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A CLASS OF ONE DIMENSIONAL PERIODIC MICROSTRUCTURES EXHIBITING EFFECTIVE TIMOSHENKO BEAM BEHAVIOR

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Abstract. We study, from a variational viewpoint, the asymptotic behavior of a planar beam with a periodic wavy shape when the amplitude and the wavelength of the shape tend to zero. We assume that the beam behaves, at the microscopic level, as a compressible Euler–Bernoulli beam and that the material properties have the same period as the geometry. We allow for distributed or concentrated bending compliance and for a non-quadratic extensional energy. The macroscopic Γ-limit that we obtain corresponds to a non-linear model of Timoshenko type.

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1. Introduction

Statics of continuum materials is generally driven by the minimization of energy. The elastic part of the energy represents the energy needed to deform the material from its configuration at rest: it is a functional of the displacement field. In standard elasticity (linear or not), the elastic energy is assumed to be represented by an energy density depending on the gradient of the displacement field. Different models, often called "generalized models", have been considered: for instance strain-gradient or second-gradient models allow the energy density to depend also on the second gradient of the displacement field. Another possibility is to introduce extra fields and to let the energy depend also on these fields. Equilibrium can theoretically be computed by minimizing first the elastic energy with respect these extra fields, getting then a non-local functional of the displacement field, and in a second step by minimizing the energy with respect to it. Depending on the context, these extra fields called "extra kinematic fields" have different tensorial nature and different physical interpretation: they can be similar to a displacement field in bi-continua models, to a field of rotations in micropolar models, to a strain field in micromorphic models but there is no limit in the variety of fields which can be used. Generalized models are generally introduced in an axiomatic way and their physical ground is often questioned. The usual way of justifying them is to say that they describe at a macroscopic level a structure which has a standard behavior but is very heterogeneous at a microscopic level: in other words, that they result from classical elasticity through a homogenization procedure. That is why rigorous homogenization results of this type are important [1–4, 10, 21].

Keywords and phrases: Γ-convergence, homogenization, Euler–Bernoulli beam, Timoshenko beam.

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The homogenization result that we establish in this paper is not completely of this type as we focus on a one-dimensional object (a beam) and consider that its behavior at the microscopic level is described by the Euler–Bernoulli model: even though very classical, the Euler–Bernoulli model of beams already involves the second derivative of the displacement field. The homogenized energy that we obtain is of Timoshenko type: it involves an extra kinematic field similar to a field of rotations. It is remarkable that our result does not need any intricate assumption on the structure at the microscopic level: we simply assume that the shape of the beam at rest is rapidly oscillating.

A Euler–Bernoulli beam is a one-dimensional object (a curve) whose elastic deformation energy varies when bent or elongated with respect to its configuration at rest. We limit here our attention to planar beams and, in order to lighten notation, we identify the plane $\mathbb{R}^2$ with $\mathbb{C}$. Hence a beam can be represented, at rest, by a curvilinear parametrization $[0, L] \to \mathbb{C},\ s \mapsto \chi^0(s)$ (using a suitable choice of the length unit, we assume without loss of generality that $L = 1$). Let $\chi(s) \in \mathbb{C}$ be the position of the corresponding point $\chi^0(s)$ of the beam once deformed. Note that s is no more a curvilinear abscissa for the deformed beam : the quantity $\rho(s) := |\chi'(s)|$ is a measure of elongation. At point $\chi^0(s)$, the unit tangent vector to the beam at rest is represented by $\chi'^0(s)$ while the unit tangent vector to the deformed beam at the corresponding point is represented by $\rho^{-1}(s)\chi'(s)$. We can introduce $\theta^0(s)$ and $\theta(s)$ in order to represent the tangent vectors by $\chi^{0\prime}(s) = \mathrm{e}^{i\theta^0(s)}$ and $\chi'(s) = \rho(s)\,\mathrm{e}^{i\theta(s)}$. The curvature at rest and once deformed is respectively $\theta^{0\prime}(s)$ and $\theta'(s)$. The difference, $\theta' - \theta^{0\prime}$, is a measure of bending. In the Euler–Bernoulli model the elastic deformation energy density is the sum of a non-negative convex function of ρ which vanishes when $\rho = 1$, and of a non-negative function of $\theta' - \theta^{0\prime}$ which vanishes when $\theta' - \theta^{0\prime} = 0$. Note that the special simple case of the inextensible Euler–Bernoulli beam model, in which ρ is fixed ($\rho = 1$) can be treated in our framework by allowing the extensional energy density to take the value $+\infty$ as soon as $\rho \neq 1$.

The homogenized model that we obtain is a non-linear Timoshenko model. In this more sophisticated model an extra independent angular kinematic parameter ϕ is introduced. In addition to the energy associated to elongation, the elastic energy contains a term related to the derivative of ϕ with respect to the reference abscissa s and a term coupling ϕ and $\theta - \theta^0$. When the coupling stiffness has a very high value, then ϕ can be expressed as a function of $\theta - \theta^0$ and its derivative with respect to the reference abscissa s is a function of the bending measure $\theta' - \theta^{0\prime} = 0$. Hence, the Euler–Bernoulli beam model can be retrieved as a limit case of the Timoshenko model. When the coupling stiffness is finite, the model is more difficult to deal with as ϕ cannot be eliminated from the energy without using a Green function representation: in terms of χ only, the model becomes non-local. The reader can refer, for instance, to [12] for a more complete review of different extensible beam models.

In the majority of sources in the scientific literature in structural mechanics, the Timoshenko model is used to describe the macroscopic behavior of structures which are highly "contrasted" at the microscopic level, *i.e.* microstructures either exhibiting high directional or space variability in stiffness, or holes, or an intricate microscopic design. The new kinematic parameter is often considered as the "rotation of the beam's cross-section". We think better to avoid such an interpretation in terms of kinematics of a beam's cross-section. Indeed, while the reader could recognize in Figure 2 such a rotation, the "rotation of a beam's cross-section" is generally not precisely defined in presence of an intricate microstructure.

