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SATURATED BOUNDARY FEEDBACK CONTROL OF QUASI-LINEAR HYPERBOLIC BALANCE LAWS WITH APPLICATION TO LWR TRAFFIC FLOW STABILIZATION

Hanxu Zhao[1,2], Jingyuan Zhan[1,2] and Liguo Zhang[1,2,*]

Abstract. The saturated boundary stabilization problem for quasi-linear hyperbolic systems of balance laws is considered under H^2-norm in this paper, where the boundary conditions of the system are subject to actuator saturations. The resulting closed-loop system is proven to be locally exponentially stable with respect to the steady states in the presence of saturations. To this end, the sector nonlinearity model is introduced to deal with the saturation term, and then sufficient conditions for ensuring the locally exponential stability are established in terms of a set of matrix inequalities by employing the Lyapunov function method along with a sector condition. Furthermore, these results are applied to the stabilization of the two-lane traffic flow dynamic represented by Lighthill–Whitham–Richards (LWR) model. By utilizing variable speed limit (VSL) devices, a saturated boundary feedback controller is designed to stabilize the two-lane traffic flow, and the exponential convergence of the quasi-linear traffic flow system in H^2 sense is validated by numerical simulations.

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1. Introduction

Hyperbolic systems of balance laws have been extensively applied to modeling the dynamics of many physical engineering processes, the typical examples of which include the Saint–Venant equation for open channels [14], Euler equations for gas pipes [12], telegrapher equations for electrical transmission lines [9], Kac–Goldstein equations for chemotaxis [21] and Lighthill–Whitham–Richards (LWR) equation for traffic flow [19]. Boundary feedback control has been the crucial technology for the stabilization of hyperbolic systems, a great deal of research results have been achieved in recent years. For linear hyperbolic partial differential equations (PDEs) with anti-collocated boundary input and output, the problem of trajectory generation and tracking was tackled by exploiting backstepping method in [16]. The Lyapunov approach was used to solve the boundary stabilization problem for linear hyperbolic systems with relaxation structure in [15]. Although scholars have established appropriate boundary conditions for both linear and quasi-linear hyperbolic PDEs in L^2 or H^2 topology spaces [18, 34], the related work subject to actuator saturations is still limited.

Keywords and phrases: Quasi-linear hyperbolic systems, saturated boundary feedback control, LWR traffic flow model, Lyapunov function, sector condition.

[1] Faculty of Information Technology, Beijing University of Technology, 100124 Beijing, PR China.
[2] Key Laboratory of Computational Intelligence and Intelligent Systems, 100124 Beijing, PR China.

[*] Corresponding author: zhangliguo@bjut.edu.cn

Saturating actuators are ubiquitous in the control process of various industrial applications due to physical or safety constraints. The existence of actuator saturations generally gives rise to adverse reactions and even instability of controlled systems, which probably results in production accidents during the actual operations [28]. In the past few years, control problems with consideration of actuator saturations have attracted increasing research attention, and fruitful results have been achieved [3, 5, 7, 8]. A finite-time command-filtered adaptive control scheme based on approximation was proposed for nonlinear MIMO systems with input constraints in [32]. To stabilize a class of nonlinear systems with external disturbances and input saturations, two robust adaptive control schemes were developed via the backstepping approach in [29]. For the purpose of utilizing the limited communication resources efficiently, a saturated threshold event-triggered control strategy was designed in [4] for multiagent systems under sensor attacks.

Great strides have been witnessed in the saturated control of ordinary differential equation (ODE) systems in recent years, while the counterpart studies for PDE systems are still limited. In [27], semi-definite programming tools were employed to address the globally exponential stability problem for linear hyperbolic systems subject to in-domain disturbances and actuator saturations. For flexible manipulators with input saturations, an adaptive boundary control laws was proposed based on the Lyapunov method in [20] and adaptive neural-network boundary controllers with update laws were designed based on the backstepping approach, by using radial basis function neural-network method to tackle the unknown input saturations, dead zones, and model uncertainties in [24]. In order to ensure the asymptotic stability of wave equations with cone-bounded nonlinearity, Lyapunov function technique combined with the sector condition was applied in [23]. The method based on separation between the fast and slow eigenmodes of the spatial differential operator was investigated in [10] to give stability conditions for reaction–diffusion equations with input constraints.

To the best of authors' knowledge, the boundary stabilization problem for quasi-linear hyperbolic systems of balance laws in the presence of input saturations has not been addressed yet. Motivated by the aforementioned discussion, this paper studies the saturated boundary feedback control for quasi-linear hyperbolic systems of balance laws under the H^2-norm. Compared with the existing research results, we provide a solution to eliminate the influence of strong nonlinearity brought by input saturations on quasi-linear hyperbolic PDEs, and the sufficient conditions for ensuring locally exponential stability of the solution are given by using the Lyapunov function method combined with the sector condition.

Furthermore, the theoretical results are applied to the two-lane LWR traffic model in the presence of saturation. LWR traffic model, which is a first-order hyperbolic PDE, describes the rate of change in average density over a road in terms of differences of the flow [25]. Nowadays, the predictor-based backstepping method and Lyapunov stability analysis were explored respectively in [30] and [31] to solve the boundary feedback control problem of LWR model. As an ineluctable fact that saturations always exist in traffic control facilities due to the limited road capacity, driving safety requirements and personal preferences, to solve the boundary feedback stabilization problem for LWR traffic flow model subject to actuator saturation is of realistic significance. We have studied the saturated boundary feedback control problem for two-lane traffic flow with lane-changing interactions in [35], where the dynamic was described by a linearized LWR model. In this paper, we further design a saturated boundary feedback controller for the quasi-linear two-lane LWR traffic flow system to drive the traffic densities of both lanes to the steady states by employing VSL devices.

The rest of this paper is organized as follows. The description of the saturated boundary feedback control problem for the quasi-linear hyperbolic system of balance laws is given in Section 2. Section 3 is devoted to presenting the main results by using the Lyapunov function technique along with the sector condition in the H^2 topology space. An application to the saturated boundary feedback stabilization problem of two-lane traffic flow with lane-changing interactions based on LWR equations is presented in Section 4. Finally, a conclusion of this paper and possible further research are presented in Section 5.

Notations: R and R_+ denote the sets of real numbers and positive real numbers, respectively. R^n and $R^{n\times n}$ represent n-order vectors and n-order square matrices, respectively. For a matrix A, $A^\top$ denotes the transpose matrix of A, and $A > (<)\ 0$ denotes that A is a positive (negative) definite matrix. $\mathrm{diag}\{a_1, ..., a_n\}$ is the diagonal matrix, where $a_i, i = 1, ..., n$ is diagonal element. For a partitioned symmetric matrix, symbol $*$ stands for the symmetric block. For a vector $\xi = (\xi_1, ..., \xi_n)^\top \in C^0([0, L]; R^n)$, we denote $|\xi|_0 = \max\{|\xi(x)|_0, x \in [0, L]\}$, with

$|\xi(x)|_0 = \max\{|\xi_i(x)|, i = 1, ..., n\}$. $H^2((0, L); R^n)$ is the Sobolev space of all n-order C^2 functions on the open set $(0, L)$, and $\xi \in H^2((0, L); R^n)$ means that $\|\xi\|^2_{H^2((0,L);R^n)} = \int_0^L (|\xi|^2 + |\xi_x|^2 + |\xi_{xx}|^2)\mathrm{d}x < \infty$. I represents the identity matrix with appropriate dimension. $C(\cdot)$ denotes the space of continuous functions. For an open $U \subset R$ and a normed linear space $V \subset R^n$, $\mathcal{H}^2(U, V) = \{f \in H^2(U, V) : f$ is a locally second order differentiable function on U, $\frac{df}{\mathrm{d}x}, \frac{d^2 f}{\mathrm{d}x^2} \in H^2(U, V)\}$ where $\frac{\mathrm{d}}{\mathrm{d}z}$ stands for the weak derivative of f.

2. Quasi-linear hyperbolic system with saturated boundary control

2.1. Quasi-linear hyperbolic balance laws

Consider the following quasi-linear hyperbolic system of balance laws

$$\partial_t \xi + \Gamma(\xi)\partial_x \xi = M\xi, \quad t \in [0, \infty), \quad x \in [0, L], \tag{2.1}$$

where $\xi : [0, L] \times [0, +\infty) \to R^n$ is the state vector and $M \in R^{n\times n}$. $\Gamma(\xi) \in R^{n\times n}$ is a diagonal matrix of C^2-functions with non-zero real diagonal entries such that $\Gamma(\xi) = \mathrm{diag}\{\Gamma^+(\xi), -\Gamma^-(\xi)\}$, in which $\Gamma^+ = \mathrm{diag}\{\gamma_1(\xi), ..., \gamma_m(\xi)\}$, $\Gamma^- = \mathrm{diag}\{\gamma_{m+1}(\xi), ..., \gamma_n(\xi)\}$, $1 \le m \le n$, and $\gamma_i(\xi) > 0$ for all $i \in \{1, ..., n\}$.

Use the notations

$$\xi^{in} = \begin{bmatrix} \xi^+(0,t) \\ \xi^-(L,t) \end{bmatrix}, \quad \xi^{out} = \begin{bmatrix} \xi^+(L,t) \\ \xi^-(0,t) \end{bmatrix}$$

to denote the input and the output of system (2.1) on the boundaries, respectively, where ξ^+ and ξ^- are defined as $\xi^+ = [\xi_1, ..., \xi_m]^\top$ and $\xi^- = [\xi_{m+1}, ..., \xi_n]^\top$.

In this paper, we consider the boundary condition of the saturated feedback control type, *i.e.*,

$$\xi^{in} = sat(u), \tag{2.2}$$

with $u = K\xi^{out}$, where $K \in R^{n\times n}$ is the control gain to be designed. The function $sat(u) = [sat(u)_1, sat(u)_2, ..., sat(u)_n]^\top$ is the symmetric decentralized saturation function with saturation levels $u_{0_1}, u_{0_2}, ..., u_{0_n} \in R_+$, whose components for each $u = [u_1, u_2, ..., u_n]^\top$ are defined as

$$sat(u)_i = sat(u_i) = \begin{cases} u_{0_i} & if\ u_i > u_{0_i} \\ u_i & if\ -u_{0_i} \le u_i \le u_{0_i}, \\ -u_{0_i} & if\ u_i < -u_{0_i}, \end{cases} \tag{2.3}$$

where $i = 1, ..., n$.

The main objective of this paper is to present the exponential stability analysis for the quasi-linear hyperbolic system (2.1) under the saturated boundary feedback control (2.2), by establishing sufficient conditions for ensuring the solution of system (2.1) converging to the steady state in the presence of strong nonlinearity induced by saturations. For convenience, we define the following dead-zone nonlinearity as stated in [28]

$$\phi(u) = sat(u) - u. \tag{2.4}$$

Further, define the initial conditions of system (2.1) as

$$\xi(x, 0) = \xi_0(x), \quad x \in [0, L]. \tag{2.5}$$

In the following, we give the definition of the locally exponential stability for system (2.1) under the saturated boundary feedback controller (2.2).

Definition 2.1. System (2.1)–(2.2) is locally exponentially stable for H^2-norm, if there exist scalars $\alpha > 0, \chi > 0$ and ε such that, for every $\|\xi_0\|_{H^2((0,L);R^n)} \leq \varepsilon$, the solution to the Cauchy problem (2.1)–(2.2) satisfies

$$\|\xi(\cdot,t)\|_{H^2((0,L);R^n)} \leq \chi \mathrm{e}^{-\alpha t}\|\xi_0\|_{H^2((0,L);R^n)}, \tag{2.6}$$

for all $t \in [0,\infty)$.

2.2. Well-posedness of the Cauchy problem

This subsection is devoted to the well-posedness of the Cauchy problem (2.1), (2.2), (2.5) and the existence of unique mild solution by using nonlinear semigroup theory as similarly in [27] and [1].

