

PROBABILISTIC REPRESENTATION OF VISCOSITY SOLUTIONS TO QUASI-VARIATIONAL INEQUALITIES WITH NON-LOCAL DRIVERS*

MAGNUS PERNINGE** 

Abstract. We consider quasi-variational inequalities (QVIs) with general non-local drivers and related systems of reflected backward stochastic differential equations (BSDEs) in a Brownian filtration. We show existence and uniqueness of viscosity solutions to the QVIs by first considering the standard (local) setting and then applying a contraction argument. In addition, the contraction argument yields existence and uniqueness of solutions to the related systems of reflected BSDEs and extends the theory of probabilistic representations of PDEs in terms of BSDEs to our specific setting.

Mathematics Subject Classification. 60G40, 93E20, 49L20.

Received October 9, 2022. Accepted March 6, 2023.

1. INTRODUCTION

We consider existence and uniqueness of viscosity solutions to the quasi-variational inequality (QVI)

$$\begin{cases} \min\{v(t, x) - \mathcal{M}v(t, x), -v_t(t, x) - \mathcal{L}v(t, x) - f(t, x, v(t, \cdot), \sigma^\top(t, x)\nabla_x v(t, x))\} = 0, \\ \forall (t, x) \in [0, T) \times \mathbb{R}^d \\ v(T, x) = \psi(x), \end{cases} \quad (1.1)$$

where for each $(t, x, z) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^d$, the map $g \mapsto f(t, x, g, z) : C(\mathbb{R}^n \rightarrow \mathbb{R}) \rightarrow \mathbb{R}$ is a functional, $\mathcal{M}v(t, x) := \sup_{b \in U} \{v(t, \Gamma(t, x, b)) - \ell(t, x, b)\}$ and

$$\mathcal{L} := \sum_{j=1}^d a_j(t, x) \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^\top(t, x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \quad (1.2)$$

*This work was supported by the Swedish Energy Agency through grant number 48405-1.

Keywords and phrases: Backward stochastic differential equations, impulse control, partial differential equations, viscosity solutions, quasi-variational inequalities

¹ Department of Physics and Electrical Engineering, Linnaeus University, Växjö, Sweden.

** Corresponding author: perninge@kth.se

is the infinitesimal generator related to the SDE

$$X_s^{t,x} = x + \int_t^s a(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r.$$

We take a probabilistic approach and extend the Feynman-Kac framework by relating solutions to (1.1) to those of the system of reflected BSDEs

$$\begin{cases} Y_s^{t,x} = \psi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, \bar{Y}(r, \cdot), Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r + K_T^{t,x} - K_s^{t,x}, & \forall s \in [t, T], \\ Y_s^{t,x} \geq \mathcal{M}\bar{Y}(s, X_s^{t,x}), \forall s \in [t, T] & \text{and} \quad \int_t^T (Y_s^{t,x} - \mathcal{M}\bar{Y}(s, X_s^{t,x}))dK_s^{t,x} = 0, \end{cases} \quad (1.3)$$

where \bar{Y} is a continuous, deterministic function such that $\bar{Y}(t, x) = Y_t^{t,x}$, \mathbb{P} -a.s.

It is well known that, under suitable conditions on the involved parameters, value functions to impulse control problems (that are treated in *e.g.* [2, 3, 20, 24]) are solutions (in viscosity sense) to standard QVIs when the driving noise process is a Brownian motion (see the seminal work in [1]) and to so called quasi-integrovariational inequalities when the driving noise is a general Lévy process [22].

Our primary motivation for studying the QVIs of the more general type in (1.1) and the corresponding systems of reflected BSDEs (1.3) is this close connection to stochastic impulse control that we exploit in the case of a local driver $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ as a means to derive an intermediate result. The non-local component of the driver allow us to model situations where the running revenue depends on the total reward in other states, for example by letting $f(t, x, v, z) = \tilde{f}(t, x, \mathcal{M}v(t, x), z)$ for some $\tilde{f} : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$. Moreover, existence and uniqueness of solutions to (1.3) appears to be a cornerstone in attempts to prove existence of solutions to systems of doubly reflected BSDEs through penalization (see [9, 10] for a complete result in a less general setup). Since the unique solution to the system of doubly reflected BSDEs in [10] appear as the value function to a switching game¹ in [15], it is natural to assume that our results can lead to an extension of the theory of stochastic differential games of impulse control (see *e.g.* [7]). Given the generality of our framework and in particular the type of non-locality in our driver it is also plausible that our results open up new perspectives for dealing with more general impulse control problems, for instance mean-field type control problems involving impulses.

An alternative Feynman-Kac representation for solutions to standard QVIs was proposed in [20], where the solution to a QVI is related to the minimal solution of a BSDE driven by a Brownian motion and a Poisson random measure with a constraint on the jump component. It should be noted that a similar approach was taken in [21] to derive a probabilistic representation for solutions to nonlinear integro-partial differential equations (IPDEs) of Hamilton-Jacobi-Bellman type.

Systems of reflected BSDEs related to non-Markovian optimal switching problems were independently considered in [18, 19]. Based on a contraction argument, the authors of [5] later extended the existence and uniqueness results for such systems of reflected BSDEs, removing certain monotonicity assumptions on the drivers. In the Markovian framework, the corresponding Feynman-Kac representation was derived in [17] under a monotonicity assumption on the driver and extended to the general setting in [16].

Despite the surge of interest in switching problems and the related systems of reflected BSDEs, for a long while no work appeared that connected impulse control problems to reflected BSDEs. However, recently [23] considered a sequential system of reflected BSDEs related to non-Markovian impulse control. The intermediate result in the present work extends that of [23] in the Markovian framework and bridges the gap left in terms of a probabilistic representation of the corresponding QVIs.

¹An optimal switching problem is a type of impulse control problem where the operator of a system seeks to maximize the revenue by switching between a set of operation modes (see [4] for a complete definition).

As noted above, our intermediate step assumes that f is a local operator and seeks a solution to the system of reflected BSDEs

$$\begin{cases} Y_s^{t,x} = \psi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r + K_T^{t,x} - K_s^{t,x}, & \forall s \in [t, T], \\ Y_s^{t,x} \geq \mathcal{M}\bar{Y}(s, X_s^{t,x}), \forall s \in [t, T] & \text{and} \quad \int_t^T (Y_s^{t,x} - \mathcal{M}\bar{Y}(s, X_s^{t,x}))dK_s^{t,x} = 0. \end{cases} \quad (1.4)$$

Existence is achieved through a Picard iteration approach and uniqueness is shown by a verification argument, relating a solution to (1.4) to the value function of a stochastic impulse control problem. Moreover, we show that if (Y, Z, K) solves (1.4), letting $v : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $v(t, x) := Y_t^{t,x}$, \mathbb{P} -a.s., then v is a viscosity solution to the QVI

$$\begin{cases} \min\{v(t, x) - \mathcal{M}v(t, x), -v_t(t, x) - \mathcal{L}v(t, x) - f(t, x, v(t, x), \sigma^\top(t, x)\nabla_x v(t, x))\} = 0, \\ \forall(t, x) \in [0, T] \times \mathbb{R}^d \\ v(T, x) = \psi(x), \end{cases} \quad (1.5)$$

and (in an Appendix) we show uniqueness of solutions to (1.5). To extend these results to the non-local setting we consider a sequence of interconnected quasi-variational inequalities

$$\begin{cases} \min\{v^k(t, x) - \mathcal{M}v^k(t, x), -v_t^k(t, x) - \mathcal{L}v^k(t, x) - f(t, x, v^{k-1}(t, \cdot), \sigma(t, x)\nabla_x v^k(t, x))\} = 0, \\ \forall(t, x) \in [0, T] \times \mathbb{R}^d, \\ v^k(T, x) = \psi(x), \end{cases} \quad (1.6)$$

with $v_0 \equiv 0$ and find a norm under which the map Φ mapping v_k to v_{k+1} is a contraction. A challenge that we leave for future research is to use our representation to obtain numerical solutions to (1.1) (see [6] for a result in the setting related to switching problems). This appears not to be straightforward as we need to deal with the non-locality in the driver in a suitable way.

The remainder of the article is organised as follows. In the next section we set all required notations and state the assumptions that hold throughout. In addition, we give some preliminary results that are repeatedly referred to in the article. Then, in Section 3 we turn to the local setting before we, in the following section, derive the complete result. Uniqueness of solutions to (1.5) appears rudimentary and resembles the corresponding results in [17]. However, since our setting is fundamentally different and for the sake of completeness, a uniqueness proof through viscosity comparison is included as an appendix.

2. PRELIMINARIES

2.1. Notations

We let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which lives a d -dimensional Brownian motion W . We denote by $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ the augmented natural filtration of W and for $t \in [0, T]$ we let $\mathbb{F}^t := (\mathcal{F}_s^t)_{t \leq s \leq T}$ denote the augmented natural filtration generated by $(W_s - W_t : t \leq s \leq T)$.

Throughout, we will use the following notations, where $n \geq 1$ is the dimension of the state-space:

- We let Π^g denote the set of all functions $\varphi : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ that are of polynomial growth in x , *i.e.* there are constants $C, \rho > 0$ such that $|\varphi(t, x)| \leq C(1 + |x|^\rho)$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$, and let Π_c^g be the subset of jointly continuous functions.
- For $p \geq 1$ and $t \in [0, T]$, we let $\mathcal{S}_{\text{cag}, t}^p$ be the set of all \mathbb{R} -valued, \mathbb{F}^t -progressively measurable càglàd processes $(Z_s : s \in [t, T])$ such that $\|Z\|_{\mathcal{S}_t^p} := \mathbb{E}[\sup_{s \in [t, T]} |Z_s|^p] < \infty$. Moreover, we let \mathcal{S}_t^p be the subset of continuous processes and $\mathcal{S}_{t,i}^p$ be the subset of continuous and non-decreasing processes with $Z_t = 0$.

- We let \mathcal{H}_t^p denote the set of all \mathbb{R}^d -valued \mathbb{F}^t -progressively measurable processes $(Z_s : s \in [t, T])$ such that $\|Z\|_{\mathcal{H}_t^p} := \mathbb{E}[(\int_t^T |Z_s|^2 ds)^{p/2}]^{1/p} < \infty$.
- We let \mathfrak{S}^p be the set of all maps $Z : \cup_{t \in [0, T]} [t, T] \times \Omega \times \{t\} \times \mathbb{R}^n \rightarrow \mathbb{R} : (s, \omega, t, x) \mapsto Z_s^{t,x}(\omega)$ such that $Z^{t,x} \in \mathcal{S}_t^p$ and there is a $v \in \Pi_c^g$ such that $v(t, x) = Z_t^{t,x}$, \mathbb{P} -a.s., for all $(t, x) \in [0, T] \times \mathbb{R}^n$. Moreover, given $Z \in \mathfrak{S}^p$, we let \bar{Z} denote this deterministic function, so that $\bar{Z}(t, x) = v(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$.
- We let \mathfrak{S}_i^p be the subset of \mathfrak{S}^p with all maps Z such that $Z^{t,x} \in \mathcal{S}_{t,i}^p$.
- We let \mathfrak{H}^p denote the set of all maps $Z : \cup_{t \in [0, T]} [t, T] \times \Omega \times \{t\} \times \mathbb{R}^n \rightarrow \mathbb{R} : (s, \omega, t, x) \mapsto Z_s^{t,x}(\omega)$ such that $Z^{t,x} \in \mathcal{H}_t^p$.
- We let \mathcal{T} be the set of all \mathbb{F} -stopping times and for each $\eta \in \mathcal{T}$ we let \mathcal{T}_η be the subset of stopping times τ such that $\tau \geq \eta$, \mathbb{P} -a.s.
- For $t \in [0, T]$, we let \mathcal{U}_t be the set of all $u = (\tau_j, \beta_j)_{1 \leq j \leq N_T^u}$, where $(\tau_j)_{j=1}^\infty$ is a non-decreasing sequence of \mathbb{F}^t -stopping times in \mathcal{T}_t , β_j is a $\mathcal{F}_{\tau_j}^t$ -measurable r.v. taking values in the compact set U and $N_T^u := \max\{j : \tau_j < T\}$, such that $\Xi_T^{t,x;u} \in L^2(\Omega, \mathcal{F}_T^t, \mathbb{P})$, where $\Xi_T^{t,x;u}$ is the total cost of impulses (see (3.5)).
- For $u \in \mathcal{U}$, we let $[u]_j := (\tau_i, \beta_i)_{1 \leq i \leq N_T^u \wedge j}$.
- For $t \in [0, T]$, we let \mathcal{A}_t denote the set of all $[-1, 1]^d$ -valued, \mathbb{F}^t -progressively measurable processes $(\alpha_s : t \leq s \leq T)$ and set $\mathcal{A} := \mathcal{A}_0$.
- For $t \in [0, T]$, we define the composition \oplus_t of $\alpha^1 \in \mathcal{A}$ and $\alpha^2 \in \mathcal{A}_t$ as $(\alpha^1 \oplus_t \alpha^2)_s := \mathbb{1}_{[0,t)}(s)\alpha_s^1 + \mathbb{1}_{[t,T]}(s)\alpha_s^2$.

We also mention that, unless otherwise specified, all inequalities between random variables are to be interpreted in the \mathbb{P} -a.s. sense.

2.2. Assumptions

Throughout, we make the following assumptions on the parameters in the reward functional where $\rho > 0$ is a fixed constant.

Assumption 2.1. i) We assume that $f : [0, T] \times \mathbb{R}^n \times C(\mathbb{R}^n \rightarrow \mathbb{R}) \times \mathbb{R}^d \rightarrow \mathbb{R}$ is such that for $v \in \Pi_c^g$, the map $(t, x) \mapsto f(t, x, v(t, \cdot), z)$ is jointly continuous, uniformly in z , f is of polynomial growth in x , *i.e.* there is a $C_f > 0$ such that

$$|f(t, x, 0, 0)| \leq C_f(1 + |x|^\rho)$$

and that there are constants $k_f, K_\Gamma > 0$ such that for any $t \in [0, T]$, $x \in \mathbb{R}^n$, $g, \tilde{g} \in C(\mathbb{R}^n \rightarrow \mathbb{R})$ and $z, \tilde{z} \in \mathbb{R}^d$ we have

$$|f(t, x, \tilde{g}, \tilde{z}) - f(t, x, g, z)| \leq k_f \left(\sup_{x' \in \Lambda(|x|)} |\tilde{g}(x') - g(x)| + |\tilde{z} - z| \right),$$

where for each $\gamma \in \mathbb{R}_+$, $\Lambda(\gamma) := \{x \in \mathbb{R}^n : \|x\| \leq \gamma \vee K_\Gamma\}$ is the closed ball of radius $\gamma \vee K_\Gamma$ centered at the origin.

ii) The terminal reward $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous and satisfies the growth condition

$$|\psi(x)| \leq C_\psi(1 + |x|^\rho)$$

for some $C_\psi > 0$.

iii) The intervention cost $\ell : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}_+$ is jointly continuous, of polynomial growth and bounded from below, *i.e.*

$$\ell(t, x, b) \geq \delta > 0,$$

iv) For each $(x, b) \in \mathbb{R}^n \times U$ we have

$$\psi(x) > \psi(\Gamma(T, x, b)) - \ell(t, x, b).$$

Moreover, we make the following assumptions on the coefficients of the forward SDE (and its impulsively controlled counterpart in Sect. 3).

