1}{2}} + \widehat{w}''_{j-\frac{1}{2}}) = \delta_x^2 \widehat{w}_j \tag{3.6}$$

Then, setting $j = N+1$ in (3.4), and combining with the fourth equation of (2.2), we obtain

$$\left[\delta_x \widehat{w}_{N+\frac{1}{2}} + \frac{h}{2}\widehat{w}''_{N+\frac{1}{2}}\right] m\widehat{w}''_{N+1} = -\alpha \widehat{w}'_{N+1} - a\left[\delta_x \widehat{w}_{N+\frac{1}{2}} + \frac{h}{2}\widehat{w}''_{N+\frac{1}{2}}\right]'. \tag{3.7}$$

By the same procedure, we eliminate the variable $\{u_j\}_j$ related terms from (3.3) to get

$$\begin{cases} \frac{1}{2}(e''_{j+\frac{1}{2}} + e''_{j-\frac{1}{2}}) = \delta_x^2 e_j, & 1 \le j \le N \\ \frac{h}{2}e''_{\frac{1}{2}} + \delta_x e_{\frac{1}{2}} = re'_0 + \beta e_0, \\ \frac{h}{2}e''_{N+\frac{1}{2}} + \delta_x e_{N+\frac{1}{2}} + me''_{N+1} = 0. \end{cases} \tag{3.8}$$

Combining (3.6)–(3.8), we get the following semi-discrete scheme for the original system(1.9):

$$\begin{cases} \frac{1}{2}(\widehat{w}''_{j+\frac{1}{2}} + \widehat{w}''_{j-\frac{1}{2}}) = \delta_x^2 \widehat{w}_j, & 1 \le j \le N, \\ \widehat{w}_0 = e_0, \\ \left[\delta_x \widehat{w}_{N+\frac{1}{2}} + \frac{h}{2}\widehat{w}''_{N+\frac{1}{2}}\right] + m\widehat{w}''_{N+1} = -\alpha \widehat{w}'_{N+1} - a\left[\delta_x \widehat{w}_{N+\frac{1}{2}} + \frac{h}{2}\widehat{w}''_{N+\frac{1}{2}}\right]' \\ \frac{1}{2}(e''_{j+\frac{1}{2}} + e''_{j-\frac{1}{2}}) = \delta_x^2 e_j, & 1 \le j \le N, \\ \frac{h}{2}e''_{\frac{1}{2}} + \delta_x e_{\frac{1}{2}} = re'_0 + \beta e_0, \\ \frac{h}{2}e''_{N+\frac{1}{2}} + \delta_x e_{N+\frac{1}{2}} + me''_{N+1} = 0. \end{cases} \tag{3.9}$$

Since systems (1.9) and (1.12) are equivalent to each other, the semi-discrete scheme (3.9) is equivalent to

$$\begin{cases} \frac{1}{2}(\widehat{w}''_{j+\frac{1}{2}} + \widehat{w}''_{j-\frac{1}{2}}) = \delta_x^2 \widehat{w}_j, & 1 \le j \le N, \\ \widehat{w}_0 = e_0, \\ \frac{h}{2}\widehat{w}''_{N+\frac{1}{2}} + \delta_x \widehat{w}_{N+\frac{1}{2}} = -\frac{m}{a}\widehat{w}'_{N+1} + \frac{1}{a}\eta, \\ \dot{\eta} = -\frac{1}{a}\eta - \frac{a\alpha - m}{a}\widehat{w}'_{N+1}, \\ \frac{1}{2}(e''_{j+\frac{1}{2}} + e''_{j-\frac{1}{2}}) = \delta_x^2 e_j, & 1 \le j \le N, \\ \frac{h}{2}e''_{\frac{1}{2}} + \delta_x e_{\frac{1}{2}} = re'_0 + \beta e_0, \\ e'_{N+1} = \eta_e, \\ \dot{\eta}_e = -\frac{1}{m}\left[\frac{h}{2}e''_{N+\frac{1}{2}} + \delta_x e_{N+\frac{1}{2}}\right]. \end{cases} \tag{3.10}$$

which is a semi-discrete scheme of system (1.12).

Remark 3.1. In (3.9), there is third order derivative with respect to time but by introducing $\eta(t)$, the third order derivative with respect to time disappears in (3.10), which is consistent with (1.2).

Set

$$G = (\widehat{w}_1, \cdots, \widehat{w}_{N+1}, \eta, e_0, \cdots, e_{N+1})^\top \tag{3.11}$$

to be the variable of (3.10). System (3.10) can be reformulated into a vectorial form:

$$M_0 G'' + M_1 G' + M_2 G = 0, \tag{3.12}$$

where $M_i, i = 0, 1, 2$, are $(2N+4) \times (2N+4)$ square matrices. The matrix M_0 is given by

$$M_0 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \tag{3.13}$$

where the matrices $A_{ij}, i, j = 1, 2, 3, 4$ are defined by

$$\begin{aligned} A_{11} &= \begin{pmatrix} \frac{2}{4} & \frac{1}{4} & & & & \\ \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & & & \\ \ddots & \ddots & \ddots & & & \\ & & \frac{1}{4} & \frac{2}{4} & \frac{1}{4} & 0 \\ & & & \frac{h}{4} & \frac{h}{4} & 0 \\ & & & & \frac{a\alpha - m}{a} & 1 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} \frac{1}{4} & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \\ A_{22} &= \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & & \\ & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & & \\ & & \ddots & & & \\ & & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \\ \frac{h}{4} & \frac{h}{4} & & & & \\ & & & & \frac{h}{4} & (\frac{h}{4} + m) \end{pmatrix}. \end{aligned} \tag{3.14}$$

The matrix M_1 can be expressed as

$$M_1 = \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix}, \tag{3.15}$$

where the matrices $B_{11}, B_{22} \in \mathbb{R}^{(N+2)\times(N+2)}$ are

$$B_{11} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & \frac{m}{a} & \\ & & & & \frac{1}{a} \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ -r & & & \ddots & \\ 0 & & & & 0 \end{pmatrix}. \tag{3.16}$$

