ESAIM: COCV 29 (2023) 48
https://doi.org/10.1051/cocv/2023016

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

NULL CONTROLLABILITY OF STRONGLY DEGENERATE PARABOLIC EQUATIONS

Antoine Benoit, Romain Loyer and Lionel Rosier*

Abstract. We consider linear one-dimensional strongly degenerate parabolic equations with measurable coefficients that may be degenerate or singular. Taking 0 as the point where the strong degeneracy occurs, we assume that the coefficient $a = a(x)$ in the principal part of the parabolic equation is such that the function $x \to x/a(x)$ is in $L^p(0,1)$ for some $p > 1$. After establishing some spectral estimates for the corresponding elliptic problem, we prove that the parabolic equation is null controllable in the energy space by using one boundary control.

Mathematics Subject Classification. 35J25, 35J70, 35K65, 35P15, 93B05.

Received September 27, 2022. Accepted March 8, 2023.

1. Introduction

We continue our investigation of the controllability of parabolic equations with measurable coefficients [24] (see also [2]) by studying the case of a *strongly degenerate* equation of the type

$$(a(x)u_x)_x + q(x)u = \rho(x)u_t, \quad x \in (0,1), \quad t \in (0,T),$$

where the nonnegative function a may vanish *strongly* at $x = 0$, and the potential q may be singular at $x = 0$. Only *weakly degenerate* (*i.e.* $1/a \in L^1(0,1)$) parabolic equations were covered by the theory developed in [24].

The null controllability of (weakly or strongly) degenerate parabolic equations was considered in *e.g.* [1, 4, 5, 9–11, 13–15, 26]. Most of the papers were concerned with a parabolic equation with $a(x) = x^{2-\varepsilon}$, which is strongly (resp. weakly) degenerate for $\varepsilon \in (0,1]$ (resp. $\varepsilon \in (1,2)$). More general choices for the coefficient a were considered in *e.g.* [14]. However, several technical assumptions (*e.g.* $x \to a(x)/x^\gamma$ nondecreasing for some exponent γ and $a \in W^{1,\infty}(0,1)$ in [14]) were required in order to derive some Carleman estimate to prove the null controllability of the parabolic equation. The purpose of this paper is to remove these technical assumptions in the derivation of the null controllability of the parabolic equation.

More precisely, we propose a general method based on the flatness approach to deal with quite general parabolic equations, displaying both a strong degeneracy of a and a singularity of the potential q at the same

Keywords and phrases: Strongly degenerate parabolic equations, null controllability, measurable and singular coefficient, spectral estimates, flatness.

[1] Université du Littoral Côte d'Opale, Laboratoire de Mathématiques Pures et Appliquées J. Liouville, B.P. 699, 62228 Calais, France

* Corresponding author: lionel.rosier@univ-littoral.fr

point, with measurable coefficients, and without any monotony assumption about a. Roughly, the main assumption is that the function $x \to x/a(x)$ is in $L^p(0,1)$ for some $p>1$. That assumption is slightly stronger (by Hölder inequality) than Trudinger assumption $1/\sqrt{a} \in L^1(0,1)$ (see *e.g.* [12, 28]) which was made in order to investigate the degenerate elliptic system

$$\begin{cases} -(au_x)_x = f, \ x \in (0,1), \\ (au_x)(0) = 0, \\ u(1) = 0. \end{cases}$$

Our method is based on the flatness approach, introduced in [19] and developed since in [22, 24] for the heat equation, in [21] for the Korteweg-de Vries equation, and in [25] for Schrödinger equation. (See also [23] for a recent study of the reachable states of the heat equation and [20] for the exact controllability of semilinear heat equations.) In [26], the flatness approach is used to derive for $\varepsilon \in (0,1)$ the null controllability of the control system

$$\begin{cases} u_t - (x^{2-\varepsilon} u_x)_x = 0, \quad x \in (0,1), \ t \in (0,T), \\ (x^{2-\varepsilon} u_x)(0,t) = 0, \quad t \in (0,T), \\ u(1,t) = h(t), \quad t \in (0,T), \\ u(x,\cdot) = u_0(x), \quad x \in (0,1). \end{cases}$$

For the corresponding elliptic problem

$$\begin{cases} -(x^{2-\varepsilon} u_x)_x = f, \ x \in (0,1), \\ (x^{2-\varepsilon} u_x)(0) = 0, \\ u(1) = 0, \end{cases}$$

the eigenfunctions and eigenvalues can be expressed in terms of Bessel functions, and the asymptotic behaviour of the eigenvalues is perfectly known [26]. For a more general function a, however, Bessel functions cannot be used and, to the best knowledge of the authors, nothing is known about the sharp asymptotic behaviour of the eigenvalues. (See [3, 17, 18] for some results in that direction.) For the application of the flatness approach, what is needed is not a spectral gap, but merely that the eigenvalues tend to infinity faster than some power of the index of the eigenvalue.

To be more precise, we are concerned with the null controllability of the system

$$(a(x)u_x)_x + q(x)u = \rho(x)u_t, \ x \in (0,1), \ t \in (0,T), \tag{1.1}$$

$$(au_x)(0,t) = 0, \ t \in (0,T), \tag{1.2}$$

$$\alpha u(1,t) + \beta (au_x)(1,t) = h(t), \ t \in (0,T), \tag{1.3}$$

$$u(x,0) = u_0(x), \ x \in (0,1) \tag{1.4}$$

where $(\alpha,\beta) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$, $u_0 \in L^2(0,1)$ is the initial state, and $h \in L^2(0,T)$ is the control input.

The given functions a, q, ρ are assumed to satisfy the following conditions:

$$a(x) > 0 \text{ and } \rho(x) > 0 \ \text{ for a.e. } x \in (0,1), \tag{1.5}$$

$$a \in L^1_{loc}(0,1), \quad \big(x \to \frac{x}{a(x)}\big) \in L^p(0,1), \tag{1.6}$$

$$\rho \in L^r(0,1), \quad \limsup_{x\to 0^+} \rho(x) < \infty, \tag{1.7}$$

$$\lim_{x\to 0^+} a(x)^{-1}\left(\int_x^1 \frac{\mathrm{d}s}{a(s)}\right)^{-2} = +\infty, \tag{1.8}$$

$$\exists v \in W^{1,1}(0,1) \text{ s.t. } \begin{cases} v(x) > 0 \text{ for all } x \in [0,1], \\ (av_x)_x + qv = 0 \text{ in } (0,1), \\ (av_x)(0) = 0, \end{cases} \tag{1.9}$$

for some numbers p, r with

$$p \in (1, +\infty], \quad r \in (p', +\infty] \tag{1.10}$$

where $p' := \frac{p}{p-1}$. As the functions a and ρ are defined a.e., the limits in equations (1.7) and (1.8) should be taken after modifying them on a zero measure set, if needed. Note that equations (1.6) and (1.8) are satisfied by any measurable function $a : (0,1) \to \mathbb{R}$ fulfilling the condition

$$C_1 x^{2-\varepsilon_1} \le a(x) \le C_2 x^{2-\varepsilon_2} \quad \forall x \in (0,1) \tag{1.11}$$

for some positive constants $C_1, C_2, \varepsilon_1, \varepsilon_2$ with $\varepsilon_1 \le \varepsilon_2 < 2\varepsilon_1 \le 4$. Such a function needs not be monotonous nor smooth. (Note that a continuous function $a : (0,1) \to \mathbb{R}_+$ satisfying both equations (1.6) and (1.8) and vanishing on a sequence $x_n \searrow 0$ could also be constructed, so that equation (1.1) can be *strongly degenerate* at 0 and also *weakly degenerate* at each x_n.) A typical example displaying both a (possibly strong) degeneracy for a and a singularity for q at $x = 0$ is the parabolic equation

$$(x^{2-\varepsilon} u_x)_x + \mu \frac{u}{x^\varepsilon} = u_t \tag{1.12}$$

for $0 < \varepsilon < 2$ and $\mu < \frac{1}{4}(1-\varepsilon)^2$.

Let us introduce some notations. For any $t_1 < t_2$ and $s \ge 0$, we denote by $G^s([t_1, t_2])$ the space of (Gevrey) functions $h \in C^\infty([t_1, t_2])$ for which there exist some positive constants M, R such that

$$|h^{(p)}(t)| \le M \frac{p!^s}{R^p} \quad \forall t \in [t_1, t_2], \ \forall p \in \mathbb{N}.$$

Let L^2_ρ denote the space of (classes of) measurable functions $f : (0,1) \to \mathbb{R}$ such that

$$\|f\|_{L^2_\rho} := \left(\int_0^1 f(x)^2 \rho(x) \mathrm{d}x\right)^{\frac{1}{2}} < \infty.$$

The main result in this paper is the following

Theorem 1.1. *Let the functions $a, q, \rho, v : (0,1) \to \mathbb{R}$ satisfy assumptions (1.5)–(1.9) for some numbers p and r as in equation (1.10). Let $(\alpha, \beta) \in \mathbb{R}^2_+ \setminus \{(0,0)\}$ and $T > 0$. Pick any $u_0 \in L^2_\rho$ and any $s \in (1, 1 + \frac{1}{p'} - \frac{1}{r})$. Then there exists a function $h \in G^s([0,T])$ such that the solution u to equations (1.1)–(1.4) satisfies $u \in C^0([0,T], L^2_\rho) \cap C^1((0,T], W^{1,1}(0,1))$, $au_x \in C^1((0,T], W^{1,r}(0,1))$, and*

$$u(x,T) = 0 \quad \textit{for all } x \in [0,1].$$

Remark 1.2. A null controllability result with a distributed control can be derived from Theorem 1.1 by using a partition of unity.

Clearly, assumptions (1.5)–(1.8) are easy to test, while assumption (1.9) is not obvious to check at first glance. We shall provide in the following propositions two classes of coefficients (a, q, ρ) satisfying equation (1.9).

Proposition 1.3. *Let the numbers p, p', r be as in equation (1.10) and let the functions a, q, ρ satisfy equations (1.5)–(1.8). Assume in addition that $q \in L^{p'}(0,1)$ and that either*

$$\int_0^1 \frac{1}{a(x)} \left(\int_0^x |q(s)| \, \mathrm{d}s \right) \mathrm{d}x < 1, \tag{1.13}$$

or

$$q(x) \leq 0 \quad \text{for a.e. } x \in (0,1). \tag{1.14}$$

Then equation (1.9) holds for some function $v \in W^{1,1}(0,1)$ with $av_x \in W^{1,r}(0,1)$, and the conclusion of Theorem 1.1 is valid for any $u_0 \in L^2_\rho$. Finally, if equation (1.14) is replaced by the condition

$$\exists K \in \mathbb{R}_+ \text{ such that } \; q(x) \leq K\rho(x) \text{ for a.e. } x \in (0,1), \tag{1.15}$$

then the conclusion of Theorem 1.1 is still valid for any $u_0 \in L^2_\rho$.

Proposition 1.4. *Let $a(x) = x^{2-\varepsilon}$, $q(x) = \mu\, x^{-\varepsilon}$ and $\rho(x) = 1$ for $x \in (0,1)$, where $0 < \varepsilon < 1$ and*

$$\frac{(1-\varepsilon)^2 - 1}{4} < \mu \leq 0. \tag{1.16}$$

Pick $v(x) := x^\delta$ with $\delta := (\varepsilon - 1 + \sqrt{(1-\varepsilon)^2 - 4\mu})/2$. Then the function $v : (0,1) \to (0,\infty)$ fulfills

$$(av_x)_x + qv = 0 \quad \text{in } (0,1), \tag{1.17}$$

$$(av_x)(0) = 0, \tag{1.18}$$

and equations (1.5)–(1.8) hold for some numbers p and r as in equation (1.10). Furthermore the conclusion of Theorem 1.1 is valid for equation (1.12) supplemented with equations (1.2)–(1.4) for any initial data $u_0 : (0,1) \to \mathbb{R}$ with $u_0 \in L^2(0,1)$.

Remark 1.5. 1. Note that the main result in [26] corresponds to the case $\mu = 0$.
2. Note that Proposition 1.4 is *not* a consequence of Proposition 1.3, since we cannot find any p in $(1, +\infty]$ with both $(x \to x^{\varepsilon-1}) \in L^p(0,1)$ and $(x \to x^{-\varepsilon}) \in L^{p'}(0,1)$.
3. Our computations suggest that for $a(x) = x^{2-\varepsilon}$ and $q(x) = \mu x^{-\kappa}$, κ should be at most ε. It would be interesting to see whether it is a necessary condition, or merely a technical assumption.

Let us say a few words about the proof of the main result. In a first step, we show that we can get rid of the term $q(x)u$ in equation (1.1) by a change of variables, using assumption (1.9). Therefore we can restrict to the simplified parabolic equation

$$(au_x)_x = \rho u_t.$$

We first prove that the boundary value problem

$$\begin{cases} -(au_x)_x = f, \\ (au_x)(0) = 0, \\ \alpha u(1) + \beta (au_x)(1) = 0 \end{cases}$$

possesses a unique solution in some weighted Sobolev space. Next, we pay some attention to the spectral properties of this boundary value problem. We show that the eigenvalue λ_n grows *at least* as a power of n, and that the L^∞-norm of the corresponding eigenfunction e_n grows *at most* as a power of λ_n. This is done by using a modified Prüfer method (see [6, 29]) introducing a phase θ_n associated with λ_n. However, since $1/a \notin L^1(0,1)$ in the interesting situation of a strong degeneracy, the classical argument relating λ_n to the variation of the phase θ_n has to be refined in using a splitting of the interval $(0,1)$ involving the frequency λ_n. Roughly, we split $(0,1)$ into $(0,A_n)\cup[A_n,1)$ with $A_n := (C\lambda_n)^{-\frac{p'r}{r-p'}}$, C denoting some positive constant. We show that $e_n(x)$ remains close to $e_n(0)$ for $x\in(0,A_n)$, so that the (bad) integral term $\int_0^{A_n}\frac{d\theta_n}{\mathrm{d}x}\mathrm{d}x$ does not contribute too much in the variation of the phase $\theta_n(1)-\theta_n(0)$.

With these spectral estimates at hand, we can prove that the eigenfunctions e_n, $n\geq 0$, can be expressed in terms of the generating functions g_i, $i\geq 0$, defined by $g_0(x)=1$ and the relation

$$g_i(x)=\int_0^x\frac{1}{a(s)}\Big(\int_0^s\rho(\sigma)g_{i-1}(\sigma)d\sigma\Big)\mathrm{d}s,\quad i\geq 1.$$

Finally, the trajectories of the control problems (1.1)–(1.4) can be expanded in the form

$$u(x,t)=\sum_{i=0}^{\infty}y^{(i)}(t)g_i(x)$$

for some function $y\in G^s([0,T])$ (as in [24]), the series being convergent thanks to the spectral estimates.

The paper is organized as follows. Section 2 is devoted to the study of the corresponding elliptic problem. We introduce the appropriate weighted Sobolev space, derive some generalized Hardy inequality and obtain some estimates for both the eigenfunctions and the eigenvalues. In Section 3, we define and investigate the generating functions. The proof of the main results are given in Section 4. Finally, in some appendix we prove that the conditions (1.6) and (1.8) are independent, and we provide a class of functions for which equation (1.8) holds.

2. Study of the elliptic problem

Through the paper, we denote $\|u\|_{L^p}$ for $\|u\|_{L^p(0,1)}$ $(1\leq p\leq\infty)$, and $\|u\|_{L^p(x_1,x_2)}$ for the L^p norm of u on an interval $(x_1,x_2)\neq(0,1)$.

In this section, we investigate the elliptic problem

$$-(au')'=\rho f\quad\text{in }(0,1),\tag{2.1}$$
$$(au')(0)=0,\tag{2.2}$$
$$\alpha u(1)+\beta(au')(1)=0,\tag{2.3}$$

where $'=\mathrm{d}/\mathrm{d}x$, the functions a and ρ satisfy equations (1.5)–(1.8) for some numbers p,p', and r as in equation (1.10), $(\alpha,\beta)\in\mathbb{R}_+^2\setminus\{(0,0)\}$.

