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EXISTENCE OF VARIATIONAL SOLUTIONS TO NONLOCAL EVOLUTION EQUATIONS *VIA* CONVEX MINIMIZATION

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Abstract. We prove existence of variational solutions for a class of nonlocal evolution equations whose prototype is the double phase equation

$$\begin{aligned}\partial_t u + P.V.\int_{\mathbb{R}^N} &\frac{|u(x,t)-u(y,t)|^{p-2}(u(x,t)-u(y,t))}{|x-y|^{N+ps}} \\ &+ a(x,y)\frac{|u(x,t)-u(y,t)|^{q-2}(u(x,t)-u(y,t))}{|x-y|^{N+qs}}\,\mathrm{d}y = 0.\end{aligned}$$

The approach of minimization of parameter-dependent convex functionals over space-time trajectories requires only appropriate convexity and coercivity assumptions on the nonlocal operator. As the parameter tends to zero, we recover variational solutions. Under further growth conditions, these variational solutions are global weak solutions. Further, this provides a direct minimization approach to approximation of nonlocal evolution equations.

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1. Introduction

1.1. The problem

In [3], Bögelein *et al.* prove existence of variational solutions for parabolic systems of the form

$$\begin{aligned}\partial_t u - \operatorname{div}(D_\xi f(x,u,Du)) &= 0 \text{ in } \Omega_\infty\\ u &= u_0 \text{ on } \partial_p\Omega_\infty\end{aligned}$$

where the convex function $f : \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^{Nn} \to \mathbb{R}$ satisfies a growth condition from below, viz,

$$C|\xi|^p \le f(x,z,\xi),$$

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for a.e. $x \in \Omega$, $z \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{Nn}$. The notation Ω_∞ stands for $\Omega \times (0,\infty)$. In particular, no growth condition from above is assumed. This is useful, for example, in the context of parabolic equations with p,q growth conditions for $p \leq q$, where weak solutions may not exist without imposing further conditions on the gap of p and q [2]. In this paper, we aim to extend the framework of variational solutions to parabolic fractional equations with time independent initial and boundary data

$$\begin{aligned} \partial_t u + P.V. \int_{\mathbb{R}^N} \frac{\mathrm{D}_\xi H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}y &= 0 \text{ in } \Omega_\infty \\ u = u_0 \text{ on } (\mathbb{R}^N \setminus \Omega) \times (0,\infty) &\cup \Omega \times \{0\} \end{aligned} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, and $\Omega_\infty = \Omega \times (0,\infty)$. The function $H = H(x,y,\xi)$ satisfies the following structure condition

$$H(x,y,\xi) \geq A\left(\frac{|\xi|}{|x-y|^s}\right)^p \tag{1.2}$$

$$H \text{ is a Caratheodory function which is convex in the variable } \xi. \tag{1.3}$$

We assume $1 < p < \infty$, $s \in (0,1)$ and $A > 0$. These structure conditions admit a variety of problems with non-standard growth such as

- $H(x,y,\xi) = \left(\frac{|\xi|}{|x-y|^s}\right)^p + a(x,y)\left(\frac{|\xi|}{|x-y|^r}\right)^q$, for $a(x,y) \geq 0$ for $1 < p < q$ and $r,s \in (0,1)$.
- $H(x,y,\xi) = \left(\frac{|\xi|}{|x-y|^s}\right)^p \log\left(1 + \left(\frac{|\xi|}{|x-y|^s}\right)\right)$.
- $H(x,y,\xi) = \left(\frac{|\xi|}{|x-y|^s}\right)^{a+b\sin\left(\log\,\log\left(\frac{|\xi|}{|x-y|^s}\right)\right)}$.

In particular, we would like to emphasize that in the double phase case, we obtain existence of variational solutions *without* any restrictions on the gap $q-p$, exactly as in [3].

1.2. Background

There has been a great surge in the study of regularity theory of fractional p-Laplace equations and their parabolic counterparts. However, a theory for parabolic fractional equations with non-standard growth, particularly with a double phase character, requires the development of an existence theory which can contend with the unbalanced growth condition given the possibility of the appearance of Lavrentiev phenomenon. The variational framework introduced in [3] seems to be the most suitable for this purpose.

The motivation for the notion of variational solutions in [3] came from [32] where a variational notion of solutions was introduced for an evolutionary minimal surface equation. Variational notions are also useful in the realm of stochastic partial differential equations, for example, see [38, 43].

Further, it is related to the De Giorgi conjectures on the construction of weak solutions to hyperbolic equations as a limit of minimizers of certain functionals [21]. Some forms of these conjectures were proved in [45, 47]. Unlike De Giorgi conjecture, the present paper, as in [3], considers parabolic problems. A minimization problem over trajectories in time is attached to the parabolic problem at hand. In particular, in the standard case, one considers the functionals

$$\mathcal{F}_\varepsilon(v) = \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} \frac{1}{2}|\partial_t v|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \frac{1}{\varepsilon} e^{-\frac{t}{\varepsilon}} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t,$$

for small $\varepsilon \in (0,1]$, over a certain class of functions $v : [0,T] \times \mathbb{R}^N \to \mathbb{R}$. This method is well-known in the physics literature where it is called *Weighted Dissipation Method*.

It turns out that the Euler-Lagrange equation associated to the minimization problem is formally found to be

$$-\varepsilon \partial_{tt} v + \partial_t v + \text{P.V.} \int\limits_{\mathbb{R}^N} \frac{|v(x,t) - v(y,t)|^{p-2}(v(x,t) - v(y,t))}{|x-y|^{N+ps}} \, \mathrm{d}y = 0.$$

Hence, formally speaking, the parabolic problem and its solutions would be recovered in the limit as $\varepsilon \to 0$. However, since we will be considering nonlocal operators with nonstandard growth conditions, we cannot pass to the Euler-Lagrange equation and we must stay at the level of minimization problems. This leads to the notion of variational solutions for nonlocal evolutionary problems, see (1.5).

The regularity theory of p, q growth problems was started by Marcellini in a series of novel papers [33–36]. There is a large body of work dealing with problems of (p,q)-growth as well as other nonstandard growth problems, for which we point to the surveys in [37, 39].

Coming back to the fractional p-Laplace equation, in the elliptic case, regularity theory of fractional p-Laplace equations has been studied extensively. For example, local boundedness and Hölder regularity in the framework of De Giorgi-Nash-Moser theory was worked out in [20, 22]. Moreover, higher regularity in Sobolev spaces with explicit integrability is obtained in [6]. On the other hand, explicit Hölder regularity of the solutions is obtained in [8]. Higher integrability by a nonlocal version of Gehring's Lemma was proved in [29]. For equations of nonstandard growth, the relevant works are [11, 14–18, 44]. For the case of linear equations, *i.e.*, $p = 2$, we refer to [12, 13, 19, 30].

In the case of parabolic counterparts of the fractional p-Laplace equations, local boundedness was proved in [48]. Hölder regularity has been proved in [1] followed by [31]. Explicit exponents in Hölder regularity have been obtained in [9].

We believe this is among the first works to deal with fractional evolutionary equations exhibiting unbalanced growth. In the companion article [41], we prove local boundedness for parabolic minimizers of (p,q)-fractional parabolic equations following [20]. In another article [25], existence of nonnegative variational solutions for doubly nonlinear nonlocal parabolic equations is proved.

1.3. Definition

Definition 1.1. Let Ω be an open bounded subset of $\mathbb{R}^N$ and $T \in (0,\infty]$. Suppose that H satisfies the assumptions (1.2) and (1.3) and let the time-independent Cauchy-Dirichlet data u_0 satisfy

$$u_0 \in W^{s,p}(\mathbb{R}^N), u_{|\Omega} \in L^2(\Omega) \text{ and } \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x,y,u_0(x) - u_0(y))}{|x-y|^N} \, \mathrm{d}x \, \mathrm{d}y < \infty. \tag{1.4}$$

By a variational solution to (1.1) we mean a function u such $u \in L^p(0,T; W^{s,p}(\mathbb{R}^N)) \cap C^0(0,T; L^2(\Omega))$, $u - u_0 \in L^p(0,T; W_0^{s,p}(\Omega))$ and

$$\int_0^T \int_\Omega \partial_t v \, (v-u) \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x,y,v(x,t) - v(y,t)) - H(x,y,u(x,t) - u(y,t))}{|x-y|^N} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t$$
$$\geq \frac{1}{2} \|(v-u)(\cdot,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2} \|(v-u)(\cdot,0)\|^2_{L^2(\Omega)}, \tag{1.5}$$

for all $v \in L^p(0,T; W^{s,p}(\mathbb{R}^N))$ and $\partial_t v \in L^2(0,T;\Omega)$ such that $v - u_0 \in L^p(0,T; W_0^{s,p}(\Omega))$.

Remark 1.2. In our definition of variational solution, both the solution and the comparison map have to match on $\Omega^c \times (0,T)$ and since we have assumed that the data on Ω^c is in $W^{s,p}(\mathbb{R}^N)$ we may cancel integrals of H over $\Omega^c \times \Omega^c$ on both sides to obtain the following equivalent form of the variational inequality:

$$\int_0^T \iint\limits_{C_\Omega} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \le \int_0^T \iint\limits_{C_\Omega} \frac{H(x,y,v(x,t)-v(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$
$$+ \int_0^T \int_\Omega \partial_t v \cdot (v-u) + \frac{1}{2}\left\|v(\cdot,0)-u_0(\cdot)\right\|^2_{L^2(\Omega)} - \frac{1}{2}\left\|v(\cdot,T)-u(\cdot,T)\right\|^2_{L^2(\Omega)}$$

where $C_\Omega = (\Omega^c \times \Omega^c)^c$.

Remark 1.3. Let us also mention that in the double phase case *i.e.* when H is of the form

$$H(x,y,\xi) = \left(\frac{|\xi|}{|x-y|^s}\right)^p + a(x,y)\left(\frac{|\xi|}{|x-y|^r}\right)^q$$

the condition on initial data (1.4) is satisfied if $u_0 \in W^{s,p}(\mathbb{R}^N) \cap L^2(\Omega)$ also satisfies:

$$\iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_0(x)-u_0(y)|^p}{|x-y|^{N+sp}} + a(x,y)\frac{|u_0(x)-u_0(y)|^q}{|x-y|^{N+rq}}\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

1.4. Main results

The following existence and uniqueness theorems are the main results of the paper.

Theorem 1.4. ***(Existence)*** *Let Ω be an open bounded subset of $\mathbb{R}^N$. Suppose that H satisfies the assumptions (1.2) and (1.3) and let the time-independent Cauchy-Dirichlet data u_0 satisfy (1.4). Then, there exists a variational solution to (1.1) in the sense of (1.5).*

Theorem 1.5. ***(Uniqueness)*** *If $\xi \mapsto H(x,y,\xi)$ is strictly convex then there is at most one variational solution in the sense of (1.5).*

Theorem 1.6. *Let Ω be an open bounded subset of $\mathbb{R}^N$. Suppose that H satisfies the assumptions (1.2) and (1.3) and let the time-independent Cauchy-Dirichlet data u_0 satisfy (1.4). Then,*

- *the variational solution u in the sense of (1.5) with $T \in (0,\infty]$, belongs to $C^{0,\frac{1}{2}}([0,T];L^2(\Omega))$. Further, $\partial_t u \in L^2(\Omega_T)$ and*
- *the following energy bounds are verified*

$$\int_0^T \int_\Omega |\partial_t u(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \le \Lambda, \text{ and} \tag{1.6}$$

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2} \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \le \Lambda. \tag{1.7}$$

for $0 \le t_1 < t_2 \le T < \infty$, where Λ is a constant depending on initial data.

Remark 1.7. The restriction $p > \dfrac{2N}{2s+N}$ is often seen as a natural lower bound on p since the proofs of existence that use the monotone operator methods require it on account of the Sobolev embedding. The proof here which uses variational techniques from [3] avoids this lower bound to obtain existence in the full range $p \in (1, \infty)$ exactly as achieved by [3] for local equations. In the paper [5], functionals with linear growth, that is, corresponding to $p = 1$ are also considered by stability methods. It would be of interest to study the stability method for the nonlocal operator.

2. Preliminaries

2.1. Notations

We begin by collecting the standard notation that will be used throughout the paper:

- We shall denote N to be the space dimension. We shall denote by $z = (x, t)$ a point in $\mathbb{R}^N \times (0, T)$.
- We shall alternately use $\dfrac{\partial f}{\partial t}, \partial_t f, f'$ to denote the time derivative of f.
- Let Ω be an open bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ and for $0 < T \leq \infty$, let $\Omega_T \coloneqq \Omega \times (0, T)$.
- Integration with respect to either space or time only will be denoted by a single integral $\displaystyle\int$ whereas integration on $\Omega \times \Omega$ or $\mathbb{R}^N \times \mathbb{R}^N$ will be denoted by a double integral $\displaystyle\iint$.