In a recent paper [8], it has been formally shown that a special periodic one-dimensional elastic curve in $\mathbb{R}^2$, described at the microscopic level by the Euler–Bernoulli beam model, can lead at the macroscopic level (*i.e.* once homogenized through formal asymptotic expansions) to a Timoshenko beam model. The microstructure considered there is a mechanism obtained by the repetition of an elementary pattern in which undeformable elements are interconnected by purely extensional or rotational springs. Specifically, the extensional springs do not accumulate any bending energy and the rotational springs make the pivot joints between extensional springs and undeformable elements not perfect. Such a microstructure, first introduced in [7], owing to its peculiar chiral geometry, has been named *duoskelion* (*i.e.* two-legged). The resulting behavior was a rather peculiar Timoshenko model in which extension was possible but not compression: a type of mechanical diode.

We remark that beam lattice structures based on a chiral pattern similar to the one employed in the duoskelion have been studied from both the experimental [15, 17, 18] and theoretical [11, 19] viewpoints in the literature.

The main difference between such structures and the duoskelion lies in the utilization of different mechanical elastic interactions. Duoskelion structures have been conceived in order to exhibit at the macroscopic level specific unconventional effects: axial-transverse coupling and co-existence of an extremely strong stiffness in compression and a relatively small stiffness in extension. In [7] and [23], duoskelion structures have been studied only numerically by using a Lagrangian intrinsically discrete model. In [8], by selecting a suitable scaling law for micro stiffnesses at the microscopic level, the Timoshenko deformation energy of a one-dimensional continuum describing the mechanical behavior of duoskelion structures at the macroscopic level is deduced via a formal asymptotic homogenization procedure.

The first goal of the present paper is to give a rigorous proof via Γ-convergence of the homogenization result obtained heuristically in [8]. The second goal is to find the general features of duoskelion structures which are essential for obtaining the observed exotic diode-like behavior and, hence, to generalize the results obtained in [8]. We still consider periodic one-dimensional elastic curves with a periodic Euler–Bernoulli behavior. We allow for very different periodic shapes (see Fig. 1). The bending compliance can be, like in [8], periodically concentrated at pivot-joints but it can also be distributed all along the beam. The extensional behavior can be non-linear, provided that some compactness condition on deformation energy is ensured. It is shown that the diode-like behavior is obtained when the extensional compliance is concentrated on the part of the curve that is orthogonal to the mean direction.

The present paper is organized as follows. The geometrical assumptions are made precise in Section 2, while the material assumptions are made precise in Section 3. The main result is described in Section 4 and Section 5 is devoted to its proof. For the sake of clarity, the proof has been split into three parts. In the first two ones, the asymptotic behaviors of bending and extension energies are studied separately, leading to two independent Γ-convergence results. The third part deals with the constraint $\chi' = \rho\, \mathrm{e}^{i\theta}$ which couples extension and bending at the microscopic level and leads at the limit to the Timoshenko model. Actually, it is not easy to recognize a Timoshenko model in our main result. That is why we reformulate the limit energy in Section 6, where the effective energy density is written in terms of a cell problem. This cell problem can be analytically solved in some particular cases. We study three such cases. One of them, in subsection 6.2, shows that, albeit the material behavior is assumed to be linear at the microscopic level, it does not remain linear after homogenization. As a consequence, a bifurcation phenomenon can arise during a simple extension test. Another case, studied in Subsection 6.3, corresponds to the problem that is dealt with in [8]. We recover the results obtained there. In the last subsection, we study the applicability range, from the physical point of view, of our result. In particular we show that, even if one assumes – which seems very natural from the physical point of view – that the material properties at the microscopic level prevent the extension to reach zero and become negative, this demand is not preserved at the macroscopic level (*i.e.* after homogenization).

2. Description of the curve at rest

Let θ^0 be a 1-periodic function on $\mathbb{R}$ whose variations are locally bounded. Let $Y := [0,1[$ be a representative period of θ^0. For $n \in \mathbb{N}^*$, let us introduce the $\frac{1}{n}$-periodic function $\theta^0_n(t) := \theta^0(nt)$. For any $t \in \Omega := [0,1]$, let us define the complex

$$\chi^0_n(t) := \int_0^t \mathrm{e}^{i\theta^0_n(s)}\, \mathrm{d}s.$$

Identifying $\mathbb{C}$ with the two-dimensional real vector space $\mathbb{R}^2$, the function χ^0_n is the curvilinear description of a periodic plane curve with periodicity vector $\frac{1}{n}\ell$ with

$$\ell := \int_Y \mathrm{e}^{i\theta^0(s)}\, \mathrm{d}s.$$

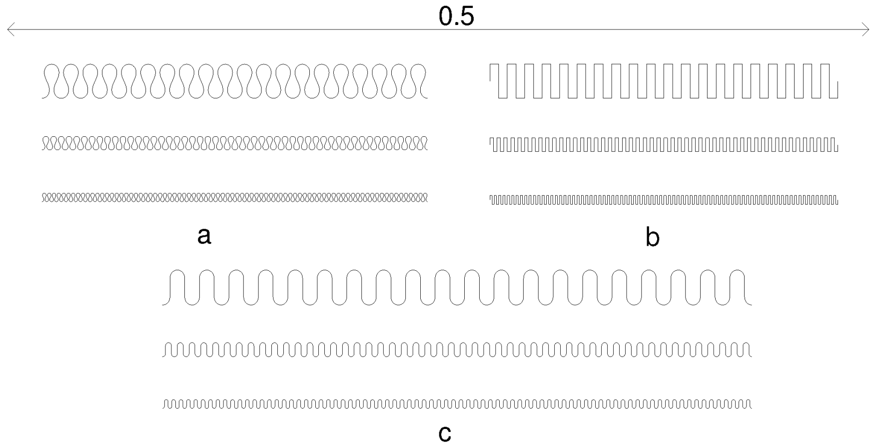

FIGURE 1. Examples of the curve χ_n^0 for increasing values of n (20, 50, 80). The curves (a) on the top left correspond to the choice $\theta^0(t) = 2\sin(2\pi t)$. The curves (c) at the bottom correspond to the same, but truncated, choice: $\theta^0(t) = \max(\min(2\sin(2\pi t), \pi/2), -\pi/2)$. For the curves (b), at the top right, the function θ^0 is chosen to be piece-wise constant, taking successively values 0, $\frac{\pi}{2}$, 0 and $-\frac{\pi}{2}$. Note that all the curves have the same length 1, but that the different local geometries lead to very different (and much smaller) apparent lengths at the limit.