Let us introduce the spaces

$$H_a:=\{u\in W^{1,1}_{loc}(0,1);\ \sqrt{a}u'\in L^2(0,1)\text{ and }u(1)=0\},\tag{2.4}$$
$$H_{a,\rho}:=\{u\in W^{1,1}_{loc}(0,1);\ \sqrt{a}u'\in L^2(0,1)\text{ and }\sqrt{\rho}u\in L^2(0,1)\}\tag{2.5}$$

endowed respectively with the norms

$$\|u\|_{H_a}:=\Big(\int_0^1a(x)u'(x)^2\mathrm{d}x\Big)^{\frac12},\quad \|u\|_{H_{a,\rho}}:=\Big(\int_0^1[a(x)u'(x)^2+\rho(x)u(x)^2]\mathrm{d}x\Big)^{\frac12}.$$

Then the following result holds.

Proposition 2.1. *The spaces H_a and $H_{a,\rho}$ are complete. Furthermore, we have $H_a \subset L^2(0,1)$, $H_{a,\rho} \subset L^2(0,1)$ and $H_{a,\rho} \subset L^2_\rho$ with continuous and compact embeddings.*

Proof. We first investigate the space H_a. By Hölder inequality, we have for all $\varepsilon \in (0,1)$

$$\int_\varepsilon^1 \frac{\mathrm{d}x}{a(x)} \leq \left(\int_\varepsilon^1 (\frac{x}{a(x)})^p \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_\varepsilon^1 x^{-p'} \mathrm{d}x\right)^{\frac{1}{p'}} < \infty,$$

so that $a^{-1} \in L^1(\varepsilon, 1)$. Thus, by Cauchy-Schwarz inequality,

$$\int_\varepsilon^1 |u'(x)| \mathrm{d}x \leq \left(\int_\varepsilon^1 a(x) u'(x)^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\varepsilon^1 \frac{\mathrm{d}x}{a(x)}\right)^{\frac{1}{2}} < \infty \tag{2.6}$$

if $\sqrt{a}u' \in L^2(0,1)$, so that $u \in W^{1,1}(\varepsilon,1) \subset C^0([\varepsilon,1])$. Therefore, the condition $u(1) = 0$ is meaningful whenever $\sqrt{a}u' \in L^2(0,1)$, and if $u \in H_a$ satisfies $\|u\|_{H_a} = 0$, then $\sqrt{a}u' = 0$ a.e., u is constant and $u = 0$ since $u(1) = 0$. Thus $\|\cdot\|_{H_a}$ is a norm on H_a, which is clearly Hilbertian.

If (u_n) is a Cauchy sequence in H_a, then by equation (2.6) and the fact that $u_n(1) = 0$, (u_n) is a Cauchy sequence in $W^{1,1}(\varepsilon,1)$ for all $\varepsilon > 0$. Therefore, there exists $u \in W^{1,1}_{loc}(0,1)$ such that $u_n \to u$ in $W^{1,1}(\varepsilon,1)$ for all $\varepsilon > 0$, hence in $\mathcal{D}'(0,1)$. There is also some $v \in L^2_a$ such that $u_n' \to v$ in L^2_a. But for any $\varphi \in \mathcal{D}(0,1) := C_c^\infty\big((0,1)\big)$,

$$\left|\int_0^1 (u_n' - v)\varphi \mathrm{d}x\right| \leq \left(\int_0^1 (u_n' - v)^2 a \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_0^1 \frac{\varphi^2}{a} \mathrm{d}x\right)^{\frac{1}{2}} \to 0,$$

that is, $u_n' \to v$ in $\mathcal{D}'(0,1)$. We infer that $u' = v \in L^2_a$, and hence, with $u(1) = 0$ (since $u_n \to u$ in $W^{1,1}(\varepsilon,1)$), $u \in H_a$ and $u_n \to u$ in H_a. Therefore H_a is complete.

Let us now show that $H_{a,\rho}$ is complete. We first need the following

Lemma 2.2. *Let $\delta \in (0,1)$. Then for any $\varepsilon > 0$, there exists some number $C_{\delta,\varepsilon} > 0$ such that*

$$\int_\delta^1 |u| \mathrm{d}x \leq \varepsilon \int_\delta^1 |u'| \mathrm{d}x + C_{\delta,\varepsilon} \left(\int_\delta^1 \rho u^2 \mathrm{d}x\right)^{\frac{1}{2}} \qquad \forall u \in H_{a,\rho}. \tag{2.7}$$

Proof of Lemma 2.2: If equation (2.7) does not hold, there exists $\varepsilon > 0$ and a sequence (u_n) in $H_{a,\rho}$ such that for all $n \geq 1$

$$1 = \int_\delta^1 |u_n| \mathrm{d}x > \varepsilon \int_\delta^1 |u_n'| \mathrm{d}x + n \left(\int_\delta^1 \rho u_n^2 \mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.8}$$

Recall that $\rho \in L^r(0,1)$ for some $r \in (p', \infty]$, by equations (1.7) and (1.10). Since (u_n) is bounded in $W^{1,1}(\delta,1)$, which is compactly embedded in $L^{2r'}(\delta,1)$ $(2 \leq 2r' < 2p \leq \infty)$, there is a subsequence (u_{n_k}) such that $u_{n_k} \to u$ in $L^{2r'}(\delta,1)$. Then, by Hölder inequality,

$$\int_\delta^1 \rho(u_{n_k} - u)^2 \mathrm{d}x \leq \left(\int_\delta^1 \rho^r \mathrm{d}x\right)^{\frac{1}{r}} \left(\int_\delta^1 |u_{n_k} - u|^{2r'} \mathrm{d}x\right)^{\frac{1}{r'}} \to 0,$$

and hence $\int_\delta^1 \rho u_{n_k}^2 \mathrm{d}x \to \int_\delta^1 \rho u^2 \mathrm{d}x$. But $\int_\delta^1 \rho u_n^2 \mathrm{d}x \to 0$ by equation (2.8), and hence $\int_\delta^1 \rho u^2 \mathrm{d}x = 0$. It follows that $u = 0$ a.e. in $(\delta, 1)$. But this contradicts the fact that $1 = \int_\delta^1 |u_{n_k}| \mathrm{d}x \to \int_\delta^1 |u| \mathrm{d}x$. Lemma 2.2 is proved. □

Combining equations (2.6) and (2.7), we obtain that for all $\delta \in (0,1)$, there is some $C_\delta > 0$ such that

$$\|u\|_{W^{1,1}(\delta,1)} \leq C_\delta \|a\|_{H_{a,\rho}} \quad \forall u \in H_{a,\rho}. \tag{2.9}$$

Proceeding as above and using the fact that $W^{1,1}(\varepsilon, 1) \subset C^0([\varepsilon, 1])$ continuously for any $\varepsilon \in (0,1)$, one can prove that $H_{a,\rho}$ is a Hilbert space.

The next result is concerned with the density of spaces of smooth functions.

Lemma 2.3. *The space $\mathcal{D}(0,1)$ is dense in H_a if $a^{-1} \notin L^1(0,1)$, while the space $\{\varphi \in C^\infty([0,1]);\ \ \varphi(1) = 0\}$ is dense in H_a if $a^{-1} \in L^1(0,1)$.*

Proof. 1. Assume first that

$$a^{-1} \notin L^1(0,1). \tag{2.10}$$

Let $u \in H_a$ with $\int_0^1 a u' \varphi' \mathrm{d}x = 0$ for all $\varphi \in \mathcal{D}(0,1)$. Note that $au' \in L^1_{loc}(0,1) \subset \mathcal{D}'(0,1)$, for

$$\int_\delta^\varepsilon |au'| \mathrm{d}x \leq \left(\int_\delta^\varepsilon a u'^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\delta^\varepsilon \frac{\mathrm{d}x}{a} \right)^{\frac{1}{2}} < \infty \quad \text{ if } 0 < \delta < \varepsilon < 1.$$

Thus $\langle (au')', \varphi \rangle_{\mathcal{D}', \mathcal{D}} = -\langle au', \varphi' \rangle_{\mathcal{D}', \mathcal{D}} = 0$ for all $\varphi \in \mathcal{D}(0,1)$ and $(au')' = 0$ in $\mathcal{D}'(0,1)$. Thus there is some number $K \in \mathbb{R}$ such that $au' = K$ a.e. in $(0,1)$. But $\frac{K}{\sqrt{a}} = \sqrt{a} u' \in L^2(0,1)$, hence $\int_0^1 \frac{K^2}{a} \mathrm{d}x < \infty$ and $K = 0$, by equation (2.10). Thus $u = 0$ and $\mathcal{D}(0,1)$ is dense in H_a.

2. Assume now that

$$a^{-1} \in L^1(0,1). \tag{2.11}$$

Pick any $u \in H_a$ with $\int_0^1 a u' \varphi' \mathrm{d}x = 0$ for all $\varphi \in C^\infty([0,1])$ with $\varphi(1) = 0$. (Such functions φ belong to H_a, for $\int_0^1 a \varphi'^2 \mathrm{d}x < \infty$.) In particular, as $\int_0^1 a u' \varphi' \mathrm{d}x = 0$ for all $\varphi \in \mathcal{D}(0,1)$, we infer that $au' = K$ a.e. in $(0,1)$ for some $K \in \mathbb{R}$. Thus

$$0 = \int_0^1 a u' \varphi' \mathrm{d}x = K \int_0^1 \varphi' \mathrm{d}x = -K\varphi(0)$$

for all $\varphi \in C^\infty([0,1])$ with $\varphi(1) = 0$. This yields again $K = 0$, $u = 0$, and the density of the space $\{\varphi \in C^\infty([0,1]); \varphi(1) = 0\}$ in H_a. □

The following result gives a generalized Hardy inequality (see [7, 16, 27]).

Lemma 2.4. *Let $a : (0,1) \to \mathbb{R}$ be as in equations (1.5) and (1.8). Extend a to $(0, \infty)$ by setting*

$$a(x) = x^2 \quad \textit{for } x \geq 1, \tag{2.12}$$

and let

$$b(x) = a(x)^{-1} \left(\int_x^\infty \frac{\mathrm{d}s}{a(s)} \right)^{-2}, \qquad x \in (0, \infty). \tag{2.13}$$

Then

$$\lim_{x\to 0^+} b(x) = +\infty \tag{2.14}$$

and

$$\int_0^1 b(x)u(x)^2\mathrm{d}x \le 4\int_0^1 a(x)u'(x)^2\mathrm{d}x \quad \forall u\in H_a. \tag{2.15}$$

Proof. First, we note that equation (2.14) follows from equation (1.8) and the fact that

$$\lim_{x\to 0^+}\frac{\int_x^1 a(s)^{-1}\mathrm{d}s}{\int_x^\infty a(s)^{-1}\mathrm{d}s} = \begin{cases} \dfrac{\int_0^1 a(s)^{-1}\mathrm{d}s}{\int_0^\infty a(s)^{-1}\mathrm{d}s}\in(0,\infty) & \text{if } \int_0^1 a(s)^{-1}\mathrm{d}s<\infty;\\ 1 & \text{if } \int_0^1 a(s)^{-1}\mathrm{d}s=\infty.\end{cases}$$

From [7, 27], if α,β,f are nonnegative measurable functions defined on $\mathbb{R}_+$, if

$$K := \sup_{r>0}\left[\left(\int_0^r \beta(x)^2\mathrm{d}x\right)^{\frac12}\left(\int_r^\infty \alpha(x)^{-2}\mathrm{d}x\right)^{\frac12}\right]<\infty \tag{2.16}$$

then

$$\left(\int_0^\infty [\beta(x)\int_x^\infty f(t)\mathrm{d}t]^2\mathrm{d}x\right)^{\frac12} \le 2K\left(\int_0^\infty [\alpha(x)f(x)]^2\mathrm{d}x\right)^{\frac12}. \tag{2.17}$$

Pick $\alpha(x) := \sqrt{a(x)}$ and $\beta(x) := \sqrt{b(x)}$. Let us check that condition (2.16) is satisfied. For $0<\varepsilon<r$, we have that

$$\begin{aligned}\int_\varepsilon^r \beta(x)^2\mathrm{d}x &= \int_\varepsilon^r a(x)^{-1}\left(\int_x^\infty a(s)^{-1}\mathrm{d}s\right)^{-2}\mathrm{d}x = \int_\varepsilon^r \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^\infty a(s)^{-1}\mathrm{d}s\right)^{-1}\mathrm{d}x\\ &= \left(\int_r^\infty a(s)^{-1}\mathrm{d}s\right)^{-1} - \left(\int_\varepsilon^\infty a(s)^{-1}\mathrm{d}s\right)^{-1}.\end{aligned}$$

Note that by equation (2.12),

$$l := \lim_{\varepsilon\to 0^+}\left(\int_\varepsilon^\infty \frac{\mathrm{d}s}{a(s)}\right)^{-1}\in[0,1)$$

always exists. It follows that

$$\int_0^1 \beta(x)^2\mathrm{d}x = 1-l<\infty.$$

Thus

$$\int_0^r \beta(x)^2\mathrm{d}x\int_r^\infty \alpha(x)^{-2}\mathrm{d}x = \left(\Big(\int_r^\infty \frac{\mathrm{d}s}{a(s)}\Big)^{-1} - l\right)\int_r^\infty \frac{\mathrm{d}s}{a(s)} \le 1 \quad \forall r\in(0,+\infty)$$

and equation (2.16) is indeed satisfied with $K \le 1$.

Pick now any $u \in H_a$. Extend u by 0 for $x \ge 1$. (Note that $u \in W^{1,1}(\varepsilon, +\infty)$ for all $\varepsilon \in (0,1)$.) Pick $f(x) = |u'(x)|$ for $x \in (0, +\infty)$. Then for $x \in (0,1)$

$$|u(x)| = \left| \int_x^\infty u'(t)\mathrm{d}t \right| \le \int_x^\infty f(t)\mathrm{d}t$$

and equation (2.15) follows from equation (2.17). □

By equation (2.14), one may pick $x_0 \in (0,1)$ such that $b(x) \ge 1$ for $0 < x < x_0$. Then equation (2.15) yields

$$\int_0^{x_0} u(x)^2 \mathrm{d}x \le 4\|u\|_{H_a}^2 \quad \forall u \in H_a.$$

Combined with equation (2.6) and the fact that $u(1) = 0$, we infer the existence of some constant $C > 0$ such that

$$\int_0^1 u(x)^2 \mathrm{d}x \le C\|u\|_{H_a}^2.$$

Thus $H_a \subset L^2(0,1)$ continuously. Actually, the embedding is also compact.

Lemma 2.5. *The embedding $H_a \subset L^2(0,1)$ is compact.*

Proof of Lemma 2.5: Let (u_n) be a sequence in H_a and let $u \in H_a$ be such that $u_n \to u$ weakly in H_a. We have to show that $u_n \to u$ strongly in $L^2(0,1)$. Since for $\delta \in (0,1)$ the embedding $W^{1,1}(\delta, 1) \subset L^2(\delta, 1)$ is compact, the map $v \in H_a \to v_{|(\delta,1)} \in L^2(\delta, 1)$ is compact for any $\delta \in (0,1)$, and hence $u_n \to u$ in $L^2(\delta, 1)$. Let $\varepsilon > 0$ be given. By equation (2.14), there exists some $\delta \in (0,1)$ such that

$$b(x) \ge B := \frac{8}{\varepsilon^2}\big(1 + 4\sup_{n\ge 0} \|u_n\|_{H_a}^2\big) \quad \forall x \in (0, \delta).$$

Using the fact that $\|u\|_{H_a} \le \sup_{n\ge 0} \|u_n\|_{H_a}$ and equation (2.15), we obtain

$$\int_0^\delta |u_n(x) - u(x)|^2 \mathrm{d}x \le B^{-1} \int_0^\delta b(x)|u_n(x) - u(x)|^2 \mathrm{d}x \le 4B^{-1}\|u_n - u\|_{H_a}^2 \le \frac{\varepsilon^2}{2}.$$

Since $u_n \to u$ in $L^2(\delta, 1)$, we have that $\int_\delta^1 |u_n(x) - u(x)|^2 \mathrm{d}x \le \varepsilon^2/2$ for $n \ge n_0$ for some $n_0 \in \mathbb{N}$. Thus $\|u_n - u\|_{L^2(0,1)} \le \varepsilon$ for $n \ge n_0$, and $u_n \to u$ in $L^2(0,1)$. □

Lemma 2.6. *The embeddings $H_{a,\rho} \subset L^2(0,1)$ and $H_{a,\rho} \subset L^2_\rho$ are compact. The space $C^\infty([0,1])$ is dense in $H_{a,\rho}$.*

Proof of Lemma 2.6: By equation (1.7) one may pick some numbers $\delta \in (0, \frac{1}{2})$ and $C > 0$ such that

$$0 \le \rho(x) \le C \quad \forall x \in (0, 2\delta). \tag{2.18}$$

Let $\theta \in C^\infty([0,1])$ be such that

$$\theta(x) = \begin{cases} 1 & \text{if } \ 0 \le x \le \delta, \\ 0 & \text{if } \ 2\delta \le x \le 1. \end{cases}$$

Let $u \in H_{a,\rho}$. Then $\theta u \in H_a$, for $(\theta u)(1) = 0$ and

$$\sqrt{a}(\theta u)' = (\sqrt{a}u')\theta + \sqrt{a}\,\theta' u \in L^2(0,1).$$

Indeed, $\sqrt{a}u' \in L^2(0,1)$, Supp $\theta' \subset [\delta, 2\delta]$, $\sqrt{a} \in L^2(\delta, 2\delta)$, and $u \in L^\infty(\delta, 2\delta)$. Furthermore,

$$\|(\theta u)'\|_{L^2_a} \le C\|u\|_{H_{a,\rho}},$$

so that the map $u \in H_{a,\rho} \to \theta u \in H_a$ is continuous. The embedding $H_a \subset L^2(0,1)$ being compact, the map $u \in H_{a,\rho} \to \theta u \in L^2(0,1)$ is compact. Clearly, the map $u \in H_{a,\rho} \to (1-\theta)u \in W^{1,1}(0,1)$ is continuous. The embedding $W^{1,1}(0,1) \subset L^2(0,1)$ being compact, we infer that the map $u \in H_{a,\rho} \to (1-\theta)u \in L^2(0,1)$ is compact. Thus the embedding $H_{a,\rho} \subset L^2(0,1)$ is compact.