2.2. Sobolev spaces

Let $1 < p < \infty$, we denote by $p' = p/(p-1)$ the conjugate exponent of p. Let Ω be an open subset of $\mathbb{R}^N$. We define the *Sobolev-Slobodekiĭ* space, which is the fractional analogue of Sobolev spaces.

$$W^{s,p}(\Omega) = \left\{ \psi \in L^p(\Omega) : [\psi]_{W^{s,p}(\Omega)} < \infty \right\}, s \in (0, 1),$$

where the seminorm $[\cdot]_{W^{s,p}(\Omega)}$ is defined by

$$[\psi]_{W^{s,p}(\Omega)} = \left(\iint\limits_{\Omega\times\Omega} \frac{|\psi(x) - \psi(y)|^p}{|x - y|^{N+ps}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{p}}.$$

The space when endowed with the norm $\|\psi\|_{W^{s,p}(\Omega)} = \|\psi\|_{L^p(\Omega)} + [\psi]_{W^{s,p}(\Omega)}$ becomes a Banach space. The space $W_0^{s,p}(\Omega)$ denotes the completion of the space of smooth compactly supported functions $C_c^\infty(\Omega)$ in the norm of $W^{s,p}(\mathbb{R}^N)$.

Let I be an interval and let V be a separable, reflexive Banach space, endowed with a norm $\|\cdot\|_V$. We denote by V^* its topological dual space. Let v be a mapping such that for a.e. $t \in I$, $v(t) \in V$. If the function $t \mapsto \|v(t)\|_V$ is measurable on I, then v is said to belong to the Banach space $L^p(I; V)$ if $\displaystyle\int_I \|v(t)\|_V^p \, \mathrm{d}t < \infty$. It is well known that the dual space $L^p(I; V)^*$ can be characterized as $L^{p'}(I; V^*)$.

2.3. Mollification in time

Throughout the paper, we will use the following mollification in time. Let Ω be an open subset of $\mathbb{R}^N$. For $T > 0$, $v \in L^1(\Omega_T)$, $v_0 \in L^1(\Omega)$ and $h \in (0, T]$, we define

$$[v]_h(\cdot, t) = e^{-\frac{t}{h}} v_0 + \frac{1}{h} \int_0^t e^{\frac{\varkappa - t}{h}} v(\cdot, \varkappa) \, d\varkappa, \tag{2.1}$$

for $t \in [0, T]$. The convergence properties of mollified functions have been collected in Appendix A.

2.4. Auxiliary results

We collect the following standard results which will be used in the course of the paper. We begin with a general result on convex minimization.

Proposition 2.1. *([42], Thm. 2.3) Let X be a closed affine subset of a reflexive Banach space and let $\mathcal{F} : X \to (-\infty, \infty]$. Assume the following:*

1. *For all $\Lambda \in \mathbb{R}$, the sublevel set $\{x \in X : \mathcal{F}[u] < \Lambda\}$ is sequentially weakly precompact, i.e., if for a sequence $(u_j) \subset X$, $\mathcal{F}[u_j] < \Lambda$, then u_j has a weakly convergent subsequence.*
2. *For all sequences $(u_j) \subset X$ with $u_j \rightharpoonup u$ in X-weak, it holds that*

$$\mathcal{F}[u] \leq \liminf_{j\to\infty} \mathcal{F}[u_j].$$

Then, the minimization problem

$$\textbf{Minimize } \mathcal{F}[\mathbf{u}] \textbf{ over all } \mathbf{u} \in \mathbf{X}$$

has a solution.

We will need the following general result on weak lower semicontinuity of functionals.

Proposition 2.2. *([10], Cor. 3.9) Let E be a Banach space. Assume that $\phi : E \to (-\infty, \infty]$ is convex and lower semicontinuous in the strong topology. Then ϕ is lower semicontinuous in the weak topology.*

We have the following Sobolev-type inequality ([23], Thm. 6.5).

Theorem 2.3 ([23])**.** *Let $s \in (0, 1)$ and $1 \leq p < \infty$, $sp < N$ and let $\kappa^* = \dfrac{N}{N - sp}$, then for any $g \in W^{s,p}(\mathbb{R}^N)$, we have*

$$\|g\|^p_{L^{\kappa^* p}} \leq C \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|g(x) - g(y)|^p}{|x - y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y. \tag{2.2}$$

We will require the compact embedding as follows.

Proposition 2.4. *([26], Prop. 2.1) Assume $N \geq 1$, $1 \leq p \leq \infty$ and $0 < s < 1$. Let Ω be a bounded extension domain. When $sp < N$, the embedding $W^{s,p}(\Omega) \mapsto L^r(\Omega)$ is compact for $r \in [1, p_*)$, and when $sp \geq N$, the same embedding is compact for $r \in [1, \infty)$. The theorem also holds for $W^{s,p}_0(\Omega)$ for any bounded domain in $\mathbb{R}^N$.*

We also need a Poincaré inequality for Gagliardo seminorms.

Proposition 2.5. *([7], Lem. 2.4) Let $1 \leq p < \infty$, $s \in (0, 1)$ and Ω is an open and bounded set in $\mathbb{R}^N$. Then, for every $u \in W^{s,p}_0(\Omega)$, it holds that*

$$||u||^p_{L^p(\Omega)} \leq C(N, s, p, \Omega)\, [u]^p_{W^{s,p}(\mathbb{R}^N)}$$

We end this subsection with the following result about parabolic Banach spaces.

Theorem 2.6. *([46], pp. 106, Prop. 1.2) Let the Banach space V be dense and continuously embedded in the Hilbert space H; identify $H = H^*$ so that $V \hookrightarrow H \hookrightarrow V^*$. The Banach space $W_p(0, T) = \{u \in L^p(0, T; V) : \partial_t v \in L^p(0, T; V^*)\}$ is contained in $C([0, T]; H)$. Moreover, if $u \in W_p(0, T)$, then $|u(\cdot)|^2_H$ is absolutely continuous on*

$[0,T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|u(t)|_H^2 = 2\partial_t u(t)u(t) \ \textit{a.e. } t \in [0,T],$$

and there is a constant C such that

$$\|u\|_{C([0,T];H)} \le C\,\|u\|_{W_p(0,T)}\,, u \in W_p.$$

Moreover, if $u, v \in W_p(0,T)$, then $\langle u(\cdot)\ , v(\cdot)\rangle_H$ is absolutely continuous on $[0,T]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\ , v(t)\rangle_H = \partial_t u(t)v(t) + \partial_t v(t)u(t), \ \textit{a.e. } t \in [0,T].$$

3. Existence of variational solutions

In this section, we will prove the existence of variational solutions to the following nonlocal evolution equation with time independent initial-boundary data

$$\begin{cases} \partial_t u + Lu = 0 & \text{in } \Omega_T \\ u = u_0 & \text{on } (\mathbb{R}^N \setminus \Omega) \times (0,T) \cup \Omega \times \{0\} \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ and $\Omega_T = \Omega \times (0,T)$. We recall that the operator L is a nonlocal operator whose explicit structure is as follows.

$$Lu(x,t) = P.V. \int_{\mathbb{R}^N} \frac{D_\xi H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}y$$

where $H = H(x,y,\xi) : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying (1.2) and (1.3). We recall the assumptions on initial-boundary data. The initial-boundary $u_0 : \mathbb{R}^N \to \mathbb{R}$ is assumed to be time independent and $u_0 \in W^{s,p}(\mathbb{R}^N) \cap L^2(\Omega)$. We further assume that

$$\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y \le \Lambda < \infty \tag{3.1}$$

for some constant $\Lambda > 0$.

Finally, we recall our definition of solution to the equation (1.1). We denote by $W^{s,p}_{u_0}(\Omega)$ the following space of functions

$$W^{s,p}_{u_0}(\Omega) = \{g : \mathbb{R}^N \to \mathbb{R} : g \in W^{s,p}(\mathbb{R}^N),\ g - u_0 \in W^{s,p}_0(\Omega)$$

We further define the space $L^p(0,T;W^{s,p}_{u_0}(\Omega))$ as the space of functions

$$L^p(0,T;W^{s,p}_{u_0}(\Omega)) = \Big\{ g \in L^p(0,T;W^{s,p}(\mathbb{R}^N)) : g - u_0 \in L^p(0,T;W^{s,p}_0(\Omega)) \\ \text{and } \|u(\cdot,\tau) - u_0(\cdot)\|_{L^2(\Omega)} \to 0 \text{ as } \tau \to 0 \Big\}. \tag{3.2}$$

We repeat the definition of variational solutions.

Definition 3.1. Let $T \in (0, \infty]$. We say that a function

$$u \in L^p(0,T; W^{s,p}_{u_0}(\Omega)) \cap C([0,T]; L^2(\Omega))$$

is a variational solution to (1.1) if for any comparison function

$$v \in L^p(0,T; W^{s,p}_{u_0}(\Omega)) \text{ with } \partial_t v \in L^2(\Omega \times (0,T))$$

the following variational inequality is verified:

$$\begin{aligned} \int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t &\le \int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,v(x,t)-v(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &+ \int_0^T \int_\Omega \partial_t v\cdot(v-u) + \frac{1}{2}\,\|v(\cdot,0)-u_0(\cdot)\|^2_{L^2(\Omega)} - \frac{1}{2}\,\|v(\cdot,T)-u(\cdot,T)\|^2_{L^2(\Omega)} \end{aligned} \tag{3.3}$$

Remark 3.2. (Initial Data) Since the initial data u_0 is a valid comparison function and $H \ge 0$, The inequality (3.3) with the admissible comparision function $v(\cdot,t) = u_0(\cdot)$ for all $t \in (0,T)$ and (3.1) implies that for any $\tau > 0$

$$0 \le \frac{1}{2}\,\|u(\cdot,\tau)-u_0(\cdot)\|^2_{L^2(\Omega)} \le \tau\Lambda \to 0 \text{ as } \tau \to 0+\,.$$

So variational solutions pick up the data at the initial time $t = 0$ in the L^2 sense.

3.1. Lower bound for the functional

In this subsection, we derive a lower bound on the nonlinear and nonlocal operator by the use of Poincaré inequality ([7], Lem. 2.4).

Note that for almost every time $t \ge 0$, $(u-u_0)(\cdot,t) \in W^{s,p}_0(\Omega)$. Therefore, by Poincaré inequality (Prop. 2.5) in the space $W^{s,p}_0(\Omega)$, we get for almost every $t \in [0,T]$,

$$\begin{aligned} ||u-u_0||^p_{L^p(\Omega)} &\le C[u-u_0]^p_{W^{s,p}(\mathbb{R}^N)} \\ &\le C[u]^p_{W^{s,p}(\mathbb{R}^N)} + C[u_0]^p_{W^{s,p}(\mathbb{R}^N)} \\ &\le C \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C[u_0]^p_{W^{s,p}(\mathbb{R}^N)}, \end{aligned}$$

where we made use of (1.2). As a result, we obtain

$$\begin{aligned} ||u||^p_{L^p(\mathbb{R}^N)} &\le ||u-u_0||^p_{L^p(\Omega)} + ||u_0||^p_{L^p(\mathbb{R}^N)} \\ &\le C_1 \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C_2||u_0||^p_{W^{s,p}(\mathbb{R}^N)}, \end{aligned} \tag{3.4}$$

for some positive constants C_1, C_2. Once again, due to (1.2), we get

$$||u||^p_{W^{s,p}(\mathbb{R}^N)} \le C_1 \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C_2||u_0||^p_{W^{s,p}(\mathbb{R}^N)}. \tag{3.5}$$

3.2. Localization in time

We will show that the restriction of a variational solution to a smaller time interval is still a variational solution *i.e.* if u is a variational solution in Ω_T then u is also a variational solution in Ω_τ for any $0 < \tau < T$. We follow the arguments given in Section 2.4 of [4] except that the admissibility criteria for test functions and passing to limits now require the mollification theorems as proved in Appendix A. We provide the proof below.