Using a suitable basis of the two-dimensional real vector space, we can assume without loss of generality that $\ell \in \mathbb{R}$.

Even if the assumption is not needed from the mathematical point of view, it is better, from the physical point of view to restrict our attention to simple curves: we assume that, for all $t_1 < t_2$, $\chi_n^0(t_2) \neq \chi_n^0(t_1)$ or equivalently

$$\forall t_1 < t_2, \int_{t_1}^{t_2} \mathrm{e}^{i\theta^0(s)}\,\mathrm{d}s \neq 0.$$

It is easy to check that, when n tends to infinity, χ_n^0 converges uniformly to the linear function $t \mapsto \ell\, t$. This result can be interpreted by saying that, "from the macroscopic point of view", the elastic beam appears at rest as a straight beam with length ℓ. This is illustrated in Figure 1: when n increases, the beams described by χ_n^0 become close to a straight but shorter line.

3. KINEMATICS AND ELASTIC ENERGY

When the beam is deformed, each point $\chi_n^0(t)$ takes a new position $\chi_n(t)$. We associate to this deformation, quantities $\rho_n(t)$ and $\theta_n(t))$ such that

$$\chi_n(t) = \chi_n(0) + \int_0^t \rho_n(s)\mathrm{e}^{i\theta_n(s)}\,\mathrm{d}s \quad \text{that is} \quad \chi_n'(t) = \rho_n(t)\mathrm{e}^{i\theta_n(t)}\,. \tag{3.1}$$

Note that the deformed configuration is in general no more periodic and that t is no more a curvilinear abscissa.

At rest, the geometry of the beams that we just described is $\frac{1}{n}$-periodic with respect to the curvilinear abscissa. We assume that the material behavior is also $\frac{1}{n}$-periodic. Moreover we assume that the elastic energy associated to a deformation contains two uncoupled parts related to the elongation of the beam and to its bending, respectively.

Let us first focus on the energy associated to the elongation of the beam. The beam can be made of a $\frac{1}{n}$-periodic succession of different materials presenting different extensional stiffnesses. That is why we assume that the extensional energy density is described by a measurable function $(y,\rho)\mapsto f(y,\rho)$ defined on $\mathbb{R}\times\mathbb{R}$, 1-periodic with respect to the first variable, convex and lower semi-continuous with respect to the second variable and vanishing at rest: $\forall y\in\mathbb{R},\ f(y,1)=0$. It takes values in $[0,+\infty]$ (the value $+\infty$ allowing to take into account the possible presence of inextensible parts of the beam). We also assume that this energy is uniformly coercive by introducing

$$\underline{f}(\rho) := \inf_{y\in\mathbb{R}} f(y,\rho) \tag{3.2}$$

and by assuming[1] that

$$\lim_{|\rho|\to\infty} \frac{\underline{f}(\rho)}{|\rho|} = +\infty. \tag{3.3}$$

The extensional elastic energy of the beam thus reads

$$E_n^{ext}(\rho_n) := \int_\Omega f(nt,\rho_n(t))\,\mathrm{d}t, \tag{3.4}$$

We define the functional E_n^{ext} on the whole space $L^1(\Omega)$ extending it by $+\infty$ when needed.

Let us now focus on the energy associated to the bending of the beam. Bending is associated to the derivative (in the sense of distributions) ϕ_n' of the difference

$$\phi_n := \theta_n - \theta_n^0. \tag{3.5}$$

We assume that the bending compliance is described by a 1-periodic non-negative measure ν or, more specifically, by $\nu_n(\mathrm{d}t)=\frac{1}{n}\nu(n\,\mathrm{d}t)$ the $\frac{1}{n}$-periodic measure resulting from the scaling of ν. We define the bending energy on the space $BV(\Omega)$: it is finite when ϕ_n' is a measure absolutely continuous with respect to ν_n, with a square integrable density and reads

$$E_n^{bend}(\phi_n) := \begin{cases} \int_\Omega \left(\frac{d(\phi_n')}{\mathrm{d}\nu_n}(t)\right)^2 \nu_n(\mathrm{d}t) & \text{if } \phi_n' << \nu_n \text{ and } \frac{d(\phi_n')}{\mathrm{d}\nu_n}\in \mathrm{L}^2_{\nu_n}, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.6}$$

We assume that $|\nu| := \nu(Y)$ is bounded. We also assume that $\nu(\{0\})=0$. This assumption is not mandatory, but it simplifies the presentation: first, as it forbids any jump of ϕ_n at $x=0$, it makes less ambiguous the embedding condition that we will introduce later on; moreover, it implies the identity $|\nu_n| := \nu_n(\Omega)=\nu(Y)=|\nu|$, which otherwise would be true only asymptotically.

Using a compliance measure for describing the bending energy may seem to the reader unnecessarily technical. Indeed, one can in a first time consider the simple case of a beam described by a smooth positive (and $\frac{1}{n}$-periodic) bending stiffness $b(nt)$: this case is simply obtained by setting $\nu(\mathrm{d}t)=\frac{1}{b(t)}\,\mathrm{d}t$. However, the description in terms of a compliance measure presents two advantages. First, it allows to consider a beam which is rigid with respect to bending in some parts of every unit-cells: it is enough to decide that the measure ν vanishes on these parts. Second, it allows to consider a beam containing pivot joints with (quadratic) rotational springs . Indeed the possible presence of a Dirac addend $c\delta_y$ in the measure ν allows for jumps $[[\theta_n-\theta_n^0]]_{ky}$ of $(\theta_n-\theta_n^0)$ at points

[1] One may reinforce this hypothesis, by assuming that $\underline{f}$ tends to infinity when ρ tends to zero, forbidding thus ρ_n to vanish and to become negative. This assumption, which is natural from the physical point of view, is not needed from the mathematical point of view.

ky, for $k \in \{1 \dots n\}$, and associates to these jumps the energy $\sum_{k=1}^{n} \frac{1}{c}[[\theta_n - \theta_n^0]]_{ky}^2$. Such a modeling is possible even if the beam is not smooth at rest and the possible pivot joints may coincide, or not, with discontinuities of θ_0. Describing the bending compliance in terms of a measure is thus mandatory for encompassing the case treated in [8].