The fact that $H_{a,\rho} \subset L^2_\rho$ continuously comes from the definition of the spaces $H_{a,\rho}$ and L^2_ρ and of their norms. Using equation (4.2) and the lines above, we infer that the map $u \in H_{a,\rho} \to \theta u \in L^2_\rho$ is compact. On the other hand, the embedding $W^{1,1}(0,1) \subset L^{2r'}(0,1)$ is compact, and by Hölder inequality

$$\int_0^1 \rho(1-\theta)^2 u^2 \mathrm{d}x \le C\left(\int_0^1 \rho^r \mathrm{d}x\right)^{\frac{1}{r}} \left(\int_0^1 u^{2r'}\mathrm{d}x\right)^{\frac{1}{r'}}.$$

It follows that the map $u \in H_{a,\rho} \to (1-\theta)u \in L^2_\rho$ is compact. Thus the embedding $H_{a,\rho} \subset L^2_\rho$ is compact.

Let us prove that $C^\infty([0,1])$ is dense in $H_{a,\rho}$. Pick any $u \in H_{a,\rho}$. If we set

$$a(x) = x,\ \ \rho(x) = 1,\ \ u(x) = (2-x)u(1) \quad \forall x \in (1,2),$$

then $u \in H_a(0,2) := \{u \in L^1_{loc}(0,2);\ \sqrt{a}u' \in L^2(0,2) \text{ and } u(2) = 0\}$. As in Lemma 2.3, $C^\infty([0,2])$ is dense in $H_a(0,2)$, so that we can pick a sequence (φ_n) in $C^\infty([0,2])$ with $\varphi_n \to u$ in $H_a(0,2)$, and also in $L^2(0,2)$. This gives

$$\int_0^1 [a(\varphi_n' - u')^2 + (\varphi_n - u)^2]\mathrm{d}x \to 0.$$

By equation (4.2), we have $\int_0^{2\delta} \rho(\varphi_n - u)^2 \mathrm{d}x \to 0$. Since $\varphi_n \to u$ in $W^{1,1}(2\delta, 1)$ and in $L^{2r'}(2\delta, 1)$, we also have that

$$\int_{2\delta}^1 \rho(\varphi_n - u)^2 \mathrm{d}x \to 0.$$

We conclude that $\varphi_n \to u$ in $H_{a,\rho}$. □

The proof of Proposition 2.1 is complete. □

Next, we investigate the elliptic problem equations (2.1)–(2.3). Introduce the symmetric bilinear form

$$a(u,v) = \int_0^1 a u' v'\,\mathrm{d}x + a_b(u,v)$$

where

$$a_b(u,v) = \begin{cases} \frac{\alpha}{\beta} u(1)v(1) & \text{if } \beta \neq 0, \\ 0 & \text{if } \beta = 0. \end{cases}$$

Let

$$H = \begin{cases} H_{a,\rho} & \text{if } \alpha\beta \neq 0, \\ \{u \in H_{a,\rho};\ u(1) = 0\} & \text{if } \beta = 0, \\ \{u \in H_{a,\rho};\ \int_0^1 u\rho\,\mathrm{d}x = 0\} & \text{if } \alpha = 0, \end{cases}$$

be endowed with the norm $\|\cdot\|_{H_{a,\rho}}$. By equation (2.9), H is a closed subspace of $H_{a,\rho}$, and the bilinear form a is continuous on $H \times H$. To prove that the bilinear form is coercive, we need the following lemma.

Lemma 2.7. *There exist a constant $C > 0$ such that*

$$\int_0^1 |u|^2\rho\,\mathrm{d}x \le C\left(\int_0^1 |u'|^2 a\,\mathrm{d}x + u(1)^2\right) \qquad \forall u \in H_{a,\rho}, \tag{2.19}$$

$$\int_0^1 |u|^2\rho\,\mathrm{d}x \le C\left(\int_0^1 |u'|^2 a\,\mathrm{d}x + \Big(\int_0^1 u\rho\,\mathrm{d}x\Big)^2\right) \qquad \forall u \in H_{a,\rho}. \tag{2.20}$$

Proof of Lemma 2.7: We prove equation (2.19) only, the proof of equation (2.20) being similar. If equation (2.19) is false, one can pick a sequence (u_n) in $H_{a,\rho}$ such that

$$1 = \int_0^1 |u_n|^2\rho\,\mathrm{d}x > n\left(\int_0^1 |u_n'|^2 a\,\mathrm{d}x + u_n(1)^2\right). \tag{2.21}$$

As the sequence (u_n) is bounded in $H_{a,\rho}$, one can extract a subsequence (u_{n_k}) such that $u_{n_k} \to u$ weakly in $H_{a,\rho}$ for some $u \in H_{a,\rho}$. By Lemma 2.6, $u_{n_k} \to u$ strongly in L^2_ρ. As $u'_{n_k} \to 0$ strongly in L^2_a by equation (2.21), we infer that (u_{n_k}) is a Cauchy sequence in $H_{a,\rho}$, and hence $u_{n_k} \to u$ strongly in $H_{a,\rho}$. Letting $n \to \infty$ in equation (2.21), we obtain

$$\int_0^1 |u'|^2 a\,\mathrm{d}x = 0, \quad u(1) = 0.$$

Thus $u = 0$, but this contradicts the condition $\int_0^1 |u|^2\rho\,\mathrm{d}x = 1$. The proof of Lemma 2.7 is complete. □

We have to prove that

$$a(u,u) \ge K\|u\|^2_{H_{a,\rho}} \quad \forall u \in H \tag{2.22}$$

for some constant $K > 0$. If $\beta = 0$ (resp. $\alpha = 0$), then equation (2.19) (resp. Eq. (2.20)) yields $\|u\|^2_{L^2_\rho} \le C\|u'\|^2_{L^2_a}$ for $u \in H$, which gives equation (2.22).
If $\alpha\beta > 0$, then equation (2.19) yields

$$\int_0^1 |u|^2\rho\,\mathrm{d}x \le C' a(u,u)$$

for some $C' > 0$, which gives again equation (2.22). Thus the bilinear form a is coercive.

Let $f \in L^2_\rho$ be given (with also $\int_0^1 f\rho\,\mathrm{d}x = 0$ if $\alpha = 0$). Since the linear form $L(v) = \int_0^1 fv\rho\,\mathrm{d}x$ is continuous on H, it follows from Lax-Milgram theorem that there exists a unique function $u \in H$ such that

$$a(u,v) = L(v) \quad \forall v \in H. \tag{2.23}$$

Assume first $\alpha\beta \neq 0$. Clearly $\mathcal{D}(0,1) \subset H$. Picking $v \in \mathcal{D}(0,1)$ in equation (2.23), we obtain equation (2.1) in the sense of distributions. Multiplying in equation (2.1) by $v \in C^\infty([0,1]) \subset H$, integrating by parts and comparing with equation (2.23), we infer that

$$(au')(1)v(1) - (au')(0)v(0) + \frac{\alpha}{\beta}u(1)v(1) = 0.$$

Picking $v(x) = 1 - x$ (resp. $v(x) = 1$) yields equation (2.2) (resp. Eq. (2.3)).

If $\beta = 0$, we have again $\mathcal{D}(0,1) \subset H$, and picking $v \in \mathcal{D}(0,1)$ in equation (2.23) yields equation (2.1) in the sense of distributions. Multiplying in equation (2.1) by $v(x) = 1 - x$ and integrating by parts yields equation (2.2). Finally equation (2.3) comes from the definition of H.

If $\alpha = 0$, then for any given $\theta \in \mathcal{D}(0,1)$ the function $\varphi(x) = \theta(x) - \int_0^1 \theta\rho \mathrm{d}s$ belongs to H. Picking such a function in equation (2.23) and using the fact that $\int_0^1 f\rho\,\mathrm{d}x = 0$, we obtain $\int_0^1 au'\theta'\mathrm{d}x = \int_0^1 f\theta\rho\,\mathrm{d}x$, so that equation (2.1) holds in the sense of distributions. Multiplying in equation (2.1) by $v(x) = 1$ (resp. by $v(x) = 1 - 2x$), we obtain equations (2.2)–(2.3).

Finally, using

$$\int_0^1 |\rho f|\mathrm{d}x \le \left(\int_0^1 \rho f^2 \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_0^1 \rho\,\mathrm{d}x\right)^{\frac{1}{2}} < \infty,$$

we infer from equation (2.1) that $au' \in W^{1,1}(0,1)$ for any value of (α,β).

We are in a position to study the spectral problem associated with equations (2.1)–(2.3).

Theorem 2.8. *Let a, ρ and (α,β) be as above. Then there are a sequence $(e_n)_{n\ge0}$ in L^2_ρ and a nondecreasing sequence $(\lambda_n)_{n\ge0}$ in $(0,+\infty)$ such that*
(i) $(e_n)_{n\ge0}$ is an orthonormal basis in L^2_ρ;
(ii) for all $n \ge 0$, $e_n \in H_{a,\rho}$, $ae_n' \in W^{1,\min(2,r)}(0,1)$, and e_n solves

$$-(ae_n')' = \lambda_n\,\rho\,e_n \quad \text{in } (0,1), \tag{2.24}$$
$$(ae_n')(0) = 0, \tag{2.25}$$
$$\alpha e_n(1) + \beta(ae_n')(1) = 0. \tag{2.26}$$

Proof. Assume first that $\alpha \neq 0$. For $f \in L^2_\rho$, let $T(f)$ denote the unique solution $u \in H$ of equation (2.23). The operator $T : f \in L^2_\rho \to u = T(f) \in L^2_\rho$ is continuous, compact, and selfadjoint by equations (2.22), (2.23), and Proposition 2.1. It is also positive definite, for

$$K\|u\|^2_{H_{a,\rho}} \le a(u,u) = (f,u)_{L^2_\rho}.$$

By the spectral theorem, there are an orthonormal basis $(e_n)_{n\ge0}$ in L^2_ρ and a sequence $(\mu_n)_{n\ge0}$ in $(0,+\infty)$ with $\mu_n \searrow 0$ such that $T(e_n) = \mu_n e_n$ for all $n \ge 0$. Then equations (2.24)–(2.26) hold with $\lambda_n := \mu_n^{-1} > 0$.

Assume now that $\alpha = 0$, and let $V := \{f \in L^2_\rho;\ \int_0^1 f\rho\,\mathrm{d}x = 0\}$. For $f \in V$, let $T(f)$ denote the unique solution $u \in H$ of equation (2.23). Again, the operator $T : f \in V \to u = T(f) \in V$ is continuous, compact, selfadjoint and positive definite. Therefore there are an orthonormal basis $(e_n)_{n\ge1}$ in V and a sequence $(\mu_n)_{n\ge1}$ in $(0,+\infty)$ with $\mu_n \searrow 0$ such that $T(e_n) = \mu_n e_n$ for all $n \ge 1$. Let $e_0 := 1$. Noticing that $V = \{u \in L^2_\rho;\ (u,e_0)_{L^2_\rho} = 0\}$, we see that $(e_n)_{n\ge0}$ is a orthonormal basis of L^2_ρ and that equations (2.24)–(2.26) hold with $\lambda_0 := 0$ and $\lambda_n := \mu_n^{-1} > 0$ for $n \ge 1$.

By equation (1.7), one can pick some numbers $C > 0$ and $\delta \in (0,1)$ such that $0 \le \rho(x) \le C$ for $0 < x < \delta$, so that $(ae_n')' = -\lambda_n\rho e_n \in L^2(0,\delta)$. Next, $e_n \in W^{1,1}(\delta,1) \subset L^\infty(\delta,1)$, and $(ae_n')' \in L^r(\delta,1)$. Thus $(ae_n')' \in L^{\min(2,r)}(0,1)$ and $ae_n' \in W^{1,\min(2,r)}(0,1)$. □

We are now interested in the asymptotic behavior of the eigenvalues λ_n, $n \geq 0$. Indeed, to apply the flatness approach, we need to prove that $\lambda_n \geq Cn^\kappa$ for some $C, \kappa > 0$ and all $n \geq 0$. The estimate we shall derive is likely not sharp, but it is sufficient for the sequel.

