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THE ε–ε PROPERTY AND THE BOUNDEDNESS OF ISOPERIMETRIC SETS WITH DIFFERENT MONOMIAL WEIGHTS

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Abstract. We consider a class of monomial weights $x^A = |x_1|^{a_1} \dots |x_N|^{a_N}$ in $\mathbb{R}^N$, where a_i is a nonnegative real number for each $i \in \{1, \dots, N\}$, and we establish the $\varepsilon - \varepsilon$ property and the boundedness of isoperimetric sets with different monomial weights for the perimeter and volume. Moreover, we present cases of nonexistence of the isoperimetric inequality when it is not possible to associate the corresponding Sobolev inequality. Finally, for $N = 2$, we developed an original type of symmetrization, which we call star-shaped Steiner symmetrization, and we apply it to a class of isoperimetric problems with different monomial weights.

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1. Introduction

This paper deals with the isoperimetric problem in $\mathbb{R}^N$ with double density. More precisely, we consider two densities as follows: let $f : \mathbb{R}^N \to \mathbb{R}^+$ and $h : \mathbb{R}^N \to \mathbb{R}^+$ be given functions, and

$$m_f(E) := \int_E f(x)\mathrm{d}x, \qquad P_h(E) := \int_{\partial^* E} h(x) d\mathcal{H}^{N-1}(x),$$

where $\partial^* E$ denotes the reduced boundary of E. The isoperimetric problem consists in searching for sets of minimal weighted perimeter $P_h(E)$ among those sets E having weighted volume $m_f(E)$ equal to a given positive constant. A set solving the problem, if it exists, is called an isoperimetric set. When $f(x) = x^B := |x_1|^{b_1} \cdot \dots \cdot |x_N|^{b_N}$ and $h(x) = x^A := |x_1|^{a_1} \cdot \dots \cdot |x_N|^{a_N}$ are monomial weights, there are three main questions which we need to understand: the $\varepsilon - \varepsilon$ property; the boundedness, and geometric properties of isoperimetric sets.

We point out that a great attention has been given to the isoperimetric inequalities with weights, see for instance [1, 7–10, 13, 14, 17, 18, 20–22, 24, 27–32, 35, 40–42] and the references therein. However, in the wide

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literature, the majority of works approach volume functional and perimeter functional carrying the same weight. For the case in which the volume and the perimeter carry two different weights, see [1, 7–9, 11, 29, 30, 41].

A fundamental tool to study isoperimetric problem is the $\varepsilon-\varepsilon$ regularity property already discussed by Almgren, Allard and Bombieri, see [2–6] and [16], which is mainly important to obtain boundedness and regularity of isoperimetric sets. This technique basically says that one can locally modify a set E by changing its volume by a small quantity ε (which may be positive or negative), while increasing the perimeter by at most a quantity $C|\varepsilon|$. This study was extended by Cinti and Pratelli in [27] and [28], they conveniently weakened the $\varepsilon-\varepsilon$ property which is still sufficient to obtain boundedness, namely, an $\varepsilon-\varepsilon^{\beta}$ version of the property, this method was applied to isoperimetric problems when volume and perimeter carry the same weights. Moreover, in [41] Pratelli and Saracco considered isoperimetric problems with double density and studied the so-called $\varepsilon-\varepsilon^{\beta}$ property, boundedness and regularity of isoperimetric sets. Here, we remember this property and results established in [41].

Definition 1.1 (The $\varepsilon-\varepsilon^{\beta}$ property, [41])**.** Let E be a set of locally finite perimeter and $\beta\in[0,1]$. We say that E possesses the $\varepsilon-\varepsilon^{\beta}$ property (relative to the densities f and h) if for any ball $\mathcal{B}$ such that $\mathcal{H}^{n-1}(\mathcal{B}\cap\partial^{*}E)>0$ there exist constants $C>0$ and $\overline{\varepsilon}>0$ such that for all $|\varepsilon|<\overline{\varepsilon}$, there exists a set F such that

$$F\Delta E\subset\subset\mathcal{B},\ m_f(F)-m_f(E)=\varepsilon,\ P_h(F)-P_h(E)\leq C|\varepsilon|^{\beta}.$$

Theorem A (The $\varepsilon-\varepsilon^{\beta}$ Property, [41])**.** *Assume that f and h are locally bounded, that h is locally α-Hölder for some $\alpha\in[0,1]$, and that $E\subset\mathbb{R}^N$ is a set of locally finite perimeter. Then, E possesses the $\varepsilon-\varepsilon^{\beta}$ property, where β is given by*

$$\beta=\beta(N,\alpha)=\frac{\alpha+(N-1)(1-\alpha)}{\alpha+N(1-\alpha)}. \tag{1.1}$$

If $\alpha=0$ (in which case locally α-Hölder precisely means locally bounded) and h is continuous, then E possesses the $\varepsilon-\varepsilon^{\frac{N-1}{N}}$ property for all constants $C>0$ (that is, given any ball $\mathcal{B}$ then the constant C of Definition 1.1 can be taken arbitrarily small, up to choosing $\overline{\varepsilon}$ small enough).

Theorem B (Boundedness, [41])**.** *Assume that there exists a constant $M>0$ such that*

$$\frac{1}{M}\leq f(x)\leq M,\ \frac{1}{M}\leq h(x)\leq M\ \forall x\in\mathbb{R}^N,$$

and that $E\subset\mathbb{R}^N$ is an isoperimetric set for which the $\varepsilon-\varepsilon^{\frac{N-1}{N}}$ property holds with an arbitrarily small constant C. Then, E is bounded.

In order to prove the boundedness results of the isoperimetric sets in [27, 28] and [41], the authors used the classical isoperimetric inequality combined with either the $\varepsilon-\varepsilon^{\beta}$ property with $\beta>(N-1)/N$ or the $\varepsilon-\varepsilon^{\frac{N-1}{N}}$ property when the constant C can be chosen arbitrarily small. Based on their arguments, it is essential to obtain the $\varepsilon-\varepsilon^{\beta}$ property with $\frac{N-1}{N}<\beta\leq1$, because they made some estimates constructed on the classical isoperimetric inequality.

It is worth noting that the Theorem B does not cover the cases $f(x)=x^B:=|x_1|^{b_1}\ldots|x_n|^{b_N}$ and $h(x)=x^A:=|x_1|^{a_1}\ldots|x_n|^{a_N}$, where the $b_i's$ and $a_i's$ are nonnegative numbers. According to [41], for functions satisfying the hypothesis of Theorems A and B, they are l.s.c and positive, and hence, they are always locally away from zero. Thus, in Theorem A, f and h are locally bounded and away from zero, whereas in Theorem B these requests are made globally. These assumptions are essentially sharp. They can be slightly relaxed to the case of "essentially bounded" or "essentially α-Hölder" functions as defined in ([28], Defs. 1.6 and 1.7). On the axes, the monomial weights do not fulfill this hypothesis. Furthermore, if the smallest power of x^A satisfies $0<a_{i_0}<1$, then h is a_{i_0}-Hölder, but not Lipschitz, and so, we may apply the Theorem A to obtain equation (1.1) with

$$\beta = \frac{a_{i_0} + (N-1)(1-a_{i_0})}{a_{i_0} + N(1-a_{i_0})}, \text{ where } a_{i_0} := \min\{a_j; j \in \{1,\dots,N\}, a_j > 0\}, \tag{1.2}$$

however, this value of β will not be big enough to get the boundedness of the isoperimetric set. So, in some ways, our results are improvements since we relaxed the way the weights behave towards zero. Here, our approach is not based on the classical isoperimetric inequality, because the densities are not locally bounded from below by a positive constant.

Throughout this paper, we will assume $N \geq 2$ and that the vectors $A = (a_1, a_2, \dots, a_N),\ B = (b_1, \dots, b_N) \in \mathbb{R}^N$ are nonnegative if all its entries are nonnegative, and we set

$$a := a_1 + a_2 + \cdots + a_N, \quad b := b_1 + b_2 + \cdots + b_N,$$

and

$$\mathbb{R}^N_A = \{(x_1, \dots, x_N) \in \mathbb{R}^N; x_i > 0, \text{ whenever } a_i > 0\}.$$

Thus, for example $\mathbb{R}^2_{(0,1)} = \{(x_1, x_2) \in \mathbb{R}^2; x_2 > 0\} = \mathbb{R}^2_+$.

In order to prove the boundedness of isoperimetric sets with different monomial weights, we are going to use the existence of isoperimetric inequalities and the *half Steiner symmetrization with weights*, already studied by the first two authors in [1]. More precisely, we are going to apply the existence of isoperimetric inequalities with different monomial weights and the $\varepsilon - \varepsilon^\beta$ property with $\frac{N+a-1}{N+b} < \beta \leq 1$; however, as was mentioned, the value of β given by equation (1.2) is not necessarily greater than $\frac{N+a-1}{N+b}$, then it is essential to prove the property with $\frac{N+a-1}{N+b} < \beta \leq 1$. In this direction, let us state our first two important results.

Theorem 1.2 (The ε–ε property). *Assume that $a_i = b_i$ for some $i \in \{1, \dots, N\}$. Then, any isoperimetric set $E \subset \mathbb{R}^N_A$, relative to the densities $h(x) = x^A$ and $f(x) = x^B$, possesses the ε–ε property.*

The primary difference between Theorem A and the preceding conclusion is that $\beta = 1$.

Theorem 1.3 (Boundedness). *Suppose that the following conditions are satisfied:*

$(H1)$ $0 \leq a_i - \frac{N+a-1}{N+b} b_i \leq \frac{N+a-1}{N+b}$ *for all* $i \in \{1, \dots, N\}$*;*
$(H2)$ $a - b < 1$*;*
$(H3)$ $b_i \leq a_i$ *for all* $i \in \{1, \dots, n\}$ *and* $a_j = b_j$ *for some* $j \in \{1, \dots, N\}$*.*

Then, any isoperimetric set E in $\mathbb{R}^N_A$, relative to the densities x^B and x^A, is bounded.

As noticed by Cabré and Ros-Oton in [21], when $a - b < 1$, the isoperimetric set is in $\mathbb{R}^N_A$. See also [1], Remark 3.4. To better understand the assumptions made in the Theorem 1.3, notice that the condition $(H1)$ comes from [1], Theorem 1.1, it is a necessary condition to the existence of isoperimetric inequality. Second, the condition $(H2)$ is also based on [1], Theorem 1.1, it concerns the sufficient condition. Moreover, the strict inequality $a - b < 1$ is required by our argument used to establish the Theorem itself. Finally, we emphasize that the condition $(H3)$ is derived from [1], Theorem 1.3, that is, the *half Steiner symmetrization with weights*, and it will be very important in our arguments.

As it was mentioned, the conditions $(H1)$ and $(H2)$ are based on [1]. Furthermore, when $a - b > 1$, the study is not complete. The next result follows the same purpose of [1], Theorem 1.1, and give us a slight improvement of cases of nonexistence of the isoperimetric inequality with $a - b > 1$.

Theorem 1.4. *Assume that the conditions*

$(H4)$ $a_i - \frac{N+a-1}{N+b} b_i = \frac{N+a-1}{N+b}$ *for* $i \in I$*, where* $I \subsetneq \{1, \cdots, N\}$*,*

$(H5)$ $a - b > 1$,

are satisfied. Then, there is no isoperimetric inequality relative to the densities x^B *e* x^A*.*

We wish to point out that, in general, to find an isoperimetric set, it is a difficult task. Some attempts to find a set E in $\mathbb{R}^N$ which minimizes the weighted perimeter $\int_{\partial E} x^A d\mathcal{H}^{N-1}$ once the weighted volume $\int_E x^B d\mathcal{H}^N$ is prescribed, it was discussed by Cabré and Ros-Oton in [21] when $A = B$, by Alvino at al. in [9], and by the first two authors in [1], Theorem 1.3 including the case $A \neq B$. Here, under some conditions, we will show that our search to the isoperimetric sets can be restricted to a specific class of sets. The main ingredient in the proof will be a Steiner symmetrization argument when the volume and the perimeter carry two different monomial weights, but unlike [1], Theorem 1.3, we will not have the restriction $a_i = b_i$ for some $i \in \{1, \ldots, N\}$. This part of the work will be developed only in the 2-dimensional Euclidean space $\mathbb{R}^2$ with the assumption $a = b + 2$. We emphasize that, for this condition, we do not even know if the corresponding isoperimetric inequality exists. Hence, our symmetrization is also intended to help in this way. Noticing that, if we assume $a = b + 2$, then the assumption $(H4)$ in the Theorem (1.4) can still be satisfied. However, if $(H4)$ holds for every $i = 1, \ldots, N$, then we arrive at the absurd, $a = a + 1$. In fact, as we will see below, our arguments permit us to suspect that, if $(H4)$ fails, then such an isoperimetric set exists. We are going to symmetrize an arbitrary set $E \subset \mathbb{R}^2$ so that the obtained set $E^{\otimes}$ has the same volume and the perimeter does not increase. As our argument is a bit more general, we extend to weights $|x|^{b_1}|y|^{b_2}(x^2 + y^2)^{l/2}$. Moreover, it is worth to mention that symmetrization methods are extremely useful in analysis and geometry, with a numerous applications. Oftentimes, these methods are associated with isoperimetric inequalities. One of the most important is the classical Steiner symmetrization introduced by Steiner in 1840. In this direction, we refer to the interested reader to [12, 23, 25, 36].

In the following, we describe the mentioned symmetrization. Let us consider $N = 2$, $A = (a_1, a_2)$, $B = (b_1, b_2)$, two nonnegative vectors, $l, k \in \mathbb{R}$, and the following functions:

$$f(x, y) \equiv |x|^{b_1}|y|^{b_2}(x^2 + y^2)^{l/2} \text{ and } h(x, y) \equiv |x|^{a_1}|y|^{a_2}(x^2 + y^2)^{k/2}.$$

Thus, in this case we study the following isoperimetric problem for smooth sets:

$$\text{minimize } P_{A,k}(E) \text{ for all sets } E \subset \mathbb{R}^2 \text{ satisfying } m_{B,l}(E) = 1.$$

where

$$P_{A,k}(E) := \int_{\partial^* E} |x|^{a_1}|y|^{a_2}(x^2 + y^2)^{k/2} d\mathcal{H}^1(x, y) \quad \text{and} \quad m_{B,l}(E) := \int_E |x|^{b_1}|y|^{b_2}(x^2 + y^2)^{l/2} d\mathcal{H}^2(x, y).$$

It is important to point out that the existence and nonexistence of isoperimetric inequalities referred above are obtained by the first two authors, for $k = l = 0$ in [1], and by Balogh-Gutiérrez-Kristály [11] for a class of homogeneous weights when it is possible to associate the corresponding Sobolev inequality.