Assumption 2.2. For any $t \in [0, T]$, $b \in U$ and $x, x' \in \mathbb{R}^n$ we have:

i) The function $\Gamma : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ is jointly continuous and satisfies the growth condition

$$|\Gamma(t, x, b)| \leq K_\Gamma \vee |x|. \quad (2.1)$$

ii) The coefficients $a : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ are jointly continuous and satisfy the growth conditions

$$|a(t, x)| + |\sigma(t, x)| \leq C_{a,\sigma}(1 + |x|),$$

for some $C_{a,\sigma} > 0$ and the Lipschitz continuity

$$|a(t, x) - a(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq k_{a,\sigma}|x' - x|,$$

for some $k_{a,\sigma} > 0$.

2.3. Viscosity solutions

We define the upper, v^* , and lower, v_* semi-continuous envelope of a function v as

$$v^*(t, x) := \limsup_{(t', x') \rightarrow (t, x), t' < T} v(t', x') \quad \text{and} \quad v_*(t, x) := \liminf_{(t', x') \rightarrow (t, x), t' < T} v(t', x')$$

Next we introduce the notion of a viscosity solution using the limiting parabolic superjet \bar{J}^+v and subjet \bar{J}^-v of a function v .

Definition 2.3. Subjets and superjets

i) For a lsc (resp. usc) function $u : [0, T] \times \mathbb{R}^n \rightarrow R$, the parabolic subjet, denote by $J^-u(t, x)$, (resp. the parabolic superjet, $J^+u(t, x)$) of u at $(t, x) \in [0, T] \times \mathbb{R}^n$, is defined as the set of triples $(p, q, M) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ satisfying

$$u(t', x') \geq (\text{resp. } \leq) u(t, x) + p(t' - t) + \langle q, x' - x \rangle + \frac{1}{2} \langle x' - x, M(x' - x) \rangle + o(|t' - t| + |x' - x|^2)$$

for all $(t', x') \in [0, T] \times \mathbb{R}^n$, where \mathbb{S}^n is the set of symmetric real matrices of dimension $n \times n$.

ii) For a lsc (resp. usc) function $u : [0, T] \times \mathbb{R}^n \rightarrow R$ we denote by $\bar{J}^-u(t, x)$ the parabolic limiting subjet (resp. $\bar{J}^+u(t, x)$ the parabolic limiting superjet) of u at $(t, x) \in [0, T] \times \mathbb{R}^n$, defined as the set of triples $(p, q, M) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n$ such that:

$$(p, q, M) = \lim_{n \rightarrow \infty} (p_n, q_n, M_n), \quad (t, x) = \lim_{n \rightarrow \infty} (t_n, x_n)$$

for some sequence $(t_n, x_n, p_n, q_n, M_n)_{n \geq 1}$ with $(p_n, q_n, M_n) \in J^-u(t_n, x_n)$ (resp. $(p_n, q_n, M_n) \in J^+u(t_n, x_n)$) for all $n \geq 1$ and $u(t, x) = \lim_{n \rightarrow \infty} u(t_n, x_n)$.

We now give the definition of a viscosity solution for the QVI in (1.1). (see also pp. 9-10 of [8]).

Definition 2.4. Let v be a locally bounded function from $[0, T] \times \mathbb{R}^n$ to \mathbb{R} . Then,

- a) It is referred to as a viscosity supersolution (resp. subsolution) to (1.1) if it is l.s.c. (resp. u.s.c.) and satisfies:
- i) $v(T, x) \geq \psi(x)$ (resp. $v(T, x) \leq \psi(x)$)
 - ii) For any $(t, x) \in [0, T] \times \mathbb{R}^d$ and $(p, q, X) \in \bar{J}^-v(t, x)$ (resp. $\bar{J}^+v(t, x)$) we have

$$\min \left\{ v(t, x) - \mathcal{M}v(t, x), -p - f(t, x, v(t, \cdot), q, X, a) \right\} \geq 0$$

(resp.

$$\min \left\{ v(t, x) - \mathcal{M}v(t, x), -p - f(t, x, v(t, \cdot), q, X, a) \right\} \leq 0).$$

- b) It is called a viscosity solution to (1.1) if v_* is a supersolution and v^* is a subsolution.

We will sometimes use the following alternative definition of viscosity supersolutions (resp. subsolutions).

Definition 2.5. A l.s.c. (resp. u.s.c.) function v is a viscosity supersolution (subsolution) to (1.1) if $v(T, x) \leq \psi(x)$ (resp. $\geq \psi(x)$) and whenever $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d \rightarrow \mathbb{R})$ is such that $\varphi(t, x) = v(t, x)$ and $\varphi - v$ has a local maximum (resp. minimum) at (t, x) , then

$$\min \left\{ v(t, x) - \mathcal{M}v(t, x), -\varphi_t(t, x) - f(t, x, v(t, \cdot), D\varphi(t, x), D^2\varphi(t, x), a) \right\} \geq 0 \ (\leq 0).$$

2.4. Reflected BSDEs and obstacle problems

We will make extensive use of the following classical result.

Theorem 2.6. (El Karoui et al. [13]) Let $h : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ be (jointly) continuous and of polynomial growth with $h(T, x) \leq \psi(x)$ for all $x \in \mathbb{R}^n$ and assume that $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ ($(t, x, y, z) \mapsto f(t, x, y, z)$) is jointly continuous in (t, x) uniformly in (y, z) , Lipschitz continuous in (z, y) uniformly in (t, x) and such that $f(\cdot, \cdot, 0, 0) \in \Pi^g$. Then, for each $(t, x) \in [0, T] \times \mathbb{R}^n$, there is a unique triple $(Y^{t,x}, Z^{t,x}, K^{t,x}) \in \mathcal{S}_t^2 \times \mathcal{H}_t^2 \times \mathcal{S}_{t,i}^2$ such that

$$\begin{cases} Y_s^{t,x} = \psi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r + K_T^{t,x} - K_s^{t,x}, & \forall s \in [t, T], \\ Y_s^{t,x} \geq h(s, X_s^{t,x}), \forall s \in [t, T] & \text{and} \quad \int_t^T (Y_s^{t,x} - h(s, X_s^{t,x})) dK_s^{t,x} = 0. \end{cases} \quad (2.2)$$

Moreover, there is a $v \in \Pi_c^g$ such that $Y_s^{t,x} = v(s, X_s^{t,x})$, \mathbb{P} -a.s. for all $s \in [t, T]$, and v is the unique viscosity solution in Π^g to the following obstacle problem

$$\begin{cases} \min \{ v(t, x) - h(t, x), -v_t(t, x) - \mathcal{L}v(t, x) - f(t, x, v(t, x), \sigma^\top(t, x) \nabla_x v(t, x)) \} = 0, \\ \forall (t, x) \in [0, T] \times \mathbb{R}^d \\ v(T, x) = \psi(x). \end{cases} \quad (2.3)$$

Furthermore, there is a constant² $C > 0$ (that only depends on the Lipschitz constants of the coefficients) such that

$$\|Y^{t,x}\|_{\mathcal{S}_t^2}^2 + \|Z^{t,x}\|_{\mathcal{H}_t^2}^2 + \|K^{t,x}\|_{\mathcal{S}_t^2}^2 \leq C \mathbb{E} \left[|\psi(X_T^{t,x})|^2 + \int_t^T |f(s, X_s^{t,x}, 0, 0)|^2 ds + \sup_{s \in [t, T]} |(h(s, X_s^{t,x}))^+|^2 \right]. \quad (2.4)$$

²Throughout, C will denote a generic positive constant that may change value from line to line.

In addition, Y can be interpreted as the Snell envelope in the following way

$$Y_s^{t,x} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_s} \mathbb{E} \left[\int_s^\tau f(r, Y_r^{t,x}, Z_r^{t,x}) dr + h(\tau, X_\tau^{t,x}) \mathbb{1}_{[\tau < T]} + \psi(X_T^{t,x}) \mathbb{1}_{[\tau = T]} \middle| \mathcal{F}_s^t \right]$$

and with $D_s^{t,x} := \inf\{r \geq s : Y_r^{t,x} = h(r, X_r^{t,x})\} \wedge T$ we have the representation

$$Y_s = \mathbb{E} \left[\int_s^{D_s^{t,x}} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr + h(D_s^{t,x}, X_{D_s^{t,x}}^{t,x}) \mathbb{1}_{[D_s^{t,x} < T]} + \psi(X_T^{t,x}) \mathbb{1}_{[D_s^{t,x} = T]} \middle| \mathcal{F}_s^t \right]$$

and $K_{D_s^{t,x}}^{t,x} - K_s^{t,x} = 0$, \mathbb{P} -a.s.

Finally, if $(\tilde{Y}, \tilde{Z}, \tilde{K})$ is the solution to the reflected BSDE with parameters $(\tilde{\psi}, \tilde{f}, \tilde{h})$, then

$$\begin{aligned} & \|\tilde{Y}_{\cdot, \sqrt{t}}^{\tilde{t}, \tilde{x}} - Y_{\cdot, \sqrt{t}}^{t,x}\|_{\mathcal{S}^2}^2 + \|\mathbb{1}_{[\tilde{t}, T]} \tilde{Z}^{\tilde{t}, \tilde{x}} - \mathbb{1}_{[t, T]} Z^{t,x}\|_{\mathcal{H}^2}^2 + \|\tilde{K}_{\cdot, \sqrt{t}}^{\tilde{t}, \tilde{x}} - K_{\cdot, \sqrt{t}}^{t,x}\|_{\mathcal{S}^2}^2 \leq C(\|\tilde{h}(\cdot \vee \tilde{t}, X_{\cdot, \sqrt{t}}^{\tilde{t}, \tilde{x}}) - h(\cdot \vee t, X_{\cdot, \sqrt{t}}^{t,x})\|_{\mathcal{S}^2} \Psi_T^{1/2} \\ & + \mathbb{E} \left[|\tilde{\psi}(X_T^{\tilde{t}, \tilde{x}}) - \psi(X_T^{t,x})|^2 + \int_0^T |\mathbb{1}_{[\tilde{t}, T]}(s) \tilde{f}(s, X_s^{\tilde{t}, \tilde{x}}, Y_s, Z_s) - \mathbb{1}_{[t, T]}(s) f(s, X_s^{t,x}, Y_s, Z_s)|^2 ds \right]), \end{aligned} \quad (2.5)$$

where

$$\begin{aligned} \Psi_T &:= \mathbb{E} \left[|\tilde{\psi}(X_T^{\tilde{t}, \tilde{x}})|^2 + |\psi(X_T^{t,x})|^2 + \int_{\tilde{t}}^T |\tilde{f}(s, X_s^{\tilde{t}, \tilde{x}}, 0, 0)|^2 ds + \int_t^T |f(s, X_s^{t,x}, 0, 0)|^2 ds \right. \\ & \left. + \sup_{s \in [\tilde{t}, T]} |(\tilde{h}(s, X_s^{\tilde{t}, \tilde{x}}))^+|^2 + \sup_{s \in [t, T]} |(h(s, X_s^{t,x}))^+|^2 \right]. \end{aligned}$$

3. THE CASE OF A LOCAL DRIVER AND THE CORRESPONDING IMPULSE CONTROL PROBLEM

In this section we consider a simplified setting in which f is a function $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying the requirements in the statement of Theorem 2.6. We show that in this case we can relate the solution to the BSDE to an impulse control problem. We recall the system of reflected BSDEs from the introduction

$$\begin{cases} Y_s^{t,x} = \psi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r + K_T^{t,x} - K_s^{t,x}, & \forall s \in [t, T], \\ Y_s^{t,x} \geq \mathcal{M}\bar{Y}(s, X_s^{t,x}), \quad \forall s \in [t, T] \quad \text{and} \quad \int_t^T (Y_s^{t,x} - \mathcal{M}\bar{Y}(s, X_s^{t,x})) dK_s^{t,x} = 0, \end{cases} \quad (3.1)$$

where $\bar{Y}(t, x) := Y_t^{t,x}$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$. For $(t, x) \in [0, T] \times \mathbb{R}^n$ and $u \in \mathcal{U}_t$ (recall that \mathcal{U}_t is the set of all \mathbb{F}^t -adapted impulse controls $u := (\tau_j, \beta_j)_{j=1}^\infty$ with $\tau_1 \geq t$, \mathbb{P} -a.s.) we let the càdlàg process $X^{t,x;u}$ solve the impulsively controlled SDE

$$X_s^{t,x;u} = x + \int_t^s a(r, X_r^{t,x;u}) dr + \int_t^s \sigma(r, X_r^{t,x;u}) dW_r \quad (3.2)$$

for $s \in [t, \tau_1)$ and

$$X_s^{t,x;u} = \Gamma(\tau_j, X_{\tau_j}^{t,x;[u]_{j-1}}, \beta_j) + \int_{\tau_j}^s a(r, X_r^{t,x;u}) dr + \int_{\tau_j}^s \sigma(r, X_r^{t,x;u}) dW_r, \quad (3.3)$$

whenever $s \in [\tau_j, \tau_{j+1})$ for $j = 1, \dots, N_T^u - 1$ and $s \in [\tau_{N_T^u}, T]$ when $j = N_T^u$. Moreover, we let the pair $(P^{t,x;u}, Q^{t,x;u}) \in \mathcal{S}_{\text{cag},t}^2 \times \mathcal{H}_t^2$ be the unique solution to the non-standard BSDE

$$P_s^{t,x;u} = \psi(X_T^{t,x;u}) + \int_s^T f(r, X_r^{t,x;u}, P_r^{t,x;u}, Q_r^{t,x;u}) dr - \int_s^T Q_r^{t,x;u} dW_r - \Xi_T^{t,x;u} + \Xi_s^{t,x;u}, \quad (3.4)$$

where the impulse cost process Ξ is defined as

$$\Xi_s^{t,x;u} := \sum_{j=1}^{N_T^u} \mathbb{1}_{[\tau_j < s]} \ell(\tau_j, X_{\tau_j}^{t,x;[u]_{j-1}}, \beta_j). \quad (3.5)$$

Remark 3.1. Since we require that $\Xi_T^{t,x;u}$ is square-integrable for all $u \in \mathcal{U}$ and $\ell \geq \delta > 0$, it follows that N_T^u is \mathbb{P} -a.s. finite implying by standard arguments that the forward SDE (3.2)–(3.3) admits a unique solution for each $(t, x) \in [0, T] \times \mathbb{R}^n$ and $u \in \mathcal{U}_t$.

