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OBSERVABILITY OF A STRING-BEAMS NETWORK WITH MANY BEAMS

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Abstract. We prove the direct and inverse observability inequality for a network connecting one string with infinitely many beams, at a common point, in the case where the lengths of the beams are all equal. The observation is at the exterior node of the string and at the exterior nodes of all the beams except one. The proof is based on a careful analysis of the asymptotic behavior of the underlying eigenvalues and eigenfunctions, and on the use of a Ingham type theorem with weakened gap condition [C. Baiocchi, V. Komornik and P. Loreti, *Acta Math. Hung.* **97** (2002) 55–95.]. On the one hand, the proof of the crucial gap condition already observed in the case where there is only one beam [K. Ammari, M. Jellouli and M. Mehrenberger, *Networks Heterogeneous Media* **4** (2009) 2009.] is new and based on elementary monotonicity arguments. On the other hand, we are able to handle both the complication arising with the appearance of eigenvalues with unbounded multiplicity, due to the many beams case, and the terms coming from the weakened gap condition, arising when at least 2 beams are present.

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1. Introduction

The present paper aims to investigate evolutive control models for networks of vibrating structures, with particular attention to bio-inspired models and to possible applications to network engineering. The control of complex networks is a deeply investigated question in network science and engineering [15]. In particular, the paper [12] studies systems with an arbitrarily large number strings and systems with a possibly infinite number of beams, with a common endpoint. For both classes of systems, exact observability conditions were established. On the other hand, [1] deals with the observability of a heterogeneous structure composed by a beam and a string. This paper presents an extension of the above two papers to composite structures composed by a string and infinitely many beams. In particular we investigate the simultaneous observability of the system, namely the existence of a one-to-one correspondence between the initial data and some observed data during the evolution of the system. The study of infinite systems is motivated by the search of uniform constants

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associated to the observability conditions. Uniformity in the constants indeed allows to cope with perturbations of the topology of the network, for instance by adding or removing a beam. The extension from "one string-one beam" system to a "one string-many beams" system comes with some technical difficulties. This is related to the appearance of eigenvalues with unbounded multiplicity, and of new terms in the observable associated to the weakened gap condition. Future perspectives include the generalization of the present approach to different boundary conditions and networks. We remark that in [12] an application of a Minkowski theorem on Diophantine approximations seems to prevent the observability of infinite strings: however the discussion of string-beam structures with many strings remains an open problem.

The paper is organized as follows. In Section 2 we introduce the main results of the paper. They are: Theorem 2.1, dealing with the well posedness of the system under exam; Theorem 2.2, providing new norm estimates for the initial energy of a "one string-one beam" system; and Theorem 2.3, stating some sufficient exact observability conditions for a "one string-(infinitely) many beams" system. In Section 3 we perform a spectral analysis of the system, we characterize the eigenvalues and related eigenspaces, and we express the solutions in terms of Fourier series, so to prove Theorem 2.1. Section 4 is devoted to the proofs of Theorem 2.2 and Theorem 2.3. More precisely, in Section 4.1 we establish a generalized gap condition on the eigenvalues; Section 4.2 contains the proof of Theorem 2.2 and some estimates of Ingham type with weakened gap conditions and Section 4.3 is devoted to the proof of Theorem 2.3. Finally, in Section 5 we present some numerical simulations.

In the remaining part of the present introduction, we present the system under exam.

We use the following notation $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$. Now, let $J \in \mathbb{N}^* \cup \{\infty\}$, $\ell > 0$ and $\alpha_1, \ldots, \alpha_J > 0$, such that $A := \sum_{j=1}^{J} \alpha_j < \infty$. We consider the following system (see [13], *e.g.* pages 80–81, for the model) coupling one string with J beams of same length ℓ:

$$
\begin{cases}
(\partial_t^2 u_0 - \partial_x^2 u_0)(t,x) = 0, & x \in (0,1), \\
(\partial_t^2 u_j + \partial_x^4 u_j)(t,x) = 0, & x \in (0,\ell),\ t > 0, \\
u_0(t,0) = 0,\ u_j(t,0) = 0,\ \partial_x^2 u_j(t,0) = 0,\ \partial_x^2 u_j(t,\ell) = 0, & t > 0, \\
u_0(t,1) = u_j(t,\ell),\ \partial_x u_0(t,1) = \displaystyle\sum_{k=1}^{J} \alpha_k \partial_x^3 u_k(t,\ell), & t > 0, \\
u_0(0,x) = u_0^0(x),\ \partial_t u_0(0,x) = u_0^1(x), & x \in (0,1), \\
u_j(0,x) = u_j^0(x),\ \partial_t u_j(0,x) = u_j^1(x), & x \in (0,\ell),
\end{cases}
\tag{1.1}
$$

for $j = 1, \ldots, J$. A stabilization problem corresponding to such system has been studied in [1] for $J = 1$, in [3] for arbitrary finite J and in [5] in a more general framework. In particular, for $J = 1$, an observability inequality is obtained thanks to Ingham's theorem, using a gap condition of the underlying eigenvalues. For arbitrary J, however, the gap does no longer hold and an Ingham type approach is more challenging, as also pointed in [4], where another generalization of the case $J = 1$ has been performed. Some numerical illustrations have been performed in [3] to suggest that such observability inequality still holds, but the proof remains to be done. The origin of this work dates back to [2], where the model was introduced, but the study was incomplete. We are here able to give a complete theoretical study and also to analyse the asymptotic behavior, where the number of beams becomes large, which was not considered in [2]. In this present case, the analysis is simplified, as the lengths of the beams are all equal, and we let the case of infinitely many beams with arbitrary length as open problem. To the best of our knowledge, the case of infinitely many beams was first considered in [12], in the different context of simultaneous observability; there, a very different behavior was observed between strings and beams.

2. Main results

We first consider an assumption on the length ℓ of the beam(s). So we define

$$\mathcal{L} := \{\ell \in \mathbb{R}_+^*, \ p\pi \neq \frac{k^2\pi^2}{\ell^2}, \ k \in \mathbb{Z}, \ p \in \mathbb{N}^*\}.$$

We state the well posedness of the system. To this end, recall the definition $A := \sum_{j=1}^J \alpha_j$ and consider the function f given by

$$f(z) := f_{\ell,A}(z) = 2\cot(z^2) + Az(\cot(z\ell) - \coth(z\ell)).$$

We prove in Lemma 3.2 that the positive zeros of f form a strictly increasing sequence that we denote by $(z_n)_{n\in\mathbb{N}^*}$. Also we use the notation $u = (u_0, u_1, \dots, u_J)$ and we consider the space

$$V = \left\{ u \in H^1(0,1) \times \prod_{j=1}^J H^2(0,\ell), \ u_0(0) = 0, \ u_j(0) = 0, \ u_0(1) = u_j(\ell), \ j = 1, \dots, J \right\}$$

and the Hilbert space

$$H := \left\{ (u,v) \in V \times \left(L^2(0,1) \times \prod_{j=1}^J L^2(0,\ell) \right), \ \|(u,v)\|_H^2 < \infty \right\},$$

with the norm

$$\|(u,v)\|_H^2 := \int_0^1 |v_0|^2 \mathrm{d}x + \sum_{j=1}^J \alpha_j \int_0^\ell |v_j|^2 \mathrm{d}x + \int_0^1 |\partial_x u_0|^2 \mathrm{d}x + \sum_{j=1}^J \alpha_j \int_0^\ell |\partial_x^2 u_j|^2 \mathrm{d}x.$$

Our first result is a series representation for the solution to (1.1).

Theorem 2.1. *Let $\ell \in \mathcal{L}$ and $J \in \mathbb{N}^* \cup \{\infty\}$. Assume that $(a_n)_{n\in\mathbb{Z}^*}, (b_{k,q})_{k\in\mathbb{Z}^*, q=1,\dots,J-1} \in \mathbb{C}$ are sequences with a finite number non zero elements, and let $(\mathrm{e}^q)_{q=1,\dots,J-1}$ be a basis of $\mathcal{C} := \{(C_1, \dots, C_J) \in \mathbb{C}^J, \sum_{j=1}^J \alpha_j C_j = 0\}$. Let $u : (t,x) \mapsto (u_0(t,x), \dots, u_J(t,x))$ be defined by*

$$u_0(t,x) := \sum_{n\in\mathbb{Z}^*}^{\infty} a_n \mathrm{e}^{\frac{n}{|n|} i z_n^2 t} \sin(z_n^2 x)$$

$$u_j(t,x) := \sum_{n\in\mathbb{Z}^*}^{\infty} a_n \mathrm{e}^{\frac{n}{|n|} i z_n^2 t} \frac{\sin(z_n^2)}{2} \left(\frac{\sinh(z_n x)}{\sinh(z_n \ell)} + \frac{\sin(z_n x)}{\sin(z_n \ell)} \right) + \sum_{k\in\mathbb{Z}^*}^{\infty} \left(\sum_{q=1}^{J-1} b_{k,q} e_j^q \right) \mathrm{e}^{\frac{k}{|k|} i \frac{\pi^2 k^2}{\ell^2} t} \sin\left(\frac{k\pi}{\ell} x \right),$$

$$j = 1, \dots, J.$$

Then, for $(u^0, u^1) = (u(0,\cdot), \partial_t u(0,\cdot))$, the function u is the unique solution of system (1.1) belonging to $C([0,\infty), H)$.

We focus now on observability results. We also use the notation $a \asymp b$, meaning that there exists constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 b$.

Let us first consider the case where $J=1$. Note that such case was already studied in [1], but, there, the observation was at the junction point with a more restrictive assumption on the length ℓ. We define the initial energy of the system (1.1) as

$$E_{0,J} := ||(u^0,u^1)||_H^2 = \int_0^1 |u_0^1(x)|^2 \mathrm{d}x + \sum_{j=1}^J \alpha_j \int_0^\ell |u_j^1(x)|^2 \mathrm{d}x + \int_0^1 |\partial_x u_0^0(x)|^2 \mathrm{d}x + \sum_{j=1}^J \alpha_j \int_0^\ell |\partial_x^2 u_j^0(x)|^2 \mathrm{d}x.$$

We get the following result, when observing on the exterior node of the string:

Theorem 2.2. *Let $\ell \in \mathcal{L}$ and $J=1$. Let $(a_n)_{n\in\mathbb{Z}^*} \in \mathbb{C}$ be a sequence with a finite number non zero elements and assume that the initial data of (1.1) are*

$$u^0(x) := \left(\sum_{n\in\mathbb{Z}^*}^{\infty} a_n \sin(z_n^2 x), \sum_{n\in\mathbb{Z}^*}^{\infty} a_n \frac{\sin(z_n^2)}{2}\left(\frac{\sinh(z_n x)}{\sinh(z_n\ell)} + \frac{\sin(z_n x)}{\sin(z_n \ell)}\right)\right)$$

and

$$u^1(x) := \left(\sum_{n\in\mathbb{Z}^*}^{\infty} a_n \frac{n}{|n|} i z_n^2 \sin(z_n^2 x), \sum_{n\in\mathbb{Z}^*}^{\infty} a_n \frac{n}{|n|} i z_n^2 \frac{\sin(z_n^2)}{2}\left(\frac{\sinh(z_n x)}{\sinh(z_n\ell)} + \frac{\sin(z_n x)}{\sin(z_n \ell)}\right)\right).$$

Then, the corresponding solution u of (1.1) satisfies for $T>2$,

$$E_{0,J} \asymp \int_0^T |\partial_x u_0(t,0)|^2 \mathrm{d}t,$$

where the related constants are independent of the initial data.

Observing again on the exterior node of the string, but also on the exterior nodes of the beams, except one, we get our main result:

Theorem 2.3. *Let $\ell \in \mathcal{L}$, $2 \le J \le \infty$ and assume that $\alpha_j \asymp \beta_j$, $j = 2,\dots,J$, together with $\alpha_1 \asymp 1$, $A = \sum_{j=1}^J \alpha_j \asymp 1$, with constants independent of J. Let Z be the set of pairs $(u^0(x), u^1(x))$ where $u^p = (u_0^p(x),\dots,u_J^p(x))$, $p = 0,1$ and*

$$u_0^0(x) = \sum_{n\in\mathbb{Z}^*}^{\infty} a_n \sin(z_n^2 x),$$

$$u_j^0(x) = \sum_{n\in\mathbb{Z}^*}^{\infty} a_n \frac{\sin(z_n^2)}{2}\left(\frac{\sinh(z_n x)}{\sinh(z_n\ell)} + \frac{\sin(z_n x)}{\sin(z_n \ell)}\right) + \sum_{k\in\mathbb{Z}^*}^{\infty}\left(\sum_{q=1}^{J-1} b_{k,q} e_j^q\right)\sin\left(\frac{k\pi}{\ell}x\right), \quad j=1,\dots,J,$$

$$u_0^1(x) = a_n \frac{n}{|n|} i z_n^2 \sin(z_n^2 x),$$

$$u_j^1(x) = \sum_{n\in\mathbb{Z}^*}^{\infty} a_n \frac{n}{|n|} i z_n^2 \frac{\sin(z_n^2)}{2}\left(\frac{\sinh(z_n x)}{\sinh(z_n\ell)} + \frac{\sin(z_n x)}{\sin(z_n \ell)}\right) + \sum_{k\in\mathbb{Z}^*}^{\infty}\left(\sum_{q=1}^{J-1} b_{k,q} e_j^q\right)\frac{k}{|k|} i \frac{\pi^2 k^2}{\ell^2}\sin\left(\frac{k\pi}{\ell}x\right), \quad j=1,\dots,J,$$

for some sequences $(a_n)_{n\in\mathbb{Z}^*}, (b_{k,q})_{k\in\mathbb{Z}^*, q=1,\dots,J-1} \in \mathbb{C}$ with a finite number of non zero elements.