Theorem 2.9. *Let $a, \rho, (\alpha, \beta)$ and the sequences $(e_n)_{n\geq 0}$, $(\lambda_n)_{n\geq 0}$ be as in Theorem 2.8. Then*
(i) $e_n \in W^{1,1}(0,1)$ and $ae_n' \in W^{1,r}(0,1)$ for all $n \geq 0$;
(ii) there exists some constant $C_1 > 0$ such that

$$\|e_n\|_{L^\infty(0,1)} \leq C_1 \lambda_n^{\frac{3}{4}(1+\frac{p'r}{r-p'})} \qquad \text{if } \lambda_n > 0; \tag{2.27}$$

(For $r = \infty$, $\frac{p'r}{r-p'} = p'$.)
(iii) let $\kappa := [\frac{1}{2} + \frac{1}{p}(\frac{p'r}{r-p'})]^{-1} > 0$ if $p < \infty$ and pick any $\kappa < 2$ if $p = \infty$. Then there exists some constant $C_2 > 0$ such that

$$\lambda_n \geq C_2 n^\kappa \quad \forall n \geq 0. \tag{2.28}$$

Remark 2.10. The estimates equations (2.27) and (2.28) are not sharp, but sufficient for our aim which is to apply the flatness approach. In the classical case when $a(x) = x^{2-\varepsilon}$, $q(x) = 0$ and $\rho(x) = 1$ for all $x \in (0,1)$ with $(\alpha, \beta) = (1, 0)$, so that $e_n(1) = 0$, it follows from the proof of Proposition 1.4 (see below) that $1 < p < (1-\varepsilon)^{-1}$ and $r = \infty$, so that

$$\frac{3}{4}(1 + \frac{p'r}{r-p'}) = \frac{3}{4}(1+p') > \frac{3}{4}(1+\varepsilon^{-1}), \tag{2.29}$$

$$\kappa = (\frac{1}{2} + \frac{p'}{p})^{-1} = (p' - \frac{1}{2})^{-1} < \varepsilon(1 - \frac{\varepsilon}{2})^{-1}. \tag{2.30}$$

On the other hand, it is well known (see *e.g.* [26]) that, letting $\nu = \varepsilon^{-1} - 1 > 0$, we have

$$e_n(x) = \frac{\sqrt{\varepsilon} x^{\frac{\varepsilon-1}{2}}}{|J_{\nu+1}(j_{\nu,n})|} J_\nu(j_{\nu,n} x^{\frac{\varepsilon}{2}}), \tag{2.31}$$

$$\lambda_n = (\frac{\varepsilon}{2} j_{\nu,n})^2 \tag{2.32}$$

where

$$J_\nu(z) := \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu} \tag{2.33}$$

is the Bessel function of order ν of the first kind, and $(j_{\nu,n})_{n\in\mathbb{N}^*}$ is the increasing sequence of zeros of J_ν, which are real and satisfy

$$\nu < j_{\nu,n} < j_{\nu,n+1} \quad \forall n \in \mathbb{N}^* \tag{2.34}$$

$$\lim_{n\to+\infty} (j_{\nu,n+1} - j_{\nu,n}) = \pi. \tag{2.35}$$

It follows from equations (2.32) and (2.35) that

$$\lambda_n \sim \left(\frac{\varepsilon\pi}{2}\right)^2 n^2 \qquad \text{as } n \to +\infty, \tag{2.36}$$

so that $\kappa < \varepsilon(1-\frac{\varepsilon}{2})^{-1}$ is far (resp. is not far) from the exponent 2 in equation (2.36) as $\varepsilon \to 0^+$ (resp. as $\varepsilon \to 1^-$). On the other hand, we know from [26] that

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}}\cos(z - \frac{\nu\pi}{2} - \frac{\pi}{4}) + O(\frac{1}{z\sqrt{z}}) \quad \text{as } z \to +\infty,$$

so that the continuous function $f_\nu(z) := z^{-\nu}J_\nu(z)$ satisfies $\lim_{z\to+\infty} f_\nu(z) = 0$. Since $\lim_{z\to 0^+} f_\nu(z) = \frac{1}{\Gamma(\nu+1)2^\nu}$, we infer that $f_\nu \in L^\infty(\mathbb{R}_+)$. Clearly

$$e_n(x) = \frac{\sqrt{\varepsilon}j_{\nu,n}^\nu}{|J_{\nu+1}(j_{\nu,n})|} f_\nu(j_{\nu,n}x^{\frac{\varepsilon}{2}})\cdot$$

But we have as $n \to +\infty$

$$\sqrt{j_{\nu,n}}|J_{\nu+1}(j_{\nu,n})| = \sqrt{\frac{2}{\pi}} + O(\frac{1}{j_{\nu,n}})$$

(see again [26]). It follows that

$$\frac{j_{\nu,n}^\nu}{|J_{\nu+1}(j_{\nu,n})|} \sim \sqrt{\frac{\pi}{2}} j_{\nu,n}^{\nu+\frac{1}{2}} = \sqrt{\frac{\pi}{2}}\left(\frac{2}{\varepsilon}\lambda_n^{\frac{1}{2}}\right)^{\nu+\frac{1}{2}}$$

and that

$$\|e_n\|_{L^\infty(0,1)} \leq C(\varepsilon)\lambda_n^{\frac{1}{2}(\varepsilon^{-1}-\frac{1}{2})}$$

for some constant $C(\varepsilon) > 0$. The sharp exponent $\frac{1}{2}(\varepsilon^{-1}-\frac{1}{2})$ of λ_n is much smaller than the lower bound $\frac{3}{4}(\varepsilon^{-1}+1)$ of the exponent of λ_n in equation (2.27).

Proof. (i) We need several lemmas.

Lemma 2.11. *Let p, p' be as in equation (1.10), let a be as in equations (1.5)–(1.6), and let $q \in L^{p'}(0,1)$. Then there exists a unique function $v \in W^{1,1}(0,1)$ with $av' \in W^{1,p'}(0,1)$ and such that*

$$(av')' = -qv \quad \textit{a.e. in } (0,1), \tag{2.37}$$
$$(av')(0) = 0, \tag{2.38}$$
$$v(0) = 1. \tag{2.39}$$

Proof of Lemma 2.11. If a function v as in Lemma 2.11 does exist, then $v \in C^0([0,1])$ and we obtain by successive integrations

$$(av')(y) = -\int_0^y (qv)(s)\mathrm{d}s \quad \forall y \in [0,1], \tag{2.40}$$

and

$$v(x) = 1 - \int_0^x \frac{\mathrm{d}y}{a(y)}\int_0^y (qv)(s)\mathrm{d}s \quad \forall x \in [0,1]. \tag{2.41}$$

Using Hardy inequality

$$\int_0^1 [\frac{1}{y}\int_0^y |q(s)|\mathrm{d}s]^{p'}\mathrm{d}y \le C\int_0^1 |q(s)|^{p'}\mathrm{d}s,$$

Hölder inequality and equation (1.6), we infer that

$$\int_0^1 \frac{\mathrm{d}y}{a(y)}\int_0^y |q(s)|\mathrm{d}s < \infty. \tag{2.42}$$

Therefore, if $v \in C^0([0,1])$ satisfies equation (2.41), then $v \in W^{1,1}(0,1)$ by equation (2.42) and $av' \in W^{1,p'}(0,1)$ by equation (2.40). Therefore, it is sufficient to prove the existence and uniqueness of a solution $v \in C^0([0,1])$ of equation (2.41).

First, one may pick some $\delta \in (0,1)$ such that for all $x_1, x_2 \in [0,1]$ with $0 \le x_2 - x_1 \le \delta$, we have

$$\int_{x_1}^{x_2} \frac{\mathrm{d}y}{a(y)}\int_0^y |q(s)|\mathrm{d}s \le \frac{1}{2}.$$

For $0 \le x_1 < x_2 \le 1$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ with $\gamma_1 = 0$ if $x_1 = 0$, we consider the map $\Gamma : C^0([x_1,x_2]) \to C^0([x_1,x_2])$ defined by

$$\Gamma(v)(x) = \gamma_2 - \int_{x_1}^{x} \frac{\mathrm{d}y}{a(y)}\left(\gamma_1 + \int_{x_1}^y (qv)(s)\mathrm{d}s\right). \tag{2.43}$$

By equations (1.6) and (2.42), the map Γ is well defined. Let us prove that it is a contraction provided that $|x_2 - x_1|$ is "small enough". For $v, w \in C^0([x_1,x_2])$, we have

$$\|\Gamma(v) - \Gamma(w)\|_{L^\infty(x_1,x_2)} \le \int_{x_1}^{x_2} \frac{\mathrm{d}y}{a(y)}\int_{x_1}^y |q(s)(v-w)(s)|\mathrm{d}s.$$

The map Γ is a contraction in $C^0([0,\delta])$ for $x_1 = 0$, $x_2 = \delta$, $\gamma_1 = 0$ and $\gamma_2 = 1$, since

$$\|\Gamma(v) - \Gamma(w)\|_{L^\infty(0,\delta)} \le \frac{1}{2}\|v - w\|_{L^\infty(0,\delta)}.$$

This gives a solution v of equation (2.41) on $[0,\delta]$ by the contraction principle. Note that

$$v(\delta) = 1 - \int_0^\delta \frac{\mathrm{d}y}{a(y)}\int_0^y (qv)(s)\mathrm{d}s. \tag{2.44}$$

If $\delta \le x_1 \le x_2 \le 1$ and $x_2 - x_1 \le \delta$, then

$$\begin{aligned}\|\Gamma(v) - \Gamma(w)\|_{L^\infty(x_1,x_2)} &\le \left(\int_{x_1}^{x_2} \frac{\mathrm{d}y}{a(y)}\int_{x_1}^y |q(s)|\mathrm{d}s\right)\|v - w\|_{L^\infty(x_1,x_2)}\\ &\le \frac{1}{2}\|v - w\|_{L^\infty(x_1,x_2)}.\end{aligned}$$

Then Γ is a contraction in $C^0([x_1,x_2])$ if $\delta \le x_1 \le x_2 \le 1$ and $x_2 - x_1 \le \delta$. Picking $x_1 = \delta$, $x_2 = x_1 + \delta$, $\gamma_1 = \int_0^\delta (qv)(s)\mathrm{d}s$, and $\gamma_2 = v(\delta)$, we obtain a solution of equation (2.41) on $[0, 2\delta]$. We can proceed in a similar way to extend v to $[0,3\delta]$, $[0,4\delta]$,... and finally to $[0,1]$. □

Corollary 2.12. *Let ρ and r be as in equations (1.7) and (1.10), and let $\lambda \in \mathbb{R}$. Then there exists a unique function $u \in W^{1,1}(0,1)$ with $au' \in W^{1,r}(0,1)$ such that*

$$(au')' = -\lambda \rho u \quad \textit{a.e. in } (0,1), \tag{2.45}$$

$$(au')(0) = 0, \tag{2.46}$$

$$u(0) = 1. \tag{2.47}$$

Furthermore, the map $\lambda \to (u, au')$ from $\mathbb{R}$ to $W^{1,1}(0,1) \times W^{1,r}(0,1)$ is continuous.

Proof. Since $\rho \in L^r(0,1) \subset L^{p'}(0,1)$, the existence and uniqueness of u follows from Lemma 2.11. The continuity of the map $\lambda \in \mathbb{R} \to u \in C^0([0,\delta])$ (or $C^0([\delta, 2\delta])$, etc.) follows from the version of the contraction principle with a parameter. Using equations (2.40) and (2.41) gives the last sentence in the statement. $\square$

Lemma 2.13. *Let $\lambda \in \mathbb{R}$. If $0 < x_0 - \delta < x_0 + \delta < 1$ and $u \in W^{1,1}(x_0 - \delta, x_0 + \delta)$ is such that $au' \in W^{1,1}(x_0 - \delta, x_0 + \delta)$ and*

$$\begin{cases} (au')' = -\lambda \rho u & \textit{in } (x_0 - \delta, x_0 + \delta), \\ u(x_0) = 0, \\ (au')(x_0) = 0, \end{cases}$$

then $u = 0$ in $(x_0 - \delta, x_0 + \delta)$.

The proof is similar to those of Lemma 2.11 by applying the contraction principle to the map Γ from $C^0([x_0 - \delta, x_0 + \delta])$ into itself defined by

$$\Gamma(u)(x) = -\lambda \int_{x_0}^{x} \frac{\mathrm{d}y}{a(y)} \int_{x_0}^{y} (\rho u)(s) \mathrm{d}s$$

when $\delta > 0$ is small enough, and by propagating the uniqueness up to $[x_0 - \delta, x_0 + \delta]$ when δ is as in the statement of the lemma. $\square$

Lemma 2.14. *Let $n \geq 0$, and let e_n and λ_n be as in Theorem 2.8. If u is the function associated with $\lambda = \lambda_n$ in Corollary 2.12, then*

$$e_n(x) = e_n(0) u(x) \quad \forall x \in (0,1), \tag{2.48}$$

so that $e_n \in W^{1,1}(0,1)$ and $ae_n' \in W^{1,r}(0,1)$.

Proof of Lemma 2.14: As $u(0) = 1$ and $u \in C^0([0,1])$, we can pick some $\varepsilon \in (0, 1/2)$ such that

$$u(x) \geq \frac{1}{2} \quad \forall x \in [0, 2\varepsilon].$$

Let

$$v(x) := u(x) \int_{\varepsilon}^{x} \frac{\mathrm{d}s}{u(s)^2 a(s)} \quad x \in (0, 2\varepsilon].$$

Since $a^{-1} \in L^1_{loc}(0,1)$, $v \in W^{1,1}_{loc}(0, 2\varepsilon)$ and

$$v'(x) = u'(x) \int_{\varepsilon}^{x} \frac{\mathrm{d}s}{u(s)^2 a(s)} + \frac{1}{u(x) a(x)} \quad \text{a.e. in } (0, 2\varepsilon). \tag{2.49}$$

Since $u, au' \in W^{1,1}(0,1)$, it follows that $av' \in W^{1,1}_{loc}(0,2\varepsilon)$ and that

$$\begin{aligned}(av')'(x) &= (au')'(x)\int_\varepsilon^x \frac{\mathrm{d}s}{u(s)^2a(s)} + \frac{(au')(x)}{u(x)^2a(x)} - \frac{u'(x)}{u(x)^2}\\ &= -\lambda(\rho u)(x)\int_\varepsilon^x \frac{\mathrm{d}s}{u(s)^2a(s)}\\ &= -\lambda(\rho v)(x) \qquad \text{a.e. in } (0,2\varepsilon).\end{aligned}$$

Note that the function u satisfies equation (2.45) for $\lambda = \lambda_n$ in $(0,2\varepsilon)$ and that it is not proportional to v (otherwise we would have $[u(s)^2a(s)]^{-1} = 0$ a.e.). Note also that $v(\varepsilon) = 0$ and $(av')(\varepsilon) = 1/u(\varepsilon) > 0$. We may therefore find some number $\mu \in \mathbb{R}$ so that the function $U(x) := e_n(x) - \frac{e_n(\varepsilon)}{u(\varepsilon)}u(x) - \mu v(x)$ satisfies the assumptions of Lemma 2.13 for $0 < \delta < x_0 = \varepsilon$. We infer from Lemma 2.13 that $U = 0$ on $(0,2\varepsilon)$, that is

$$e_n(x) = \frac{e_n(\varepsilon)}{u(\varepsilon)}u(x) + \mu v(x), \quad x \in (0,2\varepsilon). \tag{2.50}$$

If $\mu \neq 0$, then by equations (2.25), (2.46) and (2.50), we obtain $(av')(0) = 0$ which, combined to equations (2.47) and (2.49), yields

$$\Big[au'(x)\int_\varepsilon^x \frac{\mathrm{d}s}{u(s)^2a(s)}\Big](0^+) = -1.$$

But this is impossible, since $\int_\varepsilon^x \frac{\mathrm{d}s}{u(s)^2a(s)} < 0$ and $(au')(x) \le 0$ for $0 < x < \varepsilon$. Indeed, $\lambda_n \ge 0$ yields

$$(au')(x) = -\int_0^x \lambda_n\rho(x)u(x)\mathrm{d}x \le 0, \quad 0 < x < \varepsilon.$$

Thus $\mu = 0$ and $e_n(x) = \frac{e_n(\varepsilon)}{u(\varepsilon)}u(x)$ for $x \in (0,2\varepsilon)$.

Using Lemma 2.13 several times, we infer that

$$e_n(x) = \frac{e_n(\varepsilon)}{u(\varepsilon)}u(x) \quad \forall x \in (0,1).$$

Thus, by Corollary 2.12, $e_n \in W^{1,1}(0,1)$, $ae_n' \in W^{1,r}(0,1)$, and letting $x \to 0$ yields $e_n(0) = e_n(\varepsilon)/u(\varepsilon)$, so that

$$e_n(x) = e_n(0)u(x) \quad \forall x \in (0,1).$$

The proof of Lemma 2.14 is complete. This ends the proof of (i) in Theorem 2.9.

(ii) In what follows, we fix some $n \ge 0$ for which $\lambda_n > 0$ and denote $e = e_n$ and $\lambda = \lambda_n$ to simplify the writing. The letter C will denote a constant that may change from line to line.