The matrix M_2 can be expressed as

$$M_2 = \begin{pmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{pmatrix}, \tag{3.17}$$

where the matrices $C_{11}, C_{12}, C_{21}, C_{22} \in \mathbb{R}^{(N+2)\times(N+2)}$ are defined by

$$
\begin{gathered}
C_{11} = \begin{pmatrix}
\frac{2}{h^2} & -\frac{1}{h^2} & & & & \\
-\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & & & \\
\ddots & \ddots & \ddots & & & \\
& & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & \\
& & & -\frac{1}{h} & \frac{1}{h} & -\frac{1}{a} \\
& & & & & 0
\end{pmatrix}, \quad C_{12} = \begin{pmatrix}
-\frac{1}{h^2} & & & & \\
& 0 & & & \\
& & \ddots & & \\
& & & \ddots & \\
& & & & 0
\end{pmatrix}, \\
C_{22} = \begin{pmatrix}
-\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & & & \\
& -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & & \\
& & \ddots & & & \\
& & -\frac{1}{h^2} & \frac{2}{h^2} & -\frac{1}{h^2} & \\
(-\frac{1}{h} - \beta) & \frac{1}{h} & & & & \\
& & & & -\frac{1}{h} & \frac{1}{h}
\end{pmatrix}.
\end{gathered} \tag{3.18}
$$

Let $X_h = (G, \dot{G})$. Combining (3.12), the system (3.9) is described by

$$
\dot{X}_h = A_h X_h, \tag{3.19}
$$

where $A_h \in \mathbb{R}^{(4N+8)\times(4N+8)}$ is defined by

$$
A_h = \begin{pmatrix} 0 & I \\ -M_0^{-1} M_2 & -M_0^{-1} M_1 \end{pmatrix}. \tag{3.20}
$$

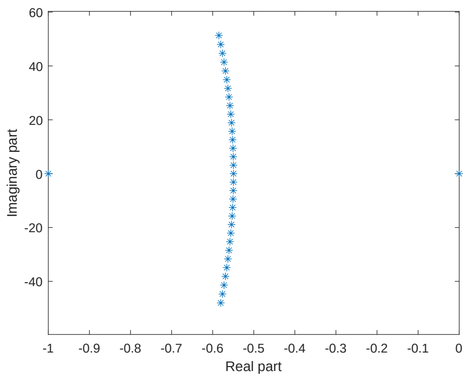
Certainly, the matrices A_h generate semigroups $\mathrm{e}^{A_h t}$. Our major task in this paper is to show that $\mathrm{e}^{A_h t}$ is uniformly exponentially stable with respect to the step size $h \to 0$.

To illustrate why the semi-discretized finite difference scheme (3.9) can preserve uniformly the exponential stability of the original couple system (1.9), we present a numerical experiment to compare our scheme with the classical finite difference based semi-discrete scheme of the following

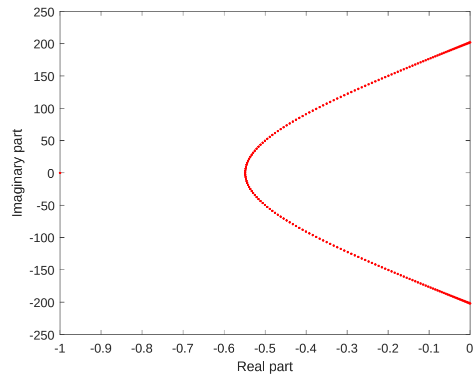
$$
\begin{cases}
\widehat{w}_j'' = \delta_x^2 \widehat{w}_j, & 1 \le j \le N, \\
\widehat{w}_0 = e_0, \\
\frac{\widehat{w}_{N+1} - \widehat{w}_N}{h} + m\widehat{w}_{N+1}'' = -\alpha \widehat{w}_{N+1}' - a\frac{\widehat{w}_{N+1}' - \widehat{w}_N'}{h}. \\
e_j'' = \delta_x^2 e_j, & 1 \le j \le N, \\
\frac{e_1 - e_0}{h} = r e_0' + \beta e_0, \\
\frac{e_{N+1} - e_N}{h} + m e_{N+1}'' = 0.
\end{cases} \tag{3.21}
$$

For simplicity of numerical experiments, we set $e_j = 0$ in both (3.9) and (3.21) because in this case, the semi-discrete schemes are for the continuous system:

$$
\begin{cases}
\widehat{w}_{tt}(x,t) = \widehat{w}_{xx}(x,t), \\
\widehat{w}(0,t) = 0, \\
\widehat{w}_x(1,t) + m\widehat{w}_{tt}(1,t) = -\alpha \widehat{w}_t(1,t) - a\widehat{w}_{xt}(1,t),
\end{cases} \tag{3.22}
$$



(a) Eigenvalue distribution of system (3.23)



(b) Eigenvalue distribution of system (3.24)

FIGURE 1. The eigenvalue distributions of systems (3.23) and (3.24) with $N = 100$.

which is just the closed-loop system considered in [18]. Our order reduction semi-discrete scheme is from (3.9) by setting all $e_j = 0$:

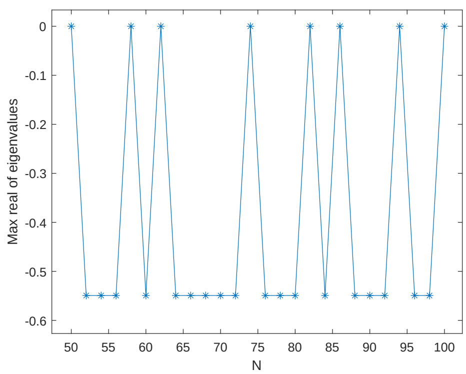
$$\begin{cases} \frac{1}{2}(\widehat{w}''_{j+\frac{1}{2}} + \widehat{w}''_{j-\frac{1}{2}}) = \delta_x^2 \widehat{w}_j, & 1 \le j \le N, \\ \widehat{w}_0 = 0, \\ \left[\delta_x \widehat{w}_{N+\frac{1}{2}} + \frac{h}{2}\widehat{w}''_{N+\frac{1}{2}}\right] + m\widehat{w}''_{N+1} = -\alpha \widehat{w}'_{N+1} - a\left[\delta_x \widehat{w}_{N+\frac{1}{2}} + \frac{h}{2}\widehat{w}''_{N+\frac{1}{2}}\right]', \end{cases} \tag{3.23}$$

while the classical semi-discrete scheme is from (3.21) by setting all $e_j = 0$:

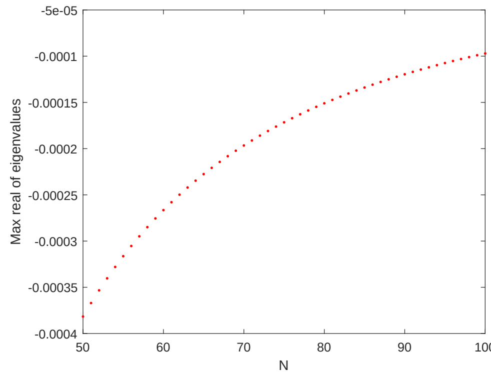
$$\begin{cases} \widehat{w}''_j = \delta_x^2 \widehat{w}_j, & 1 \le j \le N, \\ \widehat{w}_0 = 0, \\ \frac{w_{N+1} - \widehat{w}_N}{h} + m\widehat{w}''_{N+1} = -\alpha \widehat{w}'_{N+1} - a\frac{\widehat{w}'_{N+1} - \widehat{w}'_N}{h}. \end{cases} \tag{3.24}$$

We compute all eigenvalue of the semi-discrete schemes (3.23) and (3.24) for $\alpha = 2, m = 2, a = 1$. Figure 1 shows the eigenvalue distribution of the semi-discretized system (3.23) and (3.24) with $N = 100$, respectively. As shown in Figure 1, the high frequencies of semi-discretized system (3.23) tend to infinity (please note that the low eigenvalue is not null real part yet small one), while the high frequencies of the semi-discretized system (3.24) accumulate to imaginary axis. This explains why the semi-discretized system (3.9) preserves the uniform exponential stability. Figure 2 shows the maximal real parts of the eigenvalues of the semi-discretized systems (3.23) and (3.24) for N from 50 to 100, respectively. We find that the maximal real parts of system (3.24) increase as N increases and gradually tends to zero, indicating that the decay rate is also gradually tending to zero. The maximal real parts of the system (3.23) remain in a certain range, which ensure the uniformly exponential stability.

Finally, since we cancel the intermediate variables $\{z_j\}_j$ and $\{u_j\}_j$ from (3.3) to get the semi-discretized finite difference scheme (3.9), we can see that systems (3.3) and (3.9) are equivalent to each other. Same to [13], in latter sections, we use (3.3) to prove the uniformly exponential stability, because the proof for the system (3.3) is much easier than that for the system (3.9).



(a) Maximal reals of eigenvalues of system (3.23)



(b) Maximal reals of eigenvalues of system (3.24)

FIGURE 2. The maximal reals of eigenvalues of systems (3.23) and (3.24).

4. UNIFORMLY EXPONENTIAL STABILITY

In this section, we establish the uniformly exponential stability of the semi-discretized system (3.3). First, the energy of the system (3.3) is defined by

$$E_h(t) = E_h^{\widehat{w}}(t) + E_h^{e}(t). \tag{4.1}$$

where

$$\begin{cases} E_h^{\widehat{w}}(t) = \dfrac{1}{2}h\displaystyle\sum_{j=0}^{N}(\widehat{w}'_{j+\frac{1}{2}})^2 + \dfrac{1}{2}h\sum_{j=0}^{N} z_{j+\frac{1}{2}}^2 + \dfrac{1}{2}\dfrac{\eta^2}{m+\alpha a}, \ \eta = a z_{N+1} + m\widehat{w}'_{N+1}, \\ E_h^{e}(t) = \dfrac{1}{2}h\displaystyle\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2 + \dfrac{1}{2}h\sum_{j=0}^{N} u_{j+\frac{1}{2}}^2 + \dfrac{\beta}{2}e_0^2 + \dfrac{m}{2}(e'_{N+1})^2, \end{cases} \tag{4.2}$$

which are discretizations of energy functions defined by (2.4). In order to prove the uniform exponential stability of system (3.3), we need the following Lemmas 4.1–4.5.

The following Lemma 4.1 comes from [14].

Lemma 4.1. *For any grid functions $\{U_j\}_j, \{V_j\}_j, and \{W_j\}_j$ at any mesh grid $\{x_j\}_j$, the following two formulas of summation by parts:*

$$h\sum_{j=0}^{N}\delta_x U_{j+\frac{1}{2}} V_{j+\frac{1}{2}} + h\sum_{j=0}^{N} U_{j+\frac{1}{2}}\delta_x V_{j+\frac{1}{2}} = U_{N+1}V_{N+1} - U_0V_0. \tag{4.3}$$

$$\begin{aligned} &h\sum_{j=0}^{N}\delta_x U_{j+\frac{1}{2}} V_{j+\frac{1}{2}} W_{j+\frac{1}{2}} + h\sum_{j=0}^{N} U_{j+\frac{1}{2}}\delta_x V_{j+\frac{1}{2}} W_{j+\frac{1}{2}} + h\sum_{j=0}^{N} U_{j+\frac{1}{2}} V_{j+\frac{1}{2}}\delta_x W_{j+\frac{1}{2}} \\ =&U_{N+1}V_{N+1}W_{N+1} - U_0V_0W_0 - \frac{1}{4}\sum_{j=0}^{N}(U_{j+1}-U_j)(V_{j+1}-V_j)(W_{j+1}-W_j). \end{aligned} \tag{4.4}$$

Lemma 4.2. *The derivative of the discrete energy $E_h(t)$ of the system (3.3) satisfies*

$$\dot{E}_h(t) = \dot{E}_h^{\widehat{w}}(t) + \dot{E}_h^{e}(t) = -\frac{1}{m+\alpha a}\left[az_{N+1}^2 + \alpha m(\widehat{w}'_{N+1})^2\right] - z_0 e'_0 - r(e'_0)^2. \tag{4.5}$$