We first reformulate the closed-loop system as an abstract differential equation. Consider the following operator defined in the Hilbert space $H^2((0,L);R^n)$ equipped with standard inner product:

$$\begin{aligned} \mathcal{A} :&\mathcal{D}(\mathcal{A}) \to H^2((0,L);R^n), \\ &\xi \mapsto -\Gamma(\xi)\xi_x + M\xi, \end{aligned} \tag{2.7}$$

where $\mathcal{D}(\mathcal{A}) = \{\xi \in \mathcal{H}^2((0,L);R^n); \xi^{in} = K\xi^{out} + \phi(K\xi^{out})\}$, and $\xi_x = \partial_x \xi$ for simplicity, then the closed-loop dynamics can be formally written as the following abstract system with state $\xi \in H^2((0,L);R^n)$

$$\dot{\xi} = \mathcal{A}\xi. \tag{2.8}$$

We recall the following definitions of mild solution stated in Barbu(2010, Def. 4.3) and non-accretive operator in Shreim, Ferrante, and Prieur(2022, Def. 2).

Definition 2.2 ([1])**.** A mild solution of the system (2.8) with the initial condition $\xi(x,0) = \xi_0(x)$ is a function $\xi \in C([0,\infty); H^2((0,L);R^n))$ with the property that for each $\epsilon > 0$ there exists an ϵ-approximate solution x of $\dot{\xi} = \mathcal{A}\xi$ such that $\|\xi(t) - x(t)\| \leq \epsilon$ for all $t \in [0,\infty)$.

Definition 2.3 ([27])**.** An operator $\mathcal{A}$ from $\mathcal{D}(\mathcal{A})$ to $H^2((0,L);R^n)$ is said to be non-accretive with respect to an inner product $\langle\cdot,\cdot\rangle$, if for every pair $(\xi_{(1)},\xi_{(2)}) \in \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A})$, the following inequality holds:

$$\langle \mathcal{A}\xi_{(1)} - \mathcal{A}\xi_{(2)}, \xi_{(1)} - \xi_{(2)}\rangle \leq 0. \tag{2.9}$$

Based on Bastin and Coron (2016, Appendix A), let us introduce the inner product on $H^2((0,L);R^n)$ as follows

$$\langle \xi_{(1)},\xi_{(2)}\rangle_\mu = \int_0^L \xi_{(1)}^\top Q(x)\xi_{(2)}\mathrm{d}x + \int_0^L \xi_{(1)_x}^\top Q(x)\xi_{(2)_x}\mathrm{d}x + \int_0^L \xi_{(1)_{xx}}^\top Q(x)\xi_{(2)_{xx}}\mathrm{d}x \tag{2.10}$$

where $Q(x) = \mathrm{diag}\{\mathrm{e}^{\mu(x-L)}I_m, \mathrm{e}^{-\mu x}I_{n-m}\}$, $\mu > 0$, and $\xi_{xx} = \partial_{xx}\xi$ for simplicity. The well-posedness of the Cauchy problem (2.1), (2.2), (2.5) and the existence of mild solution are given in the following theorem.

Theorem 2.4. *For every initial state $\xi_0 \in H^2((0,L);R^n)$, the Cauchy problem (2.1), (2.2), (2.5) has a unique mild solution $\xi \in C([0,\infty); H^2((0,L);R^n))$ such that $\xi(0,x) = \xi_0$.*

The proof of the theorem is based on the following lemmas which provide non-accretive and range property of the operator $\mathcal{A}$.

Lemma 2.5. *There always exist $\mu > 0$ and $\kappa \in R$ such that the operator $\mathcal{A} + \kappa I$ is non-accretive (with respect to the scalar product $\langle \cdot, \cdot \rangle_\mu$).*

Proof: To prove (2.9) for the operator $\mathcal{A} + \kappa I$ with a suitable μ, we denote $\tilde{\xi} = \xi_{(1)} - \xi_{(2)} \in \mathcal{D}(\mathcal{A})$ and $\tilde{\phi} = \phi(K\xi^{out}_{(1)}) - \phi(K\xi^{out}_{(2)}) \in R^n$ for convenience, then it can be obtained that

$$\langle (\mathcal{A} + \kappa I)\tilde{\xi}, \tilde{\xi} \rangle_\mu = \langle \mathcal{A}\tilde{\xi}, \tilde{\xi} \rangle_\mu + \kappa \langle \tilde{\xi}, \tilde{\xi} \rangle_\mu, \tag{2.11}$$

where

$$\langle \mathcal{A}\tilde{\xi}, \tilde{\xi} \rangle_\mu = W_1 + W_2 + W_3$$

with

$$\begin{aligned} W_1 &= \int_0^L \tilde{\xi}^\top Q(x)(\Gamma(M\tilde{\xi} - \tilde{\xi})\tilde{\xi}_x)\mathrm{d}x, \\ W_2 &= \int_0^L \tilde{\xi}_x^\top Q(x)(\mathcal{A}\tilde{\xi})_x \mathrm{d}x, \\ W_3 &= \int_0^L \tilde{\xi}_{xx}^\top Q(x)(\mathcal{A}\tilde{\xi})_{xx} \mathrm{d}x. \end{aligned} \tag{2.12}$$

For clarity, we introduce a notation to deal with the estimates of the higher order terms. We denote by $\mathcal{O}(X; Y)$, with $X \geq 0, Y \geq 0$, the quantities for which there exist $\sigma > 0, \zeta > 0$, such that

$$(Y \leq \zeta) \Rightarrow (|\mathcal{O}(X; Y)| \leq \sigma X).$$

Using the integration by parts for W_1, W_2 and W_3, we get

$$\begin{aligned} W_1 &= -\left[\frac{1}{2}\tilde{\xi}^\top Q(x)\Gamma(\tilde{\xi})\tilde{\xi}\right]_0^L + \frac{1}{2}\int_0^L \tilde{\xi}^\top Q(x)\big(\frac{\partial \Gamma(\tilde{\xi})}{\partial \tilde{\xi}}\tilde{\xi}_x + \mu|\Gamma(\tilde{\xi})| + 2M\big)\tilde{\xi}\mathrm{d}x \\ &= \frac{1}{2}\left[\mathrm{e}^{-\mu L}(\tilde{\xi}^{in})^\top |\Gamma_1(\tilde{\xi})|\tilde{\xi}^{in} - (\tilde{\xi}^{out})^\top |\Gamma_2(\tilde{\xi})|\tilde{\xi}^{out}\right] + \frac{1}{2}\int_0^L \tilde{\xi}^\top Q(x)\big(\frac{\partial \Gamma(\tilde{\xi})}{\partial \tilde{\xi}}\tilde{\xi}_x \\ &\quad + \mu|\Gamma(\tilde{\xi})| + 2M\big)\tilde{\xi}\mathrm{d}x; \end{aligned} \tag{2.13}$$

$$\begin{aligned} W_2 &= -\left[\frac{1}{2}\tilde{\xi}_x^\top Q(x)\Gamma(\tilde{\xi})\tilde{\xi}_x\right]_0^L + \frac{1}{2}\int_0^L \tilde{\xi}_x^\top Q(x)\big(\mu|\Gamma(\tilde{\xi})| + 2M - \frac{\partial \Gamma(\tilde{\xi})}{\partial \tilde{\xi}}\tilde{\xi}_x\big)\tilde{\xi}_x\mathrm{d}x \\ &= \frac{1}{2}\left[\mathrm{e}^{-\mu L}(\tilde{\xi}_x^{in})^\top |\Gamma_1(\tilde{\xi})|\tilde{\xi}_x^{in} - (\tilde{\xi}_x^{out})^\top |\Gamma_2(\tilde{\xi})|\tilde{\xi}_x^{out}\right] + \frac{1}{2}\int_0^L \tilde{\xi}_x^\top Q(x)\big(\mu|\Gamma(\tilde{\xi})| \\ &\quad + 2M - \frac{\partial \Gamma(\tilde{\xi})}{\partial \tilde{\xi}}\tilde{\xi}_x\big)\tilde{\xi}_x\mathrm{d}x; \end{aligned} \tag{2.14}$$

$$\begin{aligned} W_3 &= -\left[\frac{1}{2}\tilde{\xi}_{xx}^\top Q(x)\Gamma(\tilde{\xi})\tilde{\xi}_{xx}\right]_0^L + \frac{1}{2}\int_0^L \tilde{\xi}_{xx}^\top Q(x)\big(\frac{\partial \Gamma(\tilde{\xi})}{\partial \tilde{\xi}}\tilde{\xi}_x + \mu|\Gamma(\tilde{\xi})|\big)\tilde{\xi}_{xx}\mathrm{d}x \\ &\quad - \int_0^L \tilde{\xi}_{xx}^\top Q(x)\big(2(\frac{\partial \Gamma(\tilde{\xi})}{\partial \tilde{\xi}}\tilde{\xi}_x)\tilde{\xi}_{xx} + (\frac{\partial \Gamma(\tilde{\xi})}{\partial \tilde{\xi}}\tilde{\xi}_x)_x\tilde{\xi}_x - M\tilde{\xi}_{xx}\big)\mathrm{d}x \\ &= \frac{1}{2}\left[\mathrm{e}^{-\mu L}(\tilde{\xi}_{xx}^{in})^\top |\Gamma_1(\tilde{\xi})|\tilde{\xi}_{xx}^{in} - (\tilde{\xi}_{xx}^{out})^\top |\Gamma_2(\tilde{\xi})|\tilde{\xi}_{xx}^{out}\right] + \frac{1}{2}\int_0^L \tilde{\xi}_{xx}^\top Q(x)\big(\mu|\Gamma(\tilde{\xi})| \end{aligned}$$

$$+ 2M - 3\frac{\partial\Gamma(\tilde{\xi})}{\partial\tilde{\xi}}\tilde{\xi}_x)\tilde{\xi}_{xx}\mathrm{d}x - \int_0^L \tilde{\xi}_{xx}^\top Q(x)(\frac{\partial\Gamma(\tilde{\xi})}{\partial\tilde{\xi}}\tilde{\xi}_x)_x\tilde{\xi}_x\mathrm{d}x, \tag{2.15}$$

where $|\Gamma_1(\tilde{\xi})| = \mathrm{diag}\{\gamma_1(\tilde{\xi}(0)), ..., \gamma_m(\tilde{\xi}(0)), \gamma_{m+1}(\tilde{\xi}(L)), ..., \gamma_n(\tilde{\xi}(L))\}$ and $|\Gamma_2(\tilde{\xi})| = \mathrm{diag}\{\gamma_1(\xi(L)), ..., \gamma_m(\tilde{\xi}(L)), \gamma_{m+1}(\tilde{\xi}(0)), ..., \gamma_n(\tilde{\xi}(0))\}$.

From the boundary condition in (2.2), it can be calculated that

$$\xi_x^{in} = ssat(\Lambda), \tag{2.16}$$

with $\Lambda = \Gamma_1^{-1}(\tilde{\xi})M\tilde{\xi}^{in} - \Gamma_1^{-1}(\tilde{\xi})KM\tilde{\xi}^{out} + \Gamma_1^{-1}(\tilde{\xi})K\Gamma_2(\tilde{\xi})\tilde{\xi}_x^{out}$, whose components for each $\Lambda = [\Lambda_1, \Lambda_2, ..., \Lambda_n]^\top$ are defined as

$$ssat(\Lambda)_i = ssat(\Lambda_i) = \begin{cases} \Lambda_i, & if\ -u_{0_i} \leq (K\xi^{out})_i \leq u_{0_i}, \\ \underline{\Lambda}_i, & else, \end{cases} \tag{2.17}$$

$i = 1, ..., n$, where $\underline{\Lambda} = \Gamma_1^{-1}(\tilde{\xi})M\tilde{\xi}^{in}$, $\Gamma_1(\tilde{\xi}) = \mathrm{diag}\{\gamma_1(\tilde{\xi}(0)), ..., \gamma_m(\tilde{\xi}(0)), -\gamma_{m+1}(\tilde{\xi}(L)), ..., -\gamma_n(\tilde{\xi}(L))\}$ and $\Gamma_2(\tilde{\xi}) = \mathrm{diag}\{\gamma_1(\xi(L)), ..., \gamma_m(\tilde{\xi}(L)), -\gamma_{m+1}(\tilde{\xi}(0)), ..., -\gamma_n(\tilde{\xi}(0))\}$.