- Assume first that $\alpha \neq 0$. We first check that

$$|e(1)| \le \sqrt{\frac{\beta}{\alpha}}\sqrt{\lambda} \tag{2.51}$$

The estimate equation (2.51) is obvious if $\beta = 0$, for $e(1) = 0$. If $\beta \neq 0$, we infer from equation (2.23) with $u = v = e$ that

$$\int_0^1 a|e'|^2 \mathrm{d}x + \frac{\alpha}{\beta} e(1)^2 = \lambda \int_0^1 e^2 \rho \, \mathrm{d}x.$$

This yields

$$|e(1)|^2 \leq \frac{\beta}{\alpha} \lambda \int_0^1 e^2 \rho \mathrm{d}x = \frac{\beta}{\alpha} \lambda.$$

Note also that

$$\int_0^1 a|e'|^2 \mathrm{d}x \leq \lambda \int_0^1 e^2 \rho \mathrm{d}x = \lambda.$$

But for $0 < x < 1$ we have that

$$|e(x) - e(1)| \leq \int_x^1 |e'(s)| \mathrm{d}s \leq \left(\int_x^1 \frac{\mathrm{d}s}{a(s)} \right)^{\frac{1}{2}} \left(\int_x^1 a e'^2 \mathrm{d}s \right)^{\frac{1}{2}} \leq \sqrt{\lambda} \left(\int_x^1 \frac{\mathrm{d}s}{a(s)} \right)^{\frac{1}{2}}$$

and

$$\int_x^1 \frac{\mathrm{d}s}{a(s)} \leq \frac{1}{x} \int_x^1 \frac{s}{a(s)} \mathrm{d}s \leq \frac{1}{x} \| \frac{s}{a(s))} \|_{L^1} \tag{2.52}$$

where $\|\frac{s}{a(s)}\|_{L^1}$ denotes $\int_0^1 \frac{s}{a(s)} \mathrm{d}s$. It follows that

$$|e(x)| \leq (\frac{\beta}{\alpha} \lambda)^{\frac{1}{2}} + \left(\| \frac{s}{a(s)} \|_{L^1} \frac{\lambda}{x} \right)^{\frac{1}{2}}, \quad 0 < x < 1. \tag{2.53}$$

On the other hand, for $0 < x < 1$, by Hölder inequality

$$\begin{aligned} \int_0^x \frac{1}{a(s)} (\int_0^s \rho(t) \mathrm{d}t) \mathrm{d}s &\leq \|\rho\|_{L^r} \int_0^x \frac{s^{1-\frac{1}{r}}}{a(s)} \mathrm{d}s \\ &\leq \|\rho\|_{L^r} \| \frac{s}{a(s)} \|_{L^p} \left(\int_0^x s^{-\frac{p'}{r}} \mathrm{d}s \right)^{\frac{1}{p'}} \\ &\leq C x^{\frac{1}{p'} - \frac{1}{r}} \end{aligned} \tag{2.54}$$

with $C := \|\rho\|_{L^r} \|\frac{s}{a(s)}\|_{L^p} (1 - \frac{p'}{r})^{-\frac{1}{p'}}$. The function $u(x)$ still denoting the solution to equations (2.45)–(2.47), we have that

$$u'(x) = -\frac{\lambda}{a(x)} \int_0^x (\rho u)(s) \mathrm{d}s \leq 0$$

if $u \geq 0$ on $[0, x]$, and hence for such x

$$0 \leq 1 - u(x) \leq \int_0^x |u'(s)| \mathrm{d}s \leq \lambda \|u\|_{L^\infty(0,x)} \int_0^x \frac{1}{a(s)} (\int_0^s \rho(t) \mathrm{d}t) \mathrm{d}s \leq C \lambda x^{\frac{1}{p'} - \frac{1}{r}}.$$

It follows that

$$\frac{1}{2} \leq u(x) \leq 1 \qquad \text{for } x \in [0, (2C\lambda)^{\frac{p'r}{p'-r}}].$$

(Note that $\lambda^{\frac{p'r}{p'-r}} \to 0$ as $\lambda \to \infty$, for $r > p'$.) Replacing e by $-e$ if needed, we can assume that $e(0) > 0$. Using Lemma 2.14, we infer that

$$\frac{e(0)}{2} \leq e(x) \leq e(0) \qquad \text{for } x \in [0, (2C\lambda)^{\frac{p'r}{p'-r}}]. \tag{2.55}$$

It follows from equation (2.24) that

$$\begin{aligned} e(0)^2 - e(1)^2 &= -2\int_0^1 ee'\mathrm{d}x = 2\lambda \int_0^1 \frac{e(x)}{a(x)}(\int_0^x \rho e ds)\mathrm{d}x \\ &= 2\lambda\Big(\int_0^{(4C\lambda)^{\frac{p'r}{p'-r}}} \cdots \mathrm{d}x + \int_{(4C\lambda)^{\frac{p'r}{p'-r}}}^1 \cdots \mathrm{d}x\Big) =: 2\lambda(I_1 + I_2). \end{aligned} \tag{2.56}$$

Then, by equations (2.54) and (2.55),

$$|I_1| \leq e(0)^2 \int_0^{(4C\lambda)^{\frac{p'r}{p'-r}}} \frac{1}{a(x)}(\int_0^x \rho(s)\mathrm{d}s)\mathrm{d}x \leq \frac{e(0)^2}{4\lambda}. \tag{2.57}$$

On the other hand, by equation (2.53), we have for $x \in ((4C\lambda)^{\frac{p'r}{p'-r}}, 1)$

$$\begin{aligned} |e(x)| &\leq (\frac{\beta}{\alpha}\lambda)^{\frac{1}{2}} + (\|\frac{s}{a(s)}\|_{L^1}(4C)^{-\frac{p'r}{p'-r}})^{\frac{1}{2}}\lambda^{\frac{1}{2}(1-\frac{p'r}{p'-r})} \\ &\leq C'\lambda^{\frac{1}{2}(1+\frac{p'r}{r-p'})} \end{aligned} \tag{2.58}$$

for $\lambda \geq 1$, where $C' := \sqrt{\frac{\beta}{\alpha}} + \|\frac{s}{a(s)}\|_{L^1}^{\frac{1}{2}}(4C)^{-\frac{p'r}{2(p'-r)}}$. Since by Cauchy-Schwarz inequality

$$\int_0^x \rho|e|\mathrm{d}s \leq \Big(\int_0^1 \rho e^2\mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_0^1 \rho\mathrm{d}x\Big)^{\frac{1}{2}} = \|\rho\|_{L^1}^{\frac{1}{2}}$$

we obtain with equation (2.52) that

$$|I_2| \leq \int_{(4C\lambda)^{\frac{p'r}{p'-r}}}^1 \frac{|e(x)|}{a(x)}(\int_0^x \rho(s)|e(s)|\mathrm{d}s)\mathrm{d}x \leq C'\frac{\lambda^{\frac{1}{2}(1+\frac{p'r}{r-p'})}}{(4C\lambda)^{\frac{p'r}{p'-r}}}\|\frac{s}{a(s)}\|_{L^1}\|\rho\|_{L^1}^{\frac{1}{2}}. \tag{2.59}$$

Gathering together equations (2.51) and (2.56)–(2.59), we infer that

$$\frac{1}{2}e(0)^2 \leq \frac{\beta}{\alpha}\lambda + \frac{2C'}{(4C)^{\frac{p'r}{p'-r}}}\|\frac{s}{a(s)}\|_{L^1}\|\rho\|_{L^1}^{\frac{1}{2}}\lambda^{\frac{3}{2}(1+\frac{p'r}{r-p'})}.$$

It follows that

$$|e(0)| \leq C'' \lambda^{\frac{3}{4}(1+\frac{p'r}{r-p'})}$$

for $\lambda \geq 1$, where $C'' = C''(p, r, \|\rho\|_{L^1}, \|\frac{s}{a(s)}\|_{L^1}) > 0$. As

$$e(y)^2 - e(1)^2 = 2\lambda \int_y^1 \frac{e(x)}{a(x)} (\int_0^x \rho e ds) \mathrm{d}x, \quad \forall y \in (0,1),$$

and $|e(1)| \leq \sqrt{\beta\lambda/\alpha}$, the same calculations as above yield

$$|e(y)| \leq C'' \lambda^{\frac{3}{4}(1+\frac{p'r}{r-p'})} \quad \forall y \in (0,1) \tag{2.60}$$

for $\lambda \geq 1$.

• Assume now that $\alpha = 0$. Then $\int_0^1 \rho e \mathrm{d}x = 0$ and hence there exists $\bar{x} \in ((2C\lambda)^{\frac{p'r}{p'-r}}, 1]$ such that $e(\bar{x}) = 0$. For any $y \in (0,1)$, we have

$$\begin{aligned} e(y)^2 &= 2\lambda \int_y^{\bar{x}} \frac{e(x)}{a(x)} (\int_0^x \rho e \mathrm{d}s) \mathrm{d}x \\ &\leq 2\lambda \left(\int_0^{(4C\lambda)^{\frac{p'r}{p'-r}}} \cdots \mathrm{d}x + \int_{(4C\lambda)^{\frac{p'r}{p'-r}}}^1 \cdots \mathrm{d}x \right) =: 2\lambda(I_1 + I_2). \end{aligned}$$

The estimate equation (2.57) for I_1 is still valid without any change. For I_2, we first notice that for $x > (4C\lambda)^{\frac{p'r}{p'-r}}$,

$$|e(x)| \leq \left| \int_x^{\bar{x}} e'(s) \, \mathrm{d}s \right| \leq \sqrt{\lambda} \left(\int_{\min(x,\bar{x})}^1 \frac{\mathrm{d}s}{a(s)} \right)^{\frac{1}{2}} \leq \left(\|\frac{s}{a(s)}\|_{L^1} \frac{\lambda}{(4C\lambda)^{\frac{p'r}{p'-r}}} \right)^{\frac{1}{2}},$$

so that equation (2.58) holds again. Therefore equations (2.59) and (2.60) hold.

• Since $\lambda_n > 0$ for $n \geq 1$ and since the number of λ_n's in $(0,1)$ is finite, we infer that equation (2.60) is still valid for all the eigenvalues $\lambda_n > 0$ by replacing the constant C'' by a larger one denoted C_1. The proof of (ii) is complete.

(iii) Let $\lambda \in [0, \infty)$, let u given by Corollary 2.12 and let $e(x) := e(0)u(x)$ with $e(0) > 0$. We use a modified Prüfer substitution (see *e.g.* [6, 29]). We set

$$ae' = \lambda^{\frac{1}{4}} R \cos\theta, \tag{2.61}$$

$$e = \lambda^{-\frac{1}{4}} R \sin\theta, \tag{2.62}$$

so that

$$\begin{aligned} R &= (\lambda^{-\frac{1}{2}} (ae')^2 + \lambda^{\frac{1}{2}} e^2)^{\frac{1}{2}}, \\ \cot\theta &= \frac{\cos\theta}{\sin\theta} = \lambda^{-\frac{1}{2}} \frac{ae'}{e}. \end{aligned}$$

We can impose that $\theta(0) = \frac{\pi}{2}$, for $(ae')(0) = 0$ and $e(0) > 0$.

We note that $R \in W^{1,1}(0,1)$, since

$$\begin{aligned} R &\leq \lambda^{-\frac{1}{4}}|ae'| + \lambda^{\frac{1}{4}}|e|, \\ |R'| &\leq R^{-1}|\lambda^{-\frac{1}{2}}(ae')(ae')' + \lambda^{\frac{1}{2}}ee'| \\ &\leq (\lambda^{-\frac{1}{2}}|(ae')'|^2 + \lambda^{\frac{1}{2}}|e'|^2)^{\frac{1}{2}} \\ &\leq \lambda^{-\frac{1}{4}}|(ae')'| + \lambda^{\frac{1}{4}}|e'| \end{aligned}$$

and $e, ae' \in W^{1,1}(0,1)$.

Since $R \in C^0([0,1])$ and $R(x) > 0$ for all $x \in [0,1]$ by Lemma 2.13, we infer that $\inf_{x\in[0,1]} R(x) > 0$ and $R^{-1} \in W^{1,1}(0,1)$. Thus $\cos\theta = \lambda^{-\frac{1}{4}}\frac{ae'}{R}$ and $\sin\theta = \lambda^{\frac{1}{4}}\frac{e}{R}$ are both in $W^{1,1}(0,1)$. The functions arcsin and arccos being both of class C^1 on $(-1,1)$, we infer that $\theta \in W^{1,1}(0,1)$.

We obtain by straightforward computations that the pair (R,θ) solves the following Cauchy problem

$$R' = \lambda^{\frac{1}{2}}R(\frac{1}{a} - \rho)\cos\theta\sin\theta, \tag{2.63}$$

$$\theta' = \lambda^{\frac{1}{2}}(\rho\sin^2\theta + \frac{1}{a}\cos^2\theta), \tag{2.64}$$

$$R(0) = \lambda^{\frac{1}{4}}e(0), \tag{2.65}$$

$$\theta(0) = \frac{\pi}{2} \tag{2.66}$$

on $(0,1)$. Conversely, if $R, \theta \in W^{1,1}(0,1)$ satisfy equations (2.63)–(2.66), then the function e, defined in equation (2.62), is in $W^{1,1}(0,1)$, equation (2.61) holds, $(ae')' = -\lambda\rho e$ a.e. in $(0,1)$, and $e(x) = e(0)u(x)$, where u is as given in Corollary 2.12.

Lemma 2.15. *The map $\lambda \to \theta(x,\lambda)$ is continuous and strictly increasing for all $x \in (0,1]$.*

Proof of Lemma 2.15: The continuity of the map $\lambda \to \theta(x,\lambda)$ follows from those of the maps $\lambda \in \mathbb{R}_+ \to u \in W^{1,1}(0,1)$ and $\lambda \in \mathbb{R}_+ \to au' \in W^{1,r}(0,1)$ and from the definition of θ.

Let us show that the map $\lambda \to \theta(x,\lambda)$ is strictly increasing for all $x \in (0,1]$. Assume $\lambda_1 < \lambda_2$, and let θ_1 and θ_2 be associated with λ_1 and λ_2, respectively. Let $w := \theta_2 - \theta_1$. Then $w(0) = 0$, and we have a.e. in $(0,1)$

$$w' = \underbrace{(\lambda_2^{\frac{1}{2}} - \lambda_1^{\frac{1}{2}})\left[\rho\sin^2\theta_2 + \frac{1}{a}\cos^2\theta_2\right]}_{=:J_1} + \underbrace{\lambda_1^{\frac{1}{2}}\left[\rho(\sin^2\theta_2 - \sin^2\theta_1) + \frac{1}{a}(\cos^2\theta_2 - \cos^2\theta_1)\right]}_{=:J_2}.$$

Then $J_1 > 0$ and $|J_2| \leq 2\lambda_1^{\frac{1}{2}}(\rho + \frac{1}{a})|w|$ a.e. in $(0,1)$, where we used the mean value theorem. It follows that

$$w' > -2\lambda_1^{\frac{1}{2}}(\rho + \frac{1}{a})|w| \quad \text{ a.e. in } (0,1).$$

Assume that there exists $d \in (0,1]$ with $w(d) < 0$. If $[c,d]$ denotes the largest segment to the left of d where $w \leq 0$, then

$$w' > 2\lambda_1^{\frac{1}{2}}(\rho + \frac{1}{a})w \quad \text{ a.e. in } (c,d)$$

and

$$\left(we^{-2\lambda_1^{\frac{1}{2}}\int_d^x(\rho+a^{-1})\mathrm{d}s}\right)' = [w' - 2\lambda_1^{\frac{1}{2}}(\rho+\frac{1}{a})w]e^{-2\lambda_1^{\frac{1}{2}}\int_d^x(\rho+a^{-1})\mathrm{d}s} > 0 \quad \text{a.e. in } (c,d),$$

so that the function $x \to w(x)e^{-2\lambda_1^{\frac{1}{2}}\int_d^x(\rho+a^{-1})\mathrm{d}s}$ is strictly increasing in $[c,d]$, and thus negative in $[c,d]$. This yields $c=0$ and $w(0)<0$, which is a contradiction. We infer that $w(x)\geq 0$ for all $x\in[0,1]$. If $w(x)=0$ for all $x\in[\tilde{c},\tilde{d}]$ for some $\tilde{c},\tilde{d}$ with $0\leq\tilde{c}<\tilde{d}\leq 1$, then

$$w' = (\lambda_2^{\frac{1}{2}} - \lambda_1^{\frac{1}{2}})[\rho\sin^2\theta_2 + \frac{1}{a}\cos^2\theta_2] > 0 \quad \text{a.e. in } (\tilde{c},\tilde{d}),$$

which yields $w(\tilde{d})>0$, a contradiction. Thus for any $\tilde{d}\in(0,1)$ one can pick some $\tilde{c}\in(0,\tilde{d})$ with $w(\tilde{c})>0$. Then

$$\left(we^{2\lambda_1^{\frac{1}{2}}\int_{\tilde{d}}^x(\rho+a^{-1})\mathrm{d}s}\right)' = [w' + 2\lambda_1^{\frac{1}{2}}(\rho+\frac{1}{a})w]e^{2\lambda_1^{\frac{1}{2}}\int_{\tilde{d}}^x(\rho+a^{-1})\mathrm{d}s} > 0 \quad \text{a.e. in } (0,1),$$

and hence $w(x)>0$ on $[\tilde{c},1]$. We conclude that $w(x)>0$ for all $x\in(0,1]$. □

Note that the map $x\to\theta(x,\lambda)$ is also strictly increasing for all $\lambda>0$, for the r.h.s. of equation (2.64) is positive a.e. on $(0,1)$, while $\theta(.,0)\equiv\frac{\pi}{2}$.

