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DIRICHLET PROBLEM FOR NONCOERCIVE NONLINEAR ELLIPTIC EQUATIONS WITH SINGULAR DRIFT TERM IN UNBOUNDED DOMAINS

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Abstract. In this paper, we study a Dirichlet problem for noncoercive nonlinear elliptic equations with first order term in an unbounded domain. We obtain Stampacchia type existence, regularity and uniqueness results, when the singular drift term is controlled through a function in a suitable functional space, strictly containing Lebesgue one. The main tools are a weak maximum principle together with some *a priori* estimates proved by contradiction.

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1. Introduction

Let Ω be an open unbounded domain in $\mathbb{R}^N$, $N > 2$. We study the following problem

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) + \mu u = B(x, \nabla u) + f(x) & \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega) \end{cases} \tag{1.1}$$

where we assume that $A : \Omega \to \mathbb{R}^{N^2}$ is a measurable matrix field such that for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$ one has

$$\alpha|\xi|^2 \leq A(x)\,\xi\xi, \quad |A(x)| \leq \beta, \tag{1.2}$$

for some α, $\beta \in \mathbb{R}_+$, and

$$\mu > 0. \tag{1.3}$$

The Carathéodory function $B : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies, for almost every $x \in \Omega$ and for every $\xi, \eta \in \mathbb{R}^N$, the following condition

$$|B(x, \xi)| \leq |b(x)||\xi|, \tag{1.4}$$

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where $b : \Omega \to \mathbb{R}^N$ is a vector field verifying

$$|b| \in L^2(\Omega) \cap M_0^N(\Omega). \tag{1.5}$$

Here, $M_0^N(\Omega)$ denotes a functional space strictly containing $L^N(\Omega)$ that was introduced in [19] within the study of certain variational elliptic problems in unbounded domains (see Sect. 2).

The real function $f : \Omega \to \mathbb{R}$ is such that

$$f \in L^{2_*}(\Omega), \text{ where } 2_* = \frac{2N}{N+2}. \tag{1.6}$$

In general, the operator $A(u) = -div(A(x)\nabla u) + \mu u - B(x, \nabla u)$ is not coercive unless μ is large enough or $\|b\|_{M^N(\Omega)}$ is small enough.

The linear counterpart of this problem

$$\begin{cases} -\text{div}(A(x)\nabla u) + \mu u = b(x)\nabla u + f(x) & \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega) \end{cases} \tag{1.7}$$

models the stationary equation of diffusion-advection problems with inhomogeneous diffusion. In the case of unbounded domains, we refer the reader to [6] for existence and uniqueness results for problem (1.7) in the coercive case. The dual problem of (1.7) in the noncoercive case has been considered in [14] (see also [1, 2]).

If Ω is bounded, the noncoercive linear problem (1.7) has been studied in [3] assuming $\mu = 0$ and $b \in L^N(\Omega)$. More precisely, the author proves, among other results, that Stampacchia theory holds for the boundary value problem (1.7). Namely, that there exists a unique weak solution $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$, if $f \in L^m(\Omega)$, with $m \geq \frac{2N}{N+2}$, while, if $m > \frac{N}{2}$, then $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. For a simple and self-contained approach to existence and uniqueness results for the same problem and for the dual one, we quote also [4], where a weaker datum f is considered. Analogous results in the nonlinear case have been obtained in [5]. We refer to [8, 9, 16] for existence results.

For the study of the coercive linear problem (1.7), we quote here the classical works [12, 17, 18].

In this paper we use some nonlinear methods allowing us to obtain existence and regularity of solutions of the noncoercive nonlinear problem (1.1) on the unbounded domain Ω, in the spirit of [3]. In particular, inspired by classical arguments of [12] (see also [16]), we prove a weak maximum principle (see Prop. 3.1) that leads to existence and uniform *a priori* bounds for weak solutions of more regular approximating noncoercive nonlinear problems. Thus, passing to the limit, we get the existence of a weak solution of problem (1.1), under assumptions (1.2)–(1.6). This is done in Theorem 4.3. Thanks to the approximating problems, we also achieve the Stampacchia type regularity results given in Theorem 5.4. Let us remark that we treat the critical case with a summable $f \in L^{N/2}(\Omega)$ as well, obtaining the expected exponential summability of the solution as in [18]. Finally, always by means of the weak maximum principle, under the additional hypothesis

$$|B(x,\xi) - B(x,\eta)| \leq |b(x)||\xi - \eta| \tag{1.8}$$

we come to the uniqueness of the solution in Theorem 6.1.

We point out that our results are obtained under the more general assumption (1.5), since, as already specified, when Ω is unbounded, the space $M_0^N(\Omega)$ strictly contains $L^N(\Omega)$ (see Ex. 2.1 and also Ex. 2.2 in Sect. 2, for more details).

The main difficulties one has to deal with when working on unbounded sets are well-known: there are no natural decreasing inclusions among Lebesgue spaces, there is a lack of compactness results and the norm in $W_0^{1,2}(\Omega)$ is not equivalent to the $L^2(\Omega)$ norm of gradient due to the fact that Poincaré inequality does not hold. This last consideration forces us to assume that μ is not null, whence hypothesis (1.3). Nevertheless, our operator is in general noncoercive since we do not require μ to be large enough or restrictions on the size of the M^N-norm of b. Related results can be found in [10].

The manuscript develops as follows: Section 2 is devoted to the definition and the main properties of the M^t spaces. An essential tool in our study is the compactness result in Lemma 2.3, contained in [19]. In Section 3, we collect some useful known results and we prove the above mentioned weak maximum principle. Section 4 contains the existence results. We start proving the existence of a solution of the approximating problems. This can be done by means of Leray–Schauder theorem finding a fixed point of suitable operators related to coercive auxiliary problems. Then, by contradiction, we prove uniform *a priori* bounds crucial to pass to the limit and get the existence result. In Section 5, we establish the regularity results, showing how the summabilty of f affects that of the solution of problem (1.1). We conclude by proving the uniqueness in Section 6.

2. The space of M^t functions

In this section, we recall a class of functional spaces, firstly introduced in [19], that in unbounded domains generalize Lebesgues ones.

We consider an unbounded open subset Ω of $\mathbb{R}^N$, $N > 2$ and we denote by $\Sigma(\Omega)$ the σ-algebra of Lebesgue measurable subsets of Ω. Given $O \in \Sigma(\Omega)$, χ_O is the characteristic function of O, $O(x, r)$ is the intersection $O \cap B(x, r)$ ($x \in \mathbb{R}^N$, $r \in \mathbb{R}_+$), where $B(x, r)$ is the open ball centered in x and with radius r and $|O|$ is the Lebesgue measure of O.

The class $\mathcal{D}(\overline{\Omega})$ contains the restrictions to $\overline{\Omega}$ of functions $\zeta \in C_0^\infty(\mathbb{R}^N)$. For $t \in [1, +\infty[$, $L^t_{loc}(\overline{\Omega})$ denotes the class of functions $g : \Omega \to \mathbb{R}$ such that $\zeta\, g \in L^t(\Omega)$ for any $\zeta \in \mathcal{D}(\overline{\Omega})$.

For every $t \in [1, +\infty[$, we set

$$M^t(\Omega) = \{g \in L^t_{loc}(\overline{\Omega}) :\ \|g\|_{M^t(\Omega)} = \sup_{x\in\Omega} \|g\|_{L^t(\Omega(x,1))} < +\infty\},$$

equipped with the norm above defined and we also consider the following subset of $M^t(\Omega)$

$$M^t_0(\Omega) = \{g \in M^t(\Omega) :\ \lim_{|x|\to+\infty} \|g\|_{L^t(\Omega(x,1))} = 0\}.$$

For every $g \in M^t(\Omega)$ the following properties are equivalent:

i) $g \in M^t_0(\Omega)$,
ii) for any $\varepsilon \in \mathbb{R}_+$ there exist $\nu_\varepsilon, \sigma_\varepsilon \in \mathbb{R}_+$ such that

$$O \in \Sigma(\Omega),\ |O(0, \sigma_\varepsilon)| \le \nu_\varepsilon \Rightarrow \|g\chi_O\|_{M^t(\Omega)} \le \varepsilon. \tag{2.1}$$

Furthermore, we point out that, differently from Lebesgue spaces defined on unbounded domains, one has the decreasing inclusions

$$M^s(\Omega) \subseteq M^t(\Omega), \quad M^s_0(\Omega) \subseteq M^t_0(\Omega) \quad \text{ if } 1 \le t \le s < +\infty.$$

As shown in the example below (see [19]), for any $t > 1$, the following strict inclusion holds

$$L^t(\Omega) \subset M^t_0(\Omega).$$

Example 2.1. The function

$$\frac{1}{1 + |x|^\alpha} \in M^t_0(\Omega), \forall\, t > 1 \text{ and } \forall\, \alpha \in \mathbb{R}_+,$$

while, if $\Omega = \mathbb{R}^N$, for $t > 1$ and $0 < \alpha < N/t$,

$$\frac{1}{1+|x|^\alpha} \notin L^t(\Omega).$$

In addition, we prove that, for all $1 \leq q < t$, one has

$$L^t(\Omega) \nsubseteq M_0^t(\Omega) \cap L^q(\Omega). \tag{2.2}$$

To this aim, in the next example we construct a function $\varphi \in M_0^t(\Omega) \cap L^q(\Omega)$ such that $\varphi \notin L^t(\Omega)$.