Proof. By virtue of (4.3) in Lemma 4.1, finding the derivative of $E_h^{\widehat{w}}(t)$ defined by (4.2) with respect to t on both sides yields

$$\begin{aligned}
\dot{E}_h^{\widehat{w}}(t) &= h\sum_{j=0}^{N} \widehat{w}'_{j+\frac12}\widehat{w}''_{j+\frac12} + h\sum_{j=0}^{N} z_{j+\frac12}z'_{j+\frac12} + \frac{\eta}{m+\alpha a}\eta' \\
&= h\sum_{j=0}^{N} \widehat{w}'_{j+\frac12}\delta_x z_{j+\frac12} + h\sum_{j=0}^{N} z_{j+\frac12}\delta_x \widehat{w}'_{j+\frac12} + \frac{az_{N+1}+m\widehat{w}'_{N+1}}{m+\alpha a}[-z_{N+1}-\alpha\widehat{w}'_{N+1}] \\
&= z_{N+1}\widehat{w}'_{N+1} - z_0\widehat{w}'_0 + \frac{-az_{N+1}^2 - a\alpha z_{N+1}\widehat{w}'_{N+1} - mz_{N+1}\widehat{w}'_{N+1} - m\alpha(\widehat{w}'_{N+1})^2}{m+\alpha a} \\
&= -\frac{1}{m+\alpha a}[az_{N+1}^2 + \alpha m(\widehat{w}'_{N+1})^2] - z_0 e'_0.
\end{aligned} \tag{4.6}$$

Similarly,

$$\dot{E}_h^{e}(t) = -r(e'_0)^2. \tag{4.7}$$

Combining (4.6) and (4.7) with (4.1) gives

$$\dot{E}_h(t) = \dot{E}_h^{\widehat{w}}(t) + \dot{E}_h^{e}(t) = -\frac{1}{m+\alpha a}[az_{N+1}^2 + \alpha m(\widehat{w}'_{N+1})^2] - z_0 e'_0 - r(e'_0)^2. \tag{4.8}$$

□

Next we construct the following Lyapunov functions $L_h(t)$ and function $V_h(t)$, which are the discrete counterparts of (2.7)

$$L_h(t) = E_{1h}(t) + \varepsilon P_{1h}(t),\ V_h(t) = tL_h(t) + P_h(t), \tag{4.9}$$

where the auxiliary functions are given by

$$\begin{cases}
E_{1h}(t) = E_h^{\widehat{w}}(t) + \delta_1 E_h^{e}(t), \\
P_{1h}(t) = 2h\sum\limits_{j=0}^{N} \widehat{w}'_{j+\frac12} z_{j+\frac12}, \\
\psi_{1h}(t) = 2h\sum\limits_{j=0}^{N} [x_{j+\frac12} - (m+1)]e'_{j+\frac12}u_{j+\frac12}, \\
\psi_{2h}(t) = e_0 h\sum\limits_{j=0}^{N} e'_{j+\frac12} + me_0 e'_{N+1}, \\
P_h(t) = NE_h^{e}(t) + \delta_1(\psi_{1h}(t) + N_e\psi_{2h}(t)) + 3h\sum\limits_{j=0}^{N} x_{j+\frac12}\widehat{w}'_{j+\frac12} z_{j+\frac12}.
\end{cases} \tag{4.10}$$

which are really the natural discretizations of the auxiliary functions defined in (2.8) for the continuous counterpart with the same parameters $\varepsilon, \delta_1, N_e, N > 0$ defined by (2.9).

The following Lemma 4.3 is a trivial fact.

Lemma 4.3. *The function $E_{1h}(t)$ defined by (4.10) is equivalent to the discrete energy $E_h(t)$ of the system (3.3), i.e., there exist two constants C_1 and C_2, such that*

$$C_1 E_{1h}(t) \leq E_h(t) \leq C_2 E_{1h}(t), \forall t \geq 0, \tag{4.11}$$

where $C_1 = \min\{1, \delta_1\}$, $C_2 = \max\{1, \delta_1\}$ and δ_1 is a positive constant selected in (2.9).

Lemma 4.4. *The function $L_h(t)$ defined by (4.9) is equivalent to $E_{1h}(t)$ defined by (2.8), i.e, for any $0 < \varepsilon < \frac{1}{2}$, it has*

$$(1 - 2\varepsilon)E_{1h}(t) \leq L_h(t) \leq (1 + 2\varepsilon)E_{1h}(t). \tag{4.12}$$

Proof. For the auxiliary function $P_{1h}(t)$ defined by (4.10), we have straightforwardly that

$$|P_{1h}(t)| \leq \left| 2h \sum_{j=0}^{N} \widehat{w}'_{j+\frac{1}{2}} z_{j+\frac{1}{2}} \right| \leq h \sum_{j=0}^{N} (\widehat{w}'_{j+\frac{1}{2}})^2 + h \sum_{j=0}^{N} z^2_{j+\frac{1}{2}} \leq 2E_{1h}(t).$$

According to the third identity of (4.10), we further obtain the required inequalities:

$$(1 - 2\varepsilon)E_{1h}(t) \leq L_h(t) \leq (1 + 2\varepsilon)E_{1h}(t).$$

for any $0 < \varepsilon < \frac{1}{2}$. □

The following Lemma 4.5 is the discrete counterpart of Lemma 2.3.