We then introduce the following impulse control problem:

Problem 1. For $(t, x) \in [0, T] \times \mathbb{R}^n$, find $u^* \in \mathcal{U}_t$ such that

$$P_t^{t,x;u^*} = \sup_{u \in \mathcal{U}_t} P_t^{t,x;u}.$$

The main result of this section is the following:

Theorem 3.2. *There exists a unique solution $(Y, Z, K) \in \mathfrak{G}^2 \times \mathfrak{H}^2 \times \mathfrak{G}_i^2$ to (3.1). Moreover, there is a $v \in \Pi_c^g$ such that $v(t, x) = Y_t^{t,x}$, \mathbb{P} -a.s., and v is the unique viscosity solution in Π^g to*

$$\begin{cases} \min\{v(t, x) - \mathcal{M}v(t, x), -v_t(t, x) - \mathcal{L}v(t, x) - f(t, x, v(t, x), \sigma^\top(t, x) \nabla_x v(t, x))\} = 0, \\ \forall (t, x) \in [0, T) \times \mathbb{R}^d \\ v(T, x) = \psi(x). \end{cases} \quad (3.6)$$

Finally, we have the representation

$$Y_t^{t,x} = \text{ess sup}_{u \in \mathcal{U}_t} P_t^{t,x;u} = P_t^{t,x;u^*}, \quad (3.7)$$

where $u^* = (\tau_j^*, \beta_j^*)_{j=1}^{N_T^{u^*}} \in \mathcal{U}_t$ is defined as:

$$\begin{aligned} - \tau_j^* &:= \inf \{s \geq \tau_{j-1}^* : v(s, X_s^{t,x;[u^*]_{j-1}}) = \mathcal{M}v(s, X_s^{t,x;[u^*]_{j-1}})\} \wedge T, \\ - \beta_j^* &\in \arg \max_{b \in U} \{v(\tau_j^*, \Gamma(\tau_j^*, X_{\tau_j^*}^{t,x;[u^*]_{j-1}}, b)) - \ell(\tau_j^*, X_{\tau_j^*}^{t,x;[u^*]_{j-1}}, b)\} \end{aligned}$$

and $N_T^{u^*} = \sup\{j : \tau_j^* < T\}$, with $\tau_0^* := t$.

Since the uniqueness of solutions to (3.6) is rather standard we postpone the proof to the appendix. The proof of the existence part is based on an approximating scheme given in the following subsection.

3.1. An approximating scheme

The existence part of the proof of Theorem 3.2 is based on an approximation routine where we restrict the number of allowed interventions. We introduce the following sequence of reflected BSDEs

$$\begin{cases} Y_s^{t,x,k} = \psi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x,k}, Z_r^{t,x,k})dr - \int_s^T Z_r^{t,x,k}dW_r + K_T^{t,x,k} - K_s^{t,x,k}, & \forall s \in [t, T], \\ Y_s^{t,x,k} \geq \mathcal{M}\bar{Y}^{k-1}(s, X_s^{t,x}), \forall s \in [t, T] & \text{and} \quad \int_t^T (Y_s^{t,x,k} - \mathcal{M}\bar{Y}^{k-1}(s, X_s^{t,x}))dK_s^{t,x,k} = 0 \end{cases} \quad (3.8)$$

and

$$Y_s^{t,x,0} = \psi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x,0}, Z_r^{t,x,0})dr - \int_s^T Z_r^{t,x,0}dW_r, \quad \forall s \in [t, T]. \quad (3.9)$$

We will relate the solution of these BSDEs to a sequence of impulse control problems, defined for $k \geq 1$:

Problem 1.k For $(t, x) \in [0, T] \times \mathbb{R}^n$, find $u^{*,k} \in \mathcal{U}_t^k$ such that

$$P_t^{t,x;u^{*,k}} = \sup_{u \in \mathcal{U}_t^k} P_t^{t,x;u}.$$

We also make use of the following lemma which is given without proof as it follows immediately from the definitions:

Lemma 3.3. *Let $u, v : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}$ be locally bounded functions. \mathcal{M} is monotone (if $u \leq v$ pointwise, then $\mathcal{M}u \leq \mathcal{M}v$). Moreover, $\mathcal{M}(u_*)$ (resp. $\mathcal{M}(u^*)$) is l.s.c. (resp. u.s.c.).*

In particular, it follows that $\mathcal{M}v$ is jointly continuous whenever v is.

Using Theorem 2.6 and the previous lemma we get the following intermediate result:

Proposition 3.4. *The system (3.8)–(3.9) admits a unique solution and there is a $v_k \in \Pi_c^g$ such that*

$$v_k(t, x) = Y_t^{t,x,k} = \sup_{u \in \mathcal{U}_t^k} P_t^{t,x;u}, \quad (3.10)$$

\mathbb{P} -a.s., for each $(t, x) \in [0, T] \times \mathbb{R}^n$ and $k \geq 0$. Moreover, an optimal control $u^{*,k} \in \mathcal{U}_t^k$ for (3.10) exists.

Proof. First note that classically (3.9) admits a unique solution $(Y^{t,x,0}, Z^{t,x,0})$ and that there is a $v_0 \in \Pi_c^g$ such that $Y_s^{t,x,0} = v_0(s, X_s^{t,x})$, \mathbb{P} -a.s. for all $s \in [t, T]$. Then, since $(t, x) \mapsto \mathcal{M}g(t, x) \in \Pi_c^g$ whenever $g \in \Pi_c^g$ by Lemma 3.3 it follows that $\mathcal{M}v_0 \in \Pi_c^g$. On the other hand, by definition

$$\mathcal{M}\bar{Y}^0(s, X_s^{t,x}) = \mathcal{M}v_0(s, X_s^{t,x}), \quad \forall s \in [t, T], \quad \mathbb{P}\text{-a.s.}$$

and we conclude by Theorem 2.6 that (3.8) admits a unique solution for $k = 1$ with $v_1 := \bar{Y}^1 \in \Pi_c^g$. Repeating this argument gives that (3.8) admits a unique solution for arbitrary k with $v_k := \bar{Y}^k \in \Pi_c^g$.

For the representation, given $(t, x) \in [0, T] \times \mathbb{R}^n$ we will use an inductive procedure to construct a control $u^{*,k}$ such that $Y_t^{t,x,k} = P_t^{t,x;u^{*,k}}$ and then establish (3.10) by showing that $u^{*,k}$ solves Problem 1.k. We thus pick $(t, x) \in [0, T] \times \mathbb{R}^n$ and define $u^{*,k} = (\tau_j^*, \beta_j^*)_{j=1}^{N_T^{u^{*,k}}}$ $\in \mathcal{U}_t^k$ as:

$$- \tau_j^* := \inf \{ s \geq \tau_{j-1}^* : v_{k+1-j}(s, X_s^{t,x;[u^{*,k}]_{j-1}}) = \mathcal{M}v_{k-j}(s, X_s^{t,x;[u^{*,k}]_{j-1}}) \} \wedge T,$$

$$- \beta_j^* \in \arg \max_{b \in U} \{v_{k-j}(\tau_j^*, \Gamma(\tau_j^*, X_{\tau_j^*}^{t,x;[u^*,k]_{j-1}}, b)) - \ell(\tau_j^*, X_{\tau_j^*}^{t,x;[u^*,k]_{j-1}}, b)\}$$

for $j = 1, \dots, k$, and $N_T^{u^*,k} := \max\{j \in \{0, \dots, k\} : \tau_j^* < T\}$, with $\tau_0^* := t$. Then, $K_{\tau_1^*}^{t,x} - K_t^{t,x} = 0$, \mathbb{P} -a.s., and we find that

$$\begin{aligned} Y_s^{t,x,k} &= \mathbb{1}_{[\tau_1^* < T]} \mathcal{M}v_{k-1}(\tau_1^*, X_{\tau_1^*}^{t,x}) + \mathbb{1}_{[\tau_1^* = T]} \psi(X_T^{t,x}) + \int_s^{\tau_1^*} f(r, X_r^{t,x}, Y_r^{t,x,k}, Z_r^{t,x,k}) dr - \int_s^{\tau_1^*} Z_r^{t,x,k} dW_r \\ &= \mathbb{1}_{[\tau_1^* < T]} \{v_{k-1}(\tau_1^*, X_{\tau_1^*}^{t,x;[u^*,k]_1}) - \ell(\tau_1^*, X_{\tau_1^*}^{t,x}, \beta_1^*)\} + \mathbb{1}_{[\tau_1^* = T]} \psi(X_T^{t,x}) \\ &\quad + \int_s^{\tau_1^*} f(r, X_r^{t,x}, Y_r^{t,x,k}, Z_r^{t,x,k}) dr - \int_s^{\tau_1^*} Z_r^{t,x,k} dW_r \end{aligned}$$

for all $s \in [t, \tau_1^*]$. To simplify notation later we let $(\mathcal{Y}^0, \mathcal{Z}^0) := (Y^{t,x}, Z^{t,x})$. Similarly, for each $j \in \{1, \dots, k-1\}$, Theorem 5.2 of [13] implies that there is a unique triple $(\mathcal{Y}^j, \mathcal{Z}^j, \mathcal{K}^j) \in \mathcal{S}_t^2 \times \mathcal{H}_t^2 \times \mathcal{S}_{t,i}^2$ that solves the reflected bsde

$$\begin{cases} \mathcal{Y}_s^j = \psi(X_T^{t,x;[u^*,k]_j}) + \int_s^T f(r, X_r^{t,x;[u^*,k]_j}, \mathcal{Y}_r^j, \mathcal{Z}_r^j) dr - \int_s^T \mathcal{Z}_r^j dW_r + \mathcal{K}_T^j - \mathcal{K}_s^j, \quad \forall s \in [t, T], \\ \mathcal{Y}_s^j \geq \mathcal{M}v_{k-j-1}(s, X_s^{t,x;[u^*,k]_j}), \quad \forall s \in [t, T] \quad \text{and} \quad \int_t^T (\mathcal{Y}_s^j - \mathcal{M}v_{k-j-1}(s, X_s^{t,x;[u^*,k]_j})) d\mathcal{K}_s^j = 0 \end{cases}$$

and continuity together with an approximation of $X^{t,x;[u^*,k]_j}$ and the stability result for reflected BSDEs in Proposition 3.6 of [13] implies that $\mathcal{Y}_s^j = v_{k-j}(s, X_s^{t,x;[u^*,k]_j})$, \mathbb{P} -a.s., for each $s \in [\tau_j^*, T]$. Moreover, the proof of Proposition 2.3 in [13] gives that $\mathcal{K}_{\tau_{j+1}^*}^j - \mathcal{K}_{\tau_j^*}^j = 0$, \mathbb{P} -a.s. and we conclude that

$$\begin{aligned} \mathcal{Y}_s^j &= \mathbb{1}_{[\tau_{j+1}^* < T]} \{v_{k-j-1}(\tau_{j+1}^*, X_{\tau_{j+1}^*}^{t,x;[u^*,k]_{j+1}}) - \ell(\tau_{j+1}^*, X_{\tau_{j+1}^*}^{t,x;[u^*,k]_j}, \beta_{j+1}^*)\} + \mathbb{1}_{[\tau_{j+1}^* = T]} \psi(X_T^{t,x;u^*,k}) \\ &\quad + \int_s^{\tau_{j+1}^*} f(r, X_r^{t,x}, \mathcal{Y}_r^j, \mathcal{Z}_r^j) dr - \int_s^{\tau_{j+1}^*} \mathcal{Z}_r^j dW_r \end{aligned}$$

for all $s \in [\tau_j^*, \tau_{j+1}^*]$. Finally, letting $(\mathcal{Y}^k, \mathcal{Z}^k)$ be the unique solution to the standard BSDE

$$\mathcal{Y}_s^k = \psi(X_T^{t,x;u^*,k}) + \int_s^T f(r, X_r^{t,x}, \mathcal{Y}_r^k, \mathcal{Z}_r^k) dr - \int_s^T \mathcal{Z}_r^k dW_r$$

we get that $\mathcal{Y}_{\tau_k^*}^k = v_0(\tau_k^*, X_{\tau_k^*}^{t,x;u^*,k})$, \mathbb{P} -a.s. and we conclude that the pair $(\mathcal{Y}, \mathcal{Z})$ defined as $\mathcal{Y} := \mathbb{1}_{[t, \tau_1^*]} \mathcal{Y}^0 + \sum_{j=1}^k \mathbb{1}_{(\tau_j^*, \tau_{j+1}^*]} \mathcal{Y}^j$ and $\mathcal{Z} := \sum_{j=0}^k \mathbb{1}_{(\tau_j^*, \tau_{j+1}^*]} \mathcal{Z}^j$ belongs to $\mathcal{S}_t^2 \times \mathcal{H}_t^2$ and satisfies

$$\mathcal{Y}_s = \psi(X_T^{t,x;u^*,k}) + \int_s^T f(r, X_r^{t,x;u^*,k}, \mathcal{Y}_r, \mathcal{Z}_r) dr - \int_s^T \mathcal{Z}_r dW_r - \sum_{j=1}^{N_T^{u^*,k}} \mathbb{1}_{[s \leq \tau_j^*]} \ell(\tau_j^*, X_{\tau_j^*}^{t,x;[u^*,k]_{j-1}}, \beta_j^*), \quad \forall s \in [t, T].$$