Example 2.2. Let $t > 1$ and Ω be an arbitrary unbounded subset of $\mathbb{R}^N$.

We can consider a sequence of points $\{y_k\}_{k\in\mathbb{N}} \subseteq \Omega$ such that, for every $k \in \mathbb{N}$,

$$|y_{k+1}| > |y_k| + 4,$$

and obviously $\lim_{k\to+\infty} |y_k| = +\infty$.

Moreover, for every $x \in \Omega$, we define the function $k : \Omega \to \mathbb{N}$ as

$$k(x) = \min\{k \in \mathbb{N} : |y_k| > |x| - 2\}.$$

One has that $\lim_{|x|\to+\infty} k(x) = +\infty$.

We remark that for every $x \in \Omega$, $k(x)$ is the unique value of $k \in \mathbb{N}$ such that the intersection $B(x,1)\cap\Omega(y_k,1)$ could be not empty, *i.e.*

$$\forall k \neq k(x), \;\; B(x,1) \cap \Omega(y_k,1) = \emptyset.$$

Indeed, for every $k \neq k(x)$, we note that if $k < k(x)$ then $|y_k| \leq |x| - 2$, and if $k \geq k(x)$ then $|y_k| > |y_{k(x)}| + 4 \geq |x| + 2$. Hence, in any case we have

$$|y_k - x| \geq ||y_k| - |x|| \geq 2.$$

At this point, for a fixed $\delta > 0$ and for every $k \in \mathbb{N}$, consider a ball $B_k \subset \Omega(y_k, 1)$ such that

$$|B_k| \leq \left(\frac{1}{k}\right)^{(1+\delta)}, \tag{2.3}$$

and we observe that the B_k are pairwise disjoint balls.

Now, for every $k \in \mathbb{N}$, let $\varphi_k \in C_0^\infty(B_k)$ be a function satisfying

$$\int_{B_k} |\varphi_k(x)|^t \mathrm{d}x = \frac{1}{k}. \tag{2.4}$$

We set $\varphi_k(x) = 0$ outside of B_k and define:

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k(x).$$

Note that, for every $x \in \mathbb{R}^N$, in the previous sum at most one term is not zero. Furthermore, the function $\varphi(x)$ belongs to $C^\infty(\mathbb{R}^N)$ with supp $\varphi \subset \cup_{k=1}^{+\infty} B_k \subset \Omega$ and, by construction, $\varphi \in L^t_{loc}(\overline{\Omega})$.

For every $x \in \mathbb{R}^N$, one has

$$\int_{B(x,1)} |\varphi|^t \mathrm{d}x = \int_{B(x,1)\cap B_{k(x)}} |\varphi|^t \mathrm{d}x \leq \int_{B_{k(x)}} |\varphi_{k(x)}|^t \mathrm{d}x = \frac{1}{k(x)} \to 0 \qquad \text{as } |x| \to +\infty,$$

so that $\varphi \in M_0^t(\mathbb{R}^N) \subseteq M_0^t(\Omega)$. Moreover, for every $1 \leq q < t$,

$$\int_{\mathbb{R}^N} |\varphi|^q \mathrm{d}x = \sum_{k=1}^{\infty} \int_{B_k} |\varphi_k|^q \mathrm{d}x \leq \sum_{k=1}^{\infty} \left(\int_{B_k} |\varphi_k|^t \mathrm{d}x \right)^{\frac{q}{t}} |B_k|^{(1-\frac{q}{t})} \leq \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{[1+\delta(1-\frac{q}{t})]} < +\infty$$

so that $\varphi \in L^q(\Omega)$.

On the other hand,

$$\int_\Omega |\varphi|^t \mathrm{d}x = \sum_{k=1}^{\infty} \int_{B_k} |\varphi_k|^t \mathrm{d}x = \sum_{k=1}^{\infty} \frac{1}{k} = +\infty.$$

This completes the proof of (2.2).

Finally, we recall the following boundedness and compactness results, useful in the sequel.

Lemma 2.3 ([20])**.** *For every $g \in M^N(\Omega)$ the product operator*

$$u \in W_0^{1,2}(\Omega) \longrightarrow g\, u \in L^2(\Omega), \tag{2.5}$$

is bounded and

$$\|g\, u\|_{L^2(\Omega)} \leq C\, \|g\|_{M^N(\Omega)} \|u\|_{W^{1,2}(\Omega)} \quad \forall\, u \in W_0^{1,2}(\Omega), \tag{2.6}$$

with $C = C\,(N)$ positive constant.

Furthermore, if $g \in M_0^N(\Omega)$, operator (2.5) is also compact.

We refer the reader to [19, 20] for the proofs of all the above mentioned results and for more details about M^t spaces.

3. Preliminary results

In this section, we give some useful results.

A key tool in the sequel is the weak maximum principle in $W_0^{1,2}(\Omega)$ below, extending the well known one contained in Chapter 8 of [12] (see also [16])

Proposition 3.1. *Assume (1.2)–(1.5). If $w \in W_0^{1,2}(\Omega)$ is such that*

$$\int_\Omega A(x)\nabla w \nabla \varphi \; \mathrm{d}x + \mu \int_\Omega w\, \varphi \; \mathrm{d}x \leq \int_\Omega |b(x)||\nabla w||\varphi| \; \mathrm{d}x, \; \forall\, \varphi \in W_0^{1,2}(\Omega), \tag{3.1}$$

then $w \leq 0$.

Proof. Let $w \in W_0^{1,2}(\Omega)$. For $k > 0$, we denote $w_k = (w-k)^+ = \max\{w-k, 0\}$ and we use it as a test function in (3.1)

$$\int_\Omega A(x)\nabla w \nabla w_k \, \mathrm{d}x + \mu \int_\Omega w \, w_k \, \mathrm{d}x \le \int_\Omega |b(x)||\nabla w| w_k \, \mathrm{d}x.$$

Then we have

$$\int_\Omega A(x)\nabla w \nabla w_k \, \mathrm{d}x + \mu \int_\Omega (w-k) w_k \, \mathrm{d}x \le \int_\Omega |b(x)||\nabla w| w_k \, \mathrm{d}x - \mu k \int_\Omega w_k \, \mathrm{d}x,$$

which implies, by (1.3),

$$\int_\Omega A(x)\nabla w \nabla w_k \, \mathrm{d}x + \mu \int_\Omega (w-k) w_k \, \mathrm{d}x \le \int_\Omega |b(x)||\nabla w| w_k \, \mathrm{d}x.$$

Now, observe that $\nabla w = \nabla w_k$, if $w > k$, thus by Hölder inequality and Lemma 2.3, we get

$$\begin{aligned} &\int_\Omega A(x)\nabla w_k \nabla w_k \, \mathrm{d}x + \mu \int_\Omega |w_k|^2 \, \mathrm{d}x \\ &\le \int_\Omega |b(x) \, w_k||\nabla w_k| \, \mathrm{d}x = \int_{E_k} |b(x) \, w_k||\nabla w_k| \, \mathrm{d}x \\ &\le \|b w_k\|_{L^2(E_k)} \|\nabla w_k\|_{L^2(E_k)} \le C \|b\|_{M^N(E_k)} \|w_k\|_{W^{1,2}(\Omega)} \|\nabla w_k\|_{L^2(\Omega)} \\ &\le C \|b\|_{M^N(E_k)} \|w_k\|^2_{W^{1,2}(\Omega)}, \end{aligned} \tag{3.2}$$

where $E_k = \{x \in \Omega : w(x) > k, \ |\nabla w(x)| > 0\}$ and $C = C(N)$ is a positive constant.

From (1.2), (1.3) and (3.2) we have

$$min(\alpha, \mu) \|w_k\|^2_{W^{1,2}(\Omega)} \le C \|b\|_{M^N(E_k)} \|w_k\|^2_{W^{1,2}(\Omega)}. \tag{3.3}$$

Now, by contradiction, let us suppose that $sup \, w > 0$ and set $M = sup \, w$. If $M = +\infty$, then

$$\lim_{k \to M} meas(E_k) = 0. \tag{3.4}$$

Otherwise, if M is finite, since $w \in W_0^{1,2}(\Omega)$, by known properties of Sobolev functions (see, *e.g.* [12]), we have that $|\nabla w(x)| = 0$ a.e. on $\{x \in \Omega : w(x) = M\}$, hence we still deduce (3.4).

By (2.1) and (3.4) we have $\lim_{k \to M} \|b\|_{M^N(E_k)} = 0$, hence there exists $k_0 < M$ such that $\|b\|_{M^N(E_k)} < \frac{min(\alpha,\mu)}{C}$ for $k \ge k_0$. So, using (3.3) we have

$$(min(\alpha, \mu) - C \|b\|_{M^N(E_{k_0})}) \|w_{k_0}\|^2_{W^{1,2}(\Omega)} \le 0,$$

hence $\|w_{k_0}\|_{W^{1,2}(\Omega)} = 0$ and therefore $w_{k_0} = 0$ a.e. in Ω, that is $w \le k_0$ a.e. in Ω. This means that $sup \, w \le k_0 < M$, getting the contradiction.