Similar to ξ_{xx}^{in}, we have

$$\xi_{xx}^{in} = ssat(\Upsilon), \tag{2.18}$$

with $\Upsilon = \Gamma_1^{-1}(\tilde{\xi})\Gamma_1^{-1}(\tilde{\xi})KMM\tilde{\xi}^{out} - \Gamma_1^{-1}(\tilde{\xi})\Gamma_1^{-1}(\tilde{\xi})\big(KM\Gamma_2(\tilde{\xi}) + K\Gamma_2(\tilde{\xi})M\big)\tilde{\xi}_x^{out} + \Gamma_1^{-1}(\tilde{\xi})\Gamma_1^{-1}(\tilde{\xi})K\Gamma_2(\tilde{\xi})\Gamma_2(\tilde{\xi})\tilde{\xi}_{xx}^{out}$ $-\Gamma_1^{-1}(\tilde{\xi})\Gamma_1^{-1}(\tilde{\xi})MM\tilde{\xi}^{in} + \Gamma_1^{-1}(\tilde{\xi})\big(\Gamma_1^{-1}(\tilde{\xi})M\Gamma_1(\tilde{\xi}) + M\big)\tilde{\xi}_x^{in} - \Gamma_1^{-1}(\tilde{\xi})\Gamma_1^{-1}(\tilde{\xi})K\frac{\partial\Gamma_2(\tilde{\xi})}{\partial\tilde{\xi}^{out}}M\tilde{\xi}^{out}\tilde{\xi}_x^{out} + \Gamma_1^{-1}(\tilde{\xi})\Gamma_1^{-1}(\tilde{\xi})\frac{\partial\Gamma_1(\tilde{\xi})}{\partial\tilde{\xi}^{in}}$ $M\tilde{\xi}^{in}\tilde{\xi}_x^{in} + \Gamma_1^{-1}(\tilde{\xi})\Gamma_1^{-1}(\tilde{\xi})K\frac{\partial\Gamma_2(\tilde{\xi})}{\partial\tilde{\xi}^{out}}M\tilde{\xi}^{out}\big(\Gamma_2(\tilde{\xi}) - I\big)\tilde{\xi}_x^{out}\tilde{\xi}_x^{out} - \Gamma_1^{-1}(\tilde{\xi})\Gamma_1^{-1}(\tilde{\xi})\big(\Gamma_1(\tilde{\xi})\frac{\partial\Gamma_1(\tilde{\xi})}{\partial\tilde{\xi}^{in}} + \frac{\partial\Gamma_1(\tilde{\xi})}{\partial\tilde{\xi}^{in}}\Gamma_1(\tilde{\xi})\big)\tilde{\xi}_x^{in}\tilde{\xi}_x^{in}$, whose components for each $\Upsilon = [\Upsilon_1, \Upsilon_2, ..., \Upsilon_n]^\top$ are defined as

$$ssat(\Upsilon)_i = ssat(\Upsilon_i) = \begin{cases} \Upsilon_i, & if\ -u_{0_i} \leq (K\xi^{out})_i \leq u_{0_i}, \\ \underline{\Upsilon}_i, & else, \end{cases} \tag{2.19}$$

$i=1, ..., n$, where $\underline{\Upsilon} = \Gamma_1^{-1}(\tilde{\xi})\Gamma_1^{-1}(\tilde{\xi})\Big(M\Gamma_1(\tilde{\xi}) + \Gamma_1(\tilde{\xi})M + \frac{\partial\Gamma_1(\tilde{\xi})}{\partial\tilde{\xi}^{in}}M\tilde{\xi}^{in}\Big)\tilde{\xi}_x^{in} - \Gamma_1^{-1}(\tilde{\xi})\Gamma_1^{-1}(\tilde{\xi})MM\tilde{\xi}^{in} - \Gamma_1^{-1}(\tilde{\xi})\Big(\frac{\partial\Gamma_1(\tilde{\xi})}{\partial\tilde{\xi}^{in}} + \Gamma_1^{-1}(\tilde{\xi})\ \frac{\partial\Gamma_1(\tilde{\xi})}{\partial\tilde{\xi}^{in}}\Gamma_1(\tilde{\xi})\Big)\tilde{\xi}_x^{in}\tilde{\xi}_x^{in}$.

Considering the dead-zone nonlinearity defined in (2.4), we obtain

$$\phi(\Lambda) = sat(\Lambda) - \Lambda, \tag{2.20}$$

and

$$\phi(\Upsilon) = sat(\Upsilon) - \Upsilon. \tag{2.21}$$

Then the inner product in (2.11) can be rewritten as

$$\begin{aligned}\langle(\mathcal{A} + \kappa I)\tilde{\xi}, \tilde{\xi}\rangle_\mu =& \frac{1}{2}\mathrm{e}^{-\mu L}\begin{bmatrix} Z^{out} \\ \Phi \end{bmatrix}^\top \begin{bmatrix} \mathcal{K}^\top\Pi\mathcal{K} - \mathrm{e}^{\mu L}\Pi & \mathcal{K}^\top\Pi \\ * & \Pi \end{bmatrix}\begin{bmatrix} Z^{out} \\ \Phi \end{bmatrix} \\ &+ \frac{1}{2}\int_0^L Z^\top \mathrm{diag}\{\mathcal{M}, \mathcal{M}, \mathcal{M}\}Z\mathrm{d}x + \mathcal{O}(\int_0^L (|\tilde{\xi}|^2 + |\tilde{\xi}_x|^2 + |\tilde{\xi}_{xx}|^2)(|\tilde{\xi}|\end{aligned}$$

$$
\begin{aligned}
&+|\tilde{\xi}_x|)\mathrm{d}x;|\tilde{\xi}|_0+|\tilde{\xi}_x|_0)+\mathcal{O}\Big((|\tilde{\xi}^{out}|^2+|\tilde{\xi}^{out}_x|^2+|\tilde{\xi}^{out}_{xx}|^2)|\tilde{\xi}^{out}|^2+|\tilde{\xi}^{out}||\tilde{\phi}|^2\\
&+|\tilde{\xi}^{out}|^2|\tilde{\phi}|+|\tilde{\phi}|^3+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^2|\tilde{\xi}^{out}|\Big((|\tilde{\xi}^{out}|+|\tilde{\phi}|)|\tilde{\xi}^{out}||\tilde{\xi}^{out}_x|(|\tilde{\xi}^{out}|\\
&+|\tilde{\xi}^{out}_x|)+|\phi(\Lambda)|(1+|\tilde{\xi}^{out}_x|)\Big)+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^5|\tilde{\xi}^{out}|^2(1+|\tilde{\xi}^{out}_x|+|\tilde{\xi}^{out}|\\
&\quad|\tilde{\xi}^{out}_x|)^2+|\phi(\Upsilon)|^2(|\tilde{\xi}^{out}|+|\tilde{\phi}|)+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^3|\tilde{\xi}^{out}||\phi(\Upsilon)|(1+|\tilde{\xi}^{out}_x|\\
&+|\tilde{\xi}^{out}||\tilde{\xi}^{out}_x|)+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^5|\tilde{\xi}^{out}_x|^2(|\tilde{\xi}^{out}|+|\tilde{\xi}^{out}||\tilde{\xi}^{out}_x|+|\tilde{\xi}^{out}_x|)^2\\
&+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^3\big((|\tilde{\xi}^{out}|+|\tilde{\phi}|)|\tilde{\xi}^{out}|(1+|\tilde{\xi}^{out}_x|)+|\phi(\Lambda)|\big)^2\Big(\big((|\tilde{\xi}^{out}|+|\tilde{\phi}|)\\
&\quad|\tilde{\xi}^{out}|(1+|\tilde{\xi}^{out}_x|)+|\phi(\Lambda)|\big)(1+|\tilde{\xi}^{out}|+|\tilde{\phi}|)^2+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^2\Big)^2\\
&+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^4\big((|\tilde{\xi}^{out}|+|\tilde{\phi}|)|\tilde{\xi}^{out}|(1+|\tilde{\xi}^{out}_x|)+|\phi(\Lambda)|\big)|\tilde{\xi}^{out}_x|\\
&\quad(|\tilde{\xi}^{out}|+|\tilde{\xi}^{out}||\tilde{\xi}^{out}_x|+|\tilde{\xi}^{out}_x|)\Big(\big((|\tilde{\xi}^{out}|+|\tilde{\phi}|)|\tilde{\xi}^{out}|(1+|\tilde{\xi}^{out}_x|)\\
&+|\phi(\Lambda)|\big)(1+|\tilde{\xi}^{out}|+|\tilde{\phi}|)^2+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^2\Big);|\tilde{\xi}^{out}|_0+|\tilde{\xi}^{out}_x|_0\Big),
\end{aligned}
\tag{2.22}
$$

where $Z=\left[\tilde{\xi}^\top,\tilde{\xi}_x^\top,\tilde{\xi}_{xx}^\top\right]^\top$, $\Phi=\left[\tilde{\phi}^\top,\phi^\top(\Lambda),\phi^\top(\Upsilon)\right]^\top$, $\Pi=\mathrm{diag}\{|\Gamma(0)||\Gamma(0)|,|\Gamma(0)|,|\Gamma(0)|\}$, $\mathcal{M}=2\kappa Q(x)+\mu Q(x)|\Gamma(0)|+2Q(x)M$, $|\Gamma(0)|=\mathrm{diag}\{|\gamma_i(0)|\}$, and

$$
\mathcal{K}=\begin{bmatrix} K & 0 & 0\\ \mathcal{K}_{21} & \mathcal{K}_{22} & 0\\ \mathcal{K}_{31} & \mathcal{K}_{32} & \mathcal{K}_{33}\end{bmatrix}
\tag{2.23}
$$

with

$$
\begin{aligned}
\mathcal{K}_{21}&=\Gamma^{-1}(0)MK-\Gamma^{-1}(0)KM,\\
\mathcal{K}_{22}&=\Gamma^{-1}(0)K\Gamma(0),\\
\mathcal{K}_{31}&=\Gamma^{-1}(0)\Gamma^{-1}(0)KMM+(\Gamma^{-1}(0)\Gamma^{-1}(0)M\Gamma(0)+\Gamma^{-1}(0)M)\Gamma^{-1}(0)MK\\
&\quad-(\Gamma^{-1}(0)\Gamma^{-1}(0)M\Gamma(0)+\Gamma^{-1}(0)M)\Gamma^{-1}(0)KM-\Gamma^{-1}(0)\Gamma^{-1}(0)MMK,\\
\mathcal{K}_{32}&=-\Gamma^{-1}(0)\Gamma^{-1}(0)KM\Gamma(0)-\Gamma^{-1}(0)\Gamma^{-1}(0)K\Gamma(0)M\\
&\quad+(\Gamma^{-1}(0)\Gamma^{-1}(0)M\Gamma(0)+\Gamma^{-1}(0)M)\Gamma^{-1}(0)K\Gamma(0),\\
\mathcal{K}_{33}&=\Gamma^{-1}(0)\Gamma^{-1}(0)K\Gamma(0)\Gamma(0).
\end{aligned}
\tag{2.24}
$$

Recalling that ϕ is Lipschitz continuous, we then get $\Phi^\top\Phi\le(\tilde{\xi}^{out})^\top K^\top K\tilde{\xi}^{out}+\Lambda^\top\Lambda+\Upsilon^\top\Upsilon$, i,e,

$$
\begin{bmatrix} Z^{out}\\ \Phi\end{bmatrix}^\top\begin{bmatrix} -\mathcal{K}^\Phi & 0\\ * & I\end{bmatrix}\begin{bmatrix} Z^{out}\\ \Phi\end{bmatrix}\le 0,
\tag{2.25}
$$

where

$$
\mathcal{K}^\Phi=\begin{bmatrix} \mathcal{K}^\Phi_{11} & \mathcal{K}^\Phi_{12} & \mathcal{K}^\Phi_{13}\\ * & \mathcal{K}^\Phi_{22} & \mathcal{K}^\Phi_{23}\\ * & * & \mathcal{K}^\Phi_{33}\end{bmatrix}
$$

with

$$\begin{aligned}
\mathcal{K}_{11}^{\Phi} &= K^{\top}K+\mathcal{M}_a^{\top}\mathcal{M}_a+(\Gamma^{-1}(0)MK-\Gamma^{-1}(0)KM)^{\top}(\Gamma^{-1}(0)MK-\Gamma^{-1}(0)KM),\\
\mathcal{K}_{12}^{\Phi} &= (\Gamma^{-1}(0)MK-\Gamma^{-1}(0)KM)^{\top}\Gamma^{-1}(0)K\Gamma(0)+\mathcal{M}_a^{\top}\mathcal{M}_b,\\
\mathcal{K}_{13}^{\Phi} &= \mathcal{M}_b^{\top}\Gamma^{-1}(0)\Gamma^{-1}(0)K\Gamma(0)\Gamma(0),\\
\mathcal{K}_{22}^{\Phi} &= \Gamma(0)K^{\top}\Gamma^{-1}(0)\Gamma^{-1}(0)K\Gamma(0)+\mathcal{M}_b^{\top}\mathcal{M}_b,\\
\mathcal{K}_{23}^{\Phi} &= \mathcal{M}_b^{\top}\Gamma^{-1}(0)\Gamma^{-1}(0)K\Gamma(0)\Gamma(0),\\
\mathcal{K}_{33}^{\Phi} &= \Gamma(0)\Gamma(0)K^{\top}\Gamma^{-1}(0)\Gamma^{-1}(0)\Gamma^{-1}(0)\Gamma^{-1}(0)K\Gamma(0)\Gamma(0),\\
\mathcal{M}_a &= \Gamma^{-1}(0)\Gamma^{-1}(0)(KMM-MKM)+\Gamma^{-1}(0)M\Gamma^{-1}(0)(MK-KM)\\
\mathcal{M}_b &= \Gamma^{-1}(0)\Gamma^{-1}(0)\big(MK\Gamma(0)-KM\Gamma(0)-K\Gamma(0)M\big)+\Gamma^{-1}(0)M\Gamma^{-1}(0)K\Gamma(0).
\end{aligned}\tag{2.26}$$