In order to consider only well-posed equilibrium problems, we assume that the considered beam is embedded at its left-hand extremity point:

$$\chi_n(0) = \chi_n^0(0) = 0 \quad \text{and} \quad \theta_n(0^+) = \theta_n^0(0^+). \tag{3.7}$$

We integrate this constraint in the definition of E_n^{bend} by setting $E_n^{bend}(\phi_n) = +\infty$ whenever $\phi_n(0) \neq 0$. Note that, owing to our assumption $\nu(\{0\}) = 0$, this constraint can be understood as a constraint on the standard (inner) trace of ϕ_n as a function of $\mathrm{BV}(\Omega)$.

We cannot introduce the total elastic energy as $E_n^{bend}(\phi_n) + E_n^{ext}(\rho_n)$. Indeed, the pair (ρ_n, θ_n), and thus also the pair (ρ_n, ϕ_n), are not uniquely defined by (3.1). A finite energy for the beam only implies that ϕ_n is bounded in $BV(\Omega, \mathbb{R})$ and jumps of ϕ_n by multiples of 2π are possible. That is why we set[2]

$$E_n(\chi_n) := \inf_{(\rho_n, \phi_n)} \big(E_n^{bend}(\phi_n) + E_n^{ext}(\rho_n)\big) \tag{3.8}$$

where the infimum is taken over all pairs (ρ_n, ϕ_n) which satisfy $\chi_n' = \rho_n \mathrm{e}^{i(\theta_n^0 + \phi_n)}$ in the sense of distributions.

When the beam is submitted to some external force field, the equilibrium displacement minimizes the total energy, that is the sum of the elastic energy and the potential of external forces[3]

$$-\int_\Omega \chi_n(t) \cdot g(\mathrm{d}t) \tag{3.9}$$

where g is a complex valued measure representing the external forces. Note that this potential can be written in terms of (ρ_n, θ_n) by using (3.1), introducing $G(t) := g([t, 1])$ and integrating by parts. We get

$$V(\chi_n) := \int_\Omega \rho_n(t)\, \mathrm{e}^{i\theta_n(t)} \cdot G(t)\, \mathrm{d}t = \int_\Omega \rho_n(t)\, \mathrm{e}^{i(\theta_n^0(t) + \phi_n(t))} \cdot G(t)\, \mathrm{d}t. \tag{3.10}$$

Hence the following minimization, on the space $C_l(\overline{\Omega}, \mathbb{C})$ of continuous functions χ_n vanishing at 0,

$$\inf_{\chi_n} \Big\{ E_n(\chi_n) + V(\chi_n) \Big\} \tag{3.11}$$

is equivalent to the minimization over all pairs $(\rho_n, \theta_n) \in \mathrm{L}^1(\Omega, \mathbb{R}) \times \mathrm{BV}(\Omega, \mathbb{R})$,

$$\inf_{(\rho_n, \phi_n)} \Big\{ E_n^{bend}(\phi_n) + E_n^{ext}(\rho_n) + \int_\Omega \rho_n(t)\, \mathrm{e}^{i(\theta_n^0(t) + \phi_n(t))} \cdot G(t)\, \mathrm{d}t \Big\}, \tag{3.12}$$

their respective solutions being related by the equation $\chi_n' = \rho_n \mathrm{e}^{i(\theta_n^0 + \phi_n)}$ in the sense of distributions.

[2]This definition is not purely technical. It has physical implications: a beam containing a pivot joint and making complete turns around it, cannot be modeled in our framework.

[3]Note that, in the identification of $\mathbb{C}$ with $\mathbb{R}^2$, the inner product $u \cdot v$ corresponds to the real part $Re(u\bar{v})$ of the product $u\bar{v}$.

4. Main result

Our main result establishes the effective behavior of the periodic beams we described in the previous section. The following theorem states that the sequence (E_n) Γ-converges for the topology of uniform convergence to the functional E defined, for any $\chi \in C_l(\overline{\Omega}, \mathbb{C})$ (*i.e.* $\chi \in C(\overline{\Omega})$ with $\chi(0) = 0$), by

$$E(\chi) := \inf_{(\bar{\rho},\phi)} \int_\Omega \left(\frac{1}{|\nu|} (\phi'(t))^2 + \int_Y f(y, \bar{\rho}(t,y))\,\mathrm{d}y \right) \mathrm{d}t \tag{4.1}$$

where the infimum is taken over all pairs $(\bar{\rho}, \phi)$ in $L^1(\Omega \times Y) \times W^{1,2}(\Omega)$ satisfying the constraints

$$\phi(0) = 0 \quad \text{and} \quad \forall t \in \Omega, \quad \mathrm{e}^{i\phi(t)} \int_Y \bar{\rho}(t,y) \mathrm{e}^{i\theta^0(y)}\,\mathrm{d}y = \chi'(t). \tag{4.2}$$

Theorem 4.1. *We have:*

1. *any sequence (χ_n) with bounded energy ($E_n(\chi_n) < M < +\infty$) is relatively compact in $C(\bar{\Omega})$;*
2. *for any sequence (χ_n) converging uniformly to χ,*

$$\liminf_n E_n(\chi_n) \geq E(\chi);$$

3. *for any χ with finite energy ($E(\chi) < +\infty$), there exists an approximating sequence (χ_n) converging uniformly to χ and satisfying*

$$\limsup_n E_n(\chi_n) \leq E(\chi).$$

Standard properties of Γ-convergence [9] imply that the previous convergence result remains true if we add to both E_n and E the same continuous functional. In particular one can take into account external forces by adding the potential V defined in (3.9). They also state that any sequence of minima of $E_n + V$ converges to a minimum of $E + V$. Therefore, E is actually the effective elastic energy, appropriate for describing the limit of the equilibrium states of the beam. Expression (4.1) is difficult to grasp from the mechanical point of view. To this end, the reader should refer to Section 6, where it is proved that it corresponds to a Timoshenko beam model and several examples are provided. Next section will instead be devoted to the proof of Theorem 4.1.