□

Now, let us recall a useful lemma proved in [6] by G.F. Bottaro and M.E. Marina in the case of unbounded domains, generalizing a well known result by G. Stampacchia, contained in [18].

Lemma 3.2. *Let G be a uniformly Lipschitz function satisfying $G(0) = 0$ and $u \in W_0^{1,2}(\Omega)$. Then $G \circ u \in W_0^{1,2}(\Omega)$.*

For $k \in \mathbb{R}_+$, we remind Stampacchia's truncates

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\frac{s}{|s|}, & \text{if } |s| > k \end{cases} \tag{3.5}$$

and let

$$G_k(s) = s - T_k(s). \tag{3.6}$$

For every $u \in W_0^{1,2}(\Omega)$, we put

$$\Omega_k = \{x \in \Omega \,:\, |u(x)| > k\}. \tag{3.7}$$

As a straightforward consequence of Lemma 3.2, we get

Lemma 3.3. *Let $u \in W_0^{1,2}(\Omega)$ and $k \in \mathbb{R}_+$. The following properties hold*

$$G_k(u) = G_k \circ u \in W_0^{1,2}(\Omega), \tag{3.8}$$

$$|G_k(u)| \leq |u|, \ \text{ a.e. in } \Omega, \tag{3.9}$$

$$|u| \leq |G_k(u)| + k, \ \text{ a.e. in } \Omega, \tag{3.10}$$

$$\nabla u \nabla G_k(u) = |\nabla G_k(u)|^2, \ \text{ a.e. in } \Omega, \tag{3.11}$$

$$u\, G_k(u) \geq |G_k(u)|^2, \ \text{ a.e. in } \Omega, \tag{3.12}$$

$$\text{supp } G_k(u) \subseteq \overline{\Omega}_k, \tag{3.13}$$

$$(G_k(u))_{x_i} = \begin{cases} u_{x_i} & \text{a.e. in } \Omega_k, \\ 0 & \text{a.e. in } \Omega \setminus \Omega_k, \ i = 1 \ldots n. \end{cases} \tag{3.14}$$

$$T_k(u) = T_k \circ u \in W_0^{1,2}(\Omega), \tag{3.15}$$

$$\nabla u \nabla T_k(u) = |\nabla T_k(u)|^2, \ \text{ a.e. in } \Omega, \tag{3.16}$$

$$u\, T_k(u) \geq |T_k(u)|^2, \ \text{ a.e. in } \Omega, \tag{3.17}$$

$$u\, \nabla T_k(u) = T_k(u) \nabla T_k(u), \ \text{ a.e. in } \Omega. \tag{3.18}$$

We shall make use of the following lemma due to G. Stampacchia, see Lemma 4.1 of [18], in order to prove our boundedness result

Lemma 3.4. *Let $k_0 > 0$ and $\varphi : [k_0, +\infty[\to \mathbb{R}_+$ be a non increasing function s.t.*

$$\varphi(h) \leq \frac{C}{(h-k)^\gamma} [\varphi(k)]^\delta \quad \forall\, h > k \geq k_0, \tag{3.19}$$

where C, γ and δ are positive constants, with $\delta > 1$. For

$$d = 2^{\frac{\delta}{\delta-1}} C^{1/\gamma} \, [\varphi(k_0)]^{\frac{\delta-1}{\gamma}} \tag{3.20}$$

one has

$$\varphi(k_0 + d) = 0. \tag{3.21}$$

We conclude this section giving the well known Leray–Schauder fixed point theorem in the following form (see [12], Thm. 11.3, p. 280). For the sake of completeness, let us remind that a continuous mapping between two Banach spaces is called compact if the images of bounded sets are precompact.

Theorem 3.5. *Let $\mathcal{F}$ be a continuous, compact mapping of a Banach space X into itself, and suppose there exists a positive constant L such that $\|x\|_X \leq L$ for all $x \in X$ and $\sigma \in [0,1]$ satisfying $x = \sigma \mathcal{F}(x)$. Then, $\mathcal{F}$ has a fixed point.*

4. Existence

In this section, we achieve the existence of a solution of problem (1.1) *via* approximation through the more regular noncoercive nonlinear problems $(\mathcal{P}_n)$ defined below.

We firstly obtain the existence of a solution sequence $\{u_n\}$ of $(\mathcal{P}_n)$ applying Leray–Schauder fixed point theorem to operators (4.3) related to coercive auxiliary problems. Then, uniform *a priori* bounds, proved by contradiction, allow us to pass to the limit.

Let us introduce the approximate problems, as in [11, 13],

$$(\mathcal{P}_n) \quad \begin{cases} -\operatorname{div}(A(x)\nabla u_n) + \mu u_n = \theta_n(x) B(x, \nabla u_n) + T_n(f(x)) & \text{in } \Omega, \\ u_n \in W_0^{1,2}(\Omega). \end{cases}$$

where, for each $n \in \mathbb{N}$ and almost every $x \in \Omega$, we set

$$\theta_n(x) = \begin{cases} \frac{T_n(b(x))}{b(x)} & \text{if } b(x) \neq 0 \\ 1 & \text{if } b(x) = 0, \end{cases} \tag{4.1}$$

and T_n denotes the truncation operator in (3.5).

We start proving, for any $n \in \mathbb{N}$, the existence of weak solutions of problems $(\mathcal{P}_n)$ together with the uniform *a priori* estimates.

Theorem 4.1. *Under assumptions (1.2)–(1.6), for any $n \in \mathbb{N}$, problem $(\mathcal{P}_n)$ admits a weak solution u_n of class $W_0^{1,2}(\Omega)$. Moreover, there exists a positive constant C, independent of n, such that*

$$\|u_n\|_{W^{1,2}(\Omega)} \leq C. \tag{4.2}$$

Proof. For any $v \in W_0^{1,2}(\Omega)$, we define the operator

$$\mathcal{F}_n : v \in W_0^{1,2}(\Omega) \to u \in W_0^{1,2}(\Omega)$$

assuming

$$\mathcal{F}_n(v) = u \Longleftrightarrow \begin{cases} -\mathrm{div}(A(x)\nabla u) + \mu u = \theta_n(x)B(x,\nabla v) + T_n(f(x)) \ \text{ in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{4.3}$$

For the sake of readiability in the proof we take $n = 1$.
Under our assumptions, by Lax–Milgram Theorem $\mathcal{F} = \mathcal{F}_1$ is well posed, indeed, for every $v \in W_0^{1,2}(\Omega)$, we have $\theta_1(x)B(x,\nabla v) \in L^2(\Omega)$.

Now, in order to prove the existence of a solution $u \in W_0^{1,2}(\Omega)$ of problem $(\mathcal{P}_1)$, we shall apply Theorem 3.5 to the operator $\mathcal{F}$. Thus, we shall prove that

i) $\mathcal{F}$ is continuous;
ii) $\mathcal{F}$ is compact;
iii) there exists a positive constant L such that

$$\|u\|_{W^{1,2}(\Omega)} \le L,$$

for every $u \in W_0^{1,2}(\Omega)$ and every $\sigma \in [0,1]$ such that $u = \sigma\mathcal{F}(u)$.

We start proving that the map $\mathcal{F}$ is continuous. Thus let

$$v_k \to v \quad \text{strongly in } W_0^{1,2}(\Omega) \tag{4.4}$$

and set

$$u_k = \mathcal{F}(v_k). \tag{4.5}$$

By taking u_k as test function in the variational formulation of the problem corresponding to (4.5) we get,

$$\int_\Omega A(x)\nabla u_k \nabla u_k \,\mathrm{d}x + \mu \int_\Omega u_k^2 \,\mathrm{d}x = \int_\Omega \theta_1(x)B(x,\nabla v_k)\, u_k \,\mathrm{d}x + \int_\Omega T_1(f)\, u_k \,\mathrm{d}x.$$

In the sequel, as before, we denote by $2_* = (2^*)' = \frac{2N}{N+2}$.
From our assumptions, Hölder and Sobolev inequalities and Lemma 2.3, we have

$$\begin{aligned} \min\{\alpha,\mu\}\|u_k\|^2_{W^{1,2}(\Omega)} &\le \|bu_k\|_{L^2(\Omega)}\|\nabla v_k\|_{L^2(\Omega)} + \|f\|_{L^{2_*}(\Omega)}\|u_k\|_{L^{2^*}(\Omega)} \\ &\le C\|b\|_{M^N(\Omega)}\|u_k\|_{W^{1,2}(\Omega)}\|\nabla v_k\|_{L^2(\Omega)} + C_S\|f\|_{L^{2_*}(\Omega)}\|u_k\|_{W^{1,2}(\Omega)}, \end{aligned}$$

where $C = C(N)$ and C_S is the Sobolev constant.
Hence

$$\|u_k\|_{W^{1,2}(\Omega)} \le C\frac{\|b\|_{M^N(\Omega)}\|\nabla v_k\|_{L^2(\Omega)} + \|f\|_{L^{2_*}(\Omega)}}{min\{\alpha,\mu\}}, \tag{4.6}$$

with $C = C(N)$. Thus, in view of (4.4), the sequence $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$. Then there exists $u \in W_0^{1,2}(\Omega)$ such that, unless to pass to a subsequence not relabeled,

$$u_k \rightharpoonup u \text{ weakly in } W_0^{1,2}(\Omega). \tag{4.7}$$

Now, we use $u_k - u$ as test function in the variational formulation of the problem corresponding to (4.5) and we get, by using (1.4),

$$\begin{aligned}
&\int_\Omega A(x)\nabla u_k \nabla(u_k - u)\,\mathrm{d}x + \mu\int_\Omega u_k(u_k - u)\,\mathrm{d}x \\
&\le \int_\Omega |b(x)||\nabla v_k||u_k - u|\,\mathrm{d}x + \int_\Omega T_1(f)(u_k - u)\,\mathrm{d}x.
\end{aligned}$$

