

UNIQUE CONTINUATION AND TIME DECAY FOR A HIGHER-ORDER WATER WAVE MODEL

ADEMIR F. PAZOTO^{*} AND MIGUEL D. SOTO VIEIRA

Abstract. This work is devoted to prove the exponential decay for the energy of solutions of a higher order Korteweg–de Vries (KdV)–Benjamin–Bona–Mahony (BBM) equation on a periodic domain with a localized damping mechanism. Following the method in [L. Rosier and B.-Y. Zhang, *J. Diff. Equ.* **254** (2013) 141–178], which combines energy estimates, multipliers and compactness arguments, the problem is reduced to prove the Unique Continuation Property (UCP) for weak solutions of the model. Then, this is done by deriving Carleman estimates for a system of coupled elliptic-hyperbolic equations.

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1. INTRODUCTION

The field of dispersive equations has received increasing attention since the pioneering works of Stokes, Boussinesq and Korteweg and de Vries in the nineteenth century. It pertains to a modern line of research which is important both scientifically as well as for potential applications. On the one hand, the mathematical theoretical research of dispersive equations is important for applied sciences since it has provided solid foundations for the verification and applicability of these models. On the other hand, this theoretical research has proved to be very valuable for mathematics itself. Such equations have presented very difficult and interesting challenges, motivating the development of many new ideas and techniques within mathematical analysis. Particularly, sets of equations were derived to describe the dynamics of the waves in some specific physical regimes and much effort has been expended on various aspects of the initial and boundary value problems. In this context, Bona, Carvajal, Panthee and Scialom [2] derived and analyzed a higher order water wave model to describe the unidirectional propagation of water waves by using the second order approximation in the two-way model, the so-called *abcd*-system introduced in [3, 4]. The model is also known as the fifth order KdV–BBM type equation and has the form

$$\begin{aligned} u_t + u_x - b_1 u_{txx} + a_1 u_{xxx} + b u_{txxxx} + a u_{xxxxx} \\ + \frac{3}{2} u u_x + \gamma (u^2)_{xxx} - \frac{7}{48} (u_x^2)_x - \frac{1}{8} (u^3)_x = 0. \end{aligned} \quad (1.1)$$

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Institute of Mathematics, Federal University of Rio de Janeiro, UFRJ, PO Box 68530, CEP 21945-970 Rio de Janeiro, RJ, Brazil.

^{*} Corresponding author: ademir@im.ufrj.br

The unknown u is a real valued function of the real variables x and t and subscripts indicate partial differentiation. The five parameters a, a_1, b, b_1 and γ are not arbitrary. Indeed, they are determined by the choice of five more fundamental parameters $\theta, \lambda, \mu, \lambda_1$ and μ_1 . The constant θ has physical significance. It is related to the height at which the horizontal velocity is specified, a dependent variable which does not appear explicitly in these unidirectional models. The parameter $\theta \in [0, 1]$ because the vertical coordinate has been scaled by the undisturbed depth h_0 . The constants λ, μ, λ_1 and μ_1 are modelling parameters and, in principle, can take any real value. However, the five parameters that appear in (1.1) are not independent and should satisfy some relations. All of them are shown in [2]. Such conditions come from the physics of the problem and we tacitly assume them to hold throughout the entire paper. Depending on the problem under study (well-posedness of the linear or nonlinear model, stabilization), additional restrictions on the sign of these parameters will be imposed later on.

The incorporation of damping mechanisms is often crucial in obtaining good agreement between experimental observations and the prediction of theoretical models describing the propagation of waves in dispersive media. The problem might be easy to solve when the underlying models have a strong intrinsic dissipative nature, but very often, as the cases we address here, the models are of conservative nature and the decay requires appropriate damping mechanisms. Obviously, for practical purposes, it is desirable to achieve this property with a minimal amount of damping both in what concerns its support and its intensity.

In this work, considerations will be given for model (1.1) on the one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, with a localized damping term. Our purpose is to investigate the dissipative effects generated by this damping when $t \rightarrow \infty$. Our analysis does not depend on particular relations between the coefficients of the system. However, in order to provide the tools needed to deal with the problem, we assume that

$$b, b_1 > 0.$$

Under the above conditions, the following closed-loop system will be considered:

$$\begin{aligned} u_t + u_x - b_1 u_{txx} + a_1 u_{xxx} + bu_{txxxx} + au_{xxxxx} \\ + \frac{3}{2}uu_x + \gamma(u^2)_{xxx} - \frac{7}{48}(u_x^2)_x - \frac{1}{8}(u^3)_x = \sigma(I - b_1\partial_x^2 + b\partial_x^4)[\sigma u], \quad (x, t) \in \mathbb{T} \times (0, T) \\ u(x, 0) = u_0(x), \quad x \in \mathbb{T}, \end{aligned} \quad (1.2)$$

where $\sigma \in C^\infty(\mathbb{T})$ is a nonzero function. We also introduce the set

$$\omega = \{x \in \mathbb{T} : \sigma(x) \neq 0\} \neq \emptyset. \quad (1.3)$$

Then, if we consider the following equivalent norm in the L^2 -based Sobolev space $H^2(\mathbb{T})$,

$$\|u(\cdot, t)\|_{H^2(\mathbb{T})}^2 = \int_{\mathbb{T}} (u^2 + b_1 u_x^2 + bu_{xx}^2) dx, \quad (1.4)$$

at least formally, we can deduce that

$$\|u(\cdot, t)\|_{H^2(\mathbb{T})}^2 - \|u_0\|_{H^2(\mathbb{T})}^2 = -2 \int_0^t \|\sigma u(s)\|_{H^2(\mathbb{T})}^2 ds. \quad (1.5)$$

Indeed, to obtain the identity above we multiply the equation in (1.2) by u and integrate by parts over the spatial domain $\mathbb{T}$. Identity (1.5) shows that the term on the right hand side plays the role of a damping mechanism. Moreover, it indicates that the H^2 -norm decreases along the trajectories of the system, which generates a flow that can be continued indefinitely in the temporal variable. Therefore, we can ask whether the solutions converge to zero, as $t \rightarrow \infty$, and at which rate they decay.

In view of (1.5), the problem of the exponential decay can be stated in the following equivalent form: To find $T > 0$ and $C > 0$, such that

$$\|u_0\|_{H^2(\mathbb{T})}^2 \leq C \int_0^T \|\sigma u(t)\|_{H^2(\mathbb{T})}^2 dt \quad (1.6)$$

holds for every finite energy solution of (1.2). Indeed, from (1.6) and (1.5), we have that $E(T) \leq \beta E(0)$ with $0 < \beta < 1$, which combined with the semigroup property, allows us to derive the exponential decay of the H^2 -norm of the solutions.

This paper is devoted to analyze this problem. Our analysis was inspired by the results proved in [11] from which we borrow the main ideas involved in our proofs. In fact, proceeding as in [11], which combine multipliers, energy estimates and compactness arguments, the problem of obtaining (1.6) is reduced to show that the unique solution of (1.2), such that $\sigma(x)u = 0$ everywhere, has to be the trivial one. This problem may be viewed as a unique continuation one since $\sigma u = 0$ implies that $u = 0$ in $\omega \times (0, T)$, for ω given by (1.3). To solve this problem we develop a Carleman inequality for weak solutions of the model as it has been done in [11] for a KdV–BBM equation. The equation is first split into a coupled system of an elliptic equation and a transport equation. Next, some Carleman estimates are derived with the same singular weights for both the elliptic and the hyperbolic equations. Finally, the unique continuation result is proved by combining these Carleman estimates with a regularization process. However, this unique continuation result cannot be applied directly due to the regularity of solutions we are dealing with. Therefore, some additional assumptions concerning the initial conditions are needed. More precisely, we assume that $0 < r \leq \|u_0\|_{H^2(\mathbb{T})}$ and $\|u_0\|_{H^3(\mathbb{T})} \leq R$, for given $r, R > 0$. These conditions seem to be technical, but, in their absence, we do not know how to derive the result. It is also important to emphasize that the choice of the damping function was motivated by the controllability results obtained in [1] for corresponding linearized system. Indeed, as an application of the controllability result, we construct feedback controls *via* some results obtained in [8, 12], such that the resulting linearized closed-loop system is shown to be exponentially stable. Such stabilization results play a crucial role in the analysis we describe above.

We remark that the unique continuation property for the BBM equation is still an open problem. Moreover, since the underlying Cauchy problem is a characteristic one, we cannot expect to apply Carleman-type estimates or the classical Holmgren uniqueness theorem. In (1.1), the presence of the higher-order KdV term u_{xxxxx} results in much better properties and allows to establish a unique continuation result.

It is also worth mentioning that, in addition to the analysis carried out in the work, the stabilization of the KdV and BBM equations have also been studied considering other types of damping mechanisms. In this sense, we refer to [10, 11] for a quite complete review of the field.

The remainder of this paper is organized as follows: in Section 2, we prove some Carleman estimates and derive a unique continuation property for the higher-order KdV–BBM equation. Section 3 is devoted to the stabilization of the damped equation. Finally, for the sake of completeness, we include in the Appendix some computations used in Section 3 and the proof a Carleman estimate derived in [11].