5. Asymptotic analysis

5.1. Convergence of bending energy

We first establish the Γ-convergence, for the strong topology of $L^\infty(\Omega)$, of the sequence (E_n^{bend}) of bending energies to the functional E^{bend} defined on $L^\infty(\Omega)$ by setting

$$E^{bend}(\phi) := |\nu|^{-1} \int_\Omega (\phi'(t))^2\,\mathrm{d}t$$

if $\phi \in W_l^{1,2}(\Omega)$, $E^{bend}(\phi) := +\infty$ otherwise. Here $W_l^{1,2}(\Omega)$ stands for the set of functions in $W^{1,2}(\Omega)$ satisfying the left boundary condition $\phi(0) = 0$.

Theorem 5.1. *Γ-convergence of bending energies.*

1. *For any sequence (ϕ_n) with bounded bending energy ($E_n^{bend}(\phi_n) \leq M < +\infty$), there exists $\phi \in W_l^{1,2}(\Omega)$ such that, up to a subsequence,*

$$\phi_n'(\mathrm{d}t) \overset{*}{\rightharpoonup} \phi'(t)\mathrm{d}t \qquad \text{and} \qquad \|\phi - \phi_n\|_{L^\infty(\Omega)} \to 0.$$

2. *For any sequence (ϕ_n) converging to ϕ for the $L^\infty(\Omega)$-norm, we have*

$$\liminf_n E_n^{bend}(\phi_n) \geq E^{bend}(\phi).$$

3. *For any ϕ satisfying $E^{bend}(\phi) < +\infty$, there exists a sequence (ϕ_n) converging for the $L^\infty(\Omega)$-norm to ϕ and satisfying*

$$\limsup_n E_n^{bend}(\phi_n) \leq E^{bend}(\phi).$$

Proof. By definition of E_n^{bend}, the bound $E_n^{bend}(\phi_n) \leq M$ implies that ϕ_n' is a measure absolutely continuous with respect to ν_n: $\phi_n' = \frac{\mathrm{d}\phi_n'}{\mathrm{d}\nu_n}(t)\nu_n(\mathrm{d}t)$. As we have assumed $\phi_n(0^+) = 0$, the quantity $\phi_n(t)$ coincides almost everywhere with $\phi_n'((0,t))$. By the Cauchy-Schwarz inequality in $\mathrm{L}^2_{\nu_n}(\Omega)$ we have

$$\left(\int_\Omega \left|\frac{\mathrm{d}\phi_n'}{\mathrm{d}\nu_n}(t)\right| \nu_n(\mathrm{d}t)\right)^2 \leq \nu_n(\Omega) \int_\Omega \left(\frac{\mathrm{d}\phi_n'}{\mathrm{d}\nu_n}(t)\right)^2 \nu_n(\mathrm{d}t) \leq M\nu_n(\Omega) = M\nu(\Omega). \tag{5.1}$$

Hence the sequence (ϕ_n) is bounded in $BV(\Omega)$: up to a subsequence, it converges weakly* to some function $\phi \in BV(\Omega)$ satisfying $\phi(0^+) = 0$. Moreover, for any test function φ in $\mathcal{D}(\Omega)$ we have, again by the Cauchy-Schwarz inequality,

$$\int_\Omega \phi(t)\varphi'(t)\,\mathrm{d}t = \lim_n \int_\Omega \phi_n(t)\varphi'(t)\,\mathrm{d}t = -\lim_n \int_\Omega \varphi(t)\frac{\mathrm{d}\phi_n'}{\mathrm{d}\nu_n}(t)\,\nu_n(\mathrm{d}t) \leq \sqrt{M}\lim_n \|\varphi\|_{\mathrm{L}^2_{\nu_n}}.$$

It is clear that the sequence (ν_n) of periodic measures weakly* converges to the measure $|\nu|\mathrm{d}t$. Hence, $\|\varphi\|^2_{\mathrm{L}^2_{\nu_n}(\Omega)}$ converges to $|\nu|\|\varphi\|^2_{\mathrm{L}^2(\Omega)}$ and we get

$$\int_\Omega \phi(t)\varphi'(t)\,\mathrm{d}t \leq \sqrt{M|\nu|}\|\varphi\|_{\mathrm{L}^2(\Omega)}.$$

Therefore, ϕ belongs to $W^{1,2}(\Omega)$. The weak convergence of ϕ_n to $\phi \in W^{1,2}(\Omega)$ in $BV(\Omega)$ clearly implies its weak* convergence in $L^\infty(\Omega)$ and the weak* convergence of $\phi_n'(\mathrm{d}t)$ to $\phi'(t)\mathrm{d}t$. In order to prove the strong convergence of ϕ_n to ϕ in $L^\infty(\Omega)$, we introduce regularized functions $\phi_{n,\delta}$ and ϕ_δ defined by

$$\phi_{n,\delta}(t) := \frac{1}{2\delta}\int_{t-\delta}^{t+\delta} \widetilde{\phi}_n(x)\mathrm{d}x, \quad \phi_\delta(t) := \frac{1}{2\delta}\int_{t-\delta}^{t+\delta} \widetilde{\phi}(x)\mathrm{d}x$$

where

$$\widetilde{\phi}(x) := \begin{cases} \phi(x) & \text{if } x \in (0,1), \\ 0 & \text{if } x \leq 0, \\ \phi(1^-) & \text{if } x \geq 1 \end{cases}$$

and $\widetilde{\phi}_n$ is defined in an analogous way.