We claim that

$$\lim_{\lambda\to+\infty}\theta(1,\lambda) = +\infty.$$

Indeed

$$\theta(1,\lambda) - \theta(\frac{1}{2},\lambda) = \lambda^{\frac{1}{2}}\int_{\frac{1}{2}}^1(\rho\sin^2\theta + \frac{1}{a}\cos^2\theta)\mathrm{d}x \geq \lambda^{\frac{1}{2}}\underbrace{\int_{\frac{1}{2}}^1\min(\rho,\frac{1}{a})\mathrm{d}x}_{>0}.$$

It follows that

$$\theta(1,[0,+\infty)) = [\frac{\pi}{2},+\infty).$$

Let $\Lambda_n := \theta(1,.)^{-1}(n\pi)$ for all $n\in\mathbb{N}^*$. We claim that for all $n\in\mathbb{N}^*$ there exists a unique $\lambda\in[\Lambda_n,\Lambda_{n+1})$ such that the function u associated with λ in Corollary 2.12 satisfies

$$\alpha u(1) + \beta(au')(1) = 0. \tag{2.67}$$

Indeed, if $\beta=0$, then the condition $u(1)=0$ is equivalent to $\sin\theta(1,\lambda)=0$ and $\lambda=\Lambda_n$ is the only solution in $[\Lambda_n,\Lambda_{n+1})$. If $\beta\neq 0$, then equation (2.67) can be written

$$\frac{\alpha}{\beta}\lambda^{-\frac{1}{2}}\sin\theta(1,\lambda) + \cos\theta(1,\lambda) = 0.$$

If $\alpha=0$, $\cos\theta(1,\lambda)=0$ gives $\theta(1,\lambda)=n\pi+\frac{\pi}{2}$ and $\lambda=\theta(1,.)^{-1}(n\pi+\frac{\pi}{2})$. If $\alpha\neq 0$, then both $\cos\theta(1,\lambda)$ and $\sin\theta(1,\lambda)$ have to be different from 0 and

$$h(\lambda) := \frac{\alpha}{\beta}\lambda^{-\frac{1}{2}} + \cot\theta(1,\lambda) = 0.$$

But the function h is continuous and strictly decreasing in $(\Lambda_n, \Lambda_{n+1})$ with $h(\Lambda_n^+) = +\infty$ and $h(\Lambda_{n+1}^-) = -\infty$. It follows that there exists a unique $\lambda \in (\Lambda_n, \Lambda_{n+1})$ such that equation (2.67) holds.

Consider now the possible solutions λ of equation (2.67) in $[0, \Lambda_1)$. If $\beta = 0$, equation (2.67) cannot hold, for $\sin\theta(1,\lambda) > 0$ (since $\pi/2 \leq \theta(1,\lambda) < \pi$). If $\alpha = 0$, equation (2.67) holds only if $\lambda = 0$. If $\alpha \neq 0$ and $\beta \neq 0$, then $\alpha/\beta > 0$ and $h(0^+) = +\infty$, $h(\Lambda_1^-) = -\infty$, so that there exists a unique $\lambda \in (0, \Lambda_1)$ with $h(\lambda) = 0$.

We conclude that the eigenvalues λ_n, $n \in \mathbb{N}$, which are all simple by Lemma 2.14, fulfill the following property:

$$\text{if } \beta = 0,\ \lambda_n = \Lambda_{n+1} \text{ for all } n \in \mathbb{N}; \tag{2.68}$$

$$\text{if } \alpha = 0,\ \lambda_0 = 0 \text{ and } \lambda_n = \theta(1,.)^{-1}(n\pi + \frac{\pi}{2}) \text{ for all } n \in \mathbb{N}^*; \tag{2.69}$$

$$\text{if } \alpha\beta \neq 0, \lambda_0 \in (0, \Lambda_1) \text{ and } \lambda_n \in (\Lambda_n, \Lambda_{n+1}) \text{ for all } \ n \in \mathbb{N}^*. \tag{2.70}$$

Since $\lambda \to \theta(1,\lambda)$ is strictly increasing and $\theta(1, \Lambda_n) = n\pi$, we infer that

$$\text{if } \beta = 0,\ \theta(1,\lambda_n) = (n+1)\pi \text{ for all } n \in \mathbb{N}; \tag{2.71}$$

$$\text{if } \alpha = 0,\ \theta(1,\lambda_n) = n\pi + \frac{\pi}{2} \text{ for all } n \in \mathbb{N}; \tag{2.72}$$

$$\text{if } \alpha\beta \neq 0,\ n\pi < \theta(1,\lambda_n) < (n+1)\pi \text{ for all } \ n \in \mathbb{N}^*. \tag{2.73}$$

Let $n \in \mathbb{N}$ and let (λ_n, e_n) be as in Theorem 2.8. Assume that $\lambda_n \neq 0$. Let (R_n, θ_n) denote the pair associated with (λ_n, e_n). Since $\theta_n = \theta(., \lambda_n) \in W^{1,1}(0,1)$, we can integrate in equation (2.64) along $(0,1)$ to obtain

$$\theta_n(1) - \frac{\pi}{2} = \lambda_n^{\frac{1}{2}} \int_0^1 \rho \sin^2\theta_n \mathrm{d}x + \lambda_n^{\frac{1}{2}} \int_0^1 \frac{\cos^2\theta_n}{a} \mathrm{d}x. \tag{2.74}$$

It follows from equations (2.71)–(2.73) that

$$\theta_n(1) - \frac{\pi}{2} \geq n\pi - \frac{\pi}{2}\cdot \tag{2.75}$$

The first term in the r.h.s. of equation (2.74) is easily estimated:

$$\lambda_n^{\frac{1}{2}} \int_0^1 \rho \sin^2\theta_n \mathrm{d}x \leq \lambda_n^{\frac{1}{2}} \|\rho\|_{L^r}.$$

To estimate the second term in the r.h.s. of equation (2.74), we split the integral into two terms as in (ii), namely

$$\lambda_n^{\frac{1}{2}} \int_0^1 \frac{\cos^2\theta_n}{a} \mathrm{d}x = \lambda_n^{\frac{1}{2}} \int_0^{(2C\lambda_n)^{\frac{p'r}{p'-r}}} \frac{\cos^2\theta_n}{a} \mathrm{d}x + \lambda_n^{\frac{1}{2}} \int_{(2C\lambda_n)^{\frac{p'r}{p'-r}}}^1 \frac{\cos^2\theta_n}{a} \mathrm{d}x =: I_1 + I_2. \tag{2.76}$$

• Assume that $p < \infty$ (so $p' > 1$). Since for all $y \in (0,1)$

$$\int_y^1 \frac{\mathrm{d}x}{a(x)} \leq \left(\int_y^1 (\frac{x}{a(x)})^p \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_y^1 x^{-p'} \mathrm{d}x\right)^{\frac{1}{p'}} \leq \|\frac{x}{a(x)}\|_{L^p} \left(\frac{y^{1-p'}-1}{p'-1}\right)^{\frac{1}{p'}},$$

we infer that

$$\begin{aligned} I_2 &\le \lambda_n^{\frac{1}{2}} \int_{(2C\lambda_n)^{\frac{p'r}{p'-r}}}^1 \frac{\mathrm{d}x}{a(x)} \\ &\le \lambda_n^{\frac{1}{2}} \frac{\|\frac{x}{a(x)}\|_{L^p}}{(p'-1)^{\frac{1}{p'}}} (2C\lambda_n)^{(\frac{1}{p'}-1)\frac{p'r}{p'-r}} \\ &\le C_2' \lambda_n^{\frac{1}{2}+\frac{1}{p}\frac{p'r}{r-p'}} \end{aligned} \tag{2.77}$$

for some constant $C_2' = C_2'(p, r, \|\frac{x}{a(x)}\|_{L^p}, C) > 0$.

If $p=\infty$, then $p'=1$, $\int_y^1 \frac{\mathrm{d}x}{a(x)} \le \|\frac{x}{a(x)}\|_{L^\infty} |\ln y|$, and

$$I_2 \le C_2' \lambda_n^{\frac{1}{2}} |\ln \lambda_n| \tag{2.78}$$

for some constant $C_2' = C_2'(r, \|\frac{x}{a(x)}\|_{L^\infty}, C) > 0$.

• Let us proceed to estimate I_1. First, recall that

$$0 \le \frac{e_n(0)}{2} \le e_n(x) \le e_n(0) \qquad \text{for } \ 0 < x < (2C\lambda_n)^{\frac{p'r}{p'-r}}.$$

Using equations (2.24)–(2.25) and (2.61), we obtain

$$I_1 = \lambda_n^{\frac{1}{2}} \int_0^{(2C\lambda_n)^{\frac{p'r}{p'-r}}} \frac{(ae_n')^2}{a(\lambda_n^{\frac{1}{4}} R_n)^2} \mathrm{d}x = \lambda_n^2 \int_0^{(2C\lambda_n)^{\frac{p'r}{p'-r}}} \frac{(\int_0^x (\rho e_n)(s)\mathrm{d}s)^2}{a(x)R_n(x)^2} \mathrm{d}x.$$

But for $0 < x < (2C\lambda_n)^{\frac{p'r}{p'-r}}$,

$$\begin{aligned} |\int_0^x (\rho e_n)(s)\mathrm{d}s| &\le \left(\int_0^x \rho^r \mathrm{d}s\right)^{\frac{1}{r}} \left(\int_0^x e_n^{r'} \mathrm{d}s\right)^{\frac{1}{r'}} \\ &\le \|\rho\|_{L^r} e_n(0) x^{\frac{1}{r'}} \end{aligned}$$

while

$$|R_n(x)|^2 \ge \lambda_n^{\frac{1}{2}} e_n(x)^2 \ge \frac{\lambda_n^{\frac{1}{2}}}{4} e_n(0)^2.$$

It follows that

$$\begin{aligned} I_1 &\le 4\|\rho\|_{L^r}^2 \lambda_n^{\frac{3}{2}} \int_0^{(2C\lambda_n)^{\frac{p'r}{p'-r}}} \frac{x^{\frac{2}{r'}}}{a(x)} \mathrm{d}x \\ &\le 4\|\rho\|_{L^r}^2 \lambda_n^{\frac{3}{2}} \|\frac{x}{a(x)}\|_{L^p} \left(\int_0^{(2C\lambda_n)^{\frac{p'r}{p'-r}}} x^{(\frac{2}{r'}-1)p'} \mathrm{d}x\right)^{\frac{1}{p'}}. \end{aligned}$$

Note that $(\frac{2}{r'}-1)p' = (\frac{1}{r'}-\frac{1}{r})p' > -p'/r > -1$ by equation (1.10). It follows that

$$\int_0^{(2C\lambda_n)^{\frac{p'r}{p'-r}}} x^{(\frac{2}{r'}-1)p'}\,\mathrm{d}x = \frac{(2C\lambda_n)^{[(\frac{2}{r'}-1)p'+1]\frac{p'r}{p'-r}}}{(\frac{2}{r'}-1)p'+1} < \infty.$$

Thus

$$I_1 \le C_2''\lambda_n^{\frac{3}{2}-[\frac{2}{r'}-1+\frac{1}{p'}]\frac{p'r}{r-p'}} \tag{2.79}$$

for some constant $C_2'' = C_2''(p, r, \|\frac{x}{a(x)}\|_{L^p}, \|\rho\|_{L^r}, C) > 0$.

It remains to compare the exponents of λ_n in equations (2.77) and (2.79). We have

$$\frac{1}{2}+\frac{1}{p}\frac{p'r}{r-p'} > \frac{3}{2} - [\frac{2}{r'}-1+\frac{1}{p'}]\frac{p'r}{r-p'}. \tag{2.80}$$

Indeed,

$$[\frac{1}{p}+\frac{2}{r'}-1+\frac{1}{p'}]p'r - (r-p') = 2\frac{p'r}{r'} - r + p' = (2p'-1)r - p' \ge r - p' > 0.$$

If $p < \infty$, we infer from equations (2.75)–(2.79) and (2.80) that if $\lambda_n \ge 1$,

$$n\pi - \frac{\pi}{2} \le C_2'''\lambda_n^{\frac{1}{2}+\frac{1}{p}\frac{p'r}{r-p'}}$$

for some constant $C_2''' > 0$. Then equation (2.28) follows.
If $p = \infty$ and $\lambda_n > 1$, we obtain for given $\kappa < 2$

$$n\pi - \frac{\pi}{2} \le C_2'''\lambda_n^{\frac{1}{2}}|\ln\lambda_n| \le C_2''''\lambda_n^{\kappa^{-1}}$$

for some constants $C_2''', C_2'''' > 0$. Then equation (2.28) follows. The proof of Theorem 2.9 is complete. □

3. Introduction of the generating functions

We shall see later that the zero-order term $q(x)u$ in equation (1.1) can be removed thanks to a change of variables. Consider the simplified system

$$(au_x)_x = \rho u_t \tag{3.1}$$
$$(au_x)(0,t) = 0 \tag{3.2}$$

and search for a solution of it in the form

$$u(x,t) = \sum_{i=0}^{\infty} y^{(i)}(t)g_i(x) \tag{3.3}$$

where y is the *flat output* and the g_i's are the *generating functions*.

A formal computation shows that

$$0 = (au_x)_x - \rho u_t = \sum_{i=0}^{\infty} y^{(i)}(t)(ag_{i,x})_x - \sum_{i=0}^{\infty} y^{(i+1)}(t)\rho(x)g_i(x).$$

It is thus natural to impose

$$(ag_{0,x})_x = 0 \tag{3.4}$$
$$(ag_{i,x})_x = \rho g_{i-1}, \quad i \geq 1, \tag{3.5}$$

together with the condition

$$(ag_{i,x})(0) = 0 \quad \forall i \geq 0. \tag{3.6}$$

We infer from equations (3.4) and (3.6) that $g_{0,x} = 0$ a.e. We pick

$$g_0(x) := 1 \quad \forall x \in [0,1]. \tag{3.7}$$

Integrating in equation (3.5) yields

$$g_{i,x}(x) = \frac{1}{a(x)} \int_0^x \rho(s) g_{i-1}(s) \mathrm{d}s.$$

We pick

$$g_i(0) = 0, \quad i \geq 1 \tag{3.8}$$

to obtain a rapid decay of $\|g_i\|_{L^\infty}$ as $i \to +\infty$, so that

$$g_i(x) := \int_0^x \frac{1}{a(s)} (\int_0^s \rho(\sigma) g_{i-1}(\sigma) d\sigma) \mathrm{d}s, \quad i \geq 1.$$

This defines formally the sequence $(g_i)_{i \geq 0}$ of generating functions. To obtain the estimate of $\|g_i\|_{L^\infty}$ which ensures the convergence of the series in equation (3.3) for $y \in G^s([0,T])$, we need the following

Proposition 3.1. *There are some constants $C, R > 0$ such that*

$$\|g_i\|_{W^{1,1}(0,1)} + \|ag_{i,x}\|_{W^{1,r}(0,1)} \leq \frac{C}{R^i (i!)^{1+\frac{1}{p'}-\frac{1}{r}}} \quad \forall i \in \mathbb{N}. \tag{3.9}$$

Proof. We need the following

Lemma 3.2. *Let $f \in L^\infty(0,1)$ and $g(x) = \displaystyle\int_0^x \frac{1}{a(s)} \left(\int_0^s \rho(\sigma) f(\sigma) d\sigma \right) \mathrm{d}s$. Then $g \in W^{1,1}(0,1)$ and $ag_x \in W^{1,r}(0,1)$. If, in addition,*

$$|f(x)| \leq C x^{\delta} \quad \textit{for a.e. } x \in (0,1), \tag{3.10}$$

for some constants $C, \delta \geq 0$, then

$$|g(x)| \leq C \frac{\|\frac{s}{a(s)}\|_{L^p}\|\rho\|_{L^r}}{p'^{\frac{1}{p'}}} \frac{x^{\delta+\omega}}{(r'\delta+1)^{\frac{1}{r'}}(\delta+\omega)^{\frac{1}{p'}}} \quad \forall x \in [0,1], \tag{3.11}$$

where $\omega := \frac{1}{r'} - 1 + \frac{1}{p'} = \frac{1}{p'} - \frac{1}{r} > 0.$