2. CARLEMAN ESTIMATES

The first part of this section is devoted to prove an appropriate Carleman estimate for the higher-order KdV–BBM equation

$$\begin{aligned} u_t + u_x - b_1 u_{txx} + a_1 u_{xxx} + b u_{txxxx} + a u_{xxxxx} \\ + \frac{3}{2} u u_x + \gamma (u^2)_{xxx} - \frac{7}{48} (u_x^2)_x - \frac{1}{8} (u^3)_x = 0, \quad (x, t) \in \mathbb{T} \times (0, T), \end{aligned} \quad (2.1)$$

or

$$u_t - b_1 u_{txx} + b u_{txxxx} + a u_{xxxxx} + q(u)u_x + p(u)u_{xxx} + r(u)u_{xx} = 0, \quad (x, t) \in \mathbb{T} \times (0, T), \quad (2.2)$$

where $q(u) = 1 + \frac{3}{2}u - \frac{3}{8}u^2$, $p(u) = a_1 + 2\gamma u$ and $r(u) = (6\gamma - \frac{7}{24})u_x$.

Next, this Carleman estimate is employed to prove the following unique continuation result:

Theorem 2.1. *Let $a, b \neq 0$, $T > \frac{2\pi b}{|a|}$ and $q, p, r \in L^\infty(0, T; L^\infty(\mathbb{T}))$. Let $\omega \subset \mathbb{T}$ be a nonempty open set. Let $u \in L^2(0, T; H^4(\mathbb{T})) \cup L^\infty(0, T; H^3(\mathbb{T}))$ satisfying (2.2) and*

$$u(x, t) = 0 \quad \text{for a.e. } (x, t) \in \omega \times (0, T). \quad (2.3)$$

Then, $u \equiv 0$ in $\mathbb{T} \times (0, T)$.

Proof. We first assume that

$$u \in L^2(0, T; H^4(\mathbb{T})). \quad (2.4)$$

and let $w = u - b_1 u_{xx} + b u_{xxxx} \in L^2(0, T; L^2(\mathbb{T}))$. Then, the pair (u, w) solves the following system

$$u - b_1 u_{xx} + b u_{xxxx} = w \quad (2.5)$$

$$w_t + \frac{a}{b} w_x = \left(\frac{a}{b} - q\right)u_x - \left(\frac{ab_1}{b} + p\right)u_{xxx} - r u_{xx}. \quad (2.6)$$

Just to make the assumption of the theorem clear, we recall the following remark, which can be found in [11]:

Remark 2.2. There is a finite speed propagation for KdV–BBM. For instance, if we assume that $q(x) = \frac{a}{b}$, $p(x) = -\frac{ab_1}{b}$ and $r(x) = 0$ for all $x \in \mathbb{T}$, where $a, b, b_1 > 0$ are given, and that $\omega = (2\pi - \epsilon, 2\pi)$ for a small $\epsilon > 0$, then the UCP fails in time $T \leq \frac{b(2\pi - 2\epsilon)}{a}$. Indeed, picking any nontrivial initial state $u_0 \in C_0^\infty(0, \epsilon)$, we easily see that the solution (u, w) of (2.5)–(2.6) is $u(x, t) = u_0(x - \frac{a}{b}t)$, $w(x, t) = w_0(x - \frac{a}{b}t)$, where $w_0 = (I - \partial_x^2)u_0$. Then, $u(x, t) = 0$ for $(x, t) \in \omega \times (0, \frac{b(2\pi - 2\epsilon)}{a})$ although $u \neq 0$. Hence, the condition $T > \frac{2b\pi}{|a|}$ in the Theorem 2.1 is sharp.

We choose a system of coordinates and identify $\mathbb{T}$ with $[0, 2\pi)$. Thus, without loss of generality, we can assume that $a > 0$, and that $\omega = (2\pi - \eta, 2\pi + \eta) \sim [0, \eta) \cup (2\pi - \eta, 2\pi)$ for some $\eta \in (0, \pi)$ (choosing the origin of the coordinates inside ω). Then, we consider T , such that

$$T > \frac{2b\pi}{a}. \quad (2.7)$$

Following the approach developed in [11], we first obtain some Carleman estimates for the elliptic equation (2.5) and the transport equation (2.6) with the same weights function. Next, we combine them to derive a single one for (2.2). In order to do that, we pick $\delta > 0$ and $\rho \in (0, 1)$, such that

$$\frac{\rho a}{b} T > 2\pi + \delta \quad (2.8)$$

and a function $\psi \in C^\infty([0, 2\pi])$ satisfying

$$\psi(x) = (x + \delta)^2 \quad \text{for } x \in [\frac{\eta}{2}, 2\pi - \frac{\eta}{2}], \quad (2.9)$$

$$\frac{d^k \psi}{dx^k}(0) = \frac{d^k \psi}{dx^k}(2\pi) \quad \text{for } k = 1, 2, 3, 4, 5, 6, 7, \quad (2.10)$$

$$2\delta \leq \frac{d\psi}{dx}(x) \leq 2(2\pi + \delta) \quad \text{for } x \in [0, 2\pi]. \quad (2.11)$$

Then, we introduce the function $\varphi \in C^\infty([0, 2\pi] \times \mathbb{R})$ as follows

$$\varphi(x, t) = \psi(x) - \rho a^2 t^2. \quad (2.12)$$

Under the above considerations, we derive the following Carleman estimate for (2.2).

Proposition 2.3. *Let ω , a and T be as above. Then, there exist some positive numbers s_2 and C_2 , such that, for all $s \geq s_2$ and all $u \in L^2(0, T; H^4(\mathbb{T}))$ satisfying (2.2), we have*

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}} [s|u_{xxxx}|^2 + s|u_{xxx}|^2 + s^3|u_{xx}|^2 + s^5|u_x|^2 + s^7|u|^2] e^{2s\varphi} dx dt \\ & \quad + s \int_{\mathbb{T}} [|u - b_1 u_{xx} + b u_{xxxx}|^2 e^{2s\varphi}]_{t=0} dx \\ & \leq C_2 \int_0^T \int_{\omega} [s|u_{xxxx}|^2 + s^3|u_{xx}|^2 + s^7|u|^2] e^{2s\varphi} dx dt. \end{aligned} \quad (2.13)$$

Proof. In order to make the reading easier, we proceed in several steps.

- Carleman estimate for the transport equation.

Lemma 2.4. *There exist $s_1 \geq s_0$ and $C_1 > 0$ such that for all $s \geq s_1$ and all $w \in L^2(\mathbb{T} \times (0, T))$ with $w_t + \frac{a}{b} w_x \in L^2(\mathbb{T} \times (0, T))$, the following holds*

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}} s|w|^2 e^{2s\varphi} dx dt + \int_{\mathbb{T}} s[|w|^2 e^{2s\varphi}]_{t=0} dx + \int_{\mathbb{T}} s[|w|^2 e^{2s\varphi}]_{t=T} dx \\ & \leq C_1 \left(\int_0^T \int_{\mathbb{T}} |w_t + \frac{a}{b} w_x|^2 e^{2s\varphi} dx dt + \int_0^T \int_{\omega} s|w|^2 e^{2s\varphi} dx dt \right). \end{aligned} \quad (2.14)$$

Proof. The result was proved in Lemma 5.5 of [11]. For the sake of completeness we have included the proof in the Appendix. $\square$

- Carleman estimate for the elliptic equation.

Lemma 2.5. *There exist $s_0 \geq 1$ and $C_0 > 0$, such that, for all $s \geq s_0$ and all $u \in H^4(\mathbb{T})$, the following holds*

$$\begin{aligned} & \int_{\mathbb{T}} [s|u_{xxx}|^2 + s^3|u_{xx}|^2 + s^5|u_x|^2 + s^7|u|^2] e^{2s\psi} dx \\ & \leq C_0 \left(\int_{\mathbb{T}} |u_{xxxx}|^2 e^{2s\psi} dx + \int_{\omega} (s^7|u|^2 + s^3|u_{xx}|^2) e^{2s\psi} dx \right). \end{aligned} \quad (2.15)$$

Proof. We start by considering a classical change of function and a appropriate differential operator. More precisely, let $v = e^{s\psi} u$ and $P = \partial_x^4$. Then, we decompose $e^{s\psi} P u$ as follows

$$e^{s\psi} P u = e^{s\psi} P (e^{-s\psi} v) = P_p v + P_n v,$$

where

$$P_p v = (s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) v + 12s^2 \psi_x \psi_{xx} v_x + 6s^2 \psi_x^2 v_{xx} + v_{xxxx}, \quad (2.16)$$

$$P_n v = -(6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx}) v - (4s^3 \psi_x^3 + 4s \psi_{xxx}) v_x - 6s \psi_{xx} v_{xx} - 4s \psi_x v_{xxx}. \quad (2.17)$$

From the above decomposition, we obtain

$$\|e^{s\psi} P u\|^2 = \|P_p v\|^2 + \|P_n v\|^2 + 2(P_p v, P_n v),$$

where $(f, g) = \int_{\mathbb{T}} f g dx$, and $\|f\| = (f, f)$.