We begin by fixing a parameter $\theta \in (0, \tau)$ which we will send to zero. Let ξ_θ be the following cutoff

$$\xi_\theta(\tau) = \chi_{[0,\tau-\theta]}(t) + \frac{\tau - t}{\theta}\chi_{(\tau-\theta,\tau]}(t).$$

For any comparison map $v \in L^p(0, \tau; W^{s,p}_{u_0}(\Omega))$, $\partial_t v \in L^2(\Omega_\tau)$ with

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x, y, v(x, \cdot) - v(y, \cdot))}{|x - y|^N}\, \mathrm{d}x\, \mathrm{d}y \in L^1(0, \tau)$$

(if it is not integrable then the variational inequality is satisfied trivially) we take

$$v_\theta = \xi_\theta v + (1 - \xi_\theta)[u]_h$$

as a comparison map - here we extend $\xi_\theta v$ by 0 to Ω_T and invoke Proposition A.1 and Lemma A.2 to obtain the admissibility of v_θ as a comparison map. Therefore,

$$\begin{aligned}\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x, y, u(x, t) - u(y, t))}{|x - y|^N}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t \le \int_0^T \int_\Omega \partial_t v_\theta \cdot (v_\theta - u)\, \mathrm{d}x\, \mathrm{d}t \\ + \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x, y, v_\theta(x, t) - v_\theta(y, t))}{|x - y|^N}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}t \\ + \frac{1}{2}\|v(0) - u_0\|^2_{L^2} - \frac{1}{2}\|[u]_h(T) - u(T)\|^2_{L^2}\end{aligned} \tag{3.6}$$

Now, we deal with the first term on the right hand side (RHS). Observe that we can write

$$\begin{aligned}\partial_t v_\theta \cdot (v_\theta - u) = \xi'_\theta(v - [u]_h) \cdot ([u]_h - u) + \xi'_\theta \xi_\theta (v - [u]_h)^2 \\ + (\xi_\theta \partial_t v + (1 - \xi_\theta)\partial_t [u]_h) \cdot (\xi_\theta (v - [u]_h) + [u]_h - u).\end{aligned}$$

As a result, we write the first term on the RHS as

$$\begin{aligned}\iint_{\Omega_T} \partial_t v_\theta \cdot (v_\theta - u)\, \mathrm{d}x\, \mathrm{d}t = \underbrace{\iint_{\Omega\times(0,\tau-\theta)} \partial_t v (v - u)\, \mathrm{d}x\, \mathrm{d}t}_{I_\theta} + \underbrace{\iint_{\Omega\times(\tau,T)} \partial_t [u]_h ([u]_h - u)\, \mathrm{d}x\, \mathrm{d}t}_{II_\theta} \\ + \underbrace{\iint_{\Omega\times(\tau-\theta,\tau)} \xi'_\theta \xi_\theta (v - [u]_h)^2\, \mathrm{d}x\, \mathrm{d}t}_{III_\theta} + \underbrace{\iint_{\Omega\times(\tau-\theta,\tau)} \xi'_\theta (v - [u]_h) \cdot ([u]_h - u)\, \mathrm{d}x\, \mathrm{d}t}_{IV_\theta}\end{aligned}$$

$$+\underbrace{\iint\limits_{\Omega\times(\tau-\theta,\tau)}(\xi_\theta\partial_t v+(1-\xi_\theta)\partial_t[u]_h)\cdot(\xi_\theta(v-[u]_h)+[u]_h-u)\,\mathrm{d}x\,\mathrm{d}t}_{V_\theta}$$

We have

$$\begin{aligned}
\lim_{\theta\downarrow 0} I_\theta &= \iint\limits_{\Omega\times(0,\tau)}\partial_t v(v-u)\,\mathrm{d}x\,\mathrm{d}t.\\
\lim_{\theta\downarrow 0} III_\theta &= \lim_{\theta\downarrow 0}\int_{\tau-\theta}^{\tau}\frac{1}{2}\partial_t(\xi_\theta^2)\int\limits_{\Omega}(v-[u]_h)^2\,\mathrm{d}x\,\mathrm{d}t\\
&= -\frac{1}{2}\int_\Omega (v-[u]_h)(x,\tau)^2\,\mathrm{d}x.\\
\lim_{\theta\downarrow 0} IV_\theta &\le \|([u]_h-u)(v-[u]_h)(\cdot,\tau)\|_{L^1}.\\
\lim_{\theta\downarrow 0} V_\theta &= 0.
\end{aligned}$$

In all, we get

$$\begin{aligned}
\limsup_{\theta\to 0+}\int_0^T\int_\Omega \partial_t v_\theta(v_\theta-u)\,\mathrm{d}x\,\mathrm{d}t \le \int_0^\tau\int_\Omega \partial_t v(v-u)\,\mathrm{d}x\,\mathrm{d}t + \int_\tau^T\int_\Omega \partial_t[u]_h([u]_h-u)\,\mathrm{d}x\,\mathrm{d}t\\
-\frac{1}{2}\left\|v(\tau)-[u]_h(\tau)\right\|_{L^2}^2 + \left\|([u]_h-u)(v-[u]_h)(\tau)\right\|_{L^1}.
\end{aligned}$$

By convexity of H we have

$$\begin{aligned}
\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,v_\theta(x,\cdot)-v_\theta(y,\cdot))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y \le \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,v(x,\cdot)-v(y,\cdot))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\
+\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,[u]_h(x,\cdot)-[u]_h(y,\cdot))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y \in L^1(\tau-\theta,\tau),
\end{aligned}$$

when restricted to the interval $(\tau-\theta,\tau)$, which implies that

$$\int_{\tau-\theta}^{\tau}\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,v_\theta(x,t)-v_\theta(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y \to 0 \text{ as } \theta\downarrow 0.$$

and hence

$$\begin{aligned}
\lim_{\theta\downarrow 0}\int_0^T\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,v_\theta(x,t)-v_\theta(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y = \int_0^\tau\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,v(x,t)-v(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\
+\int_\tau^T\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,[u]_h(x,t)-[u]_h(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Thus letting $\theta \to 0+$ in the variational inequality (3.6) yields

$$\begin{aligned}\int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \leq{}& \int_0^\tau\int_\Omega \partial_t v(v-u)\,\mathrm{d}x\,\mathrm{d}t + \int_\tau^T\int_\Omega \partial_t[u]_h([u]_h-u)\,\mathrm{d}x\,\mathrm{d}t\\ &-\frac{1}{2}\|v(\tau)-[u]_h(\tau)\|_{L^2}^2 + \|([u]_h-u)(v-[u]_h)(\tau)\|_{L^1}\\ &+\int_0^\tau \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,v(x,t)-v(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\ &+\int_\tau^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,[u]_h(x,t)-[u]_h(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\ &+\frac{1}{2}\|v(0)-u_0\|_{L^2}^2 - \frac{1}{2}\|[u]_h(T)-u(T)\|_{L^2}^2\,. \end{aligned}\tag{3.7}$$

We note that

- $\partial_t[u]_h\cdot([u]_h-u)\leq 0$ by Proposition A.1.
- $[u]_h \to u$ in $L^\infty(0,T;L^2(\Omega))$ as $h\to 0+$ by Lemma A.2. In particular, $\|v(\tau)-[u]_h(\tau)\|_{L^2}^2 \to \|v(\tau)-u(\tau)\|_{L^2}^2$ and $\|([u]_h-u)(v-[u]_h)(\tau)\|_{L^1}\to 0$ as $h\to 0+$.
- $\displaystyle\lim_{h\downarrow 0}\int_\tau^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,[u]_h(x,t)-[u]_h(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y = \int_\tau^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y$ by Theorem A.3.

Dropping the non-positive terms involving $\partial_t[u]_h\cdot([u]_h-u)$ and $\|[u]_h(T)-u(T)\|_{L^2}^2$, passing to the limit as $h\to 0$ in (3.7) and recalling that v was an arbitrary comparison map it follows that u is a variational solution in Ω_τ.

3.3. Proof of Theorem 1.6

Proof. (Proof of Thm. 1.6) Let $\tau\in\mathbb{R}\cap(0,T]$, then u is a solution in the smaller time interval $[0,\tau]$ as shown in Section 3.2. We use in (1.5) the test function $v=[u]_h$, we get, after dropping the $L^2(\Omega)$ norm at time τ and noting that the $L^2(\Omega)$ norm at time 0 is zero.

$$\begin{aligned}-\int_0^\tau\int_\Omega \partial_t[u]_h\cdot([u]_h-u)\,\mathrm{d}x\,\mathrm{d}t &\leq \int_0^\tau \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,[u]_h(x,t)-[u]_h(y,t))}{|x-y|^N} - \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ &\leq \int_0^\tau \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \left[\frac{H(x,y,u(x,\cdot)-u(y,\cdot))}{|x-y|^N}\right]_h(t) - \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ &\leq -h\int_0^\tau \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \partial_t\left[\frac{H(x,y,u(x,\cdot)-u(y,\cdot))}{|x-y|^N}\right]_h(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ &= h\left(\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y - \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\left[\frac{H(x,y,u(x,\cdot)-u(y,\cdot))}{|x-y|^N}\right]_h(\tau)\,\mathrm{d}x\,\mathrm{d}y\right). \end{aligned}$$

Now, using the identity $-([u]_h - u) = h\partial_t [u]_h$ on the left hand side (LHS) and using the lower bound (3.5), we get the uniform estimate

$$\int_0^\tau \int_\Omega (\partial_t [u]_h)^2 \,\mathrm{d}x\,\mathrm{d}t \le \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)}, \tag{3.8}$$

so that $\partial_t u$ exists with $\partial_t u \in L^2(\Omega_\tau)$ for all $\tau \in (0,T]$. Moreover, the following quantitative estimate holds.

$$\int_0^\tau \int_\Omega (\partial_t u)^2 \,\mathrm{d}x\,\mathrm{d}t \le \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)}. \tag{3.9}$$

Next, let t_1, t_2 be two positive numbers such that $0 \le t_1 < t_2 \le T$, then

$$\begin{aligned}
\|u(t_2)-u(t_1)\|^2_{L^2(\Omega)} &= \int_\Omega \left| \int_{t_1}^{t_2} \partial_t u(\cdot,t)\,\mathrm{d}t\right|^2 \mathrm{d}x \\
&\le |t_2-t_1| \int_{t_1}^{t_2}\int_\Omega |\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\overset{(3.9)}{\le} |t_2-t_1| \left(\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)}\right).
\end{aligned} \tag{3.10}$$

Therefore,

$$\begin{aligned}
\|u(\cdot,t)\|^2_{L^2(\Omega)} &\le 2\|u_0\|^2_{L^2(\Omega)} + 2\|u(t)-u(0)\|^2_{L^2(\Omega)} \\
&\overset{(3.10)}{\le} 2\|u_0\|^2_{L^2(\Omega)} + 2|t| \left(\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)}\right).
\end{aligned} \tag{3.11}$$

As a consequence, $u \in C^{0,\frac{1}{2}}([0,\tau]; L^2(\Omega))$ for all $\tau \in (0,T]$.

It remains to prove (1.7). For any $\tau \in (0,T]$ and for any $v \in L^p(0,\tau; W^{s,p}_{u_0}(\Omega))$ with $\partial_t v \in L^2(0,\tau;\Omega)$, the following variational inequality holds

$$\begin{aligned}
\int_0^\tau \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t &\le \int_0^\tau \int_\Omega \partial_t u(v-u)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^\tau \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,v(x,t)-v(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned} \tag{3.12}$$

For t_1, t_2 with $0 \le t_1 < t_2 \le \tau$, define the cutoff function $\xi(t) = \xi_{t_1,t_2}(t) := \chi_{[0,t_1]}(t) + \dfrac{t_2-t}{t_2-t_1}\chi_{(t_1,t_2)}(t)$ and choose $v = u + \xi_{t_1,t_2}([u]_h - u)$ as a companion map in the variational inequality (3.12). The function v is admissible since u is admissible and the cutoff function ξ_{t_1,t_2} is Lipschitz.