As the sequence (ϕ_n) is bounded in $BV(\Omega)$, the sequence $(\widetilde{\phi}_n)$ is bounded in $L^\infty(\mathbb{R})$: there exists $C \in \mathbb{R}$ such that $\|\widetilde{\phi}_n\|_{\mathrm{L}^\infty(\mathbb{R})} \leq C$. Let $0 \leq s < t \leq 1$, with $t - s < 2\delta$. We have

$$\begin{aligned} |\phi_{n,\delta}(t) - \phi_{n,\delta}(s)| &\leq \frac{1}{2\delta}\left|\int_{t-\delta}^{t+\delta} \widetilde{\phi}_n(x)\mathrm{d}x - \int_{s-\delta}^{s+\delta} \widetilde{\phi}_n(x)\mathrm{d}x\right| \\ &\leq \frac{1}{2\delta}\left|\int_{s+\delta}^{t+\delta} \widetilde{\phi}_n(x)\mathrm{d}x - \int_{s-\delta}^{t-\delta} \widetilde{\phi}_n(x)\mathrm{d}x\right| \\ &\leq \frac{t-s}{\delta}C. \end{aligned}$$

The sequence $(\phi_{n,\delta})_n$ is equi-continuous on $[0,1]$. As the weak* convergence in $L^\infty(\Omega)$ of ϕ_n clearly implies the pointwise convergence of $\phi_{n,\delta}$ to ϕ_δ, we get, owing to Ascoli's theorem, its uniform convergence.

On the other hand, for almost every $0 < t < 1$ and almost every $x \in (t, t+\delta)$ (the same estimate holds if $x \in (t-\delta, t)$)

$$|\widetilde{\phi}_n(x) - \widetilde{\phi}_n(t)|^2 \leq \left(\int_{[t-\delta,t+\delta]}\left|\frac{\mathrm{d}\widetilde{\phi}'_n}{\mathrm{d}\nu_n}(u)\right|\nu_n(\mathrm{d}u)\right)^2 \leq M\nu_n([t-\delta, t+\delta]).$$

By covering the interval $[t-\delta, t+\delta)$ with intervals of the type $[y, y+1/n)$ whose ν_n-measure is always $|\nu|/n$, we get the estimate $\nu_n([t-\delta, t+\delta]) \leq (2\delta + 1/n)|\nu|$. Hence

$$|\phi_{n,\delta}(t) - \phi_n(t)| = \frac{1}{2\delta}\left|\int_{t-\delta}^{t+\delta}(\widetilde{\phi}_n(x) - \widetilde{\phi}_n(t))\,\mathrm{d}x\right| \leq \sqrt{M|\nu|(2\delta + 1/n)}.$$

Finally, we remark that, as ϕ is a continuous function, ϕ_δ converges uniformly to ϕ when δ tends to zero. For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|\phi_\delta - \phi\|_{L^\infty(\Omega)} < \varepsilon/3. \tag{5.2}$$

Possibly reducing δ, we can also ensure that $\sqrt{M|\nu|2\delta} < \varepsilon/3$. From the uniform convergence of $\phi_{n,\delta}$ to ϕ_δ, we know that there exists N such that

$$n > N \implies \|\phi_{n,\delta} - \phi_\delta\|_{L^\infty(\Omega)} < \varepsilon/3. \tag{5.3}$$

Possibly increasing N we can also ensure that $\sqrt{M|\nu|(2\delta + 1/N)} < \varepsilon/3$ and, thus, that

$$n > N \implies \|\phi_{n,\delta} - \phi_n\|_{L^\infty(\Omega)} < \varepsilon/3. \tag{5.4}$$

Collecting (5.2), (5.3), (5.4), point (a) is proven by the triangle inequality.

Now, let (ϕ_n) be a sequence satisfying $\|\phi_n - \phi\|_{L^\infty(\Omega)} \to 0$ and $E_n^{bend}(\phi_n) \leq M$. Consider $\varphi \in \mathcal{D}(\Omega)$. From the obvious inequality

$$\left(\frac{\mathrm{d}\phi'_n}{d\nu_n}\right)^2 + \varphi^2 \geq 2\varphi\frac{d\phi'_n}{d\nu_n}$$

we deduce

$$\liminf_n E_n^{bend}(\phi_n) \geq \liminf_n \int_\Omega \left(2\varphi(t)\frac{\mathrm{d}\phi_n'}{\mathrm{d}\nu_n}(t) - \varphi^2(t)\right)\nu_n(\mathrm{d}t) \tag{5.5}$$

and passing to the limit

$$\liminf_n E_n^{bend}(\phi_n) \geq \int_\Omega \left(2\varphi(t)\phi'(t) - |\nu|\,\varphi^2(t)\right)\mathrm{d}t. \tag{5.6}$$

This can be rewritten

$$\liminf_n E_n^{bend}(\phi_n) \geq E^{bend}(\phi) - |\nu| \left\| \frac{1}{|\nu|}\phi' - \varphi \right\|^2_{\mathrm{L}^2(\Omega)}. \tag{5.7}$$

Point (b) is proven by invoking the density of $\mathcal{D}(\Omega)$ in $\mathrm{L}^2(\Omega)$.

Finally, let us consider $\phi \in L^\infty(\Omega)$ with $E^{bend}(\phi) \leq M < +\infty$. Hence $\phi \in W_l^{1,2}(\Omega)$. By density there exists a sequence $(\varphi_p)_p$ in $\mathcal{D}(\Omega)$ such that $\|\varphi_p - \phi'\|_{\mathrm{L}^2(\Omega)} \to 0$. Let us define $\phi_{n,p} \in BV(\Omega)$ by setting

$$\phi_{n,p}(t) := \frac{1}{|\nu|}\int_{]0,t]} \varphi_p(s)\,\nu_n(\mathrm{d}s).$$

The derivative of $\phi_{n,p}$ in the sense of distributions is the measure $\frac{1}{|\nu|}\varphi_p\,\nu_n$. Hence, recalling that $\nu_n(\mathrm{d}t) \rightharpoonup |\nu|\,\mathrm{d}t$, we get the convergence as n tends to infinity:

$$E_n^{bend}(\phi_{n,p}) = \int_\Omega \left(\frac{\varphi_p(t)}{|\nu|}\right)^2 \nu_n(\mathrm{d}t) \to \frac{1}{|\nu|}\int_\Omega (\varphi_p(t))^2\,\mathrm{d}t.$$