The main goal now is to study the perimeter inequality under a Steiner type symmetrization. To do this, we assume that $b + l + 1 > -1$ with $b = b_1 + b_2$. We define the *star-shaped Steiner symmetrization with weights* as follows: Given a set $E \subset \mathbb{R}^2$ and $\theta \in [0, \pi]$, we denote by $E_\theta := E \cap L_\theta$, the corresponding one-dimensional polar section of E, where

$$L_\theta = \{(x, y) \in \mathbb{R}^2; x = r\cos(\theta), y = r\sin(\theta) \text{ and } r \in \mathbb{R}\}.$$

The star-shaped distribution function $\mu_{B,l}$ of E is defined by

$$\mu_{B,l}(\theta) = \int_{E \cap L_\theta} |r|^{b+l+1} d\mathcal{H}^1(r), \quad \text{for all } \theta \in [0, \pi].$$

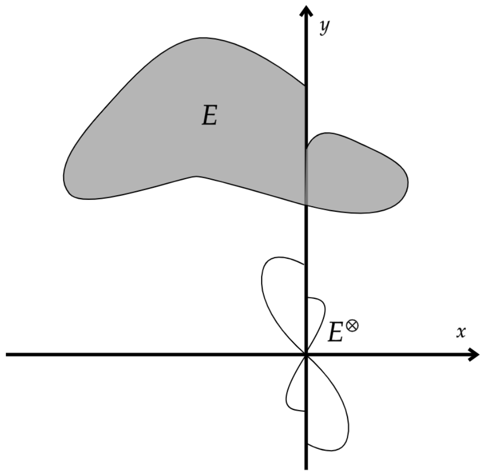


FIGURE 1. The star-shaped Steiner symmetrization of the set E.

Finally, denoting the essential star-shaped projection of E by

$$\Pi^{\otimes}(E) := \{\theta \in [0,\pi]; \mu_{B,l}(\theta) > 0\} \text{ and } \eta_{B,l}(\theta) := \left[\frac{b+l+2}{2}\mu_{B,l}(\theta)\right]^{\frac{1}{b+l+2}},$$

we set the *star-shaped Steiner symmetrization with weights of* E as being the set

$$E^{\otimes} := \left\{(r\cos(\theta), r\sin(\theta)); \theta \in \Pi^{\otimes}(E) \text{ and } -\eta_{B,l}(\theta) < r < \eta_{B,l}(\theta)\right\}.$$

By Fubini's theorem, it follows that $\mu_{B,l}$ is an $\mathcal{H}^1$- measurable function in $[0,\pi]$, $\Pi^{\otimes}(E)$ is a measurable set in $\mathbb{R}^1$ and

$$m_{B,l}(E^{\otimes}) := \int_{E^{\otimes}} |x|^{b_1}|y|^{b_2}(x^2+y^2)^{l/2} d\mathcal{H}^2(x,y) = \int_E |x|^{b_1}|y|^{b_2}(x^2+y^2)^{l/2} d\mathcal{H}^2(x,y) =: m_{B,l}(E).$$

The weighted perimeter inequality under the star-shaped Steiner symmetrization with weights reads as follows.

Theorem 1.5. *Consider l, k real numbers. Let $A = (a_1, a_2)$ be a nonnegative vector, and let $B = (b_1, b_2)$ a vector satisfying $b + l + 1 := b_1 + b_2 + l + 1 > -1$. Assume that $a + k = b + 2 + l$ and*

$$\left.\begin{array}{r}\displaystyle\int_{E_\theta^{\nu_r} := \{(r\cos(\theta), r\sin(\theta)) \in \partial^* E; \nu_r(r,\theta)=0\}} r^{b+l+1}|\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2} d\mathcal{H}^1(r) \\ \displaystyle\int_{(E_\theta^{\otimes})^{\nu_r} := \{(r\cos(\theta), r\sin(\theta)) \in \partial^* E^{\otimes}; \nu_r(r,\theta)=0\}} r^{b+l+1}|\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2} d\mathcal{H}^1(r)\end{array}\right\} = 0. \tag{1.3}$$

Then

$$P_{A,k}(E^{\otimes}) \le P_{A,k}(E).$$

Moreover, if $P_{A,k}(E^{\otimes}) = P_{A,k}(E)$, then for $\mathcal{H}^1$ – a.e. $\theta \in \Pi^{\otimes}(E)$, the set

$$E_\theta \text{ is equivalent to the segment } (-\eta_{B,l}(\theta), \eta_{B,l}(\theta)). \tag{1.4}$$

The condition (1.3) is equivalent to say that both the reduced boundary of $E^{\otimes}$ and E have no flat parts contained in L_θ whenever $\theta \in (0, \pi)$. Although it might seem artificial, if the goal is to obtain a minimizer for the isoperimetric inequality, this condition happens naturally, see Remark 5.2.

The plan of the paper is as follows. In Section 2 we define some basic elements that we will use throughout the paper. Section 3 is devoted to the proof of the Theorems 1.2 and 1.3. Section 4 discusses the proof of Theorem 1.4. Finally, Section 5 concerns on the proof of Theorem 1.5.

2. Notation and preliminaries

Most of the definitions and notations contained in this section come basically from Geometric Measure Theory, see for instance [33, 34, 38, 39], along with some further definitions.

Given a function $\omega : \mathbb{R}^N \to \mathbb{R}^+$, locally Lipschitz on $\mathbb{R}^N$, we set the P_ω-Perimeter of a measurable set E as

$$P_\omega(E) := \sup \left\{ \int_E \operatorname{div}(\omega(x)\phi(x))\mathrm{d}x ; \phi \in C_c^1(\mathbb{R}^N, \mathbb{R}^N), |\phi| \le 1 \text{ on } \mathbb{R}^N \right\}.$$

If E is a smooth bounded open set, then the weighted perimeter P_ω is equivalent to

$$P_\omega(E) = \int_{\partial E} \omega(x) d\mathcal{H}^{N-1}(x),$$

and the same happens for $\omega(x) = x^A$, where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure. We denote P_A instead of P_{x^A}.

We say that a Borel set $E \subset \mathbb{R}^N$ is a set of locally finite perimeter if its characteristic function 1_E is BV_{loc} function. We briefly recall that the reduced boundary $\partial^* E$ of a locally finite perimeter set $E \subset \mathbb{R}^N$ is the collection of points $x \in \mathbb{R}^N$ such that:

1. $\|D1_E\|(B(x,r)) > 0$ for all $r > 0$, where $B(x,r)$ denotes the ball with center x and radius r,
2. $\displaystyle\lim_{r\to 0} \frac{1}{\|D1_E\|(B(x,r))} \int_{B(x,r)} \nu_E d\|D1_E\| = \nu_E(x)$, and
3. $|\nu_E(x)| = 1$,

where $\|D1_E\|$ and ν_E are, respectively, the Radon measure on $\mathbb{R}^N$ and the $\|D1_E\|$-measurable function $\nu_E : \mathbb{R}^N \to \mathbb{R}^N$ given by the *Riesz Representation Theorem*, se for instance [33], Theorem 1.38. The vector $\nu_E(x) = (\nu_{E,1}(x), \ldots, \nu_{E,N}(x))$ is called the generalized outward normal to E at x.

Theorem C (De Giorgi)**.** *Assume that E has locally finite perimeter in $\mathbb{R}^N$.*

(i) Then

$$\partial^* E = \left(\bigcup_{j=1}^{\infty} K_j \right) \cup F,$$

where $\|D1_E\|(F) = 0$ and K_j is a compact subset of C^1-hypersurface S_j for every $j \in \mathbb{N}$.

(ii) Furthermore, $\nu_E(x)|_{S_j}$ is normal to S_j for all $j \in \mathbb{N}$, and

(iii) $\|D1_E\| = \mathcal{H}^{N-1} \llcorner \partial^* E$.

For a nonnegative measurable function $\gamma : \mathbb{R}^N \to \mathbb{R}$, we set by m_γ the Lebesgue measure with weight $\gamma(x)$, namely,

$$m_\gamma(M) = \int_M \gamma(x)\mathrm{d}x,$$

for all $\mathcal{H}^N$-measurable set M in $\mathbb{R}^N$. If $\gamma(x) = x^B := |x|^{b_1} \cdot \ldots \cdot |x_N|^{b_N}$, we denote m_B instead of m_{x^B}.

If a measurable set M satisfies $0 < m_\gamma(M) < \infty$, then $\mathcal{R}_{A,B,N}(M)$ denotes the isoperimetric quotient of M given by

$$\mathcal{R}_{A,B,N}(M) := \frac{P_A(M)}{[m_B(M)]^{\frac{N+a-1}{N+b}}}.$$

It follows that, for any smooth open set Ω with $0 < m_B(\Omega) = \int_\Omega x^B \mathrm{d}x < \infty$, we define

$$\mathcal{R}_{A,B,N}(\Omega) := \frac{\displaystyle\int_{\partial\Omega} x^A d\mathcal{H}^{N-1}(x)}{\left[\displaystyle\int_\Omega x^B \mathrm{d}x\right]^{\frac{N+a-1}{N+b}}}.$$

Thus, the best constant of the isoperimetric inequality, related to weights x^A and x^B, is defined by

$$C_{A,B,N} := \inf\left\{\mathcal{R}_{A,B,N}(\Omega) : \Omega \text{ is a smooth open set, and } 0 < m_B(\Omega) < \infty\right\}. \tag{2.1}$$

For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ a vector, and $A = (a_1, \ldots, a_N) \in \mathbb{R}^N$ a nonnegative vector, we will use the following notations:

$\overline{x}_i := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)$;
$\overline{A}_i := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_N)$;
$\overline{B}_i := (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_N)$.

3. Boundedness and $\varepsilon - \varepsilon$ property of isoperimetric sets

The plan for this section is the following: first, we show the $\varepsilon - \varepsilon$ property of isoperimetric sets based on the half Steiner symmetrization with weights. Then, we prove the boundedness of isoperimetric sets using their $\varepsilon - \varepsilon^\beta$ property with $\frac{N+a-1}{N+b} < \beta \leq 1$.

In order to prove Theorem 1.2, we need to make a small improvement on [1], Theorem 1.3. Thus, we briefly remember the half Steiner symmetrization with weights defined in [1]. Given a set $E \subset \mathbb{R}^N_A$ and $x \in \mathbb{R}^{N-1}$, we denote the corresponding one-dimensional section of E by

$$E_x = \{y \in \mathbb{R}; (x, y) \in E\}.$$

The distribution function μ of E is defined by

$$\mu(x) := \int_{E_x} |y|^{b_N} d\mathcal{H}^1(y), \ x \in \mathbb{R}^{N-1}.$$

Finally, denoting the essential projection of E by $\pi(E)^+ = \{x \in \mathbb{R}^{N-1}; \mu(x) > 0\}$ and $l(x) := ((b_N + 1)\mu(x))^{\frac{1}{b_N+1}}$, we set the *half Steiner symmetrization with weights of E with respect to the hyperplane*

$\{x_N = 0\}$ as being the set

$$E^s = \left\{(x,y) \in \mathbb{R}^{N-1} \times \mathbb{R}; x \in \pi(E)^+, 0 < y < l(x)\right\},$$

whenever $a_N \geq b_N$ and $a_N > 0$. By Fubini's theorem, it follows that μ is an $\mathcal{H}^{N-1}$- measurable function in $\mathbb{R}^{N-1}$, E^s is a measurable set in $\mathbb{R}^N$ and

$$m_B(E^s) := \int_{E^s} x^{\overline{B}_N} |y|^{b_N} d\mathcal{H}^N(x,y) = \int_E x^{\overline{B}_N} |y|^{b_N} d\mathcal{H}^N(x,y) =: m_B(E).$$

By the above equality and the following result, we can compare the isoperimetric quotients of E and E^s.

Theorem 3.1 (half Steiner symmetrization: cases of equality)**.** *Let D be an open set in $\mathbb{R}^{N-1}$, and let $E \subset \mathbb{R}^N_A$ be a set of locally finite perimeter with finite volume $m_B(E)$ and finite perimeter $P_A(E)$. Assume that*

$$\int_{(D\times\mathbb{R})\cap E^s_{\{\nu_N=0\}}} x^{\overline{A}_N} |y|^{a_N} d\mathcal{H}^{N-1}(x,y) = \int_{(D\times\mathbb{R})\cap E_{\{\nu_N=0\}}} x^{\overline{A}_N} |y|^{a_N} d\mathcal{H}^{N-1}(x,y) = 0, \tag{3.1}$$

where $E^s_{\{\nu_N=0\}} = \{(x,y) \in \partial^* E^s; \nu_{E^s,N}(x,y) = 0\}$, and $E_{\{\nu_N=0\}} = \{(x,y) \in \partial^* E; \nu_{E,N}(x,y) = 0\}$. Then, for $a_N = b_N > 0$,

$$P_A(E^s, D\times\mathbb{R}) \leq P_A(E, D\times\mathbb{R}). \tag{3.2}$$

Moreover, if

$$P_A(E^s, D\times\mathbb{R}) = P_A(E, D\times\mathbb{R}), \tag{3.3}$$

then for $\mathcal{H}^{N-1}$ – a.e. $x \in \pi(E)^+ \cap D$

$$E_x \text{ is equivalent to the segment } (0, l(x)). \tag{3.4}$$

Here, the inequality (3.2) comes from [1], Theorem 1.3, while equation (3.4) is the improvement.