We then have

$$\int_0^T |\partial_x u_0(t,0)|^2 \mathrm{d}t + \sum_{j=2}^{J} \beta_j \int_0^T |\partial_x^3 u_j(t,0)|^2 \mathrm{d}t \asymp \sum_{n\in\mathbb{Z}^*} |z_n|^4 |a_n|^2 \left(1 + (A-\alpha_1)\left|\frac{z_n \sin(z_n^2)}{\sin(z_n \ell)}\right|^2\right) + \sum_{k\in\mathbb{Z}^*} \left(\frac{k\pi}{\ell}\right)^6 \sum_{j=1}^{J} \alpha_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2,$$

and the underlying constants do not depend on the initial data nor on J.

3. Spectral analysis

The spectral analysis of such system has already been performed in [1, 3], for $J = 1$ or finite J. We suppose here that $1 \le J \le \infty$. We use the notation $w = (w_0, w_1, \dots, w_J)$.

The system then rewrites $U' = \mathcal{A}U,\ U(0) = U^0$, with $U = (u, \partial_t u)$ and $U^0 = (u^0, u^1)$, where the operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$ is defined by

$$\mathcal{A}(u,v) = \left(v, \partial_x^2 u_0, -\partial_x^4 u_1, \dots, -\partial_x^4 u_J\right),\ (u,v) \in D(\mathcal{A})$$

with

$$D(\mathcal{A}) = \Bigg\{ (u,v) \in H \cap \left(\left(H^2(0,1) \times \prod_{j=1}^{J} H^4(0,\ell) \right) \times \left(L^2(0,1) \times \prod_{j=1}^{J} L^2(0,\ell) \right) \right), \ \mathcal{A}(u,v) \in H,\ (u,v) \text{ satisfies } (3.1),(3.2) \Bigg\},$$

the additional boundary conditions being given by

$$\partial_x^2 u_j(0) = 0,\ \partial_x^2 u_j(\ell) = 0,\ \partial_x u_0(1) = \sum_{j=1}^{J} \alpha_j \partial_x^3 u_j(\ell), \tag{3.1}$$

and the additional assumption given by

$$\sum_{j=1}^{J} \alpha_j \|u_j\|^2_{H^4(0,\ell)} < \infty. \tag{3.2}$$

We first have a skew symmetric property for $\mathcal{A}$, writing $\langle \cdot,\cdot \rangle_H$ the scalar products corresponding to the norm of H.

Proposition 3.1. *We have*

(i) *If $U \in H$ and $\langle U, U\rangle_H = 0$, then $U = 0$.*
(ii) *$\langle \mathcal{A}U, \tilde{U}\rangle_H = -\langle U, \mathcal{A}\tilde{U}\rangle_H$, for all $U, \tilde{U} \in D(\mathcal{A})$.*
(iii) *If Φ is an eigenvector of $\mathcal{A}$ with corresponding eigenvalue λ (i.e., $\Phi \in D(\mathcal{A})$, $\Phi \neq 0$ and $\mathcal{A}\Phi = \lambda\Phi$), then $\lambda \in i\mathbb{R}^*$.*

Proof. (i) If $U \in H$ and $\langle U, U\rangle_H = 0$, we get $u_0 \in H^1(0,1)$ and $\partial_x u_0 = 0$. Then $u_0(x) = u_0(0) = 0$. From the continuity condition we get $u_j(\ell) = u_0(0) = 0$. Moreover $u_j(0) = 0$ and $\partial_x^2 u_j = 0$, with $u_j \in H^2(0,\ell)$, we also get $u_j = 0$ and thus $U = 0$.

(ii) We write $U = (u, v), \tilde{U} = (\tilde{u}, \tilde{v})$. We preliminarily remark that $U, \tilde{U} \in D(\mathcal{A})$ implies that $\tilde{u}$ satisfies (3.1) and that $v \in V$, so that

$$[v_0(x)\overline{\partial_x \tilde{u}_0(x)}]_{x=0}^{x=1} + \sum_{j=1}^{J} \alpha_j [\partial_x v_j(x)\overline{\partial_x^2 \tilde{u}_j(x)}]_{x=0}^{x=\ell} - \sum_{j=1}^{J} \alpha_j [v_j(x)\overline{\partial_x^3 \tilde{u}_j(x)}]_{x=0}^{x=\ell} = 0.$$

Therefore by integrating by parts we have the identity

$$\int_0^1 \partial_x v_0(x)\overline{\partial_x \tilde{u}_0(x)}\mathrm{d}x + \sum_{j=1}^{J} \alpha_j \int_0^\ell \partial_x^2 v_j(x)\overline{\partial_x^2 \tilde{u}_j(x)}\mathrm{d}x = -\int_0^1 v_0(x)\overline{\partial_x^2 \tilde{u}_0(x)}\mathrm{d}x + \sum_{j=1}^{J} \alpha_j \int_0^\ell v_j(x)\overline{\partial_x^4 \tilde{u}_j(x)}\mathrm{d}x,$$

and, consequently,

$$\begin{aligned}\langle \mathcal{A}(u,v), (\tilde{u},\tilde{v})\rangle_H = &\int_0^1 \partial_x^2 u_0(x)\overline{\tilde{v}_0(x)}\mathrm{d}x - \sum_{j=1}^{J} \alpha_j \int_0^\ell \partial_x^4 u_j(x)\overline{\tilde{v}_j(x)}\mathrm{d}x \\ &- \int_0^1 v_0(x)\overline{\partial_x^2 \tilde{u}_0(x)}\mathrm{d}x + \sum_{j=1}^{J} \alpha_j \int_0^\ell v_j(x)\overline{\partial_x^4 \tilde{u}_j}(x)\mathrm{d}x.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\langle (u,v), \mathcal{A}(\tilde{u},\tilde{v})\rangle_H = &\int_0^1 v_0(x)\overline{\partial_x^2 \tilde{u}_0(x)}\mathrm{d}x - \sum_{j=1}^{J} \alpha_j \int_0^\ell v_j(x)\overline{\partial_x^4 \tilde{u}_j(x)}\mathrm{d}x \\ &+ \int_0^1 \partial_x u_0(x)\overline{\partial_x \tilde{v}_0(x)}\mathrm{d}x + \sum_{j=1}^{N} \alpha_j \int_0^\ell \partial_x^2 u_j(x)\overline{\partial_x^2 \tilde{v}_j(x)}\mathrm{d}x,\end{aligned}$$

and using that u also satisfies (3.1) whereas $\tilde{v} \in V$ we get

$$\int_0^1 \partial_x u_0(x)\overline{\partial_x \tilde{v}_0(x)}\mathrm{d}x + \sum_{j=1}^{J} \alpha_j \int_0^\ell \partial_x^2 u_j(x)\overline{\partial_x^2 \tilde{v}_j(x)}\mathrm{d}x = -\int_0^1 \partial_x^2 u_0(x)\overline{\tilde{v}_0(x)}\mathrm{d}x + \sum_{j=1}^{J} \alpha_j \int_0^\ell \partial_x^4 u_j(x)\overline{\tilde{v}_j(x)}\mathrm{d}x,$$

and this leads to $\langle \mathcal{A}(u,v), (\tilde{u},\tilde{v})\rangle_H = -\langle (u,v), \mathcal{A}(\tilde{u},\tilde{v})\rangle_H$.

(iii) As consequence of the previous steps, we get that if $\Phi \neq 0$ is eigenvector then $\langle \Phi, \Phi\rangle_H \neq 0$ and the corresponding eigenvalue λ satisfies

$$\lambda\langle \Phi, \Phi\rangle_H = \langle \mathcal{A}\Phi, \Phi\rangle_H = -\langle \Phi, \mathcal{A}\Phi\rangle_H = -\overline{\lambda}\langle \Phi, \Phi\rangle_H.$$

Consequently λ is purely imaginary. Now suppose that 0 is an eigenvalue and let $\Phi = (u, v)$ be a corresponding eigenvector. We get from $\mathcal{A}(u, v) = 0$ that $v = 0$ and $\partial_x^2 u_0 = 0$ and $\partial_x^4 u_j = 0,\ j = 1, \dots, J$. Moreover, as $(u, v) \in \mathcal{D}(\mathcal{A})$, we get $\partial_x^2 u_j(0) = \partial_x^2 u_j(\ell) = 0$ leading to $\partial_x^2 u_j = 0$, for all $j = 1, \dots, J$. From (3.1), we get $\partial_x u_0(1) = 0$. As $u \in V$, we also have $u_0(0) = 0$. This leads to $u_0 = 0$. For $j = 1, \dots, J$, from

$\partial_x^2 u_j = 0$ and $u_j(0) = 0$ and $u_j(\ell) = u_0(\ell) = 0$, we get $u_j = 0$, and finally $\Phi = 0$, which contradicts the fact that Φ is an eigenvector.

□

In order to characterize the eigenvalues of $\mathcal{A}$ we shall need some notation. Recall the function f given by

$$f(z) := 2\cot(z^2) + Az(\cot(z\ell) - \coth(z\ell)).$$

Define

$$\mathcal{S} = \mathcal{S}(\ell) := \{p\pi,\ p \in \mathbb{N}\} \cup \left\{ \frac{k^2\pi^2}{\ell^2},\ k \in \mathbb{N}^* \right\}$$

and rewrite $\mathcal{S}$ as a (unique) strictly increasing sequence $(s_n)_{n\in\mathbb{N}}$. We prove below that the positive zeros of f form a strictly increasing sequence that we denote by $(z_n)_{n\in\mathbb{N}^*}$. We finally consider

$$\Lambda_1 := \{\pm z_n^2,\, n \in \mathbb{N}^*\}; \quad \Lambda_2 := \{\pm \frac{k^2\pi^2}{\ell^2},\, k \in \mathbb{N}^*\}; \quad \Lambda := \Lambda_1 \cup \Lambda_2$$

and we arrange the elements of Λ in a strictly increasing sequence $(\omega_m)_{m\in\mathbb{Z}^*}$. We define the *upper density* $D^+(X)$ of a set $X \subset \mathbb{R}$ as

$$D^+(X) := \lim_{r\to+\infty} \frac{n^+(X,r)}{r} \in [0,+\infty]$$

where $n^+(X,r)$ denotes the largest number of elements of X contained in an interval of length r.

Lemma 3.2. *We suppose that $\ell \in \mathcal{L}$.*

(i) $f(z)$ is well defined if and only if $z^2 \notin \mathcal{S}$.
(ii) f is strictly decreasing for $\sqrt{s_n} < z < \sqrt{s_{n+1}}$, for each $n \in \mathbb{N}$.
(iii) The strictly positive zeros of f form a strictly increasing, diverging sequence $(z_n)_{n\in\mathbb{N}^*}$, with $s_{n-1} < z_n^2 < s_n$, for all $n \in \mathbb{N}^*$.
(iv) The upper density of Λ_1 is $\frac{1}{\pi}$, the upper density of Λ_2 is 0 and the upper density of Λ is equal to $\frac{1}{\pi}$.

Proof. (i) It suffices to remark that $z^2 \in \mathcal{S}$ if and only if either $\sin(z^2) = 0$ or $\sin(z\ell) = 0$. In particular $z^2 \notin \mathcal{S}$ if and only if $f(z)$ is well defined and C^∞.

(ii) Let $g(z) = z(\cot(z) - \coth(z))$ and note that $f(z) = 2\cot(z^2) + Ag(z\ell)/\ell$. As $\cot(\cdot)$ is a strictly decreasing function and $\ell > 0$, it suffices to prove that $g(z)$ is decreasing too. We show in particular that $g'(z) \le 0$. Indeed

$$\begin{aligned} g'(z) &= \cot(z) - \frac{z}{\sin^2(z)} - \coth(z) + \frac{z}{\sinh^2(z)} = \frac{\cos(z)\sin(z) - z}{\sin^2(z)} + \frac{-\cosh(z)\sinh(z) + z}{\sinh^2(z)} \\ &= \frac{\sin(2z) - 2z}{2\sin^2(z)} + \frac{-\sinh(2z) + 2z}{2\sinh^2(z)} \le 0 \end{aligned}$$

for all positive z in the domain of g.

(iii) Note that $\lim_{z\to\sqrt{s_n}^\pm} f(z) = \mp\infty$ for all $n \in \mathbb{N}$. We apply the mean value theorem for the existence and the strict monotonicity of f for unicity, and use the fact that $s_0 = 0$.

(iv) On an interval $[p_1\pi,(p_1+r_1)\pi]$, we have at least r_1 élements of Λ_1 and at most r_1+1, if $r_1\pi<\delta_k:=\frac{(k+1)^2\pi^2}{\ell^2}-\frac{k^2\pi^2}{\ell^2}$. As the size δ_k tends to infinity, when k goes to infinity, we deduce that $D^+(\Lambda_1)=\frac{1}{\pi}$. Also we have $D^+(\Lambda_2)=0$ and then $D^+(\Lambda)=\frac{1}{\pi}$.

□

Now, we want to give a more precise description of the eigenvectors and eigenvalues. For convenience, in the sequel, we define $z_{-n}=-z_n,\ n\in\mathbb{N}^*$. Note that f is an even function, then $f(z_{-n})=f(-z_n)=f(z_n)=0$, since by definition z_n are the positive zeros of f.