As $\int_\Omega (\varphi_p(t))^2\,\mathrm{d}t$ converges to $\int_\Omega (\phi'(t))^2\,\mathrm{d}t = |\nu|\,E^{bend}(\phi)$ when p tends to infinity, there exists a diagonal sequence $(\varphi_{n_p,p})_p$ which satisfies point (c). □

5.2. Convergence of extension energy

Double-scale convergence [5, 20] is a powerful tool for describing the limit behavior of sequences in a periodic framework. Our asymptotic result concerning the extension energy is established in this framework. Introducing the effective extensional energy E^{ext} as the functional from $\mathrm{L}^1(\Omega \times Y)$ to $[0, +\infty]$, defined by

$$E^{ext}(\bar\rho) := \int_{\Omega\times Y} f(y, \bar\rho(t,y))\,\mathrm{d}t\,\mathrm{d}y,$$

this result reads:

Theorem 5.2. Convergence of extension energies.

1. *Let ρ_n be a sequence in $L^1(\Omega)$ satisfying $E_n^{ext}(\rho_n) \leq M < +\infty$. Then there exist $\bar\rho \in L^1(\Omega \times Y)$ such that, up to a subsequence, ρ_n double-scale converges to $\bar\rho$.*
2. *If (ρ_n) is a sequence in $\mathrm{L}^1(\Omega)$ double-scale converging to $\bar\rho \in \mathrm{L}^1(\Omega \times \mathrm{Y})$, then*

$$\liminf_n E_n^{ext}(\rho_n) \geq E^{ext}(\bar\rho).$$

3. For any $\bar\rho \in L^1(\Omega\times Y)$ with finite energy $E^{ext}(\bar\rho) < +\infty$, there exists a sequence (ρ_n) in $L^1(\Omega)$ that double-scale converges to $\bar\rho$ and satisfies

$$\limsup_n E_n^{ext}(\rho_n) \le E^{ext}(\bar\rho).$$

The proof is divided in three lemmas established in next subsections. Let us first recall the definition of double-scale convergence and give some of its properties.

5.2.1. Double-scale convergence

Recall that the structure we consider is contained in the domain $\Omega =]0,1[$ and is $\frac{1}{n}Y$-periodic where $Y = [0,1[$ is the rescaled periodic cell. Let us recall the definition of double-scale convergence.

In the sequel, we call "test function" any function $\psi \in C^0(\overline{\Omega}\times\mathbb{R})$ 1-periodic with respect to the second variable.

Definition 5.3. We say that a sequence of functions (u_n) in $L^1(\Omega)$ double-scale converges to $\bar u \in L^1(\Omega\times Y)$ if, for any test function ψ, the following convergence holds:

$$\int_\Omega \psi(t,nt)u_n(t)\,\mathrm{d}t \to \int_{\Omega\times Y} \psi(t,y)\bar u(t,y)\,\mathrm{d}t\,\mathrm{d}y.$$

Remark 5.4. Considering test functions depending only on the first variable, we see that the double-scale convergence of u_n to $\bar u$ implies the weak* convergence of the sequence of measures $u_n(t)\mathrm{d}t$ to the measure $\Big(\int_Y \bar u(t,y)\,\mathrm{d}y\Big)\,\mathrm{d}t$.

Remark 5.5. We denote by $L^\infty_\#(\mathbb{R})$ the set of 1-periodic functions in $L^\infty(\mathbb{R})$. If $\bar\varphi \in C^0(\overline{\Omega}, \mathrm{L}^\infty_\#(\mathbb{R}))$, then the function $t \mapsto \bar\varphi(t,nt)$ double-scale converges to $\bar\varphi$. This applies in particular when $\bar\varphi$ is a test function.

Proof. Indeed, the integral $\int_\Omega \bar\varphi(t,nt)\,\mathrm{d}t$ can be written, by dividing Ω in small parts and changing variables, under the form $\int_Y \frac{1}{n}\sum_{k=1}^n \bar\varphi\left(\frac{k-1}{n}+\frac{y}{n}, y\right)\,\mathrm{d}y$. For almost every $y\in Y$, the integrand converges by Riemann integration to $\int_\Omega \bar\varphi\,(t,y)\,\mathrm{d}t$. The convergence of $\int_\Omega \bar\varphi(t,nt)\,\mathrm{d}t \to \int_{\Omega\times Y}\bar\varphi\,(t,y)\,\mathrm{d}t\,\mathrm{d}y$ is then a consequence of the Lebesgue dominated convergence theorem. It is then enough to remark that, if $\bar\varphi \in C^0(\overline{\Omega}, \mathrm{L}^\infty_\#(\mathbb{R}))$ and if ψ is a test function, then the product $\bar\varphi\psi$ belongs to $C^0(\overline{\Omega}, \mathrm{L}^\infty_\#(\mathbb{R}))$. The previous result can be applied and the double-scale convergence of $t\mapsto \bar\varphi(t,nt)$ to $\bar\varphi$ is proven. □

Lemma 5.6. *Let (u_n) be a sequence double-scale converging to $\bar u \in L^1(\Omega\times Y)$ and satisfying the condition*

$$\lim_{q\to\infty}\sup_n \int_{|u_n|>q} |u_n(t)|\,\mathrm{d}t = 0. \tag{5.8}$$

Then, for any functions $g\in L^\infty(\Omega)$ and $h \in L^\infty_\#(\mathbb{R})$, the sequence of functions $t\mapsto u_n(t)g(t)h(nt)$ double-scale converges to $(t,y)\mapsto \bar u(t,y)g(t)h(y)$.