By uniqueness of solutions to (3.4) and since $\mathcal{Y}_t^0 = Y_t^{t,x,k}$, we thus conclude that $v_k(t, x) = Y_t^{t,x,k} = P_t^{t,x;u^*,k}$. Now, suppose that $\hat{u} \in \mathcal{U}_t^k$ is another impulse control, then

$$\begin{aligned} Y_s^{t,x,k} &= \mathbb{1}_{[\hat{\tau}_1 < T]} \{v_{k-1}(\hat{\tau}_1, \Gamma(\hat{\tau}_1, X_{\hat{\tau}_1}^{t,x})) - \ell(\hat{\tau}_1, X_{\hat{\tau}_1}^{t,x}, \hat{\beta}_1)\} + \mathbb{1}_{[\hat{\tau}_1 = T]} \psi(X_T^{t,x}) \\ &\quad + \int_s^{\hat{\tau}_1} f(r, X_r^{t,x}, Y_r^{t,x,k}, Z_r^{t,x,k}) dr - \int_s^{\hat{\tau}_1} Z_r^{t,x,k} dW_r + K_{\hat{\tau}_1}^{t,x,k} - K_s^{t,x,k} \end{aligned}$$

for all $s \in [t, \hat{\tau}_1]$. Arguing as above gives that there is a sequence $(\hat{\mathcal{Y}}^j, \hat{\mathcal{Z}}^j, \hat{\mathcal{K}}^j)_{j=0}^k \subset \mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{S}_i^2$ such that letting $\hat{\mathcal{Y}} := \mathbb{1}_{[t, \hat{\tau}_1]} \hat{\mathcal{Y}}^0 + \sum_{j=0}^k \mathbb{1}_{(\hat{\tau}_j, \hat{\tau}_{j+1}]} \hat{\mathcal{Y}}^j$, $\hat{\mathcal{Z}} := \sum_{j=0}^k \mathbb{1}_{(\hat{\tau}_j, \hat{\tau}_{j+1}]} \hat{\mathcal{Z}}^j$ and $\hat{\mathcal{K}}_s := \sum_{j=0}^k \mathbb{1}_{[\hat{\tau}_j < s]} \{\hat{\mathcal{K}}_{s \wedge \hat{\tau}_{j+1}}^j - \hat{\mathcal{K}}_{\hat{\tau}_j}^j\}$, with $\hat{\tau}_0 := t$, implies that $(\hat{\mathcal{Y}}, \hat{\mathcal{Z}}, \hat{\mathcal{K}}) \subset \mathcal{S}_{\text{cag}, t}^2 \times \mathcal{H}_t^2 \times \mathcal{S}_{i, t}^2$ satisfies

$$\begin{aligned} \hat{\mathcal{Y}}_s &= \psi(X_T^{t, x; \hat{u}}) + \int_s^T f(r, X_r^{t, x; \hat{u}}, \hat{\mathcal{Y}}_r, \hat{\mathcal{Z}}_r) dr - \int_s^T \hat{\mathcal{Z}}_r dW_r - \sum_{j=1}^{N_T^{\hat{u}}} \mathbb{1}_{[s \leq \hat{\tau}_j]} \ell(\hat{\tau}_j, X_{\hat{\tau}_j}^{t, x; [\hat{u}]_{j-1}}, \hat{\beta}_j) \\ &\quad + \hat{\mathcal{K}}_T - \hat{\mathcal{K}}_s, \quad \forall s \in [t, T] \end{aligned}$$

and $\hat{\mathcal{Y}}_t = Y_t^{t, x, k}$. Now, comparison gives that $P_t^{t, x; \hat{u}} \leq \hat{\mathcal{Y}}_t$ and we conclude that $P_t^{t, x; \hat{u}} \leq Y_t^{t, x, k}$. \square

Before we proceed to give the proof of Theorem 3.2 we need some preliminary estimates, which we give in the following subsection.

3.2. Some preliminary estimates

Proposition 3.5. *For each $p \geq 1$, there is a $C > 0$ such that*

$$\mathbb{E} \left[\sup_{s \in [\zeta, T]} |X_s^{t, x; u}|^p \middle| \mathcal{F}_\zeta^t \right] \leq C(1 + |X_\zeta^{t, x; u}|^p), \quad (3.11)$$

\mathbb{P} -a.s. for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $(\zeta, u) \in [t, T] \times \mathcal{U}_t$.

Proof. We use the shorthand $X^j := X^{t, x; [u]^j}$. By Assumption 2.2.(i) we get for $s \in [\tau_j, T]$, using integration by parts, that

$$\begin{aligned} |X_s^j|^2 &= |X_{\zeta \vee \tau_j}^j|^2 + 2 \int_{(\zeta \vee \tau_j)_+}^s X_r^j dX_r^j + \int_{(\zeta \vee \tau_j)_+}^s d[X^j, X^j]_r \\ &\leq K_\Gamma^2 \vee |X_{\zeta \vee \tau_j}^{j-1}|^2 + 2 \int_{(\zeta \vee \tau_j)_+}^s X_r^j dX_r^j + \int_{(\zeta \vee \tau_j)_+}^s d[X^j, X^j]_r. \end{aligned}$$

Now, either $|X_{\tau_j}^{j-1}| \leq K_\Gamma$ in which case

$$|X_s^j|^2 \leq |X_\zeta^j|^2 \vee K_\Gamma^2 + 2 \int_{(\zeta \vee \tau_j)_+}^s X_r^j dX_r^j + \int_{(\zeta \vee \tau_j)_+}^s d[X^j, X^j]_r.$$

or $|X_{\tau_j}^{j-1}| > K_\Gamma$ implying that

$$\begin{aligned} |X_s^j|^2 &\leq K_\Gamma^2 \vee |X_{\zeta \vee \tau_{j-1}}^{j-2}|^2 + 2 \int_{(\zeta \vee \tau_{j-1})_+}^{\tau_j} X_r^{j-1} dX_r^{j-1} + \int_{(\zeta \vee \tau_{j-1})_+}^{\tau_j} d[X^{j-1}, X^{j-1}]_r \\ &\quad + 2 \int_{(\zeta \vee \tau_j)_+}^s X_r^j dX_r^j + \int_{(\zeta \vee \tau_j)_+}^s d[X^j, X^j]_r. \end{aligned}$$

In the latter case the same argument can be repeated and we conclude that

$$|X_s^j|^2 \leq |X_\zeta^j|^2 \vee K_\Gamma^2 + \sum_{i=j_0}^j \left\{ 2 \int_{(\zeta \vee \hat{\tau}_i)_+}^{s \wedge \hat{\tau}_{i+1}} X_r^i dX_r^i + \int_{(\zeta \vee \hat{\tau}_i)_+}^{s \wedge \hat{\tau}_{i+1}} d[X^i, X^i]_r \right\}, \quad (3.12)$$

where $\tilde{\tau}_0 = -1$, $\tilde{\tau}_i = \tau_i$ for $i = 1, \dots, j$ and $\tilde{\tau}_{j+1} = \infty$ and $j_0 := \max\{i \in \{1, \dots, j\} : |X_{\tilde{\tau}_i}^{i-1}| \leq K_\Gamma\} \vee 0$.
Now, since X^i and X^j coincide on $[0, \tau_{i+1} \wedge j+1)$ we have

$$\sum_{i=j_0}^j \int_{(\zeta \vee \tilde{\tau}_i)^+}^{s \wedge \tilde{\tau}_{i+1}} X_r^i dX_r^i = \int_{\zeta \vee \tau_{j_0}}^s X_r^j a(r, X_r^j) dr + \int_{\zeta \vee \tau_{j_0}}^s X_r^j \sigma(r, X_r^j) dW_r,$$

and

$$\sum_{i=j_0}^j \int_{(\zeta \vee \tilde{\tau}_i)^+}^{s \wedge \tilde{\tau}_{i+1}} d[X^i, X^i]_r = \int_{\zeta \vee \tau_{j_0}}^s \sigma^2(r, X_r^j) dr.$$

Inserted in (3.12) this that

$$\begin{aligned} |X_s^j|^2 &\leq |X_\zeta^j|^2 \vee K_\Gamma^2 + \int_{\tau_{j_0}}^s (2X_s^j a(r, X_r^j) + \sigma^2(r, X_r^j)) dr + 2 \int_{\tau_{j_0}}^s X_r^j \sigma(r, X_r^j) dW_r \\ &\leq |X_\zeta^j|^2 + C \left(1 + \int_\zeta^s |X_r^j|^2 dr + \sup_{\eta \in [\zeta, s]} \left| \int_\zeta^\eta X_r^j \sigma(r, X_r^j) dW_r \right| \right) \end{aligned} \quad (3.13)$$

for all $s \in [\zeta, T]$. The Burkholder-Davis-Gundy inequality combined with Grönwall's lemma gives that for $p \geq 4$,

$$\mathbb{E} \left[\sup_{s \in [\zeta, T]} |X_s^j|^p \middle| \mathcal{F}_\zeta^t \right] \leq C(1 + |X_\zeta^j|^p),$$

\mathbb{P} -a.s., where the constant $C = C(T, p)$ does not depend on u or j and (3.11) follows by letting $j \rightarrow \infty$ on both sides and using Fatou's lemma. The result for general $p \geq 1$ follows by Jensen's inequality. \square

For $(t, x) \in [0, T] \times \mathbb{R}^n$ and $u \in \mathcal{U}_t$ we let $(\check{P}^{t,x;u}, \check{Q}^{t,x;u})$ be the unique solution to the following standard BSDE

$$\check{P}_s^{t,x;u} = \psi(X_T^{t,x;u}) + \int_s^T f(r, X_r^{t,x;u}, \check{P}_r^{t,x;u}, \check{Q}_r^{t,x;u}) dr - \int_s^T \check{Q}_r^{t,x;u} dW_r. \quad (3.14)$$

Combining classical results (see *e.g.* [14]) with Proposition 3.5, we have

$$\begin{aligned} &\mathbb{E} \left[\sup_{s \in [t, T]} |\check{P}_s^{t,x;u}|^2 + \int_t^T |\check{Q}_s^{t,x;u}|^2 ds \right] \\ &\leq C \mathbb{E} \left[|\psi(X_T^{t,x;u})|^2 + \int_t^T |f(r, X_r^{t,x;u}, 0, 0)|^2 dr \right] \leq C(1 + |x|^{2\rho}), \end{aligned} \quad (3.15)$$

for all $u \in \mathcal{U}_t$.

Using the comparison principle we easily deduce the following moment estimates:

Proposition 3.6. *We have,*

$$\left| \sup_{u \in \mathcal{U}_t} P_t^{t,x;u} \right| \leq C(1 + |x|^\rho) \quad (3.16)$$

and for each $k \geq 0$, there is a $C > 0$ such that

$$\mathbb{E} \left[\sup_{s \in [t, T]} |P_s^{t, x; u}|^2 + \int_t^T |Q_s^{t, x; u}|^2 ds \right] \leq C(1 + |x|^{2\rho}), \quad (3.17)$$

\mathbb{P} -a.s. for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $u \in \mathcal{U}_t^k$.

Proof. The first statement follows by repeated application of the comparison principle which gives that $\check{P}_t^{t, x; \emptyset} \leq \sup_{u \in \mathcal{U}_t} P_t^{t, x; u} \leq \sup_{u \in \mathcal{U}_t} \check{P}_t^{t, x; u}$ and using (3.15).

The second statement follows by noting that for fixed $k \geq 0$, there is a $C > 0$ such that

$$\mathbb{E}[|\Xi_T^{t, x; u}|^2] \leq C(1 + \mathbb{E}[\sup_{s \in [t, T]} |X_s^{t, x; u}|^{2\rho}]) \leq C(1 + |x|^{2\rho})$$

for all $u \in \mathcal{U}_t^k$. □

In the following lemma we use the above estimates to derive a bound on the expected number of interventions in an optimal control for Problem 1.k. This bound plays an important role in the convergence analysis employed later on.

Lemma 3.7. *There is a $C > 0$ such that whenever $u^{*, k}$ is an optimal control to Problem 1.k for some $k \geq 0$, then $\mathbb{E}[N_T^{u^{*, k}}] \leq C(1 + |x|^\rho)$.*

Proof. To simplify notation we let $(X, P, Q) := (X^{t, x; u^{*, k}}, P^{t, x; u^{*, k}}, Q^{t, x; u^{*, k}})$ and $X^j = X^{t, x; [u^{*, k}]_j}$ and get that

$$P_s = \psi(X_T) + \int_s^T f(r, X_r, P_r, Q_r) dr - \int_s^T Q_r dW_r - \sum_{\tau_j \geq s} \ell(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*).$$

Letting

$$\zeta_1(s) := \frac{f(s, X_s, P_s, Q_s) - f(s, X_s, 0, Q_s)}{P_s} \mathbb{1}_{[P_s \neq 0]}$$

and

$$\zeta_2(s) := \frac{f(s, X_s, 0, Q_s) - f(s, X_s, 0, 0)}{|Q_s|^2} (Q_s)^\top$$

we have by the Lipschitz continuity of f that $|\zeta_1(s)| \vee |\zeta_2(s)| \leq k_f$. Using Ito's formula we find that

$$P_s = R_{s, T} \psi(X_T) + \int_s^T R_{s, r} f(r, X_r, 0, 0) dr - \int_s^T R_{s, r} Q_r dW_r - \sum_{j=1}^{N_T^{u^{*, k}}} R_{s, \tau_j^*} \mathbb{1}_{[\tau_j^* \geq s]} \ell(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*)$$

with $R_{s, r} := e^{\int_s^r (\zeta_1(v) - \frac{1}{2} |\zeta_2(v)|^2) dv + \frac{1}{2} \int_s^r \zeta_2(v) dW_v}$. Since the intervention costs are positive, taking the conditional expectation on both sides and using Proposition 3.5 gives

$$P_s \leq \mathbb{E} \left[R_{s, T} \psi(X_T) + \int_s^T R_{s, r} f(r, X_r, 0, 0) dr \middle| \mathcal{F}_s^t \right]$$

$$\begin{aligned} &\leq C \left(1 + \mathbb{E}[R_{s,T}^2 | \mathcal{F}_s^t]^{1/2} \mathbb{E} \left[\sup_{r \in [s,T]} |X_r|^{2\rho} | \mathcal{F}_s^t \right]^{1/2} \right) \\ &\leq C(1 + |X_s|^\rho) \end{aligned}$$

On the other hand, since $u^{*,k}$ is an optimal control,

$$P_s \geq v_0(s, X_s) \geq -C(1 + |X_s|^\rho),$$

\mathbb{P} -a.s., for some $C > 0$ (independent of (t, s, x) and k). Proposition 3.5 then gives

$$\mathbb{E} \left[\sup_{s \in [t, T]} |P_s|^2 \right] \leq C(1 + |x|^{2\rho}),$$

where $C > 0$ does not depend on k . Next, we derive a bound on the \mathcal{H}_t^2 -norm of Q . Applying Ito's formula to $|P_s|^2$ we get

$$\begin{aligned} |P_t|^2 + \int_t^T |Q_s|^2 ds &= \psi^2(X_T) + 2 \int_t^T P_s f(s, X_s, P_s, Q_s) ds - 2 \int_t^T P_s Q_s dW_s \\ &\quad - \sum_{j=1}^{N_T^{u^{*,k}}} (2P_{\tau_j^*}^{j-1} \ell(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*) + \ell^2(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*)), \end{aligned} \quad (3.18)$$

where P^{j-1} is P without the $j-1$ first intervention costs. Since the intervention costs are nonnegative, we have

$$\begin{aligned} - \sum_{j=1}^{N_T^{u^{*,k}}} (2P_{\tau_j^*}^{j-1} \ell(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*) + \ell^2(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*)) &\leq 2 \sup_{s \in [t, T]} |P_s| \sum_{j=1}^{N_T^{u^{*,k}}} \ell(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*) \\ &\leq \kappa \sup_{s \in [t, T]} |P_s|^2 + \frac{1}{\kappa} \left(\sum_{j=1}^{N_T^{u^{*,k}}} \ell(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*) \right)^2 \end{aligned}$$

for any $\kappa > 0$. Inserted in (3.18) and using the Lipschitz property of f this gives

$$\begin{aligned} |P_t|^2 + \int_t^T |Q_s|^2 ds &\leq \psi^2(X_T) + (C + \kappa) \sup_{s \in [t, T]} |P_s|^2 + \int_t^T (|f(s, X_s, 0, 0)|^2 + \frac{1}{2}|Q_s|^2) ds \\ &\quad - 2 \int_t^T P_s Q_s dW_s + \frac{1}{\kappa} \left(\sum_{j=1}^{N_T^{u^{*,k}}} \ell(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*) \right)^2. \end{aligned} \quad (3.19)$$