Proposition 3.3. *We suppose that $\ell\in\mathcal{L}$.*

(i) The set of eigenvalues of $\mathcal{A}$ is $i\Lambda$, that is, if λ is an eigenvalue of $\mathcal{A}$ then $\lambda=i\omega_m$ for some $m\in\mathbb{Z}^*$. In particular either $\lambda=\pm iz_n^2$ for some $n\in\mathbb{N}^*$ or $\lambda=\pm i\frac{k^2\pi^2}{\ell^2}$ for some $k\in\mathbb{N}^*$. If Φ is an eigenvector corresponding to λ, then $\Phi=(\phi,\lambda\phi)$ for some ϕ in V satisfying (3.1).

(ii) For $\lambda=\pm iz_n^2$, the eigenvector space is of dimension 1 and if $(\phi,\lambda\phi)$ is an eigenvector, it writes $\phi_0(x)=C\sin(z_n^2x)$, $\phi_j(x)=C\frac{\sin(z_n^2)}{2}\left(\frac{\sinh(z_nx)}{\sinh(z_n\ell)}+\frac{\sin(z_nx)}{\sin(z_n\ell)}\right)$, with $j=1,\dots,J$ and $C\neq 0$.

(iii) For $\lambda=\pm i\frac{k^2\pi^2}{\ell^2}$, the eigenspace is of dimension $J-1$, and if $(\phi,\lambda\phi)$ is an eigenvector, it writes $\phi_0(x)=0$ and $\phi_j(x)=C_j\sin(\frac{k\pi}{\ell}x)$, with $\sum_{j=1}^J\alpha_jC_j=0$, and $(C_1,\dots,C_J)\neq(0,\dots,0)$.

Proof. We have seen in Proposition 3.1-(iii) that $\lambda\in i\mathbb{R}^*$ and we can thus already write $\lambda=\pm iz^2$, with $z>0$. Let $\Phi=(\phi,\psi)$ be an eigenvector. We know that ϕ satisfies (3.1) and $\psi_0=\lambda\phi_0,\ \psi_j=\lambda\phi_j$ (these identities imply $\psi=\lambda\phi$), $\partial_x^2\phi_0=\lambda\psi_0$ on $(0,1)$, $-\partial_x^4\phi_j=\lambda\psi_j$, on $(0,\ell)$, $j=1,\dots,J$. In particular one has that $\psi=(\phi_0,\phi_1,\dots,\phi_J)$ sastisfies

$$\begin{cases}\partial_x^2\phi_0=-z^4\phi_0; & \phi_0(0)=0\\ \partial_x^4\phi_j=z^4\phi_j; & \phi_j(0)=0;\ \partial_x^2\phi_j(0)=0,\ j=1,\dots,J.\end{cases}\tag{3.3}$$

Solving the first equation in (3.3), we have that $\phi_0(x)=A_1\mathrm{e}^{iz^2x}+A_2\mathrm{e}^{-iz^2x}$ for some $A_1,A_2\in\mathbb{C}$. Imposing the condition $\phi_0(0)=0$ we deduce $A_1=-A_2$ and, consequently,

$$\phi_0(x)=C\sin(z^2x)\quad\text{for some } C\in\mathbb{C}.\tag{3.4}$$

The second line of (3.3), implies $\partial_x^4\phi_j=z^4\phi_j$ for $j=1,\dots,J$, whose general solution is $\phi_j(x)=B_{j,1}\mathrm{e}^{zx}+B_{j,2}\mathrm{e}^{-zx}+C_{j,1}\mathrm{e}^{izx}+C_{j,2}\mathrm{e}^{-izx}$, since $z\neq 0$. Imposing the boundary conditions $\phi_j(0)=0$ and $\partial_x^2\phi_j(0)=0$ yields respectively

$$B_{j,1}+B_{j,2}+C_{j,1}+C_{j,2}=0,\qquad B_{j,1}+B_{j,2}-C_{j,1}-C_{j,2}=0,\quad\text{for } j=1,\dots,J$$

from which we deduce $B_{j,1}+B_{j,2}=C_{j,1}+C_{j,2}=0$. Then

$$\phi_j(x)=B_j\sinh(zx)+C_j\sin(zx)\quad\text{for some } B_j,C_j\in\mathbb{C},\ \forall j=1,\dots,J.\tag{3.5}$$

Using (3.4), (3.5) and the boundary and transmission conditions, we investigate the properties of z. First we remark that from $\partial_x^2\phi_j(\ell)=0$, we get

$$B_j\sinh(z\ell)=C_j\sin(z\ell)\quad j=1,\dots,J.\tag{3.6}$$

Using $\phi_j(\ell) = \phi_0(1)$, we derive from (3.4) and (3.5) that $2B_j \sinh(z\ell) = C \sin(z^2)$ and, since $z \neq 0$, then

$$B_j = C \frac{\sin(z^2)}{2\sinh(z\ell)} \quad \forall j = 1, \dots, J. \tag{3.7}$$

We claim that $\sin(z^2) \neq 0$. Indeed suppose on the contrary that $\sin(z^2) = 0$. Then $z^2 = 0 \mod \pi$, and this implies $B_j = 0$, and $C_j \sin(z\ell) = 0$. We cannot have $\sin(z\ell) = 0$, since if it is the case, we get $z^2 = p\pi$ and $z\ell = k\pi$, for some $k, p \in \mathbb{Z}^*$ leading to $p\pi = \frac{k^2\pi^2}{\ell^2}$, which is not possible, as $\ell \in \mathcal{L}$. Hence, we have $C_j = 0$, and thus $\phi_j = 0$. But then $\partial_x \phi_0(1) = Cz^2 \cos(z^2) = 0$, which implies $C = 0$ and $\Phi = 0$. Since Φ is an eigenvector this gives the required contradiction.

We finally distinguish the cases $C \neq 0$ and $C = 0$.

a) If $C \neq 0$, then (3.7) and $\sin(z^2) \neq 0$ imply $B_j \neq 0$ for $j = 1, \dots, J$. Using (3.6) we deduce $\sin(z\ell) \neq 0$ and $C_j = C \frac{\sin(z^2)\sinh(z\ell)}{2\sinh(z\ell)\sin(z\ell)}$. Replacing in (3.5) the values of B_j and C_j, and differentiating, we deduce

$$\partial_x^3 \phi_j(\ell) = z^3 C \left(\frac{\sin(z^2)}{2\sinh(z\ell)} \cosh(z\ell) - \frac{\sin(z^2)}{2\sin(z\ell)} \cos(z\ell) \right) \quad \forall j = 1, \dots, J$$

whereas (3.4) implies $\partial_x \phi_0(1) = Cz^2 \cos(z^2)$. Using the transmission condition in (3.1) and the fact that both C and z are not zero, we deduce

$$\cos(z^2) = z \sum_{j=1}^{J} \alpha_j \left(\frac{\sin(z^2)}{2\sinh(z\ell)} \cosh(z\ell) - \frac{\sin(z^2)}{2\sin(z\ell)} \cos(z\ell) \right),$$

and, since $\sin(z^2) \neq 0$ then $f(z) = 0$. Also, (3.4) and (3.5) imply

$$\phi_0(x) = C \sin(z_n^2 x), \qquad \phi_j(x) = C \frac{\sin(z^2)}{2} \left(\frac{\sinh(z_n x)}{\sinh(z_n \ell)} + \frac{\sin(z_n x)}{\sin(z_n \ell)} \right),$$

with $j = 1, \dots, J$ and, consequently, the point (ii) of the claim.

b) If $C = 0$, then respectively by (3.4), (3.7) and (3.6) we have $\phi_0 = 0$, $B_j = 0$ and $C_j \sin(z\ell) = 0$. If $\sin(z\ell) \neq 0$, we get $C_j = 0$ for all j and thus $\Phi = 0$, which is a contradiction. Then $\sin(z\ell) = 0$ and this implies $z^2 \in \Lambda_2$ and, consequently, $\lambda \in i\Lambda$. This concludes the proof of the point (i) of the claim.

   Now we want to show the point (iii) of the claim. By (3.5) we obtain $\phi_j(x) = C_j \sin(zx)$ and $\partial_x^3 \phi_j(\ell) = -C_j z^3 \cos(z\ell)$. Imposing the transmission condition in (3.1) and using the fact that $\partial_x \phi_0(1) = 0$ (due to $\phi_0 = 0$), $z \neq 0$ we deduce that $\sum_{j=1}^J \alpha_j C_j \cos(z\ell) = 0$. This, together with $\sin(z\ell) = 0$ implies ($\cos(z\ell) \neq 0$ and) $\sum_{j=1}^J \alpha_j C_j = 0$. In view of above arguments, remarking that $z = \frac{k\pi}{\ell}$ for some $k \in \mathbb{Z}^*$, replacing $C = 0$ in (3.4) and $B_j = 0$ in (3.5) we get $\phi_0(x) = 0$ and $\phi_j(x) = C_j \sin(\frac{k\pi}{\ell} x)$, with $\sum_{j=1}^J \alpha_j C_j = 0$, and $(C_1, \dots, C_J) \neq (0, \dots, 0)$, and this completes the proof.

□

Once we have the eigenvalues and eigenvectors, we can readily express the solution of (1.1) as a nonharmonic Fourier series, proving Theorem 2.1.

Proof of Theorem 2.1. Let $u = (u_0, \dots, u_J)$ be as in the statement. The claim readily follows by Proposition 3.3, characterizing the eigenvalues and the eigenspaces of the operator $\mathcal{A}$, and by Proposition 3.1, implying that $u \in H$.

□

Remark 3.4. Note that the solution belongs also to $C([0,\infty), D(\mathcal{A})) \cap C^1([0,\infty), H)$, since the initial data are in $D(\mathcal{A})$. By density arguments, the well posedness extends to initial data of the form of the previous theorem, but with infinite number of coefficients. For $J = 1$ we can consider such initial data satisfying that $E_{0,J} < +\infty$, which, as we shall see, gives the explicit condition $\sum_{n\in\mathbb{Z}^*} |z_n|^4 |a_n|^2 < \infty$.

We can consider a basis e_j^q, with only a finite number of indices j such that $\mathrm{e}_j^q \neq 0$ for a given q, giving non ambiguous convergence of the series $\sum_{q=1}^{J-1} b_{k,q}\mathrm{e}_j^q$. The property of the completeness of a basis of eigenvectors is not considered here; we refer to [3] for example, for the proof of the property that $\mathcal{A}$ is a skew adjoint operator with compact resolvant for finite J, leading to the completeness of a basis of eigenvectors. We note that the resolvant is no more compact for $J = \infty$, since we have then eigenvalues with geometrical infinite multiplicity. The completeness of a basis of eigenvectors permits to have well posedness for arbitrary initial data in H ($E_{0,J}$ is then finite), with solution in $C([0,\infty), H)$.

4. Proof of the observability results

In this section we prove Theorem 2.2 and Theorem 2.3. The proof relies on non-harmonic analysis methods and on Ingham type inequalities. An ad hoc analysis is needed in view of the peculiarity of the system and, in particular, its transmission conditions.

4.1. A gap condition

We locate more precisely the eigenvalues characterized in Proposition 3.3, see also Figure 1. The following proposition tells us that there is a gap.

Proposition 4.1. *Suppose that $\ell \in \mathcal{L}$. Then there exists $\gamma' > 0$ such that $z_{n+1}^2 - z_n^2 > \gamma'$, for all $n \in \mathbb{N}^*$.*

Proof. Recall from Lemma 3.2 that (z_n) is strictly increasing, and the definition $\mathcal{S} = \{p\pi,\ p \in \mathbb{N}\} \cup \{\frac{k^2\pi^2}{\ell^2},\ k \in \mathbb{N}^*\} = (s_n)_{n\in\mathbb{N}}$. In particular the strictly increasing sequence $(s_n)_{n\in\mathbb{N}}$ represents the squares of the singularities of f. Lemma 3.2 also implies

$$s_{n-1} < z_n^2 < s_n < z_{n+1}^2 < s_{n+1}. \tag{4.1}$$

Let $f_1(z) = 2\cot(z^2)$ and $f_2(z) = Az(\cot(z\ell) - \coth(z\ell))$. We have $f(z) = f_1(z) + f_2(z)$ and $f(z_n) = f(z_{n+1}) = 0$. We assume that n is sufficiently large to have $Az_n \geq 4$. We distinguish the cases in which $s_n = p\pi$ and $s_n = (k\pi/\ell)^2$.