Lemma 4.5. *For the parameters ε, δ and δ_1 selected by (2.9), the derivative of the $L_h(t)$ defined by (4.10) satisfies*

$$\dot{L}_h(t) \leq 0. \tag{4.13}$$

Proof. Combining (4.6) and (4.7), the derivative of the auxiliary function $E_{1h}(t)$ defined by (4.10) satisfies

$$\dot{E}_{1h}(t) = -\frac{1}{m + \alpha a}[a z^2_{N+1} + \alpha m (\hat{w}'_{N+1})^2] - z_0 e'_0 - \delta_1 r (e'_0)^2. \tag{4.14}$$

By virtue of (4.4) in Lemma 4.1, the derivative of the auxiliary function $P_{1h}(t)$ defined by (4.10) along the solution of (3.3) satisfies

$$\begin{aligned} \dot{P}_{1h}(t) &= 2h \sum_{j=0}^{N} \widehat{w}''_{j+\frac{1}{2}} z_{j+\frac{1}{2}} + 2h \sum_{j=0}^{N} \widehat{w}'_{j+\frac{1}{2}} z'_{j+\frac{1}{2}} \\ &= 2h \sum_{j=0}^{N} \delta_x z_{j+\frac{1}{2}} z_{j+\frac{1}{2}} + 2h \sum_{j=0}^{N} \widehat{w}'_{j+\frac{1}{2}} \delta_x \widehat{w}'_{j+\frac{1}{2}} \\ &= z^2_{N+1} + (\widehat{w}'_{N+1})^2 - [z_0^2 + (\widehat{w}'_0)^2]. \end{aligned} \tag{4.15}$$

Therefore, the derivative of $L_h(t)$ defined by (4.10) satisfies

$$\begin{aligned}
\dot{L}_h(t) &= \dot{E}_{1h}(t) + \varepsilon \dot{P}_{1h}(t) \\
&= -\frac{1}{m+\alpha a}[a z_{N+1}^2 + \alpha m (\widehat{w}'_{N+1})^2] - z_0 e'_0 - \delta_1 r (e'_0)^2 + \varepsilon [z_{N+1}^2 + (\widehat{w}'_{N+1})^2] \\
&\quad -\varepsilon [z_0^2 + (\widehat{w}'_0)^2] \\
&\le -\frac{a}{m+\alpha a} z_{N+1}^2 - \frac{\alpha m}{m+\alpha a}(\widehat{w}'_{N+1})^2 + \frac{\delta}{2} z_0^2 + \frac{1}{2\delta}(e'_0)^2 - \delta_1 r (e'_0)^2 \\
&\quad +\varepsilon [z_{N+1}^2 + (\widehat{w}'_{N+1})^2] - \varepsilon [z_0^2 + (\widehat{w}'_0)^2] \\
&\le -\left(\frac{a}{m+\alpha a} - \varepsilon\right) z_{N+1}^2 - (\frac{\alpha m}{m+\alpha a} - \varepsilon)(\widehat{w}'_{N+1})^2 \\
&\quad -\left(\varepsilon - \frac{\delta}{2}\right) z_0^2 - (\delta_1 r - \frac{1}{2\delta})(e'_0)^2.
\end{aligned} \tag{4.16}$$

Since

$$\begin{cases} 0 < \varepsilon < \min\left\{\dfrac{a}{m+\alpha a}, \dfrac{\alpha m}{m+\alpha a}\right\}, \\ 0 < \delta < 2\varepsilon,\ \delta_1 > \frac{1}{2\delta r}. \end{cases} \tag{4.17}$$

which are selected by (2.9), we arrive at

$$\dot{L}_h(t) \le 0. \tag{4.18}$$

□

Lemma 4.6. *The functions $V_h(t)$ defined by (4.9) and the Lyapunov functions $L_h(t)$ defined by (4.10) have the relations: There exist constant $C > 0$ such that*

$$(t - C)L_h(t) \le V_h(t) \le (t + C)L_h(t), \forall t > C, \tag{4.19}$$

where

$$C = \frac{C_3}{1-2\varepsilon},\ C_3 = \max\left\{3, \frac{N + 2\delta_1(m+1) + \delta_1 N_e \max\left\{1, \frac{m+1}{\beta}\right\}}{\delta_1}\right\}.$$

Proof. By the $P_h(t)$ defined by (4.10), we write

$$\begin{aligned}
P_h(t) &= N E_h^e(t) + \delta_1(\psi_{1h}(t) + N_e \psi_{2h}(t)) + 3h \sum_{j=0}^{N} x_{j+\frac{1}{2}} \widehat{w}'_{j+\frac{1}{2}} z_{j+\frac{1}{2}} \\
&= I_{1h}(t) + I_{2h}(t),
\end{aligned} \tag{4.20}$$

where

$$I_{1h}(t) = NE_h^e(t) + \delta_1(\psi_{1h}(t) + N_e\psi_{2h}(t)),\ I_{2h}(t) = 3h\sum_{j=0}^{N} x_{j+\frac{1}{2}}\widehat{w}'_{j+\frac{1}{2}}z_{j+\frac{1}{2}}. \tag{4.21}$$

The $\psi_{1h}(t)$ and $\psi_{2h}(t)$ defined by (4.10) satisfy

$$\begin{aligned} |\psi_{1h}(t)| &= 2h\left|\sum_{j=0}^{N}[x_{j+\frac{1}{2}} - (m+1)]e'_{j+\frac{1}{2}}u_{j+\frac{1}{2}}\right| \\ &\le (m+1)\left[\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2 + \sum_{j=0}^{N}u^2_{j+\frac{1}{2}}\right] \\ &\le 2(m+1)E_h^e(t), \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} |\psi_{2h}(t)| &= \left|e_0 h\sum_{j=0}^{N}e'_{j+\frac{1}{2}} + me_0e'_{N+1}\right| \\ &\le \frac{1}{2}h\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2 + \frac{1}{2}e_0^2 + \frac{m}{2}(e'_{N+1})^2 + \frac{m}{2}e_0^2 \\ &= \frac{1}{2}h\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2 + \frac{m}{2}(e'_{N+1})^2 + (\frac{1}{2} + \frac{m}{2})e_0^2 \\ &\le \max\left\{1, \frac{m+1}{\beta}\right\}E_h^e(t). \end{aligned} \tag{4.23}$$

Therefore

$$|\delta_1(\psi_{1h}(t) + N_e\psi_{2h}(t))| \le \delta_1\left(2(m+1) + N_e\max\left\{1, \frac{m+1}{\beta}\right\}\right)E_h^e(t), \tag{4.24}$$

and hence

$$|I_{1h}(t)| \le \left(N + 2\delta_1(m+1) + \delta_1 N_e max\left\{1, \frac{m+1}{\beta}\right\}\right)E_h^e(t). \tag{4.25}$$