Combining with equation (2.22) and picking $\iota>0$ such that $\Pi-\iota I\le -I$, results in

$$\begin{aligned}
\langle(\mathcal{A}+\kappa I)\tilde{\xi},\tilde{\xi}\rangle_{\mu}\le&\frac{1}{2}\mathrm{e}^{-\mu L}\begin{bmatrix} Z^{out}\\ \Phi\end{bmatrix}^{\top}\begin{bmatrix}\mathcal{K}^{\top}\Pi\mathcal{K}-\mathrm{e}^{\mu L}\Pi+\iota\mathcal{K}^{\Phi} & \mathcal{K}^{\top}\Pi\\ * & -I\end{bmatrix}\begin{bmatrix} Z^{out}\\ \Phi\end{bmatrix}\\
&+\frac{1}{2}\int_0^L Z^{\top}\mathrm{diag}\{\mathcal{M},\mathcal{M},\mathcal{M}\}Z\mathrm{d}x+\mathcal{O}(\int_0^L(|\tilde{\xi}|^2+|\tilde{\xi}_x|^2+|\tilde{\xi}_{xx}|^2)(|\tilde{\xi}|\\
&+|\tilde{\xi}_x|)\mathrm{d}x;|\tilde{\xi}|_0+|\tilde{\xi}_x|_0)+\mathcal{O}\bigg((|\tilde{\xi}^{out}|^2+|\tilde{\xi}_x^{out}|^2+|\tilde{\xi}_{xx}^{out}|^2)|\tilde{\xi}^{out}|^2+|\tilde{\xi}^{out}||\tilde{\phi}|^2\\
&+|\tilde{\xi}^{out}|^2|\tilde{\phi}|+|\tilde{\phi}|^3+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^2|\tilde{\xi}^{out}|\Big((|\tilde{\xi}^{out}|+|\tilde{\phi}|)|\tilde{\xi}^{out}||\tilde{\xi}_x^{out}|(|\tilde{\xi}^{out}|\\
&+|\tilde{\xi}_x^{out}|)+|\phi(\Lambda)|(1+|\tilde{\xi}_x^{out}|)\Big)+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^5|\tilde{\xi}^{out}|^2(1+|\tilde{\xi}_x^{out}|+|\tilde{\xi}^{out}|\\
&\quad|\tilde{\xi}_x^{out}|)^2+|\phi(\Upsilon)|^2(|\tilde{\xi}^{out}|+|\tilde{\phi}|)+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^3|\tilde{\xi}^{out}||\phi(\Upsilon)|(1+|\tilde{\xi}_x^{out}|\\
&+|\tilde{\xi}^{out}||\tilde{\xi}_x^{out}|)+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^5|\tilde{\xi}_x^{out}|^2(|\tilde{\xi}^{out}|+|\tilde{\xi}^{out}||\tilde{\xi}_x^{out}|+|\tilde{\xi}_x^{out}|)^2\\
&+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^3\big((|\tilde{\xi}^{out}|+|\tilde{\phi}|)|\tilde{\xi}^{out}|(1+|\tilde{\xi}_x^{out}|)+|\phi(\Lambda)|\big)^2\Big(\big((|\tilde{\xi}^{out}|+|\tilde{\phi}|)\\
&\quad|\tilde{\xi}^{out}|(1+|\tilde{\xi}_x^{out}|)+|\phi(\Lambda)|\big)(1+|\tilde{\xi}^{out}|+|\tilde{\phi}|)^2+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^2\Big)^2\\
&+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^4\big((|\tilde{\xi}^{out}|+|\tilde{\phi}|)|\tilde{\xi}^{out}|(1+|\tilde{\xi}_x^{out}|)+|\phi(\Lambda)|\big)|\tilde{\xi}_x^{out}|\\
&\quad(|\tilde{\xi}^{out}|+|\tilde{\xi}^{out}||\tilde{\xi}_x^{out}|+|\tilde{\xi}_x^{out}|)\Big(\big((|\tilde{\xi}^{out}|+|\tilde{\phi}|)|\tilde{\xi}^{out}|(1+|\tilde{\xi}_x^{out}|)\\
&+|\phi(\Lambda)|\big)(1+|\tilde{\xi}^{out}|+|\tilde{\phi}|)^2+(|\tilde{\xi}^{out}|+|\tilde{\phi}|)^2\Big);|\tilde{\xi}^{out}|_0+|\tilde{\xi}_x^{out}|_0\bigg).
\end{aligned}\tag{2.27}$$

According to the Schur-complement lemma in [33], if and only if the following inequalities hold

$$\begin{aligned}
&\mathcal{K}^{\top}\Pi\mathcal{K}-\mathrm{e}^{\mu L}\Pi+\iota\mathcal{K}^{\Phi}<0,\\
&-I-(\Pi\mathcal{K})(\mathcal{K}^{\top}\Pi\mathcal{K}-\mathrm{e}^{\mu L}\Pi+\iota\mathcal{K}^{\Phi})^{-1}(\mathcal{K}^{\top}\Pi)<0,
\end{aligned}\tag{2.28}$$

then

$$\Psi_0=\begin{bmatrix}\mathcal{K}^{\top}\Pi\mathcal{K}-\mathrm{e}^{\mu L}\Pi+\iota\mathcal{K}^{\Phi} & \mathcal{K}^{\top}\Pi\\ * & -I\end{bmatrix}<0.$$

Thus, there always exist μ and κ satisfying

$$\kappa < -\|\frac{\mu}{2}|\Gamma(0)| + M\|, \tag{2.29}$$

$$\frac{1}{L}\ln\left(\|\mathcal{K}^\top\Pi\mathcal{K}\Pi^{-1} + \iota\mathcal{K}^\Phi\Pi^{-1}\|\right) < \mu < \frac{1}{L}\ln\left(\|\mathcal{K}^\top\Pi\mathcal{K}\Pi^{-1} + \iota\mathcal{K}^\Phi\Pi^{-1} + \|\mathcal{K}^\top\Pi\|^2\Pi^{-1}\|\right), \tag{2.30}$$

and then it can be get that $\Psi_0 < 0$ and $\text{diag}\{\mathcal{M},\mathcal{M},\mathcal{M}\} < 0$.

Since $\Psi_0 < 0$, there exist $\varrho_1 > 0$ and $\psi_1 > 0$ such that when $|\tilde{\xi}^{out}|_0 + |\tilde{\xi}_x^{out}|_0 < \psi_1$, the above formula yields that

$$\begin{aligned}\langle(\mathcal{A}+\kappa I)\tilde{\xi},\tilde{\xi}\rangle_\mu \leq &-\varrho_1\left|[(Z^{out})^\top,\Phi^\top]^\top\right|^2 + \frac{1}{2}\int_0^L Z^\top\text{diag}\{\mathcal{M},\mathcal{M},\mathcal{M}\}Z\text{d}x\\ &+\mathcal{O}(\int_0^L(|\tilde{\xi}|^2+|\tilde{\xi}_x|^2+|\tilde{\xi}_{xx}|^2)(|\tilde{\xi}|+|\tilde{\xi}_x|)\text{d}x;|\tilde{\xi}|_0+|\tilde{\xi}_x|_0).\end{aligned} \tag{2.31}$$

Similarly, there exist $\varrho_2 > 0$ and $\psi_2 > 0$ such that when $|\tilde{\xi}|_0 + |\tilde{\xi}_x|_0 < \psi_2$, it holds that

$$\langle(\mathcal{A}+\kappa I)\tilde{\xi},\tilde{\xi}\rangle_\mu \leq -\varrho_1\left|[(Z^{out})^\top,\Phi^\top]^\top\right|^2 - \varrho_2\int_0^L|Z|^2\text{d}x < 0. \tag{2.32}$$

This completes the proof of Lemma 2.5.

Lemma 2.6. *There exists $\kappa \in R$ such that the following range property holds*

$$\text{Ran}(I+\lambda(\mathcal{A}+\kappa I)) = H^2((0,L);R^n) \tag{2.33}$$

for all $\lambda > 0$, where Ran *stands for the range.*

Proof: It's obviously that $\text{Ran}(I+\lambda(\mathcal{A}+\kappa I)) \subset H^2((0,L);R^n)$, now we prove that $\text{Ran}(I+\lambda(\mathcal{A}+\kappa I)) \supset H^2((0,L);R^n)$. Choose any $f \in H^2((0,L);R^n)$ with $f = [(f^+)^\top,(f^-)^\top]^\top$, $f^+ \in H^2((0,L);R^m)$, $f^- \in H^2((0,L);R^{(n-m)})$, there exists $\xi \in \mathcal{D}(\mathcal{A})$ holds that

$$(I+\lambda(\mathcal{A}+\kappa I))\xi = f. \tag{2.34}$$

The above statement is equivalent to checking the existence of solution for the boundary value problem as follows

$$\begin{cases}(I_\kappa+\lambda M)\xi(x) - \lambda\Gamma(\xi)\xi_x = f(x), \quad \forall x \in (0,L),\\ \xi^{in} = K\xi^{out} + \phi(K\xi^{out}),\end{cases} \tag{2.35}$$

where $I_\kappa = (1+\lambda\kappa)I \in R^{n\times n}$. The solution of the first equation above is given by

$$\begin{cases}\xi^+(x) = \text{e}^{\frac{1}{\lambda}\Gamma^{-1}(\xi^+)I_\kappa^+x}\xi^+(0) + \int_0^x \text{e}^{\frac{1}{\lambda}\Gamma^{-1}(\xi^+)I_\kappa^+(x-s)}\frac{1}{\lambda}\Gamma^{-1}(\xi^+)(\lambda M^+\xi - f^+(s))\text{d}s,\\ \xi^-(x) = \text{e}^{-\frac{1}{\lambda}\Gamma^{-1}(\xi^-)I_\kappa^-x}\xi^-(L) - \int_L^x \text{e}^{-\frac{1}{\lambda}\Gamma^{-1}(\xi^-)I_\kappa^-(x-s)}\frac{1}{\lambda}\Gamma^{-1}(\xi^-)(\lambda M^-\xi - f^-(s))\text{d}s,\end{cases} \tag{2.36}$$

$\forall x \in (0,L)$, where $M^+ \in R^{m\times n}$ and $M^- \in R^{(n-m)\times n}$ denote the first m and the last $(n-m)$ rows of the matrix M, respectively. $I_\kappa^+ = (1+\lambda\kappa)I \in R^{m\times m}$ and $I_\kappa^- = (1+\lambda\kappa)I \in R^{(n-m)\times(n-m)}$.