Hence, by assumptions (1.2), (1.3), Lemma 2.3 and Hölder inequality, we have

$$\begin{aligned}
&\min\{\alpha,\mu\}\|u_k - u\|^2_{W^{1,2}(\Omega)} \\
&\le C\|b(u_k - u)\|_{L^2(\Omega)}\|\nabla v_k\|_{L^2(\Omega)} + \int_\Omega T_1(f)(u_k - u)\,\mathrm{d}x \\
&- \int_\Omega A(x)\nabla u \nabla(u_k - u)\,\mathrm{d}x - \mu\int_\Omega u(u_k - u)\,\mathrm{d}x,
\end{aligned} \tag{4.8}$$

with $C = C(N)$. Note that by (4.7) and by the compactness result in Lemma 2.1, up to a subsequence, we have

$$b(u_k - u) \to 0 \qquad \text{strongly in } L^2(\Omega). \tag{4.9}$$

Thus, by (4.4), (4.7) and (4.9) the right hand side of (4.8), unless to pass to a subsequence, tends to zero. This gives, for a subsequence, that

$$u_k = \mathcal{F}(v_k) \to u \quad \text{strongly in } W_0^{1,2}(\Omega). \tag{4.10}$$

Now, observe that, in view of convergence (4.4), up to a subsequence, we have

$$\nabla v_k(x) \to \nabla v(x) \text{ a.e. in } \Omega. \tag{4.11}$$

Therefore, since B is a Carathéodory function, we have

$$\theta_1(x)B(x,\nabla v_k) \to \theta_1(x)B(x,\nabla v) \text{ a.e. in } \Omega. \tag{4.12}$$

Note that, since $v_k \to v$ strongly in $W_0^{1,2}(\Omega)$, then, by Vitali Theorem (see, for instance, [15]), one has that for every $\varepsilon > 0$ there exists $\Omega_\varepsilon \subset \Omega$ s.t. $|\Omega_\varepsilon| < +\infty$ and

$$\int_{\Omega\setminus\Omega_\varepsilon} |\nabla v_k|^2\mathrm{d}x < \varepsilon, \qquad \text{uniformly with respect to } k. \tag{4.13}$$

Thus, by (1.4), Hölder inequality, Lemma 2.3 and (4.13), for every $\varphi \in W_0^{1,2}(\Omega)$, we have that

$$\int_{\Omega\setminus\Omega_\varepsilon} |\theta_1(x)B(x,\nabla v_k)\varphi(x)|\mathrm{d}x < \|b\varphi\|_{L^2(\Omega)}\varepsilon^{1/2} \quad \forall k \in \mathbb{N} \tag{4.14}$$

and, by (4.4)

$$\int_A |\theta_1(x)B(x,\nabla v_k)\varphi(x)|\mathrm{d}x \le \|b\varphi\|_{L^2(A)}\|\nabla v_k\|_{L^2(\Omega)} \le C\|b\varphi\|_{L^2(A)},$$

uniformly with respect to k.

Therefore

$$\lim_{|A|\to 0}\int_A |\theta_1(x)B(x,\nabla v_k)\varphi(x)|\mathrm{d}x \to 0, \tag{4.15}$$

uniformly with respect to k.

Hence, by (4.14) and (4.15), using again Vitali Theorem, we get

$$\theta_1(x)B(x,\nabla v_k)\varphi(x) \to \theta_1(x)B(x,\nabla v)\varphi(x) \ \text{ in } L^1(\Omega), \quad \forall\varphi \in W_0^{1,2}(\Omega). \tag{4.16}$$

Thus, passing to the limit in the variational formulation of the problem corresponding to (4.5), namely

$$\textstyle\int_\Omega A(x)\nabla u_k\nabla\varphi \,\mathrm{d}x + \mu\int_\Omega u_k\varphi \,\mathrm{d}x = \int_\Omega \theta_1(x)B(x,\nabla v_k)\,\varphi \,\mathrm{d}x + \int_\Omega T_1(f)\varphi \,\mathrm{d}x,$$

$\forall\varphi \in W_0^{1,2}(\Omega)$, by (4.7) and (4.16), we obtain

$$\mathcal{F}(v) = u. \tag{4.17}$$

On the other hand, since any subsequence of u_k has a subsequence converging to $u = \mathcal{F}(v)$, then, by uniqueness, u is the limit of the whole sequence $u_k = \mathcal{F}(v_k)$. This gives the continuity of $\mathcal{F}$.

To prove that $\mathcal{F}$ is also compact we can follow exactly the same argument used to prove the continuity of $\mathcal{F}$. More precisely, we have to show that if

$$\|v_k\|_{W^{1,2}(\Omega)} \le C, \tag{4.18}$$

then, up to a subsequence, $u_k = \mathcal{F}(v_k)$ strongly converges in $W_0^{1,2}(\Omega)$. To this aim, as before, by taking u_k as test function in the variational of problem corresponding to (4.5) we get (4.6), hence $\{u_k\}$ is bounded in $W_0^{1,2}(\Omega)$. So we have that there exists $u \in W_0^{1,2}(\Omega)$ such that, up to subsequence, one has (4.7). Again, using $u_k - u$ as test function in the variational problem corresponding to (4.5) we get (4.10), so that $\mathcal{F}$ is compact.

To show iii) we argue by contradiction. Hence, let us assume that

$$\begin{aligned}&\text{for any } k \in \mathbb{N}, \text{ there exists } \sigma_k \in [0,1]\\ &\text{such that } u_k = \sigma_k\mathcal{F}(u_k) \text{ and } \|u_k\|_{W^{1,2}(\Omega)} > k.\end{aligned}$$

This means that

$$\|u_k\|_{W^{1,2}(\Omega)} \to +\infty. \tag{4.19}$$

Being $u_k = \sigma_k \mathcal{F}(u_k)$, one has

$$-\operatorname{div}(A(x)\nabla u_k) + \mu u_k = \sigma_k \theta_1(x) B(x, \nabla u_k) + \sigma_k T_1(f(x)). \tag{4.20}$$

Now set $w_k = \frac{u_k}{\|u_k\|_{W^{1,2}(\Omega)}}$, from (4.20) we get

$$\begin{aligned} \int_\Omega A(x)\nabla w_k \nabla \varphi \, \mathrm{d}x + \mu \int_\Omega w_k \varphi \, \mathrm{d}x &= \tfrac{\sigma_k}{\|u_k\|_{W^{1,2}(\Omega)}} \int_\Omega \theta_1(x) B(x, \nabla u_k)\, \varphi \, \mathrm{d}x \\ &+ \tfrac{\sigma_k}{\|u_k\|_{W^{1,2}(\Omega)}} \int_\Omega T_1(f) \varphi \, \mathrm{d}x, \qquad \forall \varphi \in W_0^{1,2}(\Omega). \end{aligned} \tag{4.21}$$

Let us observe that, since $\|w_k\|_{W^{1,2}(\Omega)} = 1$, and since $\sigma_k \in [0,1]$, up to a subsequence, one has that there exist $\bar{w} \in W_0^{1,2}(\Omega)$ and $\bar{\sigma} \in [0,1]$ such that

$$w_k \rightharpoonup \bar{w} \quad \text{weakly in } W_0^{1,2}(\Omega) \tag{4.22}$$

$$\sigma_k \to \bar{\sigma}. \tag{4.23}$$

Now, we use $\varphi = w_k - \bar{w}$ as test function in (4.21) to get, by assumptions (1.2), (1.3), (1.4), Hölder inequality and Lemma 2.3,

$$\begin{aligned} &\min\{\alpha, \mu\} \|w_k - \bar{w}\|^2_{W^{1,2}(\Omega)} \\ &\le \int_\Omega |b(x)||\nabla w_k||w_k - \bar{w}| \, \mathrm{d}x + \tfrac{\sigma_k}{\|u_k\|_{W^{1,2}(\Omega)}} \int_\Omega T_1(f)(w_k - \bar{w}) \, \mathrm{d}x \\ &\quad - \int_\Omega A(x)\nabla \bar{w} \nabla (w_k - \bar{w}) \, \mathrm{d}x - \mu \int_\Omega \bar{w}(w_k - \bar{w}) \, \mathrm{d}x \\ &\le C\|b(w_k - \bar{w})\|_{L^2(\Omega)} \|\nabla w_k\|_{L^2(\Omega)} + \tfrac{1}{\|u_k\|_{W^{1,2}(\Omega)}} \int_\Omega T_1(f)(w_k - \bar{w}) \, \mathrm{d}x \\ &\quad - \int_\Omega A(x)\nabla \bar{w} \nabla (w_k - \bar{w}) \, \mathrm{d}x - \mu \int_\Omega \bar{w}(w_k - \bar{w}) \, \mathrm{d}x, \end{aligned} \tag{4.24}$$

with $C = C(N)$.