Proof of Theorem 3.1. The first part follows from [1], Theorem 1.3. For the benefit of the reader we are going to write a brief demonstration here. We have that

$$\begin{aligned}
P_A\big(E^s; D\times\mathbb{R}\big) &= \int_{\partial^* E^s\cap(D\times\mathbb{R})} x^{\overline{A}_N} |y|^{a_N} d\mathcal{H}^{N-1}(x,y) \\
&= \int_D \sqrt{x^{2\overline{A}_N} (l(x))^{2a_N} + |l(x)|^{2(a_N-b_N)} \sum_{j=1}^{N-1} \left(\frac{x^{\overline{A}_N} |l(x)|^{b_N} \nu_{E^s,j}(x,l(x))}{\nu_{E^s,N}(x,l(x))}\right)^2}\, d\mathcal{H}^{N-1}(x) \\
&= \int_D x^{\overline{A}_N} \sqrt{(l(x))^{2a_N} + \sum_{j=1}^{N-1} \left(\int_{(\partial^* E)_x} |l(x)|^{(a_N-b_N)} |y|^{b_N} \frac{\nu_{E,j}(x,y)}{|\nu_{E,N}(x,y)|} d\mathcal{H}^0(y)\right)^2}\, d\mathcal{H}^{N-1}(x).
\end{aligned} \tag{3.5}$$

Note that, for each $x \in \pi(E)^+$, there exists $z \in (\partial^* E)_x$ such that $l(x) \le z$ and, therefore,

$$x^{2\overline{A}_N} l(x)^{2a_N} + \sum_{j=1}^{N-1} \left(\int_{(\partial^* E)_x} x^{\overline{A}_N} |y|^{a_N} \frac{\nu_{E,j}(x,y)}{|\nu_{E,N}(x,y)|} d\mathcal{H}^0(y) \right)^2$$
$$\le x^{2\overline{A}_N} \left(\int_{(\partial^* E)_x} |y|^{a_N} d\mathcal{H}^0(y) \right)^2 + \sum_{j=1}^{N-1} \left(\int_{(\partial^* E)_x} x^{\overline{A}_N} |y|^{a_N} \frac{\nu_{E,j}(x,y)}{|\nu_{E,N}(x,y)|} d\mathcal{H}^0(y) \right)^2 .$$

It follows from equation (3.5) and the discrete Minkowski's inequality that

$$P_A\big(E^s; D\times\mathbb{R}\big) \le \int_D \sqrt{ x^{2\overline{A}_N} \left(\int_{(\partial^* E)_x} |y|^{a_N} d\mathcal{H}^0(y) \right)^2 + \sum_{j=1}^{N-1} \left(\int_{(\partial^* E)_x} x^{\overline{A}_N} |y|^{a_N} \frac{\nu_{E,j}(x,y)}{|\nu_{E,N}(x,y)|} d\mathcal{H}^0(y) \right)^2 }\, d\mathcal{H}^{N-1}(x)$$
$$\le \int_D \int_{(\partial^* E)_x} \sqrt{ x^{2\overline{A}_N} |y|^{2a_N} + \sum_{j=1}^{N-1} \left(x^{\overline{A}_N} |y|^{a_N} \frac{\nu_{E,j}(x,y)}{|\nu_{E,N}(x,y)|} \right)^2 }\, d\mathcal{H}^0(y) d\mathcal{H}^{N-1}(x)$$
$$= \int_D \int_{(\partial^* E)_x} \frac{x^{\overline{A}_N} |y|^{a_N}}{|\nu_{E,N}(x,y)|} d\mathcal{H}^0(y) d\mathcal{H}^{N-1}(x,y) = \int_{\partial^* E \cap (D\times\mathbb{R})} x^{\overline{A}_N} |y|^{a_N} d\mathcal{H}^{N-1}(x,y) = P_A\left(E; D\times\mathbb{R}\right). \tag{3.6}$$

This ends the first part.

Now we will show the second part of the theorem. Since $P_A(E)$ is finite, it follows from [43] that E_x is rectifiable. Thus, up to a set of null $(N-1)$-dimensional Hausdorff measure, for any fixed x there exist $k \in \mathbb{N}$ and real numbers $d_j = d_j(x)$ with $d_j \le d_{j+1}$ such that

$$E_x = (d_0, d_1) \cup (d_2, d_3) \cup (d_4, d_5) \ldots (d_k, d_{k+1}). \tag{3.7}$$

On the other hand, by equations (3.3), (3.5) and (3.6) we have that

$$l(x)^{a_N} = \int_{(\partial^* E)_x} |y|^{a_N} \mathcal{H}^0(y),$$

i.e.

$$\left[(a_N+1) \int_{E_x} |y|^{a_N} d\mathcal{H}^1(y) \right]^{\frac{a_N}{a_N+1}} = \int_{(\partial^* E)_x} |y|^{a_N} \mathcal{H}^0(y). \tag{3.8}$$

From equations (3.7) and (3.8) we deduce that

$$\left[(a_N+1) \int_{E_x} |y|^{a_N} d\mathcal{H}^1(y) \right]^{\frac{a_N}{a_N+1}} = \left[\sum_{0\le i\le k, i \text{ even}} \left(d_{i+1}^{a_N+1} - d_i^{a_N+1} \right) \right]^{\frac{a_N}{a_N+1}},$$
$$\int_{(\partial^* E)_x} |y|^{a_N} \mathcal{H}^0(y) = \sum_{i=0}^{k+1} d_i^{a_N},$$

and consequently,

$$\left[\sum_{0\le i\le k, i \text{ even}} \left(d_{i+1}^{a_N+1} - d_i^{a_N+1}\right)\right]^{\frac{a_N}{a_N+1}} = d_0^{a_N} + \sum_{i=0}^{k} d_{i+1}^{a_N}. \tag{3.9}$$

It is easy to check that $k = 0$ and $d_0 = 0$. Indeed, from the elementary inequality $(a+b)^\tau < a^\tau + b^\tau$, $(\tau \in (0,1)$, $a > 0$, $b > 0)$, if $k > 0$ or $d_0 \neq 0$, then we readily obtain that

$$\begin{aligned}\left[\sum_{0\le i\le k, i \text{ even}} \left(d_{i+1}^{a_N+1} - d_i^{a_N+1}\right)\right]^{\frac{a_N}{a_N+1}} &< \sum_{0\le i\le k, i \text{ even}} \left(d_{i+1}^{a_N+1} - d_i^{a_N+1}\right)^{\frac{a_N}{a_N+1}} \\ &< \sum_{0\le i\le k, i \text{ even}} \left(d_{i+1}^{a_N+1} + d_i^{a_N+1}\right)^{\frac{a_N}{a_N+1}} < d_0^{a_N} + \sum_{i=0}^{k} d_{i+1}^{a_N},\end{aligned}$$

which are both in contradiction with equation (3.9). Therefore, the theorem follows from equation (3.7).

□

Proof of Theorem 1.2. First of all, notice that it suffices to consider $i = N$. Second, we will consider the case $a_N = b_N > 0$, because the case $a_N = b_N = 0$ follows from similar arguments using the classical Steiner symmetrization (see Rem. 3.2).

According to Theorem 3.1 and a similar argument as in [1], Remark 5.3, it follows that any isoperimetric set relative to the densities x^B and x^A is a graph of a function $u \in W^{1,\infty}_{loc}(\pi_N(E)^+)$, *i.e.*

$$E = \{(\overline{x}_N, x_N) \in \pi_N(E)^+ \times \mathbb{R};\ 0 < x_N < u(\overline{x}_N)\}.$$

In order to prove the $\varepsilon - \varepsilon$ property, we take a ball $\mathcal{B}$ in $\mathbb{R}^N$ such that $\mathcal{H}^{N-1}(\mathcal{B} \cap \partial^* E) > 0$ and we consider only the case $\mathcal{B} \subset \mathbb{R}^N_A$, since the case $\mathcal{B} \cap \partial\mathbb{R}^N_A \neq \emptyset$ follows from similar arguments. Thus, there are $x^0 = (x_1^0, x_2^0, \dots, x_{N-1}^0) \in \mathbb{R}^{N-1}_A$ and $\delta > 0$ small such that

$$Q^\delta := \{x \in \mathbb{R}^{N-1}_A;\ x_i^0 - \delta < x_i < x_i^0 + \delta,\ \forall i \in \{1, \dots, N-1\}\} \subset \pi_N(E)^+.$$

Let us now define $X = (\overline{x}_N, x_N) \in \mathbb{R}^N_A$,

$$\begin{aligned}&F_\delta := \left\{X;\ \text{either } X \in E \text{ or } \overline{x}_N \in Q^\delta \text{ and } u(\overline{x}_N) \le x_N < u(\overline{x}_N) + \delta\right\},\ \text{(see Fig. 2 below)},\\ &Q_i^\delta := \{x \in \mathbb{R}^{N-1}_A;\ x_i = x_i^0 \text{ and } x_j^0 - \delta < x_j < x_j^0 + \delta \text{ for all } j \neq i\},\\ &x_{i0(-\delta)} = (x_1, x_2, \dots, x_{i-1}, x_i^0 - \delta, x_{i+1}, \dots, x_{N-1}),\ \text{and}\\ &x_{i0(+\delta)} = (x_1, x_2, \dots, x_{i-1}, x_i^0 + \delta, x_{i+1}, \dots, x_{N-1}),\ \text{where } i \in \{1, \dots, N-1\}.\end{aligned}$$

By definition of F_δ, see Figure 2, we have that

$$\begin{aligned}m_B(F_\delta) &= m_B(E) + \int_{Q^\delta}\int_{u(\overline{x}_N)}^{u(\overline{x}_N)+\delta} x_N^{b_N}\overline{x}_N^{\overline{B}_N}\,\mathrm{d}x_N d\overline{x}_N\\ &= m_B(E) + \int_{Q^\delta} \overline{x}_N^{\overline{B}_N}\frac{(u(\overline{x}_N)+\delta)^{b_N+1} - (u(\overline{x}_N))^{b_N+1}}{b_N+1} d\overline{x}_N\end{aligned}$$

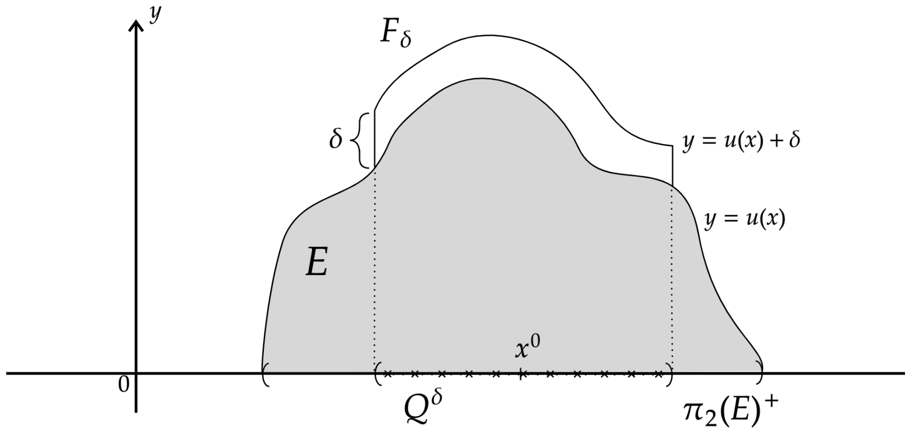


FIGURE 2. The sets F_δ's.

$$= m_B(E) + \delta \int_{Q^\delta} \overline{x}_N^{\overline{B}_N} \left(u(\overline{x}_N) + \varrho(\overline{x}_n)\delta\right)^{b_N} d\overline{x}_N =: m_B(E) + \delta \cdot \gamma(\delta), \tag{3.10}$$

where the function $\varrho : Q^\delta \to (0,1)$ is given by the Mean Value Theorem and γ is a bounded function for $0 < \delta < 1$.

On the other hand, it follows from Area Formula and basic inequalities that

$$\begin{aligned}
&P_A(F_\delta) = P_A(E) + \int_{Q^\delta} \left[(u(\overline{x}_N) + \delta)^{a_N} - (u(\overline{x}_N))^{a_N}\right] \overline{x}_N^{\overline{A}_N} \sqrt{1 + |\nabla u(\overline{x}_N)|^2} d\overline{x}_N \\
&+ \sum_{i=1}^{N-1} (x_i^0 + \delta)^{a_i} \int_{Q_i^\delta} \frac{\left(u(x_{i0(-\delta)}) + \delta\right)^{a_N+1} - \left(u(x_{i0(-\delta)})\right)^{a_N+1}}{a_N + 1} \overline{x}_{iN}^{\overline{A}_{iN}} d\overline{x}_{iN} \\
&+ \sum_{i=1}^{N-1} (x_i^0 - \delta)^{a_i} \int_{Q_i^\delta} \frac{\left(u(x_{i0(+\delta)}) + \delta\right)^{a_N+1} - \left(u(x_{i0(+\delta)})\right)^{a_N+1}}{a_N + 1} \overline{x}_{iN}^{\overline{A}_{iN}} d\overline{x}_{iN} \\
&\le P_A(E) + \left(1 + \|u\|_{L^\infty(Q^\delta)}\right)^{a_N} \sqrt{1 + \|\nabla u\|^2_{L^\infty(Q^\delta)}} \cdot \int_{Q^\delta} \overline{x}_N^{\overline{A}_N} d\overline{x}_N \\
&+ 2 \sum_{i=1}^{N-1} \left(|x_i^0| + 1\right)^{a_i} \left(\|u\|_{L^\infty(Q^\delta)} + 1\right)^{a_N} \cdot \delta \cdot \int_{Q_i^\delta} \overline{x}_{iN}^{\overline{A}_{iN}} d\overline{x}_{iN}
\end{aligned}$$

Then

$$P_A(F_\delta) \le P_A(E) + C(N,A) \left(1 + \|u\|_{L^\infty(Q^\delta)}\right)^{a_N} \left(1 + \|\nabla u\|^2_{L^\infty(Q^\delta)}\right)^{\frac{1}{2}} \cdot \delta + 2C(N,A) \left(\|u\|_{L^\infty(Q^\delta)} + 1\right)^{a_N} \cdot \delta, \tag{3.11}$$

where $C(N,A)$ is a constant which depends only on N and A.