The next steps are devoted to analyze the inner product in the identity above. First, observe that, (2.16) and (2.17) lead to

$$\begin{aligned} (P_p v, P_n v) &= ((s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) v, -(6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx}) v) \\ &\quad + ((s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) v, -(4s^3 \psi_x^3 + 4s \psi_{xxx}) v_x) \\ &\quad + ((s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) v, -6s \psi_{xx} v_{xx}) \\ &\quad + ((s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) v, -4s \psi_x v_{xxx}) + (12s^2 \psi_x \psi_{xx} v_x, -(6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx}) v) \\ &\quad + (12s^2 \psi_x \psi_{xx} v_x, -(4s^3 \psi_x^3 + 4s \psi_{xxx}) v_x) + (12s^2 \psi_x \psi_{xx} v_x, -6s \psi_{xx} v_{xx}) \\ &\quad + (12s^2 \psi_x \psi_{xx} v_x, -4s \psi_x v_{xxx}) + (6s^2 \psi_x^2 v_{xx}, -(6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx}) v) \\ &\quad + (6s^2 \psi_x^2 v_{xx}, -(4s^3 \psi_x^3 + 4s \psi_{xxx}) v_x) + (6s^2 \psi_x^2 v_{xx}, -6s \psi_{xx} v_{xx}) + (6s^2 \psi_x^2 v_{xx}, -4s \psi_x v_{xxx}) \\ &\quad + (v_{xxxx}, -(6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx}) v) + (v_{xxxx}, -(4s^3 \psi_x^3 + 4s \psi_{xxx}) v_x) + (v_{xxxx}, -6s \psi_{xx} v_{xx}) \\ &\quad + (v_{xxxx}, -4s \psi_x v_{xxx}) \\ &= \sum_{n=1}^{16} I_n. \end{aligned}$$

Next, we compute each term I_n . This is done by employing (2.10) and by the integration by parts in x :

$$\begin{aligned} I_1 &= - \int_{\mathbb{T}} (s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) (6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx}) v^2 dx \\ I_2 &= - \int_{\mathbb{T}} (s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) (4s^3 \psi_x^3 + 4s \psi_{xxx}) v_x v dx \\ &= \int_{\mathbb{T}} [(s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) (2s^3 \psi_x^3 + 2s \psi_{xxx})]_x v^2 dx \\ I_3 &= - \int_{\mathbb{T}} (s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) (6s \psi_{xx}) v v_{xx} dx \\ &= \int_{\mathbb{T}} [(s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) (6s \psi_{xx})]_x v v_{xx} dx \\ &\quad + \int_{\mathbb{T}} [(s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) (6s \psi_{xx})] v_x^2 dx \\ &= - \int_{\mathbb{T}} [(s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) (3s \psi_{xx})]_{xx} v^2 dx \\ &\quad + \int_{\mathbb{T}} [(s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) (6s \psi_{xx})] v_x^2 dx \end{aligned}$$

$$\begin{aligned}
I_4 &= - \int_{\mathbb{T}} (s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) 4s \psi_x v v_{xxx} dx \\
&= \int_{\mathbb{T}} [(s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) 4s \psi_x]_x v v_{xx} dx \\
&\quad + \int_{\mathbb{T}} (s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) 4s \psi_x v_x v_{xx} dx \\
&= - \int_{\mathbb{T}} [(s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) 4s \psi_x]_{xx} v v_x dx \\
&\quad - \int_{\mathbb{T}} [(s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) 4s \psi_x]_x v_x^2 dx \\
&\quad - \int_{\mathbb{T}} [(s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) 2s \psi_x]_x v_x^2 dx \\
&= \int_{\mathbb{T}} [(s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) 2s \psi_x]_{xxx} v^2 dx \\
&\quad - \int_{\mathbb{T}} [(s^4 \psi_x^4 + 3s^2 \psi_{xx}^2 + 4s^2 \psi_{xxx} \psi_x) 6s \psi_x]_x v_x^2 dx \\
I_5 &= - \int_{\mathbb{T}} 12s^2 \psi_x \psi_{xx} (6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx}) v v_x dx \\
&= \int_{\mathbb{T}} [6s^2 \psi_x \psi_{xx} (6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx})]_x v^2 dx \\
I_6 &= - \int_{\mathbb{T}} 12s^2 \psi_x \psi_{xx} (4s^3 \psi_x^3 + 4s \psi_{xxx}) v_x^2 dx \\
I_7 &= - \int_{\mathbb{T}} 12s^2 \psi_x \psi_{xx} 6s \psi_{xx} v_x v_{xx} dx = \int_{\mathbb{T}} [36s^3 \psi_x \psi_{xx}]_x v_x^2 dx \\
I_8 &= - \int_{\mathbb{T}} 12s^2 \psi_x \psi_{xx} 4s \psi_x v_x v_{xxx} dx = - \int_{\mathbb{T}} 48s^3 \psi_x^2 \psi_{xx} v_x v_{xxx} dx \\
&= \int_{\mathbb{T}} [48s^3 \psi_x^2 \psi_{xx}]_x v_x v_{xx} dx + \int_{\mathbb{T}} 48s^3 \psi_x^2 \psi_{xx} v_{xx}^2 dx \\
&= - \int_{\mathbb{T}} [24s^3 \psi_x^2 \psi_{xx}]_{xx} v_x^2 dx + \int_{\mathbb{T}} 48s^3 \psi_x^2 \psi_{xx} v_{xx}^2 dx \\
I_9 &= - \int_{\mathbb{T}} 6s^2 \psi_x^2 (6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx}) v v_{xx} dx \\
&= \int_{\mathbb{T}} [6s^2 \psi_x^2 (6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx})]_x v v_x dx + \int_{\mathbb{T}} 6s^2 \psi_x^2 (6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx}) v_x^2 dx \\
&= - \int_{\mathbb{T}} [3s^2 \psi_x^2 (6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx})]_{xx} v^2 dx + \int_{\mathbb{T}} 6s^2 \psi_x^2 (6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx}) v_x^2 dx \\
I_{10} &= - \int_{\mathbb{T}} 6s^2 \psi_x^2 (4s^3 \psi_x^3 + 4s \psi_{xxx}) v_x v_{xx} dx = \int_{\mathbb{T}} [3s^2 \psi_x^2 (4s^3 \psi_x^3 + 4s \psi_{xxx})]_x v_x^2 dx \\
I_{11} &= - \int_{\mathbb{T}} 6s^2 \psi_x^2 6s \psi_{xx} v_{xx}^2 dx = - \int_{\mathbb{T}} 36s^3 \psi_x^2 \psi_{xx} v_{xx}^2 dx \\
I_{12} &= - \int_{\mathbb{T}} 6s^2 \psi_x^2 4s \psi_x v_{xx} v_{xxx} dx = \int_{\mathbb{T}} [12s^3 \psi_x^3]_x v_{xx}^2 dx \\
I_{13} &= - \int_{\mathbb{T}} (6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx}) v v_{xxx} dx = \int_{\mathbb{T}} [6s^3 \psi_x^2 \psi_{xx} + s \psi_{xxxx}]_x v v_{xxx} dx
\end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{T}} (6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v_x v_{xxx} dx \\
& = - \int_{\mathbb{T}} [6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}]_{xx} v v_{xx} dx - \int_{\mathbb{T}} [6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}]_x v_x v_{xx} dx \\
& \quad - \int_{\mathbb{T}} [6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}]_x v_x v_{xx} dx - \int_{\mathbb{T}} (6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v_{xx}^2 dx \\
& = \int_{\mathbb{T}} [6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}]_{xxx} v v_x dx + \int_{\mathbb{T}} [6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}]_{xx} v_x^2 dx \\
& \quad + \int_{\mathbb{T}} [6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}]_{xx} v_x^2 dx - \int_{\mathbb{T}} (6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v_{xx}^2 dx \\
& = - \int_{\mathbb{T}} \frac{1}{2} [6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}]_{xxx} v^2 dx + \int_{\mathbb{T}} 2[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}]_{xx} v_x^2 dx \\
& \quad - \int_{\mathbb{T}} (6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})v_{xx}^2 dx \\
I_{14} & = - \int_{\mathbb{T}} (4s^3\psi_x^3 + 4s\psi_{xxx})v_x v_{xxx} dx = \int_{\mathbb{T}} [(4s^3\psi_x^3 + 4s\psi_{xxx})]_x v_x v_{xxx} dx \\
& \quad + \int_{\mathbb{T}} (4s^3\psi_x^3 + 4s\psi_{xxx})v_{xx} v_{xxx} dx \\
& = - \int_{\mathbb{T}} [(4s^3\psi_x^3 + 4s\psi_{xxx})]_{xx} v_x v_{xx} dx - \int_{\mathbb{T}} [(4s^3\psi_x^3 + 4s\psi_{xxx})]_x v_{xx}^2 dx \\
& \quad - \int_{\mathbb{T}} [(2s^3\psi_x^3 + 2s\psi_{xxx})]_x v_{xx}^2 dx \\
& = \int_{\mathbb{T}} [(2s^3\psi_x^3 + 2s\psi_{xxx})]_{xxx} v_x^2 dx - \int_{\mathbb{T}} 3[(2s^3\psi_x^3 + 2s\psi_{xxx})]_x v_{xx}^2 dx \\
I_{15} & = - \int_{\mathbb{T}} 6s\psi_{xx} v_{xx} v_{xxxx} dx = \int_{\mathbb{T}} 6s\psi_{xxx} v_{xx} v_{xxx} dx + \int_{\mathbb{T}} 6s\psi_{xx} v_{xxx}^2 dx \\
& = - \int_{\mathbb{T}} 3s\psi_{xxx} v_{xx}^2 dx + \int_{\mathbb{T}} 6s\psi_{xx} v_{xxx}^2 dx \\
I_{16} & = - \int_{\mathbb{T}} 4s\psi_x v_{xxx} v_{xxxx} dx = \int_{\mathbb{T}} 2s\psi_{xx} v_{xxx}^2 dx.
\end{aligned}$$