Now, in view of (4.22) Lemma 2.3 applies, thus, unless to pass to a subsequence, we have

$$b\,(w_k - \bar{w}) \to 0 \quad \text{strongly in } L^2(\Omega). \tag{4.25}$$

Hence, by (4.19), (4.22) and (4.32) we have that the right hand side of previous inequality goes to zero. Hence we get, up to a subsequence,

$$w_k \to \bar{w} \quad \text{strongly in } W_0^{1,2}(\Omega). \tag{4.26}$$

By (4.19) and (4.26), passing to the limit as $k \to \infty$ in (4.21), we deduce by (1.4) that the following inequality holds

$$\int_\Omega A(x)\nabla \bar{w} \nabla \varphi \mathrm{d}x + \int_\Omega \mu \bar{w} \varphi \, \mathrm{d}x \le \int_\Omega |b(x)||\nabla \bar{w}||\varphi| \, \mathrm{d}x, \qquad \forall \varphi \in W_0^{1,2}(\Omega).$$

Using Proposition 3.1 we have that $\bar{w} \leq 0$. Repeating the same argument for $-\bar{w}$ we obtain that $\bar{w} \geq 0$ and hence $\bar{w} = 0$. Thus $\|\bar{w}\|_{W^{1,2}(\Omega)} = 0$. This gives a contradiction, in view of (4.26) and since $\|w_k\|_{W^{1,2}(\Omega)} = 1$.

Since also condition iii) holds, $\mathcal{F}$ has a fixed point $u = \mathcal{F}(u) \in W_0^{1,2}(\Omega)$ and then u solves problem $(\mathcal{P}_1)$.

Finally, if $\{u_n\}$ is a solution sequence of $(\mathcal{P}_n)$, *i.e.* $u_n = \mathcal{F}_n(u_n)$ for every $n \in \mathbb{N}$, we have that (4.2) holds. Indeed, if on the contrary,

$$\|u_n\|_{W^{1,2}(\Omega)} \to +\infty,$$

following along the lines the analogous argument used in the previous step, starting by (4.20) with $\sigma_n = \bar{\sigma} = 1$, we get a contradiction. □

Next result is an immediate consequence of estimate (4.2).

Corollary 4.2. *Assume* (1.2)–(1.6)*. Let $\{u_n\}$ be a solution sequence of $(\mathcal{P}_n)$. Then, for any $\varepsilon > 0$, there exists k_ε, independent of n, such that*

$$|\Omega_{n,k}| \leq \varepsilon, \quad \forall\, k > k_\varepsilon, \tag{4.27}$$

where

$$\Omega_{n,k} = \{x \in \Omega : |u_n(x)| > k\}. \tag{4.28}$$

Proof. Thanks to (4.2), one easily has that there exists a positive constant C, independent of n, such that

$$k^2|\Omega_{n,k}| \leq \int_{\Omega_{n,k}} |u_n(x)|^2 \,\mathrm{d}x \leq C,$$

which concludes the proof. □

Now we are in position to prove the existence result for problem (1.1).

Theorem 4.3 (Existence)**.** *Assume* (1.2)–(1.6)*. Then there exists a solution $u \in W_0^{1,2}(\Omega)$ of problem* (1.1)*.*

Proof. Let $\{u_n\}$ be a solution sequence of $(\mathcal{P}_n)$, that is

$$\int_\Omega A(x)\nabla u_n \nabla\varphi \,\mathrm{d}x + \mu \int_\Omega u_n\varphi \,\mathrm{d}x = \int_\Omega \theta_n(x) B(x,\nabla u_n)\,\varphi \,\mathrm{d}x + \int_\Omega T_n(f)\,\varphi \,\mathrm{d}x, \tag{4.29}$$

for every $\varphi \in W_0^{1,2}(\Omega)$.

In order to prove our result, we want to pass to the limit in (4.29). To this aim, observe that, by estimate (4.2), the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Then, there exists $u \in W_0^{1,2}(\Omega)$ such that, unless to pass to a subsequence not relabeled,

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,2}(\Omega). \tag{4.30}$$

On the other hand, using $u_n - u$ as test function in the variational formulation corresponding to problems $(\mathcal{P}_n)$, by assumptions (1.2), (1.3), (1.4), Hölder inequality and Lemma 2.3, we get

$$\begin{aligned} &\min\{\alpha,\mu\}\|u_n-u\|^2_{W^{1,2}(\Omega)} \\ &\leq C\|b\,(u_n-u)\|_{L^2(\Omega)}\|\nabla u_n\|_{L^2(\Omega)}+\textstyle\int_\Omega T_n(f)\,(u_n-u)\,\mathrm{d}x \\ &-\textstyle\int_\Omega A(x)\nabla u\nabla(u_n-u)\,\mathrm{d}x-\mu\int_\Omega u(u_n-u)\,\mathrm{d}x, \end{aligned} \tag{4.31}$$

with $C = C(N)$.

Thus, unless to pass to a subsequence, by the compactness result in Lemma 2.3, we have

$$b\,(u_n - u) \to 0 \quad \text{strongly in } L^2(\Omega). \tag{4.32}$$

Moreover,

$$T_n(f) \to f \text{ strongly in } L^{2_*}(\Omega). \tag{4.33}$$

Therefore, combining (4.2), (4.30), (4.32), (4.33) and (4.31), up to a subsequence, we have

$$u_n \to u \;\text{ strongly in } W_0^{1,2}(\Omega).$$

Then, arguing as done to obtain (4.15) by (4.11), by Vitali Theorem, we get

$$\int_\Omega \theta_n(x)B(x,\nabla u_n)\,\varphi\,\mathrm{d}x \to \int_\Omega B(x,\nabla u)\,\varphi\,\mathrm{d}x, \quad \forall\varphi\in W_0^{1,2}(\Omega). \tag{4.34}$$

Hence, finally, using (4.30), (4.33) and (4.34), we can pass to the limit, as $n \to +\infty$ in (4.29), obtaining the claimed existence result. □

In the sequel, will be useful the next convergence result, proved in Theorem 4.3

Remark 4.4. Assume (1.2)–(1.6). Let $\{u_n\}$ be a solution sequence of $(\mathcal{P}_n)$. Then, up to a subsequence,

$$u_n \to u \;\text{ strongly in } W_0^{1,2}(\Omega),$$

where u is a solution of problem (1.1).

5. Regularity

In this section, we show how the summability of f affects the summability of a solution u of problem (1.1). By Sobolev embeddings, the function u and the solutions u_n of the approximating problems $(\mathcal{P}_n)$ belong to $L^{2^*}(\Omega)$. To obtain further regularity, we start proving three preliminary lemmas for the u_n. Then, we show the claimed regularity results for the solution u of (1.1) obtained in Theorem 4.3.

5.1. Regularity of the solutions of the approximating problems

This first lemma concerns the uniform boundedness of the u_n.

Lemma 5.1. *Assume (1.2)–(1.5) and $f \in L^{2_*}(\Omega) \cap L^r(\Omega)$, with $r > \frac{N}{2}$. Let $\{u_n\}$ be a solution sequence of problems $(\mathcal{P}_n)$. Then, there exists a positive constant $C_r(\alpha, \mu, N, b, f)$, independent of n, such that*

$$\|u_n\|_{L^\infty(\Omega)} \le C_r\left(\alpha, \mu, N, \|b\|_{M^N(\Omega)}, \|f\|_{L^r(\Omega)}\right). \tag{5.1}$$

Proof. By (3.8) we can take $G_k(u_n)$ as test function in the variational formulations of $(\mathcal{P}_n)$. Then by (1.2), (1.3), (1.4), (3.11), (3.12), (3.13), (4.28) and Young inequality one gets

$$\alpha \int_\Omega |\nabla G_k(u_n)|^2 \,\mathrm{d}x + \mu \int_\Omega |G_k(u_n)|^2 \,\mathrm{d}x$$

$$\le \int_{\Omega_{n,k}} |T_n(b(x))||\nabla G_k(u_n)||G_k(u_n)| \,\mathrm{d}x + \int_{\Omega_{n,k}} |T_n(f)||G_k(u_n)| \,\mathrm{d}x$$

$$\le \frac{\alpha}{2} \int_{\Omega_{n,k}} |\nabla G_k(u_n)|^2 \,\mathrm{d}x + \frac{1}{2\alpha} \int_{\Omega_{n,k}} |T_n(b(x))|^2 |G_k(u_n)|^2 \,\mathrm{d}x$$

$$+ \int_{\Omega_{n,k}} |T_n(f)||G_k(u_n)| \,\mathrm{d}x.$$

Therefore, by (2.6) of Lemma 2.3 and Hölder inequality, we get

$$\frac{\alpha}{2} \int_\Omega |\nabla G_k(u_n)|^2 \,\mathrm{d}x + \mu \int_\Omega |G_k(u_n)|^2 \,\mathrm{d}x$$

$$\le \frac{1}{2\alpha} \int_{\Omega_{n,k}} |b(x)|^2 |G_k(u_n)|^2 \,\mathrm{d}x + \int_{\Omega_{n,k}} |f||G_k(u_n)| \,\mathrm{d}x$$

$$\le C\|b\|^2_{M^N(\Omega_{n,k})} \|G_k(u_n)\|^2_{W^{1,2}(\Omega)} + \|f\|_{L^{2_*}(\Omega_{n,k})} \|G_k(u_n)\|_{L^{2^*}(\Omega_{n,k})},$$

where $C = C(N, \alpha)$.