Proof of Lemma 3.2: We have

$$\left|\int_0^s \rho(\sigma) f(\sigma) d\sigma\right| \leq \|f\|_{L^\infty} \left(\int_0^s \rho(\sigma)^r d\sigma\right)^{\frac{1}{r}} s^{\frac{1}{r'}}$$

and

$$\|s^{-1}\int_0^s \rho(\sigma) f(\sigma) d\sigma\|_{L^{p'}} \leq \|f\|_{L^\infty}\|\rho\|_{L^r} \left(\int_0^1 s^{(\frac{1}{r'}-1)p'} \mathrm{d}s\right)^{\frac{1}{p'}}. \tag{3.12}$$

But $(\frac{1}{r'}-1)p' = -p'/r > -1$, since $r > p'$. Thus $(s \to s^{-1}\int_0^s \rho f d\sigma) \in L^{p'}(0,1)$ and $(s \to a(s)^{-1}\int_0^s \rho f d\sigma) \in L^1(0,1)$ by Hölder inequality. Therefore $g \in W^{1,1}(0,1)$ and $(x \to (a g_x)(x) = \int_0^x (\rho f)(s)\mathrm{d}s) \in W^{1,r}(0,1)$. Assume now that equation (3.10) holds. Then

$$\left|\int_0^s \rho(\sigma) f(\sigma) d\sigma\right| \leq C\|\rho\|_{L^r} \left(\int_0^s \sigma^{r'\delta} d\sigma\right)^{\frac{1}{r'}}$$

and

$$\begin{aligned}
|g(x)| &\leq \|\frac{s}{a(s)}\|_{L^p(0,x)}\|s^{-1}\int_0^s \rho f d\sigma\|_{L^{p'}(0,x)} \\
&\leq C\|\frac{s}{a(s)}\|_{L^p}\|\rho\|_{L^r} \frac{\left(\int_0^x s^{(\delta+\frac{1}{r'}-1)p'}\right)^{\frac{1}{p'}}}{(r'\delta+1)^{\frac{1}{r'}}} \\
&\leq C\|\frac{s}{a(s)}\|_{L^p}\|\rho\|_{L^r} \frac{x^{\delta+\frac{1}{r'}-1+\frac{1}{p'}}}{(r'\delta+1)^{\frac{1}{r'}}((\delta+\frac{1}{r'}-1)p'+1)^{\frac{1}{p'}}} \\
&\leq C \frac{\|\frac{s}{a(s)}\|_{L^p}\|\rho\|_{L^r}}{p'^{\frac{1}{p'}}} \frac{x^{\delta+\omega}}{(r'\delta+1)^{\frac{1}{r'}}(\delta+\omega)^{\frac{1}{p'}}} \qquad \forall x \in [0,1].
\end{aligned}$$

The proof of Lemma 3.2 is complete. □

Using equations (3.7) and (3.11), we obtain by an easy induction that for all $i \in \mathbb{N}$,

$$|g_i(x)| \leq \left(\frac{\|\frac{s}{a(s)}\|_{L^p}\|\rho\|_{L^r}}{p'^{\frac{1}{p'}}}\right)^i \frac{x^{i\omega}}{[\prod_{j=1}^i (1+(j-1)r'\omega)]^{\frac{1}{r'}}[\prod_{j=1}^i (j\omega)]^{\frac{1}{p'}}} \quad \forall x \in [0,1]. \tag{3.13}$$

Since

$$[\prod_{j=1}^i (1+(j-1)r'\omega)]^{\frac{1}{r'}}[\prod_{j=1}^i (j\omega)]^{\frac{1}{p'}} \geq \frac{[(r'\omega)^{\frac{1}{r'}}\omega^{\frac{1}{p'}}]^i}{(ir'\omega)^{\frac{1}{r'}}}(i!)^{\frac{1}{r'}+\frac{1}{p'}}$$

with $\frac{1}{r'} + \frac{1}{p'} = 1 + \frac{1}{p'} - \frac{1}{r} > 1$ and $i \leq 2^i$ for all $i \geq 0$, we infer that

$$\|g_i\|_{L^\infty} \leq \frac{C}{R^i} \frac{1}{(i!)^{1+\frac{1}{p'}-\frac{1}{r}}} \quad \forall i \geq 0 \tag{3.14}$$

for some positive constants C and R.

On the other hand $(ag_{i,x})_x = \rho g_{i-1}$ and $g_{i,x}(x) = \frac{1}{a(x)} \int_0^x \rho g_{i-1} \mathrm{d}s$, so that, with equation (3.12)

$$\|(ag_{i,x})_x\|_{L^r} \leq \|\rho\|_{L^r} \|g_{i-1}\|_{L^\infty}, \quad \|g_{i,x}\|_{L^1} \leq C\|\rho\|_{L^r} \|\frac{x}{a(x)}\|_{L^p} \|g_{i-1}\|_{L^\infty}. \tag{3.15}$$

Finally, equation (3.9) follows from equations (3.6), (3.8), (3.14) and (3.15). □

Next, we show that the eigenfunctions can be expressed in terms of the generating functions.

Proposition 3.3. *Let $n \in \mathbb{N}$ and let e_n and λ_n be as in Theorem 2.8. Then*

$$e_n = e_n(0) \sum_{i \geq 0} (-\lambda_n)^i g_i \quad \text{in } W^{1,1}(0,1).$$

Proof. Fix some $n \in \mathbb{N}$ and set

$$\tilde{e} = \sum_{i \geq 0} (-\lambda_n)^i g_i. \tag{3.16}$$

The series in equation (3.16) converges in $W^{1,1}(0,1)$ by equation (3.9). It follows that

$$a\tilde{e}_x = \sum_{i \geq 0} (-\lambda_n)^i a g_{i,x} \quad \text{in } L^1(0,1).$$

Using equation (3.9) again, we deduce that $a\tilde{e}_x \in W^{1,r}(0,1)$, and that we have in $L^r(0,1)$

$$\begin{aligned} (a\tilde{e}_x)_x &= \sum_{i \geq 0} (-\lambda_n)^i (ag_{i,x})_x \\ &= \sum_{i \geq 1} (-\lambda_n)^i \rho g_{i-1} \\ &= -\lambda_n \rho \tilde{e}. \end{aligned}$$

On the other hand, by equations (3.6), (3.7) and (3.8), we have that

$$\tilde{e}(0) = 1, \quad (a\tilde{e}_x)(0) = 0.$$

It follows from Corollary 2.12 that $\tilde{e} = u$ for $\lambda = \lambda_n$, and from Lemma 2.14 that

$$e_n(x) = e_n(0) \sum_{i \geq 0} (-\lambda_n)^i g_i(x) \quad \forall x \in [0,1].$$

□

We are in a position to prove the main results in the paper.

4. Proof of the main results

4.1. Proof of Theorem 1.1

4.1.1. Step 1: Reduction to the case $q = 0$

Let $\hat{u}(x,t) = u(x,t)/v(x)$, where u satisfies equation (1.1) and (1.2) and v is as in equation (1.9). Then

$$v^2 a\hat{u}_x = v^2 a(\frac{u_x}{v} - \frac{uv_x}{v^2}) = a(u_x v - uv_x)$$

and

$$\begin{aligned}(v^2 a\hat{u}_x)_x &= (au_x)_x v + au_x v_x - ((av_x)_x u + au_x v_x) \\ &= [(au_x)_x + qu]v \\ &= \rho u_t v \\ &= \rho v^2 \hat{u}_t.\end{aligned}$$

Let $\hat{a}(x) := v(x)^2 a(x)$ and $\hat{\rho}(x) := v(x)^2\rho(x)$. Then $\hat{u}$ satisfies

$$\begin{aligned}(\hat{a}\hat{u}_x)_x &= \hat{\rho}\hat{u}_t \quad x \in (0,1),\ t \in (0,T), \\ (\hat{a}\hat{u}_x)(0) &= 0, \quad t \in (0,T)\end{aligned}$$

with $\hat{a}$ and $\hat{\rho}$ satisfying equations (1.5)–(1.8). Indeed, using equation (1.9), one may pick some constants C_1 and C_2 such that

$$0 < C_1 \le v(x) \le C_2 \quad \forall x \in [0,1].$$

We infer from $u = \hat{u}v$ and equation (1.3) that

$$\big(\alpha v(1) + \beta(av_x)(1)\big)\hat{u}(1,t) + \beta v(1)(a\hat{u}_x)(1,t) = h(t).$$

Setting $\hat{\alpha} := \alpha v(1) + \beta(av_x)(1)$ and $\hat{\beta} := \beta/v(1)$, we arrive to

$$\hat{\alpha}\hat{u}(1,t) + \hat{\beta}(\hat{a}\hat{u}_x)(1,t) = h(t).$$

Let $\hat{u}_0 := u_0/v$. Then $\hat{u}_0 \in L^2_{\hat{\rho}} \iff u_0 \in L^2_\rho$.

4.2. Step 2: Flatness approach

We follow closely [24]. We assume that $q = 0$ in equation (1.1). Let $u_0 \in L^2_\rho$. As $(e_n)_{n\ge 0}$ is an orthonormal basis in L^2_ρ, we can expand u_0 as a series of the e_n's:

$$u_0 = \sum_{n=0}^{\infty} c_n e_n \quad \text{ in } L^2_\rho,$$

where the sequence $(c_n)_{n\ge 0}$ of real numbers satisfies $\sum_{n=0}^\infty c_n^2 < \infty$.

Using equations (2.27) and (2.28), we notice that the map $z \to \sum_{n\ge 0} c_n e_n(0)e^{-\lambda_n z}$ is analytic in the set $\{z = t + it';\ t > 0, t' \in \mathbb{R}\}$. It follows that the map $t \to \sum_{n\ge 0} c_n e_n(0)e^{-\lambda_n t}$ is real analytic in $(0,\infty)$, and its restriction to $[\epsilon, T]$ belongs to $G^1([\epsilon, T]) \subset G^s([\epsilon, T])$ for all $\epsilon \in (0, T)$ and all $s \in (1, 2)$. Pick $s \in (1, 1 + \frac{1}{p'} - \frac{1}{r})$ and $\varphi \in G^s([0, T])$ with

$$\varphi(t) = \begin{cases} 1 & \text{if } t \leq \frac{T}{3}, \\ 0 & \text{if } t \geq \frac{2T}{3}. \end{cases}$$

Let

$$y(t) := \varphi(t) \sum_{n=0}^{\infty} c_n e_n(0) e^{-\lambda_n t}, \qquad \text{for } 0 < t < T,$$
$$u(x,t) := \begin{cases} u_0(x) & \text{if } t = 0, \\ \sum_{i=0}^{\infty} y^{(i)}(t) g_i(x) & \text{if } 0 < t \leq T. \end{cases}$$

Pick any $\epsilon \in (0, T/3)$. Then $y \in G^s([\epsilon, T])$, and there exist some positive numbers $\tilde{M}, \tilde{R}$ such that

$$|y^{(i)}(t)| \leq \tilde{M} \frac{(i!)^s}{\tilde{R}^i} \quad \forall t \in [\epsilon, T].$$

Combined with equation (3.9), this yields

$$u \in C^1([\epsilon, T], W^{1,1}(0,1)), \quad a u_x \in C^1([\epsilon, T], W^{1,r}(0,1)).$$

Furthermore,

$$\rho u_t = \sum_{i=0}^{\infty} y^{(i+1)}(t) \rho g_i(x) = \sum_{i=0}^{\infty} y^{(i+1)}(t) (a g_{i+1,x})_x = (a u_x)_x$$

and $a u_x(0, t) = 0$. We notice that for $0 < t \leq T/3$, we have that $y(t) = \sum_{n=0}^{\infty} c_n e_n(0) e^{-\lambda_n t}$ and

$$\begin{aligned} u(x,t) &= \sum_{i=0}^{\infty} \partial_t^i [\sum_{n=0}^{\infty} c_n e_n(0) e^{-\lambda_n t}] g_i(x) \\ &= \sum_{i=0}^{\infty} [\sum_{n=0}^{\infty} c_n e_n(0) (-\lambda_n)^i e^{-\lambda_n t}] g_i(x) \\ &= \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} e_n(0) \sum_{i=0}^{\infty} (-\lambda_n)^i g_i(x) \\ &= \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} e_n(x) \end{aligned} \tag{4.1}$$

for all $x \in [0, 1]$. The above computations are valid, since for $0 < \delta \leq t \leq T$ and $0 \leq x \leq 1$

$$\sum_{i \geq 0, n \geq 0} |c_n e_n(0) (-\lambda_n)^i e^{-\lambda_n t} g_i(x)| \leq \sum_{i \geq 0, n \geq 0} |c_n e_n(0)| \frac{e^{\frac{\delta}{2}\lambda_n}}{(\frac{\delta}{2})^i} (i!) e^{-\delta \lambda_n} \frac{C}{R^i (i!)^{1 + \frac{1}{p'} - \frac{1}{r}}}$$

$$\begin{aligned}
&\leq \left(|c_0| + C_1 \sum_{n;\lambda_n>0}^{\infty} |c_n| \frac{\lambda_n^{\frac{3}{4}(1+\frac{p'r}{r-p'})}}{e^{\frac{\delta}{2}\lambda_n}} \right) \sum_{i=0}^{\infty} \frac{1}{(\frac{R\delta}{2})^i (i!)^{\frac{1}{p'}-\frac{1}{r}}} \\
&< \infty
\end{aligned}$$

where we used the estimate $x^i/i! \leq e^x$ for $x = \delta\lambda_n/2 \geq 0$ and $i \in \mathbb{N}$.

It follows from equation (4.1) that u is the free evolution (*i.e.* with a null control) of the parabolic equation for $0 < t \leq T/3$. Therefore

$$\lim_{t\to 0^+} u(.,t) = u_0 \quad \text{ in } L^2_\rho.$$

We pick as control input

$$h(t) := \begin{cases} 0 & \text{for } t = 0, \\ \sum_{i=0}^{\infty} y^{(i)}(t)[\alpha g_i(1) + \beta(a g_{i,x})(1)] & \text{for } 0 < t \leq T. \end{cases}$$

It follows from equation (4.1) that $h(t) = 0$ for $0 < t \leq T/3$ and from equation (3.9) combined with the choice of s that $h \in C^\infty([0,T])$ with $h^{(j)}(t) = \sum_{i=0}^{\infty} y^{(i+j)}(t)[\alpha g_i(1) + \beta(a g_{i,x})(1)]$. Let us check that $h \in G^s([0,T])$. We have for $t \in [\epsilon, T]$

$$\begin{aligned}
|h^{(j)}(t)| &\leq \sum_{i\geq 0} |y^{(i+j)}(t)[\alpha g_i(1) + \beta(a g_{i,x})(1)]| \\
&\leq C \sum_{i\geq 0} \frac{(i+j)!^s}{\tilde{R}^{i+j}} \frac{1}{R^i i!^{1+\frac{1}{p'}-\frac{1}{r}}} \\
&\leq C (\frac{2^s}{\tilde{R}})^j \left(\sum_{i\geq 0} (\frac{2^s}{R\tilde{R}})^i \frac{1}{i!^{1+\frac{1}{p'}-\frac{1}{r}-s}} \right) j!^s
\end{aligned}$$

for some constant C which does not depend on j and t, where we used $(i+j)! \leq 2^{i+j} i! j!$. As $h(t) = 0$ for $0 \leq t \leq T/3$, we conclude that $h \in G^s([0,T])$. Finally

$$u(x,t) = 0 \quad \forall (x,t) \in [0,1] \times [\frac{2T}{3}, T].$$

The proof of Theorem 1.1 is complete.

Remark 4.1. We stress that assumption (1.9) was used only in Step 1 the get rid of the term $q(x)u$ in equation (1.1). If $q \equiv 0$ in equation (1.1), then Theorem 1.1 is still valid with the assumptions (1.5)–(1.8).