By combining the identities above, we get

$$\begin{aligned}
\|e^{s\psi} P u\|^2 & = \|P_p v\|^2 + \|P_n v\|^2 + 2 \int_{\mathbb{T}} h_1(\psi) v^2 dx + 2 \int_{\mathbb{T}} h_2(\psi) v_x^2 dx + \\
& \quad 2 \int_{\mathbb{T}} h_3(\psi) v_{xx}^2 dx + 2 \int_{\mathbb{T}} h_4(\psi) v_{xxx}^2 dx,
\end{aligned}$$

where

$$\begin{aligned}
h_1(\psi) & = [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(2s^3\psi_x^3 + 2s\psi_{xxx})]_x \\
& \quad - (s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}) \\
& \quad - [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(3s\psi_{xx})]_{xx} \\
& \quad + [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)2s\psi_x]_{xxx} \\
& \quad + [12s^2\psi_x\psi_{xx}(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_x - [3s^2\psi_x^2(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx})]_{xx}
\end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2}[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}]]_{xxx} \\
 h_2(\psi) &= [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)(6s\psi_{xx})] - [(s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)6s\psi_x]_x \\
 & - 12s^2\psi_x\psi_{xx}(4s^3\psi_x^3 + 4s\psi_{xxx}) + [36s^3\psi_x\psi_{xx}^2]_x - [24s^3\psi_x^2\psi_{xx}]_{xx} \\
 & + [3s^2\psi_x^2(4s^3\psi_x^3 + 4s\psi_{xxx})]_x \\
 & + 2[6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}]_{xx} + [(2s^3\psi_x^3 + 2s\psi_{xxx})]_{xxx} + 6s^2\psi_x^2(6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}) \\
 h_3(\psi) &= 48s^3\psi_x^2\psi_{xx} - 36s^3\psi_x^2\psi_{xx} + [12s^3\psi_x^3]_x - (6s^3\psi_x^2\psi_{xx} + s\psi_{xxxx}) \\
 & - 3[(2s^3\psi_x^3 + 2s\psi_{xxx})]_x - 3s\psi_{xxxx} \\
 h_4(\psi) &= 6s\psi_{xx} + 2s\psi_{xx} = 8s\psi_{xx}.
 \end{aligned}$$

The choice of the function ψ given by (2.9) allows us to conclude that there exist $s_0 \geq 1$, $K > 0$ and $K_1 > 0$, such that, for all $s \geq s_0$,

$$\begin{aligned}
 2h_1(\psi) &\geq Ks^7 \quad \text{for } (x, t) \in (\frac{\eta}{2}, 2\pi - \frac{\eta}{2}) \times (0, T), \\
 2h_2(\psi) &\geq Ks^5 \quad \text{for } (x, t) \in (\frac{\eta}{2}, 2\pi - \frac{\eta}{2}) \times (0, T), \\
 2h_3(\psi) &\geq Ks^3 \quad \text{for } (x, t) \in (\frac{\eta}{2}, 2\pi - \frac{\eta}{2}) \times (0, T), \\
 2h_4(\psi) &\geq Ks \quad \text{for } (x, t) \in (\frac{\eta}{2}, 2\pi - \frac{\eta}{2}) \times (0, T),
 \end{aligned}$$

and, if $x \in \omega_0 = [0, \frac{\eta}{2}) \cup (2\pi - \frac{\eta}{2}, 2\pi)$,

$$\begin{aligned}
 |2h_1(\psi)| &\leq K_1s^7 \quad \text{for } (x, t) \in \omega_0 \times (0, T), \\
 |2h_2(\psi)| &\leq K_1s^5 \quad \text{for } (x, t) \in \omega_0 \times (0, T), \\
 |2h_3(\psi)| &\leq K_1s^3 \quad \text{for } (x, t) \in \omega_0 \times (0, T), \\
 |2h_4(\psi)| &\leq K_1s \quad \text{for } (x, t) \in \omega_0 \times (0, T).
 \end{aligned}$$

Consequently, for $s \geq s_0$, we obtain $C > 0$ satisfying

$$\begin{aligned}
 & \|P_p v\|^2 + \int_{\mathbb{T}} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \\
 &= \|P_p v\|^2 + \int_{\omega_0} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \\
 &+ \int_{\mathbb{T} \setminus \omega_0} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \\
 &\leq \|P_p v\|^2 + \int_{\omega_0} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \\
 &+ C \int_{\mathbb{T} \setminus \omega_0} [2h_1(\psi)|v|^2 + 2h_2(\psi)|v_x|^2 + 2h_3(\psi)|v_{xx}|^2 + 2h_4(\psi)|v_{xxx}|^2] dx \\
 &\leq \|P_p v\|^2 + \int_{\omega_0} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \\
 &+ C \int_{\mathbb{T}} [2h_1(\psi)|v|^2 + 2h_2(\psi)|v_x|^2 + 2h_3(\psi)|v_{xx}|^2 + 2h_4(\psi)|v_{xxx}|^2] dx
 \end{aligned}$$

$$\leq C \left(\|e^{s\psi} Pu\|^2 + \int_{\omega_0} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \right),$$

that is,

$$\begin{aligned} & \|P_p v\|^2 + \int_{\mathbb{T}} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \\ & \leq C \left(\|e^{s\psi} Pu\|^2 + \int_{\omega_0} [s^7|v|^2 + s^5|v_x|^2 + s^3|v_{xx}|^2 + s|v_{xxx}|^2] dx \right). \end{aligned} \quad (2.18)$$

Observe that $\int_{\mathbb{T}} s^{-1}|v_{xxx}|^2 dx$ is less than the right hand side of (2.18). Indeed,

$$\begin{aligned} & \int_{\mathbb{T}} s^{-1}|v_{xxx}|^2 dx = \int_{\mathbb{T}} s^{-1} |P_p v - (s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x)v - 12s^2\psi_x\psi_{xx}v_x - 6s^2\psi_x^2v_{xx}|^2 dx \\ & \leq C \int_{\mathbb{T}} s^{-1} (|P_p v|^2 + |s^4\psi_x^4 + 3s^2\psi_{xx}^2 + 4s^2\psi_{xxx}\psi_x|^2|v|^2 + |12s^2\psi_x\psi_{xx}|^2|v_x|^2 + |6s^2\psi_x^2|^2|v_{xx}|^2) dx \\ & \leq C s^{-1} \left(\|P_p v\|^2 + \int_{\mathbb{T}} (s^8|v|^2 + s^4|v_x|^2 + s^4|v_{xx}|^2) \right) \\ & = C \left(s^{-1} \|P_p v\|^2 + \int_{\mathbb{T}} (s^7|v|^2 + s^3|v_x|^2 + s^3|v_{xx}|^2) \right), \end{aligned}$$

for some $C > 0$. The above estimate and (2.18) leads to

$$\begin{aligned} & \int_{\mathbb{T}} \{s^{-1}|v_{xxx}|^2 + s|v_{xxx}|^2 + s^3|v_{xx}|^2 + s^5|v_x|^2 + s^7|v|^2\} dx \\ & \leq C \left(s^{-1} \|P_p v\|^2 + \int_{\omega_0} (s^7|v|^2 + s^3|v_x|^2 + s^3|v_{xx}|^2) \right. \\ & \quad \left. + \int_{\omega_0} (s|v_{xxx}|^2 + s^3|v_{xx}|^2 + s^5|v_x|^2 + s^7|v|^2) \right) \\ & \leq C \left(s^{-1} \|P_p v\|^2 + \int_{\omega_0} (s|v_{xxx}|^2 + s^3|v_{xx}|^2 + s^5|v_x|^2 + s^7|v|^2) \right), \end{aligned} \quad (2.19)$$

where C does not depend on s and v . In order to absorb the terms v_x and v_{xxx} above, let us consider $\xi \in C_0^\infty(\omega)$ with $0 \leq \xi \leq 1$ for $x \in \omega_0$. Then, observe that

$$\begin{aligned} \int_{\omega_0} |v_x|^2 dx & \leq \int_{\omega} \xi |v_x|^2 dx = \int_{\omega} \xi v_x v_x dx = - \int_{\omega} (\xi_x v_x + \xi v_{xx}) v dx \\ & = \frac{1}{2} \int_{\omega} \xi_{xx} v^2 dx - \int_{\omega} \xi v_{xx} v dx, \end{aligned}$$

which allows us to deduce that

$$2 \int_{\omega_0} s |v_x|^2 dx \leq \|\xi_{xx}\|_{L^\infty(\mathbb{T})} \int_{\omega} s |v|^2 dx + \kappa \int_{\omega} s^{-1} |v_{xx}|^2 dx + \int_{\omega} s^3 |v|^2 dx \quad (2.20)$$

and

$$\int_{\omega_0} |v_{xxx}|^2 dx \leq \int_{\omega} \xi |v_{xxx}|^2 dx = \int_{\omega} \xi v_{xxx} v_{xxx} dx = - \int_{\omega} (\xi_x v_{xxx} + \xi v_{xxxx}) v_{xx} dx$$

$$= \frac{1}{2} \int_{\omega} \xi_{xx} v_{xx}^2 dx - \int_{\omega} \xi v_{xxxx} v_{xx} dx.$$

Consequently,

$$2 \int_{\omega_0} s |v_{xxx}|^2 dx \leq \|\xi_{xx}\|_{L^\infty(\mathbb{T})} \int_{\omega} s |v_{xx}|^2 dx + \kappa \int_{\omega} s^{-1} |v_{xxxx}|^2 dx + \kappa^{-1} \int_{\omega} s^3 |v_{xx}|^2 dx, \quad (2.21)$$

for any $\kappa > 0$. Combining (2.19), (2.20) and (2.21) with κ small enough and $s \geq s_0$ sufficiently large, for some constant C that does not depend on s and v we get

$$\begin{aligned} \int_{\mathbb{T}} \{s^{-1} |v_{xxxx}|^2 + s |v_{xxx}|^2 + s^3 |v_{xx}|^2 + s^5 |v_x|^2 + s^7 |v|^2\} dx \\ \leq C \left(\|e^{s\psi} Pu\|^2 + \int_{\omega} (s^7 |v|^2 + s^3 |v_{xx}|^2) dx \right). \end{aligned} \quad (2.22)$$