With v as chosen and by using the convexity of H, we get

$$\begin{aligned}\int_0^{t_2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t &\le \int_0^{t_2}\int_\Omega \xi_{t_1,t_2}\partial_t u([u]_h-u)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^{t_2}(1-\xi_{t_1,t_2})\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ &\quad+\int_0^{t_2}\xi_{t_1,t_2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,[u]_h(x,t)-[u]_h(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ &\le \int_0^{t_2}\int_\Omega \xi_{t_1,t_2}\partial_t u([u]_h-u)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^{t_2}(1-\xi_{t_1,t_2})\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ &\quad+\int_0^{t_2}\xi_{t_1,t_2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\left[\frac{H(x,y,u(x,\cdot)-u(y,\cdot))}{|x-y|^N}\right]_h(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\end{aligned}$$

As a consequence, we have

$$\begin{aligned}0&\le\int_0^{t_2}\xi_{t_1,t_2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\left[\frac{H(x,y,u(x,\cdot)-u(y,\cdot))}{|x-y|^N}\right]_h(t)-\frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ &\qquad\int_0^{t_2}\int_\Omega\xi_{t_1,t_2}\partial_t u([u]_h-u)\,\mathrm{d}x\,\mathrm{d}t\\ &\le -h\int_0^{t_2}\xi_{t_1,t_2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\partial_t\left[\frac{H(x,y,u(x,\cdot)-u(y,\cdot))}{|x-y|^N}\right]_h(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\ &\qquad-h\int_0^{t_2}\int_\Omega\xi_{t_1,t_2}\partial_t u\partial_t[u]_h\,\mathrm{d}x\,\mathrm{d}t\\ &= h\int_0^{t_2}\xi'_{t_1,t_2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\left[\frac{H(x,y,u(x,\cdot)-u(y,\cdot))}{|x-y|^N}\right]_h(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t+h\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad-h\int_0^{t_2}\int_\Omega\xi_{t_1,t_2}\partial_t u\partial_t[u]_h\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Rearranging the previous inequality, we get

$$\begin{aligned}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\left[\frac{H(x,y,u(x,\cdot)-u(y,\cdot))}{|x-y|^N}\right]_h(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t &\le \iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad-\int_0^{t_2}\int_\Omega\xi_{t_1,t_2}\partial_t u\partial_t[u]_h\,\mathrm{d}x\,\mathrm{d}t. \qquad (3.13)\end{aligned}$$

Passing to the limit as $h \to 0$ in (3.13) and using (3.9), we obtain

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x, y, u(x,t) - u(y,t))}{|x-y|^N} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \leq 2 \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x, y, u_0(x) - u_0(y))}{|x-y|^N} \,\mathrm{d}x\,\mathrm{d}y + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)}, \tag{3.14}$$

which completes the proof of Theorem 1.6. □

3.4. Convex minimization

Let $T \in (0, \infty]$. Define the reflexive Banach space

$$K := \left\{ v \in L^p(0, T; W^{s,p}(\mathbb{R}^N)) : \partial_t v \in L^2(0, T; \Omega) \right\},$$

with norm

$$\|v\|_K := \|v\|_{L^p(0,T;\mathbb{R}^N)} + \|v\|_{L^p(0,T;W^{s,p}(\mathbb{R}^N))} + \|\partial_t v\|_{L^2(\Omega_T)}.$$

Define a subclass K_{u_0} of K as

$$K_{u_0} := \left\{ v \in L^p(0, T; W^{s,p}_{u_0}(\Omega)) : \partial_t v \in L^2(0, T; \Omega) \right\},$$

where $L^p(0, T; W^{s,p}_{u_0}(\Omega))$ is defined in (3.2).

For $\varepsilon \in (0, 1]$ we define the weighted energy

$$F_\varepsilon(v) := \frac{1}{2} \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} |\partial_t v|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon} \int_0^T \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} e^{-\frac{t}{\varepsilon}} \frac{H(x, y, v(x,t) - v(y,t))}{|x-y|^N} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

Let $K^\varepsilon_{u_0}$ denote the subclass of K_{u_0} with finite F_ε-energy, then $K^\varepsilon_{u_0}$ is non-empty since the time-independent extension of u_0 satisfies

$$\begin{aligned} F_\varepsilon(u_0) &:= \frac{1}{\varepsilon} \int_0^T \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} e^{-\frac{t}{\varepsilon}} \frac{H(x, y, u_0(x) - u_0(y))}{|x-y|^N} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &= \left(1 - e^{-\frac{T}{\varepsilon}}\right) \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x, y, u_0(x) - u_0(y))}{|x-y|^N} \,\mathrm{d}x\,\mathrm{d}y < \infty. \end{aligned}$$

We note that $F_\varepsilon : K_{u_0} \to (-\infty, \infty]$ satisfies the following properties on the subset $K^\varepsilon_{u_0}$

- **Strict convexity of F_ε:** F_ε is strictly convex due to convexity of H in ξ and the strict convexity of $Z \mapsto |Z|^2$.
- **Weak lower-semicontinuity of F_ε:** In order to show weak lower-semicontinuity of F_ε, we will make use of Proposition 2.2, that is, we will show that F_ε is strongly lower semicontinuous and convex. Convexity of F_ε was shown in the previous step. It remains to show that F_ε is strongly lower semicontinuous.
  We begin by writing

$$F_\varepsilon(v) = \frac{1}{2} \int_0^T \int_\Omega e^{-\frac{t}{\varepsilon}} |\partial_t v|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon} \int_0^T \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} e^{-\frac{t}{\varepsilon}} \frac{H(x, y, v(x,t) - v(y,t))}{|x-y|^N} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$:= F_1(v) + F_2(v).$$

Let u_j be strongly convergent to u in K_ε. Then $w_j \in L^1(0,T;\Omega)$ defined by $w_j(t,x) := e^{-\frac{t}{\varepsilon}}|\partial_t u_j(x,t)|^2$ converges strongly in $L^1(0,T;\Omega)$. Therefore, we obtain

$$F_1(u) = \lim_{j\to\infty} F_1(u_j) = \liminf_{j\to\infty} F_1(u_j). \tag{3.15}$$

On the other hand, u_j converging strongly to u in K_ε also implies that the sequence $\tilde{w}_j \in L^p(0,T;\mathbb{R}^N)$ defined by $\tilde{w}_j(t,x) := e^{-\frac{t}{p\varepsilon}}u_j$ converges strongly to the function $\tilde{w} = e^{-\frac{t}{p\varepsilon}}u$ in $L^p(0,T;\mathbb{R}^N)$. As a result, after multiplying by $e^{\frac{t}{p\varepsilon}}$, $u_j(t,x) \to u(t,x)$ for a.e. $t \in (0,T)$ and $x \in \mathbb{R}^N$, for a subsequence. By continuity of H in the last variable, we have

$$\frac{H(x,y,u_j(x,t)-u_j(x,t))}{|x-y|^N} \to \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N} \; a.e.\, x,y \in \mathbb{R}^N, t \in (0,\infty).$$

Finally, by Fatou's lemma,

$$\begin{aligned} F_2(u) &= \frac{1}{\varepsilon}\int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} e^{-\frac{t}{\varepsilon}} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &\le \frac{1}{\varepsilon}\liminf_{j\to\infty}\int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} e^{-\frac{t}{\varepsilon}} \frac{H(x,y,u_j(x,t)-u_j(x,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \liminf_{j\to\infty} F_2(u_j). \end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16), we get

$$\begin{aligned} F_\varepsilon(u) &= F_1(v) + F_2(v) \\ &\le \liminf_{j\to\infty} F_1(u_j) + \liminf_{j\to\infty} F_2(u_j) \\ &\le \liminf_{j\to\infty} F_\varepsilon(u_j), \end{aligned}$$

where in the last inequality, we have used the super-additivity of lim inf.

We want to prove that the functional F_ε has a minimizer in the class $K^\varepsilon_{u_0}$.
We begin by proving an estimate for elements of $K^\varepsilon_{u_0}$ in terms of their F_ε-energy and u_0.
Let $u \in K^\varepsilon_{u_0}$. Integrating (3.4) in time over $(0,T)$, we obtain

$$\begin{aligned} ||u||^p_{L^p(0,T;\mathbb{R}^N)} &\le C_1 \int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C_2\,T\,||u_0||^p_{W^{s,p}(\mathbb{R}^N)} \\ &\le C_1 \int_0^T \frac{e^{-\frac{t}{\varepsilon}}}{\varepsilon} \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C_2\,T\,||u_0||^p_{W^{s,p}(\mathbb{R}^N)} \\ &\le C_1 F_\varepsilon(u) + C_2\,T\,||u_0||^p_{W^{s,p}(\mathbb{R}^N)}. \end{aligned} \tag{3.17}$$

Further

$$\int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y \le F_\varepsilon(u). \tag{3.18}$$

Combining (3.17) and (3.18), we obtain

$$\|u\|_{L^p(0,T;W^{s,p}(\mathbb{R}^N))} \le C_1 F_\varepsilon(u) + C_2\,T\,||u_0||^p_{W^{s,p}(\mathbb{R}^N)}. \tag{3.19}$$

Further, combining (3.19) with (1.6), we receive

$$\|u\|_K \le \|u\|_{L^p(0,T;W^{s,p}(\mathbb{R}^N))} + \|\partial_t u\|_{L^2(\Omega_T)} \le C_1 F_\varepsilon(u) + C_2\,T\,||u_0||^p_{W^{s,p}(\mathbb{R}^N)}. \tag{3.20}$$

Now, we consider a minimizing sequence $(u_j) \subset K^\varepsilon_{u_0}$. Then,

$$\begin{aligned}\lim_{j\to\infty} F_\varepsilon(u_j) = \inf_{v\in K^\varepsilon_{u_0}} F_\varepsilon(v) \le F_\varepsilon(u_0) \\ &= \frac{1}{\varepsilon}\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{e^{-\frac{t}{\varepsilon}} H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &= (1-e^{-\frac{T}{\varepsilon}}) \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{e^{-\frac{t}{\varepsilon}} H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y \\ &< (1-e^{-\frac{T}{\varepsilon}})\Lambda. \end{aligned} \tag{3.21}$$

As a consequence of (3.20) and (3.21), we obtain the uniform-in-j bound for the sequence u_j in $\|\cdot\|_K$. Therefore, due to the reflexivity of K, we can obtain a subsequence (still labelled with j) so that u_j weakly converges to some $u_\varepsilon \in K$.

Before proving that u_ε is the desired minimizer, we need to prove that $u_\varepsilon \in K^\varepsilon_{u_0}$. Since, the function sequence u_j belongs to $K^\varepsilon_{u_0}$, the sequence and hence its limit remains fixed for all $(x,t) \in \Omega^c \times (0,T)$. Hence, the limit u_ε satisfies the boundary condition.

We also need to show that u_ε satisfies the initial condition in the L^2-sense. For this purpose, we observe that

$$\|u_j(t_2)-u_j(t_1)\|^2_{L^2(\Omega)} \le (t_2-t_1)\|\partial_t u_j\|^2_{L^2(\Omega_T)} \tag{3.22}$$

$$\le C(t_2-t_1) \tag{3.23}$$

for any $0 \le t_1 < t_2 \le T$ where the last inequality is due to the uniform bound in K for u_j. Choosing $t_1 = 0$ and $t_2 = t$ and integrating over $(0,h)$ for some $h > 0$, we get

$$\begin{aligned}\frac{1}{h}\int_0^h \|u(t)-u_0\|^2_{L^2(\Omega)}\,\mathrm{d}t &\le \liminf_{j\to\infty}\frac{1}{h}\int_0^h \|u_j(t)-u_j(0)\|^2_{L^2(\Omega)}\,\mathrm{d}t \\ &\le C\frac{1}{h}\int_0^h t\,\mathrm{d}t = O(h) \to 0 \text{ as } h\to 0, \end{aligned}\tag{3.24}$$

which proves the condition.

Now, since F_ε is weakly lower semicontinuous, it holds that

$$F_\varepsilon(u_\varepsilon) \le \liminf_{j\to\infty} F_\varepsilon(u_j) = \inf_{v\in K^\varepsilon_{u_0}} F_\varepsilon(v). \tag{3.25}$$

This proves that u_ε is the desired minimizer. We record this as a lemma below.

Lemma 3.3. *For any $\varepsilon \in (0,1]$, the functional F_ε admits a unique minimizer $u_\varepsilon \in K^\varepsilon_{u_0}$.*

3.5. A reformulation of minimality condition

In this subsection we provide a reformulation of the minimality criteria for use in energy estimates below as in Section 5.2 of [4]. We begin by fixing $\epsilon \in (0,1]$ and considering a test function $\phi \in L^p(0,T;W_0^{s,p}(\Omega))$ such that $\partial_t \phi \in L^2(\Omega_T)$ and

$$\int_0^T \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x,y,(u_\epsilon+\phi)(x,t)-(u_\epsilon+\phi)(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t < \infty. \tag{3.26}$$

Let $\xi \in W^{1,\infty}(0,T), 0 \le \xi \le 1$ and $0 < \delta \le e^{-T/\epsilon}$. Put $\sigma(t) = \delta e^{t\epsilon}$ and

$$\tilde{\phi}_{\epsilon,\delta}(x,t) = \sigma(t)\phi(x,t).$$

We assume that either $\phi(0) = 0$ or $\xi(0) = 0$ and set

$$v_{\epsilon,\delta}(x,t) = u_\epsilon(x,t) + \tilde{\phi}_{\epsilon,\delta}(x,t).$$