Let us define that

$$\begin{aligned}\xi^+(L) &= \mathrm{e}^{\frac{1}{\lambda}\Gamma^{-1}(\xi^+(L))I_\kappa^+ L}\xi^+(0)+\int_0^L \mathrm{e}^{\frac{1}{\lambda}\Gamma^{-1}(\xi^+(L))I_\kappa^+(L-s)}\frac{1}{\lambda}\Gamma^{-1}(\xi^+(L))\big(\lambda M^+\xi(L)-f^+(s)\big)\mathrm{d}s\\ &=: g_\lambda^+(\xi^+(0)), \qquad (2.37)\\ \xi^-(0) &= \xi^-(L) - \int_L^0 \mathrm{e}^{-\frac{1}{\lambda}\Gamma^{-1}(\xi^-(0))I_\kappa^-(-s)}\frac{1}{\lambda}\Gamma^{-1}(\xi^-(0))(\lambda M^-\xi(0)-f^-(s))\mathrm{d}s\\ &=: g_\lambda^-(\xi^-(L)). \qquad (2.38)\end{aligned}$$

Thus, the boundary condition can be rewritten as

$$\xi^+(0) = K_1\begin{bmatrix} g_\lambda^+(\xi^+(0)) \\ \xi^-(0)\end{bmatrix} + \phi(K_1\begin{bmatrix} g_\lambda^+(\xi^+(0)) \\ \xi^-(0)\end{bmatrix}), \tag{2.39}$$

$$\xi^-(L) = K_2\begin{bmatrix} \xi^+(L) \\ g_\lambda^-(\xi^-(L))\end{bmatrix} + \phi(K_2\begin{bmatrix} \xi^+(L) \\ g_\lambda^-(\xi^-(L))\end{bmatrix}), \tag{2.40}$$

where $K_1 \in R^{m\times n}$ and $K_2 \in R^{(n-m)\times n}$ denotes the first m and the last $n-m$ rows of the matrix K, respectively. Therefore, (2.35) has a solution if and only if there exist $\xi^+(0)$ and $\xi^-(L)$ satisfying (2.39) and (2.40).

Now, we introduce the map

$$\begin{aligned}&\mathcal{T}^+ : R^m \to R^m\\ &\qquad c \mapsto K_1\begin{bmatrix} g_\lambda^+(c) \\ \xi^-(0)\end{bmatrix} + \phi(K_1\begin{bmatrix} g_\lambda^+(c) \\ \xi^-(0)\end{bmatrix}); \qquad (2.41)\\ &\mathcal{T}^- : R^{n-m} \to R^{n-m}\\ &\qquad d \mapsto K_2\begin{bmatrix} \xi^+(L) \\ g_\lambda^-(d)\end{bmatrix} + \phi(K_2\begin{bmatrix} \xi^+(L) \\ g_\lambda^-(d)\end{bmatrix}). \qquad (2.42)\end{aligned}$$

Then we use Banach fixed point theorem in [13] for $\mathcal{T}^+$, $\mathcal{T}^-$. In order to show that $\mathcal{T}^+$ and $\mathcal{T}^-$ are contractions, it's easy to write that

$$g_\lambda^+(c_1) - g_\lambda^+(c_2) = \mathrm{e}^{\frac{1}{\lambda}\Gamma^{-1}(\xi^+(L))I_\kappa^+ L}(c_1 - c_2), \quad \forall c_1,\ c_2 \in R^m, \tag{2.43}$$

$$g_\lambda^-(d_1) - g_\lambda^-(d_2) = d_1 - d_2, \qquad \forall d_1,\ d_2 \in R^{n-m}. \tag{2.44}$$

Since ϕ is a Lipschitz continuous function, by using (2.43) and (2.44) it follows that

$$\left\|\phi(K_1\begin{bmatrix} g_\lambda^+(c_1) \\ \xi^-(0)\end{bmatrix}) - \phi(K_1\begin{bmatrix} g_\lambda^+(c_2) \\ \xi^-(0)\end{bmatrix})\right\| \le \|K_1^{'}\mathrm{e}^{\frac{1}{\lambda}\Gamma^{-1}(\xi^+(L))I_\kappa^+ L}\|\|c_1 - c_2\|, \tag{2.45}$$

$$\left\|\phi(K_2\begin{bmatrix} \xi^+(L) \\ g_\lambda^-(d_1)\end{bmatrix}) - \phi(K_2\begin{bmatrix} \xi^+(L) \\ g_\lambda^-(d_2)\end{bmatrix})\right\| \le \|K_2^{'}\|\|d_1 - d_2\|, \tag{2.46}$$

for all $c_1, c_2 \in R^m$, $d_1, d_2 \in R^{n-m}$, where $K_1^{'} \in R^{m\times m}$ and $K_2^{'} \in R^{(n-m)\times(n-m)}$ denotes the first m and the last $n-m$ columns of the matrix K_1 and K_2, respectively.

Based on the above formulas, we finally have

$$\begin{aligned}\big\|\mathcal{T}^+(c_1) - \mathcal{T}^+(c_2)\big\| &\le 2\|K_1^{'}\mathrm{e}^{\frac{1}{\lambda}\Gamma^{-1}(\xi^+(L))I_\kappa^+ L}\|\|c_1 - c_2\|,\\ &\le 2\|\mathrm{e}^{\frac{\kappa L}{\lambda_{max}(\Gamma^{-1}(\xi^+(L)))}}K_1^{'}\|\|c_1 - c_2\|, \qquad (2.47)\end{aligned}$$

$$\big\|\mathcal{T}^-(d_1)-\mathcal{T}^-(d_2)\big\| \le 2\|K_2'\|\|d_1-d_2\|, \tag{2.48}$$

where $\lambda_{max}(\Gamma^{-1}(\xi^+(L)))$ is the largest eigenvalue of the matrix $\Gamma^{-1}(\xi^+(L))$. By choosing $\|K_2'\| < \frac{1}{2}$ and $\kappa \in R$ such that $\mathrm{e}^{\frac{\kappa L}{\lambda_{max}(\Gamma^{-1}(\xi^+(L)))}}\|K_1'\| < \frac{1}{2}$, it holds that

$$\|\mathcal{T}^+(c_1)-\mathcal{T}^+(c_2)\| \le \|c_1-c_2\|, \tag{2.49}$$
$$\|\mathcal{T}^-(d_1)-\mathcal{T}^-(d_2)\| \le \|d_1-d_2\|. \tag{2.50}$$

This completes the proof of Lemma 2.6.

Proof of Theorem 2.4: The choice of $\xi_0 \in H^2_\mu((0,L);R^n)$ is equivalent to $\xi_0 \in H^2((0,L);R^n)$ where H^2_μ is defined by the norm induced by the scalar product in (2.10). By means of Lemma 2.5 and Lemma 2.6, the operator $\mathcal{A}+\kappa I$ is non-accretive and satisfies the range condition, thus the Cauchy problem (2.1), (2.2), (2.5) has a unique mild solution according to Theorem 4.3 in [1].

3. Saturated boundary stabilization of quasi-linear hyperbolic balance laws

In this section, sufficient conditions for guaranteeing the locally exponential stability of quasi-linear hyperbolic system of balance laws (2.1) under the saturated boundary feedback control (2.2) are derived by employing the Lyapunov function method along with a sector condition.

Due to hard discontinuities of the saturation function, it is difficult to make the stability analysis by directly using Lyapunov-like inequalities. Hence, appropriate models are particularly important to take saturation effects into account when deriving stabilization conditions. Before giving the main results of this paper, the sector condition is given to relax stabilization conditions by introducing the dead-zone nonlinearity (2.4).

Lemma 3.1 ([28])**.** *For all $u \in R^n$, the nonlinearity $\phi(u)$ satisfies the inequality*

$$\phi^\top(u)\Theta(\phi(u)+u) \le 0, \tag{3.1}$$

for any diagonal positive definite matrix $\Theta \in R^{n\times n}$.

The sector condition used is globally valid here. Considering $u = K\xi^{out}$, we can easily obtain a sector condition with respect to ξ^{out} following Lemma 3.1.

Lemma 3.2. *For all $K\xi^{out} \in R^n$, it holds that*

$$\phi^\top(K\xi^{out})\Theta(\phi(K\xi^{out})+K\xi^{out}) \le 0, \tag{3.2}$$

where $\Theta \in R^{n\times n}$ is a diagonal positive definite matrix.

Then the nonlinear influence caused by saturations can be disposed in the condition of convergence based on Lemma 3.2 by transforming the saturation term $sat(K\xi^{out})$ into the form of dead-time nonlinearity (2.4). The main result of the paper is shown as follows.

Theorem 3.3. *System (2.1) is locally exponentially stable for H^2-norm under saturated boundary feedback controller (2.2), if there exist diagonal positive matrices $P \in R^{n\times n}$ and $\Theta \in R^{n\times n}$ such that the following inequalities hold for all $x \in [0,L]$:*

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} < 0, \tag{3.3}$$

$$\Xi = -\mu|\Gamma(0)|P(x) + P(x)M + M^\top P(x) < 0, \tag{3.4}$$

where

$$\begin{aligned}\Psi_{11} &= \mathrm{e}^{\mu L}K^\top|\Gamma(0)|PK - |\Gamma(0)|P,\\ \Psi_{12} &= \mathrm{e}^{\mu L}K^\top|\Gamma(0)|P - K^\top\Theta,\\ \Psi_{22} &= \mathrm{e}^{\mu L}|\Gamma(0)|P - 2\Theta,\end{aligned}$$

with $P(x) = diag\{\mathrm{e}^{\mu(L-x)}I_m, \mathrm{e}^{\mu x}I_{n-m}\}P$, $|\Gamma(0)| = diag\{|\gamma_i(0)|\}$.

Remark 3.4. Recalling the work [6] on the boundary control for quasi-linear hyperbolic PDE systems, sufficient conditions for ensuring exponential stability were obtained by using an explicit strict Lyapunov function. As an extension of the results in [6], this paper takes the boundary input saturation into account, and then the stability conditions need to be relaxed by the sector condition when using the Lyapunov-like inequalities, which is presented in the following Proof.

Proof: Consider the Lyapunov function candidate V as follows

$$V = V_1(\xi) + V_2(\xi_t) + V_3(\xi_{tt}) \tag{3.5}$$

with

$$\begin{aligned}V_1(\xi) &= \int_0^L \xi^\top P(x)\xi \mathrm{d}x,\\ V_2(\xi_t) &= \int_0^L \xi_t^\top P(x)\xi_t \mathrm{d}x,\\ V_3(\xi_{tt}) &= \int_0^L \xi_{tt}^\top P(x)\xi_{tt} \mathrm{d}x,\end{aligned}$$

where $\xi_t = \partial_t\xi$ and $\xi_{tt} = \partial_{tt}\xi$ for simplicity.

The proof of Theorem 3.3 mainly relies on the analysis of estimates for the time derivatives of $V_i, i = 1, 2, 3$, along solutions of system (2.1)–(2.2), by expanding the analysis to the dynamics of ξ_{tt} with the assumption that solutions ξ are of class C^2.

• *Analysis of the first term* $V_1(\xi)$.

Taking the time derivative of $V_1(\xi)$ along the solution of system (2.1), we have

$$\begin{aligned}\dot V_1(\xi) &= \int_0^L (\xi_t^\top P(x)\xi + \xi^\top P(x)\xi_t)\mathrm{d}x\\ &= \int_0^L ((\xi^\top M^\top - \xi_x^\top \Gamma(\xi)^\top)P(x)\xi + \xi^\top P(x)(M\xi - \Gamma(\xi)\xi_x))\mathrm{d}x\\ &= -\left[\xi^\top\Gamma(\xi)P(x)\xi\right]_0^L + \int_0^L \xi^\top\Big(\big(\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_x - \mu|\Gamma(\xi)|\big)P(x)\\ &\quad + P(x)M + M^\top P(x)\Big)\xi\mathrm{d}x\\ &= \xi^\top(0,t)\Gamma(\xi(0))P(0)\xi(0,t) - \xi^\top(L,t)\Gamma(\xi(L))P(L)\xi(L,t)\\ &\quad + \int_0^L \xi^\top\big((\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_x - \mu|\Gamma(\xi)| + M^\top)P(x) + P(x)M\big)\xi\mathrm{d}x\end{aligned}$$

$$= (\xi^{in})^\top |\Gamma_1(\xi)| P \mathrm{e}^{\mu L} \xi^{in} - (\xi^{out})^\top |\Gamma_2(\xi)| P \xi^{out} + \int_0^L \xi^\top \big((M^\top - \mu|\Gamma(\xi)| + \frac{\partial \Gamma(\xi)}{\partial \xi}\xi_x)P(x) + P(x)M\big)\xi \mathrm{d}x,$$

with $|\Gamma_1(\xi)| = \mathrm{diag}\{\gamma_1(\xi(0)), ..., \gamma_m(\xi(0)), \gamma_{m+1}(\xi(L)), ..., \gamma_n(\xi(L))\}$ and $|\Gamma_2(\xi)| = \mathrm{diag}\{\gamma_1(\xi(L)), ..., \gamma_m(\xi(L)), \gamma_{m+1}(\xi(0)), ..., \gamma_n(\xi(0))\}$.