Therefore, taking $\varepsilon = \gamma(\delta)\delta$, the theorem follows from equations (3.10) and (3.11). □

Remark 3.2. In the case $a_N = b_N = 0$, we can use the perimeter inequality under Steiner symmetrization: cases of equality, due to Chlebík-Cianchi-Fusco [25], to conclude that any isoperimetric set E is locally a graph of a function, *i.e.*, for each $\overline{x}_N^0 \in \pi_N(E)^+ := \{\overline{x}_N \in \mathbb{R}_A^{N-1}; \mathcal{H}^{N-1}(E_{\overline{x}_N}) > 0\}$, $E_{\overline{x}_N} := \{y \in \mathbb{R}; (\overline{x}_N, y) \in E\}$, there exists a neighborhood $V(\overline{x}_N^0) \subset \mathbb{R}^N_{\overline{A}_N}$ and functions $u, v \in W^{1,\infty}_{loc}(V(\overline{x}_N^0))$ such that $u(\overline{x}_N) < v(\overline{x}_N)$, $\forall \overline{x}_N \in V(\overline{x}_N^0)$ and

$$(V(\overline{x}_N^0) \times \mathbb{R}) \cap E = \{(\overline{x}_N, x_N);\ \overline{x}_N \in V(\overline{x}_N^0),\ u(\overline{x}_N) < x_N < v(\overline{x}_N)\},$$

see [25], Theorem 1.1. Thus, we can use the same sets $Q^\delta \subset V(\overline{x}_N^0)$ and F_δ in the above proof to demonstrate that the isoperimetric set E satisfies the $\varepsilon - \varepsilon$ property.

Proof of Theorem 1.3. Let E be an isoperimetric set relative to the densities x^B and x^A, *i.e.*,

$$\mathcal{R}_{A,B,N}(E) = \inf\left\{\mathcal{R}_{A,B,N}(M); 0 < m_B(M) < +\infty\right\}.$$

By Theorem 1.2, we may choose a ball $\mathcal{B} \subset\subset \mathbb{R}^N_A$ and some $\overline{\varepsilon} > 0$ such that for every $|\varepsilon| < \overline{\varepsilon}$ there exists a set F satisfying $F \Delta E \subset\subset \mathcal{B}$,

$$\int_F x^B \mathrm{d}x = \int_E x^B \mathrm{d}x + \varepsilon, \tag{3.12}$$

$$\int_{\partial^* F} x^A d\mathcal{H}^{N-1}(x) \leq \int_{\partial^* E} x^A d\mathcal{H}^{N-1}(x) + \varepsilon.$$

Since $\frac{N+a-1}{N+b} < 1$, then we have, for ε sufficiently small,

$$\int_{\partial^* F} x^A d\mathcal{H}^{N-1}(x) \leq \int_{\partial^* E} x^A d\mathcal{H}^{N-1}(x) + \varepsilon \leq \int_{\partial^* E} x^A d\mathcal{H}^{N-1}(x) + \frac{C_{A,B,N}}{2} \varepsilon^{\frac{N+a-1}{N+b}}. \tag{3.13}$$

Setting $B_{R_\infty} := \{x \in \mathbb{R}^N_A; 0 < |x| < R_\infty\}$, with $R_\infty > 0$ big enough, one has $\mathcal{B} \subset B_{R_\infty}$ and $m_B(E \backslash B_{R_\infty}) < \overline{\varepsilon}$.

Let us now define $\varphi : (R_\infty, +\infty) \to [0, +\infty)$ by

$$\varphi(R) := m_B(E \backslash B_R), \text{ where } B_R := \{x \in \mathbb{R}^N_A; 0 < |x| < R\}.$$

Thus φ is a bounded and locally Lipschitz decreasing function, and in particular $\varphi \in W^{1,1}_{loc}(\mathbb{R})$. Therefore, we can assume that φ is a locally absolutely continuous function. Notice that, if $\varphi(R) = 0$ for some $R > 0$, then the set E is bounded, and the proof follows. Hence, assume by contradiction that $\varphi(R) > 0$, for any $R > 0$.

It follows from equation (3.12), with $\varepsilon = \varphi(R)$, that $m_B(F \cap B_R) = m_B(E)$. Since E is an isoperimetric set, we then get

$$\begin{aligned} P_A(E) &\leq P_A(F \cap B_R) = P_A(F) + P_A(E \cap B_R) - P_A(E) \\ &\leq P_A(F) - P_A(E \backslash B_R) + 2\int_{\partial^* B_R \cap E} x^A \mathcal{H}^{N-1}(x). \end{aligned} \tag{3.14}$$

On the other hand, by the Coarea formula, for almost every $R > 0$, we have

$$\varphi'(R) = -\int_{\partial B_R \cap E} x^B \mathcal{H}^{N-1}(x) \tag{3.15}$$

Hence, making the following change of variable

$$\begin{cases} \psi(\theta_1,\dots,\theta_{N-1}) = (x_1, x_2, \dots, x_N), \text{ where} \\ x_1 = R\prod_{k=1}^{N-1}\sin(\theta_k) \\ x_m = R\cos(\theta_{m-1})\prod_{k=m}^{N-1}\sin(\theta_k), \; 2\le m\le N-1, \\ x_N = R\cos(\theta_{N-1}), \\ \text{with } \theta_1\in[0,2\pi) \text{ and } \theta_i\in(0,\pi) \text{ for all } i\in\{2,\dots,N-1\}, \end{cases}$$

and setting $\mathcal{V}_R := \psi^{-1}(\partial B_R\cap E)$, we get from equation (3.15) that

$$\begin{aligned} -\varphi'(R) &= \int_{\partial B_R\cap E} x^B\mathcal{H}^{N-1}(x) \\ &= \int_{\mathcal{V}_R} R^b\prod_{i=1}^{N-1}\left(|\cos(\theta_i)|^{b_{i+1}}|\sin(\theta_i)|^{b_1+\dots+b_i}\right)\left[R^{N-1}\prod_{k=2}^{N-1}|\sin(\theta_k)|^{k-1}\right]d\mathcal{H}^{N-1}(\theta). \end{aligned}$$

From $(H3)$ we get

$$\begin{aligned} -\varphi'(R) &\ge \int_{\mathcal{V}_R} R^b\prod_{i=1}^{N-1}\left(|\cos(\theta_i)|^{a_{i+1}}|\sin(\theta_i)|^{a_1+\dots+a_i}\right)\left[R^{N-1}\prod_{k=2}^{N-1}|\sin(\theta_k)|^{k-1}\right]d\mathcal{H}^{N-1}(\theta) \\ &= R^{b-a}\int_{\partial B_R\cap E} x^A d\mathcal{H}^{N-1}. \end{aligned} \tag{3.16}$$

It follows from equations (3.12)–(3.16) that

$$\begin{aligned} P_A(E) &\le P_A(F) - P_A(E\backslash B_R) + 2\int_{\partial^* B_R\cap E} x^A\mathcal{H}^{N-1}(x) \\ &\le P_A(E) + \frac{C_{A,B,N}}{2}\varphi(R)^{\frac{N+a-1}{N+b}} - P_A(E\backslash B_R) - 2R^{a-b}\varphi'(R). \end{aligned}$$

By the isoperimetric inequality with weights x^B and x^A proved in [1], Theorem 1.1, we obtain

$$C_{A,B,N}\left(m_B(E\backslash B_R)\right)^{\frac{N+a-1}{N+b}} \le P_A(E\backslash B_R) \le \frac{C_{A,B,N}}{2}\varphi(R)^{\frac{N+a-1}{N+b}} - 2R^{a-b}\varphi'(R).$$

Hence

$$\varphi(R)^{\frac{N+a-1}{N+b}} \le -\frac{2}{C_{A,B,N}}R^{a-b}\varphi'(R). \tag{3.17}$$

Observe now that equation (3.17) can be rewritten as

$$-\frac{\mathrm{d}}{dR}\left[\varphi(R)^{1-\frac{N+a-1}{N+b}}\right] \ge \frac{b-a+1}{N+b}\frac{C_{A,B,2}}{2}R^{b-a}, \text{ whenever } \varphi(R)>0.$$

Thus

$$\varphi(t_0)^{1-\frac{N+a-1}{N+b}} - \varphi(t)^{1-\frac{N+a-1}{N+b}} \geq \frac{C_{A,B,2}}{2(N+b)}\left(t^{b-a+1} - t_0^{b-a+1}\right)$$

for all $t > t_0$, which leads to a contradiction if $\varphi(t) > 0$. In fact, this inequality implies the existence of some $R_1 > R_\infty$ such that $\varphi(R) = 0\ \forall R \geq R_1$, *i.e.*, E is a bounded set. Hence, the proof of the theorem is completed. □

4. Nonexistence of the isoperimetric inequality

This section is devoted to prove Theorem 1.4. The main step of the proof is to show that there exists a sequence of sets $(\Omega_\varepsilon)_{\varepsilon>0}$ such that the weighted perimeter $P_A(\Omega_\varepsilon) \to 0$, while the weighted volume $m_B(\Omega_\varepsilon) \to l \neq 0$ as $\varepsilon \to 0$.

First of all, we consider $a = b + k$, $k > 1$, and

$$a_N = \frac{N+a-1}{N+b}(1+b_N) = 1 + b_N + \frac{(k-1)(1+b_N)}{N+b},$$

i.e. $\frac{a_N}{1+b_N} - 1 = \frac{k-1}{N+b} > 0$. Let us define the sequence of sets $(\Omega_\varepsilon)_{\varepsilon>0}$ (see Fig. 3) as follows:

$$\Omega_\varepsilon = \left\{ (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N \;\middle|\; \begin{array}{c} 1 \leq x_1 < \beta_\varepsilon, \\ 0 \leq x_N \leq \varphi_\varepsilon(x_{N-1}),\ \text{for } N \geq 2, \\ x_i \leq x_{i+1} < \beta_\varepsilon,\ i = 1, \cdots, N-2 \ \text{ when } N \geq 3. \end{array} \right\}, \tag{4.1}$$

where $\beta_\varepsilon \to +\infty$, $\beta_\varepsilon^{-\varepsilon} \to c \in (0,1)$ as $\varepsilon \to 0$, and $\varphi_\varepsilon(t) = \dfrac{\varepsilon^{\frac{1}{1+b_N}}}{t^{\frac{\varepsilon+N-1+b-b_N}{1+b_N}}}$, $t > 0$.

The boundary of Ω_ε is given by

$$\partial\Omega_\varepsilon = \Gamma_\varepsilon^0 \cup \left(\bigcup_{i=1}^{N-2} \Gamma_\varepsilon^i\right) \cup \Gamma_\varepsilon^{N-1} \cup \Gamma_\varepsilon^N \cup \Gamma_\varepsilon^{N+1},$$

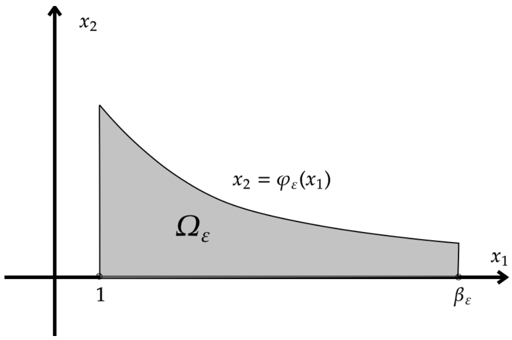
where

$$\begin{aligned}
\Gamma_\varepsilon^0 &= \left\{1 = x_1,\ x_i \leq x_{i+1} \leq \beta_\varepsilon,\ 0 \leq x_N \leq \varphi_\varepsilon(x_{N-1});\ i = 1, \cdots, N-2\right\},\\
\Gamma_\varepsilon^i &= \left\{ \begin{array}{c} 1 \leq x_1 \leq \beta_\varepsilon,\ x_i = x_{i+1},\ x_j \leq x_{j+1} < \beta_\varepsilon, 0 \leq x_N \leq \varphi_\varepsilon(x_{N-1}) \\ j = 1, \cdots, N-2,\ j \neq i \end{array} \right\},\\
\Gamma_\varepsilon^{N-1} &= \left\{ \begin{array}{c} 1 \leq x_1 \leq \beta_\varepsilon,\ x_j \leq x_{j+1} \leq \beta_\varepsilon,\ x_{N-1} = \beta_\varepsilon,\ 0 \leq x_N \leq \varphi_\varepsilon(x_{N-1}) \\ j = 1, \cdots, N-3, \end{array} \right\},\\
\Gamma_\varepsilon^N &= \left\{1 \leq x_1 \leq \beta_\varepsilon,\ x_j \leq x_{j+1} \leq \beta_\varepsilon,\ x_N = \varphi_\varepsilon(x_{N-1});\ j = 1, \cdots, N-2, \right\},\\
\Gamma_\varepsilon^{N+1} &= \left\{1 \leq x_1 \leq \beta_\varepsilon,\ x_j \leq x_{j+1} \leq \beta_\varepsilon,\ x_N = 0;\ j = 1, \cdots, N-2, \right\}.
\end{aligned}$$

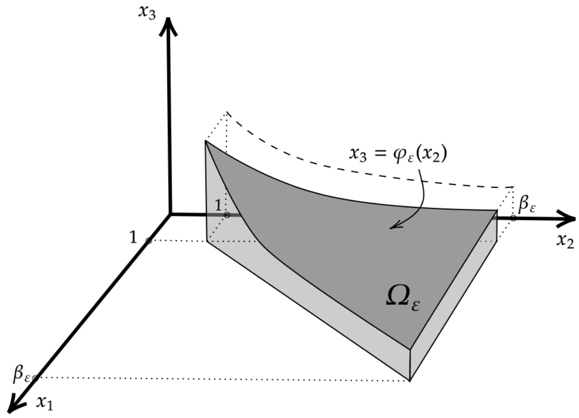
In order to simplify the calculations we define, for $n = 0, 1, \cdots, N-1$:

$$l_{n,\varepsilon} =: \varepsilon - n + N - 1 + b - \sum_{i=N-n}^{N} b_i,$$

$$s_{n,\varepsilon} =: -\frac{l_{0,\varepsilon}}{b_N+1}(a_N+1) + n + \sum_{i=1}^{n} a_{N-i}, \text{ and}$$



(a)



(b)

FIGURE 3. The set Ω_ε, for (a) $N = 2$, and (b) $N = 3$.

$$t_{n,\varepsilon} =: -\frac{l_{0,\varepsilon}}{b_N + 1} a_N + n + \sum_{i=1}^{n} a_{N-i}.$$