Then $v_{\epsilon,\delta} \in L^p(0,T;W^{s,p}_{u_0}(\Omega))$ and $\partial_t v_{\epsilon,\delta} \in L^2(\Omega_T)$. Further, by construction, $v_{\epsilon,\delta}$ is a convex combination of u_ϵ and $u_\epsilon + \phi$ on every time slice; this together with (3.26) shows that $F_\epsilon(v_{\epsilon,\delta}) < \infty$. It follows that $v_{\epsilon,\delta} \in K^\epsilon_{u_0}$. Indeed, the convexity of H implies that

$$\begin{aligned}
&\int_0^T e^{-\frac{t}{\epsilon}} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x,y,v_\epsilon(x,t)-v_\epsilon(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad \le \int_0^T e^{-\frac{t}{\epsilon}} \iint_{\mathbb{R}^N \times \mathbb{R}^N} (1-\sigma(t))\frac{H(x,y,u_\epsilon(x,t)-u_\epsilon(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad\quad + \int_0^T e^{-\frac{t}{\epsilon}} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \sigma(t)\frac{H(x,y,(u_\epsilon+\phi)(x,t)-(u_\epsilon+\phi)(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad \le \int_0^T e^{-\frac{t}{\epsilon}} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x,y,u_\epsilon(x,t)-u_\epsilon(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad\quad + \int_0^T \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x,y,(u_\epsilon+\phi)(x,t)-(u_\epsilon+\phi)(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad < \infty.
\end{aligned}$$

Next, we invoke the minimality of u_ϵ to get

$$F_\epsilon(u_\epsilon) \le F_\epsilon(v_{\epsilon,\delta}) < \infty$$

which we rewrite using the convexity as above of H to get

$$\frac{\delta}{\epsilon}\int_0^T \xi(t) \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_\epsilon(x,t)-u_\epsilon(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \leq \int_0^T e^{-\frac{t}{\epsilon}} \int_\Omega \frac{1}{2}\delta^2|\partial_t(e^{\frac{t}{\epsilon}}\xi\phi)|^2 + \delta\partial_t u_\epsilon \partial_t(e^{\frac{t}{\epsilon}}\xi\phi)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\frac{\delta}{\epsilon}\int_0^T \xi(t) \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,(u_\epsilon+\phi)(x,t)-(u_\epsilon+\phi)(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

We multiply the preceding estimate by ϵ/δ and let $\delta \to 0+$ to get

$$\begin{aligned}\int_0^T \xi(t) \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_\epsilon(x,t)-u_\epsilon(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ \leq \int_0^T \xi(t) \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,(u_\epsilon+\phi)(x,t)-(u_\epsilon+\phi)(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ + \int_0^T \int_\Omega \xi\partial_t u_\epsilon \phi\,\mathrm{d}x\,\mathrm{d}t + \epsilon\int_0^T\int_\Omega \xi'\partial_t u_\epsilon\phi + \xi\partial_t u_\epsilon \partial_t\phi\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{3.27}$$

We will use this reformulation in the next subsection.

3.6. Energy estimates

In this subsection, we will obtain uniform-in-ε bounds for the minimizers u_ε of F_ε in $K^\varepsilon_{u_0}$ so that we may obtain a subsequence that converges to a variational solution.

We begin by defining $[u_\varepsilon]_h$ with initial data u_0, then it holds that $[u_\varepsilon]_h \in L^p(0,T;W^{s,p}_{u_0}(\Omega))$ and $\partial_t[u_\varepsilon]_h \in L^2(\Omega_T)$. Moreover, we have the identities

$$\begin{aligned}\partial_t[u_\varepsilon]_h &= \frac{1}{h}(u_\varepsilon - [u_\varepsilon]_h),\\ \partial_t[u_\varepsilon]_h(0) &= \frac{1}{h}(u(0)-[u_\varepsilon]_h(0)) = 0.\end{aligned}$$

Also, notice that by Theorem A.3, it holds that

$$\begin{aligned}\int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,(u_\varepsilon-h\partial_t[u_\varepsilon]_h)(x,t)-(u_\varepsilon-h\partial_t[u_\varepsilon]_h)(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ = \int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,[u_\varepsilon]_h(x,t)-[u_\varepsilon]_h(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t < \infty.\end{aligned}$$

This motivates us to take $\phi := -h\partial_t[u_\varepsilon]_h$ as a test function in the reformulation (3.27). This yields the following chain of arguments.

$$h\int_0^T\int_\Omega \{(\xi+\varepsilon\xi')\partial_t u_\varepsilon\cdot\partial_t[u_\varepsilon]_h + \underbrace{\varepsilon\xi\partial_t u_\varepsilon\cdot\partial_{tt}[u_\varepsilon]_h}_{M}\}\,\mathrm{d}x\,\mathrm{d}t$$

$$\begin{aligned}
&\le \int_0^T \xi(t) \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,[u_\varepsilon]_h(x,t)-[u_\varepsilon]_h(y,t)) - H(x,y,u_\varepsilon(x,t)-u_\varepsilon(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\overset{(A.6)}{\le} \int_0^T \xi(t) \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \left[\frac{H(x,y,u_\varepsilon(x,\cdot)-u_\varepsilon(y,\cdot))}{|x-y|^N}\right]_h(t) - \frac{H(x,y,u_\varepsilon(x,t)-u_\varepsilon(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&= - h\int_0^T \xi(t) \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \partial_t\left[\frac{H(x,y,u_\varepsilon(x,\cdot)-u_\varepsilon(y,t\cdot))}{|x-y|^N}\right]_h(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \qquad (3.28)
\end{aligned}$$

Meanwhile

$$\begin{aligned}
M &= \varepsilon\xi\partial_t u_\varepsilon\cdot\partial_{tt}[u_\varepsilon]_h \\
&= \varepsilon\xi\partial_t[u_\varepsilon]_h\cdot\partial_{tt}[u_\varepsilon]_h + \varepsilon\xi\partial_t(u_\varepsilon-[u_\varepsilon]_h)\cdot\partial_{tt}[u_\varepsilon]_h \\
&= \frac{\varepsilon}{2}\xi\partial_t\left|\partial_t[u_\varepsilon]_h\right|^2 + \frac{\varepsilon}{h}\xi\left|\partial_t[u_\varepsilon]_h-\partial_t u_\varepsilon\right|^2 \\
&\ge \frac{\varepsilon}{2}\xi\partial_t\left|\partial_t[u_\varepsilon]_h\right|^2. \qquad (3.29)
\end{aligned}$$

Combining (3.28) and (3.29) followed by dividing by h, we receive

$$\begin{aligned}
&\int_0^T\int\limits_\Omega \{(\xi+\varepsilon\xi')\partial_t u_\varepsilon\cdot\partial_t[u_\varepsilon]_h + \frac{\varepsilon}{2}\xi\partial_t\left|\partial_t[u_\varepsilon]_h\right|^2\}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\le -\int_0^T \xi(t) \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \partial_t\left[\frac{H(x,y,u_\varepsilon(x,\cdot)-u_\varepsilon(y,t\cdot))}{|x-y|^N}\right]_h(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \qquad (3.30)
\end{aligned}$$

Now, in (3.30), we choose $\xi(t)\equiv 1$ to get

$$\begin{aligned}
&\int_0^T\int\limits_\Omega \partial_t u_\varepsilon\cdot\partial_t[u_\varepsilon]_h\,\mathrm{d}x\,\mathrm{d}t \\
&\le -\int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \partial_t\left[\frac{H(x,y,u_\varepsilon(x,\cdot)-u_\varepsilon(y,\cdot))}{|x-y|^N}\right]_h(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \int_0^T\int\limits_\Omega \frac{\varepsilon}{2}\partial_t\left|\partial_t[u_\varepsilon]_h\right|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\le \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N}\left[\frac{H(x,y,u_\varepsilon(x,\cdot)-u_\varepsilon(y,\cdot))}{|x-y|^N}\right]_h(T)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \int\limits_\Omega \frac{\varepsilon}{2}\left|\partial_t[u_\varepsilon]_h(0)\right|^2\,\mathrm{d}x - \int\limits_\Omega \frac{\varepsilon}{2}\left|\partial_t[u_\varepsilon]_h(T)\right|^2\,\mathrm{d}x \\
&\le \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)}. \qquad (3.31)
\end{aligned}$$

where in the last inequality, we use the fact that $\partial_t[u_\varepsilon]_h(0)=0$, we drop the negative term and we also use the lower bound on the functional from (3.5). Now, passing to the limit as $h\to 0$ by using Lemma A.2, we obtain the following bound for $\partial_t u_\varepsilon$:

$$\int_0^T \int_\Omega |\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \le \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)}. \tag{3.32}$$

Further, by the same arguments as in the proof of Theorem 1.6 (see the inequalities (3.10) and (3.11)), we get uniform bounds for u_ε in $L^2(\Omega_T)$ as well as $C^{0,\frac{1}{2}}([0,T];L^2(\Omega))$ that come along with the following estimates.

For t_1, t_2 two positive numbers such that $0 \le t_1 < t_2 \le T$, we have

$$\|u_\varepsilon(\cdot,t_2)-u_\varepsilon(\cdot,t_1)\|^2_{L^2(\Omega)} \le |t_2-t_1| \left(\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)} \right). \tag{3.33}$$

Therefore,

$$\|u_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} \le 2\|u_0\|^2_{L^2(\Omega)} + 2|t| \left(\iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)} \right). \tag{3.34}$$

Now, we want to establish uniform bounds for u_ε in the space $L^p(0,T;W^{s,p}(\mathbb{R}^N))$. For this purpose, we will choose $\xi(t) = \xi_{t_1,t_2}(t) := \chi_{[0,t]}(t) + \dfrac{t_2-t}{t_2-t_1}\chi_{(t_1,t_2)}(t)$ in the inequality (3.30) and integrate by parts the LHS and the second term in the RHS to get

$$\begin{aligned}\int_0^T \int_\Omega \xi_{t_1,t_2}(t)\partial_t u_\varepsilon \cdot \partial_t [u_\varepsilon]_h \,\mathrm{d}x\,\mathrm{d}t \le{}& \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_0^T \xi'_{t_1,t_2}(t) \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \left[\frac{H(x,y,u_\varepsilon(x,\cdot)-u_\varepsilon(y,\cdot))}{|x-y|^N}\right]_h (t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &+ \int_0^T \xi'_{t_1,t_2}(t) \int_\Omega \left\{ \frac{\varepsilon}{2}|\partial_t [u_\varepsilon]_h|^2 - \varepsilon \partial_t u_\varepsilon \cdot \partial_t [u_\varepsilon]_h \right\} \mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.35}$$

Taking limits as $h \downarrow 0$, we get

$$\begin{aligned}\int_0^T \int_\Omega \xi_{t_1,t_2}(t)|\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \le{}& \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y \\ &+ \int_0^T \xi'_{t_1,t_2}(t) \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_\varepsilon(x,t)-u_\varepsilon(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &+ \int_0^T \xi'_{t_1,t_2}(t) \int_\Omega \frac{\varepsilon}{2}|\partial_t u_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.36}$$

This gives

$$\begin{aligned}\int_{t_1}^{t_2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u_\varepsilon(x,t)-u_\varepsilon(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t &\le (t_2-t_1)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+\int_{t_1}^{t_2}\int_\Omega\frac{\varepsilon}{2}|\partial_t u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\overset{(3.9)}{\le}\Big(t_2-t_1+\frac{\varepsilon}{2}\Big)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad+C\varepsilon\,\|u_0\|_{W^{s,p}(\mathbb{R}^N)},\end{aligned}\tag{3.37}$$

where C is some generic constant. Now, using (3.5), we get

$$\begin{aligned}\|u_\varepsilon\|_{L^p(0,T;W^{s,p}(\mathbb{R}^N))}\le C_1\Big(t_2-t_1+\frac{\varepsilon}{2}\Big)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\\ +C_2\varepsilon\|u_0\|_{W^{s,p}(\mathbb{R}^N)},\end{aligned}\tag{3.38}$$

where again C_1, C_2 are some generic constants. Hence, we have also obtained a uniform bound for u_ε in $L^p(0,T;W^{s,p}(\mathbb{R}^N))$.

3.7. Weak limit of minimizers

In this subsection, we will give the proof of Theorem 1.4.