Now, as $u^{*,k} \in \mathcal{U}^k$, it follows that the stochastic integral is uniformly integrable and thus a martingale. To see this, note that the Burkholder-Davis-Gundy inequality gives

$$\mathbb{E} \left[\sup_{s \in [t, T]} \left| \int_t^s P_r Q_r dW_r \right| \right] \leq C \mathbb{E} \left[\left(\int_t^T |P_s Q_s|^2 ds \right)^{1/2} \right] \leq C \mathbb{E} \left[\sup_{s \in [t, T]} |P_s|^2 + \int_t^T |Q_s|^2 ds \right]$$

where the right-hand side is finite by (3.17). Taking expectations on both sides of (3.19) thus gives

$$\mathbb{E}\left[\int_t^T |Q_s|^2 ds\right] \leq C(1 + \kappa)(1 + |x|^{2\rho}) + \frac{2}{\kappa} \mathbb{E}\left[\left(\sum_{j=1}^{N_T^{u^*,k}} \ell(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*)\right)^2\right].$$

Finally,

$$\mathbb{E}[N_T^{u^*,k}] \leq \frac{1}{\delta} \mathbb{E}\left[\left(\sum_{j=1}^{N_T^{u^*,k}} \ell(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*)\right)^2\right]^{1/2}$$

and

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{j=1}^{N_T^{u^*,k}} \ell(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*)\right)^2\right] &\leq C\mathbb{E}\left[|P_t|^2 + |\psi(X_T)|^2 + \int_t^T |f(r, X_r, P_r, Q_r)|^2 dr + \int_t^T |Q_r|^2 dr\right] \\ &\leq C\mathbb{E}\left[|\psi(X_T)|^2 + \sup_{s \in [t, T]} |P_s|^2 + \int_t^T (|f(s, X_s, 0, 0)|^2 + |Q_s|^2) ds\right] \\ &\leq C(1 + \kappa)(1 + |x|^{2\rho}) + \frac{C}{\kappa} \mathbb{E}\left[\left(\sum_{j=1}^{N_T^{u^*,k}} \ell(\tau_j^*, X_{\tau_j^*}^{j-1}, \beta_j^*)\right)^2\right] \end{aligned}$$

and the lemma follows by choosing κ sufficiently large. \square

Lemma 3.8. *There is a $C > 0$ such that*

$$|v_k(t, x) - v_{k-1}(t, x)| \leq \frac{C}{\sqrt{k}}(1 + |x|^{2\rho}) \quad (3.20)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and $k \geq 1$.

Proof. We let $u^{*,k} \in \mathcal{U}_t^k$ be an optimal control for Problem 1.k. To simplify notation we let $(P, Q) = (P^{t,x;u^{*,k}}, Q^{t,x;u^{*,k}})$ and set $(\hat{P}, \hat{Q}) = (P^{t,x;[u^{*,k}]_{k-1}}, Q^{t,x;[u^{*,k}]_{k-1}})$. Then, with $X := X^{t,x;u^{*,k}}$ and $\hat{X} := X^{t,x;[u^{*,k}]_{k-1}}$, we have $\hat{X}_s = X_s$ for all $s \in [0, \tau_k^*) \cap [0, T]$. This gives

$$\begin{aligned} P_t - \hat{P}_t &= \psi(X_T) - \psi(\hat{X}_T) + \int_t^T (f(s, X_s, P_s, Q_s) - f(s, \hat{X}_s, \hat{P}_s, \hat{Q}_s)) ds \\ &\quad - \int_t^T (Q_s - \hat{Q}_s) dW_s + \Xi_T^{t,x;[u^{*,k}]_{k-1}} - \Xi_T^{t,x;u^{*,k}} \\ &\leq \mathbb{1}_{[N_T^{u^*,k}=k]} \left(R_{t,T}(\psi(X_T) - \psi(\hat{X}_T)) + \int_t^T R_{t,s}(f(s, X_s, P_s, Q_s) - f(s, \hat{X}_s, P_s, Q_s)) ds \right) \\ &\quad - \int_t^T R_{t,s}(Q_s - \hat{Q}_s) dW_s \end{aligned}$$

for some $R_{t,s} := e^{\int_t^s (\zeta_1 - \frac{1}{2}|\zeta_2(r)|^2)dr + \frac{1}{2} \int_t^s \zeta_2(r)dW_r}$, with $|\zeta_1(r)| \vee |\zeta_2(r)| \leq k_f$. Taking expectation on both sides and using the Cauchy-Schwartz inequality gives

$$\begin{aligned} \mathbb{E}[P_t - \hat{P}_t] &\leq \mathbb{E}\left[\mathbb{1}_{[N_T^{u^*,k}=k]} \left(R_{t,T}(\psi(X_T) - \psi(\hat{X}_T)) + \int_t^T R_{t,s}(f(s, X_s, P_s, Q_s) - f(s, \hat{X}_s, P_s, Q_s))ds \right)\right] \\ &\leq C(1 + |x|^\rho)\mathbb{E}\left[\mathbb{1}_{[N_T^{u^*,k}=k]}\right]^{1/2}. \end{aligned}$$

Now, Lemma 3.7 implies that

$$\mathbb{E}\left[\mathbb{1}_{[N_T^{u^*,k}=k]}\right] \leq \frac{C(1 + |x|^\rho)}{k}$$

from which (3.20) follows. \square

3.3. Proof of Theorem 3.2

Proof of Theorem 3.2 Since the sequence $(v_k)_{k \geq 0}$ is non-decreasing and uniformly bounded by a polynomial, there is a $v \in \Pi^g$ such that $v_k \nearrow v$ as $k \rightarrow \infty$. Moreover, by Lemma 3.8 the convergence is uniform on compacts which implies that the limit is jointly continuous, thus $v \in \Pi_c^g$.

Now, by Theorem 2.6 there is, for each $(t, x) \in [0, T] \times \mathbb{R}^n$, a unique triple $(Y^{t,x}, Z^{t,x}, K^{t,x}) \in \mathcal{S}_t^2 \times \mathcal{H}_t^2 \times \mathcal{S}_{i,t}^2$ such that

$$\begin{cases} Y_s^{t,x} = \psi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r + K_T^{t,x} - K_s^{t,x}, & \forall s \in [t, T], \\ Y_s^{t,x} \geq \mathcal{M}v(X_s^{t,x}), & \forall s \in [t, T] \quad \text{and} \quad \int_t^T (Y_s^{t,x} - \mathcal{M}v(X_s^{t,x}))dK_s^{t,x} = 0. \end{cases} \quad (3.21)$$

On the other hand, by once more invoking Lemma 3.8 we get \mathbb{P} -a.s. convergence of the barrier in (3.8) and we easily deduce by (2.5) that $Y^{t,x} = \lim_{k \rightarrow \infty} Y^{t,x,k}$ pointwisely, and we conclude that $Y_t^{t,x} = v(t, x)$, \mathbb{P} -a.s. In particular, this implies that $(Y, Z, K) \in \mathfrak{S}^2 \times \mathfrak{H}^2 \times \mathfrak{S}_i^2$ solves (3.1).

To prove the representation we use ideas similar to those in the proof of Theorem 1 of [11]. We get by repeating the proof of Proposition 3.4 that, whenever $(Y, Z, K) \in \mathfrak{S}^2 \times \mathfrak{H}^2 \times \mathfrak{S}_i^2$ satisfies (3.1), there is a unique pair $(\mathcal{Y}, \mathcal{Z}) \in \mathcal{S}_{\text{cag},t}^2 \times \mathcal{H}_t^2$ such that $\mathcal{Y}_t = Y_t^{t,x}$, \mathbb{P} -a.s., and

$$\begin{aligned} \mathcal{Y}_s &= \int_s^{\tau_k^*} f(r, X_r^{t,x;u^*}, \mathcal{Y}_r, \mathcal{Z})dr - \int_s^{\tau_k^*} \mathcal{Z}_r dW_r - \sum_{j=1}^{k \wedge N^*} \mathbb{1}_{[s \leq \tau_j^*]} \ell(\tau_j^*, X_{\tau_j^*}^{t,x;[u^*]_{j-1}}, \beta_j^*) \\ &\quad + \mathbb{1}_{[\tau_k^*=T]} \psi(X_T^{t,x;u^*}) + \mathbb{1}_{[\tau_k^* < T]} v(\tau_k^*, X_{\tau_k^*}^{t,x;[u^*]_k}), \end{aligned}$$

where u^* is now the impulse control in the statement of the theorem. The sequence $v(\tau_k^*, X_{\tau_k^*}^{t,x;[u^*]_k})$ is uniformly bounded in $L^2(\mathbb{P})$ and we conclude that $u^* \in \mathcal{U}_t$. In particular, this implies that $N_T^{u^*}$ is \mathbb{P} -a.s. finite. Taking the limit as $k \rightarrow \infty$ and using that (3.4) admits a unique solution, we conclude that $Y_t^{t,x} = P_t^{t,x;u^*}$. Repeating the comparison part in the proof of Proposition 3.4 then gives that u^* is an optimal control for Problem 1. Since this holds for any $(Y, Z, K) \in \mathfrak{S}_c^2 \times \mathfrak{H}^2 \times \mathfrak{S}_i^2$ that solves (3.1), uniqueness of solutions to (3.1) follows.

Moreover, from Theorem 2.6 it immediately follows that v solves (3.6) and by the comparison result for viscosity solutions to (3.6) in Proposition A.4 the solution is unique. \square

Remark 3.9. Letting \mathcal{U}_s^t be the subset of \mathcal{U}_t with $\tau_1 \geq s$, \mathbb{P} -a.s., we may extend the representation to $Y_s^{t,x} = \text{ess sup}_{u \in \mathcal{U}_s^t} P_s^{t,x;u}$ for all $s \in [t, T]$.

4. THE GENERAL SETTING

We now turn to the general setting of a non-local driver. Existence will again follow by an approximation routine and we introduce the following sequence of systems of BSDEs

$$\begin{cases} Y_s^{t,x,k} = \psi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, \bar{Y}^{k-1}(r, \cdot), Z_r^{t,x,k})dr - \int_s^T Z_r^{t,x,k}dW_r + K_T^{t,x,k} - K_s^{t,x,k}, \quad \forall s \in [t, T] \\ Y_s^{t,x,k} \geq \mathcal{M}\bar{Y}^k(s, X_s^{t,x}), \quad \forall s \in [t, T] \quad \text{and} \quad \int_0^T (Y_s^{t,x,k} - \mathcal{M}\bar{Y}^k(s, X_s^{t,x}))dK_s^{t,x,k} = 0. \end{cases} \quad (4.1)$$

for $k \geq 1$, with $\bar{Y}^0 \equiv 0$.

Proposition 4.1. *There is a sequence $((Y^k, Z^k, K^k) \in \mathfrak{G}^2 \times \mathfrak{H}^2 \times \mathfrak{G}_i^2)_{k \geq 0}$ that satisfies the recursion in (4.1).*

Proof. We need to show that for each $k \geq 1$, there is a $v_{k-1} \in \Pi_c^g$ such that $Y_t^{t,x,k-1} = v_{k-1}(t, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$. However, for $k = 1$ this is immediate by the definition. Now, the result follows by using Theorem 3.2 and induction. \square

For $(t, \gamma) \in [0, T] \times \mathbb{R}_+$ and $\alpha \in \mathcal{A}_t$, we let $(\Psi^{t,\gamma;\alpha}, \Theta^{t,\gamma;\alpha}) \in \mathcal{S}_t^2 \times \mathcal{S}_{i,t}^2$ (with $\Theta_t^{t,\gamma;\alpha} = 0$) solve the one-dimensional reflected SDE

$$\begin{cases} \Psi_s^{t,\gamma;\alpha} = \gamma^2 \vee K_\Gamma^2 + (4C_{a,\sigma} + 2C_{a,\sigma}^2) \int_t^s (1 + \Psi_r^{t,\gamma;\alpha})dr + 4C_{a,\sigma} \int_t^s (1 + \Psi_r^{t,\gamma;\alpha})\alpha_r dW_r + \Theta_s^{t,\gamma;\alpha}, \quad \forall s \in [t, T] \\ \Psi_s^{t,\gamma;\alpha} \geq \gamma^2 \vee K_\Gamma^2 \quad \text{and} \quad \int_t^T (\Psi_s^{t,\gamma;\alpha} - (\gamma^2 \vee K_\Gamma^2))d\Theta_s^{t,\gamma;\alpha} = 0, \end{cases} \quad (4.2)$$

and set $\Psi_s^{t,\gamma;\alpha} := \gamma^2 \vee K_\Gamma^2$ for $s \in [0, t]$. For existence and uniqueness of solutions to reflected SDEs the interested reader is directed to [12]. For $(t, \gamma) \in [0, T] \times \mathbb{R}_+$, we then set $R_s^{t,\gamma;\alpha} := \sqrt{\Psi_s^{t,\gamma;\alpha}}$ and note that classically, we have

$$\mathbb{E} \left[\sup_{s \in [t, T]} |R_s^{t,\gamma;\alpha}|^p \right] \leq C(1 + |\gamma \vee K_\Gamma|^p),$$

for all $p \geq 2$.