If a and b verify $(H4)$ and $(H5)$, then a direct computation shows that

$$l_{n,\varepsilon} > 0, \quad s_{n,\varepsilon} + N - n - 2 < 0, \quad t_{n,\varepsilon} < 0, \quad t_{N-1,\varepsilon} = -\frac{\varepsilon a_N}{1 + b_N},$$

$$\lim_{\varepsilon\to 0} l_{n,\varepsilon} \neq 0, \ \lim_{\varepsilon\to 0} s_{n,\varepsilon} \neq 0, \ \lim_{\varepsilon\to 0} t_{n,\varepsilon} \neq 0, \ \text{ for } n = 1, 2, ..., N-2, \tag{4.2}$$

$$\lim_{\varepsilon\to 0} s_{n,\varepsilon} - 1 \neq 0, \quad \text{for } n = 1, 2, ..., N-1. \tag{4.3}$$

Also, we have the following estimate:

$$\begin{aligned}
&\varepsilon \int_1^{\beta_\varepsilon} \int_{x_1}^{\beta_\varepsilon} \cdots \int_{x_{N-j-1}}^{\beta_\varepsilon} \int_{x_{N-j}}^{\beta_\varepsilon} x_1^{b_1} x_2^{b_2} \cdots x_{N-j}^{b_{N-j}} x_{N-j+1}^{b_{N-j+1}} \beta_\varepsilon^{-l_{j-2,\varepsilon}} \mathrm{d}x_{N-j+1} \mathrm{d}x_{N-j} \cdots \mathrm{d}x_2 \mathrm{d}x_1 \\
&\le 2\varepsilon \int_1^{\beta_\varepsilon} \int_{x_1}^{\beta_\varepsilon} \cdots \int_{x_{N-j-1}}^{\beta_\varepsilon} x_1^{b_1} x_2^{b_2} \cdots x_{N-j}^{b_{N-j}} \beta_\varepsilon^{1+b_{N-j+1}} \beta_\varepsilon^{-l_{j-2,\varepsilon}} \mathrm{d}x_{N-j} \cdots \mathrm{d}x_2 \mathrm{d}x_1 \\
&= 2\varepsilon \int_1^{\beta_\varepsilon} \int_{x_1}^{\beta_\varepsilon} \cdots \int_{x_{N-j-1}}^{\beta_\varepsilon} x_1^{b_1} x_2^{b_2} \cdots x_{N-j}^{b_{N-j}} \beta_\varepsilon^{-l_{j-1,\varepsilon}} \mathrm{d}x_{N-j} \cdots \mathrm{d}x_2 \mathrm{d}x_1 \\
&\le 2^2\varepsilon \int_1^{\beta_\varepsilon} \int_{x_1}^{\beta_\varepsilon} \cdots \int_{x_{N-j-2}}^{\beta_\varepsilon} x_1^{b_1} x_2^{b_2} \cdots x_{N-j-1}^{b_{N-j-1}} \beta_\varepsilon^{-l_{j,\varepsilon}} \mathrm{d}x_{N-j-1} \cdots \mathrm{d}x_2 \mathrm{d}x_1 \\
&\le \varepsilon 2^N \int_1^{\beta_\varepsilon} x_1^{b_1} \beta_\varepsilon^{-l_{N-2,\varepsilon}} \mathrm{d}x_1 \le \varepsilon 2^N \beta_\varepsilon^{-\varepsilon} = o(1), \ \text{ as } \varepsilon \to 0.
\end{aligned} \tag{4.4}$$

In order to be able to prove Theorem 1.4 we need to show the following:
Claim 1. $m_B(\Omega_\varepsilon) \to l$ as $\varepsilon \to 0$, for some $l > 0$.
Claim 2. $P_A(\Omega_\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof of Claim 1. From the definition of Ω_ε, equations (4.2) and (4.4), it follows that

$$\begin{aligned}(b_N+1)m_B(\Omega_\varepsilon) &= (b_N+1)\int_1^{\beta_\varepsilon}\int_{x_1}^{\beta_\varepsilon}\cdots\int_{x_{N-2}}^{\beta_\varepsilon}\int_0^{\varphi_\varepsilon(x_{N-1})} x^B \mathrm{d}x_N\mathrm{d}x_{N-1}\mathrm{d}x_{N-2}\cdots\mathrm{d}x_2\mathrm{d}x_1\\
&= \int_1^{\beta_\varepsilon}\int_{x_1}^{\beta_\varepsilon}\cdots\int_{x_{N-3}}^{\beta_\varepsilon}\int_{x_{N-2}}^{\beta_\varepsilon} x_1^{b_1}x_2^{b_2}\cdots x_{N-2}^{b_{N-2}}x_{N-1}^{b_{N-1}}\varphi_\varepsilon(x_{N-1})^{b_N+1}\mathrm{d}x_{N-1}\mathrm{d}x_{N-2}\cdots\mathrm{d}x_2\mathrm{d}x_1\\
&= \varepsilon\int_1^{\beta_\varepsilon}\int_{x_1}^{\beta_\varepsilon}\cdots\int_{x_{N-3}}^{\beta_\varepsilon}\int_{x_{N-2}}^{\beta_\varepsilon} x_1^{b_1}x_2^{b_2}\cdots x_{N-2}^{b_{N-2}}x_{N-1}^{-l_{1,\varepsilon}-1}\mathrm{d}x_{N-1}\mathrm{d}x_{N-2}\cdots\mathrm{d}x_2\mathrm{d}x_1\\
&= \frac{\varepsilon}{l_{1,\varepsilon}}\int_1^{\beta_\varepsilon}\int_{x_1}^{\beta_\varepsilon}\cdots\int_{x_{N-3}}^{\beta_\varepsilon} x_1^{b_1}x_2^{b_2}\cdots x_{N-2}^{b_{N-2}}\left(x_{N-2}^{-l_{1,\varepsilon}}-\beta_\varepsilon^{-l_{1,\varepsilon}}\right)\mathrm{d}x_{N-2}\cdots\mathrm{d}x_2\mathrm{d}x_1\\
&= \frac{\varepsilon}{l_{1,\varepsilon}}\int_1^{\beta_\varepsilon}\int_{x_1}^{\beta_\varepsilon}\cdots\int_{x_{N-3}}^{\beta_\varepsilon} x_1^{b_1}x_2^{b_2}\cdots x_{N-2}^{-l_{2,\varepsilon}-1}\mathrm{d}x_{N-2}\cdots\mathrm{d}x_2\mathrm{d}x_1+o(1)\\
&= \frac{\varepsilon}{l_{1,\varepsilon}l_{2,\varepsilon}}\int_1^{\beta_\varepsilon}\int_{x_1}^{\beta_\varepsilon}\cdots\int_{x_{N-4}}^{\beta_\varepsilon} x_1^{b_1}x_2^{b_2}\cdots x_{N-3}^{b_{N-3}}x_{N-3}^{-l_{2,\varepsilon}}\mathrm{d}x_{N-3}\cdots\mathrm{d}x_2\mathrm{d}x_1+o(1)\\
&= \frac{\varepsilon}{l_{1,\varepsilon}l_{2,\varepsilon}\cdots l_{N-3,\varepsilon}}\int_1^{\beta_\varepsilon}\int_{x_1}^{\beta_\varepsilon} x_1^{b_1}x_2^{b_2}x_2^{-l_{N-3,\varepsilon}}\mathrm{d}x_2\mathrm{d}x_1+o(1)\\
&= \frac{\varepsilon}{l_{1,\varepsilon}l_{2,\varepsilon}\cdots l_{N-3,\varepsilon}l_{N-2,\varepsilon}}\int_1^{\beta_\varepsilon} x_1^{b_1}x_1^{-l_{N-2,\varepsilon}}\mathrm{d}x_1+o(1)\\
&= \frac{1}{l_{1,\varepsilon}l_{2,\varepsilon}\cdots l_{N-3,\varepsilon}l_{N-2,\varepsilon}}(1-\beta_\varepsilon^{-\varepsilon})+o(1)\to l\neq 0 \text{ as } \varepsilon\to 0.\end{aligned}$$

□

Proof of Claim 2. Since

$$P_A(\Omega_\varepsilon)\leq \int_{\Gamma_\varepsilon^0} x^A d\mathcal{H}^{N-1}+\sum_{i=1}^{N-2}\int_{\Gamma_\varepsilon^i} x^A d\mathcal{H}^{N-1}+\int_{\Gamma_\varepsilon^{N-1}} x^A d\mathcal{H}^{N-1}+\int_{\Gamma_\varepsilon^{N}} x^A d\mathcal{H}^{N-1}+\int_{\Gamma_\varepsilon^{N+1}} x^A d\mathcal{H}^{N-1}, \quad (4.5)$$

it is suffices show that

$$\int_{\Gamma_\varepsilon^i} x^A d\mathcal{H}^{N-1}\to 0, \quad \text{as } \varepsilon\to 0, \quad \text{for each} \quad i=0,1,2,\cdots,N+1.$$

From the definition of Γ_ε^{N+1}, we have that

$$\int_{\Gamma_\varepsilon^{N+1}} x^A d\mathcal{H}^{N-1}=0, \quad \text{for all } \varepsilon>0 \text{ small}. \quad (4.6)$$

From the definitions of $l_{0,\varepsilon}$, $s_{n,\varepsilon}$ and by equation (4.2), it follows that

$$\begin{aligned}
\int_{\Gamma_\varepsilon^0} x^A d\mathcal{H}^{N-1} &\le \int_1^{+\infty}\cdots\int_{x_{N-2}}^{+\infty}\int_0^{\varphi_\varepsilon(x_{N-1})} 1^{a_1}x_2^{a_2}\cdots x_{N-1}^{a_{N-1}}x_N^{a_N}\,\mathrm{d}x_N\mathrm{d}x_{N-1}\cdots\mathrm{d}x_2\\
&= \frac{1}{a_N+1}\int_1^{+\infty}\cdots\int_{x_{N-2}}^{+\infty} x_2^{a_2}\cdots x_{N-1}^{a_{N-1}}\varphi_\varepsilon(x_{N-1})^{a_N+1}\mathrm{d}x_{N-1}\cdots\mathrm{d}x_2\\
&= \frac{\varepsilon^{\frac{a_N+1}{1+b_N}}}{a_N+1}\int_1^{+\infty}\cdots\int_{x_{N-2}}^{+\infty} x_2^{a_2}\cdots x_{N-1}^{a_{N-1}}x_{N-1}^{-\frac{l_{0,\varepsilon}}{1+b_N}(a_N+1)}\mathrm{d}x_{N-1}\cdots\mathrm{d}x_2\\
&= \frac{\varepsilon^{\frac{a_N+1}{1+b_N}}}{a_N+1}\int_1^{+\infty}\cdots\int_{x_{N-2}}^{+\infty} x_2^{a_2}\cdots x_{N-1}^{a_{N-1}}x_{N-1}^{s_{0,\varepsilon}}\mathrm{d}x_{N-1}\cdots\mathrm{d}x_2\\
&= \frac{\varepsilon^{\frac{a_N+1}{1+b_N}}}{a_N+1}\int_1^{+\infty}\cdots\int_{x_{N-2}}^{+\infty} x_2^{a_2}\cdots x_{N-1}^{s_{1,\varepsilon}-1}\mathrm{d}x_{N-1}\cdots\mathrm{d}x_2\\
&= \frac{\varepsilon^{\frac{a_N+1}{1+b_N}}}{(a_N+1)}\frac{1}{|s_{1,\varepsilon}|}\int_1^{+\infty}\cdots\int_{x_{N-3}}^{+\infty} x_2^{a_2}\cdots x_{N-2}^{a_{N-2}}x_{N-2}^{s_{1,\varepsilon}}\mathrm{d}x_{N-2}\cdots\mathrm{d}x_2\\
&= \frac{\varepsilon^{\frac{a_N+1}{1+b_N}}}{(a_N+1)}\frac{1}{|s_{1,\varepsilon}|}\int_1^{+\infty}\cdots\int_{x_{N-3}}^{+\infty} x_2^{a_2}\cdots x_{N-2}^{s_{2,\varepsilon}-1}\mathrm{d}x_{N-2}\cdots\mathrm{d}x_2\\
&= \frac{\varepsilon^{\frac{a_N+1}{1+b_N}}}{(a_N+1)}\frac{1}{|s_{1,\varepsilon}s_{2,\varepsilon}|}\int_1^{+\infty}\cdots\int_{x_{N-4}}^{+\infty} x_2^{a_2}\cdots x_{N-3}^{a_{N-3}}x_{N-3}^{s_{2,\varepsilon}}\mathrm{d}x_{N-3}\cdots\mathrm{d}x_2\\
&= \frac{\varepsilon^{\frac{a_N+1}{1+b_N}}}{(a_N+1)}\frac{1}{|s_{1,\varepsilon}s_{2,\varepsilon}\cdots s_{N-2,\varepsilon}|}\to 0 \text{ as } \varepsilon\to 0. \qquad (4.7)
\end{aligned}$$