According to the dead-zone nonlinearity defined in (2.4), the boundary condition (2.2) equals to

$$\xi^{in} = \phi(K\xi^{out}) + K\xi^{out}. \tag{3.6}$$

Then the time derivative $\dot{V}_1(\xi)$ can be written as

$$\begin{aligned} \dot{V}_1(\xi) = &\int_0^L \xi^\top \big((\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_x - \mu|\Gamma(\xi)| + M^\top)P(x) + P(x)M\big)\xi \mathrm{d}x \\ &+ (\phi^\top(K\xi^{out}) + (\xi^{out})^\top K^\top)|\Gamma_1(\xi)| P \mathrm{e}^{\mu L}(\phi(K\xi^{out}) \\ &+ K\xi^{out}) - (\xi^{out})^\top |\Gamma_2(\xi)| P \xi^{out}. \end{aligned} \tag{3.7}$$

By combining with the sector condition in Lemma 3.2, we further have

$$\begin{aligned} \dot{V}_1(\xi) &\le \dot{V}_1(\xi) - 2\phi^\top(K\xi^{out})\Theta(\phi(K\xi^{out}) + K\xi^{out}) \\ &\le \begin{bmatrix} \xi^{out} \\ \phi(K\xi^{out}) \end{bmatrix}^\top \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} \begin{bmatrix} \xi^{out} \\ \phi(K\xi^{out}) \end{bmatrix} + \int_0^L \xi^\top \Xi \xi \mathrm{d}x \\ &\quad + \mathcal{O}(\int_0^L (|\xi|^3 + |\xi|^2|\xi_t|)\mathrm{d}x; |\xi|_0) + \mathcal{O}(|\phi(K\xi^{out})|^2|\xi^{out}| + |\xi^{out}|^2 \\ &\quad |\phi(K\xi^{out})| + |\xi^{out}|^3 + |\phi(K\xi^{out})|^3; |\xi^{out}|_0 + |\phi(K\xi^{out})|_0). \end{aligned} \tag{3.8}$$

• *Analysis of the second term $V_2(\xi_t)$.*

Under the assumption that ξ is the class of C^2, then $\xi_t : [0, L] \times [0, T) \to R^n$ satisfies the following hyperbolic equation

$$\xi_{tt} + \Gamma(\xi)\xi_{tx} + [\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_t]\xi_x = M\xi_t, \tag{3.9}$$

with the boundary condition

$$(\xi_t^{in})_i = \begin{cases} (K\xi_t^{out})_i, & if \ -u_{0_i} \le (K\xi^{out})_i \le u_{0_i}, \\ 0, & else, \end{cases} \tag{3.10}$$

for $i = 1, ..., n$.

Then the time derivative of $V_2(\xi_t)$ along the solutions of equations (3.9)–(3.10) is calculated as

$$\begin{aligned} \dot{V}_2(\xi_t) &= \int_0^L (\xi_{tt}^\top P(x)\xi_t + \xi_t^\top P(x)\xi_{tt})\mathrm{d}x \\ &= \int_0^L \Big\{[M\xi_t - \Gamma(\xi)\xi_{tx} - (\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_t)\Gamma^{-1}(\xi)(M\xi - \xi_t)]^\top P(x)\xi_t \end{aligned}$$

$$
\begin{aligned}
&+ \xi_t^\top P(x)[M\xi_t - \Gamma(\xi)\xi_{tx} - (\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_t)\Gamma^{-1}(\xi)(M\xi - \xi_t)]\Big\}\mathrm{d}x \\
= & -\Big[\xi_t^\top \Gamma(\xi)P(x)\xi_t\Big]_0^L + \int_0^L \xi_t^\top(-\mu|\Gamma(\xi)|P(x) + M^\top P(x) \\
&+ P(x)M)\xi_t \mathrm{d}x - \int_0^L \xi_t^\top P(x)(\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_t)\Gamma^{-1}(\xi)(M\xi - \xi_t)\mathrm{d}x \\
&+ \int_0^L \Big\{\xi_t^\top(\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_x) - \big[(\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_t)\Gamma^{-1}(\xi)(M\xi - \xi_t)\big]^\top\Big\}P(x)\xi_t \mathrm{d}x \\
= & (\xi_t^{in})^\top |\Gamma_1(\xi)|P\mathrm{e}^{\mu L}\xi_t^{in} - (\xi_t^{out})^\top|\Gamma_2(\xi)|P\xi_t^{out} \\
&+ \int_0^L \xi_t^\top(-\mu|\Gamma(\xi)|P(x) + M^\top P(x) + P(x)M)\xi_t \mathrm{d}x \\
&+ \int_0^L \Big\{\xi_t^\top(\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_x) - \big[(\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_t)\Gamma^{-1}(\xi)(M\xi - \xi_t)\big]^\top\Big\}P(x)\xi_t \mathrm{d}x \\
&- \int_0^L \xi_t^\top P(x)(\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_t)\Gamma^{-1}(\xi)(M\xi - \xi_t)\mathrm{d}x. \qquad (3.11)
\end{aligned}
$$

Based on (3.10), it can be obtained that $(\xi_t^{in})_i^2 \leq (K\xi_t^{out})_i^2$. Hence, we get the estimate of time derivative $\dot{V}_2(\xi_t)$ as

$$
\begin{aligned}
\dot{V}_2(\xi_t) \leq & (\xi_t^{out})^\top \Psi_{11}\xi_t^{out} + \int_0^L \xi_t^\top \Xi \xi_t \mathrm{d}x + \mathcal{O}(\int_0^L (|\xi_t|^3|\xi| + |\xi_t|^2|\xi|^2 \\
&+ |\xi_t|^3 + |\xi_t|^2|\xi|)\mathrm{d}x; |\xi|_0 + |\xi_t|_0) + \mathcal{O}(|\xi_t^{out}|^2(|\xi^{out}| \\
&+ |\phi(K\xi^{out})|); |\xi^{out}|_0 + |\phi(K\xi^{out})|_0). \qquad (3.12)
\end{aligned}
$$

- *Analysis of the third term $V_3(\xi_{tt})$.*

For the same reason, by time differentiation of equations (3.9)–(3.10), the dynamic of $\xi_{tt} : [0, L] \times [0, T) \to R^n$ is governed by the hyperbolic equation

$$
\xi_{ttt} + \Gamma(\xi)\xi_{ttx} + 2[\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_t]\xi_{tx} + [\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_t]_t \xi_x = M\xi_{tt}, \qquad (3.13)
$$

with the boundary condition

$$
(\xi_{tt}^{in})_i = \begin{cases} (K\xi_{tt}^{out})_i, & if \ -u_{0_i} \leq (K\xi^{out})_i \leq u_{0_i}, \\ 0, & else, \end{cases} \qquad (3.14)
$$

for $i = 1, ..., n$.

The time derivative of $V_3(\xi_{tt})$ along the solutions of equations (3.13)–(3.14) is calculated as

$$
\begin{aligned}
\dot{V}_3(\xi_{tt}) = & \int_0^L (\xi_{ttt}^\top P(x)\xi_{tt} + \xi_{tt}^\top P(x)\xi_{ttt})\mathrm{d}x \\
= & \int_0^L \Big\{[M\xi_{tt} - \Gamma(\xi)\xi_{ttx} - 2\Gamma^{-1}(\xi)(\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_t)(M\xi_t - \xi_{tt} \\
&- (\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_t)\xi_x) - (\frac{\partial \Gamma(\xi)}{\partial \xi}\xi_t)_t\Gamma^{-1}(\xi)(M\xi - \xi_t)]^\top P(x)\xi_{tt}
\end{aligned}
$$

$$+ \xi_{tt}^\top P(x)[M\xi_{tt} - \Gamma(\xi)\xi_{ttx} - 2\Gamma^{-1}(\xi)(\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)(M\xi_t - \xi_{tt} - (\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)\xi_x) - (\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)_t\Gamma^{-1}(\xi)(M\xi - \xi_t)]\Big\}\mathrm{d}x.$$

Employing the integration by parts, we then get

$$\begin{aligned}
\dot{V}_3(\xi_{tt}) = & -\Big[\xi_{tt}^\top\Gamma(\xi)P(x)\xi_{tt}\Big]_0^L + \int_0^L \xi_{tt}^\top\big(-\mu|\Gamma(\xi)|P(x) + M^\top P(x) \\
& + P(x)M\big)\xi_{tt}\mathrm{d}x + \int_0^L \xi_{tt}^\top\big(\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_x\big)P(x)\xi_{tt}\mathrm{d}x + \int_0^L \big\{[-2\Gamma^{-1}(\xi) \\
& (\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)(M\xi_t - \xi_{tt} - (\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)\xi_x) - (\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)_t\Gamma^{-1}(\xi)(M\xi \\
& - \xi_t)]^\top P(x)\xi_{tt} + \xi_{tt}^\top P(x)[-2\Gamma^{-1}(\xi)(\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)(M\xi_t - \xi_{tt} \\
& - (\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)\xi_x) - (\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)_t\Gamma^{-1}(\xi)(M\xi - \xi_t)]\Big\}\mathrm{d}x \\
= & (\xi_{tt}^{in})^\top|\Gamma_1(\xi)|P\mathrm{e}^{\mu L}\xi_{tt}^{in} - (\xi_{tt}^{out})^\top|\Gamma_2(\xi)|P\xi_{tt}^{out} \\
& + \int_0^L \xi_{tt}^\top\big(-\mu|\Gamma(\xi)|P(x) + M^\top P(x) + P(x)M\big)\xi_{tt}\mathrm{d}x \\
& + \int_0^L \xi_{tt}^\top\big(\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_x\big)P(x)\xi_{tt}\mathrm{d}x + \int_0^L \big\{[-2\Gamma^{-1}(\xi)(\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)(M\xi_t \\
& - \xi_{tt} - (\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)\xi_x) - (\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)_t\Gamma^{-1}(\xi)(M\xi - \xi_t)]^\top P(x)\xi_{tt} \\
& + \xi_{tt}^\top P(x)[-2\Gamma^{-1}(\xi)(\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)(M\xi_t - \xi_{tt} - (\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)\xi_x) \\
& - (\frac{\partial\Gamma(\xi)}{\partial\xi}\xi_t)_t\Gamma^{-1}(\xi)(M\xi - \xi_t)]\Big\}\mathrm{d}x. \qquad (3.15)
\end{aligned}$$

Based on (3.14), it can be obtained that $(\xi_{tt}^{in})_i^2 \leq (K\xi_{tt}^{out})_i^2$, then we get the estimate of time derivative $\dot{V}_3(\xi_{tt})$ as

$$\begin{aligned}
\dot{V}_3(\xi_{tt}) \leq & (\xi_{tt}^{out})^\top\Psi_{11}\xi_{tt}^{out} + \int_0^L \xi_{tt}^\top\Xi\xi_{tt}\mathrm{d}x + \mathcal{O}(|\xi_{tt}^{out}|^2(|\xi^{out}| + |\phi(K\xi^{out})|); |\xi^{out}|_0 \\
& + |\phi(K\xi^{out})|_0) + \mathcal{O}(\int_0^L \big(|\xi||\xi_t|^2|\xi_{tt}|(1+|\xi|+|\xi_t|) + |\xi_{tt}|^2(|\xi|+|\xi_t|+|\xi|^2 \\
& + |\xi||\xi_t|)\big)\mathrm{d}x; |\xi|_0 + |\xi_t|_0). \qquad (3.16)
\end{aligned}$$