Lemma 4.2. *For each $(t, x) \in [0, T] \times \mathbb{R}^d$ and $u \in \mathcal{U}_t$, there is an $\alpha \in \mathcal{A}_t$ such that $|X_s^{t,x,u}| \leq R_s^{t,|x|;\alpha}$ for all $s \in [t, T]$, \mathbb{P} -a.s.*

Proof. Since

$$|2xa(r, x) + \sigma^2(r, x)| \leq (4C_{a,\sigma} + 2C_{a,\sigma}^2)(1 + |x|^2),$$

it follows from (3.13) that we can always choose $\alpha \in \mathcal{A}_t$ such that

$$2C_{a,\sigma}(1 + \Psi_r^{t,\gamma,\xi;\alpha})\alpha_r = X_r^j \sigma(r, X_r^j)$$

and the statement holds by (3.13). \square

For $\varphi \in \Pi_c^g$ we let $(Y^\varphi, Z^\varphi, K^\varphi) \in \mathfrak{G}_c^2 \times \mathfrak{H}^2 \times \mathfrak{G}_i^2$ be the unique solution to

$$\begin{cases} Y_s^{t,x,\varphi} = \psi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, \varphi(r, \cdot), Z_r^{t,x,\varphi})dr - \int_s^T Z_r^{t,x,\varphi}dW_r + K_T^{t,x,\varphi} - K_s^{t,x,\varphi}, \quad \forall s \in [t, T], \\ Y_s^{t,x,\varphi} \geq \mathcal{M}\bar{Y}^\varphi(s, X_s^{t,x}), \quad \forall s \in [t, T] \quad \text{and} \quad \int_0^T (Y_s^{t,x,\varphi} - \mathcal{M}\bar{Y}^\varphi(s, X_s^{t,x}))dK_s^{t,x,\varphi} = 0, \end{cases} \quad (4.3)$$

where $\bar{Y}^\varphi(t, x) := Y_t^{t,x,\varphi}$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$, and note that letting $(P^{t,x,\varphi;u}, Q^{t,x,\varphi;u}) \in \mathcal{S}_{\text{cag},t}^2 \times \mathcal{H}_t^2$ solve

$$P_s^{t,x,\varphi;u} = \psi(X_T^{t,x;u}) + \int_s^T f(r, X_r^{t,x;u}, \varphi(r, \cdot), Q_r^{t,x,\varphi;u}) dr - \int_s^T Q_r^{t,x,\varphi;u} dW_r - \Xi_T^{t,x;u} + \Xi_s^{t,x;u}, \quad (4.4)$$

Theorem 3.2 gives that

$$Y_t^{t,x,\varphi} = \sup_{u \in \mathcal{U}_t} P_t^{t,x,\varphi;u}.$$

Proposition 4.3. *There is a $\kappa > 0$ such that for all $\varphi, \tilde{\varphi} \in \Pi_c^g$ and $\gamma > 0$, we have*

$$\sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T e^{\kappa t} \sup_{x \in \Lambda(R_t^0, \gamma; \alpha)} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^\varphi(t, x)|^2 dt \right] \leq \frac{1}{2} \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T e^{\kappa t} \sup_{x \in \Lambda(R_t^0, \gamma; \alpha)} |\tilde{\varphi}(t, x) - \varphi(t, x)|^2 dt \right]. \quad (4.5)$$

Furthermore, there is a $C > 0$ such that

$$\sup_{t \in [0, T]} \sup_{x \in \Lambda(\gamma)} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^\varphi(t, x)|^2 \leq C \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T \sup_{x \in \Lambda(R_t^0, \gamma; \alpha)} |\tilde{\varphi}(t, x) - \varphi(t, x)|^2 dt \right] \quad (4.6)$$

for each $\gamma > 0$.

Proof. Let $u^*, \tilde{u}^* \in \mathcal{U}_t$ be optimal strategies for $P^{t,x,\varphi;u}$ and $P^{t,x,\tilde{\varphi};u}$, respectively, so that

$$Y_t^{t,x,\varphi} = P_t^{t,x,\varphi;u^*} \quad \text{and} \quad Y_t^{t,x,\tilde{\varphi}} = P_t^{t,x,\tilde{\varphi};\tilde{u}^*}.$$

Then

$$\begin{aligned} Y_t^{t,x,\varphi} - Y_t^{t,x,\tilde{\varphi}} &= P_t^{t,x,\varphi;u^*} - P_t^{t,x,\tilde{\varphi};\tilde{u}^*} \\ &\leq P_t^{t,x,\varphi;u^*} - P_t^{t,x,\tilde{\varphi};u^*}. \end{aligned}$$

Since a similar inequality holds in the opposite direction, we find that

$$|Y_t^{t,x,\varphi} - Y_t^{t,x,\tilde{\varphi}}| \leq \sup_{u \in \mathcal{U}_t} |P_t^{t,x,\varphi;u} - P_t^{t,x,\tilde{\varphi};u}|. \quad (4.7)$$

For $u \in \mathcal{U}_t$, let $(P, Q) := (P^{t,x,\varphi;u}, Q^{t,x,\varphi;u})$ and $(\tilde{P}, \tilde{Q}) := (P^{t,x,\tilde{\varphi};u}, Q^{t,x,\tilde{\varphi};u})$ and note that for $\kappa > 0$, Itô's formula applied to $e^{\kappa \cdot} |\tilde{P} - P|^2$ gives

$$\begin{aligned} e^{\kappa t} |\tilde{P}_t - P_t|^2 + \int_t^T e^{\kappa s} |\tilde{Q}_s - Q_s|^2 ds &= -2 \int_t^T e^{\kappa s} (\tilde{P}_s - P_s)(\tilde{Q}_s - Q_s) dW_s - \kappa \int_t^T e^{\kappa s} |\tilde{P}_s - P_s|^2 ds \\ &\quad + 2 \int_t^T e^{\kappa s} (\tilde{P}_s - P_s) (f(s, X_s^{t,x;u}, \tilde{\varphi}(s, \cdot), \tilde{Q}_s) - f(s, X_s^{t,x;u}, \varphi(s, \cdot), Q_s)) ds. \end{aligned}$$

By assumption

$$|f(s, X_s^{t,x;u}, \varphi(s, \cdot), Q_s) - f(s, X_s^{t,x;u}, \tilde{\varphi}(s, \cdot), \tilde{Q}_s)| \leq k_f \left(\sup_{x' \in \Lambda(|X_s^{t,x;u}|)} |\tilde{\varphi}(s, x') - \varphi(s, x')| + |\tilde{Q}_s - Q_s| \right).$$

Hence, taking the expectation and using inequalities $2Cxy \leq (Cx)^2 + y^2$ and $2xy \leq x^2/\sqrt{\kappa} + \sqrt{\kappa}y^2$ gives

$$\begin{aligned} e^{\kappa t} |\tilde{P}_t - P_t|^2 &\leq (C^2 + C\sqrt{\kappa} - \kappa) \mathbb{E} \left[\int_t^T e^{\kappa s} |\tilde{P}_s - P_s|^2 ds \right] \\ &\quad + \frac{C}{\sqrt{\kappa}} \mathbb{E} \left[\int_t^T e^{\kappa s} \sup_{x' \in \Lambda(|X_s^t, x; u|)} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right]. \end{aligned}$$

Now, pick $\kappa_0 > 0$ such that $\kappa_0 \geq C^2 + C\sqrt{\kappa_0}$ and note that for each $\kappa \geq \kappa_0$, we have

$$\begin{aligned} e^{\kappa t} |P_t - \tilde{P}_t|^2 &\leq \frac{C}{\sqrt{\kappa}} \mathbb{E} \left[\int_t^T e^{\kappa s} \sup_{x' \in \Lambda(|X_s^t, x; u|)} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right] \\ &\leq \frac{C}{\sqrt{\kappa}} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E} \left[\int_t^T e^{\kappa s} \sup_{x' \in \Lambda(|R_s^t, |x|; \alpha|)} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right], \end{aligned}$$

where the last inequality follows from Lemma 4.2. Since the right-hand side is non-decreasing in $|x|$ and independent of u , (4.7) now gives that

$$e^{\kappa t} \sup_{x \in \Lambda(\gamma)} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x)|^2 \leq \frac{C}{\sqrt{\kappa}} \sup_{\alpha \in \mathcal{A}_t} \mathbb{E} \left[\int_t^T e^{\kappa s} \sup_{x' \in \Lambda(R_s^t, \gamma; \alpha)} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right], \quad (4.8)$$

for any $\gamma \geq 0$. In particular, as both sides are continuous in γ a standard dynamic programming argument gives that for any $\alpha_1 \in \mathcal{A}$, we have

$$\mathbb{E} \left[e^{\kappa t} \sup_{x \in \Lambda(R_t^{0, \gamma; \alpha_1})} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x)|^2 \right] \leq \frac{C}{\sqrt{\kappa}} \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_t^T e^{\kappa s} \sup_{x' \in \Lambda(R_s^{0, \gamma; \alpha_1 \oplus t \alpha})} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right].$$

Taking the supremum with respect to α_1 on the right hand side and once again relying on a standard dynamic programming argument gives that

$$\mathbb{E} \left[e^{\kappa t} \sup_{x \in \Lambda(R_t^{0, \gamma; \alpha_1})} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x)|^2 \right] \leq \frac{C}{\sqrt{\kappa}} \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T e^{\kappa s} \sup_{x' \in \Lambda(R_s^{0, \gamma; \alpha})} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right].$$

Integrating with respect to time and using Fubini's theorem, we find that

$$\mathbb{E} \left[\int_0^T e^{\kappa t} \sup_{x \in \Lambda(R_t^{0, \gamma; \alpha_1})} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x)|^2 dt \right] \leq \frac{CT}{\sqrt{\kappa}} \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T e^{\kappa s} \sup_{x' \in \Lambda(R_s^{0, \gamma; \alpha})} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right]$$

after which taking the supremum with respect to $\alpha_1 \in \mathcal{A}$ and choosing $\kappa \geq 2(CT)^2 \vee \kappa_0$ gives the first inequality. To get (4.6) we note that comparison gives that $R_s^{0, \gamma; \alpha} \geq R_s^{t, \gamma; \alpha}$ for all $s \in [t, T]$ and $\alpha \in \mathcal{A}$. From (4.8) we thus get that

$$\sup_{x \in \Lambda(\gamma)} |\bar{Y}^{\tilde{\varphi}}(t, x) - \bar{Y}^{\varphi}(t, x)|^2 \leq C \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T \sup_{x' \in \Lambda(R_s^{0, \gamma; \alpha})} |\tilde{\varphi}(s, x') - \varphi(s, x')|^2 ds \right]$$

from which (4.6) is immediate since the right hand side is independent of t . \square

We now introduce the norm $\|\cdot\|_\gamma$ on the space of jointly continuous functions of polynomial growth, Π_c^g , defined as

$$\|\varphi\|_\gamma^2 := \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T e^{\kappa t} \sup_{x \in \Lambda(R_t^{0,\gamma;\alpha})} |\varphi(t, x)|^2 dt \right],$$

with $\kappa > 0$ as in Proposition 4.3 and note that under $\|\cdot\|_\gamma$, the map $\Phi : \Pi_c^g \rightarrow \Pi_c^g$ that maps φ to \bar{Y}^φ is a contraction.

Corollary 4.4. *There are constants $C > 0$ and $p \geq 0$ such that $|\bar{Y}^k(t, x)| \leq C(1 + |x|^p)$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and all $k \geq 0$.*

Proof. First, we note that (4.5) and the triangle inequality implies that

$$\|\bar{Y}^k\|_\gamma \leq \|\bar{Y}^k - \bar{Y}^{k-1}\|_\gamma + \|\bar{Y}^{k-1}\|_\gamma \leq \frac{1}{2} \|\bar{Y}^{k-1} - \bar{Y}^{k-2}\|_\gamma + \|\bar{Y}^{k-1}\|_\gamma \leq \frac{1}{2^{k-1}} \|\bar{Y}^1 - \bar{Y}^0\|_\gamma + \|\bar{Y}^{k-1}\|_\gamma.$$

However, as a similar scheme holds for $\|\bar{Y}^{k-1}\|_\gamma$ and since $\bar{Y}^0 \equiv 0$ we conclude that

$$\|\bar{Y}^k\|_\gamma \leq \sum_{j=1}^k \frac{1}{2^{j-1}} \|\bar{Y}^1\|_\gamma \leq 2 \|\bar{Y}^1\|_\gamma.$$

On the other hand, as $\bar{Y}^1 \in \Pi_c^g$ there are constants $C > 0$ and $p \geq 2$ such that $|\bar{Y}^1(t, x)| \leq C(1 + |x|^p)$ and we conclude that

$$\begin{aligned} \|\bar{Y}^1\|_\gamma^2 &= \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T e^{\kappa t} \sup_{x \in \Lambda(R_t^{0,\gamma;\alpha})} |\bar{Y}^1(t, x)|^2 dt \right] \\ &\leq C \left(1 + \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\sup_{t \in [0, T]} |R_t^{0,\gamma;\alpha}|^{2p} \right] \right) \\ &\leq C(1 + |\gamma|^{2p}) \end{aligned}$$

implying the existence of a $C > 0$ such that $\|\bar{Y}^k\|_\gamma \leq C(1 + |\gamma|^p)$ for all $k \geq 0$. Now, (4.6) gives that

$$\begin{aligned} \sup_{t \in [0, T]} \sup_{x \in \Lambda(\gamma)} |\bar{Y}^k(t, x) - \bar{Y}^1(t, x)|^2 &\leq C \sup_{\alpha \in \mathcal{A}} \mathbb{E} \left[\int_0^T \sup_{x \in \Lambda(R_t^{0,\gamma;\alpha})} |\bar{Y}^{k-1}(t, x)|^2 dt \right] \\ &\leq C(1 + |\gamma|^{2p}) \end{aligned}$$

where the constants $C > 0$ and $p \geq 2$ do not depend on k and the desired bound follows. \square

Letting $v_k(t, x) := \bar{Y}^k(t, x)$, Proposition 4.3 and Corollary 4.4 implies that there is a $v \in \Pi^g$ such that for each $\gamma > 0$ we have $\|v_k - v\|_\gamma \rightarrow 0$ as $k \rightarrow \infty$.

Theorem 4.5. *v is the unique viscosity solution in Π_c^g to (1.1).*

Proof. First, (4.6) implies that the convergence is uniform on compact subsets of $[0, T] \times \mathbb{R}^n$ and since v_k is jointly continuous for each $k \geq 0$ we conclude that v is also jointly continuous. This in turn gives that $\Phi(v)$ is well defined and we conclude that $\Phi(v) = v$ establishing existence of a solution to (1.3). Moreover, if $(\tilde{Y}, \tilde{Z}, \tilde{K})$ is another solution, then $\tilde{v}(t, x) := \tilde{Y}_t^{t,x}$ must also satisfy $\Phi(\tilde{v}) = \tilde{v}$. However, then repeated use of the contraction

property in (4.5) gives that $\|\tilde{v} - v\|_{\gamma, \kappa} = 0$ and by continuity we conclude that $\tilde{v} = v$ implying by uniqueness of solutions to (3.1) as obtained in Theorem 3.2 that (1.3) admits a unique solution.