By the same argument, the $I_{2h}(t)$ defined by (4.21) satisfies

$$\begin{aligned} |I_{2h}(t)| &= \left|3h\sum_{j=0}^{N}x_{j+\frac{1}{2}}\widehat{w}'_{j+\frac{1}{2}}z_{j+\frac{1}{2}}\right| \le \frac{3}{2}h\sum_{j=0}^{N}(\widehat{w}'_{j+\frac{1}{2}})^2 + \frac{3}{2}h\sum_{j=0}^{N}z^2_{j+\frac{1}{2}} \\ &\le 3E_h^{\widehat{w}}(t). \end{aligned} \tag{4.26}$$

By (4.25) and (4.26), we can easily obtain that

$$
\begin{aligned}
|P_h(t)| &= |I_{1h}(t) + I_{2h}(t)| \\
&\leq \left(N + 2\delta_1(m+1) + \delta_1 N_e \max\left\{1, \frac{m+1}{\beta}\right\}\right) E_h^e(t) + 3E_h^{\widehat{w}}(t) \\
&\leq C_3 E_{1h}(t).
\end{aligned}
\tag{4.27}
$$

From Lemma 4.4, it follows that

$$
|P_h(t)| \leq C_3 E_{1h}(t) \leq \frac{C_3}{1-2\varepsilon} L_h(t). \tag{4.28}
$$

The functions $V_h(t)$ defined by (4.9) satisfy

$$
(t - C)L_h(t) \leq V_h(t) \leq (t + C)L_h(t), \tag{4.29}
$$

where

$$
C = \frac{C_3}{1-2\varepsilon},\ C_3 = \max\left\{3, \frac{N + 2\delta_1(m+1) + \delta_1 N_e \max\left\{1, \frac{m+1}{\beta}\right\}}{\delta_1}\right\}.
$$

□

Lemma 4.7. *There exists a constant $T > 0$ such that the derivative of $V_h(t)$ satisfies*

$$
\dot{V}_h(t) \leq 0,\ t \geq T. \tag{4.30}
$$

Proof. Finding the derivative of $\psi_{1h}(t)$ defined by (4.10) with respect to t gives

$$
\dot{\psi}_{1h}(t) = 2h\sum_{j=0}^{N}[jh - (m+1)]e''_{j+\frac{1}{2}}u_{j+\frac{1}{2}} + 2h\sum_{j=0}^{N}[jh - (m+1)]e'_{j+\frac{1}{2}}u'_{j+\frac{1}{2}}. \tag{4.31}
$$

The first term on the right-hand side of (4.31) can be expressed as

$$
\begin{aligned}
2h\sum_{j=0}^{N}[jh - (m+1)]e''_{j+\frac{1}{2}}u_{j+\frac{1}{2}} &= 2h\sum_{j=0}^{N}[jh - (m+1)]\delta_x u_{j+\frac{1}{2}}u_{j+\frac{1}{2}} \\
&= [x_{N+1} - (m+1)]u_{N+1}^2 - [x_0 - (m+1)]u_0^2 - h\sum_{j=0}^{N}u_{j+\frac{1}{2}}^2 \\
&= -mu_{N+1}^2 + (m+1)u_0^2 - h\sum_{j=0}^{N}u_{j+\frac{1}{2}}^2.
\end{aligned}
\tag{4.32}
$$

The second term on the right-hand side of (4.31) can be expressed as

$$
\begin{aligned}
2h\sum_{j=0}^{N}[jh-(m+1)]e'_{j+\frac{1}{2}}u'_{j+\frac{1}{2}} &= 2h\sum_{j=0}^{N}[jh-(m+1)]\delta_x e'_{j+\frac{1}{2}}e'_{j+\frac{1}{2}} \\
&= [x_{N+1}-(m+1)](e'_{N+1})^2-[x_0-(m+1)](e'_0)^2-h\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2 \\
&= -m(e'_{N+1})^2+(m+1)(e'_0)^2-h\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2.
\end{aligned} \tag{4.33}
$$

Plug (4.32) and (4.33) into (4.31) to get

$$
\begin{aligned}
&\dot{\psi}_{1h}(t) = -mu_{N+1}^2+(m+1)u_0^2-m(e'_{N+1})^2+(m+1)(e'_0)^2-h\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2-h\sum_{j=0}^{N}u_{j+\frac{1}{2}}^2 \\
&\le -h\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2-h\sum_{j=0}^{N}u_{j+\frac{1}{2}}^2-m(e'_{N+1})^2+(m+1)(e'_0)^2+(m+1)u_0^2 \\
&= -h\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2-h\sum_{j=0}^{N}u_{j+\frac{1}{2}}^2-m(e'_{N+1})^2+(m+1)(1+r^2)(e'_0)^2 \\
&+(m+1)\beta^2e_0^2+2(m+1)r\beta e_0e'_0.
\end{aligned} \tag{4.34}
$$

Finding the derivative of $\psi_{2h}(t)$ defined by (4.10) with respect to t gives

$$
\begin{aligned}
N_e\dot{\psi}_{2h}(t) = \quad & N_ee'_0h\sum_{j=0}^{N}e'_{j+\frac{1}{2}}+N_ee_0h\sum_{j=0}^{N}e''_{j+\frac{1}{2}}+N_eme'_0e'_{N+1}+N_eme_0e''_{N+1} \\
&= N_ee'_0h\sum_{j=0}^{N}e'_{j+\frac{1}{2}}+N_ee_0(u_{N+1}-u_0)+N_eme'_0e'_{N+1}+N_eme_0e''_{N+1} \\
&= N_ee'_0h\sum_{j=0}^{N}e'_{j+\frac{1}{2}}+N_eme'_0e'_{N+1}-N_ee_0u_0 \\
&\le \frac{1}{2}h\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2+\frac{N_e^2}{2}(e'_0)^2+\frac{m}{2}(e'_{N+1})^2+\frac{m}{2}N_e^2(e'_0)^2-N_ee_0u_0 \\
&= \frac{1}{2}h\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2+\frac{m}{2}(e'_{N+1})^2+\frac{N_e^2}{2}(1+m)(e'_0)^2-N_ere_0e'_0-N_e\beta e_0^2.
\end{aligned} \tag{4.35}
$$