Proof. (Proof of Thm. 1.4)

The sequence (u_ε) is bounded in $L^2(\Omega_T)$, $C^{0,\frac{1}{2}}([0,T];L^2(\Omega))$ and in $L^p(0,T;W^{s,p}(\mathbb{R}^N))$. The sequence of derivatives $(\partial_t u_\varepsilon)$ is also bounded uniformly in $L^2(\Omega_T)$. Therefore, there is $u\in L^2(\Omega_T)\cap C^{0,\frac{1}{2}}([0,T];L^2(\Omega))\cap L^p(0,T;W^{s,p}(\mathbb{R}^N))$ with $\partial_t u\in L^2(\Omega_T)$ such that

$$\begin{aligned}u_\varepsilon &\rightharpoonup u \text{ in } L^2(\Omega_T)-\text{weak},\\ u_\varepsilon &\rightharpoonup u \text{ in } L^p_t(W^{s,p}_x)-\text{weak},\\ \partial_t u_\varepsilon &\rightharpoonup \partial_t u \text{ in } L^2(\Omega_T)-\text{weak}.\end{aligned}$$

Then, by weak lower semicontinuity in $L^2(\Omega_T)$, we have

$$\begin{aligned}\int_0^T\int_\Omega|\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}t &\le \liminf_{\varepsilon\downarrow 0}\int_0^T\int_\Omega|\partial_t u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\\ &\overset{(3.32)}{\le}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t+C\|u_0\|_{W^{s,p}(\mathbb{R}^N)}.\end{aligned}\tag{3.39}$$

Similarly for $0\le t_1<t_2\le T$, we have

$$\int_{t_1}^{t_2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\le\liminf_{\varepsilon\downarrow 0}\int_{t_1}^{t_2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u_\varepsilon(x,t)-u_\varepsilon(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\tag{3.40}$$

$$\overset{(3.37)}{\leq} \left(t_2 - t_1 + \frac{\varepsilon}{2}\right) \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x, y, u_0(x) - u_0(y))}{|x-y|^N} \,\mathrm{d}x\,\mathrm{d}y + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)}. \tag{3.41}$$

Finally,

$$\|u\|_{L^p_t W^{s,p}_x} \leq \liminf_{\varepsilon \downarrow 0} \|u_\varepsilon\|_{L^p_t W^{s,p}_x} \leq C_1 \left(t_2 - t_1 + \frac{\varepsilon}{2}\right) \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x, y, u_0(x) - u_0(y))}{|x-y|^N} \,\mathrm{d}x\,\mathrm{d}y + C_2\|u_0\|_{W^{s,p}(\mathbb{R}^N)}, \tag{3.42}$$

where again C_1, C_2 are some generic constants.

Further, $u(0) = u_0$ holds in the L^2 sense due to uniform Hölder bound in $C^{0,\frac{1}{2}}_t L^2_x$. Also, $u(x,t) = u_0(x)$ for all $t \in (0, T]$ and $x \in \mathbb{R}^N \setminus \Omega$.

Therefore, it remains to prove that the weak limit u is, in fact, a variational solution. For this purpose, we will choose a companion map $v \in L^p(0, T; W^{s,p}_{u_0}(\Omega))$ with $\partial_t v \in L^2(\Omega_T)$ and

$$\int_0^T \iint\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x, y, v(x,t) - v(y,t))}{|x-y|^N} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t < \infty.$$

Now, for a fixed $\theta \in \left(0, \dfrac{T}{2}\right)$, define

$$\xi_\theta(t) = \begin{cases} \dfrac{t}{\theta} & \text{if } t \in [0, \theta), \\ 1 & \text{if } t \in [\theta, T-\theta], \\ \dfrac{1}{\theta}(T-t) & \text{if } t \in [T-\theta, T]. \end{cases}$$

Fix ε and consider $\phi = v - u_\varepsilon$, then $\phi \in L^p_t W^{s,p}_{0,x}$ with $\partial_t \phi \in L^2(\Omega_T)$. We will use ϕ and $\xi = \xi_\theta$ in the reformulated variational inequality (3.27).

$$\begin{aligned}
\int_0^T \int\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x, y, u_\varepsilon(x,t) - u_\varepsilon(y,t))}{|x-y|^N} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t &\leq \underbrace{\int_0^T (1 - \xi_\theta(t)) \int\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x, y, u_\varepsilon(x,t) - u_\varepsilon(y,t))}{|x-y|^N} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t}_{I_\varepsilon} \\
&\leq \underbrace{\int_0^T \int_\Omega \xi_\theta\, \partial_t u_\varepsilon \cdot (v - u_\varepsilon) \,\mathrm{d}x\,\mathrm{d}t}_{II_\varepsilon} \\
&\leq \underbrace{\int_0^T \xi_\theta(t) \int\limits_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x, y, v(x,t) - v(y,t))}{|x-y|^N} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t}_{III} \\
&\leq \underbrace{\epsilon \int_0^T \int_\Omega \xi' \partial_t u_\epsilon (v - u_\varepsilon) + \xi \partial_t u_\epsilon \partial_t (v - u_\varepsilon) \,\mathrm{d}x\,\mathrm{d}t}_{IV_\varepsilon}.
\end{aligned}$$

We need to pass to the limit as $\varepsilon \to 0$ in each of the labeled terms on the RHS. Observe that III is independent of ε so it remains fixed. Whereas, for IV_ε, due to the uniform bounds on $\partial_t u_\varepsilon$ and u_ε, the integral is bounded so that the whole term goes to zero due to presence of ε in the front, *i.e.*, $\lim_{\varepsilon\downarrow 0} IV_\varepsilon = 0$. It remains to look at I_ε and II_ε.

Now, for $\theta > \varepsilon$, we have

$$\begin{aligned}
I_\varepsilon &\leq \int_{[0,\theta]\cup[T-\theta,T]} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_\varepsilon(x,t)-u_\varepsilon(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\overset{(3.11)}{\leq} C_1\left(2\theta+\frac{\varepsilon}{2}\right)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C_2\varepsilon\|u_0\|_{W^{s,p}(\mathbb{R}^N)} \\
&\leq C_1\theta \iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C_2\theta\|u_0\|_{W^{s,p}(\mathbb{R}^N)},
\end{aligned} \tag{3.43}$$

where C_1 and C_2 are some generic constants. As for II_ε, we have

$$\begin{aligned}
II_\varepsilon &= \int_0^T\int_\Omega \xi_\theta\partial_t v\cdot(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_0^T\int_\Omega \xi_\theta\partial_t|v-u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T\int_\Omega \xi_\theta\partial_t v\cdot(v-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2\theta}\int_{T-\theta}^T\int_\Omega |v-u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{3.44}$$

where in the last equality we have integrated by parts in time.

We estimate the second term on the RHS of (3.44) using Minkowski's inequality as

$$\begin{aligned}
\frac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t &\leq \left\{\left(\frac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_0|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} + \left(\frac{1}{2\theta}\int_0^\theta\int_\Omega |u_0-u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}}\right\}^2 \\
&\overset{(3.33)}{\leq} \left\{\left(\frac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_0|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} + \left(\frac{\theta}{4}\left(\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)}\right)\right)^{\frac{1}{2}}\right\}^2
\end{aligned} \tag{3.45}$$

Hence by (3.44) and (3.45) along with weak convergence in L^2, we can pass to the limit as $\varepsilon \to 0$ to get

$$\begin{aligned}
\liminf_{\varepsilon\downarrow 0} II_\varepsilon &\leq \int_0^T\int_\Omega \xi_\theta\partial_t v\cdot(v-u)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2\theta}\int_{T-\theta}^T\int_\Omega |v-u|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \left\{\left(\frac{1}{2\theta}\int_0^\theta\int_\Omega |v-u_0|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{2}} + \left(\frac{\theta}{4}\left(\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)}\right)\right)^{\frac{1}{2}}\right\}^2.
\end{aligned} \tag{3.46}$$

Combining all these components

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad\le \liminf_{\varepsilon\downarrow 0} \int_0^T \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_\varepsilon(x,t)-u_\varepsilon(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad\le \int_0^T \int_\Omega \xi_\theta \partial_t v\cdot(v-u)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^T \xi_\theta(t) \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,v(x,t)-v(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad + C_1\theta \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C_2\theta\|u_0\|_{W^{s,p}(\mathbb{R}^N)} \\
&\qquad - \frac{1}{2\theta}\int_{T-\theta}^T \int_\Omega |v-u_0|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \left\{ \left(\frac{1}{2\theta}\int_0^\theta \int_\Omega |v-u_0|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac12} + \left(\frac{\theta}{4}\left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_0(x)-u_0(y))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y + C\|u_0\|_{W^{s,p}(\mathbb{R}^N)}\right)\right)^{\frac12} \right\}^2 .
\end{aligned} \tag{3.47}$$

Now, we obtain the variational inequality for u when we pass to the limit as $\theta \to 0$ where we use Lebesgue's differentiation theorem in t. This concludes the proof of Theorem 1.4.

□

4. Uniqueness of variational solutions

Let $T \in (0,\infty]$. In this section, we will show that a variational solution is unique provided that the function $\xi \mapsto H(x,y,\xi)$ is strictly convex proving Theorem 1.5.

To this end, we suppose that $u_1, u_2 \in L^p(0,T;W^{s,p}(\mathbb{R}^N)) \cap C^0([0,T];L^2(\Omega))$ are two different variational solutions. We add the two variational inequalities (1.5) corresponding to u_1 and u_2, to obtain

$$\begin{aligned}
\sum_{i=1}^2 \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_i(x,t)-u_i(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t &\le 2\int_0^T \int_\Omega \partial_t v\,(v-w)\,\mathrm{d}x\,\mathrm{d}t \\
&+ 2\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,v(x,t)-v(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \|v(\cdot,0)-u_0\|^2_{L^2(\Omega)} ,
\end{aligned} \tag{4.1}$$

where we have dropped the negative term from the right hand side and we have defined $w = \dfrac{u_1+u_2}{2}$. We would like to use w as a comparison function v, however the definition of variational solutions does not admit the requisite regularity of $\partial_t w \in L^2(0,T;\Omega)$.

To overcome this, we make use of $v = [w]_h$ with $v_0 = u_0$. Then, by Lemma A.2, we get

$$[w]_h \in L^p(0,T;W^{s,p}(\mathbb{R}^N)) \text{ with } \partial_t [w]_h \in L^2(0,T;\Omega)$$

and $[w]_h = u_0$ on $(\Omega^c \times (0,T)) \cup (\Omega \times \{0\})$.

Now, choosing $v = [w]_h$ in (4.1), we get

$$\begin{aligned}\sum_{i=1}^{2} \int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_i(x,t)-u_i(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t &\le 2\int_0^T \int_\Omega \partial_t v\,(v-w)\,\mathrm{d}x\,\mathrm{d}t \\ &+ 2\int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,[w]_h(x,t)-[w]_h(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 2I_h + 2II_h. \end{aligned} \tag{4.2}$$

By Lemma A.2, we have

$$I_h = -\frac{1}{h}\int_0^T \int_\Omega |[w]_h - w|^2\,dz \le 0. \tag{4.3}$$

For II_h, by convexity, we have

$$II_h \le \frac{1}{2}\int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,[u_1]_h(x,t)-[u_1]_h(y,t)) + H(x,y,[u_2]_h(x,t)-[u_2]_h(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t < \infty.$$

Therefore, $\dfrac{H(x,y,[w]_h(x,t)-[w]_h(y,t))}{|x-y|^N} \in L^1(0,T;\mathbb{R}^N\times\mathbb{R}^N)$. As a result, by Theorem A.3, we get

$$\lim_{h\to 0} II_h = \int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H\left(x,y,\frac{(u_1+u_2)(x,t)}{2} - \frac{(u_1+u_2)(y,t)}{2}\right)}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \tag{4.4}$$

Combining (4.2), (4.2) and (4.4), we get by passing to the limit as $h\to 0$,

$$\sum_{i=1}^{2} \int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_i(x,t)-u_i(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \tag{4.5}$$

$$\le 2\int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H\left(x,y,\frac{(u_1+u_2)(x,t)}{2} - \frac{(u_1+u_2)(y,t)}{2}\right)}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

$$< \sum_{i=1}^{2} \int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u_i(x,t)-u_i(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \tag{4.6}$$

In the last step, we have used strict convexity of H in ξ which leads to the desired contradiction. This finishes the proof of the uniqueness of variational solutions to (1.1).

Remark 4.1. In the case of the nonlocal parabolic p-Laplace equation, the strict convexity is guaranteed by the condition on the exponent, viz., $p > 1$, which we also assume for our nonlocal operator with nonstandard growth. However, the condition on the exponent $p > 1$ is not sufficient to ensure strict convexity of the nonlocal operator with nonstandard growth. For a two sided p-growth condition, strict convexity would be ensured provided the nonlocal integrand is also known to be twice differentiable in ξ. However, the differentiability requirement on the integrand can be weakened. In the local case, a classic paper that considers nonconvex integrands is [24]. Certain equivalent conditions for uniform convexity of the integrand in the local case are given in Proposition 2.2 of [24]. It would be of interest to consider corresponding results for integrands in the nonlocal case.

5. Variational solutions are parabolic minimizers

A notion of parabolic Q-minimizers was introduced by Weiser in [49], which has been useful to study regularity such as higher integrability [40] for parabolic equations, even in measure metric settings [28]. We define here a notion of parabolic minimizer for nonlocal parabolic equations of the form (1.1) and prove that variational solutions as in (1.5) are parabolic minimizers, although the converse might require additional time regularity.