As first case, we consider the most common situation where $s_n = p\pi$. Since $(p-1)\pi, (p+1)\pi \in \mathcal{S}$, then $(p-1)\pi \leq s_{n-1}$ and $(p+1)\pi \geq s_{n+1}$. This, together (4.1), implies $z_n^2 \in](p-1)\pi, p\pi[$ and $z_{n+1}^2 \in]p\pi, (p+1)\pi[$. We distinguish the following three subcases, also depicted in Figure 2:

- (i) $f_2(z_{n+1}) > 0$. This implies that $f_1(z_{n+1}) < 0$, because $0 = f(z_{n+1}) = f_1(z_{n+1}) + f_2(z_{n+1})$. In particular $f_1(z_n) = 2\cot(z_n^2) < 0$ and, since $z_{n+1}^2 \in]p\pi, (p+1)\pi)[$, then $z_{n+1}^2 \in](p+1/2)\pi, (p+1)\pi[$. Consequently, $z_{n+1}^2 - z_n^2 \geq \pi/2$ so we get a gap.
- (ii) $f_2(z_n) < 0$. Arguing as above we deduce that $f_1(z_n) > 0$ and $z_n^2 \in](p-1)\pi, (p-1/2)\pi[$. Again, we deduce $z_{n+1}^2 - z_n^2 \geq \pi/2$ and we get a gap.
- (iii) $f_2(z_n) \geq 0$ and $f_2(z_{n+1}) \leq 0$. Since f_2 is defined in $[z_n, z_{n+1}]$ then there are no singularities of f_2 (which are of the form $k\pi/\ell$ with $k \in \mathbb{N}^*$) in the interval $[z_n, z_{n+1}]$. In particular there exists a unique k_n such that $\frac{k_n\pi}{\ell} < z_n < z_{n+1} < \frac{(k_n+1)\pi}{\ell}$.
- (iii.1) If $|f_2(z_n)| \leq 2$ then $2\cot(z_n^2) = f_1(z_n) = -f_2(z_n) \geq -2$. This implies that $z_n^2 \in](p-1)\pi, (p-1/4)\pi]$. Since $z_{n+1}^2 > p\pi$ then we get a gap.
- (iii.2) $|f_2(z_{n+1})| \leq 2$, then, arguing as above, $f_1(z_{n+1}) \leq 2$ which implies $z_{n+1}^2 \in [(p+1/4)\pi, (p+1)\pi[$ and, together with $z_n^2 < p\pi$, a gap.

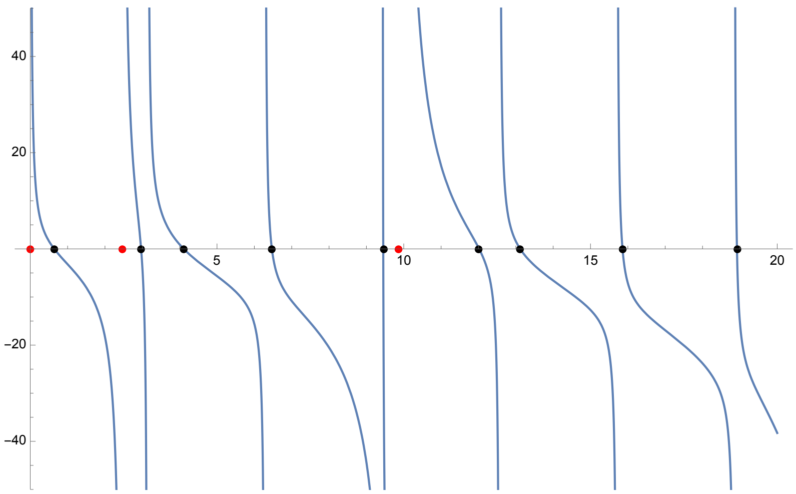

FIGURE 1. Function f and the moduli of the eigenvalues. If $\pm iz^2$ is an eigenvalue then, it is either an element of the sequence z_n^2 (black dots) or z^2 is equal to $(k\pi/\ell)^2$ (red dots).

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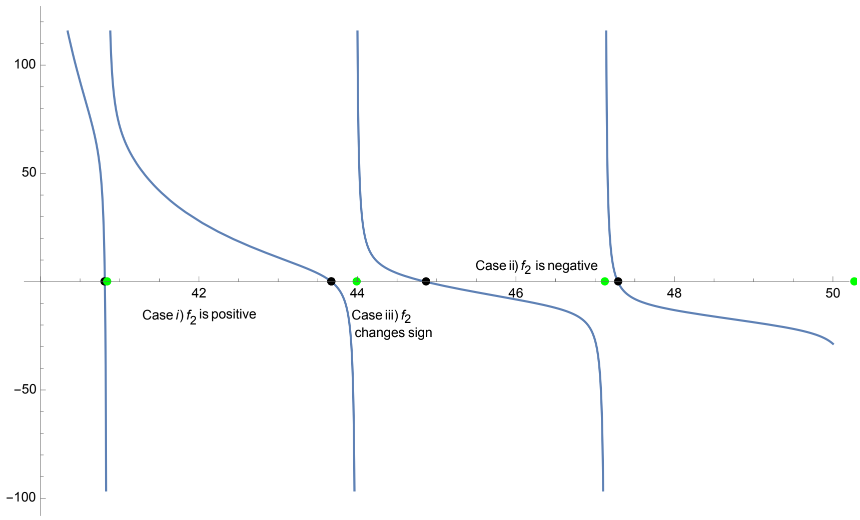

FIGURE 2. Sketch of some of the cases described in the proof of Proposition 4.1 in the case $J = 3$ and $\ell = 2$. The green dots are s_n's, *i.e.*, the squares of singularities of the function f of the form $p\pi$. The black dots are z_n^2 for $n = 18, 19, 20, 21$.

(iii.3) If $|f_2(z_n)| > 2$ and $|f_2(z_{n+1})| > 2$ we have that $f_2(z_n) > 2$ and $f_2(z_{n+1}) < -2$. Define

$$\alpha_k^- := \frac{1}{\ell}(k + \frac{1}{4})\pi - \frac{1}{2\pi Ak}; \qquad \alpha_k^+ := \frac{1}{\ell}(k + \frac{1}{4})\pi + \frac{1}{2\pi Ak}.$$

Since k_n diverges to $+\infty$ as $n \to \infty$, then

$$\begin{aligned}\lim_{n\to+\infty} f_2(\alpha^{\pm}_{k_n}) &= \lim_{k\to+\infty} f_2\left(\frac{1}{\ell}\left(k+\frac{1}{4}\right)\pi \pm \frac{1}{2\pi Ak}\right)\\ &= \lim_{k\to\infty} A\left(\frac{1}{\ell}\left(k+\frac{1}{4}\right)\pi \pm \frac{1}{2\pi Ak}\right)\left(\cot\left(\frac{\pi}{4}\pm\frac{\ell}{2\pi Ak}\right) - \coth\left(\left(k+\frac{1}{4}\right)\pi \pm \frac{\ell}{2\pi Ak}\right)\right)\\ &= \mp 1.\end{aligned}$$

Then, for a sufficiently large n one has $f_2(\alpha^-_{k_n}) < 2$ and $f_2(\alpha^+_{k_n}) > -2$. These inequalities together with the fact that f_2 is decreasing, imply, respectively, $z_n < \alpha^-_{k_n}$ and $z_{n+1} > \alpha^+_{k_n}$. Then $z^2_{n+1} - z^2_n > (\alpha^+_{k_n})^2 - (\alpha^-_{k_n})^2 > \frac{2}{A\ell}$ and we get the required gap also in this case.

Now it remains to consider the situation where $s_n = \left(\frac{k\pi}{\ell}\right)^2$. If n is big enough, we can assume that

(a) $z_n A \geq 1$;
(b) $s_{n-1} = p\pi$, $s_{n+1} = (p+1)\pi$.

We distinguish the following cases

(i) $f_2(z_n) \geq -1$. Then, by (a), one has $f_2(z_n) \geq -1 \geq -Az_n$ and, by the definition of f_2, one deduces $\cot(z_n\ell) - \coth(z_n\ell) \geq -1$. Since $\coth(z) \geq 1$ for all $z > 0$, then $\cot(z_n\ell) \geq 0$. Since $s_{n-1} < z_n^2 < s_n = \frac{k^2\pi^2}{\ell^2}$ and since $s_{n-1} \geq \frac{(k-1)^2\pi^2}{\ell^2}$, then $z_n^2 \in [\frac{(k-1)^2\pi^2}{\ell^2}, \frac{k^2\pi^2}{\ell^2}]$, which implies $z_n \in [\frac{(k-1)\pi}{\ell}, \frac{k\pi}{\ell}]$. This, together with $\cot(z_n\ell) \geq 0$ implies $z_n \in [\frac{(k-1)\pi}{\ell}, \frac{(k-1/2)\pi}{\ell}]$ and, consequently, $z^2_{n+1} - z^2_n \geq \frac{k^2\pi^2}{\ell^2} - \frac{(k-1/2)^2\pi^2}{\ell^2} \geq \frac{\pi^2}{4\ell^2}$.
(ii) $f_2(z_{n+1}) \leq 1$. In this case, by (a), one has $f_2(z_{n+1}) \leq 1 \leq Az_{n+1}$ and, by the definition of f_2, one deduces $\cot(z_n\ell) - \coth(z_n\ell) \leq 1$. Since $\coth(z) \geq 1$ for all $z > 0$, then $\cot(z_{n+1}\ell) \geq 0$. Arguing as above we deduce that $z^2_{n+1} \geq \frac{(k+1/2)^2\pi^2}{\ell^2}$ and, consequently, $z^2_{n+1} - z^2_n \geq \frac{(k+1/2)^2\pi^2}{\ell^2} - \frac{k^2\pi^2}{\ell^2} \geq \frac{\pi^2}{4\ell^2}$.
(iii) $f_2(z_n) < -1$ and $f_2(z_{n+1}) > 1$. Then $f_1(z_n) > 1$ and $f_1(z_{n+1}) < -1$, which implies, together with (b), that $z_n^2 \in]p\pi, (p+\frac{1}{4})\pi[$ and $z^2_{n+1} \in](p+\frac{3}{4})\pi, (p+1)\pi[$, leading to a gap.

□

4.2. Applying Ingham's theorem

A direct application of the gap condition established above permits to prove Theorem 2.2 by applying Ingham's theorem with weakened gap condition [7, 8]. Note that similar Ingham type theorems were also published by Avdonin et al. (see [6] and references therein).

Proof of Theorem 2.2. We first remark that, by Theorem 2.1, the solution $(u, \partial_t u)$ of (1.1) is of the form

$$u_0(t,x) = \sum_{n\in\mathbb{Z}^*}^{\infty} a_n e^{\frac{n}{|n|}iz_n^2 t}\sin(z_n^2 x), \qquad u_1(t,x) = \sum_{n\in\mathbb{Z}^*}^{\infty} a_n e^{\frac{n}{|n|}iz_n^2 t}\frac{\sin(z_n^2)}{2}\left(\frac{\sinh(z_n x)}{\sinh(z_n\ell)} + \frac{\sin(z_n x)}{\sin(z_n\ell)}\right)$$

with the coefficients a_n $(n \in \mathbb{Z}^*)$ identically zero but a finite number of them. In particular

$$\partial_x u_0(t,x) = \sum_{n\in\mathbb{Z}^*}^{\infty} a_n z_n^2 e^{\frac{n}{|n|}iz_n^2 t}\cos(z_n^2 x).$$

Then

$$\int_0^T |\partial_x u_0(t,0)|^2 \mathrm{d}t = \int_0^T |\sum_{n\in\mathbb{Z}^*}^{\infty} z_n^2 a_n \mathrm{e}^{\frac{n}{|n|} i z_n^2 t}|^2 \mathrm{d}t.$$

Since the sequence $(\frac{n}{|n|} z_n^2)$ is discrete by Proposition 4.1 and since the underlying upper density is $\frac{1}{\pi}$, from Lemma 3.2, then we can apply Ingham's theorem [10] (see also [9, 11]) and obtain for $T > 2$,

$$\int_0^T |\partial_x u(t,0)|^2 \mathrm{d}t \asymp \sum_{n\in\mathbb{Z}^*}^{\infty} |z_n|^4 |a_n|^2.$$

Now, to establish the equivalence with the initial energy $E_{0,1}$, we need to express also the latter one in terms of sums of squares of the Fourier coefficients. We set

$$\phi_n(x) := \left(\sin(z_n^2 x), \frac{\sin(z_n^2)}{2}\left(\frac{\sinh(z_n x)}{\sinh(z_n \ell)} + \frac{\sin(z_n x)}{\sin(z_n \ell)}\right)\right).$$

Using this notation, we have

$$u(t,x) := (u_0(t,x), u_1(t,x)) = \sum_{n\in\mathbb{Z}^*} a_n \mathrm{e}^{\frac{n}{|n|} i z_n^2 t} \phi_n(x).$$

Note that the eigenvalue λ_n of $\mathcal{A}$ associated to the eigenvector $\Phi_n := (\phi_n, \lambda_n \phi_n)$ is of the form $\frac{n}{|n|} i z_n^2$. By Proposition 3.1, for $m, n \in \mathbb{Z}^*$, one has

$$\frac{m}{|m|} i z_m^2 \langle \Phi_m, \Phi_n\rangle_H = \langle \frac{m}{|m|} i z_m^2 \Phi_m, \Phi_n\rangle_H = \langle \mathcal{A}\Phi_m, \Phi_n\rangle_H = -\langle \Phi_m, \mathcal{A}\Phi_n\rangle_H = -\langle \Phi_m, \frac{n}{|n|} i z_n^2 \Phi_n\rangle_H = \frac{n}{|n|} i z_n^2 \langle \Phi_m, \Phi_n\rangle_H.$$

Then $\langle \Phi_m, \Phi_n\rangle_H = 0$ and for all $n, m \in \mathbb{Z}^*$ such that $n \neq m$. This implies that $\langle \mathrm{e}^{\frac{m}{|m|} i z_m^2 t} \Phi_m, \mathrm{e}^{\frac{n}{|n|} i z_n^2 t} \Phi_n\rangle_H = 0$ whenever $n \neq m$ and for all $t \geq 0$, then

$$\begin{aligned} E_1(t) &= \int_0^1 |\partial_t u(t,x)|^2 \mathrm{d}x + \alpha_1 \int_0^\ell |\partial_t u_1(t,x)|^2 \mathrm{d}x + \int_0^1 |\partial_x u(t,x)|^2 \mathrm{d}x + \alpha_1 \int_0^\ell |\partial_x^2 u_1(t,x)|^2 \mathrm{d}x \\ &= ||x \mapsto (\vec{u}(t,x), \partial_t \vec{u}(t,x))||_H^2 = \sum_{n\in\mathbb{Z}} |a_n|^2 ||\Phi_n||_H^2 \qquad \forall t \geq 0. \end{aligned}$$