Similarly, from the definition of Γ_ε^i, for $i=1,2,\cdots,N-2$, and following the above calculations, we have that

$$\begin{aligned}
&\int_{\Gamma_\varepsilon^i} x^A d\mathcal{H}^{N-1}\\
&\le \int_1^{+\infty}\cdots\int_{x_{i-1}}^{+\infty}\int_{x_i}^{+\infty}\int_{x_{i+2}}^{+\infty}\cdots\\
&\quad\cdots\int_{x_{N-2}}^{+\infty}\int_0^{\varphi_\varepsilon(x_{N-1})} x_1^{a_1}\cdots x_i^{a_i}x_{i+1}^{a_{i+1}}x_{i+2}^{a_{i+2}}x_{i+3}^{a_{i+3}}\cdots x_{N-1}^{a_{N-1}}x_N^{a_N}\,\mathrm{d}x_N\mathrm{d}x_{N-1}\cdots\mathrm{d}x_{i+3}\mathrm{d}x_{i+2}\mathrm{d}x_i\cdots\mathrm{d}x_1\\
&= \int_1^{+\infty}\cdots\int_{x_{i-1}}^{+\infty}\int_{x_i}^{+\infty}\int_{x_{i+2}}^{+\infty}\cdots\\
&\quad\cdots\int_{x_{N-2}}^{+\infty} x_1^{a_1}\cdots x_i^{a_i+a_{i+1}}x_{i+2}^{a_{i+2}}x_{i+3}^{a_{i+3}}\cdots x_{N-1}^{a_{N-1}}x_{N-1}^{s_{0,\varepsilon}}\mathrm{d}x_{N-1}\cdots\mathrm{d}x_{i+3}\mathrm{d}x_{i+2}\mathrm{d}x_i\cdots\mathrm{d}x_1\\
&= \frac{\varepsilon^{\frac{a_N+1}{1+b_N}}}{(a_N+1)}\frac{1}{|s_{1,\varepsilon}s_{2,\varepsilon}\cdots s_{N-(i+4),\varepsilon}|}\\
&\quad\times\int_1^{+\infty}\cdots\int_{x_{i-1}}^{+\infty}\int_{x_i}^{+\infty}\int_{x_{i+2}}^{+\infty} x_1^{a_1}\cdots x_i^{a_i+a_{i+1}}x_{i+2}^{a_{i+2}}x_{i+3}^{a_{i+3}}x_{i+3}^{s_{N-(i+4),\varepsilon}}\mathrm{d}x_{i+3}\mathrm{d}x_{i+2}\mathrm{d}x_i\cdots\mathrm{d}x_1,
\end{aligned}$$

and so

$$\int_{\Gamma_\varepsilon^i} x^A d\mathcal{H}^{N-1}$$
$$\leq \frac{\varepsilon^{\frac{a_N+1}{1+b_N}}}{(a_N+1)} \frac{1}{|s_{1,\varepsilon}s_{2,\varepsilon}\cdots s_{N-(i+3),\varepsilon}|} \int_1^{+\infty}\cdots\int_{x_{i-1}}^{+\infty}\int_{x_i}^{+\infty} x_1^{a_1}\cdots x_i^{a_i+a_{i+1}} x_{i+2}^{a_{i+2}} x_{i+2}^{s_{N-(i+3),\varepsilon}} \mathrm{d}x_{i+2}\mathrm{d}x_i\cdots\mathrm{d}x_1.$$

Since $x_i = x_{i+1} \leq x_{i+2}$ in Γ_ε^i, that is, x_{i+2} depends on x_i, it follows that

$$\begin{aligned}
\int_{\Gamma_\varepsilon^i} x^A d\mathcal{H}^{N-1} &\leq \frac{\varepsilon^{\frac{a_N+1}{1+b_N}}}{(a_N+1)} \frac{1}{|s_{1,\varepsilon}s_{2,\varepsilon}\cdots s_{N-(i+2),\varepsilon}|} \int_1^{+\infty}\cdots\int_{x_{i-1}}^{+\infty} x_1^{a_1}\cdots x_i^{a_i+a_{i+1}} x_i^{s_{N-(i+2),\varepsilon}} \mathrm{d}x_i\cdots\mathrm{d}x_1\\
&= \frac{\varepsilon^{\frac{a_N+1}{1+b_N}}}{(a_N+1)} \frac{1}{|s_{1,\varepsilon}s_{2,\varepsilon}\cdots s_{N-(i+2),\varepsilon}|} \int_1^{+\infty}\cdots\int_{x_{i-1}}^{+\infty} x_1^{a_1}\cdots x_i^{s_{N-i,\varepsilon}-2} \mathrm{d}x_i\cdots\mathrm{d}x_1\\
&= \frac{\varepsilon^{\frac{a_N+1}{1+b_N}}}{(a_N+1)} \frac{1}{|s_{1,\varepsilon}s_{2,\varepsilon}\cdots s_{N-(i+4),\varepsilon}(s_{N-i,\varepsilon}-1)(s_{N-i+1,\varepsilon}-1)\cdots(s_{N-2,\varepsilon}-1)(s_{N-1,\varepsilon}-1)|},
\end{aligned}$$

and by equations (4.2)–(4.3),

$$\int_{\Gamma_\varepsilon^i} x^A d\mathcal{H}^{N-1} \to 0 \text{ as } \varepsilon \to 0, \text{ for } i = 1, 2, \cdots, N-2. \tag{4.8}$$

Now, by the definitions of Γ_ε^N, $t_{n,\varepsilon}$ and by equation (4.2), we have that

$$\begin{aligned}
\int_{\Gamma_\varepsilon^N} x^A d\mathcal{H}^{N-1} &\leq \int_1^{+\infty}\cdots\int_{x_{N-2}}^{+\infty} x_1^{a_1}\cdots x_{N-1}^{a_{N-1}} \varphi_\varepsilon(x_{N-1})^{a_N}\sqrt{1+\varphi_\varepsilon'(x_{N-1})^2}\mathrm{d}x_{N-1}\cdots\mathrm{d}x_1\\
&\leq \varepsilon^{\frac{a_N}{1+b_N}} \int_1^{+\infty}\cdots\int_{x_{N-3}}^{+\infty}\int_{x_{N-2}}^{+\infty} x_1^{a_1}\cdots x_{N-2}^{a_{N-2}} x_{N-1}^{a_{N-1}} x_{N-1}^{-\frac{l_{0,\varepsilon}}{1+b_N}a_N} \mathrm{d}x_{N-1}\mathrm{d}x_{N-2}\cdots\mathrm{d}x_1\\
&= \varepsilon^{\frac{a_N}{1+b_N}} \int_1^{+\infty}\cdots\int_{x_{N-3}}^{+\infty}\int_{x_{N-2}}^{+\infty} x_1^{a_1}\cdots x_{N-2}^{a_{N-2}} x_{N-1}^{a_{N-1}} x_{N-1}^{t_{0,\varepsilon}} \mathrm{d}x_{N-1}\mathrm{d}x_{N-2}\cdots\mathrm{d}x_1\\
&\leq \varepsilon^{\frac{a_N}{1+b_N}} \frac{1}{|t_{1,\varepsilon}|} \int_1^{+\infty}\cdots\int_{x_{N-3}}^{+\infty} x_1^{a_1}\cdots x_{N-2}^{a_{N-2}} x_{N-2}^{t_{1,\varepsilon}} \mathrm{d}x_{N-2}\cdots\mathrm{d}x_1\\
&= \frac{\varepsilon^{\frac{a_N}{1+b_N}}}{|t_{1,\varepsilon}t_{2,\varepsilon}\cdots t_{N-2,\varepsilon}|}\frac{1}{|t_{N-1,\varepsilon}|} = \frac{(1+b_N)\varepsilon^{\frac{a_N}{1+b_N}-1}}{|t_{1,\varepsilon}t_{2,\varepsilon}\cdots t_{N-2,\varepsilon}|a_N} \to 0, \text{ as } \varepsilon \to 0.
\end{aligned} \tag{4.9}$$

From equation (4.1) we have that $\beta_\varepsilon \to +\infty$, and $\beta_\varepsilon^{-\varepsilon} \to c \in (0,1)$, as $\varepsilon \to 0$. Hence, we readily deduce that

$$\begin{aligned}
\int_{\Gamma_\varepsilon^{N-1}} x^A d\mathcal{H}^{N-1} &= \int_1^{\beta_\varepsilon}\int_{x_1}^{\beta_\varepsilon}\cdots\int_{x_{N-3}}^{\beta_\varepsilon}\int_0^{\varphi_\varepsilon(\beta_\varepsilon)} x_1^{a_1}x_2^{a_2}\cdots x_{N-2}^{a_{N-2}}\beta_\varepsilon^{a_{N-1}} x_N^{a_N}\mathrm{d}x_N\mathrm{d}x_{N-2}\cdots\mathrm{d}x_2\mathrm{d}x_1\\
&\leq \beta_\varepsilon^{a-a_N}\int_1^{\beta_\varepsilon}\int_1^{\beta_\varepsilon}\cdots\int_1^{\beta_\varepsilon}\int_0^{\varphi_\varepsilon(\beta_\varepsilon)} x_N^{a_N}\mathrm{d}x_N\mathrm{d}x_{N-2}\cdots\mathrm{d}x_2\mathrm{d}x_1\\
&\leq 2^N\beta_\varepsilon^{a-a_N+N-2}\varepsilon^{\frac{a_N+1}{1+b_N}}\beta_\varepsilon^{-\frac{\varepsilon+N-1+b-b_N}{1+b_N}(a_N+1)}\\
&= 2^N\varepsilon^{\frac{a_N+1}{1+b_N}}\beta_\varepsilon^{-\frac{\varepsilon(a_N+1)+N+b}{1+b_N}} = 2^N\varepsilon^{\frac{a_N+1}{1+b_N}}\left(\beta_\varepsilon^{-\varepsilon}\right)^{\frac{a_N+1}{1+b_N}}(\beta_\varepsilon)^{-\frac{N+b}{1+b_N}} \to 0 \text{ as } \varepsilon \to 0.
\end{aligned} \tag{4.10}$$

Therefore, Claim 2 follows from equations (4.5)–(4.10). □

Proof of Theorem 1.4. Notice that, it suffices to consider the case $i = N$. We consider the sequence of sets $(\Omega_\varepsilon)_{\varepsilon>0}$ defined in equation (4.1). By Claim 1 and Claim 2, we have that

$$C_{N,A,B} \leq \frac{P_A(\Omega_\varepsilon)}{[m_B(\Omega_\varepsilon)]^{\frac{N+a-1}{N+b}}} \to 0 \text{ as } \varepsilon \to 0.$$

Therefore, $C_{N,A,B} = 0$, which implies that there is no isoperimetric inequality. □

5. Star-shaped Steiner symmetrization

It is well known that the Steiner symmetrization is one of the simplest and most powerful symmetrization processes ever introduced in analysis, which has a number of remarkable applications to geometric and analytical problems. For a more general type of symmetrization, we refer to the reader [13, 15, 19, 23, 25, 37], and references therein.

In this section, we will study the perimeter inequality under a slightly refined version of symmetrization, namely, star-shaped Steiner symmetrization, as follows: Let l, b be real numbers satisfying $b+l+1 > -1$. Given a set $E \subset \mathbb{R}^2$ and $\theta \in [0, \pi]$, we denote by $E_\theta := E \cap L_\theta$ the corresponding one-dimensional polar section of E, where

$$L_\theta = \{(x, y) \in \mathbb{R}^2; x = r\cos(\theta), y = r\sin(\theta) \text{ and } r \in \mathbb{R}\}.$$

The distribution function $\mu_{B,l}$ of E is defined by setting for all $\theta \in [0, \pi]$

$$\mu_{B,l}(\theta) = \int_{E\cap L_\theta} |r|^{b+l+1} d\mathcal{H}^1(r).$$

Finally, denoting the essential projection of E by

$$\Pi^{\otimes}(E) := \{\theta \in [0, \pi]; \mu_{B,l}(\theta) > 0\} \text{ and } \eta_{B,l}(\theta) := \left[\frac{b+l+2}{2}\mu_{B,l}(\theta)\right]^{\frac{1}{b+l+2}},$$

we set the *star-shaped Steiner symmetrization with weights of E* as being the set

$$E^{\otimes} := \left\{(r\cos(\theta), r\sin(\theta)); \theta \in \Pi^{\otimes}(E) \text{ and } -\eta_{B,l}(\theta) < r < \eta_{B,l}(\theta)\right\},$$

see Figure 4.

To prove the theorem, let us now fix some notations and basic consequences. Based on the change of variable $x = r\cos(\theta)$, $y = r\sin(\theta)$, we set

$$\vec{e}_r := \cos(\theta)\vec{i} + \sin(\theta)\vec{j},$$
$$\vec{e}_\theta := -\sin(\theta)\vec{i} + \cos(\theta)\vec{j},$$

where $\vec{i} := (1, 0)$ and $\vec{j} := (0, 1)$. It is easy to see that

$$\vec{i} = \cos(\theta)\vec{e}_r - \sin(\theta)\vec{e}_\theta,$$
$$\vec{j} = \sin(\theta)\vec{e}_r + \cos(\theta)\vec{e}_\theta.$$

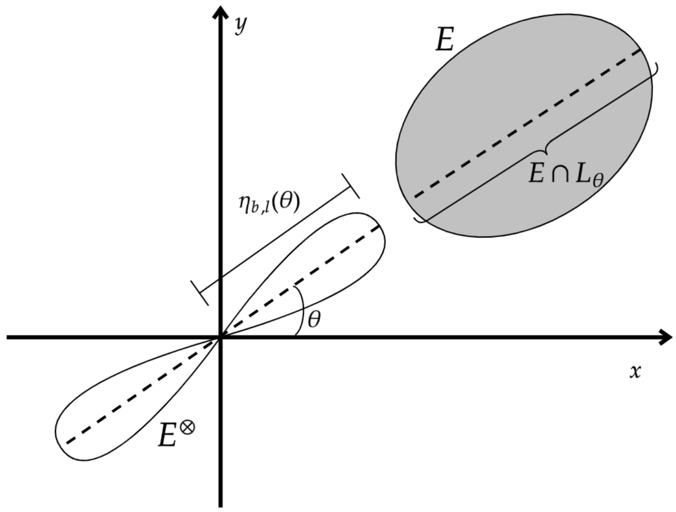


FIGURE 4. The star-shaped Steiner symmetrization of the set E.

Moreover, if $u : \mathbb{R}^2 \to \mathbb{R}$ is a function of class $C^1(\mathbb{R}^2\backslash\{(0,0)\})$ then $u(x,y) = u(r\cos(\theta), r\sin(\theta)) =: \overline{u}(r,\theta)$ satisfies

$$\nabla\overline{u}(r,\theta) = \frac{\partial\overline{u}}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\overline{u}}{\partial\theta}\vec{e}_\theta.$$

Now, let $E \subset \mathbb{R}^2$ be a set of locally finite perimeter. The outward normal of E at (r,θ) will be denote by $\nu_E(r,\theta) = \nu_r\vec{e}_r + \nu_\theta\vec{e}_\theta$. Thus, for $u(r\cos(\theta), r\sin(\theta)) = \theta$, $\theta \in [0,\pi)$, $r \in \mathbb{R}$, we have

$$\left\|\nabla^{\partial^* E}\overline{u}\right\| = \frac{|\nu_r|}{|r|}, \text{ where } \nabla^{\partial^* E}\overline{u} := \nabla\overline{u}(r,\theta) - (\nabla\overline{u}(r,\theta)\cdot\nu_E(r,\theta))\,\nu_E(r,\theta)$$

is the tangential gradient; see [38], Chapter 18. The following lemma will be crucial to our arguments.