Then, by (1.3),

$$\min\left\{\frac{\alpha}{2}, \mu\right\} \|G_k(u_n)\|^2_{W_0^{1,2}(\Omega)}$$

$$\le C\|b\|^2_{M^N(\Omega_{n,k})} \|G_k(u_n)\|^2_{W^{1,2}(\Omega)} + \|f\|_{L^{2_*}(\Omega_{n,k})} \|G_k(u_n)\|_{L^{2^*}(\Omega_{n,k})}.$$

In view of (2.1) and Corollary 4.2, there exists $k_0 \in \mathbb{R}_+$, independent of n, such that

$$\|b\|^2_{M^N(\Omega_{n,k})} < \frac{min\{\frac{\alpha}{2}, \mu\}}{C} \qquad \forall k \ge k_0.$$

Thus by Sobolev inequality we get

$$\|G_k(u_n)\|^2_{L^{2^*}(\Omega)} \leq C\|f\|_{L^{2_*}(\Omega_{n,k})}\|G_k(u_n)\|_{L^{2^*}(\Omega)},$$

where $C = C(N, \alpha, \mu, \|b\|_{M^N(\Omega)})$.

Thus, by Hölder inequality we obtain

$$\|G_k(u_n)\|_{L^{2^*}(\Omega)} \leq C\|f\|_{L^{2_*}(\Omega_{n,k})} \leq C\|f\|_{L^r(\Omega)]}||\Omega_{n,k}|^{\frac{1}{2_*}-\frac{1}{r}}. \tag{5.2}$$

On the other hand, by (3.10) and (4.28), for every $h > 0$, one has

$$h|\Omega_{n,h}|^{\frac{1}{2^*}} = \Big(\int_{\Omega_{n,h}} |h|^{2^*}\Big)^{\frac{1}{2^*}} \leq \|u_n\|_{L^{2^*}(\Omega_{n,h})} \leq \|G_k(u_n)\|_{L^{2^*}(\Omega_{n,h})} + k|\Omega_{n,h}|^{\frac{1}{2^*}}.$$

Thus

$$(h-k)|\Omega_{n,h}|^{\frac{1}{2^*}} \leq \|G_k(u_n)\|_{L^{2^*}(\Omega_{n,h})}, \quad \forall h > k. \tag{5.3}$$

Putting together (5.2) and (5.3), we get

$$|\Omega_{n,h}| \leq C\frac{|\Omega_{n,k}|^{2^*\left(\frac{1}{2_*}-\frac{1}{r}\right)}}{(h-k)^{2^*}}, \quad \forall h > k \geq k_0,$$

with $C = C(N, \alpha, \mu, \|b\|_{M^N(\Omega)}, \|f\|_{L^r(\Omega)})$. Finally, since $r > \frac{N}{2}$, one has $2^*\left(\frac{1}{2_*} - \frac{1}{r}\right) > 1$. Thus, Lemma 3.4 applies and therefore there exists $d \in \mathbb{R}_+$ such that $|\Omega_{k_0+d}| = 0$. This gives (5.1). □

Now we prove a further regularity result.

Lemma 5.2. *Assume (1.2)–(1.5) and $f \in L^1(\Omega) \cap L^m(\Omega)$. If $2_* < m < \frac{N}{2}$, then, there exists a positive constant $C = C(N, \alpha, \mu, m, ||b||_{M^N(\Omega)}, ||b||_{L^2(\Omega)}, ||f||_{L^m(\Omega)})$, independent of n, such that*

$$\|u_n\|_{L^{m^{**}}(\Omega)} \leq C. \tag{5.4}$$

Proof. We divide the proof of (5.4) into three steps. For $k \in \mathbb{R}_+$, we have $u_n = T_k(u_n) + G_k(u_n)$. We shall firstly prove that the sequence $\{T_k(u_n)\}$ is bounded in $L^{m^{**}}(\Omega)$, and then we shall prove that there exists $k_0 > 0$ such that the sequence $\{G_k(u_n)\}$ is bounded in $L^{m^{**}}(\Omega)$, for every $k \geq k_0$. Finally we conclude.

Step 1: Observe that the function $|t|^{2(\lambda-1)}t$, with $\lambda > 1$, satisfies the hypotheses of Lemma 3.2, provided $|t| \leq M$, for some $M > 0$. Thus, we can take $\dfrac{|T_k(u_n)|^{2(\lambda-1)}T_k(u_n)}{2\lambda-1}$, with $\lambda = \frac{m^{**}}{2^*}$, as test function in the variational formulation of problem $(\mathcal{P}_n)$.

Then, taking into account the definition (4.28), by (1.2), (1.4), (3.16), (3.18) and Young inequality we get

$$\begin{aligned}
&\alpha\int_\Omega |T_k(u_n)|^{2(\lambda-1)}|\nabla T_k(u_n)|^2\,\mathrm{d}x+\frac{\mu}{2\lambda-1}\int_\Omega |T_k(u_n)|^{2\lambda}\mathrm{d}x\\
&\le \frac{1}{2\lambda-1}\int_\Omega |T_k(u_n)|^{2\lambda-1}|T_n(b(x))||\nabla u_n|\mathrm{d}x+\frac{1}{2\lambda-1}\int_\Omega |T_n(f)||T_k(u_n)|^{2\lambda-1}\mathrm{d}x\\
&\le \frac{1}{2\lambda-1}\int_{\Omega\setminus\Omega_{n,k}} |T_k(u_n)|^{2\lambda-1}|T_n(b(x))||\nabla T_k(u_n)|\mathrm{d}x\\
&+\frac{1}{2\lambda-1}\int_{\Omega_{n,k}} |T_k(u_n)|^{2\lambda-1}|T_n(b(x))||\nabla G_k(u_n)|\mathrm{d}x\\
&+\frac{1}{2\lambda-1}\int_\Omega |T_n(f)||T_k(u_n)|^{2\lambda-1}\mathrm{d}x=\frac{1}{2\lambda-1}(I_1+I_2+I_3).
\end{aligned}\tag{5.5}$$

We obviously have:

$$I_3\le k^{2\lambda-1}\int_\Omega |T_n(f)|\mathrm{d}x.$$

On the other hand, by Young inequality:

$$I_1\le \varepsilon\int_{\Omega\setminus\Omega_{n,k}} |T_k(u_n)|^{2(\lambda-1)}|\nabla T_k(u_n)|^2\mathrm{d}x+C(\varepsilon)\int_{\Omega\setminus\Omega_{n,k}} |b(x)|^2|T_k(u_n)|^{2\lambda}\mathrm{d}x,$$

with $\varepsilon=\varepsilon(\alpha,\lambda)$ positive real number small enough.

Then, in view of (1.3) and combining previous estimates, we get

$$\begin{aligned}
&\int_\Omega |T_k(u_n)|^{2(\lambda-1)}|\nabla T_k(u_n)|^2\,\mathrm{d}x\\
&\le C\Big\{k^{2\lambda}\int_{\Omega\setminus\Omega_{n,k}}|b(x)|^2\,\mathrm{d}x+k^{2\lambda-1}\int_\Omega |f|\,\mathrm{d}x\\
&+\int_{\Omega_{n,k}}|T_k(u_n)|^{2\lambda-1}|T_nb(x)||\nabla G_k(u_n)|\,\mathrm{d}x\Big\},
\end{aligned}\tag{5.6}$$

with $C=C(\alpha,\lambda)$. Thanks to Sobolev inequality, we obtain

$$\begin{aligned}
\left(\int_\Omega |T_k(u_n)|^{\lambda 2^*}\,\mathrm{d}x\right)^{\frac{2}{2^*}}&\le C_S\int_\Omega |\nabla(|T_k(u_n)|^\lambda)|^2\,\mathrm{d}x\\
&=C_S\int_\Omega |T_k(u_n)|^{2(\lambda-1)}|\nabla T_k(u_n)|^2\,\mathrm{d}x,
\end{aligned}\tag{5.7}$$

where C_S denotes the Sobolev constant. Taking into account (4.2), by (5.6) and (5.7) we obtain

$$\begin{aligned}\left(\int_\Omega |T_k(u_n)|^{m^{**}}\,\mathrm{d}x\right)^{\frac{2}{2^*}} &\le C\Big\{k^{2\lambda}\int_\Omega |b(x)|^2\,\mathrm{d}x + k^{2\lambda-1}\int_\Omega |f|\,\mathrm{d}x \\ &\quad + \int_{\Omega_{k,n}} |T_n(b(x))|\,|T_k(u_n)|^{2\lambda-1}|\nabla G_k(u_n)|\,\mathrm{d}x\Big\} \\ &\le C\left(k^{2\lambda}\int_\Omega |b(x)|^2\,\mathrm{d}x + k^{2\lambda-1}\int_\Omega |f|\,\mathrm{d}x + k^{2\lambda-1}\|b\|_{L^2(\Omega)}\right),\end{aligned} \tag{5.8}$$

with C positive constant independent of n.