Replacing v by $e^{s\psi} u$ in (2.22) we obtain (2.15). The proof of Lemma 2.5 is complete. $\square$

We can now complete the proof of Proposition 2.3. Let $u \in L^2(0, T; H^4(\mathbb{T}))$ satisfying (2.2) and let $w = u - b_1 u_{xx} + b u_{xxxx} \in L^2(0, T; L^2(\mathbb{T}))$. Then,

$$w_t + \frac{a}{b} w_x = \left(\frac{a}{b} - q\right) u_x - \left(\frac{ab_1}{b} + p\right) u_{xxx} - r u_{xx} \in L^2(0, T; L^2(\mathbb{T})). \quad (2.23)$$

Combining (2.5), (2.6) (multiplied by $e^{-2s\rho c^2 t^2}$ and next integrated over $(0, T)$) and (2.15), it follows that, for $s \geq s_1$, the following estimate holds

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}} [s |u_{xxx}|^2 + s^3 |u_{xx}|^2 + s^5 |u_x|^2 + s^7 |u|^2 + s |u - b_1 u_{xx} + b u_{xxxx}|^2] e^{2s\varphi} dx dt \\ & + \int_{\mathbb{T}} [s |u - b_1 u_{xx} + b u_{xxxx}|^2 e^{2s\varphi}]_{t=0} dx \\ & \leq C \int_0^T \int_{\mathbb{T}} [|u_{xxxx}|^2 + |(\frac{a}{b} - q) u_x - (\frac{ab_1}{b} + p) u_{xxx} - r u_{xx}|^2] e^{2s\varphi} dx dt \\ & + C \int_0^T \int_{\omega} [s |u - b_1 u_{xx} + b u_{xxxx}|^2 + s^7 |u|^2 + s^3 |u_{xx}|^2] e^{2s\varphi} dx dt. \end{aligned}$$

Then, choosing $s_2 \geq s_1$ and $C_2 > C$ large enough, we obtain (2.13) for any $s \geq s_2$ and any $u \in L^2(0, T; H^4(\mathbb{T}))$ satisfying (2.2). $\square$

Finally, we can prove Theorem 2.1.

- If $u \in L^2(0, T; H^4(\mathbb{T}))$ satisfies (2.2) and (2.3), it follows from the Carleman estimate (2.13) that $u = 0$ in $\mathbb{T} \times (0, T)$.
- If $u \in L^\infty(0, T; H^3(\mathbb{T}))$ we have to employ a slightly different approach, since Lemmas 2.5 and 2.4, employed to prove (2.13), cannot be applied directly. This is due to the lack of regularity of u and $w = u - b_1 u_{xx} + b u_{xxxx}$, which solves (2.23). In order to overcome this difficulty, we first smooth them by making use of Proposition 2.6, stated below. More precisely, for any function g and any $h > 0$, we set

$$g^{[h]}(x, t) = \frac{1}{h} \int_t^{t+h} g(x, s) ds.$$

Then, taking into account the regularity of the functions $u^{[h]}$ and $w^{[h]}$, as well as the equations of which they are solutions, we proceed as in the previous case and derive a Carleman estimate for the function $u^{[h]}$. Finally, the unique continuation property is obtained by passing to the limit in this new estimate, as $h \rightarrow 0$. In order to do that, we employ Proposition 2.6 again.

We start the proof by stating Proposition 2.6 mention above (see, for instance, Prop. 1.4.29 in [6] and [11]):

Proposition 2.6. *If X is a Banach space and $g \in L^p(0, T, X)$, with $1 \leq p \leq \infty$, then for any $h > 0$ the function given by*

$$g^{[h]}(x, t) = \frac{1}{h} \int_t^{t+h} g(x, s) ds,$$

satisfies

- (i) $g^{[h]} \in W^{1,p}(0, T-h; X)$,
- (ii) $\|g^{[h]}\|_{L^p(0, T-h; X)} \leq \|g\|_{L^p(0, T; X)}$,
- (iii) $g^{[h]} \rightarrow g$ in $L^p(0, T'; X)$, as $h \rightarrow 0$, for $p < \infty$ and $T' < T$.

Under the above conditions, pick any $T' \in (\frac{2b\pi}{a}, T)$. Then, for any positive number $h < h_0 = T - T'$, the function $u^{[h]} \in W^{1,\infty}(0, T'; H^3(\mathbb{T}))$ and solves the equation

$$\begin{aligned} u_t^{[h]} - b_1 u_{txx}^{[h]} + b u_{txxxx}^{[h]} + a u_{xxxxx}^{[h]} \\ + (q(u)u_x)^{[h]} + (p(u)u_{xx})^{[h]} + (r(u)u_{xx})^{[h]} = 0 \text{ in } L^\infty(0, T'; H^{-2}(\mathbb{T})), \end{aligned} \quad (2.24)$$

where $v_t^{[h]}$ denotes $(v^{[h]})_t$, $v_x^{[h]}$ denotes $(v^{[h]})_x$, etc. Moreover,

$$u^{[h]}(x, t) = 0 \quad \text{in } (x, t) \in \omega \times (0, T'). \quad (2.25)$$

From (2.24), we infer that

$$u_{xxxxx}^{[h]} = a^{-1}(-u_t^{[h]} + b_1 u_{txx}^{[h]} - b u_{txxxx}^{[h]} - (q(u)u_x)^{[h]} - (p(u)u_{xx})^{[h]} - (r(u)u_{xx})^{[h]}).$$

Hence, since the right hand side of the above identity belongs to $L^\infty(0, T'; H^{-1}(\mathbb{T}))$,

$$u^{[h]} \in L^\infty(0, T'; H^4(\mathbb{T})). \quad (2.26)$$

This yields, with (2.5) and (2.6),

$$w^{[h]} = u^{[h]} - b_1 u_{xx}^{[h]} + b u_{xxxx}^{[h]} \in L^\infty(0, T'; L^2(\mathbb{T})) \quad (2.27)$$

$$w_t^{[h]} + \frac{a}{b} w_x^{[h]} = [(\frac{a}{b} - q)u_x]^{[h]} - [(\frac{ab_1}{b} + p)u_{xxx}]^{[h]} - [ru_{xx}]^{[h]} \in L^\infty(0, T'; L^2(\mathbb{T})). \quad (2.28)$$

Thus, from (2.24)–(2.28), Lemma 2.5 and 2.4, we obtain constants $s_1 > 0$ and $C_1 > 0$, such that, for all $s \geq s_1$ and all $h \in (0, h_0)$, we have

$$\begin{aligned}
 & \int_0^{T'} \int_{\mathbb{T}} [s|u_{xxx}^{[h]}|^2 + s^3|u_{xx}^{[h]}|^2 + s^5|u_x^{[h]}|^2 + s^7|u^{[h]}|^2 + s|u_{xxxx}^{[h]}|^2] e^{2s\varphi} dx dt \\
 & \leq C \int_0^{T'} \int_{\mathbb{T}} [|u_{xxxx}^{[h]}|^2 + \left| \left(\frac{a}{b} - q \right) u_x^{[h]} - \left(\frac{ab_1}{b} + p \right) u_{xxx}^{[h]} - [ru_{xx}]^{[h]} \right|^2] e^{2s\varphi} dx dt \\
 & \leq C \int_0^{T'} \int_{\mathbb{T}} [|u_{xxxx}^{[h]}|^2 + \left| \left(\frac{a}{b} - q \right) u_x^{[h]} \right|^2 + \left| \left(\frac{ab_1}{b} + p \right) u_{xxx}^{[h]} \right|^2 + |[ru_{xx}]^{[h]}|^2] e^{2s\varphi} dx dt \\
 & \leq C \int_0^{T'} \int_{\mathbb{T}} [|u_{xxxx}^{[h]}|^2 + \left| \left(\frac{a}{b} - q \right) u_x^{[h]} \right|^2 + \left| \left(\frac{ab_1}{b} + p \right) u_{xxx}^{[h]} \right|^2 + |[ru_{xx}]^{[h]}|^2] e^{2s\varphi} dx dt \\
 & \quad + \int_0^{T'} \int_{\mathbb{T}} \left| \left(\frac{a}{b} - q \right) u_x^{[h]} - \left(\frac{a}{b} - q \right) u_x^{[h]} \right|^2 e^{2s\varphi} dx dt \\
 & \quad + \int_0^{T'} \int_{\mathbb{T}} \left| \left(\frac{ab_1}{b} + p \right) u_{xxx}^{[h]} - \left(\frac{ab_1}{b} + p \right) u_{xxx}^{[h]} \right|^2 e^{2s\varphi} dx dt \\
 & \quad + \int_0^{T'} \int_{\mathbb{T}} |[ru_{xx}]^{[h]} - ru_{xx}^{[h]}|^2 e^{2s\varphi} dx dt.
 \end{aligned} \tag{2.29}$$