Utilizing, once more, the connection between reflected BSDEs and obstacle problems we conclude that v solves (1.1). Suppose now that there exists another function $\tilde{v} \in \Pi_c^g$ that solves (1.1) and let $\bar{v} = \Phi(\tilde{v})$, then by Theorem 3.2 we conclude that $\bar{v} \in \Pi_c^g$ is the unique solution to

$$\begin{cases} \min\{\bar{v}(t, x) - \mathcal{M}\bar{v}(t, x), -\bar{v}_t(t, x) - \mathcal{L}\bar{v}(t, x) - f(t, x, \tilde{v}(t, \cdot), \sigma^\top(t, x)\nabla_x \bar{v}(t, x))\} = 0, \\ \quad \forall (t, x) \in [0, T) \times \mathbb{R}^d \\ \bar{v}(T, x) = \psi(x), \end{cases}$$

But then $\bar{v} = \tilde{v}$ implying that \tilde{v} is a fixed point of Φ and since v is the only fixed point of Φ in the set of jointly continuous functions of polynomial growth we conclude that $\tilde{v} = v$. \square

Corollary 4.6. *(Y^v, Z^v, K^v) is the unique solution in $\mathfrak{S}^2 \times \mathfrak{H}^2 \times \mathfrak{S}_i^2$ to (1.3).*

APPENDIX A. UNIQUENESS OF VISCOSITY SOLUTIONS IN THE LOCAL FRAMEWORK

By Theorem 3.2, there is $v \in \Pi_c^g$ that solves (in viscosity sense) the quasi-variational inequality

$$\begin{cases} \min\{v(t, x) - \mathcal{M}v(t, x), -v_t(t, x) - \mathcal{L}v(t, x) - f(t, x, v(t, x), \sigma^\top(t, x)\nabla_x v(t, x))\} = 0, \\ \quad \forall (t, x) \in [0, T) \times \mathbb{R}^d \\ v(T, x) = \psi(x). \end{cases} \quad (\text{A.1})$$

In this section we show that v is the only viscosity solution in Π^g to (A.1). We need the following lemma:

Lemma A.1. *Let v be a supersolution to (1.1) satisfying*

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad |v(t, x)| \leq C(1 + |x|^{2\varrho})$$

for some $\varrho > 0$. Then there is a $\gamma_0 > 0$ such that for any $\gamma > \gamma_0$ and $\theta > 0$, the function $v + \theta e^{-\gamma t}(1 + (||x| - K_\Gamma)^+)^{2\varrho+2}$ is also a supersolution to (1.1).

Proof. With $w := v + \theta e^{-\gamma t}(1 + (||x| - K_\Gamma)^+)^{2\varrho+2}$ we note that, since v is a supersolution and $\theta e^{-\gamma T}(1 + (||x| - K_\Gamma)^+)^{2\varrho+2} \geq 0$, we have $w(T, x) \geq v(T, x) \geq \psi(x)$ so that the terminal condition holds. Moreover, we have

$$\begin{aligned} & w(t, x) - \sup_{b \in U} \{w(t, \Gamma(t, x, b)) - \ell(t, x, b)\} \\ &= v(t, x) + \theta e^{-\gamma t}(1 + (||x| - K_\Gamma)^+)^{2\varrho+2} \\ &\quad - \sup_{b \in U} \{v(t, \Gamma(t, x, b)) + \theta e^{-\gamma t}(1 + (|\Gamma(t, x, b)| - K_\Gamma)^+)^{2\varrho+2} - \ell(t, x, b)\} \\ &\geq v(t, x) - \sup_{b \in U} \{v(t, \Gamma(t, x, b)) - \ell(t, x, b)\} \\ &\quad + \theta e^{-\gamma t} \{1 + (||x| - K_\Gamma)^+)^{2\varrho+2} - \sup_{b \in U} (1 + (|\Gamma(t, x, b)| - K_\Gamma)^+)^{2\varrho+2}\}. \end{aligned}$$

Since v is a supersolution, we have

$$v(t, x) - \sup_{b \in U} \{v(t, \Gamma(t, x, b)) - \ell(t, x, b)\} \geq 0$$

Now, either $|x| \leq K_\Gamma$ in which case it follows by (2.1) that $|\Gamma(t, x, b)| \leq K_\Gamma$ or $|x| > K_\Gamma$ and (2.1) gives that $|\Gamma(t, x, b)| \leq |x|$. We conclude that

$$w(t, x) - \sup_{b \in U} \{w(t, \Gamma(t, x, b)) - \ell(t, x, b)\} \geq 0.$$

Next, let $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d \rightarrow \mathbb{R})$ be such that $\varphi - w$ has a local maximum of 0 at (t_0, x_0) with $t_0 < T$. Then $\tilde{\varphi}(t, x) := \varphi(t, x) - \theta e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2} \in C^{1,2}([0, T] \times \mathbb{R}^d \rightarrow \mathbb{R})$ and $\tilde{\varphi} - v$ has a local maximum of 0 at (t_0, x_0) . Since v is a viscosity supersolution, we have

$$\begin{aligned} & -\partial_t(\varphi(t, x) - \theta e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2}) - \mathcal{L}(\varphi(t, x) - \theta e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2}) \\ & - f(t, x, \varphi(t, x) - \theta e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2}), \sigma^\top(t, x) \nabla_x(\varphi(t, x) - \theta e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2})) \geq 0. \end{aligned}$$

Consequently,

$$\begin{aligned} & -\partial_t \varphi(t, x) - \mathcal{L} \varphi(t, x) - f(t, x, \varphi(t, x), \sigma^\top(t, x) \nabla_x \varphi(t, x)) \\ & \geq \theta \gamma e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2} - \theta \mathcal{L} e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2} \\ & \quad f(t, x, \varphi(t, x) - \theta e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2}), \sigma^\top(t, x) \nabla_x(\varphi(t, x) - \theta e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2})) \\ & \quad - f(t, x, \varphi(t, x), \sigma^\top(t, x) \nabla_x \varphi(t, x)) \\ & \geq \theta \gamma e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2} - \theta C(1 + \varrho) e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2} \\ & \quad - k_f(1 + \varrho) \theta e^{-\gamma t}(1 + (|x| - K_\Gamma)^+)^{2\varrho+2}, \end{aligned}$$

where the right hand side is non-negative for all $\theta > 0$ and all $\gamma > \gamma_0$ for some $\gamma_0 > 0$. \square

We have the following result, the proof of which we omit since it is classical:

Lemma A.2. *For any $\lambda \in \mathbb{R}$, a locally bounded function $v : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a viscosity supersolution (resp. subsolution) to (1.1) if and only if $\tilde{v}(t, x) := e^{\lambda t} v(t, x)$ is a viscosity supersolution (resp. subsolution) to*

$$\begin{cases} \min\{\tilde{v}(t, x) - \sup_{b \in U} \{\tilde{v}(t, \Gamma(t, x, b)) - e^{\lambda t} \ell(t, x, b)\}, -\tilde{v}_t(t, x) + \lambda \tilde{v}(t, x) - \mathcal{L} \tilde{v}(t, x) \\ - e^{\lambda t} f(t, x, e^{-\lambda t} \tilde{v}(t, x), e^{-\lambda t} \sigma^\top(t, x) \nabla_x \tilde{v}(t, x))\} = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d \\ \tilde{v}(T, x) = e^{\lambda T} \psi(x). \end{cases} \quad (\text{A.2})$$

Remark A.3. Here, it is important to note that $\tilde{\ell}(t, x) := e^{\lambda t} \ell(t, x)$, $\tilde{f}(t, x, y, z) := -\lambda y + e^{\lambda t} f(t, x, e^{-\lambda t} y, e^{-\lambda t} z)$ and $\tilde{\psi}(x) := e^{\lambda T} \psi(x)$ satisfy Assumption 2.1. In particular, this implies that Lemma A.1 holds for supersolutions to (A.2) as well.

We have the following comparison result for viscosity solutions in Π^g :

Proposition A.4. *Let v (resp. u) be a supersolution (resp. subsolution) to (1.1). If $u, v \in \Pi^g$, then $u \leq v$.*

Proof. First, we note that it is sufficient to show that the statement holds for solutions to (A.2) for some $\lambda \in \mathbb{R}$. We thus assume that v (resp. u) is a viscosity supersolution (resp. subsolution) to (A.2) for $\lambda \in \mathbb{R}$ specified below. Furthermore, we may without loss of generality assume that v is l.s.c. and u is u.s.c.

By assumption, $u, v \in \Pi^g$, which implies that there are $C > 0$ and $\varrho > 0$ such that

$$|v(t, x)| + |u(t, x)| \leq C(1 + |x|^{2\varrho}). \quad (\text{A.3})$$

Now, for any $\gamma > 0$ we only need to show that

$$\begin{aligned} w(t, x) &= w^{\theta, \gamma}(t, x) := v(t, x) + \theta e^{-\gamma t} (1 + (|x| - K_\Gamma)^+)^{2e+2} \\ &\geq u(t, x) \end{aligned}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and any $\theta > 0$. Then the result follows by taking the limit $\theta \rightarrow 0$. We know from Lemma A.1 that there is a $\gamma_0 > 0$ such that w is a supersolution to (A.2) for each $\gamma \geq \gamma_0$ and $\theta > 0$. We thus assume that $\gamma \geq \gamma_0$.

We search for a contradiction and assume that there is a $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ such that $u(t_0, x_0) > w(t_0, x_0)$. By (A.3), there is for each $\theta > 0$ a $R \geq K_\Gamma$ such that

$$w(t, x) > u(t, x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d, |x| > R.$$

Our assumption thus implies that there is a point $(\bar{t}, \bar{x}) \in [0, T] \times B_R$ (the open unit ball of radius R centered at 0) such that

$$\begin{aligned} \max_{(t, x) \in [0, T] \times \mathbb{R}^d} (u(t, x) - w(t, x)) &= \max_{(t, x) \in [0, T] \times B_R} (u(t, x) - w(t, x)) \\ &= u(\bar{t}, \bar{x}) - w(\bar{t}, \bar{x}) = \eta > 0. \end{aligned}$$

We first show that there is at least one point $(t^*, x^*) \in [0, T] \times B_R$ such that

- a) $u(t^*, x^*) - w(t^*, x^*) = \eta$ and
- b) $u(t^*, x^*) > \sup_{b \in U} \{u(t^*, \Gamma(t^*, x^*, b)) - \ell(t^*, b)\}$.

We again argue by contradiction and assume that $u(t, x) = \sup_{b \in U} \{u(t, \Gamma(t, x, b)) - \ell(t, b)\}$ for all $(t, x) \in A := \{(s, y) \in [0, T] \times \mathbb{R}^d : u(s, y) - w(s, y) = \eta\}$. Indeed, as u is u.s.c. and Γ is continuous, there is a b_1 such that

$$u(\bar{t}, \bar{x}) = \sup_{b \in U} \{u(\bar{t}, \Gamma(\bar{t}, \bar{x}, b)) - \ell(\bar{t}, b)\} = u(\bar{t}, \Gamma(\bar{t}, \bar{x}, b_1)) - \ell(\bar{t}, b_1). \quad (\text{A.4})$$

Now, set $x_1 = \Gamma(\bar{t}, \bar{x}, b_1)$ and note that since

$$|\Gamma(t, x, b)| \leq R, \quad \forall (t, x, b) \in [0, T] \times \bar{B}_R \times U$$

it follows that $x_1 \in \bar{B}_R$. Moreover, as w is a supersolution it satisfies

$$w(\bar{t}, \bar{x}) - (w(\bar{t}, \Gamma(\bar{t}, \bar{x}, b_1)) - \ell(\bar{t}, \bar{x}, b_1)) \geq 0$$

or

$$-w(\bar{t}, x_1) \geq -w(\bar{t}, \bar{x}) - \ell(\bar{t}, \bar{x}, b_1)$$

and we conclude from (A.4) that

$$\begin{aligned} u(\bar{t}, x_1) - w(\bar{t}, x_1) &\geq u(\bar{t}, \bar{x}) + \ell(\bar{t}, \bar{x}, b_1) - (w(\bar{t}, \bar{x}) + \ell(\bar{t}, \bar{x}, b_1)) \\ &= u(\bar{t}, \bar{x}) - w(\bar{t}, \bar{x}) = \eta. \end{aligned}$$

Hence, $(\bar{t}, x_1) \in A$ and by our assumption it follows that there is a $b_2 \in U$ such that

$$u(\bar{t}, x_1) = u(\bar{t}, \Gamma(\bar{t}, x_1, b_2)) - \ell(\bar{t}, b_2)$$

and a corresponding $x_2 := \Gamma(\bar{t}, x_1, b_2) \in B_R$. Now, this process can be repeated indefinitely to find a sequence $(x_j, b_j)_{j \geq 1}$ in $B_R \times U$ such that for any $l \geq 0$ we have

$$u(\bar{t}, \bar{x}) = u(\bar{t}, x_l) - \sum_{j=1}^l \ell(\bar{t}, x_{j-1}, b_j),$$

with $x_0 := \bar{x}$. However, as $\ell \geq \delta > 0$ we get a contradiction by letting $l \rightarrow \infty$ while noting that $|u(t, x)|$ is bounded on $[0, T] \times \bar{B}_R$. We can thus find a $(t^*, x^*) \in [0, T] \times B_R$ such that a) and b) above holds.

Since f is Lipschitz in y and z for $(t, x) \in [0, T] \times \bar{B}_R$, the remainder of the proof follows along the lines of the proof of Proposition 4.1 in [17] and is included only for the sake of completeness.