Now, by a direct computation $||\Phi_n||_H^2 = z_n^4 c_n$ where

$$\begin{aligned} c_n :&= \int_0^1 \sin^2(z_n^2 x) + \cos^2(z_n^2 x) \mathrm{d}x + \frac{\sin^2(z_n^2)}{4} \int_0^\ell \left(\frac{\sinh(z_n x)}{\sinh(z_n \ell)} + \frac{\sin(z_n x)}{\sin(z_n \ell)}\right)^2 + \left(\frac{\sinh(z_n x)}{\sinh(z_n \ell)} - \frac{\sin(z_n x)}{\sin(z_n \ell)}\right)^2 \mathrm{d}x. \\ &= 1 + \frac{\sin^2(z_n^2)}{2} \int_0^\ell \left|\frac{\sinh(z_n x)}{\sinh(z_n \ell)}\right|^2 + \left|\frac{\sin(z_n x)}{\sin(z_n \ell)}\right|^2 \mathrm{d}x \end{aligned}$$

In particular

$$1 \leq c_n \leq 1 + \frac{\ell}{2}\left(1 + \left|\frac{\sin(z_n^2)}{\sin(z_n \ell)}\right|^2\right).$$

Since $c_n = c_{-n}$, we can assume without loss of generality that $n \in \mathbb{N}^*$ and remark that, by definition, $f(z_n) = 0$, that is $2\cot(z_n^2) = \alpha_1 z_n(\coth(z_n\ell) - \cot(z_n\ell))$. This leads to

$$\frac{2\cos(z_n^2)}{\alpha_1 z_n} = \coth(z_n\ell)\sin(z_n^2) - \cos(z_n\ell)\frac{\sin(z_n^2)}{\sin(z_n\ell)},$$

which implies

$$\left|\frac{\sin(z_n^2)}{\sin(z_n\ell)}\right| \leq 2\left|\frac{\cos(z_n^2)}{\cos(z_n\ell)\alpha_1 z_n}\right| + \left|\frac{\coth(z_n\ell)\sin(z_n^2)}{\cos(z_n\ell)}\right| \leq \frac{C}{|\cos(z_n\ell)|}$$

where $C := 2/(\alpha_1 z_1) + \coth(z_1\ell) > 1$. Then

$$\left|\frac{\sin(z_n^2)}{\sin(z_n\ell)}\right| \leq \min\left\{\frac{1}{|\sin(z_n\ell)|}, \frac{C}{|\cos(z_n\ell)|}\right\} \leq \frac{C}{\sqrt{2}},$$

leading to $c_n \asymp 1$. We finally have

$$\int_0^T |\partial_x u(t,0)|^2 \mathrm{d}t \asymp \sum_{n\in\mathbb{Z}^*}^{\infty} |z_n|^4|a_n|^2 \asymp \sum_{n\in\mathbb{Z}^*}^{\infty} |z_n|^4|a_n|^2 c_n = E_1(0) = E_{0,1}.$$

□

In the case $J \geq 2$, we have no more a gap condition. We recall that we have a sequence $\omega_m,\ m \in \mathbb{Z}$, which is a strictly increasing sequence formed by the sets $\Lambda_1 = \{\pm z_n^2, n = 1, \dots\}$ and $\Lambda_2 = \{\pm\frac{k^2\pi^2}{\ell^2},\ k = 1, \dots\}$. We remark that a weakened gap condition on (ω_m) is satisfied: there exists $\gamma \in (0, \min\{\gamma', \pi/4\})$, where γ' is like in Proposition 4.1, such that $\omega_{m+2} - \omega_m > 2\gamma$ for all $m \in \mathbb{Z}$.

We define $\mathcal{M}_1 := \{m \in \mathbb{Z},\ \omega_{m+1} - \omega_m > \gamma\}$ and $\mathcal{M}_2 = \mathbb{Z} \setminus \mathcal{M}_1$. For γ small enough, if $m \in \mathcal{M}_2$, then $\omega_m \in \Lambda_1$ and $\omega_{m+1} \in \Lambda_2$ (or $\omega_m \in \Lambda_2$ and $\omega_{m+1} \in \Lambda_1$), since both Λ_1 and Λ_2 have a gap. We define also $\Gamma_j := \{m \in \mathbb{Z},\ \omega_m \in \Lambda_j\}$, for $j = 1, 2$.

Now, using these notations and Proposition 2.1, we can rewrite the observables in terms of the ω_m. In particular, by a direct computation,

$$\partial_x^3 u_j(t,0) = \sum_{m\in\mathbb{Z}^*} c_{m,j}\mathrm{e}^{i\omega_m t} \qquad j = 1, \dots, J,\ t \geq 0$$

where

$$c_{m,j} := \begin{cases} z_n^3 \dfrac{\sin(z_n^2)}{2}\left(\dfrac{1}{\sinh(z_n\ell)} - \dfrac{1}{\sin(z_n\ell)}\right) a_n & \text{if } \omega_m = \frac{n}{|n|} z_n^2 \text{ for some } n \in \mathbb{Z}^* \\ \left(\dfrac{k\pi}{\ell}\right)^3 \left(\sum_{q=1}^{J-1} b_{k,q}\mathrm{e}_j^q\right), & \text{if } \omega_m = \frac{k}{|k|}\frac{k^2\pi^2}{\ell^2} \text{ for some } k \in \mathbb{Z}^* \end{cases}$$

Now we are in position to directly apply Ingham's theorem with weakened gap condition [8]

$$\begin{aligned}\int_0^T |\partial_x^3 u_j(t,0)|^2 \mathrm{d}t &= \int_0^T |\sum_{m\in\mathbb{Z}^*} c_{m,j} \mathrm{e}^{i\omega_m t}|^2 \mathrm{d}t \\ &\asymp \sum_{m\in\mathcal{M}_1} |c_{m,j}|^2 + \sum_{m\in\mathcal{M}_2} |c_{m,j}+c_{m+1,j}|^2 + |\omega_{m+1}-\omega_m|^2 \left(|c_{m,j}|^2 + |c_{m+1,j}|^2\right).\end{aligned} \tag{4.2}$$

4.3. Proof of Theorem 3

We will need the following estimation of the coefficients.

Proposition 4.2. *Let $m\in\mathbb{Z}^*$, $\omega_m\in\Lambda_1$ and $n\in\mathbb{Z}^*$ be such that $\omega_m = \frac{n}{|n|} z_n^2$. Then*

$$|c_{m,j}|^2 \asymp |z_n|^6 \frac{\sin^2(z_n^2)}{\sin^2(z_n \ell)} |a_n|^2$$

Moreover if in addition $m\in\mathcal{M}_2$ then $|c_{m,j}|^2 \asymp |z_n|^4 |a_n|^2$.

Proof. We first remark that, as $\left|\frac{\sin(z_n\ell)}{\sinh(z_n\ell)}\right| \to 0$ as $n\to\infty$, then $\left|\frac{\sin(z_n\ell)}{\sinh(z_n\ell)} - 1\right| \asymp 1$. Then

$$\left|\frac{1}{\sinh(z_n\ell)} - \frac{1}{\sin(z_n\ell)}\right| = \left|\frac{1}{\sin(z_n\ell)}\right| \left|\frac{\sin(z_n\ell)}{\sinh(z_n\ell)} - 1\right| \asymp \left|\frac{1}{\sin(z_n\ell)}\right|.$$

This, together with the definition of $c_{m,j}$ which implies $|c_{m,j}|^2 = |z_n|^6 \frac{\sin^2(z_n^2)}{4} \left|\frac{1}{\sinh(z_n\ell)} - \frac{1}{\sin(z_n\ell)}\right|^2 |a_n|^2$, proves the first part of the claim.

If moreover $m\in\mathcal{M}_2$, we have to show that $\left|\frac{z_n \sin(z_n^2)}{\sin(z_n\ell)}\right| \asymp 1$. We have $f(z_n)=0$, that is $2\cot(z_n^2) = Az_n(\coth(z_n\ell) - \cot(z_n\ell))$, where $A = \sum_{j=1}^J \alpha_j$. This leads, after a direct computation, to

$$\left|\frac{z_n \sin(z_n^2)}{\sin(z_n\ell)}\right| = \frac{2}{A} \left|\frac{\cos(z_n^2)}{\cos(z_n\ell) - \coth(z_n\ell)\sin(z_n\ell)}\right|. \tag{4.3}$$

Now, since $m\in\mathcal{M}_2$ and $\omega_m\in\Lambda_1$, then $\omega_{m+1}\in\Lambda_2$ and, in particular, there exists a $k_n\in\mathbb{N}^*$ such that $\omega_{m+1} = \frac{k_n^2\pi^2}{\ell^2}$ and

$$\left|z_n^2 - \frac{k_n^2\pi^2}{\ell^2}\right| \le \gamma. \tag{4.4}$$

Since $z_n\to+\infty$ as $n\to+\infty$, then the above upper bound implies[1] $|z_n\ell - k_n\pi| \le \frac{\gamma}{|z_n\ell + k_n|} \to 0$ as $n\to+\infty$ and, consequently, $|\cos(z_n\ell) - \coth(z_n\ell)\sin(z_n\ell)| \to 1$ as $n\to+\infty$. On the other hand, recalling that by definition $\gamma < \pi/4$ we deduce that $\cos(z_n^2) \in (\sqrt{2}/2, 1]$ and this, together with the identity (4.3) concludes the proof. □

We need a technical lemma.

[1] More precisely, definining m_r the strictly increasing subsequence of $\mathcal{M}_2$ such that $\omega_{m_r}\in\Lambda_1$ for all r and defining n_r such that $\omega_{m_r} = z_{n_r}^2$ and k_r such that $\omega_{m_r+1} = k_r^2\pi^2/\ell^2$ one has that $|z_{n_r} - k_r^2\pi^2/\ell^2| \to 0$ as $r\to+\infty$.

Lemma 4.3. *For all $x, y \in \mathbb{C}$ and for all $\beta > 0$ one has*

$$|x+y|^2 \geq (c_\beta - 2\beta)|x|^2 + c_\beta |y|^2 \tag{4.5}$$

where $c_\beta := 1 + \beta - \sqrt{1+\beta^2} \in (0,1)$. In particular $c_\beta \downarrow 0$ (i.e., c_β monotonically converges to 0 from the above) as $\beta \to 0^+$ and the following estimate hold $\sqrt{2\beta} + c_\beta - 2\beta > 0$ for all $\beta \in (0,1)$.

Proof. If $x = 0$ then the claim follows straightforwardly. Assume now $x \neq 0$ and let $z := y/x$. Note that

$$|1+z|^2 - c_\beta(1+|z|^2) = (1+|z|^2)(1-c_\beta) + 2\Re(z) \geq (1-c_\beta)(1+\Re(z)^2) + 2\Re(z) \geq -2\beta.$$

Indeed, by a direct computation, the real valued function $g(t) := (1-c_\beta)(1+t^2) + 2t$ attains its global minimum in -2β. Multiplying both sides of the above inequalities by $|x|^2$ yields (4.5). To prove that $\beta \in (0,1)$ implies $\sqrt{2\beta} + c_\beta - 2\beta > 0$, it suffices to remark that when $\beta \in (0,1)$ this is equivalent to $(1-\beta+\sqrt{2\beta})^2 > 1+\beta^2$ which readily follows by a direct computation.

□

Using the previous proposition, we are able to get the following estimates for the observable.

Proposition 4.4. *We suppose that $\ell \in \mathcal{L}$. Then for $T > 2$, we have*

$$\int_0^T |\partial_x u_0(t,0)|^2 \mathrm{d}t + \sum_{j=2}^J \beta_j \int_0^T |\partial_x^3 u_j(t,0)|^2 \mathrm{d}t \asymp \sum_{n\in\mathbb{Z}^*} |z_n|^4 |a_n|^2 \left(1 + B\left|\frac{z_n \sin(z_n^2)}{\sin(z_n \ell)}\right|^2\right) + \sum_{k\in\mathbb{Z}^*} \left(\frac{k\pi}{\ell}\right)^6 \sum_{j=2}^J \beta_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2,$$

where we have set $B := \sum_{j=2}^J \beta_j$. The underlying constants do not depend on J.

Proof. We first have for $T > 2$, as in the proof of Theorem 2.1,

$$\int_0^T |\partial_x u_0(t,0)|^2 \mathrm{d}t = \int_0^T |\sum_{n\in\mathbb{Z}^*}^\infty z_n^2 a_n \mathrm{e}^{-\frac{n}{|n|} i z_n^2 t})|^2 \mathrm{d}t \asymp \sum_{n\in\mathbb{Z}^*}^\infty |z_n|^4 |a_n|^2. \tag{4.6}$$

On the other hand, as showed in (4.2), we also have

$$\int_0^T |\partial_x^3 u_j(t,0)|^2 \mathrm{d}t = \int_0^T |\sum_{m\in\mathbb{Z}^*} c_{m,j} \mathrm{e}^{i\omega_m t}|^2 \mathrm{d}t \asymp \sum_{m\in\mathcal{M}_1} |c_{m,j}|^2 + \sum_{m\in\mathcal{M}_2} |c_{m,j} + c_{m+1,j}|^2 + |\omega_{m+1} - \omega_m|^2 \left(|c_{m,j}|^2 + |c_{m+1,j}|^2\right).$$

We use the symbols $x \preceq y$ and $x \succeq y$ to respectively denote $x \leq C_1 y$ and $x \geq C_2 y$ for some positive constants C_1 and C_2. Clearly $x \preceq y$ and $x \succeq y$ is equivalent to $x \asymp y$.