Definition 5.1. Let $T \in (0, \infty]$. A measurable function $u : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ is called a parabolic minimizer for the equation (1.1) if $u \in L^p(0,T; W^{s,p}(\mathbb{R}^N))$ and $u - u_0 \in L^p(0,T; W_0^{s,p}(\Omega))$, and the following minimality condition holds:

$$\int_0^T \int_\Omega u \partial_t \phi \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t \\ \leq \int_0^T \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{H(x,y,(u+\phi)(x,t)-(u+\phi)(y,t))}{|x-y|^N} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t,$$

whenever $\phi \in C_0^\infty(\Omega \times (0,T))$

Theorem 5.2. *If u is a variational solution of (1.1) in the sense of 1.5, it is also a parabolic minimizer in the sense of Definition 5.1.*

Proof. Let $T > 0$ be fixed and $\phi \in C_0^\infty(\Omega_T)$. We would like to choose $v = u + \eta\phi$ for small $\eta \in [0,1]$ as a comparison function in (1.5) but this may not be admissible. We overcome this by mollification in time. Choose $v = v_h = [u]_h + \eta[\phi]_h$ with u_0 and $\phi_0 = 0$ respectively.

Now, observe that

$$\begin{aligned}
&\int_0^T \int_\Omega \partial_t v \,(v_h - u) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad = \int_0^T \int_\Omega \partial_t [u]_h \,([u]_h - u) \,\mathrm{d}x \,\mathrm{d}t + \eta \int_0^T \int_\Omega \partial_t [u]_h \,[\phi]_h \,\mathrm{d}x \,\mathrm{d}t + \eta \int_0^T \int_\Omega \partial_t [\phi]_h \,([v]_h - u) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad = -\frac{1}{h} \int_0^T \int_\Omega |[u]_h - h|^2 - \eta \int_0^T \int_\Omega [u]_h \,\partial_t [\phi]_h \,\mathrm{d}x \,\mathrm{d}t \\
&\qquad + \eta \int_0^T \int_\Omega \partial_t [\phi]_h \,([v]_h - u) \,\mathrm{d}x \,\mathrm{d}t + \eta \int_\Omega [u]_h(T) \,[\phi]_h(T) \,\mathrm{d}x \\
&\quad \leq \eta \int_0^T \int_\Omega \partial_t [\phi]_h \,(\eta [\phi]_h - u) \,\mathrm{d}x \,\mathrm{d}t + \eta \int_\Omega [u]_h(T) \,[\phi]_h(T) \,\mathrm{d}x
\end{aligned}$$

In the second equality, we have used Lemma A.2 and integration by parts. Plugging this in the minimality condition, we get

$$\begin{aligned}
&\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad\le \eta \int_0^T \int_\Omega \partial_t[\phi]_h\,(\eta[\phi]_h - u)\,\mathrm{d}x\,\mathrm{d}t + \eta \int_\Omega [u]_h(T)\,[\phi]_h(T)\,\mathrm{d}x \\
&\qquad + \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,v_h(x,t)-v_h(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \frac{1}{2}\left\|(v_h-u)(\cdot,T)\right\|^2_{L^2(\Omega)}.
\end{aligned}$$

Dropping the non-positive term from the right hand side and passing to the limit as $h\to 0$ by Lemma A.2 and Theorem A.3, we get

$$\begin{aligned}
&\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad\le \eta \int_0^T \int_\Omega \partial_t\phi\,(\eta\phi - u)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,(u+\eta\phi)(x,t)-(u+\eta\phi)(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad\le \eta \int_0^T \int_\Omega \partial_t\phi\,(\eta\phi - u)\,\mathrm{d}x\,\mathrm{d}t + (1-\eta)\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad + \eta \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,((u+\phi)(x,t)-(u+\phi)(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}$$

In the second inequality, we have used convexity of $H(x,y,\xi)$ in ξ. Now, we subtract the second term on the right hand side from both sides, followed by dividing by η, to obtain

$$\begin{aligned}
&\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad\le \int_0^T \int_\Omega \partial_t\phi\,(\eta\phi - u)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,((u+\phi)(x,t)-(u+\phi)(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned}$$

Finally, we let $\eta\to 0$ to get the desired conclusion.

□

6. Comparison with weak solutions

In this section, we compare the notion of weak solutions to that of variational solutions.

We will first show that if the function $\xi\mapsto H(x,y,\xi)$ is C^1 and satisfies a comparable growth condition from above namely

$$H(x,y,\xi) \leq C\left(\frac{|\xi|}{|x-y|^s}\right)^p \tag{6.1}$$

then a variational solution is a weak solution.

Definition 6.1. Let Ω be an open bounded subset of $\mathbb{R}^N$. Suppose that H satisfies the assumptions (1.2) (1.3) and (6.1) and let the time-independent Cauchy-Dirichlet data u_0 satisfy (1.4).

We say that

$$u \in L^p(0,T;W^{s,p}(\mathbb{R}^N)) \cap C^0(0,T;L^2(\Omega)),\ \text{such that}\ u-g \in L^p(0,T;W_0^{s,p}(\Omega))$$

and $\partial_t u \in L^{p'}(0,T;(W_0^{s,p}(\Omega))^*)$, is a weak solution to (1.1) if for every $\tau \in (0,T]$, we have

$$\int_0^\tau \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\partial_\xi H(x,y,u(x,t)-u(y,t))(\phi(x,t)-\phi(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \int_0^\tau \langle \partial_t u\ ,\phi\rangle_{W^{s,p}(\mathbb{R}^N)}\,\mathrm{d}t = 0, \tag{6.2}$$

for all $\phi \in C_0^\infty(\Omega_\tau)$.

6.1. Time derivative of variational solutions

In this subsection, we will prove that a variational solution u has a time derivative

$$\partial_t u \in L^{p'}(0,T;(W_0^{s,p}(\Omega))^*).$$

We will use the fact that variational solutions are parabolic minimizers as proved in Theorem 5.2. Now, for $\phi \in C_0^\infty(\Omega_T)$, taking $\eta\phi$, $\eta \in [0,1]$ instead of ϕ in 5.1, we obtain

$$\begin{aligned} &\left|\int_0^T \int_\Omega u\cdot\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\qquad \leq \left|\int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{1}{\eta}\frac{H(x,y,(u+\eta\phi)(x,t)-(u+\eta\phi)(y,t))-H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\right| \end{aligned} \tag{6.3}$$

By passing to the limit as $\eta \to 0$, we get

$$\begin{aligned} \left|\int_0^T \int_\Omega u\cdot\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t\right| &\leq \int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\partial_\xi H(x,y,u(x,t)-u(y,t))|\,|(\phi(x,t)-\phi(y,t))|}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &\leq C\int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x,t)-u(y,t)|^{p-1}\,|(\phi(x,t)-\phi(y,t))|}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\ &\leq C\,\|u\|^{p-1}_{L^p(0,T;W^{s,p}(\mathbb{R}^N))}\,\|\phi\|_{L^p(0,T;W^{s,p}(\mathbb{R}^N))} \\ &\leq C\,\|u\|^{p-1}_{L^p(0,T;W^{s,p}(\mathbb{R}^N))}\,\|\phi\|_{L^p(0,T;W_0^{s,p}(\Omega))}. \end{aligned}$$

In the second inequality, we have used the fact that if H satisfies 1.2 and 6.1, then $\partial_\xi H$ satisfies

$$\partial_\xi H(x,y,\xi) \le C\frac{|\xi|^{p-1}}{|x-y|^{sp}}.$$

For a proof, see Lemma 2.2 of [33]. In the third inequality, we have used Hölder's inequality. Now, since $C_0^\infty(\Omega_T)$ is dense in $L^p(0,T;W_0^{s,p}(\Omega))$, we get that $\partial_t u \in L^{p'}(0,T;(W_0^{s,p}(\Omega))^*)$.

6.2. Passage to weak solutions

In this subsection, we prove that under (6.1), variational solutions are weak solutions as well. Once again, we use the definition of parabolic minimizers as it is more convenient. In particular, taking $\eta\phi$, $\eta \in [0,1]$ instead of ϕ in Definition 5.1, and taking limit as $\eta \to 0$, we get

$$\begin{aligned} -\int_0^T\int_\Omega \partial_t u\,\phi\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_\Omega u\,\partial_t\phi\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^T \iint\limits_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\partial_\xi H(x,y,u(x,t)-u(y,t))(\phi(x,t)-\phi(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t. \end{aligned}$$

If we take $-\eta\phi$ instead of ϕ, we get the reverse inequality. This completes the proof.

6.3. Passage from weak to variational

We will now show that for the functional,

$$H(x,y,\xi) = \frac{1}{p}\left(\frac{|\xi|}{|x-y|^s}\right)^p \tag{6.4}$$

any weak solution with $\partial_t u \in L^{p'}(0,T;(W_0^{s,p}(\Omega))^*)$ must necessarily be a variational solution in one of two cases, either we have $p > \dfrac{2N}{2s+N}$ or we have $\partial_t u \in L^2(\Omega_T)$.

The condition on time derivative $\partial_t u \in L^{p'}(0,T;(W_0^{s,p}(\Omega))^*)$ is always satisfied as per the standard theory of monotone operators in Banach space [48]. To proceed with the argument, let $\phi = v-u$ be our test function where v is a comparison function. The function ϕ is a valid test function since $C_c^\infty(\Omega_T)$ is dense in $L^p(0,T;W_0^{s,p}(\Omega))$. We note that

$$|(u+\phi)(x)-(u+\phi)(y)|^p - |u(x)-u(y)|^p \ge p|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y)) \tag{6.5}$$

and compute

$$\begin{aligned} \int_0^T \langle \partial_t u, \phi\rangle &= \int_0^T \langle \partial_t v, v-u\rangle - \int_0^T \langle \partial_t(v-u), v-u\rangle \\ &= \int_0^T \langle \partial_t v, v-u\rangle - \frac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\|v-u\|_{L^2}^2 \\ &= \int_0^T \langle \partial_t v, v-u\rangle - \|(v-u)(\cdot,T)\|_{L^2}^2 + \|(v-u)(\cdot,0)\|_{L^2}^2. \end{aligned} \tag{6.6}$$

The second equality holds in one of two cases. Either $p > \dfrac{2N}{2s+N}$ which ensures the existence of the Gelfand triple $W_0^{s,p}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow W^{-s,p'}(\Omega)$, so that Theorem 2.6 can be applied. Or we assume that the weak solution u satisfies $\partial_t u \in L^2(\Omega_T)$.

Now, by the definition of weak solution (6.2), (6.6) and applying (6.5), we obtain that

$$\int_0^T \langle \partial_t v, v-u\rangle - \|(v-u)(\cdot,T)\|_{L^2}^2 + \|(v-u)(\cdot,0)\|_{L^2}^2 \geq \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,u(x,t)-u(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t - \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H(x,y,v(x,t)-v(y,t))}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

Rearranging the above inequality shows that u is a variational solution.

Appendix A. Mollification in Time

In the definition of variational solutions, the test functions or comparison functions have additional time regularity compared to the solutions, therefore the variational solutions themselves cannot be used as comparison functions. To fix this, we need a smoothing in time. Let Ω be an open subset of $\mathbb{R}^N$. For $T>0$, $v \in L^1(\Omega_T)$, $v_0 \in L^1(\Omega)$ and $h \in (0,T]$, we define

$$[v]_h(\cdot,t) = e^{-\frac{t}{h}} v_0 + \frac{1}{h}\int_0^t e^{\frac{\varkappa - t}{h}} v(\cdot,\varkappa)\,d\varkappa, \tag{A.1}$$

for $t \in [0,T]$. The basic properties of time mollification were proved earlier in [2, 27]. We state them below for easy reference and prove the ones that are new.

Proposition A.1. *(Lemma B.1 of [2]) Let X be a Banach space and assume that $v_0 \in X$, and moreover $v \in L^r(0,T;X)$ for some $1 \leq r \leq \infty$. Then, the mollification in time defined by (A.1) belongs to $L^r(0,T;X)$ and*

$$||[v]_h||_{L^r(0,T;X)} \leq ||v||_{L^r(0,t_0;X)} + \left(\frac{h}{r}\left(1-e^{-\frac{t_0 r}{h}}\right)\right)^{\frac{1}{r}} ||v_0||_X, \tag{A.2}$$

for any $t_0 \in (0,T)$. Moreover, we have

$$\partial_t [v]_h \in L^r(0,T;X) \text{ and } \partial_t[v]_h = -\frac{1}{h}([v]_h - v). \tag{A.3}$$

Lemma A.2. *([3], Lem. 2.2) Let Ω be an open subset of $\mathbb{R}^N$. Suppose that $v \in L^1(\Omega_T)$ and $v_0 \in L^1(\Omega)$. Then, the mollification in time as defined in (A.1) satisfies the following properties:*

(i) Assume that $v \in L^p(\Omega_T)$ and $v_0 \in L^p(\Omega)$ for some $p \geq 1$. Then, it holds true that $[v]_h \in L^p(\Omega_T)$ and the following quantitative bound holds.

$$||[v]_h||_{L^p(\Omega_T)} \leq ||v||_{L^p(\Omega_T)} + h^{1/p}||v_0||_{L^p(\Omega)}. \tag{A.4}$$

Moreover, $[v]_h \to v$ in $L^p(\Omega_T)$ as $h \to 0$.