Comparing the powers of s in (2.29), we obtain the following estimate for $s \geq s_3 > s_1, h \in (0, h_0)$ and some constant $C_3 > C_1$ (that does not depend on s, h):

$$\begin{aligned}
 & \int_0^{T'} \int_{\mathbb{T}} [s|u_{xxx}^{[h]}|^2 + s^3|u_{xx}^{[h]}|^2 + s^5|u_x^{[h]}|^2 + s^7|u^{[h]}|^2 + s|u_{xxxx}^{[h]}|^2] e^{2s\varphi} dx dt \\
 & \leq C \int_0^{T'} \int_{\mathbb{T}} \left| \left(\frac{a}{b} - q \right) u_x^{[h]} - \left(\frac{a}{b} - q \right) u_x^{[h]} \right|^2 e^{2s\varphi} dx dt \\
 & \quad + C \int_0^{T'} \int_{\mathbb{T}} \left| \left(\frac{ab_1}{b} + p \right) u_{xxx}^{[h]} - \left(\frac{ab_1}{b} + p \right) u_{xxx}^{[h]} \right|^2 e^{2s\varphi} dx dt \\
 & \quad + C \int_0^{T'} \int_{\mathbb{T}} |[ru_{xx}]^{[h]} - ru_{xx}^{[h]}|^2 e^{2s\varphi} dx dt.
 \end{aligned} \tag{2.30}$$

In order to pass to the limit in the right hand side of (2.30), we observe that, as $h \rightarrow 0$,

$$\begin{aligned}
 & \left(\frac{a}{b} - q \right) u_x^{[h]} \rightarrow \left(\frac{a}{b} - q \right) u_x \quad \text{in } L^2(0, T'; L^2(\mathbb{T})), \\
 & \left(\frac{a}{b} - q \right) u_x^{[h]} \rightarrow \left(\frac{a}{b} - q \right) u_x \quad \text{in } L^2(0, T'; L^2(\mathbb{T})), \\
 & \left(\frac{ab_1}{b} + p \right) u_{xxx}^{[h]} \rightarrow \left(\frac{ab_1}{b} + p \right) u_{xxx} \quad \text{in } L^2(0, T'; L^2(\mathbb{T})), \\
 & \left(\frac{ab_1}{b} + p \right) u_{xxx}^{[h]} \rightarrow \left(\frac{ab_1}{b} + p \right) u_{xxx} \quad \text{in } L^2(0, T'; L^2(\mathbb{T})), \\
 & [ru_{xx}]^{[h]} \rightarrow ru_{xx} \quad \text{in } L^2(0, T'; L^2(\mathbb{T})), \\
 & ru_{xx}^{[h]} \rightarrow ru_{xx} \quad \text{in } L^2(0, T'; L^2(\mathbb{T})),
 \end{aligned}$$

while $e^{2s_3\varphi} \in L^\infty(\mathbb{T} \times (0, T'))$. Consequently, for fixed s , we get

$$\int_0^{T'} \int_{\mathbb{T}} \left| \left(\frac{a}{b} - q \right) u_x^{[h]} - \left(\frac{a}{b} - q \right) u_x^{[h]} \right|^2 e^{2s_3\varphi} dx dt \rightarrow 0, \text{ as } h \rightarrow 0,$$

$$\begin{aligned} \int_0^{T'} \int_{\mathbb{T}} \left| \left(\frac{ab_1}{b} + p \right) u_{xxx}^{[h]} - \left(\frac{ab_1}{b} + p \right) u_{xxx}^{[h]} \right|^2 e^{2s_3\varphi} dx dt &\rightarrow 0, \text{ as } h \rightarrow 0, \\ \int_0^{T'} \int_{\mathbb{T}} \left| [ru_{xx}]^{[h]} - ru_{xx}^{[h]} \right|^2 e^{2s_3\varphi} dx dt &\rightarrow 0, \text{ as } h \rightarrow 0. \end{aligned}$$

The convergences above and (2.30) allow us to conclude that, as $h \rightarrow 0$,

$$\int_0^{T'} \int_{\mathbb{T}} |u^{[h]}|^2 e^{2s_3\varphi} dx dt \rightarrow 0.$$

Since $u^{[h]} \rightarrow u$ in $L^2(0, T'; L^2(\mathbb{T}))$,

$$\int_0^{T'} \int_{\mathbb{T}} |u^{[h]}|^2 e^{2s_3\varphi} dx dt \rightarrow \int_0^{T'} \int_{\mathbb{T}} |u|^2 e^{2s_3\varphi} dx dt,$$

as $h \rightarrow 0$. Therefore, we can conclude that $u = 0$ in $\mathbb{T} \times (0, T')$. As T' may be taken arbitrarily close to T , we infer that $u = 0$ in $\mathbb{T} \times (0, T)$. $\square$

3. EXPONENTIAL STABILIZATION

Having the unique continuation result in hands, we derive the exponential decay of the solutions of (1.2) in the energy space $H^2(\mathbb{T})$, as $t \rightarrow \infty$. This is done under suitable assumptions on the initial data.

Before going into the stabilization problem, let us explain the expression of the damping σ . We first write the linearized system of (1.2) as

$$\begin{cases} u_t = Au + Bk, \\ u(0) = u_0, \end{cases} \quad (3.1)$$

where $A = -(I - b_1\partial_x^2 + b\partial_x^4)^{-1}(\partial_x + a_1\partial_x^3 + a\partial_x^5)$, $k(t) = (I - b_1\partial_x^2 + b\partial_x^4)^{-1}h(t) \in L^2(0, T; H^s(\mathbb{T}))$ is a control input and

$$B = (I - b_1\partial_x^2 + b\partial_x^4)^{-1}\sigma(I - b_1\partial_x^2 + b\partial_x^4).$$

We know from [1] that A is skew-adjoint in $H^s(\mathbb{T})$, and that (3.1) is exactly controllable in $H^s(\mathbb{T})$. Moreover, if we choose the simple feedback law (see [8, 12])

$$k = -B^{*,s}u \quad (3.2)$$

the resulting closed-loop system

$$\begin{cases} u_t = Au - BB^{*,s}u, \\ u(0) = u_0 \end{cases} \quad (3.3)$$

is exponentially stable in $H^s(\mathbb{T})$, where $B^{*,s}$ denotes the adjoint of B in $\mathcal{L}(H^s(\mathbb{T}))$. It is shown in the Appendix that $B^{*,s}$ is given by

$$B^{*,s} = (1 - b\partial_x^2 + b_1\partial_x^4)^{1-\frac{s}{2}}\sigma(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1}. \quad (3.4)$$

We also deduce that

$$B^{*,2}u = \sigma u.$$

In order to make the results stated above more precise, let $\tilde{A} = A - BB^{*,2}$, where $(BB^{*,2})u = (I - b_1\partial_x^2 + b\partial_x^4)[\sigma(I - b_1\partial_x^2 + b\partial_x^4)(\sigma u)]$. Since $BB^{*,2} \in \mathcal{L}(H^s(\mathbb{T}))$ and A is skew adjoint in $H^s(\mathbb{T})$, $\tilde{A}$ is the infinitesimal generator of a group $\{W_a(t)\}_{t \in \mathbb{R}}$ on $H^s(\mathbb{T})$ (see [9], Thm. 3.4).

Then, we have the well-posedness and the following exponentially stabilization result for (3.1).

Lemma 3.1. *Let $\sigma \in C^\infty(\mathbb{T})$ with $\sigma \neq 0$. Then, there exist a constant $\beta > 0$, such that, for $s \geq 2$, one can find a constant $C_s > 0$ for which the following holds for all $u_0 \in H^s(\mathbb{T})$:*

$$\|W_a(t)u_0\|_{H^s} \leq C_s e^{-\beta t} \|u_0\|_{H^s}, \quad \forall \quad t \geq 0. \quad (3.5)$$

Adding the feedback law $k = -B^{*,2}u = -\sigma u$ in the nonlinear equation (1.1) gives the closed-loop system (1.2). Then, Lemma 3.1 and the analysis developed in [5] give the following well-posedness result in the space $H^s(\mathbb{T})$ for $s \geq 1$:

Theorem 3.2. *Let $s \geq 2$ and $T > 0$ be given. For any $u_0 \in H^s(\mathbb{T})$, the system (1.2) admits a unique solution $u \in C([0, T]; H^s(\mathbb{T}))$.*

The next steps are devoted to show that (1.2) is globally exponentially stable in the space $H^2(\mathbb{T})$. In order to do that, we start by proving the following observability inequality.

Proposition 3.3. *Let $R_0 > 0$ and $r > 0$ be given. Then, there exist two positive number T and θ , such that, for any $u_0 \in H^3(\mathbb{T})$ satisfying*

$$0 < r \leq \|u_0\|_{H^2(\mathbb{T})} \quad \text{and} \quad \|u_0\|_{H^3(\mathbb{T})} \leq R_0,$$

the corresponding solution u of (1.2) satisfies

$$\|u_0\|_{H^2(\mathbb{T})}^2 \leq \theta \int_0^T \|\sigma u(t)\|_{H^2(\mathbb{T})}^2 dt. \quad (3.6)$$

Proof. Let $T > \frac{2b\pi}{|a|}$. We argue by contradiction and suppose that (3.6) is not true. In this case, for any $n \geq 1$, (1.2) admits a solution $u_n \in C([0, T]; H^3(\mathbb{T}))$ satisfying

$$0 < r \leq \|u_{0,n}\|_{H^2(\mathbb{T})} \quad \text{and} \quad \|u_{0,n}\|_{H^3(\mathbb{T})} \leq R_0 \quad (3.7)$$

and

$$\int_0^T \|\sigma u_n(t)\|_{H^2(\mathbb{T})}^2 dt \leq \frac{1}{n} \|u_{0,n}\|_{H^2(\mathbb{T})}^2, \quad (3.8)$$

where $u_{0,n} = u_n(0)$. Since $\alpha_n = \|u_{0,n}\|_{H^2(\mathbb{T})} \leq \|u_{0,n}\|_{H^3(\mathbb{T})} \leq R_0$, we can choose a subsequence of (α_n) , still denoted by $(\alpha_n)_{n \in \mathbb{N}}$, such that $\lim_{n \rightarrow \infty} \alpha_n = \alpha$. Observe that, from (3.8), we obtain $\alpha_n > 0$.