Note that $m \in \mathcal{M}_2$ implies $|\omega_{m+1} - \omega_m| \leq \gamma$ and, due to the gap condition, $m+1 \in \mathcal{M}_1$. This implies

$$\begin{aligned}
\sum_{m\in\mathcal{M}_1} |c_{m,j}|^2 + \sum_{m\in\mathcal{M}_2} |c_{m,j} + c_{m+1,j}|^2 + &|\omega_{m+1} - \omega_m|^2 \left(|c_{m,j}|^2 + |c_{m+1,j}|^2\right) \\
&\leq \sum_{m\in\mathcal{M}_1} |c_{m,j}|^2 + \sum_{m\in\mathcal{M}_2} (1+\gamma^2)(|c_{m,j}|^2 + |c_{m+1,j}|^2) \\
&\leq (2+\gamma^2) \sum_{m\in\mathcal{M}_1} |c_{m,j}|^2 + \sum_{m\in\mathcal{M}_2} (1+\gamma^2)|c_{m,j}|^2 \\
&\leq (2+\gamma^2) \sum_{m\in\mathbb{Z}^*} |c_{m,j}|^2.
\end{aligned}$$

Hence

$$\int_0^T |\partial_x^3 u_j(t,0)|^2 \mathrm{d}t \preceq \sum_{m\in\mathbb{Z}^*} |c_{m,j}|^2.$$

By Proposition 4.2, if $m \in \Gamma_1$ then $|c_{m,j}|^2 \asymp |z_n|^6 \frac{\sin^2(z_n^2)}{\sin^2(z_n\ell)}|a_n|^2$. If otherwise $m \in \Gamma_2$ then $|c_{m,j}|^2 = \left(\frac{k\pi}{\ell}\right)^6 \left|\sum_{q=1}^{J-1} b_{k,q}\mathrm{e}_j^q\right|^2$. Then one has the direct inequality

$$\begin{aligned}\int_0^T |\partial_x u(t,0)|^2 \mathrm{d}t + \sum_{j=2}^J \beta_j \int_0^T |\partial_x^3 u_j(t,0)|^2 \mathrm{d}t &\preceq \sum_{n\in\mathbb{Z}^*}^{\infty} |z_n|^4|a_n|^2 + \sum_{j=2}^J \beta_j \sum_{m\in\mathbb{Z}^*} |c_{m,j}|^2 \\ &\asymp \sum_{n\in\mathbb{Z}^*} |z_n|^4|a_n|^2 \left(1 + B\left|\frac{z_n\sin(z_n^2)}{\sin(z_n\ell)}\right|^2\right) + \sum_{k\in\mathbb{Z}^*}\left(\frac{k\pi}{\ell}\right)^6 \sum_{j=2}^J \beta_j \left|\sum_{q=1}^{J-1} b_{k,q}\mathrm{e}_j^q\right|^2.\end{aligned}$$

To complete the proof we need to prove the inverse inequality. We preliminarly remark that the gap condition $\omega_{m+2}-\omega_m \geq 2\gamma$ (and the fact that Λ_1 and Λ_2 are discrete sets) implies that if $m \in \mathcal{M}_2\cap\Gamma_1$ then $m+1 \in \mathcal{M}_1\cap\Gamma_2$ and

$$m \in \mathcal{M}_2\cap\Gamma_2 \quad\Rightarrow\quad m+1 \in \mathcal{M}_1\cap\Gamma_1. \tag{4.7}$$

We then apply Lemma 4.3 by deducing that for all $\beta > 0$

$$\begin{aligned}\sum_{m\in\mathcal{M}_2} |c_{m,j}+c_{m+1,j}|^2 =& \sum_{m\in\mathcal{M}_2\cap\Gamma_1} |c_{m,j}+c_{m+1,j}|^2 + \sum_{m\in\mathcal{M}_2\cap\Gamma_2} |c_{m,j}+c_{m+1,j}|^2 \\ \geq& (c_\beta - 2\beta)\sum_{m\in\mathcal{M}_2\cap\Gamma_1} |c_{m,j}|^2 + c_\beta \sum_{m\in\mathcal{M}_2\cap\Gamma_1} |c_{m+1,j}|^2 \\ &+ c_\beta \sum_{m\in\mathcal{M}_2\cap\Gamma_2} |c_{m,j}|^2 + (c_\beta - 2\beta)\sum_{m\in\mathcal{M}_2\cap\Gamma_2} |c_{m+1,j}|^2 \\ \geq& (c_\beta - 2\beta)\sum_{m\in\mathcal{M}_2\cap\Gamma_1} |c_{m,j}|^2 + c_\beta \sum_{m\in\mathcal{M}_2\cap\Gamma_2} |c_{m,j}|^2 + (c_\beta - 2\beta)\sum_{m\in\mathcal{M}_1\cap\Gamma_1} |c_{m,j}|^2 \\ =& (c_\beta - 2\beta)\sum_{m\in\Gamma_1} |c_{m,j}|^2 + c_\beta \sum_{m\in\mathcal{M}_2\cap\Gamma_2} |c_{m,j}|^2\end{aligned}$$

where the last inequality follows from $c_\beta - 2\beta < 0$ and (4.7). By Proposition 4.2, $|c_{m,j}|^2 \asymp |z_n|^4|a_n|^2$ for all $m \in \mathcal{M}_2\cap\Gamma_1$ (where n is such that $\omega_m = n/|n|z_n^2$) and, in particular, $|c_{m,j}|^2 \leq \hat{C}|z_n|^4|a_n|^2$ for some positive $\hat{C}$ (independent from j, because of the definition of $c_{m,j}$ when $m \in \Gamma_1$) and for all $j = 2,\dots,J$. Similarly, again by Proposition 4.2, $|c_{m,j}|^2 \geq \hat{D}|z_n|^6 \frac{\sin^2(z_n^2)}{\sin^2(z_n\ell)}|a_n|^2$ for some positive $\hat{D}$ and for all $j = 2,\dots,J$. Therefore

$$\begin{aligned}\sum_{m\in\mathcal{M}_1} |c_{m,j}|^2 +& \sum_{m\in\mathcal{M}_2} |c_{m,j}+c_{m+1,j}|^2 + |\omega_{m+1}-\omega_m|^2\left(|c_{m,j}|^2 + |c_{m+1,j}|^2\right) \\ &\geq \sum_{m\in\mathcal{M}_1} |c_{m,j}|^2 + \sum_{m\in\mathcal{M}_2} |c_{m,j}+c_{m+1,j}|^2 \\ &= \sum_{m\in\mathcal{M}_1\cap\Lambda_1} |c_{m,j}|^2 + \sum_{m\in\mathcal{M}_1\cap\Lambda_2} |c_{m,j}|^2 + \sum_{m\in\mathcal{M}_2} |c_{m,j}+c_{m+1,j}|^2\end{aligned}$$

$$
\begin{aligned}
&\geq \Gamma(1+c_\beta-2\beta)\sum_{m\in\mathcal{M}_1\cap\Gamma_1}|c_{m,j}|^2+c_\beta\sum_{m\in\Gamma_2}|c_{m,j}|^2+(c_\beta-2\beta)\sum_{m\in\mathcal{M}_2\cap\Gamma_1}|c_{m,j}|^2\\
&> (\sqrt{2\beta}+c_\beta-2\beta)\sum_{m\in\mathcal{M}_1\cap\Gamma_1}|c_{m,j}|^2+c_\beta\sum_{m\in\Gamma_2}|c_{m,j}|^2+(c_\beta-2\beta)\sum_{m\in\mathcal{M}_2\cap\Gamma_1}|c_{m,j}|^2\\
&= (\sqrt{2\beta}+c_\beta-2\beta)\sum_{m\in\Gamma_1}|c_{m,j}|^2+c_\beta\sum_{m\in\Gamma_2}|c_{m,j}|^2-\sqrt{2\beta}\sum_{m\in\mathcal{M}_2\cap\Gamma_1}|c_{m,j}|^2\\
&\geq B\hat{D}(\sqrt{2\beta}+c_\beta-2\beta)\sum_{n\in\mathbb{Z}^*}|z_n|^6\frac{\sin^2(z_n^2)}{\sin^2(z_n\ell)}|a_n|^2\\
&\quad+c_\beta\sum_{k\in\mathbb{Z}^*}\left(\frac{k\pi}{\ell}\right)^6\sum_{j=2}^{J}\beta_j\left|\sum_{q=1}^{J-1}b_{k,q}\mathrm{e}_j^q\right|^2-B\hat{C}\sqrt{2\beta}\sum_{n\in\mathbb{Z}^*}|a_n|^2|z_n|^4
\end{aligned}
$$

Now, using (4.6), let $K_0>0$ be such that

$$\int_0^T|\partial_x u_0(t,0)|^2\mathrm{d}t=\int_0^T|\sum_{n\in\mathbb{Z}^*}^{\infty}z_n^2a_n\mathrm{e}^{-\frac{n}{|n|}iz_n^2t}|^2\mathrm{d}t\geq K_0\sum_{n\in\mathbb{Z}^*}^{\infty}|z_n|^4|a_n|^2$$

and let $K_1>0$ be such that

$$\int_0^T|\partial_x^3u_j(t,0)|^2\mathrm{d}t\geq K_1\left(\sum_{m\in\mathcal{M}_1}|c_{m,j}|^2+\sum_{m\in\mathcal{M}_2}|c_{m,j}+c_{m+1,j}|^2+|\omega_{m+1}-\omega_m|^2\left(|c_{m,j}|^2+|c_{m+1,j}|^2\right)\right).$$

We choose $\beta\in(0,1/2)$ sufficiently small to have $K_0-K_1B\hat{C}\sqrt{2\beta}>0$. Using above estimates and recalling that $\beta<1$ implies $\sqrt{2\beta}+c_\beta-2\beta>0$, we conclude

$$
\begin{aligned}
\int_0^T|\partial_xu(t,0)|^2\mathrm{d}t+\sum_{j=2}^J\beta_j\int_0^T|\partial_x^3u_j(t,0)|^2\mathrm{d}t &\geq (K_0-K_1B\hat{C}\sqrt{2\beta})\sum_{n\in\mathbb{Z}^*}|z_n|^4|a_n|^2\\
&\quad+K_1B\hat{D}(\sqrt{2\beta}+c_\beta-2\beta)\sum_{n\in\mathbb{Z}^*}|z_n|^6\frac{\sin^2(z_n^2)}{\sin^2(z_n\ell)}|a_n|^2\\
&\quad+K_1c_\beta\sum_{k\in\mathbb{Z}^*}\left(\frac{k\pi}{\ell}\right)^6\sum_{j=2}^J\beta_j\left|\sum_{q=1}^{J-1}b_{k,q}\mathrm{e}_j^q\right|^2\\
&\succeq\sum_{n\in\mathbb{Z}^*}|z_n|^4|a_n|^2\left(1+B\left|\frac{z_n\sin(z_n^2)}{\sin(z_n\ell)}\right|^2\right)\\
&\quad+\sum_{k\in\mathbb{Z}^*}\left(\frac{k\pi}{\ell}\right)^6\sum_{j=2}^J\beta_j\left|\sum_{q=1}^{J-1}b_{k,q}\mathrm{e}_j^q\right|
\end{aligned}
$$

and this completes the proof. □

Proposition 4.5. *We suppose that $\ell\in\mathcal{L}$. Then, for $\alpha_j\asymp\beta_j,\ j=2,\dots,J$, we have*

$$\sum_{k\in\mathbb{Z}^*}\left(\frac{k\pi}{\ell}\right)^6\sum_{j=1}^J\alpha_j\left|\sum_{q=1}^{J-1}b_{k,q}\mathrm{e}_j^q\right|^2\asymp\sum_{k\in\mathbb{Z}^*}\left(\frac{k\pi}{\ell}\right)^6\sum_{j=2}^J\beta_j\left|\sum_{q=1}^{J-1}b_{k,q}\mathrm{e}_j^q\right|^2,$$

where $A = \sum_{j=1}^{J} \alpha_j$, $B := \sum_{j=2}^{J} \beta_j$ and the underlying constants do not depend on J, assuming that $\sum_{j=1}^{J} \alpha_j$ is bounded independently of J and $\alpha_1 \geq c$, with a constant $c > 0$ independent of J (we recall also that $\alpha_j > 0$ for all j).