Differentiating (4.21) and using (4.7), (4.34) and (4.35) gives

$$
\begin{aligned}
\dot{I}_{1h}(t) \ &= N\dot{E}_h^e(t)+\delta_1(\dot{\psi}_{1h}(t)+N_e\dot{\psi}_{2h}(t)) \\
&\le -Nr(e'_0)^2-\delta_1h\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2-\delta_1h\sum_{j=0}^{N}u_{j+\frac{1}{2}}^2-\delta_1m(e'_{N+1})^2+\delta_1(m+1)(1+r^2)(e'_0)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\delta_1(m+1)\beta^2 e_0^2+\delta_1 2(m+1)r\beta e_0 e_0' \\
&+\delta_1\frac{1}{2}h\sum_{j=0}^{N}(e'_{j+\frac12})^2+\delta_1\frac{m}{2}(e'_{N+1})^2+\delta_1\frac{N_e^2}{2}(1+m)(e_0')^2-\delta_1 N_e r e_0 e_0'-\delta_1 N_e\beta e_0^2 \\
&\le -\delta_1\frac{1}{2}h\sum_{j=0}^{N}(e'_{j+\frac12})^2-\delta_1 h\sum_{j=0}^{N}u^2_{j+\frac12}-\delta_1[N_e\beta-(m+1)\beta^2]e_0^2-\delta_1\frac{m}{2}(e'_{N+1})^2 \\
&-\delta_1[N_e r-2(m+1)r\beta]e_0e_0'-[Nr-\delta_1(m+1)(1+r^2)-\delta_1\frac{N_e^2}{2}(1+m)](e_0')^2 \\
&\le -\delta_1 E_h^e(t)-\delta_1[N_e\beta-(m+1)\beta^2-\frac{\beta}{2}]e_0^2-\delta_1[N_e r-2(m+1)r\beta]e_0e_0' \\
&-[Nr-\delta_1(m+1)(1+r^2)-\delta_1\frac{N_e^2}{2}(1+m)](e_0')^2.
\end{aligned}
$$

Since

$$
\begin{cases} N_e>\max\left\{(m+1)\beta+\frac12, 2(m+1)\beta\right\}, \\ N>\delta_1(m+1)\frac{2(1+r^2)+N_e^2}{2r}. \end{cases} \tag{4.36}
$$

which are defined in (2.9), we can get

$$
\dot I_{1h}(t)\le -\delta_1 E_h^e(t). \tag{4.37}
$$

Finding the derivative of $I_{2h}(t)$ defined by (4.21) with respect to t gives

$$
\dot I_{2h}(t)=3h\sum_{j=0}^{N}x_{j+\frac12}\widehat{w}''_{j+\frac12}z_{j+\frac12}+3h\sum_{j=0}^{N}x_{j+\frac12}\widehat{w}'_{j+\frac12}z'_{j+\frac12}. \tag{4.38}
$$

The first term on the right-hand side of (4.38) can be expressed as

$$
3h\sum_{j=0}^{N}x_{j+\frac12}\widehat{w}''_{j+\frac12}z_{j+\frac12}=3h\sum_{j=0}^{N}x_{j+\frac12}\delta_x z_{j+\frac12}z_{j+\frac12}=\frac32 z_{N+1}^2-3h\sum_{j=0}^{N}z^2_{j+\frac12}. \tag{4.39}
$$

The second term on the right-hand side of (4.38) can be expressed as

$$
3h\sum_{j=0}^{N}x_{j+\frac12}\widehat{w}'_{j+\frac12}z'_{j+\frac12}=3h\sum_{j=0}^{N}x_{j+\frac12}\widehat{w}'_{j+\frac12}\delta_x\widehat{w}'_{j+\frac12}=\frac32(\widehat{w}'_{N+1})^2-3h\sum_{j=0}^{N}(\widehat{w}'_{j+\frac12})^2. \tag{4.40}
$$

Plug (4.39) and (4.40) into (4.38) to get

$$
\dot I_{2h}(t)=\frac32(\widehat{w}'_{N+1})^2+\frac32 z_{N+1}^2-3h\sum_{j=0}^{N}(\widehat{w}'_{j+\frac12})^2-3h\sum_{j=0}^{N}z^2_{j+\frac12}. \tag{4.41}
$$

From (4.20), we arrive at

$$\dot{P}_h(t) = \dot{I}_{1h}(t) + \dot{I}_{2h}(t) \leq -\delta_1 E_h^e(t) + \frac{3}{2}(\widehat{w}'_{N+1})^2 + \frac{3}{2} z_{N+1}^2 - 3h \sum_{j=0}^{N} (\widehat{w}'_{j+\frac{1}{2}})^2 - 3h \sum_{j=0}^{N} z_{j+\frac{1}{2}}^2. \tag{4.42}$$