Next, we assume without loss of generality that $\varrho \geq 2$ and define

$$\Phi_n(t, x, y) := u(t, x) - w(t, x) - \varphi_n(t, x, y),$$

where

$$\varphi_n(t, x, y) := \frac{n}{2}|x - y|^{2\varrho} + |x - x^*|^2 + |y - \bar{y}|^2 + (t - t^*)^2.$$

Since u is u.s.c. and w is l.s.c. there is a triple $(t_n, x_n, y_n) \in [0, T] \times \bar{B}_R \times \bar{B}_R$ (with \bar{B}_R the closure of B_R) such that

$$\Phi_n(t_n, x_n, y_n) = \max_{(t, x, y) \in [0, T] \times \bar{B}_R \times \bar{B}_R} \Phi_n(t, x, y).$$

Now, the inequality $2\Phi_n(t_n, x_n, y_n) \geq \Phi_n(t_n, x_n, x_n) + \Phi_n(t_n, y_n, y_n)$ gives

$$n|x_n - y_n|^{2\varrho} \leq u(t_n, x_n) - u(t_n, y_n) + w(t_n, x_n) - w(t_n, y_n).$$

Consequently, $n|x_n - y_n|^{2\varrho}$ is bounded (since u and w are bounded on $[0, T] \times \bar{B}_R \times \bar{B}_R$) and $|x_n - y_n| \rightarrow 0$ as $n \rightarrow \infty$. We can, thus, extract subsequences n_l such that $(t_{n_l}, x_{n_l}, y_{n_l}) \rightarrow (\tilde{t}, \tilde{x}, \tilde{x})$ as $l \rightarrow \infty$. Since

$$u(t^*, x^*) - w(t^*, x^*) \leq \Phi_n(t_n, x_n, y_n) \leq u(t_n, x_n) - w(t_n, y_n),$$

it follows that

$$\begin{aligned} u(t^*, x^*) - w(t^*, x^*) &\leq \limsup_{l \rightarrow \infty} \{u(t_{n_l}, x_{n_l}) - w(t_{n_l}, y_{n_l})\} \\ &\leq u(\tilde{t}, \tilde{x}) - w(\tilde{t}, \tilde{x}) \end{aligned}$$

and as the righthand side is dominated by $u(t^*, x^*) - w(t^*, x^*)$ we conclude that

$$u(\tilde{t}, \tilde{x}) - w(\tilde{t}, \tilde{x}) = u(t^*, x^*) - w(t^*, x^*).$$

In particular, this gives that $\lim_{l \rightarrow \infty} \Phi_n(t_{n_l}, x_{n_l}, y_{n_l}) = u(\tilde{t}, \tilde{x}) - w(\tilde{t}, \tilde{x})$ which implies that

$$\limsup_{l \rightarrow \infty} n_l |x_{n_l} - y_{n_l}|^{2\varrho} = 0$$

and

$$(t_{n_l}, x_{n_l}, y_{n_l}) \rightarrow (t^*, x^*, x^*).$$

We can thus extract a subsequence $(\tilde{n}_l)_{l \geq 0}$ of $(n_l)_{l \geq 0}$ such that $t_{\tilde{n}_l} < T$, $|x_{\tilde{n}_l}| < R$ and

$$u(t_{\tilde{n}_l}, x_{\tilde{n}_l}) - w(t_{\tilde{n}_l}, x_{\tilde{n}_l}) \geq \frac{\eta}{2}.$$

Moreover, since $\sup_{b \in U} \{u(t, \Gamma(t, x, b)) - \tilde{\ell}(t, b)\}$ is u.s.c. (see Lem. 3.3) and $u(t_{\tilde{n}_l}, x_{\tilde{n}_l}) \rightarrow u(t^*, x^*)$ there is an $l_0 \geq 0$ such that

$$u(t_{\tilde{n}_l}, x_{\tilde{n}_l}) - \sup_{b \in U} \{u(t_{\tilde{n}_l}, \Gamma(t_{\tilde{n}_l}, x_{\tilde{n}_l}, b)) - \tilde{\ell}(t_{\tilde{n}_l}, b)\} > 0,$$

for all $l \geq l_0$. To simplify notation we will, from now on, denote $(\tilde{n}_l)_{l \geq l_0}$ simply by n .

By Theorem 8.3 of [8] there are $(p_n^u, q_n^u, M_n^u) \in \bar{J}^{2,+}u(t_n, x_n)$ and $(p_n^w, q_n^w, M_n^w) \in \bar{J}^{2,+}w(t_n, y_n)$, where $\bar{J}^{2,+}$ is the limiting superjet, such that

$$\begin{cases} p_n^u - p_n^w = \partial_t \varphi_n(t_n, x_n, y_n) = 2(t_n - t^*) \\ q_n^u = D_x \varphi_n(t_n, x_n, y_n) = n \varrho(x - y) |x - y|^{2\varrho - 2} + 2(x - x^*) \\ q_n^w = -D_y \varphi_n(t_n, x_n, y_n) = n \varrho(x - y) |x - y|^{2\varrho - 2} + 2(x - x^*) \end{cases}$$

and for every $\epsilon > 0$,

$$\begin{bmatrix} M_x^n & 0 \\ 0 & -M_y^n \end{bmatrix} \leq B(t_n, x_n, y_n) + \epsilon B^2(t_n, x_n, y_n),$$

where $B(t_n, x_n, y_n) := D_{(x,y)}^2 \varphi_n(t_n, x_n, y_n)$. Now, we have

$$D_{(x,y)}^2 \varphi_n(t, x, y) = \begin{bmatrix} D_x^2 \varphi_n(t, x, y) & D_{yx}^2 \varphi_n(t, x, y) \\ D_{xy}^2 \varphi_n(t, x, y) & D_y^2 \varphi_n(t, x, y) \end{bmatrix} = \begin{bmatrix} n\xi(x, y) + 2I & -n\xi(x, y) \\ -n\xi(x, y) & n\xi(x, y) + 2I \end{bmatrix}$$

where I is the identity-matrix of suitable dimension and

$$\xi(x, y) := \varrho |x - y|^{2\varrho - 4} \{ |x - y|^2 I + 2(\varrho - 1)(x - y)(x - y)^\top \}.$$

In particular, since x_n and y_n are bounded, choosing $\epsilon := \frac{1}{n}$ gives that

$$\tilde{B}_n := B(t_n, x_n, y_n) + \epsilon B^2(t_n, x_n, y_n) \leq Cn |x_n - y_n|^{2\varrho - 2} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix} + CI. \quad (\text{A.5})$$

By the definition of viscosity supersolutions and subsolutions we have that

$$\begin{aligned} & -p_n^u + \lambda u(t_n, x_n) - a^\top(t_n, x_n) q_n^u - \frac{1}{2} \text{Tr}[\sigma^\top(t_n, x_n) M_n^u \sigma(t_n, x_n)] \\ & - e^{\lambda t_n} f(t_n, x_n, e^{-\lambda t_n} u(t_n, x_n), e^{-\lambda t_n} \sigma^\top(t_n, x_n) q_n^u) \leq 0 \end{aligned}$$

and

$$\begin{aligned} & -p_n^w + \lambda w(t_n, y_n) - a^\top(t_n, y_n)q_n^w - \frac{1}{2}\text{Tr}[\sigma^\top(t_n, y_n)M_n^w\sigma(t_n, y_n)] \\ & - e^{\lambda t_n} f(t_n, y_n, e^{-\lambda t_n} w(t_n, y_n), e^{-\lambda t_n} \sigma^\top(t_n, x_n)q_n^w) \geq 0. \end{aligned}$$

Combined, this gives that

$$\begin{aligned} \lambda(u(t_n, x_n) - w(t_n, y_n)) & \leq p_n^u + a^\top(t_n, x_n)q_n^u + \frac{1}{2}\text{Tr}[\sigma^\top(t_n, x_n)M_n^u\sigma(t_n, x_n)] \\ & \quad + e^{\lambda t_n} f(t_n, x_n, e^{-\lambda t_n} u(t_n, x_n), e^{-\lambda t_n} \sigma^\top(t_n, x_n)q_n^u) \\ & \quad - p_n^w - a^\top(t_n, y_n)q_n^w - \frac{1}{2}\text{Tr}[\sigma^\top(t_n, y_n)M_n^w\sigma(t_n, y_n)] \\ & \quad - e^{\lambda t_n} f(t_n, y_n, e^{-\lambda t_n} w(t_n, y_n), e^{-\lambda t_n} \sigma^\top(t_n, x_n)q_n^w) \end{aligned}$$

Collecting terms we have that

$$p_n^u - p_n^w = 2(t_n - t^*)$$

and since a is Lipschitz continuous in x and bounded on \bar{B}_R , we have

$$\begin{aligned} a^\top(t_n, x_n)q_n^u - a^\top(t_n, y_n)q_n^w & \leq (a^\top(t_n, x_n) - a^\top(t_n, y_n))n\varrho(x_n - y_n)|x_n - y_n|^{2\varrho-2} \\ & \quad + C(|x_n - x^*| + |y_n - x^*|) \\ & \leq C(n|x_n - y_n|^{2\varrho} + |x_n - x^*| + |y_n - x^*|), \end{aligned}$$

where the right-hand side tends to 0 as $n \rightarrow \infty$. Let s_x denote the i^{th} column of $\sigma(t_n, x_n)$ and let s_y denote the i^{th} column of $\sigma(t_n, y_n)$ then by the Lipschitz continuity of σ and (A.5), we have

$$\begin{aligned} s_x^\top M_n^u s_x - s_y^\top M_n^w s_y & = \begin{bmatrix} s_x^\top & s_y^\top \end{bmatrix} \begin{bmatrix} M_n^u & 0 \\ 0 & -M_n^w \end{bmatrix} \begin{bmatrix} s_x \\ s_y \end{bmatrix} \\ & \leq \begin{bmatrix} s_x^\top & s_y^\top \end{bmatrix} \bar{B}_n \begin{bmatrix} s_x \\ s_y \end{bmatrix} \\ & \leq C(n|x_n - y_n|^{2\varrho} + |x_n - y_n|) \end{aligned}$$

and we conclude that

$$\limsup_{n \rightarrow \infty} \frac{1}{2}\text{Tr}[\sigma^\top(t_n, x_n)M_n^u\sigma(t_n, x_n) - \sigma^\top(t_n, y_n)M_n^w\sigma(t_n, y_n)] \leq 0.$$

Finally, we have that

$$\begin{aligned} & e^{\lambda t_n} f(t_n, x_n, e^{-\lambda t_n} u(t_n, x_n), e^{-\lambda t_n} \sigma^\top(t_n, x_n)q_n^u) - e^{\lambda t_n} f(t_n, y_n, e^{-\lambda t_n} w(t_n, y_n), e^{-\lambda t_n} \sigma^\top(t_n, x_n)q_n^w) \\ & \leq k_f(u(t_n, x_n) - w(t_n, y_n) + |\sigma^\top(t_n, x_n)q_n^u - \sigma^\top(t_n, x_n)q_n^w|) \\ & \quad + e^{\lambda t_n} |f(t_n, x_n, e^{-\lambda t_n} u(t_n, x_n), e^{-\lambda t_n} \sigma^\top(t_n, x_n)q_n^u) - f(t_n, y_n, e^{-\lambda t_n} u(t_n, x_n), e^{-\lambda t_n} \sigma^\top(t_n, x_n)q_n^u)| \end{aligned}$$

Repeating the above argument and using that f is jointly continuous in (t, x) uniformly in (y, z) we get that the upper limit of the right-hand side when $n \rightarrow \infty$ is bounded by $k_f(u(t_n, x_n) - w(t_n, y_n))$. Put together, this

gives that

$$(\lambda - k_f) \limsup_{n \rightarrow \infty} (u(t_n, x_n) - w(t_n, y_n)) \leq 0$$

and choosing $\lambda > k_f$ gives a contradiction. \square

REFERENCES

- [1] A. Bensoussan and J. Lions, Impulse Control and Quasivariational inequalities, Gauthier-Villars, Montrouge, France (1984).
- [2] B. Bouchard, A stochastic target formulation for optimal switching problems in finite horizon. *Stochastics* **81** (2009) 171–197.
- [3] B. Bruder and H. Pham, Impulse control problem on finite horizon with execution delay. *Stoch Proc Appl.* **81** (2009) 1436–1469.
- [4] R. Carmona and M. Ludkovski, Pricing asset scheduling flexibility using optimal switching. *Appl. Math. Finance* **15** (2008) 405–447.
- [5] J.F. Chassagneux, R. Elie and I. Kharroubi, A note on existence and uniqueness for solutions of multidimensional reflected BSDEs. *Electron. Commun. Probab.* **16** (2011) 120–128.
- [6] J.-F. Chassagneux and A. Richou, Rate of convergence for the discrete-time approximation of reflected BSDEs arising in switching problems. *Stochastic Process. Appl.* **129** (2019) 4597–4637.
- [7] A. Cosso, Stochastic differential games involving impulse controls and double-obstacle quasi-variational inequalities. *SIAM J. Control Optim.* **3** (2013) 2102–2131.
- [8] M.G. Crandall, H. Ishii and P.L. Lions, Users guide to viscosity solutions of second order partial differential equations. *Bull. Am. Math. Soc.* **27** (1992) 1–67.
- [9] B. Djehiche, S. Hamadène and M. Morlais, Viscosity solutions of systems of variational inequalities with interconnected bilateral obstacles. *Funkcialaj Ekvacioj* **58** (2015) 135–175.
- [10] B. Djehiche, S. Hamadène, M.-A. Morlais and X. Zhao, On the equality of solutions of max-min and min-max systems of variational inequalities with interconnected bilateral obstacles. *J. Math. Anal. Appl.* **452** (2017) 148–175.
- [11] B. Djehiche, S. Hamadène and A. Popier, A finite horizon optimal multiple switching problem. *SIAM J. Control Optim.* **47** (2009) 2751–2770.
- [12] P. Dupuis and H. Ishii, SDEs with oblique reflection on nonsmooth domains. *Ann. Probab.* **21** (1993) 554–580.
- [13] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng and M.C. Quenez, Reflected solutions of backward SDEs and related obstacle problems for PDEs. *Ann. Probab.* **25** (1997) 702–737.
- [14] N. El Karoui, S. Peng and M.C. Quenez, Backward stochastic differential equations in finance. *Math. Finance* **7** (1997) 1–71.
- [15] S. Hamadène, R. Martyr and J. Moriarty, A probabilistic verification theorem for the finite horizon two-player zero-sum optimal switching game in continuous time. *Adv. Appl. Probab.* **51** (2019) 425–442.
- [16] S. Hamadène, M. Mnif and S. Neffati, Viscosity solutions of systems of PDEs with interconnected obstacles and switching problem without monotonicity condition. *Asymp. Anal.* **113** (2019) 123–136.
- [17] S. Hamadène and M.A. Morlais, Viscosity solutions of systems of PDEs with interconnected obstacles and switching problem. *Appl. Math. Optim.* **67** (2013) 163–196.
- [18] S. Hamadène and J. Zhang, Switching problem and related system of reflected backward SDEs. *Stochastic Process. Appl.* **120** (2010) 403–426.
- [19] Y. Hu and S. Tang, Multi-dimensional BSDE with oblique reflection and optimal switching. *Prob. Theory Related Fields* **147** (2008) 89–121.
- [20] I. Kharroubi, J. Ma, H. Pham and J. Zhang, Backward sdes with constrained jumps and quasi-variational inequalities. *Ann. Probab.* **38** (2010) 794–840.
- [21] I. Kharroubi and H. Pham, Feynman-Kac representation for Hamilton-Jacobi-Bellman IPDE. *Ann. Probab.* **43** (2015) 1823–1865.
- [22] B. Øksendal and A. Sulem, Applied Stochastic Control of Jump Diffusions. Springer (2007).
- [23] M. Perninge, Sequential systems of reflected backward stochastic differential equations with application to impulse control. *Appl. Math. Optim.* **86** (2022) 1–59.

- [24] S. Tang and J. Yong, Finite horizon stochastic optimal switching and impulse controls with a viscosity solution. *Stoch. Stoch. Reports* **45** (1999) 145–176.



Please help to maintain this journal in open access!

This journal is currently published in open access under the Subscribe to Open model (S2O). We are thankful to our subscribers and supporters for making it possible to publish this journal in open access in the current year, free of charge for authors and readers.

Check with your library that it subscribes to the journal, or consider making a personal donation to the S2O programme by contacting subscribers@edpsciences.org.

More information, including a list of supporters and financial transparency reports, is available at <https://edpsciences.org/en/subscribe-to-open-s2o>.