Step 2: Observe that, since by Lemma 5.1 the function $u_n \in L^\infty(\Omega)$, Lemma 3.2 applies and then we can take $\dfrac{|G_k(u_n)|^{2(\lambda-1)}G_k(u_n)}{2\lambda-1}$ as test function in the variational formulation of problem $(\mathcal{P}_n)$. By (1.2), (1.4) and (3.11) we obtain

$$\begin{aligned}&\alpha\int_\Omega |G_k(u_n)|^{2(\lambda-1)}|\nabla G_k(u_n)|^2\mathrm{d}x + \frac{\mu}{2\lambda-1}\int_\Omega |G_k(u_n)|^{2\lambda}\,\mathrm{d}x \\ &\le \frac{1}{2\lambda-1}\left(\int_\Omega |T_n(b(x))|\nabla G_k(u_n)||G_k(u_n)|^{2\lambda-1}\mathrm{d}x \right.\\ &\quad\left. + \int_\Omega |T_n(f)||G_k(u_n)|^{2\lambda-1}\mathrm{d}x\right).\end{aligned} \tag{5.9}$$

Using Young inequality, we have

$$\begin{aligned}&\alpha\int_\Omega |G_k(u_n)|^{2(\lambda-1)}|\nabla G_k(u_n)|^2\,\mathrm{d}x + \frac{\mu}{2\lambda-1}\int_\Omega |G_k(u_n)|^{2\lambda}\,\mathrm{d}x \\ &\le \frac{1}{2\lambda-1}\left(\varepsilon\int_\Omega |\nabla G_k(u_n)|^2|G_k(u_n)|^{2(\lambda-1)}\,\mathrm{d}x\right. \\ &\left. + C(\varepsilon)\int_{\Omega_{k,n}} |T_n b(x)|^2|G_k(u_n)|^{2\lambda}\,\mathrm{d}x + \int_\Omega |T_n(f)||G_k(u_n)|^{2\lambda-1}\mathrm{d}x\right),\end{aligned} \tag{5.10}$$

with $\varepsilon = \varepsilon(\alpha,\lambda)$ positive real number small enough.

By (1.3), Lemma 2.3 and Hölder inequality, we get

$$\begin{aligned}&\||G_k(u_n)|^\lambda\|^2_{W^{1,2}(\Omega)} \\ &\le C\left(\|b\|^2_{M^N(\Omega_{n,k})}\||G_k(u_n)|^\lambda\|^2_{W^{1,2}(\Omega)}\right. \\ &\left. + \|f\|_{L^m(\Omega)}\left(\int_\Omega |G_k(u_n)|^{(2\lambda-1)m'}\mathrm{d}x\right)^{\frac{1}{m'}}\right),\end{aligned} \tag{5.11}$$

$C = C(N,\alpha,\mu,\lambda)$.

Therefore, arguing as in Lemma 5.1, using (2.1) and Corollary 4.2 we obtain that there exists $k_0 \in \mathbb{R}_+$, independent of n, such that

$$\||G_k(u_n)|^\lambda\|^2_{W^{1,2}(\Omega)} \le C\|f\|_{L^m(\Omega)}\Big(\int_\Omega |G_k(u_n)|^{(2\lambda-1)m'}\mathrm{d}x\Big)^{\frac{1}{m'}}, \quad \forall k \ge k_0, \tag{5.12}$$

with $C = C(N, \alpha, \mu, \lambda, ||b||_{M^N(\Omega)})$.

Since $2^*\lambda = (2\lambda - 1)m' = m^{**}$, by Sobolev inequality

$$\Big(\int_\Omega |G_k(u_n)|^{m^{**}}\mathrm{d}x\Big)^{\frac{2}{2^*}} \le C\|f\|_{L^m(\Omega)}\Big(\int_\Omega |G_k(u_n)|^{m^{**}}\mathrm{d}x\Big)^{\frac{1}{m'}}, \quad \forall k \ge k_0, \tag{5.13}$$

with $C = C(N, \alpha, \mu, \lambda, ||b||_{M^N(\Omega)})$.

Therefore, since $m < \frac{N}{2}$, one has $\frac{2}{2^*} - \frac{1}{m'} > 0$ and then

$$\Big(\int_\Omega |G_k(u_n)|^{m^{**}}\mathrm{d}x\Big)^{\frac{2}{2^*}-\frac{1}{m'}} \le C\|f\|_{L^m(\Omega)}, \quad \forall k \ge k_0. \tag{5.14}$$

Step 3: Putting together (5.8) with (5.14), we obtain

$$\begin{aligned}&\textstyle\int_\Omega |u_n|^{m^{**}}\mathrm{d}x\\ &\le C\Bigg[\|f\|^{\frac{2^*m'}{2m'-2^*}}_{L^m(\Omega)} + \Big(k_0^{2\lambda}\int_\Omega |b(x)|^2\,\mathrm{d}x + k_0^{2\lambda-1}\int_\Omega |f|\,\mathrm{d}x + k_0^{2\lambda-1}\|b\|_{L^2(\Omega)}\Big)^{\frac{2^*}{2}}\Bigg],\end{aligned} \tag{5.15}$$

with C independent of n. This concludes the proof of Theorem 5.2. □

In the spirit of [7] and the references therein, let us finally prove this last lemma concerning the *exponential summability* of the u_n entaling that of u.

Lemma 5.3. *Assume (1.2)–(1.5) and $f \in L^1(\Omega) \cap L^{N/2}(\Omega)$. Then, for any $\lambda > 0$, there exists a positive constant C, independent of n, such that*

$$\int_\Omega \Big(\mathrm{e}^{\lambda|u_n|} - 1\Big)^{2^*}\,\mathrm{d}x \le C. \tag{5.16}$$

As a consequence $\mathrm{e}^{|u_n|} \in M^p(\Omega)$, for all $p \ge 1$.

Proof. As in Lemma 5.2, we prove the result by means of three steps.

Step 1: Let us start proving that, for any $\lambda > 0$, there exist $k_0 > 0$ and a positive constant C, independent of n, such that one has

$$\int_\Omega \Big(\mathrm{e}^{\lambda|G_{k_0}(u_n)|} - 1\Big)^{2^*}\,\mathrm{d}x \le C. \tag{5.17}$$

In view of Lemma 3.2 and Lemma 5.1, we can take $\big(\mathrm{e}^{2\lambda|G_k(u_n)|} - 1\big)\,\mathrm{sgn}(G_k(u_n))$ as test function in the variational formulation of problem $(\mathcal{P}_n)$, getting, by (1.2) and (1.4),

$$\begin{aligned}&2\lambda\alpha\int_\Omega |\nabla G_k(u_n)|^2\,\mathrm{e}^{2\lambda|G_k(u_n)|}\,\mathrm{d}x + \mu\int_\Omega u_n\Big(\mathrm{e}^{2\lambda|G_k(u_n)|} - 1\Big)\,\mathrm{sgn}(G_k(u_n))\mathrm{d}x\\ &\le \int_{\Omega_{n,k}} |b(x)||\nabla G_k(u_n)|\Big(\mathrm{e}^{2\lambda|G_k(u_n)|} - 1\Big)\,\mathrm{d}x + \int_\Omega |f|\Big(\mathrm{e}^{2\lambda|G_k(u_n)|} - 1\Big)\,\mathrm{d}x.\end{aligned}$$

Recalling that, for any $t \geq 0$ and any $D > 1$, the following obvious inequality holds

$$|t^2 - 1| \leq D(t-1)^2 + \frac{1}{D-1},$$

and using (1.3), Young and Hölder inequalities, one has

$$\begin{aligned} &2\lambda\alpha \int_\Omega |\nabla G_k(u_n)|^2 \,\mathrm{e}^{2\lambda|G_k(u_n)|}\,\mathrm{d}x \\ &\leq D \int_{\Omega_{n,k}} |b(x)||\nabla G_k(u_n)| \left(\mathrm{e}^{\lambda|G_k(u_n)|} - 1\right)^2 \,\mathrm{d}x \\ &+\frac{1}{D-1} \int_{\Omega_{n,k}} |b(x)||\nabla G_k(u_n)|\,\mathrm{d}x \\ &+D \int_{\Omega_{n,k}} |f| \left(\mathrm{e}^{\lambda|G_k(u_n)|} - 1\right)^2 \,\mathrm{d}x + \frac{1}{D-1}\int_\Omega |f|\,\mathrm{d}x \\ &\leq C_{\alpha\lambda D} \int_{\Omega_{n,k}} |b(x)|^2 \left(\mathrm{e}^{\lambda|G_k(u_n)|} - 1\right)^2 \,\mathrm{d}x \\ &+\lambda\alpha \int_{\Omega_{n,k}} |\nabla G_k(u_n)|^2 \left(\mathrm{e}^{\lambda|G_k(u_n)|} - 1\right)^2 \,\mathrm{d}x \\ &+\frac{1}{D-1}\, \|b(x)\|_{L^2(\Omega)}\|\nabla u_n\|_{L^2(\Omega)} \\ &+D\|f\|_{L^{N/2}(\Omega_{n,k})}\|\mathrm{e}^{\lambda|G_k(u_n)|} - 1\|^2_{L^{2^*}(\Omega)} + \frac{1}{D-1}\|f\|_{L^1(\Omega)}. \end{aligned} \tag{5.18}$$

Observe that

$$\int_\Omega \left|\nabla\left(\mathrm{e}^{\lambda|G_k(u_n)|} - 1\right)\right|^2 \,\mathrm{d}x = \lambda^2 \int_\Omega |\nabla G_k(u_n)|^2 \,\mathrm{e}^{2\lambda|G_k(u_n)|}\,\mathrm{d}x. \tag{5.19}$$

From now on, C denotes a different positive constant, independent of n, that can vary from line to line.