Following the notation introduced above, we consider the function $v_n = \frac{u_n}{\alpha_n}$, for all $n \geq 1$. Then, v_n satisfies

$$\begin{aligned} v_{n,t} + v_{n,x} - b_1 v_{n,txx} + a_1 v_{n,xxx} + b v_{n,txxxx} + a v_{n,xxxxx} \\ + \frac{3}{2} \alpha_n v_n v_{n,x} - \gamma \alpha_n (v_n^2)_{xxx} - \frac{7}{48} \alpha_n (v_{n,x}^2)_x - \frac{1}{8} \alpha_n^2 (v_n^3)_x = -\sigma(I - b_1 \partial_x^2 + b \partial_x^4) \sigma v_n \end{aligned}$$

and

$$\int_0^T \|\sigma v_n(t)\|_{H^2(\mathbb{T})}^2 dt \leq \frac{1}{n}. \quad (3.9)$$

Moreover,

$$\|v_n(0)\|_{H^2(\mathbb{T})} = 1. \quad (3.10)$$

Since

$$\|v_n(0)\|_{H^3(\mathbb{T})} = \left\| \frac{u_n(0)}{\alpha_n} \right\|_{H^3(\mathbb{T})} = \frac{\|u_n(0)\|_{H^3(\mathbb{T})}}{\alpha_n} \leq \frac{R}{r},$$

for all $n \in \mathbb{N}$, the sequence $(v_n)_{n \in \mathbb{N}}$ is bounded in $L^\infty(0, T; H^3(\mathbb{T}))$ and $(v_{n,t})_{n \in \mathbb{N}}$ is bounded in $L^\infty(0, T; H^2(\mathbb{T}))$. Then, from Aubin-Lions lemma, we deduce that $(v_n)_{n \in \mathbb{N}}$ is bounded in $C(0, T; H^s(\mathbb{T}))$ for $2 < s < 3$. Therefore, we can extract a subsequence of $(v_n)_{n \in \mathbb{N}}$, still denoted by $(v_n)_{n \in \mathbb{N}}$, such that

$$v_n \rightarrow v \quad \text{in} \quad C(0, T; H^s(\mathbb{T})), \quad (3.11)$$

$$v_n \rightarrow v \quad \text{in} \quad L^\infty(0, T; H^3(\mathbb{T})) \quad \text{weak-}^*, \quad (3.12)$$

for some $v \in L^\infty(0, T; H^3(\mathbb{T})) \cap C([0, T]; H^s(\mathbb{T}))$, for all $2 < s < 3$. Consequently, from (3.11) and (3.12), we have that

$$\begin{aligned} \alpha_n v_n v_{n,x} &\rightarrow \alpha v v_x \quad \text{in} \quad L^\infty(0, T, L^2(\mathbb{T})) \quad \text{weak-}^*, \\ \alpha_n (v_n^2)_{xxx} &\rightarrow \alpha (v^2)_{xxx} \quad \text{in} \quad L^\infty(0, T, L^2(\mathbb{T})) \quad \text{weak-}^*, \\ \alpha_n (v_{n,x}^2)_x &\rightarrow \alpha (v_x^2)_x \quad \text{in} \quad L^\infty(0, T, L^2(\mathbb{T})) \quad \text{weak-}^*, \\ \alpha_n^2 (v_n^3)_x &\rightarrow \alpha^2 (v^3)_x \quad \text{in} \quad L^\infty(0, T, L^2(\mathbb{T})) \quad \text{weak-}^*. \end{aligned}$$

Furthermore, by (3.9),

$$\int_0^T \|\sigma v\|_{H^2(\mathbb{T})}^2 dt \leq \liminf_{n \rightarrow \infty} \int_0^T \|\sigma v_n\|_{H^2(\mathbb{T})}^2 dt = 0. \quad (3.13)$$

Thus, v solves

$$\begin{aligned} &v_t + v_x - b_1 v_{txx} + a_1 v_{xxx} + b v_{txxxx} + a v_{xxxxx} \\ &+ \frac{3}{2} \alpha v v_x - \gamma \alpha (v^2)_{xxx} - \frac{7}{48} (v_x^2)_x - \frac{1}{8} \alpha^2 (v^3)_x = -\sigma (I - b_1 \partial_x^2 + b \partial_x^4) \sigma v, \quad (x, t) \in \mathbb{T} \times (0, T), \end{aligned}$$

and, in addition,

$$v = 0, \quad \text{in} \quad \omega \times (0, T).$$

From the UCP proved in Theorem 2.1 we conclude that $v \equiv 0$ in $\mathbb{T} \times (0, T)$.

In order to obtain a contradiction, we first claim that $(v_n)_{n \in \mathbb{N}}$ is linearizable in the sense of Proposition 9 in [7]. This is to say that, if $(w_n)_{n \in \mathbb{N}}$ denotes the sequence of solutions of the linear higher-order KdV-BBM

equation with the same initial data as follows

$$\begin{aligned} w_{n,t} + w_{n,x} - b_1 w_{n,txx} + a_1 w_{xxx} + b w_{txxxx} + a w_{xxxxx} &= -\sigma(I - b_1 \partial_x^2 + b \partial_x^4)[\sigma w_n], \\ w_n(x, 0) &= v_n(x, 0), \end{aligned} \quad (3.14)$$

then

$$\sup_{0 \leq t \leq T} \|v_n(t) - w_n(t)\|_{H^2(\mathbb{T})} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (3.15)$$

Indeed, if $d_n = v_n - w_n$, then d_n solves

$$\begin{aligned} d_{n,t} + d_{n,x} - b_1 d_{n,txx} + a_1 d_{n,xxx} + b d_{n,txxxx} + a d_{n,xxxxx} \\ = -\sigma(I - b_1 \partial_x^2 + b \partial_x^4)[\sigma d_n] - \frac{3}{2} \alpha_n v_n v_{n,x} - \gamma \alpha_n (v_n^2)_{xxx} + \frac{7}{48} \alpha_n (v_{n,x}^2)_x + \frac{1}{8} \alpha_n^2 (v_n^3)_x \\ d_n(0) = 0. \end{aligned}$$

Since $\|W_a(t)\|_{\mathcal{L}(H^2(\mathbb{T}))} \leq M e^{\omega t} \leq M e^{\omega T}$, with $\omega, M > 0$, from Duhamel formula we have that, for $t \in [0, T]$,

$$\begin{aligned} \|d_n(t)\|_{H^2(\mathbb{T})} &\leq M e^{\omega T} \left(\int_0^T \|(I - b_1 \partial_x^2 + b \partial_x^4)^{-1} \frac{3}{2} \alpha_n v_n v_{n,x}\|_{H^2(\mathbb{T})} dt \right. \\ &\quad + \int_0^T \|(I - b_1 \partial_x^2 + b \partial_x^4)^{-1} \gamma \alpha_n (v_n^2)_{xxx}\|_{H^2(\mathbb{T})} dt \\ &\quad + \int_0^T \|(I - b_1 \partial_x^2 + b \partial_x^4)^{-1} \frac{7}{48} \alpha_n (v_{n,x}^2)_x\|_{H^2(\mathbb{T})} dt \\ &\quad \left. + \int_0^T \|(I - b_1 \partial_x^2 + b \partial_x^4)^{-1} \frac{1}{8} \alpha_n^2 (v_n^3)_x\|_{H^2(\mathbb{T})} dt \right). \end{aligned}$$

The above estimate combined with (3.11)–(3.12) and the fact that $v \equiv 0$ give us (3.15).

The next steps are devoted to prove that $\|v_n(0)\|_{H^2(\mathbb{T})}^2 \rightarrow 0$, as $n \rightarrow \infty$. In fact, from Lemma 3.1 we have that

$$\|w_n(t)\|_{H^2(\mathbb{T})} \leq c_1 e^{-\beta t} \|w_n(0)\|_{H^2(\mathbb{T})}, \quad \text{for all } t \geq 0, \quad (3.16)$$

and from the energy identity for (3.14), we get

$$\|w_n(t)\|_{H^2(\mathbb{T})}^2 - \|w_n(0)\|_{H^2(\mathbb{T})}^2 = -2 \int_0^t \|\sigma w_n(s)\|_{H^2(\mathbb{T})}^2 ds$$

or

$$\|w_n(0)\|_{H^2(\mathbb{T})}^2 = \|w_n(t)\|_{H^2(\mathbb{T})}^2 + 2 \int_0^t \|\sigma w_n(s)\|_{H^2(\mathbb{T})}^2 ds.$$

Therefore, from (3.16) it follows that

$$\|w_n(0)\|_{H^2(\mathbb{T})}^2 \leq 2(1 - c_1^2 e^{-2\beta T})^{-1} \left[\int_0^T \|\sigma w_n(t) - \sigma v_n(t)\|_{H^2(\mathbb{T})}^2 dt + \int_0^T \|\sigma v_n(t)\|_{H^2(\mathbb{T})}^2 dt \right]. \quad (3.17)$$

Estimate (3.17) combined with (3.9) and (3.15) yields $\|v_n(0)\|_{H^2(\mathbb{T})}^2 = \|w_n(0)\|_{H^2(\mathbb{T})}^2 \rightarrow 0$, as $n \rightarrow \infty$, which contradicts (3.10). $\square$