Proof. We already have

$$\sum_{k\in\mathbb{Z}^*} \left(\frac{k\pi}{\ell}\right)^6 \sum_{j=2}^{J} \beta_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2 \asymp \sum_{k\in\mathbb{Z}^*} \left(\frac{k\pi}{\ell}\right)^6 \sum_{j=2}^{J} \alpha_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2,$$

since $\alpha_j \asymp \beta_j,\ j = 2, \dots, J$. We know that $\alpha_1 \mathrm{e}_1^q = -\sum_{j=2}^{J} \alpha_j \mathrm{e}_j^q$. Thus, we have, using Cauchy-Schwarz inequality

$$\alpha_1^2 \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_1^q\right|^2 = \left|\sum_{q=1}^{J-1} b_{k,q} \sum_{j=2}^{J} \alpha_j \mathrm{e}_j^q\right|^2 = \left|\sum_{j=2}^{J} \alpha_j^{1/2} \alpha_j^{1/2} \sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2 \leq (A - \alpha_1) \sum_{j=2}^{J} \alpha_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2,$$

and thus

$$\sum_{j=1}^{J} \alpha_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2 \leq \left(\frac{A - \alpha_1}{\alpha_1} + 1\right) \sum_{j=2}^{J} \alpha_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2 = \frac{A}{\alpha_1} \sum_{j=2}^{J} \alpha_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2,$$

and so

$$\frac{\alpha_1}{A} \sum_{j=1}^{J} \alpha_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2 \leq \sum_{j=2}^{J} \alpha_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2 \leq \sum_{j=1}^{J} \alpha_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2,$$

leading to

$$\sum_{k\in\mathbb{Z}^*} \left(\frac{k\pi}{\ell}\right)^6 \sum_{j=2}^{J} \alpha_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2 \asymp \sum_{k\in\mathbb{Z}^*} (\frac{k\pi}{\ell})^6 \sum_{j=1}^{J} \alpha_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2$$

and to the result. □

Proof of Theorem 2.3. Let $(u^0, u^1) \in Z$ and let (a_n) and $(b_{k,q})$ be the corresponding coefficients. In view of Proposition 4.4 and Proposition 4.5, for $T > 2$ we have

$$\int_0^T |\partial_x u(t,0)|^2 \mathrm{d}t + \sum_{j=2}^{J} \beta_j \int_0^T |\partial_x^3 u_j(t,0)|^2 \mathrm{d}t \asymp \sum_{n\in\mathbb{Z}^*} |z_n|^4 |a_n|^2 \left(1 + (A - \alpha_1) \left|\frac{z_n \sin(z_n^2)}{\sin(z_n \ell)}\right|^2\right)$$
$$+ \sum_{k\in\mathbb{Z}^*} \left(\frac{k\pi}{\ell}\right)^6 \sum_{j=1}^{J} \alpha_j \left|\sum_{q=1}^{J-1} b_{k,q} \mathrm{e}_j^q\right|^2,$$

which concludes the proof. □

5. Numerical results

The code that is used for the numerical results is available here

https://github.com/mehrenbe/InghamWaveBeam.

On Figure 3, we represent the gap $z_{n+1}^2 - z_n^2$ with respect to $\sqrt{n}$, for $n \in \{1, \dots, 10^4\}$, for different values of $\ell > 0$ belonging to $\mathcal{L}$, and taking $A = \frac{1}{\ell}$. We remark that the gap oscillates between a value $\gamma_{\min} \simeq 2$ and π for large values of n. The gap is almost always around π, but almost periodically, with a period proportional to $1/\ell$, it falls down near $\gamma_{\min}$. The behavior at the beginning is different. In particular, when ℓ is large, the gap becomes very small, even if it still remains strictly positive. These results are in agreement with the gap result in Proposition 4.1.

On Figure 6, we represent now the gap $\omega_{m+1} - \omega_m$ with respect to $\sqrt{m}$. The results are then quite different, and we can see on these numerical results that we no longer have a gap.

In order to have an idea of the constants $c_1(T), c_2(T)$ such that

$$c_1(T) \sum_k |a_k|^2 \le \int_0^T \left| \sum_k a_k \mathrm{e}^{iz_k^2 t} \right|^2 \mathrm{d}t \le c_2(T) \sum_k |a_k|^2,$$

we look for constants $c_{1,n,N_{\mathrm{loc}}}(T), c_{2,n,N_{\mathrm{loc}}}(T)$ such that

$$c_{1,n,N_{\mathrm{loc}}}(T) \sum_{k=n}^{n+N_{\mathrm{loc}}} |a_k|^2 \le \int_0^T \left| \sum_{k=n}^{n+N_{\mathrm{loc}}} a_k \mathrm{e}^{iz_k^2 t} \right|^2 \mathrm{d}t \le c_{2,n,N_{\mathrm{loc}}}(T) \sum_{k=n}^{n+N_{\mathrm{loc}}} |a_k|^2,$$

which are obtained by looking at the minimal (on Fig. 4) and maximal (on Fig. 5) eigenvalues of the matrix $(\int_0^T \mathrm{e}^{i(z_j^2 - z_k^2)t} \mathrm{d}t)_{j,k=n}^{n+N_{loc}}$. We vary the value of N_{loc}; the larger it is, the better is the result. We remark that the constant $c_{1,n,N_{\mathrm{loc}}}(T)$ can be quite small for low values of n, in the case where ℓ is large; this is coherent with the previous result, as the gap is very small for n small (low frequencies). The results are then better by increasing the value of T. There are some oscillations in the graphs which are pushed at later n, taking a larger N_{loc}. Finally, we do the same for

$$c_{3,n,N_{\mathrm{loc}}}(T) \sum_{k=n}^{n+N_{\mathrm{loc}}} |a_k|^2 \le \int_0^T \left| \sum_{k=n}^{n+N_{\mathrm{loc}}} a_k \mathrm{e}^{i\omega_k^2 t} \right|^2 \mathrm{d}t \le c_{4,n,N_{\mathrm{loc}}}(T) \sum_{k=n}^{n+N_{\mathrm{loc}}} |a_k|^2,$$

by looking at the minimal (on Fig. 7) and maximal (on Fig. 8) eigenvalues of the matrix $(\int_0^T \mathrm{e}^{i(\omega_j^2 - \omega_k^2)t} \mathrm{d}t)_{j,k=n}^{n+N_{loc}}$ We see that $c_{3,n,N_{\mathrm{loc}}}(T)$ is no more minored by a strictly positive constant, which is coherent, because there is no longer an asymptotic gap. On the other hand, when we express in the basis of divided differences, we observe a minoration by a strictly positive constant (for small n, the value is still very small, especially for big value of ℓ, as for the case for (z_k^2), but it's getting better by increasing the value of T) which is coherent with the weakened gap condition.

Note that the theoretical part is based on such estimates, for which we now have a numerical illustration of the behavior of its underlying constants. In particular, we observe that time of observation, even if it is enough to take it greater than π, we should take it bigger for having not too small constants, which can occur when taking ℓ large.

6. Data availability statement

The code connected with this article is available in a public data repository, at the following link https://github.com/mehrenbe/InghamWaveBeam, and referred as [14].

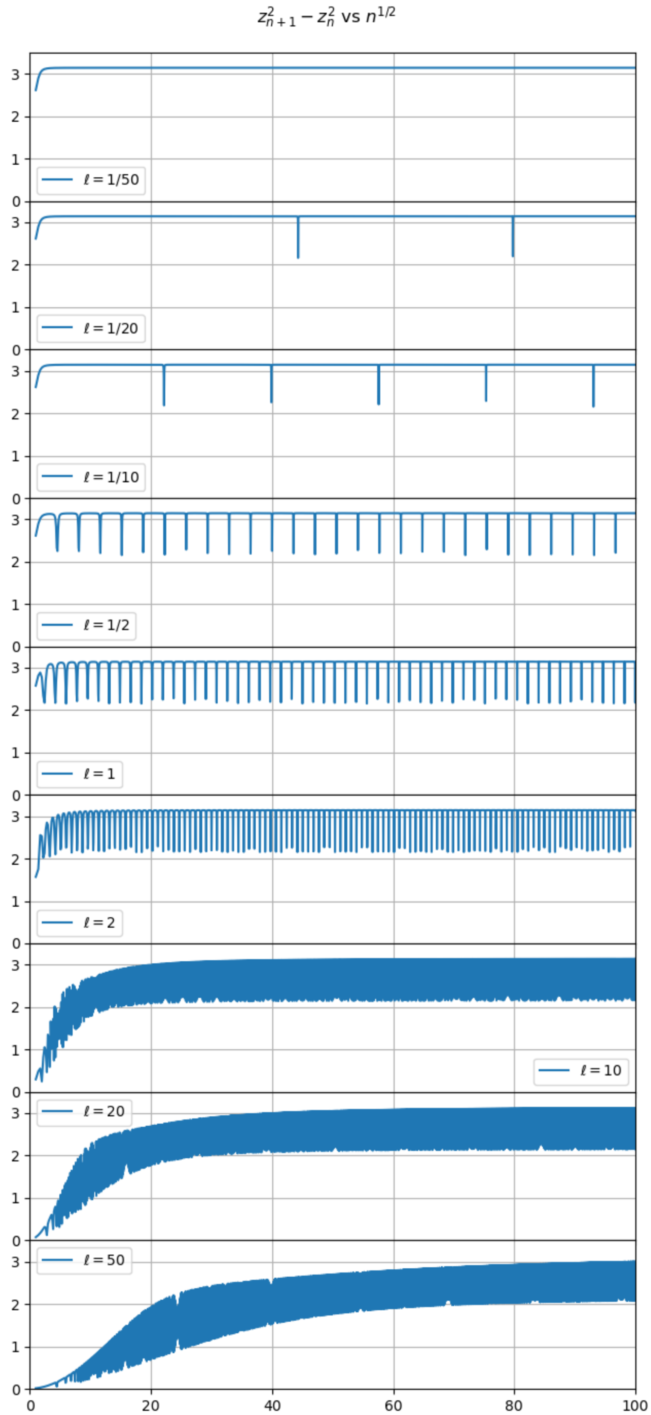

FIGURE 3. Gap $z_{n+1}^2 - z_n^2$ for $A = \frac{1}{\ell}$ vs $\sqrt{n}$, for $1 \leq n \leq 10^4$ and $\ell \in \{1/50, 1/20, 1/2, 1, 2, 10, 20, 50\}$.

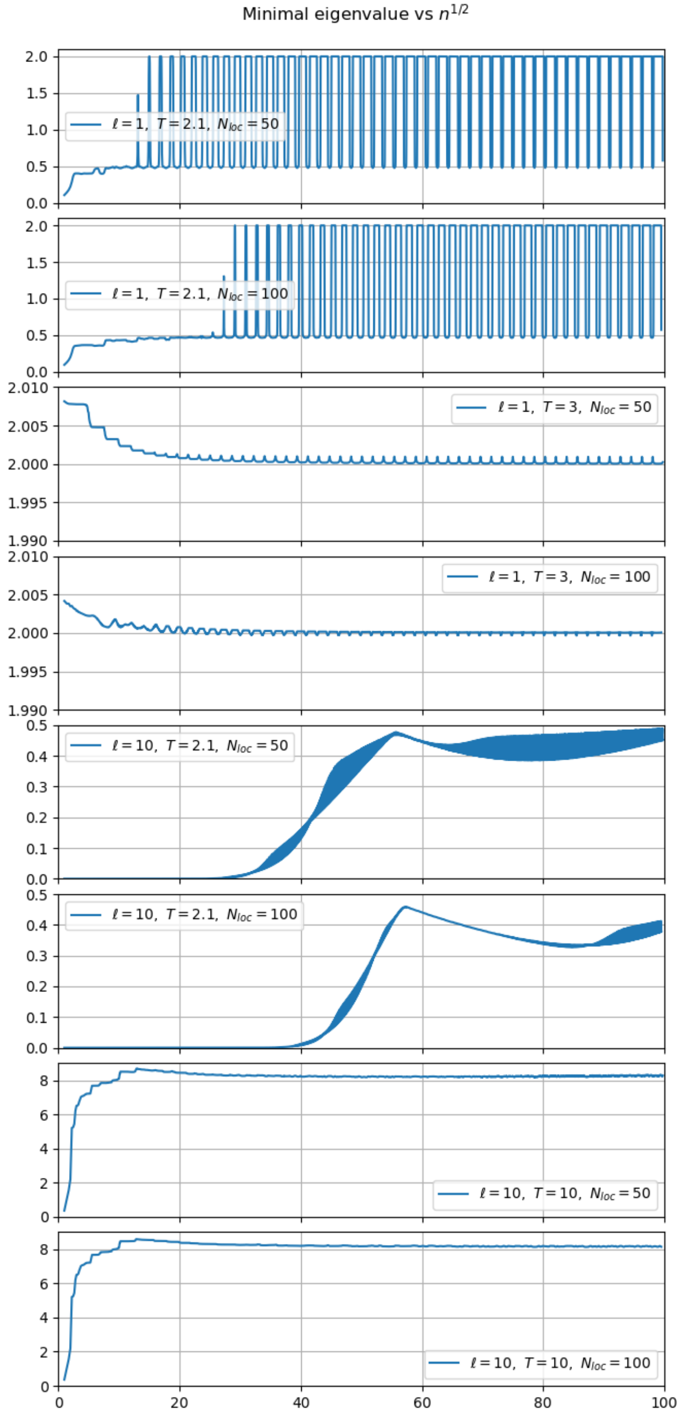

FIGURE 4. Minimal eigenvalue of $(\int_0^T \mathrm{e}^{i(z_j^2-z_k^2)t}\mathrm{d}t)_{j,k=n}^{n+N_{loc}}$ for $A=\frac{1}{\ell}$ vs $\sqrt{n}$, with $1 \le n \le 10^4$, $(\ell,T) \in \{(1,2.1),\ (1,3),\ (10,2.1),\ (10,10)\}$ and $N_{loc} \in \{50,100\}$.