By Lemma 2.3 we have

$$\begin{aligned} &\int_\Omega \left|\nabla\left(\mathrm{e}^{\lambda|G_k(u_n)|} - 1\right)\right|^2 \,\mathrm{d}x \\ &\leq C\|b(x)\|^2_{M^N(\Omega_{n,k})} \left(\|\nabla(\mathrm{e}^{\lambda|G_k(u_n)|} - 1)\|^2_{L^2(\Omega)} + \|\mathrm{e}^{\lambda|G_k(u_n)|} - 1\|^2_{L^2(\Omega)}\right) \\ &+C\, \|b(x)\|_{L^2(\Omega)}\|\nabla u_n\|_{L^2(\Omega)} \\ &+C\|f\|_{L^{N/2}(\Omega_{n,k})}\|\mathrm{e}^{\lambda|G_k(u_n)|} - 1\|^2_{L^{2^*}(\Omega)} + C\|f\|_{L^1(\Omega)}. \end{aligned} \tag{5.20}$$

Thus, in view of (2.1) and Corollary 4.2 there exists k_0, independent of n, such that using Sobolev embeddings and Hölder inequality, we get

$$\begin{aligned} &\|\mathrm{e}^{\lambda|G_{k_0}(u_n)|}-1\|^2_{L^{2^*}(\Omega)} \\ &\le C|\Omega_{n,k_0}|^{1-\frac{2}{2^*}}\|b(x)\|^2_{M^N(\Omega)}\|\mathrm{e}^{\lambda|G_{k_0}(u_n)|}-1\|^2_{L^{2^*}(\Omega)} \\ &+C\ \|b(x)\|_{L^2(\Omega)}\|\nabla u_n\|_{L^2(\Omega)} \\ &+C\|f\|_{L^{N/2}(\Omega_{n,k_0})}\|\mathrm{e}^{\lambda|G_{k_0}(u_n)|}-1\|^2_{L^{2^*}(\Omega)}+C\|f\|_{L^1(\Omega)}. \end{aligned} \tag{5.21}$$

Hence, by (4.2) and using again Corollary 4.2, unless to enlarge the value of k_0 we have

$$\|\mathrm{e}^{\lambda|G_{k_0}(u_n)|}-1\|^2_{L^{2^*}(\Omega)} \le C\left(\|b(x)\|_{L^2(\Omega)}+\|f\|_{L^1(\Omega)}\right), \tag{5.22}$$

with C independent of n.

Step 2: Now, let us show that, for any $\lambda>0$ and any $k>0$ there exists a positive constant C, independent of n, such that

$$\int_\Omega \left(\mathrm{e}^{\lambda|T_k(u_n)|}-1\right)^{2^*}\ \mathrm{d}x \le C. \tag{5.23}$$

Again, by Lemma 3.2 we can choose $\left(\mathrm{e}^{2\lambda|T_k(u_n)|}-1\right)\mathrm{sgn}(T_k(u_n))$ as test function in the variational formulation of $(\mathcal{P}_n)$, obtaining, by (1.2) and (1.4),

$$\begin{aligned} &2\lambda\alpha\int_\Omega |\nabla T_k(u_n)|^2\,\mathrm{e}^{2\lambda|T_k(u_n)|}\ \mathrm{d}x+\mu\int_\Omega u_n\left(\mathrm{e}^{2\lambda|T_k(u_n)|}-1\right)\mathrm{sgn}(T_k(u_n))\mathrm{d}x \\ &\le \int_\Omega |b(x)||\nabla u_n|\left(\mathrm{e}^{2\lambda|T_k(u_n)|}-1\right)\ \mathrm{d}x+\int_\Omega |f|\left(\mathrm{e}^{2\lambda|T_k(u_n)|}-1\right)\ \mathrm{d}x. \end{aligned}$$

Therefore, by the analogous of (5.19) and Sobolev embeddings, we get

$$\|\mathrm{e}^{\lambda|T_k(u_n)|}-1\|^2_{L^{2^*}(\Omega)} \le C(\mathrm{e}^{2\lambda k}-1)\left(\|b(x)\|_{L^2(\Omega)}\|\nabla u_n\|_{L^2(\Omega)}+\|f\|_{L^1(\Omega)}\right), \tag{5.24}$$

that gives (5.23), in view of (4.2).

Step 3: We have

$$\begin{aligned} &\int_\Omega \left(\mathrm{e}^{\lambda|u_n|}-1\right)^{2^*}\ \mathrm{d}x \le \int_\Omega \left(\mathrm{e}^{\lambda(|T_{k_0}(u_n)|+|G_{k_0}(u_n)|)}-1\right)^{2^*}\ \mathrm{d}x \\ &\le \int_\Omega \left[\mathrm{e}^{\lambda k_0}\left(\mathrm{e}^{\lambda|G_{k_0}(u_n)|}-1\right)+\left(\mathrm{e}^{\lambda|T_{k_0}(u_n)|}-1\right)\right]^{2^*}\ \mathrm{d}x \\ &\le C\left(\mathrm{e}^{2^*\lambda k_0}\int_\Omega \left(\mathrm{e}^{\lambda|G_{k_0}(u_n)|}-1\right)^{2^*}\ \mathrm{d}x+\int_\Omega \left(\mathrm{e}^{\lambda|T_{k_0}(u_n)|}-1\right)^{2^*}\ \mathrm{d}x\right). \end{aligned} \tag{5.25}$$

This, together with (5.17) and (5.23), gives (5.16). □

5.2. Regularity results

Here, we prove the claimed regularity results for a solution u of (1.1) obtained in Theorem 4.3.

Theorem 5.4. *Assume* (1.2)–(1.5). *Then,*

1. *if* $f \in L^1(\Omega) \cap L^m(\Omega)$, $2_* < m < \frac{N}{2}$, *then there exists a weak solution* $u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$ *of problem (1.1).*
2. *if* $f \in L^{2_*}(\Omega) \cap L^m(\Omega)$, $m > \frac{N}{2}$, *then there exists a weak solution* $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *of problem (1.1).*
3. *if* $f \in L^1(\Omega) \cap L^{N/2}(\Omega)$, *then there exists a weak solution* $u \in W_0^{1,2}(\Omega)$ *of problem (1.1) such that* $(\mathrm{e}^{\lambda|u|} - 1) \in L^{2^*}(\Omega)$ *for any* $\lambda > 0$. *Moreover,* $\mathrm{e}^{|u|} \in M^p(\Omega)$, *for all* $p \geq 1$.

Proof. In view of Remark 4.4, in the three case above we always have that, up to a subsequence, not relabeled,

$$u_n \to u \text{ strongly in } W_0^{1,2}(\Omega). \tag{5.26}$$

Thus, up to a second subsequence,

$$u_n \to u \text{ a.e. in } \Omega. \tag{5.27}$$

Now, if $f \in L^1(\Omega) \cap L^m(\Omega)$, $2_* < m < \frac{N}{2}$, by Lemma 5.2, from the subsequence $\{u_n\}$ of (5.26) we can extract another subsequence, such that

$$u_n \rightharpoonup u' \text{ weakly in } L^{m^{**}}(\Omega). \tag{5.28}$$

Putting together (5.26) and (5.28), we obtain that $u' = u \in W_0^{1,2}(\Omega) \cap L^{m^{**}}(\Omega)$.
Then, by (5.4)

$$\|u\|_{L^{m^{**}}(\Omega)} \leq \liminf_{n\to+\infty} \|u_n\|_{L^{m^{**}}(\Omega)} \leq C. \tag{5.29}$$

If $f \in L^{2_*}(\Omega) \cap L^m(\Omega)$, $m > \frac{N}{2}$, by (5.1) and (5.27), we conclude that

$$\|u\|_{L^\infty(\Omega)} \leq C. \tag{5.30}$$

Finally, if $f \in L^1(\Omega) \cap L^{N/2}(\Omega)$, by (5.16), (5.27) and Fatou Lemma we get the claimed regularity result. □

6. Uniqueness

In this section, under the additional assumption

$$|B(x,\xi) - B(x,\eta)| \leq |b(x)||\xi - \eta|, \tag{6.1}$$

we prove the following uniqueness result

Theorem 6.1 (Uniqueness). *Assume* (1.2)–(1.6). *Let* u_1, u_2 *be solutions of* (1.1). *Then, we have* $u_1 = u_2$ *almost everywhere in* Ω.

Proof. Set $w = u_1 - u_2$. Then, by (1.1) written for $u = u_1$ and $u = u_2$ and subtracting, by (6.1) we get that w satisfies the following inequality

$$\int_\Omega A(x)\nabla w \nabla\varphi \,\mathrm{d}x + \mu \int_\Omega w\varphi \,\mathrm{d}x \le \int_\Omega |b(x)|\, |\nabla w|\, |\varphi| \,\mathrm{d}x \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

Applying Proposition 3.1 we have that $w \le 0$ a.e. in Ω. Repeating the same argument for $-w$ we get $w \ge 0$ a.e. in Ω. Thus $w = 0$ and hence $u_1 = u_2$ a.e. in Ω.

□

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