(ii) *Assume that $v \in L^p(0,T;W^{s,p}(\Omega))$ and $v_0 \in W^{s,p}(\Omega)$ for some $p>1$ and $s \in (0,1]$. Then, it holds true that $[v]_h \in L^p(0,T;W^{s,p}(\Omega))$ and the following quantitative bound holds.*

$$||[v]_h||_{L^p(0,T;W^{s,p}(\Omega))} \leq ||v||_{L^p(0,T;W^{s,p}(\Omega))} + h^{1/p}||v_0||_{W^{s,p}(\Omega)}. \tag{A.5}$$

Moreover, $[v]_h \to v$ in $L^p(0,T;W^{s,p}(\Omega))$ as $h \to 0$.

(iii) *Suppose that $v \in L^p(0,T;W_0^{s,p}(\Omega))$ and $v_0 \in W_0^{s,p}(\Omega)$ for some $p>1$ and $s \in (0,1]$. Then, it holds true that $[v]_h \in L^p(0,T;W_0^{s,p}(\Omega))$.*

(iv) *Suppose that $v \in C^0(0,T;L^2(\Omega))$ and $v_0 \in L^2(\Omega)$. Then, it holds true that $[v]_h \in C^0(0,T;L^2(\Omega))$, $[v]_h(\cdot,0)=v_0$. Moreover, $[v]_h \to v$ in $C^0(0,T;L^2(\Omega))$ as $h \to 0$.*

(v) *Suppose that $v \in L^\infty(0,T;L^2(\Omega))$ and $v_0 \in L^2(\Omega)$. Then, it holds true that $\partial_t[v]_h \in L^\infty(0,T;L^2(\Omega))$. Moreover, $\partial_t[v]_h = -\frac{1}{h}([v]_h - v)$.*

(vi) *Suppose that $\partial_t v \in L^2(\Omega_T)$ then $\partial_t[v]_h \to \partial_t v$ in $L^2(\Omega_T)$ as $h \to 0$. Moreover, the inequality $||\partial_t[v]_h||_{L^2(\Omega_T)} \leq ||\partial_t v||_{L^2(\Omega_T)}$ holds true.*

Proof. The proofs of statements $(i),(iv),(v)$ and (vi) are the same as in Lemma B.2 of [2]. We indicate the modification required for (ii) and (iii).

Proof of (ii) The case $s=1$ is covered in Lemma B.2 of [2]. Therefore, we assume that $s \in (0,1)$. To begin with, $[v]_h \in L^p(0,T;W^{s,p}(\Omega))$ follows from Proposition A.1. To prove the convergence, observe that the inclusion $[v]_h \in L^p(0,T;W^{s,p}(\Omega))$ implies that the function defined by $f_h(x,y,t) = \frac{|[v]_h(x,t)-[v]_h(y,t)|}{|x-y|^{\frac{N}{p}+s}}$ satisfies $f_h \in L^p(0,T;L^p(\Omega\times\Omega))$.

The convergence of $[v]_h$ to v in $L^p(0,T;\mathbb{R}^N)$ as $h \to 0$ guarantees pointwise a.e. convergence for a subsequence. Since this is true for any subsequence, we get the convergence $f_h(x,y,t)^p \to \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+ps}}$ as $h \to 0$ pointwise a.e. $x,t \in \mathbb{R}^N \times (0,T)$ due to continuity of f_h in v.

Now, we will interpret the mollification as a mean with respect to the measure $e^{\frac{\varkappa - t}{h}}\,ds$ similar to Lemma 2.3 of [3]. This will allow us to use the convexity of f_h^p as the p^{th} power of the ratio $\frac{|[v]_h(x,t)-[v]_h(y,t)|}{|x-y|^{\frac{N}{p}+s}}$.

Observe the following chain of inequalities:

$$\begin{aligned}
f_h^p(x,y,t) &= \frac{|[v]_h(x,t)-[v]_h(y,t)|^p}{|x-y|^{N+sp}} = \frac{\left|e^{-t/h}(v_0(x)-v_0(y)) + \frac{1-e^{-t/h}}{h(1-e^{-t/h})}\int_0^t (v(x,\varkappa)-v(y,\varkappa))e^{\frac{\varkappa-t}{h}}\,d\varkappa\right|^p}{|x-y|^{N+ps}} \\
&\leq e^{-t/h}\frac{|v_0(x)-v_0(y)|^p}{|x-y|^{N+ps}} + \left(1-e^{-t/h}\right)\frac{\left|\frac{1}{h(1-e^{-t/h})}\int_0^t (v(x,\varkappa)-v(y,\varkappa))e^{\frac{\varkappa-\varkappa}{h}}\,d\varkappa\right|^p}{|x-y|^{N+ps}} \\
&\leq e^{-t/h}\frac{|v_0(x)-v_0(y)|^p}{|x-y|^{N+ps}} + \frac{1}{h}\frac{\int_0^t \left|(v(x,\varkappa)-v(y,\varkappa))e^{\frac{\varkappa-t}{h}}\right|^p\,d\varkappa}{|x-y|^{N+ps}} = \left[\frac{|v(x,\cdot)-v(y,\cdot)|^p}{|x-y|^{N+ps}}\right]_h(t),
\end{aligned}$$

where we have used the Jensen's inequality in the last inequality.

Since $\frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+ps}} \in L^1(0,T;L^1(\Omega\times\Omega))$ and $\frac{|v_0(x)-v_0(y)|^p}{|x-y|^{N+ps}} \in L^1(\Omega\times\Omega)$, we conclude that

$$\left[\frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+ps}}\right]_h \in L^1(0,T;L^1(\Omega\times\Omega))$$

by Proposition A.1 with the additional bound

$$\left\|\left[\frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+ps}}\right]_h\right\|_{L^1(0,T;L^1(\Omega\times\Omega))} \le \left\|\frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+ps}}\right\|_{L^1(0,T;L^1(\Omega\times\Omega))} + h\left\|\frac{|v_0(x)-v_0(y)|^p}{|x-y|^{N+ps}}\right\|_{L^1(\Omega\times\Omega)}.$$

Since $h\left\|\frac{|v_0(x)-v_0(y)|^p}{|x-y|^{N+ps}}\right\|_{L^1(\Omega\times\Omega)} \to 0$ as $h\to 0$, by a version of dominated convergence theorem, we conclude that

$$\lim_{h\to 0}\int_0^T \iint_{\Omega\times\Omega} f_h^p(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \int_0^T \iint_{\Omega\times\Omega} \frac{|v(x,t)-v(y,t)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.$$

This finishes the proof of (ii).

Proof of (iii) The statement may be proved by density of $C_0^\infty(\Omega)$ functions in $W_0^{s,p}(\Omega)$. In particular, if $\phi_\varepsilon \in L^p(0,T;C_0^\infty(\Omega))$ is an approximating sequence for a function $v\in L^p(0,T;W^{s,p}(\Omega))$, then $[\phi_\varepsilon]_h$ is an approximating sequence for the function $[v]_h \in L^p(0,T;W^{s,p}(\Omega))$. □

We finish this appendix by proving the following theorem.

Theorem A.3. *Let $T>0$, and assume that $v\in L^1(0,T;W^{s,1}(\mathbb{R}^N))$ with*

$$\frac{H(x,y,v(x,t)-v(y,t))}{|x-y|^N} \in L^1(0,T;L^1(\mathbb{R}^N\times\mathbb{R}^N)),$$

and $v_0\in W^{s,1}(\mathbb{R}^N)$, with

$$\frac{H\left(x,y,v_0(x)-v_0(y)\right)}{|x-y|^N} \in L^1(\mathbb{R}^N\times\mathbb{R}^N).$$

Then, we have

$$\frac{H\left(x,y,[v]_h(x,t)-[v]_h(y,t)\right)}{|x-y|^N} \in L^1(0,T;L^1(\mathbb{R}^N\times\mathbb{R}^N)).$$

Moreover,

$$\lim_{h\to 0}\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H\left(x,y,[v]_h(x,t)-[v]_h(y,t)\right)}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H\left(x,y,v(x,t)-v(y,t)\right)}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

Proof. The proof is the same as that of Lemma 2.3 in [3]. The only difference is we use continuity and convexity of $\xi\to H(x,y,\xi)$ and use convergence in $L^1(0,T;L^1(\mathbb{R}^N\times\mathbb{R}^N))$. Indeed, the proof is similar to the one given for Lemma A.2 (ii). Nevertheless, we repeat the argument for consistency.

As in the proof of Lemma 2.3 in [3], we will interpret the mollification $[v]_h$ as a mean with respect to the measure $e^{\frac{\varkappa - t}{h}}$. This will allow us to use convexity of H and Jensen's inequality in the following chain of estimates.

$$\begin{aligned}
\frac{H(x,y,[v]_h(x,t)-[v]_{y,t})}{|x-y|^N} &= \frac{H\left(x,y,e^{-t/h}(v_0(x)-v_0(y)) + \frac{1-e^{-t/h}}{h(1-e^{-t/h})}\int_0^t (v(x,\varkappa)-v(y,\varkappa))e^{\frac{\varkappa-t}{h}}\,d\varkappa\right)}{|x-y|^N} \\
&\le e^{-t/h}\frac{H(x,y,v_0(x)-v_0(y))}{|x-y|^N} + (1-e^{-t/h})\frac{H\left(x,y,\frac{1}{h(1-e^{-t/h})}\int_0^t (v(x,\varkappa)-v(y,\varkappa))e^{\frac{\varkappa-t}{h}}\,d\varkappa\right)}{|x-y|^N} \\
&\le e^{-t/h}\frac{H(x,y,v_0(x)-v_0(y))}{|x-y|^N} + \frac{1}{h}\int_0^t \frac{H\left(x,y,v(x,\varkappa)-v(y,\varkappa)\right)}{|x-y|^N} e^{\frac{\varkappa-t}{h}}\,d\varkappa \\
&= \left[\frac{H(x,y,v(x,\cdot)-v(y,\cdot))}{|x-y|^N}\right]_h (t), \qquad \text{(A.6)}
\end{aligned}$$

where in the second inequality, we have used convexity of $\xi \to H(x,y,\xi)$, in the third inequality, we use Jensen's inequality. The expression $\left[\frac{H(x,y,v(x,\cdot)-v(y,\cdot))}{|x-y|^N}\right]_h$ is interpreted with initial data $\frac{H(x,y,v_0(x)-v_0(y))}{|x-y|^N}$.

Now, since the function defined as $(t,x,y) \mapsto \frac{H(x,y,v(x,t)-v(y,t))}{|x-y|^N} \in L^1(0,T;L^1(\mathbb{R}^N\times\mathbb{R}^N))$ and the function defined as $(x,y) \mapsto \frac{H(x,y,v_0(x)-v_0(y))}{|x-y|^N} \in L^1(\mathbb{R}^N\times\mathbb{R}^N)$, we obtain by Proposition A.1 that the function defined by $(x,y,t) \mapsto \left[\frac{H(x,y,v(x,t)-v(y,t))}{|x-y|^N}\right]_h$ belongs to $L^1(0,T;L^1(\mathbb{R}^N\times\mathbb{R}^N))$ and so does the function $(x,y,t) \mapsto \frac{H(x,y,[v]_h(x,t)-[v]_h(y,t))}{|x-y|^N}$ due to the pointwise bound (A.6).

Moreover, we obtain the estimate

$$\begin{aligned}
\left\|\frac{H(x,y,[v]_h(x,t)-[v]_h(y,t))}{|x-y|^N}\right\|_{L^1(0,T;L^1(\mathbb{R}^N\times\mathbb{R}^N))} &\le \left\|\frac{H(x,y,v(x,t)-v(y,t))}{|x-y|^N}\right\|_{L^1(0,T;L^1(\mathbb{R}^N\times\mathbb{R}^N))} \\
&\quad + h\left\|\frac{H(x,y,v_0(x)-v_0(y))}{|x-y|^N}\right\|_{L^1(\mathbb{R}^N\times\mathbb{R}^N)}.
\end{aligned}$$

Since $h\left\|\frac{H(x,y,v_0(x)-v_0(y))}{|x-y|^N}\right\|_{L^1(\mathbb{R}^N\times\mathbb{R}^N)} \to 0$ as $h \to 0$, by a version of dominated convergence theorem, we conclude that

$$\lim_{h\to 0}\int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H\left(x,y,[v]_h(x,t)-[v]_h(y,t)\right)}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \int_0^T \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{H\left(x,y,v(x,t)-v(y,t)\right)}{|x-y|^N}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

□

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