The main result of this section reads as follows:

Theorem 3.4. *Let $\sigma \in C^\infty(\mathbb{T})$ with $\sigma \neq 0$, and $\beta > 0$ be as given in Lemma 3.1. Then, for any $R_0, r > 0$, there exists a constant $C > 0$, such that, for any $u_0 \in H^3(\mathbb{T})$ with $0 < r \leq \|u_0\|_{H^2(\mathbb{T})}$ and $\|u_0\|_{H^3(\mathbb{T})} \leq R_0$, the corresponding solution u of (1.2) satisfies*

$$\|u(\cdot, t)\|_{H^2(\mathbb{T})} \leq C e^{-\beta t} \|u_0\|_{H^2(\mathbb{T})}.$$

Proof. From Proposition 3.3 and the energy identity

$$\|u(t)\|_{H^2(\mathbb{T})}^2 = \|u(0)\|_{H^2(\mathbb{T})}^2 - 2 \int_0^t \|\sigma u(\tau)\|_{H^2(\mathbb{T})}^2 d\tau, \quad t \geq 0,$$

we have

$$\|u(T)\|_{H^2(\mathbb{T})}^2 \leq (1 - 2\theta^{-1}) \|u(0)\|_{H^2(\mathbb{T})}^2.$$

Thus,

$$\|u(kT)\|_{H^2(\mathbb{T})}^2 \leq (1 - 2\theta^{-1})^k \|u(0)\|_{H^2(\mathbb{T})}^2, \quad k \in \mathbb{N},$$

which, by the semigroup property, leads to

$$\|u(t)\|_{H^2(\mathbb{T})}^2 \leq C e^{-\kappa t} \|u(0)\|_{H^2(\mathbb{T})}^2, \quad \text{for all } t \geq 0.$$

$\square$

APPENDIX A.

Proof of Lemma 2.4: (see [11], Lem. 5.5):

We first assume that $w \in H^1(\mathbb{T} \times (0, T))$. Let $v = e^{s\varphi}$ and $P = \partial_t + \frac{a}{b} \partial_x$. Then

$$\begin{aligned} e^{s\varphi} Pw &= e^{s\varphi} P(e^{-s\varphi} v) \\ &= (-s\varphi_t v - \frac{a}{b} s\varphi_x v) + (v_t + \frac{a}{b} v_x) \\ &= P_n v + P_p v. \end{aligned}$$

It follows that

$$\|e^{s\varphi} Pw\|_{L^2(\mathbb{T} \times (0, T))}^2 = \|P_p v\|_{L^2(\mathbb{T} \times (0, T))}^2 + \|P_n v\|_{L^2(\mathbb{T} \times (0, T))}^2 + 2(P_p v, P_n v)_{L^2(\mathbb{T} \times (0, T))}. \quad (\text{A.1})$$

After some integrations by parts in t and x in the last term in (A.1), we obtain

$$\begin{aligned} 2(P_p v, P_n v)_{L^2(\mathbb{T} \times (0, T))}^2 &= \int_0^T \int_{\mathbb{T}} s(\varphi_{tt} + 2\frac{a}{b}\varphi_{xt} + \frac{a^2}{b^2}\varphi_{xx})v^2 dx dt \\ &\quad - \int_{\mathbb{T}} s(\varphi_t + \frac{a}{b}\varphi_x)v^2|_0^T dx - \int_0^T \frac{a}{b}s(\varphi_t + \frac{a}{b}\varphi_x)v^2|_0^{2\pi} dt. \end{aligned} \quad (\text{A.2})$$

Using (2.10)–(2.12) and the fact that $v(0, t) = v(2\pi, t)$, we notice that the last term in (A.2) is null. From (2.9)–(2.12), we infer that

$$\begin{aligned} \varphi_{tt} + 2\frac{a}{b}\varphi_{tx} + \frac{a^2}{b^2}\varphi_{xx} &= 2(1 - \rho)\frac{a^2}{b^2} > 0 \quad \text{for } (x, t) \in (\frac{\eta}{2}, 2\pi - \frac{\eta}{2}) \times (0, T), \\ -(\varphi_t + \frac{a}{b}\varphi_x) &\geq 2\frac{a}{b}(\frac{aT\rho}{b} - 2\pi - \delta) > 0 \quad \text{for } x \in (0, 2\pi), t = T \\ \varphi_t + \frac{a}{b}\varphi_x &\geq 2\frac{a}{b}\delta > 0 \quad \text{for } x \in (0, 2\pi), t = 0. \end{aligned}$$

Thus,

$$\int_0^T \int_{\mathbb{T}} s|v|^2 dx dt + \int_{\mathbb{T}} s(|v|_{t=0}^2 + |v|_{t=T}^2) dx \leq C \left(\int_0^T \int_{\mathbb{T}} |e^{s\varphi} Pw|^2 dx dt + \int_0^T \int_{\omega} s|v|^2 dx dt \right),$$

which gives, at once, (2.14) by replacing v by $e^{s\varphi}w$. The proof of Lemma 2.4 is achieved when $w \in H^1(\mathbb{T} \times (0, T))$. We now claim that Lemma 2.4 is still true when w and f are in $L^2(0, T; L^2(\mathbb{T}))$. Indeed, in the case $w \in C([0, T]; L^2(\mathbb{T}))$, and if (w_n^0) and f_n are two sequences in $H^1(\mathbb{T})$ and $L^2(0, T; H^1(\mathbb{T}))$, respectively, such that

$$\begin{aligned} w_n^0 &\rightarrow w(0) \quad \text{in } L^2(\mathbb{T}) \\ f_n &\rightarrow f \quad \text{in } L^2(0, T; L^2(\mathbb{T})), \end{aligned}$$

then the solution $w^n \in C([0, T]; H^1(\mathbb{T}))$ of

$$\begin{aligned} w_t^n + \frac{a}{b}w_x^n + f^n, \\ w^n(0) = w_0^n \end{aligned}$$

satisfies $w^n \in H^1(\mathbb{T} \times (0, T))$ and $w^n \rightarrow w$ in $C([0, T]; L^2(\mathbb{T}))$, so that we can apply (2.14) to w^n and then pass to the limit $n \rightarrow \infty$. The proof of Lemma 2.4 is complete.

Proof of (3.4):

Observe that

$$c_{b,b_1}(1 + bx^2 + b_1x^4)^{\frac{s}{2}} \leq [(1 + x^2)^2]^{\frac{s}{2}} \leq C_{b,b_1}(1 + bx^2 + b_1x^4)^{\frac{s}{2}},$$

for $s \geq 2$ and some positive constants c_{b,b_1}, C_{b,b_1} . Then, we can define the following equivalent inner product in $H^s(\mathbb{T})$ as

$$(u, v)_s = \int_{\mathbb{T}} (1 + bx^2 + b_1x^4)^{\frac{s}{2}} \mathcal{F}u(x) \overline{\mathcal{F}v}(x) dx,$$

where $\mathcal{F}\varphi$ denote the Fourier transform of φ . Hence, employing Plancherel Theorem, we get

$$\begin{aligned}
(B\varphi, \psi)_s &= \int_{\mathbb{T}} (1 + bx^2 + b_1x^4)^{\frac{s}{2}} \mathcal{F}[(1 - b\partial_x^2 + b_1\partial_x^4)^{-1} \sigma(x)(1 - b\partial_x^2 + b_1\partial_x^4) \varphi(x)] \overline{\mathcal{F}\psi(x)} dx \\
&= \int_{\mathbb{T}} (1 + bx^2 + b_1x^4)^{\frac{s}{2}-1} \mathcal{F}[\sigma(x)(1 - b\partial_x^2 + b_1\partial_x^4) \varphi(x)] \overline{\mathcal{F}\psi(x)} dx \\
&= \int_{\mathbb{T}} \mathcal{F}[\sigma(x)(1 - b\partial_x^2 + b_1\partial_x^4) \varphi(x)] \overline{\mathcal{F}[(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1} \psi(x)]} dx \\
&= (\sigma(x)(1 - b\partial_x^2 + b_1\partial_x^4) \varphi, (1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1} \psi)_{L^2(\mathbb{T})} \\
&= ((1 - b\partial_x^2 + b_1\partial_x^4) \varphi, \sigma(x)(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1} \psi)_{L^2(\mathbb{T})} \\
&= \int_{\mathbb{T}} \mathcal{F}[(1 - b\partial_x^2 + b_1\partial_x^4) \varphi(x)] \overline{\mathcal{F}[\sigma(x)(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1} \psi(x)]} dx \\
&= \int_{\mathbb{T}} (1 + bx^2 + b_1x^4)^{\frac{s}{2}} \mathcal{F}\varphi(x) (1 + bx^2 + b_1x^4)^{1-\frac{s}{2}} \overline{\mathcal{F}[\sigma(x)(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1} \psi(x)]} dx \\
&= \int_{\mathbb{T}} (1 + bx^2 + b_1x^4)^{\frac{s}{2}} \mathcal{F}\varphi(x) \overline{\mathcal{F}[(1 - b\partial_x^2 + b_1\partial_x^4)^{1-\frac{s}{2}} \sigma(x)(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1} \psi(x)]} dx \\
&= (\varphi, (1 - b\partial_x^2 + b_1\partial_x^4)^{1-\frac{s}{2}} \sigma(x)(1 - b\partial_x^2 + b_1\partial_x^4)^{\frac{s}{2}-1} \psi)_s,
\end{aligned}$$

for all $\varphi, \psi \in H^s(\mathbb{T})$. From the computation above we deduce (3.4).

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