Lemma 5.1 (See [26], Lem. 6.3)**.** *Let $\Sigma \subset \mathbb{R}^2$ be an open cone and let $E \subseteq \Sigma$ be a measurable set such that $P_A(E,\Omega) < \infty$ for any $\Omega \subset\subset \Sigma$. Then, for any $\psi \in L^1(E)$, we have*

$$\int_E \psi d\mathcal{H}^2 = \int_{\Pi^{\otimes}(E)}\int_{E_\theta} r\psi(r\theta)drd\mathcal{H}^1(\theta).$$

Moreover, for $\mathcal{H}^1-$almost every $\theta \in \Pi^{\otimes}(E)$, $E_\theta \subseteq \mathbb{R}$ is a $1-$dimensional set of locally finite perimeter such that the Vol'pert property $\partial^* E_\theta \cap \{r > 0\} = \{r > 0 \;:\; L_\theta \cap \partial^* E \neq \emptyset\}$ holds.

Now we are ready to prove one of our main results.

Proof of Theorem 1.5. The first part of the proof is divided in two steps.

Step one: The function $\mu_{B,l}(\theta) = \int_{E\cap L_\theta} |r|^{b+l+1}d\mathcal{H}^1(r)$ is such that $\mu_{B,l} \in W^{1,1}_{loc}(\mathbb{R})$, with

$$\frac{d\mu_{B,l}}{d\theta}(\theta) = -\int_{\partial^* E\cap L_\theta}\frac{|r|^{b+l+3}\nu_\theta}{|\nu_r|}d\mathcal{H}^0(r), \text{ for } \mathcal{H}^1-\text{a.e. } \theta \in \mathbb{R}.$$

Proof of the Step one. Indeed, given $\varphi \in C_c^1((0,\pi))$, we have

$$\begin{aligned}\int_{\Pi^{\otimes}(E)} \varphi'(\theta)\mu_{B,l}(\theta)d\theta &= \int_{\Pi^{\otimes}(E)} \varphi'(\theta)\int_{E_\theta} |r|^{b+l+1} d\mathcal{H}^1(r)d\theta \\ &= \int_{\Pi^{\otimes}(E)\times\mathbb{R}} 1_E(r\cos(\theta), r\sin(\theta))\varphi'(\theta)|r|^{b+l+1}d\mathcal{H}^2(\theta,r) \\ &= \int_{\Pi^{\otimes}(E)\times\mathbb{R}} 1_E(r\cos(\theta), r\sin(\theta))\mathrm{div}\left[\varphi(\theta)r|r|^{b+l+1}e_\theta\right]d\mathcal{H}^2(\theta,r).\end{aligned}$$

It follows from Gauss-Green theorem that

$$\int_{\Pi^{\otimes}(E)} \varphi'(\theta)\mu_{B,l}(\theta)d\theta = \int_{\partial^* E} \varphi(\theta)|r|^{b+l+2}\nu_\theta d\mathcal{H}^1(r,\theta).$$

Now, by Coarea formula on locally 1-rectifiable sets [38], Theorem 18.8, we get

$$\int_{\Pi^{\otimes}(E)} \varphi'(\theta)\mu_{B,l}(\theta)d\theta = \int_{\Pi^{\otimes}(E)}\int_{\partial^* E\cap L_\theta} \varphi(\theta)|r|^{b+l+3}\frac{\nu_\theta}{|\nu_r|}d\mathcal{H}^0(r)d\mathcal{H}^1(\theta).$$

Therefore, $\mu_{B,l} \in W^{1,1}_{loc}(\mathbb{R})$, with

$$\begin{aligned}\frac{d\mu_{B,l}}{d\theta}(\theta) &= -\int_{\partial^* E\cap L_\theta} \frac{|r|^{b+l+3}\nu_\theta}{|\nu_r|}d\mathcal{H}^0(r) \\ &= -\int_{\partial^* E^{\otimes}\cap L_\theta} \frac{|r|^{b+l+3}\nu_\theta}{|\nu_r|}d\mathcal{H}^0(r) \\ &= -\left[|\eta_{B,l}(\theta)|^{b+l+3}\frac{\nu_\theta(-\eta_{B,l}(\theta),\theta)}{|\nu_r(-\eta_{B,l}(\theta),\theta)|} + |\eta_{B,l}(\theta)|^{b+l+3}\frac{\nu_\theta(\eta_{B,l}(\theta),\theta)}{|\nu_r(\eta_{B,l}(\theta),\theta)|}\right] \\ &= -2|\eta_{B,l}(\theta)|^{b+l+3}\frac{\nu_\theta(\eta_{B,l}(\theta),\theta)}{|\nu_r(\eta_{B,l}(\theta),\theta)|} \text{ for } \mathcal{H}^1-\text{a.e. } \theta\in\mathbb{R}\end{aligned}$$

□

Step two: $P_{A,k}(E^{\otimes}) \le P_{A,k}(E)$.

Proof of the Step two. Before we start the proof let us recall a generalization of the Minkowski's inequality, which, in its classical form is

$$\left(\sum_{j=1}^m (v_j+w_j)^p\right)^{1/p} \le \left(\sum_{j=1}^m v_j^p\right)^{1/p} + \left(\sum_{j=1}^m w_j^p\right)^{1/p}, \text{ for } m\in\mathbb{N}, v_j\ge 0, w_j\ge 0, \text{ and } p\ge 1.$$

This, by repeated application, can be extended to the form

$$\left(\sum_{j=1}^m \left(\sum_{i=1}^t v_{i,j}\right)^p\right)^{1/p} \le \sum_{i=1}^t\left(\sum_{j=1}^m v_{i,j}^p\right)^{1/p}, \text{ for } m,t\in\mathbb{N}, v_{i,j}\ge 0, \text{ and } p\ge 1.$$

Returning to the proof, from assumption equation (1.3) and Lemma 5.1 we deduce that the set $\partial^* E \cap L_\theta$ is almost everywhere discrete. Hence, from the previous inequality we obtain

$$\begin{aligned}
&\sqrt{\left(\int_{\partial^* E\cap L_\theta} |r|^{a+k+1} d\mathcal{H}^0(r)\right)^2 + \left(\int_{\partial^* E\cap L_\theta} |r|^{a+k+1}\frac{\nu_\theta}{\nu_r} d\mathcal{H}^0(r)\right)^2} \\
&= \sqrt{\left(\sum_{r\in\partial^* E\cap L_\theta} |r|^{a+k+1}\right)^2 + \left(\sum_{\partial^* E\cap L_\theta} |r|^{a+k+1}\frac{\nu_\theta}{\nu_r}\right)^2} \leq \sum_{\partial^* E\cap L_\theta} \sqrt{(|r|^{a+k+1})^2 + \left(|r|^{a+k+1}\frac{\nu_\theta}{\nu_r}\right)^2} \\
&= \int_{\partial^* E\cap L_\theta} \sqrt{(|r|^{a+k+1})^2 + \left(|r|^{a+k+1}\frac{\nu_\theta}{\nu_r}\right)^2} d\mathcal{H}^0(r).
\end{aligned}$$

Now, it follows from Coarea formula on locally 1-rectifiable sets, and the above inequality that

$$\begin{aligned}
P_{A,k}(E) &= \int_{\partial^* E} |x|^{a_1}|y|^{a_2}\left[x^2+y^2\right]^{k/2} d\mathcal{H}^1(x,y) \\
&= \int_{\Pi^{\otimes}(E)} |\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2} \int_{\partial^* E\cap L_\theta} \frac{|r|^{a+k+1}}{|\nu_r|} d\mathcal{H}^0(r) d\mathcal{H}^1(\theta) \\
&= \int_{\Pi^{\otimes}(E)} |\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2} \int_{\partial^* E\cap L_\theta} |r|^{a+k+1}\sqrt{1+\left(\frac{\nu_\theta}{\nu_r}\right)^2} d\mathcal{H}^0(r) d\mathcal{H}^1(\theta) \\
&= \int_{\Pi^{\otimes}(E)} |\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2} \int_{\partial^* E\cap L_\theta} \sqrt{|r|^{2(a+k+1)} + \left(|r|^{a+k+1}\frac{\nu_\theta}{\nu_r}\right)^2} d\mathcal{H}^0(r) d\mathcal{H}^1(\theta) \\
&\geq \int_{\Pi^{\otimes}(E)} |\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2} \sqrt{\left(\int_{\partial^* E\cap L_\theta} |r|^{a+k+1} d\mathcal{H}^0(r)\right)^2 + \left(\int_{\partial^* E\cap L_\theta} |r|^{a+k+1}\frac{\nu_\theta}{\nu_r} d\mathcal{H}^0(r)\right)^2} d\mathcal{H}^1(\theta),
\end{aligned} \tag{5.1}$$

Now, we are going to use isoperimetric inequalities on $\mathbb{R}$ with different weights in the perimeter and volume which are powers of the distance function to the origin cf. [7], Theorem 6.1. More precisely, since

$$(b+l+1)+1 \leq a+k+1 \quad \text{and} \quad \int_{E\cap L_\theta} |r|^{b+l+1} d\mathcal{H}^1(r) = \int_{E^{\otimes}\cap L_\theta} |r|^{b+l+1} d\mathcal{H}^1(r),$$

then

$$\int_{\partial^* E\cap L_\theta} |r|^{a+k+1} d\mathcal{H}^0(r) \geq \int_{\partial^* E^{\otimes}\cap L_\theta} |r|^{a+k+1} \mathcal{H}^0(r). \tag{5.2}$$

By equations (5.1) and (5.2), we obtain

$$\begin{aligned}
P_{A,k}(E) &\geq \int_{\Pi^{\otimes}(E)} |\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2} \sqrt{\left(\int_{\partial^* E\cap L_\theta} |r|^{a+k+1} d\mathcal{H}^0(r)\right)^2 + \left(\int_{\partial^* E\cap L_\theta} |r|^{a+k+1}\frac{\nu_\theta}{\nu_r} d\mathcal{H}^0(r)\right)^2} d\mathcal{H}^1(\theta) \\
&\geq \int_{\Pi^{\otimes}(E)} |\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2} \sqrt{\left(\int_{\partial^* E^{\otimes}\cap L_\theta} |r|^{a+k+1} d\mathcal{H}^0(r)\right)^2 + \left(\int_{\partial^* E\cap L_\theta} |r|^{a+k+1}\frac{\nu_\theta}{\nu_r} d\mathcal{H}^0(r)\right)^2} d\mathcal{H}^1(\theta).
\end{aligned} \tag{5.3}$$

It follows from step one, $a+k=b+l+2$ and equation (5.3) that

$$\begin{aligned}
P_{A,k}(E) &\geq \int_{\Pi^{\otimes}(E)} |\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2}\sqrt{\left(\int_{\partial^* E^{\otimes}\cap L_\theta}|r|^{a+k+1}d\mathcal{H}^0(r)\right)^2+\left(\frac{d\mu_{B,l}}{d\theta}(\theta)\right)^2}\,d\mathcal{H}^1(\theta)\\
&= \int_{\Pi^{\otimes}(E)} |\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2}\sqrt{\left(\int_{\partial^* E^{\otimes}\cap L_\theta}|r|^{a+k+1}d\mathcal{H}^0(r)\right)^2+\left(\int_{\partial^*(E^{\otimes})\cap L_\theta}|r|^{a+k+1}\frac{\nu_\theta}{\nu_r}d\mathcal{H}^0(r)\right)^2}\,d\mathcal{H}^1(\theta)\\
&= \int_{\Pi^{\otimes}(E)} |\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2}\sqrt{\left(\begin{array}{c}\left(|\eta_{B,l}(\theta)|^{a+k+1}+|-\eta_{B,l}(\theta)|^{a+k+1}\right)^2+\\ \left(|\eta_{B,l}(\theta)|^{a+k+1}\dfrac{\nu_\theta(\eta_{B,l}(\theta),\theta)}{\nu_r(\eta_{B,l}(\theta),\theta)}+|\eta_{B,l}(\theta)|^{a+k+1}\dfrac{\nu_\theta(-\eta_{B,l}(\theta),\theta)}{\nu_r(-\eta_{B,l}(\theta),\theta)}\right)^2\end{array}\right)}\,d\mathcal{H}^1(\theta)\\
&= 2\int_{\Pi^{\otimes}(E)} |\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2}\sqrt{|\eta_{B,l}(\theta)|^{2(a+k+1)}+\left(|\eta_{B,l}(\theta)|^{a+k+1}\frac{\nu_\theta(\eta_{B,l}(\theta),\theta)}{\nu_r(\eta_{B,l}(\theta),\theta)}\right)^2}\,d\mathcal{H}^1(\theta)\\
&= 2\int_{\Pi^{\otimes}(E)} |\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2}\frac{|\eta_{B,l}(\theta)|^{a+k+1}}{|\nu_r(\eta_{B,l}(\theta),\theta)|}d\mathcal{H}^1(\theta)\\
&= \int_{\Pi^{\otimes}(E)} |\cos(\theta)|^{a_1}|\sin(\theta)|^{a_2}\left[\frac{|\eta_{B,l}(\theta)|^{a+k+1}}{|\nu_r(\eta_{B,l}(\theta),\theta)|}+\frac{|-\eta_{B,l}(\theta)|^{a+k+1}}{|\nu_r(-\eta_{B,l}(\theta),\theta)|}\right]d\mathcal{H}^1(\theta)\\
&= P_{A,k}(E^{\otimes}).
\end{aligned}\tag{5.4}$$

Therefore, the first part of the theorem follows from Step 2. □

To prove the second part of the theorem, we use step two and [7], Theorem 6.1. Indeed, by the equality

$$P_{A,k}(E^{\otimes})=P_{A,k}(E),$$

and equations (5.2)–(5.4), we can observe that

$$\int_{\partial^* E\cap L_\theta}|r|^{a+k+1}d\mathcal{H}^0(r)=\int_{\partial^* E^{\otimes}\cap L_\theta}|r|^{a+k+1}d\mathcal{H}^0(r)\quad\text{for } \mathcal{H}^1-\text{ a.e. } \theta\in\Pi^{\otimes}(E).$$

By [7], Theorem 6.1, we then get

$$E_\theta=(-\eta_{B,l}(\theta),\eta_{B,l}(\theta)).$$

Thus, the proof of equation (1.4) is assured and, consequently, the Theorem 1.5.