Finally, combining (4.10), (4.16) and (4.42), we obtain

$$\begin{aligned}
\dot{V}_h(t) &= t\dot{L}_h(t) + L_h(t) + \dot{P}_h(t) \\
&= -t\left(\frac{a}{m+\alpha a} - \varepsilon\right) z_{N+1}^2 - t\left(\frac{\alpha m}{m+\alpha a} - \varepsilon\right)(\widehat{w}'_{N+1})^2 - t z_0 e'_0 - t\delta_1 r (e'_0)^2 \\
&\quad -t\varepsilon[z_0^2 + (\widehat{w}'_0)^2] + \frac{1}{2}h\sum_{j=0}^{N}(\widehat{w}'_{j+\frac{1}{2}})^2 + \frac{1}{2}h\sum_{j=0}^{N} z_{j+\frac{1}{2}}^2 + \frac{1}{2}\frac{\eta^2}{m+\alpha a} + \delta_1 E_h^e(t) \\
&\quad +2h\varepsilon\sum_{j=0}^{N}\widehat{w}'_{j+\frac{1}{2}} z_{j+\frac{1}{2}} + \dot{P}_h(t) \\
&\leq -t\left(\frac{a}{m+\alpha a} - \varepsilon\right) z_{N+1}^2 - t\left(\frac{\alpha m}{m+\alpha a} - \varepsilon\right)(\widehat{w}'_{j+\frac{1}{2}})^2 + t\frac{\delta}{2}z_0^2 + t\frac{1}{2\delta}(e'_0)^2 - t\delta_1 r(e'_0)^2 \\
&\quad -t\varepsilon z_0^2 + \frac{1}{2}h\sum_{j=0}^{N}(\widehat{w}'_{j+\frac{1}{2}})^2 + \frac{1}{2}h\sum_{j=0}^{N} z_{j+\frac{1}{2}}^2 + \frac{a^2}{m+\alpha a} z_{N+1}^2 + \frac{m^2}{m+\alpha a}(\widehat{w}'_{N+1})^2 + \delta_1 E_h^e(t) \\
&\quad +h\sum_{j=0}^{N}(\widehat{w}'_{j+\frac{1}{2}})^2 + h\sum_{j=0}^{N} z_{j+\frac{1}{2}}^2 + \dot{P}_h(t) \\
&\leq -\left[t\left(\frac{a}{m+\alpha a} - \varepsilon\right) - \frac{a^2}{m+\alpha a} - \frac{3}{2}\right] z_{N+1}^2 - \left[t\left(\frac{\alpha m}{m+\alpha a} - \varepsilon\right) - \frac{m^2}{m+\alpha a} - \frac{3}{2}\right](\widehat{w}'_{N+1})^2 \\
&\quad -t\left(\varepsilon - \frac{\delta}{2}\right) z_0^2 - t\left(\delta_1 r - \frac{1}{2\delta}\right)(e'_0)^2.
\end{aligned} \tag{4.43}$$

Since $\varepsilon, \delta, \delta_1$ are chosen from (2.9), there exists a constant $T > 0$ such that

$$\dot{V}_h(t) \leq 0, \ t \geq T. \tag{4.44}$$

□

Now we are in a position to state the main result of this paper.

Theorem 4.8. *There exist strictly positive constants M and ω which are independent of the initial value such that the energy $F_h(t)$ of the semi-discretized finite difference scheme (3.9) decays uniformly exponentially:*

$$F_h(t) \leq M\mathrm{e}^{-\omega t} F_h(0), \ \forall t \geq 0, \tag{4.45}$$

where $F_h(t) = F_h^{\widehat{w}}(t) + F_h^e(t)$ *with*

$$\begin{cases} F_h^{\widehat{w}}(t) = \dfrac{1}{2}h\sum_{j=0}^{N}(\widehat{w}'_{j+\frac{1}{2}})^2 + \dfrac{1}{2}h\sum_{j=0}^{N}\delta_x\widehat{w}^2_{j+\frac{1}{2}} + \dfrac{1}{2}\dfrac{\eta^2}{m+\alpha a},\ \eta = a\delta_x\widehat{w}_{N+1} + m\widehat{w}'_{N+1}, \\ F_h^e(t) = \dfrac{1}{2}h\sum_{j=0}^{N}(e'_{j+\frac{1}{2}})^2 + \dfrac{1}{2}h\sum_{j=0}^{N}\delta_x e^2_{j+\frac{1}{2}} + \dfrac{\beta}{2}e_0^2 + \dfrac{m}{2}(e'_{N+1})^2. \end{cases} \tag{4.46}$$

Proof. For $T > C$, by (4.19), (4.13) and (4.30),

$$(t-C)L_h(t) \le V_h(t) \le V_h(T) \le (T+C)L_h(T) \le (T+C)L_h(0), \forall t > T.$$

Hence,

$$L_h(t) \le \frac{T+C}{t-C}L_h(0), \forall t > T. \tag{4.47}$$

This, together with Lemmas 4.3 and 4.4, gives

$$\begin{aligned} E_h(t) &\le C_2E_{1h}(t) \le \frac{C_2}{1-2\varepsilon}L_h(t) \le \frac{C_2}{1-2\varepsilon}\frac{T+C}{t-C}L_h(0) \\ &\le \frac{C_2}{1-2\varepsilon}\frac{T+C}{t-C}(1+2\varepsilon)E_{1h}(0) \le \frac{C_2}{1-2\varepsilon}\frac{T+C}{t-C}(1+2\varepsilon)C_1E_h(0). \end{aligned}$$

Hence, there exists T^* and $0 < \gamma < 1$ which are independent of h such that

$$E_h(t) \le \gamma E_h(0), \forall t > T^* > T.$$

Since $F_h(t) = E_h(t)$, we have

$$F_h(t) \le \gamma F_h(0), \forall t > T^* > T. \tag{4.48}$$

By the semigroup theory, there are $M, \omega > 0$ independent of h such that

$$F_h(t) \le M\mathrm{e}^{-\omega t}F_h(0),\ \forall t > 0. \tag{4.49}$$

□

It should be pointed out that the argument (4.48) to uniform exponential stability is an indirect Lyapunov argument by the property of the C_0-semigroups, which is very different to the direct Lyapunov functional approach used in [21].

5. Concluding remarks

In this paper, the uniformly exponential approximation for a vibration cable with tip mass under non-collocated boundary control is investigated. With only one pointwise non-collocated measurement, an observer-based stabilizing feedback control that makes the closed-loop a coupled PDE system is considered. By order reduction approach, we transform the closed-loop system into a low order differential algebraic system with introduced intermediate variables. It is found that a global Lyapunov functional can be constructed for this differential algebraic system. As a result, the exponential stability of the closed-loop system is then proved, which simplifies significantly the proof of a recent paper [19] and pave the way for the study of the semi-discretized models. We then discretize this differential algebraic system by finite-difference method. Having cancelled the intermediate variables, we obtain finally a semi-discrete scheme for the original closed-loop system. As a consequence, two semi-discrete finite difference schemes are equivalent and the semi-discrete scheme for differential algebraic system is much easy to be handled by the Lyapunov approach than the semi-discrete scheme for the original system, which has been demonstrated in previous studies. The uniform exponential stability of the semi-discrete schemes is finally established by an indirect Lyapunov functional method through a C_0-semigroup property, which is parallel to the proof of the continuous differential algebraic counterpart.

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