For every solution $\xi : [0, L] \times [0, T) \to R^n$, it follows from (3.8), (3.12) and (3.16) that

$$\begin{aligned}
\dot{V} \leq & \dot{V} - 2\phi^\top(K\xi^{out})\Theta(\phi(K\xi^{out}) + K\xi^{out}) \\
\leq & \int_0^L \xi^\top\Xi\xi\mathrm{d}x + \int_0^L \xi_t^\top\Xi\xi_t\mathrm{d}x + \int_0^L \xi_{tt}^\top\Xi\xi_{tt}\mathrm{d}x + (\xi_{tt}^{out})^\top\Psi_{11}\xi_{tt}^{out} \\
& + (\xi_t^{out})^\top\Psi_{11}\xi_t^{out} + \begin{bmatrix} \xi^{out} \\ \phi(K\xi^{out}) \end{bmatrix}^\top \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} \begin{bmatrix} \xi^{out} \\ \phi(K\xi^{out}) \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&+\mathcal{O}(\int_0^L \big(|\xi|^2(|\xi|+|\xi_t|)+|\xi_t|^2(|\xi|+|\xi_t|+|\xi||\xi_t|+|\xi|^2)+|\xi||\xi_t|^2\\
&\quad |\xi_{tt}|(1+|\xi|+|\xi_t|)+|\xi_{tt}|^2(|\xi|+|\xi_t|+|\xi|^2+|\xi||\xi_t|)\big)\mathrm{d}x;|\xi|_0+|\xi_t|_0)\\
&+\mathcal{O}((|\xi^{out}|^2+|\xi_t^{out}|^2+|\xi_{tt}^{out}|^2+|\phi(K\xi^{out})|^2)(|\xi^{out}|+|\phi(K\xi^{out})|);\\
&\quad |\xi^{out}|_0+|\phi(K\xi^{out})|_0).
\end{aligned}
\tag{3.17}
$$

Since $\Psi<0$, there exist $\varpi_1>0$ and $\varepsilon_1>0$ such that when $|\xi^{out}|_0+|\phi(K\xi^{out})|_0<\varepsilon_1$, the above formula yields that

$$
\begin{aligned}
\dot V \le\, &-\varpi_1\big(|[(\xi^{out})^\top,\phi^\top(K\xi^{out})]^\top|^2+|\xi_t^{out}|^2+|\xi_{tt}^{out}|^2\big)\\
&+\int_0^L \xi^\top\Xi\xi\mathrm{d}x+\int_0^L \xi_t^\top\Xi\xi_t\mathrm{d}x+\int_0^L \xi_{tt}^\top\Xi\xi_{tt}\mathrm{d}x\\
&+\mathcal{O}(\int_0^L \big(|\xi|^2(|\xi|+|\xi_t|)+|\xi_t|^2(|\xi|+|\xi_t|+|\xi||\xi_t|+|\xi|^2)+|\xi||\xi_t|^2\\
&\quad |\xi_{tt}|(1+|\xi|+|\xi_t|)+|\xi_{tt}|^2(|\xi|+|\xi_t|+|\xi|^2+|\xi||\xi_t|)\big)\mathrm{d}x;|\xi|_0+|\xi_t|_0).
\end{aligned}
\tag{3.18}
$$

For every $\varsigma>0$, using the Young's inequality, we have

$$
\int_0^L|\xi_t|^2|\xi_{tt}|\mathrm{d}x\le|\xi_t|_0\int_0^L|\xi_t||\xi_{tt}|\mathrm{d}x\le\frac{1}{4\varsigma}|\xi_t|_0\int_0^L|\xi_t|^2\mathrm{d}x+\varsigma|\xi_t|_0\int_0^L|\xi_{tt}|^2\mathrm{d}x. \tag{3.19}
$$

It implies that there exist $\varpi_2>0$ and $\varepsilon_2>0$ such that if $|\xi|_0+|\xi_t|_0<\varepsilon_2$, then

$$
\begin{aligned}
&\mathcal{O}(\int_0^L \big(|\xi|^2(|\xi|+|\xi_t|)+|\xi_t|^2(|\xi|+|\xi_t|+|\xi||\xi_t|+|\xi|^2)+|\xi||\xi_t|^2|\xi_{tt}|\\
&\quad (1+|\xi|+|\xi_t|)+|\xi_{tt}|^2(|\xi|+|\xi_t|+|\xi|^2+|\xi||\xi_t|)\big)\mathrm{d}x;|\xi|_0+|\xi_t|_0)\\
&\le\varpi_2 V.
\end{aligned}
\tag{3.20}
$$

Moreover, since $\Xi<0$, there exists $\varpi_3>0$ such that

$$
-\varpi_1\big(|[(\xi^{out})^\top,\phi^\top(K\xi^{out})]^\top|^2+|\xi_t^{out}|^2+|\xi_{tt}^{out}|^2\big)+\int_0^L\xi^\top\Xi\xi\mathrm{d}x+\int_0^L\xi_t^\top\Xi\xi_t\mathrm{d}x+\int_0^L\xi_{tt}^\top\Xi\xi_{tt}\mathrm{d}x\le-\varpi_3V. \tag{3.21}
$$

There exist $\varepsilon\le\min\{\varepsilon_1,\varepsilon_2\}$ and $\varpi>0$ such that if $|\xi|_0+|\xi_t|_0<\varepsilon$, then

$$
\dot V\le(-\varpi_3+\varpi_2)V\le-\varpi V. \tag{3.22}
$$

From the system of equations (2.1) and (3.9), and the definition of Lyapunov function (3.5), we can directly obtain that there always exists a sufficiently large constant $\tau>0$, such that if $|\xi|_0+|\xi_t|_0<\varepsilon$, the following inequality holds

$$
\frac{1}{\tau}\Big(\int_0^L(|\xi|^2+|\xi_x|^2+|\xi_{xx}|^2)\mathrm{d}x\Big)\le V\le\tau\Big(\int_0^L(|\xi|^2+|\xi_x|^2+|\xi_{xx}|^2)\mathrm{d}x\Big). \tag{3.23}
$$

In the above discussion, estimates (3.22) and (3.23) are obtained under the assumption that ξ is in class of C^2, while the selection of ϖ,ε and τ depends on $C^0([0,T];H^2((0,L);R^n))$-norm of ξ (see Comment 4.6 in [2]).

Hence, using the density argument, the estimates (3.22) and (3.23) remain valid in the distribution sense with ξ being only of class C^1.

For every ξ in the Sobolev space $H^2((0,L);R^n)$, by utilizing the Sobolev inequality in [22], there exists $\varphi_1 > 0$ such that

$$|\xi|_0 + |\xi_x|_0 \leq \varphi_1 \|\xi\|_{H^2((0,L);R^n)}. \tag{3.24}$$

In order to take the time derivative instead of the space derivative and use (3.22)–(3.23), we directly recall the results in [6] that there exists $\varphi_2 > 0$ and $\varepsilon > 0$ such that if $|\xi|_0 + |\xi_t|_0 < \varepsilon$, then

$$|\xi_t| \leq \varphi_2(|\xi| + |\xi_x|), \tag{3.25}$$
$$|\xi_{tt}| \leq \varphi_2(|\xi| + |\xi_x| + |\xi_{xx}|). \tag{3.26}$$

Therefore, there exists $\varphi_0 > \varphi_1$ such that

$$|\xi|_0 + |\xi_t|_0 \leq \varphi_0 \|\xi\|_{H^2((0,L);R^n)}, \tag{3.27}$$

for all $|\xi|_0 + |\xi_t|_0 < \varepsilon$. Let

$$\delta = \min\{\frac{\varepsilon}{2\varphi_0 \tau}, \frac{\varepsilon_0}{\tau}\}, \tag{3.28}$$

with $\tau \geq 1$, $\varepsilon_0 > 0$, and hence $\delta \leq \varepsilon \leq \varepsilon_0$. Based on (3.22)–(3.24), (3.27)–(3.28), the following implications hold for every $t \in [0,T]$:

$$\|\xi(\cdot,t)\|_{H^2((0,L);R^n)} \leq \delta \Rightarrow |\xi|_0 + |\xi_x|_0 \leq \frac{\varepsilon}{2} \ \ and \ \ V \leq \tau\delta^2, \tag{3.29}$$
$$\Rightarrow |\xi|_0 + |\xi_t|_0 \leq \frac{\varepsilon}{2} \ \ and \ \ \|\xi(\cdot,t)\|_{H^2((0,L);R^n)} \leq \varepsilon_0, \tag{3.30}$$
$$\Rightarrow \dot{V} \leq 0. \tag{3.31}$$

Let $\|\xi_0\|_{H^2((0,L);R^n)} < \delta$ and $\xi \in C^2([0,T^*);\ H^2((0,L);R^n))$ be the maximal solution of the Cauchy problem (2.1), (2.2), (2.5). Based on the implication (3.29)–(3.31) for $T \in [0,T^*]$, we obtain that

$$\|\xi(\cdot,t)\|_{H^2((0,L);R^n)} \leq \varepsilon_0, \quad \forall t \in [0,T^*), \tag{3.32}$$
$$|\xi(\cdot,t)|_0 + |\xi_t(\cdot,t)|_0 \leq \varepsilon, \quad \forall t \in [0,T^*). \tag{3.33}$$

We have that $T = +\infty$ by applying the above result to $[0,T^*]$, $[T^*,2T^*]$, $[2T^*,3T^*]$, ..., with T^* given in $\big(0, \min\{\frac{L}{\gamma_1}, ..., \frac{L}{\gamma_n}\}\big)$. Using (3.22)–(3.23) and (3.33), we finally obtain that

$$\|\xi(\cdot,t)\|^2_{H^2((0,L);R^n)} \leq \tau V \leq \tau \mathrm{e}^{-\varpi t} V(0) \leq \tau^2 \mathrm{e}^{-\varpi t} \|\xi_0\|^2_{H^2((0,L);R^n)}. \tag{3.34}$$

This completes the proof of Theorem 3.3.

4. Application to traffic flow stabilization

In this section, we apply the proposed theoretical result to the saturated boundary feedback control of two-lane traffic flow with lane-changing interactions. VSL devices are located at boundaries to limit the driving speed of vehicles in order to regulate the traffic densities of two lanes. Traffic densities are limited for the reason of limited speed requirements, and then the saturated boundary feedback control is proposed for ensuring the traffic densities converge to the steady states.

4.1. Two-lane traffic flow based on LWR model

The dynamics of two-lane traffic flow with lane-changing interactions are described by the LWR model, which are hyperbolic system of balance laws as

$$\begin{cases} \partial_t \rho_1 + \partial_x(\rho_1 v_1) = -\sigma_1 \rho_1 + \sigma_2 \rho_2, \\ \partial_t \rho_2 + \partial_x(\rho_2 v_2) = \sigma_1 \rho_1 - \sigma_2 \rho_2, \end{cases} \tag{4.1}$$

where traffic density $\rho_i(x,t)$ and speed $v_i(x,t)$, $i = 1,2$, are defined in $x \in [0, L]$ for position and $t \in [0, +\infty)$ for time, and L is the length of the road. The above hyperbolic PDEs consist of two LWR models, each of which describes one-lane traffic dynamics. Lane-changing interactions and drivers' behavior adapting to the traffic appear as source terms on the right hand side of PDEs, in which the parameter σ_i describes the drivers' preference for remaining in lane i, which relates to the density and speed of both lanes.

The density and speed relationship is given by the Greenshield's model in [11]:

$$v_i(x,t) = v_f\left(1 - \frac{\rho_i(x,t)}{\rho_m}\right), \tag{4.2}$$

where v_f is the maximum speed, ρ_m is the maximum density. Substituting the density-speed relation (4.2) into system (4.1), we obtain

$$\begin{cases} \partial_t \rho_1 + (v_f - 2a\rho_1)\partial_x \rho_1 = -\sigma_1 \rho_1 + \sigma_2 \rho_2, \\ \partial_t \rho_2 + (v_f - 2a\rho_2)\partial_x \rho_2 = \sigma_1 \rho_1 - \sigma_2 \rho_2, \end{cases} \tag{4.3}$$

with $a = v_f/\rho_m$.

According to the sign of the characteristic eigenvalue $v_f - 2a\rho_i$, $i = 1, 2$ of system (4.3), the dynamics of the two-line traffic are divided into the free-flow regime and the congestion regime, respectively.

In this paper, we assume that the first lane of the two-lane traffic is in the free-flow regime, *i.e.* $v_f - 2a\rho_1 > 0$, indicating that the speed information is transmitted from upstream to downstream. While the second lane is in the congestion regime, *i.e.* $v_f - 2a\rho_2 < 0$, in which case the speed information is transmitted from downstream to upstream.