□

Remark 5.2. Let us consider the nonnegative vectors $A=(a_1,a_2)$ and $B=(b_1,b_2)$, together with the case $l=k=0$. Let E be an open rectifiable set in $\mathbb{R}^2$. If E minimizes the isoperimetric quotient

$$\frac{P_A(E)}{[m_B(E)]^{\frac{a+1}{b+2}}},$$

then the reduced boundary $\partial^* E$ has no flat parts contained in L_θ for $\theta\in(0,\pi/2)$. Indeed, assume by contradiction that such a flat part exists in L_{θ_0} for some $\theta_0\in(0,\pi/2)$. Since E is a rectifiable set, and its boundary $\partial^* E\in C^1$ (see [41], Thm. C), we can suppose that exist r_1,r_2 positive numbers with $r_1<r_2$ such that

$$\{(r\cos(\theta_0),r\sin(\theta_0));r_1<r<r_2\}\subset\partial^* E.$$

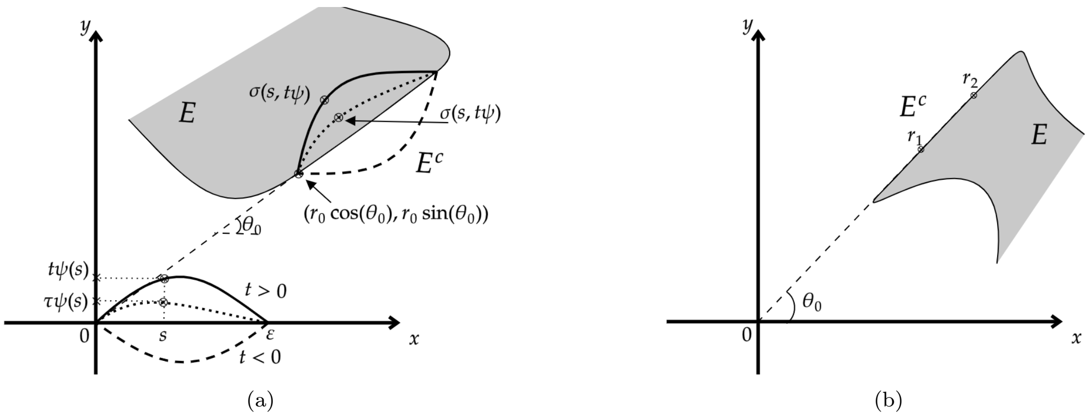


FIGURE 5. The set E with flat part on the boundary and the set $E(\psi, t)$.

Therefore, in a neighborhood of the flat portion in the reduced boundary $\partial^* E$, the set E is on one side of this flat portion, see Figure 5 – cases (a) and (b). We will consider the case (a). The case (b) follows in a similar manner. For $\varepsilon > 0$ small enough, let $r_0, r_0 + \varepsilon \in (r_1, r_2)$. Given $t_0 > 0$, we will take $t \in (-t_0, t_0)$, and $s \in [0, \varepsilon]$. Defining

$$\sigma(s,t) = (\sigma_1(s,t), \sigma_2(s,t)) := r_0(\cos(\theta_0), \sin(\theta_0)) + \begin{pmatrix} \cos(\theta_0) & \sin(\theta_0) \\ -\sin(\theta_0) & \cos(\theta_0) \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} \in \mathbb{R}^2,$$

we have owed the flat portion in the reduced boundary $\partial^* E$, that our inner and outer variations can be taken symmetrically, which allow us to consider the set

$$E(\psi, t) = \begin{cases} E \backslash \{\sigma(s, \tau\psi(s)) \in \mathbb{R}^2;\ 0 \le s \le \varepsilon,\ 0 \le \tau < t\} & \text{for } t > 0, \\ E \cup \{\sigma(s, \tau\psi(s)) \in \mathbb{R}^2;\ 0 \le s \le \varepsilon,\ t < \tau \le 0\} & \text{for } t < 0, \end{cases}$$

where $\psi \in C^1([0,\varepsilon]; \mathbb{R})$, $0 \le \psi \le 1$, $\psi(0) = 0 = \psi(\varepsilon)$, and $\psi \ne 0$. The following figures illustrates the situation. Thus,

$$P_A(E(\psi,t)) = P_A(E) - \int_{r_0}^{r_0+\varepsilon} r^a cos(\theta_0)^{a_1} \sin(\theta_0)^{a_2} dr + \int_0^{\varepsilon} |\sigma_1(s, t\psi(s))|^{a_1} |\sigma_2(s, t\psi(s))|^{a_2} \left| \frac{\mathrm{d}}{\mathrm{d}s} \sigma(s, t\psi(s)) \right| \mathrm{d}s,$$

and

$$m_B(E(\psi,t)) = \begin{cases} m_B(E) - \displaystyle\int_0^{\varepsilon} \int_0^{t\psi(s)} |\sigma_1(s,\tau)|^{b_1} |\sigma_2(s,\tau)|^{b_2} d\tau \mathrm{d}s, & \text{for } t > 0, \\ m_B(E) + \displaystyle\int_0^{\varepsilon} \int_{t\psi(s)}^{0} |\sigma_1(s,\tau)|^{b_1} |\sigma_2(s,\tau)|^{b_2} d\tau \mathrm{d}s, & \text{for } t < 0, \end{cases}$$

Since that $m_B(E) > 0$, the function $F : (-t_0, t_0) \to \mathbb{R}$ given by

$$F(t) := \frac{P_A(E(\psi,t))}{m_B(E(\psi,t))^{\frac{a+1}{b+2}}}. \tag{5.5}$$

is well-defined, once, if necessary we can take t_0 sufficiently small. Calculating the one-sided derivatives of F at $t = 0$, we obtain

$$\begin{aligned} F'_+(0) =& \frac{1}{m_B(E)^{\frac{a+1}{b+2}}} \left(P'_A(E(\varphi, 0)) + \frac{(a+1)}{(b+2)} \frac{P_A(E)}{m_B(E)} \int_0^\varepsilon |\sigma_1(s,0)|^{b_1} |\sigma_2(s,0)|^{b_2} \psi(s) \mathrm{d}s \right), \\ F'_-(0) =& \frac{1}{m_B(E)^{\frac{a+1}{b+2}}} \left(P'_A(E(\varphi, 0)) + \frac{(a+1)}{(b+2)} \frac{P_A(E)}{m_B(E)} \int_0^\varepsilon |\sigma_1(s,0)|^{b_1} |\sigma_2(s,0)|^{b_2} \psi(s) \mathrm{d}s \right). \end{aligned}$$

Since $F(t)$ has a minimum in $t = 0$, we conclude that $F'_-(0) = F'_+(0) = 0$. Applying the derivative in equation (5.5) at $t = 0$, we get

$$\begin{aligned} &- \frac{(a+1)}{(b+2)} \frac{P_A(E)}{m_B(E)} \int_0^\varepsilon |\sigma_1(s,0)|^{b_1} |\sigma_2(s,0)|^{b_2} \psi(s) \mathrm{d}s \\ &= \int_0^\varepsilon \left[a_1 \sigma_1(s,0)^{a_1 - 1} \sigma_2(s,0)^{a_2} \sin(\theta_0) + a_2 \sigma_1(s,0)^{a_1} \sigma_2(s,0)^{a_2 - 1} \cos(\theta_0) \right] \psi(s) \mathrm{d}s. \end{aligned}$$

Noticing that the right-hand side is positive, while the left-hand side is negative, which leads to a contradiction if the set E has flat parts.

We conclude this section with the following remark.

Remark 5.3. In the case $a = b = 0$ and $k = l + 2 > 0$, the isoperimetric constant is given by

$$C_{k,l,2} := \inf \left\{ \frac{\displaystyle\int_{\partial^* E} (x^2 + y^2)^{k/2} d\mathcal{H}^1(x,y)}{\left[\displaystyle\int_E (x^2 + y^2)^{l/2} d\mathcal{H}^2(x,y) \right]^{(k+1)/(l+2)}};\ E \subset \mathbb{R}^2,\ \text{open and bounded} \right\}. \tag{5.6}$$

As an application of Theorem 1.5, we can show that the balls with center at the origin are isoperimetric sets for $k > 0$ and, furthermore, $C_{k,l,2} = (2\pi)^{-1/k} k^{(k+1)/k}$. In fact, by [41], Theorem C and Theorem 1.5, we can consider the sets E as regular and star-shaped Steiner symmetric. Following the same argument of Remark 5.2, $\partial^* E$ has no flat parts contained in L_θ for $\theta \in (0, \pi)$. Thus, $\partial^* E$ can be parameterized by C^1 curves. For every $0 \leq \alpha_1 < \alpha_2 \leq \frac{\pi}{2} \leq \beta_1 < \beta_2 \leq \pi$, let us define the set

$$\Gamma := \left\{ \gamma \in C^1((\alpha_1, \alpha_2) \cup (\beta_1, \beta_2));\ \begin{array}{l} \gamma|_{(\alpha_1, \alpha_2)} = r \in C^1((\alpha_1, \alpha_2)), r \geq 0, \\ \gamma|_{(\beta_1, \beta_2)} = \rho \in C^1((\beta_1, \beta_2)), \rho \geq 0 \end{array} \right\}.$$

Finally, let $G(\gamma) = \left\{ (\gamma(\theta)\cos(\theta), \gamma(\theta)\sin(\theta)) \in \mathbb{R}^2; \theta \in (\alpha_1, \alpha_2) \cup (\beta_1, \beta_2) \right\}$. Thus

$$C_{k,l,2} = \inf_{\substack{\gamma \in \Gamma \\ \forall E \subset \mathbb{R}^2,\ \text{open and bounded} \\ G(\gamma) = \partial^* E \cap \mathbb{R}^2_+ \setminus \{(0,t), t > 0\}}} \left\{ \frac{2 \displaystyle\int_{\alpha_1}^{\alpha_2} r^k(\theta) \sqrt{r(\theta)^2 + (r'(\theta))^2} d\theta + 2 \int_{\beta_1}^{\beta_2} \rho^k(\theta) \sqrt{\rho(\theta)^2 + (\rho'(\theta))^2} d\theta}{\left(\frac{2}{l+2}\right)^{(l+3)/(l+2)} \left[\displaystyle\int_{\alpha_1}^{\alpha_2} r^{l+2} d\theta + \int_{\beta_1}^{\beta_2} \rho^{l+2} d\theta \right]^{(l+3)/(l+2)}} \right\}.$$

Given a parametrization $\gamma \in \Gamma$ of the reduced boundary of an admissible set E, let us fix a positive constant R such that

$$\int_{\alpha_1}^{\alpha_2} r(\theta)^{l+2} d\theta + \int_{\beta_1}^{\beta_2} \rho(\theta)^{l+2} d\theta = \int_0^{\pi} R^{l+2} d\theta.$$

On the one hand, by construction, we have $\int_E \left(x^2+y^2\right)^{l/2} d\mathcal{H}^2(x,y) = \int_{B_R(0)} \left(x^2+y^2\right)^{l/2} d\mathcal{H}^2(x,y)$; on the other hand, by Jensen's inequality (applied to $w \mapsto w^{\frac{l+3}{l+2}}$), we get

$$\begin{aligned}
&\int_{\partial^* E} \left(x^2+y^2\right)^{\frac{k}{2}} d\mathcal{H}^1(x,y) = 2\int_{\alpha_1}^{\alpha_2} r^k(\theta)\sqrt{r(\theta)^2+(r'(\theta))^2}d\theta + 2\int_{\beta_1}^{\beta_2} \rho^k(\theta)\sqrt{\rho(\theta)^2+(\rho'(\theta))^2}d\theta \\
&\geq 2\int_{\alpha_1}^{\alpha_2} r^{l+3}(\theta)d\theta + 2\int_{\beta_1}^{\beta_2} \rho^{l+3}(\theta)d\theta \geq 2|\alpha_2-\alpha_1|^{-\frac{1}{l+2}} \left(\int_{\alpha_1}^{\alpha_2} r^{l+2}(\theta)d\theta\right)^{\frac{l+3}{l+2}} + 2|\beta_2-\beta_1|^{-\frac{1}{l+2}} \left(\int_{\beta_1}^{\beta_2} \rho^{l+2}(\theta)d\theta\right)^{\frac{l+3}{l+2}} \\
&\geq 2\left(\frac{\pi}{2}\right)^{-\frac{1}{l+2}} \left[\left(\int_{\alpha_1}^{\alpha_2} r^{l+2}(\theta)d\theta\right)^{\frac{l+3}{l+2}} + \left(\int_{\beta_1}^{\beta_2} \rho^{l+2}(\theta)d\theta\right)^{\frac{l+3}{l+2}}\right] \\
&\geq 2\left(\frac{\pi}{2}\right)^{-\frac{1}{l+2}} 2^{-\left(\frac{l+3}{l+2}-1\right)} \left[\int_{\alpha_1}^{\alpha_2} r^{l+2}(\theta)d\theta + \int_{\beta_1}^{\beta_2} \rho^{l+2}(\theta)d\theta\right]^{\frac{l+3}{l+2}} = \left(\frac{\pi}{2}\right)^{-\frac{1}{l+2}} 2^{2-\frac{l+3}{l+2}} \left[\int_0^{\pi} R^{l+2}d\theta\right]^{\frac{l+3}{l+2}} = 2\pi R^{l+3} \\
&= \int_{\partial B_R(0)} \left(x^2+y^2\right)^{\frac{k}{2}} d\mathcal{H}^1(x,y),
\end{aligned}$$

where in the fourth inequality we have used that

$$(s+t)^{p+1} \leq 2^p(s^{p+1}+t^{p+1}) \text{ for all } s>0,\ t>0 \text{ and } p>0.$$

This establishes the equality, which concludes the proof.

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