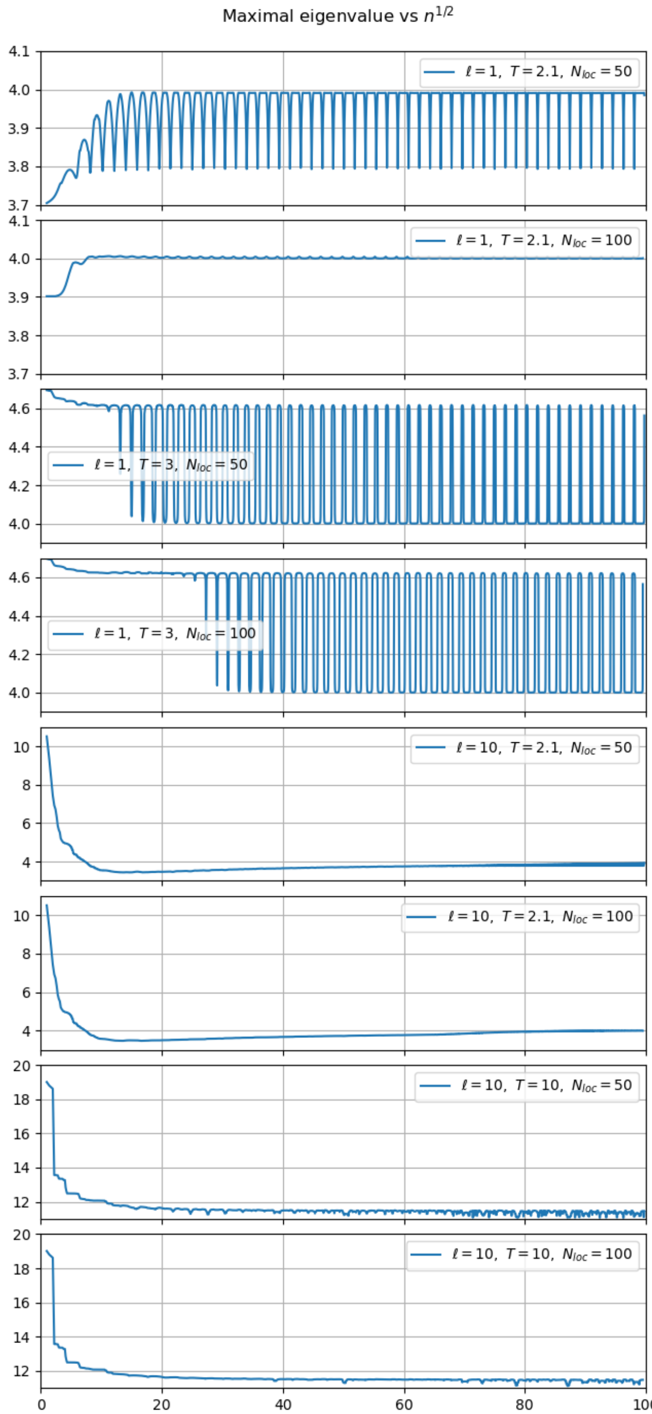

FIGURE 5. Maximal eigenvalue of $(\int_0^T e^{i(z_j^2 - z_k^2)t} dt)_{j,k=n}^{n+N_{loc}}$ for $A = \frac{1}{\ell}$ vs $\sqrt{n}$, with $1 \leq n \leq 10^4$, $(\ell, T) \in \{(1, 2.1),\ (1, 3),\ (10, 2.1),\ (10, 10)\}$ and $N_{loc} \in \{50, 100\}$.

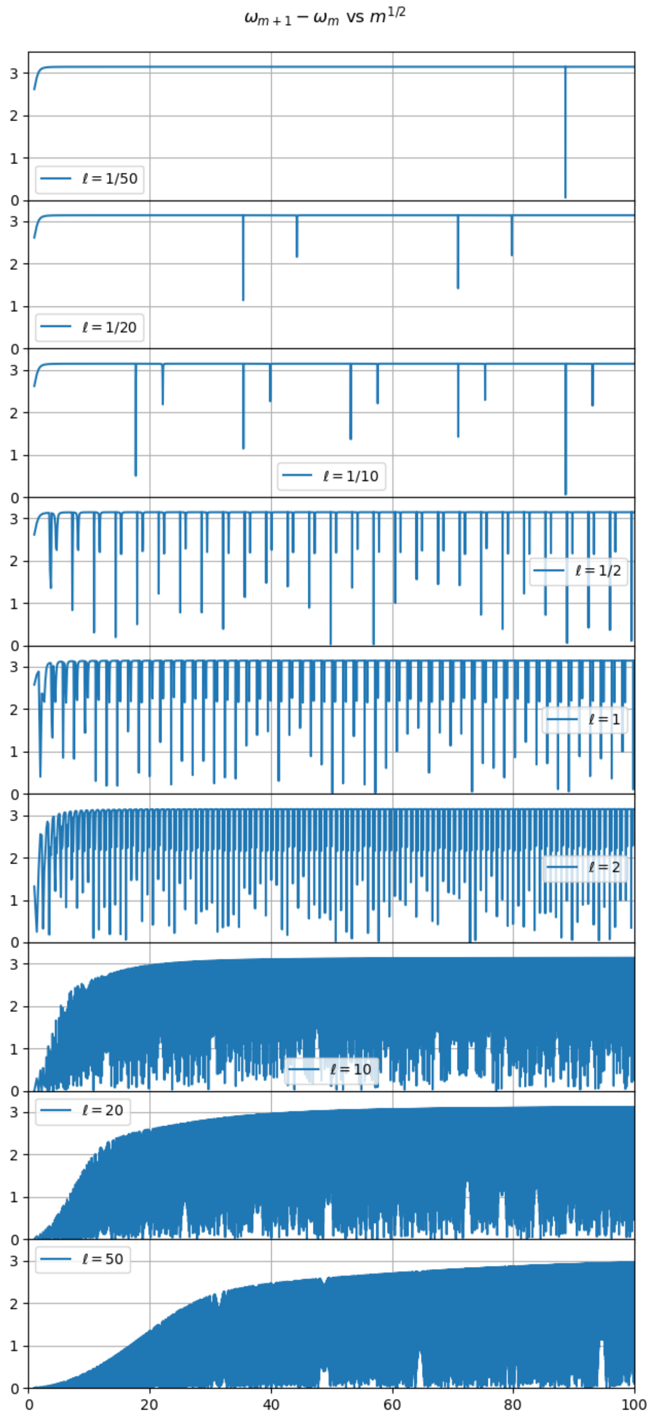

FIGURE 6. Gap $\omega_{n+1} - \omega_n$ for $A = \frac{1}{\ell}$ vs $\sqrt{n}$, for $1 \leq n \leq 10^4$ and $\ell \in \{1/50, 1/20, 1/2, 1, 2, 10, 20, 50\}$.

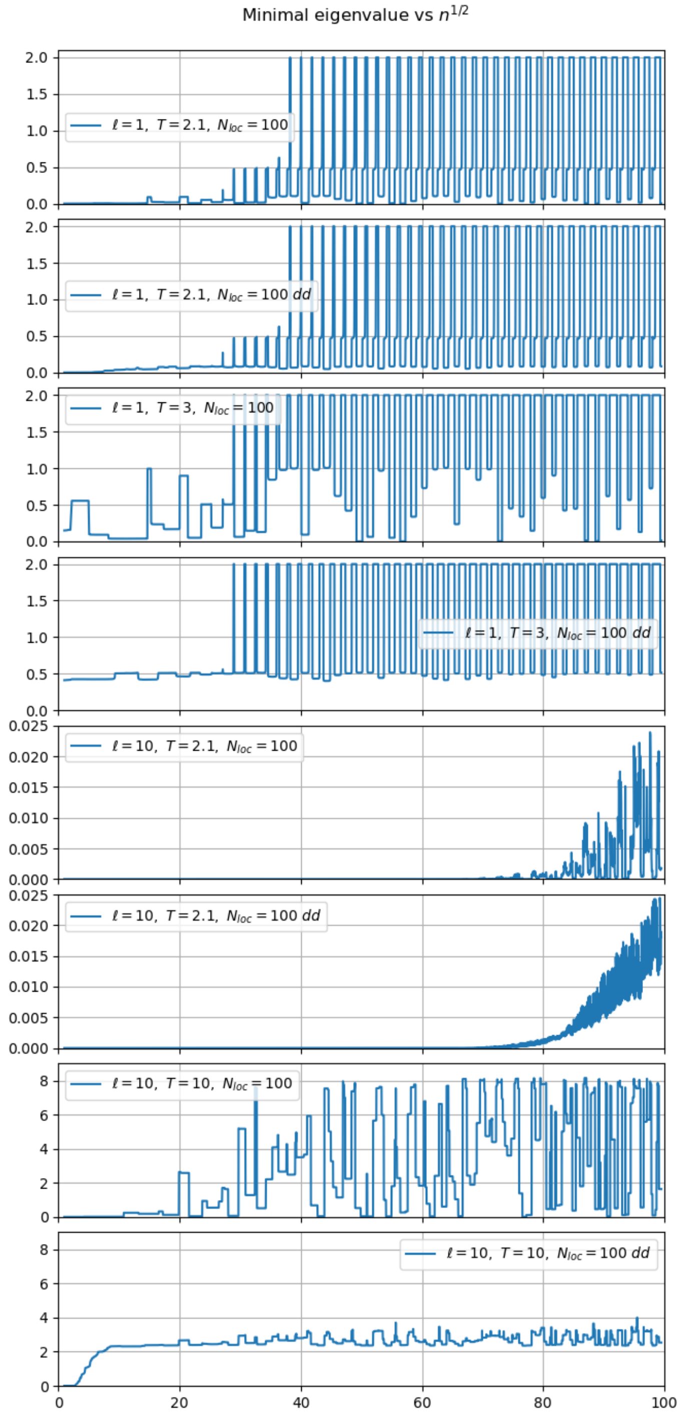

FIGURE 7. Minimal eigenvalue of $(\int_0^T e^{i(\omega_j-\omega_k)t}dt)_{j,k=n}^{n+N_{loc}}$ for $A = \frac{1}{\ell}$ vs $\sqrt{n}$; dd stands for expressing in the basis of divided differences.

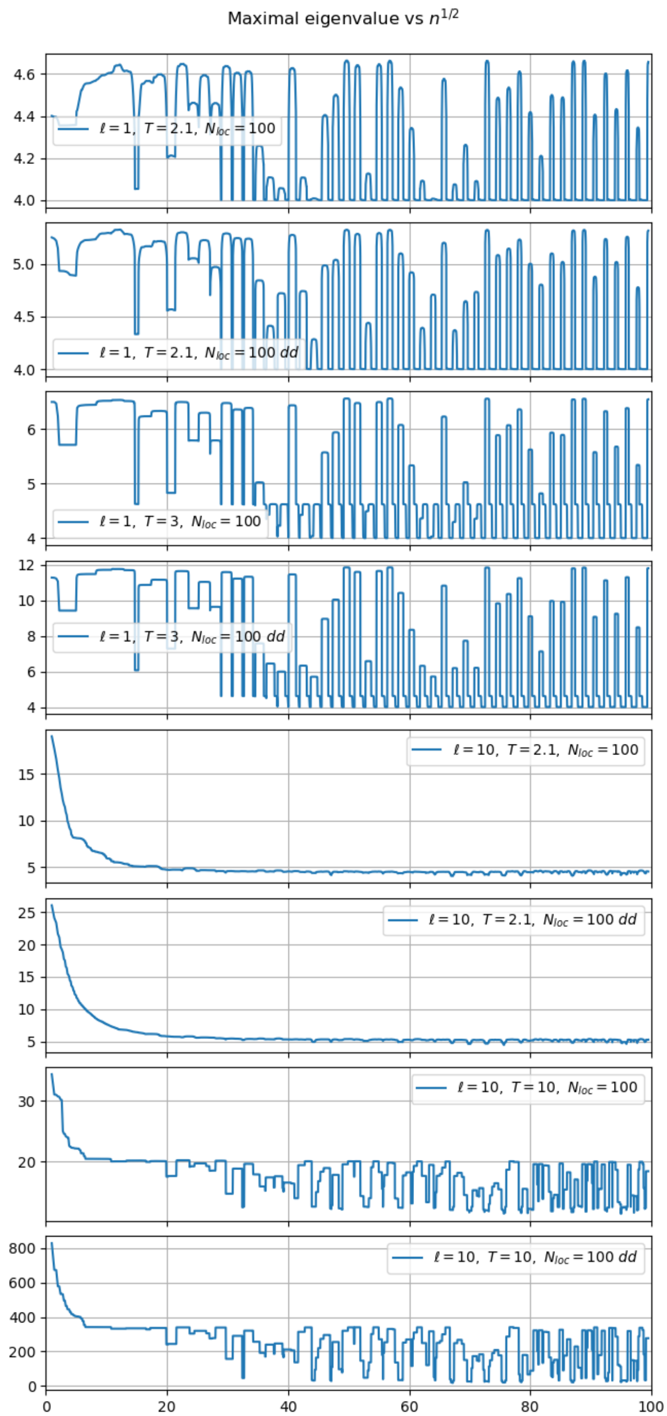

FIGURE 8. Maximal eigenvalue $(\int_0^T e^{i(\omega_j-\omega_k)t}dt)_{j,k=n}^{n+N_{loc}}$ for $A=\frac{1}{\ell}$ vs $\sqrt{n}$; dd stands for expressing in the basis of divided differences.

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