Denote (ρ_1^*, ρ_2^*) as the steady state of the two-lane traffic flow system (4.3), which satisfies the balance condition of lane-changing that $\sigma_1/\sigma_2 = \rho_2^*/\rho_1^*$. The deviation of traffic density ρ_i from the steady state is defined as $\tilde{\rho}_i = \rho_i - \rho_i^*$, $i = 1, 2$. Letting $\xi = [\tilde{\rho}_1, \tilde{\rho}_2]^\top$, the two-lane LWR quasi-linear hyperbolic system of balance laws is given as

$$\partial_t \xi + \Gamma(\xi)\partial_x \xi = M\xi, \quad t \in [0, \infty), \quad x \in [0, L], \tag{4.4}$$

in which the system matrices are

$$\Gamma(\xi) = \begin{bmatrix} \gamma_1(\tilde{\rho}_1) & 0 \\ 0 & -\gamma_2(\tilde{\rho}_2) \end{bmatrix}, \quad M = \begin{bmatrix} -\sigma_1 & \sigma_2 \\ \sigma_1 & -\sigma_2 \end{bmatrix},$$

with $\gamma_i(\tilde{\rho}_i) = v_f - 2a\rho_i^* - 2a\tilde{\rho}_i, i = 1, 2$.

4.2. Saturated boundary feedback control

In this subsection, we design the saturated boundary feedback controller to stabilize the traffic densities of the quasi-linear LWR traffic flow model (4.4). Selecting two VSL devices as the practical actuators for the boundary feedback control of two-lane traffic flow, which are located at the boundaries $x = 0$ and $x = L$ of the road, respectively.

Due to the fact that the management of vehicles by VSLs makes vehicle speed in two lanes subject to saturation constraints, the minimum speed limit for lane i is $v_i^* - u_{0_i}^v$ and the maximum one is $v_i^* + u_{0_i}^v$. Then we have

$$\begin{cases} v_1(0,t) = v_1^* + sat(k_1(v_1(L,t) - v_1^*)), \\ v_2(L,t) = v_2^* + sat(k_2(v_2(0,t) - v_2^*)), \end{cases} \tag{4.5}$$

with the saturated levels $u_{0_i}^v > 0,\ i = 1,2$, where $k_1, k_2 \in R$ are tuning gains.

With consideration of the linear density-speed relationship of the Greenshield's model (4.2), the velocities at the boundaries of two lanes $v_1(0,t), v_2(L,t)$ can be transformed into the inflow density $\rho_1(0,t)$ of lane 1 and outflow density $\rho_2(L,t)$ of lane 2:

$$\begin{cases} v_1(0,t) = v_f - a\rho_1(0,t), \\ v_2(L,t) = v_f - a\rho_2(L,t). \end{cases} \tag{4.6}$$

Substituting (4.6) into (4.5), we have

$$\begin{cases} \rho_1(0,t) = \rho_1^* + sat(k_1(\rho_1(L,t) - \rho_1^*)), \\ \rho_2(L,t) = \rho_2^* + sat(k_2(\rho_2(0,t) - \rho_2^*)), \end{cases} \tag{4.7}$$

with the saturated levels $u_{0_i} = u_{0_i}^v/a,\ i = 1,2$.

We then obtain the following saturated boundary condition for two-lane LWR model:

$$\begin{bmatrix} \tilde{\rho_1}(0,t) \\ \tilde{\rho_2}(L,t) \end{bmatrix} = sat(K \begin{bmatrix} \tilde{\rho_1}(L,t) \\ \tilde{\rho_2}(0,t) \end{bmatrix}), \tag{4.8}$$

in which the control gain is $K = \text{diag}\{k_1, k_2\}$, and the saturated level $u_0 = [u_{0_1}, u_{0_2}]^\top$.

4.3. Numerical simulations

The purpose of this section is to verify the effectiveness of the local exponential stability conditions proposed in Theorem 3.3. The existence of the spatial variable $x \in [0, L]$ makes the number of matrix inequalities infinite, hence, the overapproximation technique is introduced in order to numerically check it. Then the numerical simulations are given to show the validity of the theoretical results.

4.3.1. Overapproximation technique

Here we provide a way to numerically verify conditions of Theorem 3.3 by using overapproximation technique, which means that the original constraints are embedded in a larger set with nice structural properties to be exploited.

For fixed $\mu \in R$ and $P \in R^{n\times n}$, define the following matrix

$$P_{ij} = P(x) = \text{diag}\{e^{\mu(L-i)} I_m, e^{\mu j} I_{n-m}\}P, \quad i,j = 0, L, \tag{4.9}$$

TABLE 1. Traffic parameters and steady states in the simulation.

Parameters	Values	Unit
ρ_1^*	70	veh./km
v_1^*	67.5	km/hr
ρ_2^*	90	veh./km
v_2^*	52.5	km/hr
ρ_m	160	veh./km
v_f	120	km/hr
a	0.75	km^2/veh./hr

which satisfies the property as stated in the following lemma.

Lemma 4.1 ([17]). *For all $x \in [0, L]$, $P(x)$ lies in the convex hull of $\{P_{00}, P_{L0}, P_{LL}\}$ when $\mu > 0$.*

Then we have the following results which reduce the infinite matrix inequalities to a finite number of matrix inequalities by using polytopic embedding.

Proposition 4.2. *If there exist diagonal positive matrices $P \in R^{n\times n}$ and $\Theta \in R^{n\times n}$ such that the following inequalities*

$$\Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ * & \Psi_{22} \end{bmatrix} < 0, \tag{4.10}$$

$$-\mu|\Gamma(0)|P_{ij} + P_{ij}M + M^\top P_{ij} < 0, \tag{4.11}$$

hold for all $(i, j) \in \{(0, 0), (L, 0), (L, L)\}$, where Ψ and $|\Gamma(0)|$ are the same as those in Theorem 3.3, μ is a real positive scalar, then conditions (3.3) and (3.4) are satisfied for all $x \in [0, L]$.

4.3.2. Simulation results

The proposed saturated boundary feedback control law (4.7) for stabilizing two-lane traffic with lane-changing interactions is now verified with the numerical simulations. Select two lanes with length of $L = 1$ kilometer. The traffic parameters and the steady states of the two lanes are shown in Table 1. The steady states $(\rho_1^*, \rho_2^*) = (70, 90)$ satisfy the LWR equations with the characteristic eigenvalue $v_f - 2a\rho_1^* = 15 > 0$ for the free-flow regime and $v_f - 2a\rho_2^* = -15 < 0$ for the congestion regime, respectively. Select the boundary feedback control gains in (4.7) as $k_1 = 0.38, k_2 = 0.42$ with the actuator upper bound $u_{0_1} = 2.84$, $u_{0_2} = 2.76$. Take $\mu = 0.1$ and the lane-changing parameters as $\sigma_1 = 0.32, \sigma_2 = 0.2489$.

All the constraints of the matrix inequalities in Theorem 3.3 are enclosed by the overapproximation with the polytope described by $\{P_{00}, P_{L0}, P_{LL}\}$, and then we obtain the following diagonal matrices by solving the conditions (4.10) and (4.11),

$$P = \begin{bmatrix} 2.5676 & 0 \\ 0 & 2.5913 \end{bmatrix}, \qquad \Theta = \begin{bmatrix} 32.7919 & 0 \\ 0 & 33.0526 \end{bmatrix}.$$

To compute the solutions of system (4.4), we discretize them by using the two-step variant of Lax–Wendroff method in [26]. The initial states associated with the steady states (ρ_1^*, ρ_2^*) are given as

$$\begin{cases} \rho_1(x, 0) = \rho_1^* + 10cos(5\pi x), \\ \rho_2(x, 0) = \rho_2^* + 10cos(5\pi x). \end{cases}$$

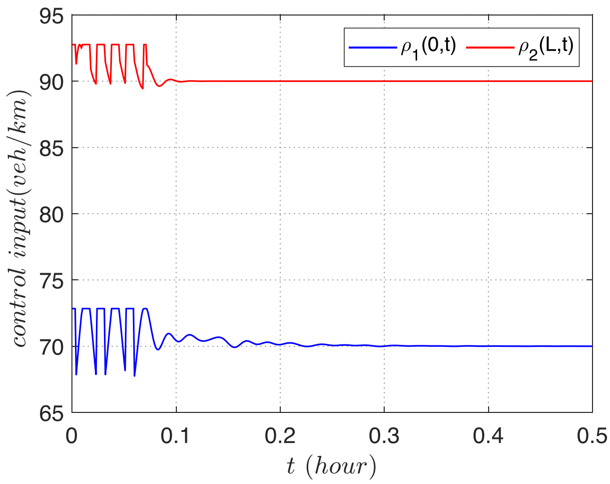

FIGURE 1. The evolution of saturated boundary feedback controller.

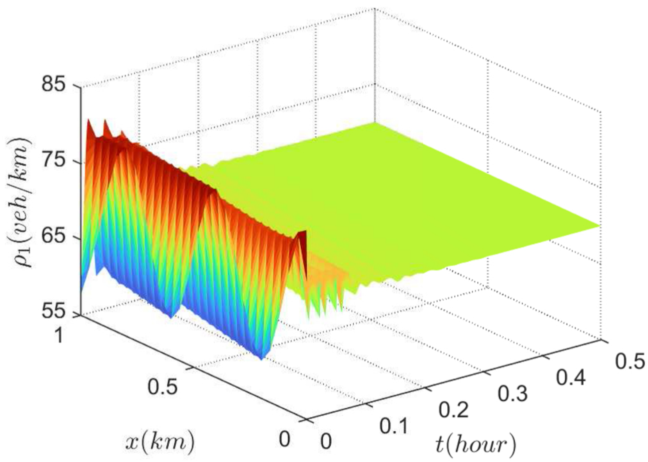

FIGURE 2. The evolution of traffic density ρ_1 of system (4.3) under the saturated boundary feedback control (4.7) with steady state $\rho_1^* = 70$.

Figure 1 shows the curves of the saturated boundary feedback controller (4.7) for the two-lane LWR traffic model with lane-changing behaviors. It's clearly indicated that although the control signals are saturated at the initial moment and reach saturations several times before 0.1h, there are enough control capacities to drive the system states to the steady values within 0.5h.

Figure 2 and Figure 3 show the evolution of traffic densities of the two lanes. It can be observed that the densities in the free-flow regime and the congestion regime can converge to $\rho_1^* = 70$ $veh./km$ and $\rho_2^* = 90$ $veh./km$, respectively. Despite the presence of strong nonlinearity induced by saturation, system (4.4) is exponentially stable under the introduced saturated boundary feedback control. As revealed in the simulation results, the sufficient conditions for ensuring the exponential stability provided in Theorem 3.3 are numerically validated.

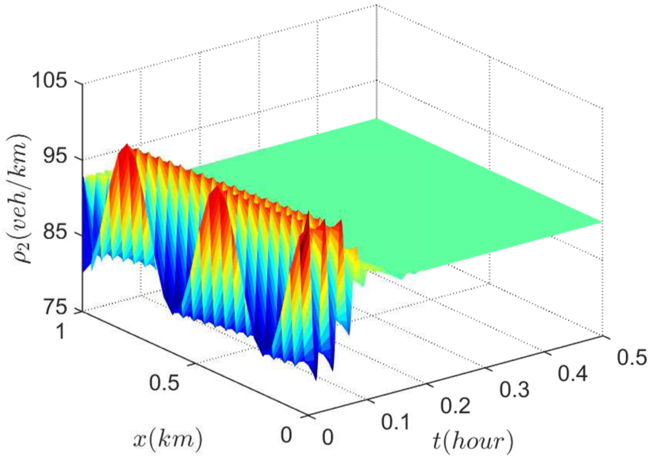

FIGURE 3. The evolution of traffic density ρ_2 of system (4.3) under the saturated boundary feedback control (4.7) with steady state $\rho_2^* = 90$.

5. CONCLUSIONS

This paper has addressed the saturated boundary feedback control problem for the quasi-linear hyperbolic system of balance laws in H^2-norm. By employing the Lyapunov function method along with a sector condition dealing with the saturation term, sufficient conditions are given for ensuring the classical Cauchy solution exponentially converges. The main contribution is Theorem 3.3, in which the exponential stability conditions are established in terms of a set of matrix inequalities. The theoretical result has been applied to the boundary control for the two-lane traffic flow based on LWR model, in which a saturated boundary feedback controller is designed to stabilize the traffic densities along two lanes by employing VSL devices. For the future work, considering the stabilization for congested traffic in the presence of saturations and a moving shockwave would be of authors' interests, where the difficulty for designing the saturated boundary controller lies in the fact that the moving boundary of the hyperbolic PDE is generated due to the existence of a moving shockwave in traffic flow.

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