4.3. Proof of Proposition 1.3

Since $q \in L^{p'}(0,1)$, we infer from Lemma 2.11 the existence and uniqueness of a function $v \in W^{1,1}(0,1)$ with $a v_x \in W^{1,r}(0,1)$ such that equations (2.37)–(2.39) hold. The only property still to establish is the fact that $v(x) > 0$ for all $x \in [0,1]$. We know that v satisfies the integral equation (2.41). Therefore, we have

$$v(x) \geq 1 - \|v\|_{L^\infty(0,x)} \int_0^x \frac{\mathrm{d}y}{a(y)} \int_0^y |q(s)| \mathrm{d}s \quad \forall x \in (0,1).$$

If equation (1.13) holds, we claim that $v(x) > 0$ for all $x \in [0,1]$. Otherwise, there would exist some $x_0 \in (0,1]$ such that $v(x_0) = 0$, and we can assume that it is the least, so that $\|v\|_{L^\infty(0,x_0)} = 1$. But this yields

$$v(x_0) \geq 1 - \int_0^{x_0} \frac{\mathrm{d}y}{a(y)} \int_0^y |q(s)|\mathrm{d}s > 0,$$

a contradiction. If we assume that $q(x) \leq 0$ for a.e. $x \in (0,1)$, then we can prove in much the same way that $v(x) \geq 1$ for all $x \in [0,1]$. Finally, if equation (1.15) holds, the change of unknowns $\tilde{u}(x,t) = e^{-Kt}u(x,t)$ transforms equation (1.5) into

$$(a(x)\tilde{u}_x)_x + [q(x) - K\rho(x)]\tilde{u} = \rho(x)\tilde{u}_t,$$

and the conclusion follows from the previous case with equation (1.14) satisfied. □

4.4. Proof of Proposition 1.4

Plugging $v(x) = x^\delta$ in equation (1.17) results in $[\delta(1+\delta-\varepsilon)+\mu]x^{\delta-\varepsilon} = 0$ for $x \in (0,1)$, so that the choice $\delta = \frac{1}{2}[\varepsilon - 1 + ((1-\varepsilon)^2 - 4\mu)^{\frac{1}{2}}]$ gives equation (1.17). We note that equation (1.18) holds, for

$$(av_x)(x) = \delta x^{\frac{1-\varepsilon+\sqrt{(1-\varepsilon)^2-4\mu}}{2}}$$

with $1 - \varepsilon + \sqrt{(1-\varepsilon)^2 - 4\mu} > 0$. Using equation (1.16), we infer that $\delta \geq 0$ and that $\sqrt{(1-\varepsilon)^2-4\mu} < 1$. Therefore, we can pick some $p \in (1,\infty)$ so that

$$1 > \frac{1}{p} > \max(1-\varepsilon, \sqrt{(1-\varepsilon)^2 - 4\mu}). \tag{4.2}$$

Then the functions $x \to x/a(x)$ and $x \to x/(a(x)v(x)^2)$ are in $L^p(0,1)$. Applying Lemma A.1 (see below), we see that equation (1.8) and

$$\lim_{x\to 0^+} [a(x)v(x)^2]^{-1} \left(\int_x^1 \frac{\mathrm{d}s}{a(s)v(s)^2}\right)^{-2} = +\infty$$

hold. As $\delta \geq 0$, $v \in C^0([0,1])$ so that $\rho v^2 \in L^r(0,1)$ with $r = \infty$ and $\limsup_{x\to 0^+}[\rho(x)v(x)^2] < \infty$.

For the application of Theorem 1.1, we do again the change of unknown $\hat{u}(x,t) := u(x,t)/v(x)$, and we set $\hat{a} = av^2$, $\hat{\rho} = \rho v^2$, and $\hat{u}_0 = u_0/v$. From

$$\hat{a}\hat{u}_x = a(u_x v - uv_x) \tag{4.3}$$

we see readily that $(au_x)(0,t) = 0$ implies $(\hat{a}\hat{u}_x)(0,t) = 0$. Let

$$\hat{\alpha} = \alpha v(1) + \beta(av_x)(1) = \alpha + \beta\delta,$$
$$\hat{\beta} = \frac{\beta}{v(1)} = \beta.$$

An application of Theorem 1.1 to the simplified system

$$\begin{cases}(\hat{a}\hat{u}_x)_x = \hat{\rho}\hat{u}_t,\\ (\hat{a}\hat{u}_x)(0,t) = 0,\\ \hat{\alpha}\hat{u}(1,t) + \hat{\beta}(\hat{a}\hat{u}_x)(1,t) = h(t),\\ \hat{u}(x,0) = \hat{u}_0(x)\end{cases} \tag{4.4}$$

yields the existence of a control input $h \in G^s([0,T])$ such that the solution $\hat{u}$ of equation (4.4) satisfies $\hat{u}(.,T) = 0$.

Going back to the original dependent variable $u(x,t) = \hat{u}(x,t)v(x)$, we see that u satisfies equations (1.1), (1.3)–(1.4) and $u(.,T) = 0$. It remains to show that equation (1.2) holds. Note that equation (1.2) is not a direct consequence of equations (4.3)–(4.4), for $v(0) = 0$ if $\delta > 0$. We infer from equation (4.3) that

$$(au_x)(0^+,t) = (\frac{\hat{a}\hat{u}_x}{v})(0^+,t) + (\frac{u}{v}av_x)(0^+,t).$$

The second term is easy to handle. Indeed, $(av_x/v)(x) = \delta x^{1-\varepsilon}$, so that $(uav_x/v)(0^+,t) = 0$. To deal with the first term, we write

$$\hat{u}(x,t) = \sum_{i\geq 0} y^{(i)}(t)g_i(x)$$

where the functions g_i's and y are respectively the generating functions and the flat output for system (4.4). It follows that

$$\frac{\hat{a}(x)\hat{u}_x(x,t)}{v(x)} = \sum_{i\geq 0} y^{(i)}(t)\frac{\hat{a}(x)g_{i,x}(x)}{v(x)}.$$

But $g_{0,x} = 0$ and for $i \geq 1$

$$(\hat{a}g_{i,x})(x) = \int_0^x \hat{\rho}(s)g_{i-1}(s)\mathrm{d}s = \int_0^x s^{2\delta}g_{i-1}(s)\mathrm{d}s.$$

This yields

$$\left|\frac{(\hat{a}g_{i,x})(x)}{v(x)}\right| \leq \frac{x^{\delta+1}}{2\delta+1}\|g_{i-1}\|_{L^\infty} \leq \frac{C}{(2\delta+1)R^{i-1}[(i-1)!]^{1+\frac{1}{p'}-\frac{1}{r}}}x^{\delta+1}$$

where we used equation (3.9). It follows that $(\hat{a}\hat{u}_x/v)(0^+,t) = 0$. We conclude that $(au_x)(0^+,t) = 0$ for $0 < t < T$. $\square$

Appendix A.

The conditions (1.6) and (1.8) are independent

1. Pick $a(x) = x^2|\ln x|^\alpha$ for some $\alpha > 0$. Then equation (1.6) is never satisfied, since $\int_0^1 x^{-p}|\ln x|^{-\alpha p}\mathrm{d}x = +\infty$ for all $p > 1$. On the other hand, by integrating by parts, we have for any $\varepsilon \in (0,1)$

$$\int_x^{1-\varepsilon}\frac{\mathrm{d}s}{s^2|\ln s|^\alpha} = \alpha\int_x^{1-\varepsilon}\frac{\mathrm{d}s}{s^2|\ln s|^{\alpha+1}} + [-s^{-1}|\ln s|^{-\alpha}]_x^{1-\varepsilon}$$

and for $1-\varepsilon$ small enough and for $x \in (0, 1-\varepsilon)$

$$\left|\alpha \int_x^{1-\varepsilon} \frac{\mathrm{d}s}{s^2|\ln s|^{\alpha+1}}\right| \le \frac{1}{2}\int_x^{1-\varepsilon} \frac{\mathrm{d}s}{s^2|\ln s|^{\alpha}}$$

so that for some positive constants C_1, C_2 and $x > 0$ small enough

$$C_1 x^{-1}|\ln x|^{-\alpha} \le \int_x^{1-\varepsilon} \frac{\mathrm{d}s}{s^2|\ln s|^{\alpha}} \le C_2 x^{-1}|\ln x|^{-\alpha}.$$

This yields $a(x)^{-1}\left(\int_x^1 \frac{\mathrm{d}s}{a(s)}\right)^{-2} \ge C_2^{-2}|\ln x|^{\alpha}$ for $0 < x \ll 1$, and equation (1.8) is satisfied.
2. Pick $\Omega = \bigcup_{n\ge 1}(2^{-n} - 10^{-n}, 2^{-n} + 10^{-n}) \subset (0,1)$, and $a(x) = x^{2-\varepsilon}1_{\Omega^c}(x) + 1_{\Omega}(x)$ for some $\varepsilon \in (0,1)$. Then equation (1.6) is satisfied for some $p > 1$, since $x/a(x) \le x^{\varepsilon-1}$ for $0 < x < 1$. On the other hand,

$$\int_\Omega \frac{\mathrm{d}x}{x^{2-\varepsilon}} \le \sum_{n=1}^{\infty} \frac{2.10^{-n}}{(2^{-n} - 10^{-n})^{2-\varepsilon}} < \infty = \int_0^1 \frac{\mathrm{d}x}{x^{2-\varepsilon}},$$

which yields $\int_{\Omega^c} \frac{\mathrm{d}x}{x^{2-\varepsilon}} = +\infty$ and $1/a \notin L^1(0,1)$. It follows that

$$\lim_{n\to+\infty} a(2^{-n})\left(\int_{2^{-n}}^1 \frac{\mathrm{d}s}{a(s)}\right)^2 = +\infty$$

and that equation (1.8) fails.

Lemma A.1. *Assume that $a(x) = [xg(x)]^2$ for some function $g \in C^1((0,1),(0,+\infty))$ satisfying for some $\varepsilon \in (0, \frac{1}{2})$ and some $x_0 \in (0,1)$*

$$\begin{aligned}
&\left|\frac{g'(x)}{g(x)}\right| \le \frac{\varepsilon}{x} \quad \textit{for} \quad 0 < x < x_0,\\
&g(0^+) = +\infty,\\
&(xg(x))(0^+) = 0,\\
&\int_{x_0}^1 \frac{\mathrm{d}x}{g(x)^2} < \infty.
\end{aligned}$$

Then equation (1.8) is satisfied.

Proof of Lemma A.1: We obtain by an integration by parts that

$$\begin{aligned}
\int_x^{x_0} \frac{\mathrm{d}s}{s^2 g(s)^2} &= [-s^{-1}g(s)^{-2}]_x^{x_0} - 2\int_x^{x_0} \frac{g'(s)}{sg(s)^3}\mathrm{d}s\\
&\le \frac{1}{xg(x)^2} - \frac{1}{x_0 g(x_0)^2} + 2\varepsilon \int_x^{x_0} \frac{\mathrm{d}s}{s^2 g(s)^2},
\end{aligned}$$

which yields

$$\sqrt{a(x)}\int_x^{x_0} \frac{\mathrm{d}s}{a(s)} \le (1-2\varepsilon)^{-1}\left(\frac{1}{g(x)} - \frac{\sqrt{a(x)}}{x_0 g(x_0)^2}\right) \to 0 \quad \text{as } x \to 0^+.$$

□

Example A.2. 1. Let $g(x) = x^{-\varepsilon}$ for some $\varepsilon \in (0, \frac{1}{2})$. Then $|g'(x)/g(x)| = \varepsilon/x$ and Lemma A.1 may be applied for any $x_0 \in (0,1)$.

2. Let $g(x) = |\ln x|^{\alpha}$ for some $\alpha > 0$. Then $|g'(x)/g(x)| = \alpha/(x|\ln x|)$ and Lemma A.1 may be applied for *e.g.* $\varepsilon = 1/4$ and $x_0 \in (0,1)$ small enough.

References

[1] F. Alabau-Boussouira, P. Cannarsa and F. Fragnelli, Carleman estimates for degenerate parabolic operators with applications to null controllability. *J. Evol. Equ.* **6** (2006) 161–204.

[2] G. Alessandrini and L. Escauriaza, Null-controllability of one-dimensional parabolic equations. *ESAIM Control Optim. Calc. Var.* **14** (2008) 284–293.

[3] F.V. Atkinson and C.T. Fulton, Asymptotics of Sturm-Liouville eigenvalues for problems on a finite interval with one limit-circle singularity. I. *Proc. Roy. Soc. Edinburgh Sect. A* **99** (1984) 51–70.

[4] K. Beauchard, P. Cannarsa and R. Guglielmi, Null controllability of Grushin-type operators in dimension two. *J. Eur. Math. Soc. (JEMS)* **16** (2014) 67–101.

[5] K. Beauchard, J. Dardé and S. Ervedoza, minimal time issues for the observability of Grushin-type equations. *Ann. Inst. Fourier (Grenoble)* **70** (2020) 247–312.

[6] G. Birkhoff and G.-C. Rota, Ordinary Differential Equations, Ginn-Blaisdell, Boston (1962).

[7] J.S. Bradley, Hardy inequalities with mixed norms. *Canad. Math. Bull.* **21** (1978) 405–408.

[8] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext. Springer, New York (2011).

[9] P. Cannarsa, P. Martinez and J. Vancostenoble, Carleman estimates for a class of degenerate parabolic operators. *SIAM J. Control Optim.* **47** (2008) 1–19.

[10] P. Cannarsa, P. Martinez and J. Vancostenoble, Global Carleman estimates for degenerate parabolic operators with applications. *Mem. Amer. Math. Soc.* **239** (2016) 209.

[11] P. Cannarsa, P. Martinez and J. Vancostenoble, The cost of controlling strongly degenerate parabolic equations. *ESAIM Control Optim. Calc. Var.* **26** (2020) 50.

[12] D.M. Duc, A class of strongly degenerate elliptic operators, *Bull. Austral. Math. Soc.* **39** (1989) 177–200.

[13] G. Fragnelli and D. Mugnai, Carleman estimates, observability inequalities and null controllability for interior degenerate nonsmooth parabolic equations. *Mem. Amer. Math. Soc.* **242** (2016) 84.

[14] G. Fragnelli and D. Mugnai, Singular parabolic equations with interior degeneracy and non smooth coefficients: the Neumann case. *Discrete Contin. Dyn. Syst. Ser. S* **13** (2020) 1495–1511.

[15] G. Fragnelli and D. Mugnai, Control of Degenerate and Singular Equations – Carleman Estimates and Observability, SpringerBriefs in Mathematics, BCAM SpringerBriefs, Springer, Cham (2021).

[16] G. Hardy, J.E. Littlewood and G. Pólya, Inequalities, Cambridge University Press, Cambridge (1952).

[17] B.J. Harris, Asymptotics of eigenvalues for regular Sturm-Liouville problems. *J. Math. Anal. Appl.* **183** (1994) 25–36.

[18] B.J. Harris and D. Race, Asymptotics for Sturm-Liouville problems with an interior singularity. *J. Diff. Equ.* **116** (1995) 88–118.

[19] B. Laroche, P. Martin and P. Rouchon, Motion planning for the heat equation. *Int. J. Robust Nonlinear Control* **10** (2000) 629–643.

[20] C. Laurent and L. Rosier, Exact controllability of semilinear heat equations in spaces of analytic functions. *Ann. Inst. H. Poincaré C. Anal. Non Linéaire* **37** (2020) 1047–1073.

[21] P. Martin, I. Rivas, L. Rosier and P. Rouchon, Exact controllability of a linear Korteweg-de Vries equation by the flatness approach. *SIAM J. Control Optim.* **57** (2019) 2467–2486.

[22] P. Martin, L. Rosier and P. Rouchon, Null controllability of the heat equation using flatness. *Automatica* **50** (2014)3067–3076.

[23] P. Martin, L. Rosier and P. Rouchon, On the reachable states for the boundary control of the heat equation. *Appl. Math. Res. Express.* (2016) 181–216.

[24] P. Martin, L. Rosier and P. Rouchon, Null controllability of one-dimensional parabolic equations by the flatness approach. *SIAM J. Control Optim.* **54** (2016) 198–220.

[25] P. Martin, L. Rosier and P. Rouchon, Controllability of the 1D Schrödinger equation using flatness. *Automatica* **91** (2018) 208–216.

[26] I. Moyano, Flatness for a strongly degenerate 1-D parabolic equation. *Math. Control Signals Syst.* **28** (2016) 22.

[27] B. Muckenhoupt, Hardy's inequality with weights. *Studia Math.* **44** (1972) 31–38.

[28] N.S. Trudinger, Linear elliptic operators with measurable coefficients. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **27** (1973) 265–308.

[29] W. Walter, *Ordinary Differential Equations*, Graduate Texts in Mathematics, 182. Readings in Mathematics, Springer-Verlag, New York, 1998.