Proof. Without loss of generality, we can assume $\|g\|_{L^\infty(\Omega)} = \|h\|_{L^\infty(\mathbb{R})} = 1$. Let $\delta > 0$. By Lusin theorem, there exist one-periodic functions $g_\delta \in C(\overline{\Omega})$ and $h_\delta \in C(\mathbb{R})$ such that

$$\|g_\delta\|_{L^\infty(\Omega)} = \|h_\delta\|_{L^\infty(\mathbb{R})} = 1, \quad |\{g_\delta \ne g\}| < \delta \quad \text{and} \quad |\{h_\delta \ne h\}\cap Y| < \delta.$$

Let us introduce the sets

$$X_\delta := \{(t,y) \in \Omega \times Y :\; g_\delta(t)\, h_\delta(y) \neq g(t)\, h(y)\}$$
$$X_{\delta,n} := \{t \in \Omega :\; g_\delta(t)\, h_\delta(nt) \neq g(t)\, h(nt)\}.$$

We clearly have $|X_\delta| < 2\delta$ and $|X_{\delta,n}| < 2\delta$. Let $\varepsilon > 0$ and $q \in \mathbb{N}^*$ such that $\sup_n \int_{|u_n|>q} |u_n(t)|\,\mathrm{d}t < \frac{\varepsilon}{12}$. Let $\delta > 0$ such that $\delta \leq \frac{\varepsilon}{24q}$ and $\int_{X_\delta} |\bar{u}(t,y)|\,\mathrm{d}t\,\mathrm{d}y \leq \frac{\varepsilon}{6}$. Let ψ be a test function with $\|\psi\|_{L^\infty(\Omega)} = 1$. We have

$$\left|\int_{\Omega\times Y} \psi(t,y)\,(g(t)h(y) - g_\delta(t)h_\delta(y))\,\bar{u}(t,y)\,\mathrm{d}t\,\mathrm{d}y\right| \leq 2\int_{X_\delta} |\bar{u}(t,y)|\,\mathrm{d}t\,\mathrm{d}y \leq \frac{\varepsilon}{3}$$

and, for any n,

$$\begin{aligned}\left|\int_{\Omega} \psi(t,nt)\big(g(t)h(nt) - g_\delta(t)h_\delta(nt)\big)u_n(t)\,\mathrm{d}t\right| &\leq 2\int_{X_{\delta,n}} |u_n(t)|\,\mathrm{d}t \\ &\leq 2q|X_{\delta,n}| + 2\int_{|u_n|>q} |u_n(t)|\,\mathrm{d}t \leq \frac{\varepsilon}{3}.\end{aligned}$$

The function $(t,y) \mapsto \psi(t,y)g_\delta(t)h_\delta(y)$ is also a test function: for n large enough,

$$\left|\int_{\Omega} \psi(t,nt)g_\delta(t)h_\delta(nt)u_n(t)\,\mathrm{d}t - \int_{\Omega\times Y} \psi(t,y)g_\delta(t)h_\delta(y)\bar{u}(t,y)\,\mathrm{d}t\,\mathrm{d}y\right| \leq \frac{\varepsilon}{3}.$$

By the triangle inequality, we get the desired result:

$$\left|\int_{\Omega} \psi(t,nt)g(t)h(nt)u_n(t)\,\mathrm{d}t - \int_{\Omega\times Y} \psi(t,y)g(t)h(y)\bar{u}(t,y)\,\mathrm{d}t\,\mathrm{d}y\right| \leq \varepsilon.$$

□

Remark 5.7. As a consequence, $t \mapsto 1_{[0,s]}(t)\mathrm{e}^{i\theta^0_n(t)}$ double-scale converges to the function $(t,y) \mapsto 1_{[0,s]}(t)\mathrm{e}^{i\theta^0(y)}$. Hence $\chi^0_n(s)$ converges to $s\ell$. The convergence of χ^0_n to the function $s \mapsto \ell\, s$ is actually uniform since the sequence χ^0_n is equi-continuous. We recover the already mentioned fact that, from the macroscopic point of view, the beam at rest is a straight segment of length ℓ.

5.2.2. Compactness induced by the extensional energy

Lemma 5.8. *Let ρ_n be a sequence in $L^1(\Omega)$ satisfying $E^{ext}_n(\rho_n) \leq M < +\infty$. For any $q \in \mathbb{N}^*$, we set*

$$\rho_{n,q} := \begin{cases} 1-q & \text{if } \rho_n < 1-q, \\ \rho_n & \text{if } 1-q \leq \rho_n \leq 1+q, \\ 1+q & \text{if } 1+q < \rho_n. \end{cases}$$

a) For any $q \in \mathbb{N}^*$, $f(y,\rho_{n,q}) \leq f(y,\rho_n)$.

Moreover there exist $\bar{\rho}_q \in L^\infty(\Omega\times Y)$ and $\bar{\rho} \in L^1(\Omega\times Y)$ such that, up to subsequences,

b) the sequences $\rho_{n,q}$ double-scale converge to $\bar{\rho}_q$ as n tends to infinity;
c) the sequence $\bar{\rho}_q$ converges to $\bar{\rho}$ almost everywhere and for the $L^1(\Omega\times Y)$-norm;
d) the sequence ρ_n double-scale converges to $\bar{\rho}$.

Proof. By convexity, $f(y,\rho)$ is a non-increasing function of ρ on $]-\infty,1]$ and non-decreasing on $[1,+\infty[$. Point (a) is then obvious. Hence

$$\int_\Omega \underline{f}(\rho_{n,q}(t))\,\mathrm{d}t \le E_n^{ext}(\rho_{n,q}) \le E_n^{ext}(\rho_n) \le M. \tag{5.9}$$

For any q, the sequence $(\rho_{n,q})_n$ satisfies $\|\rho_{n,q}\|_{L^\infty(\Omega)} \le q+1$ and, thus, it is bounded in $L^2(\Omega)$. It double-scale converges [5, 20] up to a subsequence. By the standard procedure of successive extractions and diagonalization, there exists a subsequence $(n_m)_m$ such that, for any q, $\rho_{n_m,q}$ double-scale converges to some $\bar\rho_q \in L^2(\Omega\times Y)$. Point (b) is proven.

The coerciveness assumption (3.3) and the bound (5.9) imply that there exists a constant C such that $\|\rho_{n,q}\|_{L^1(\Omega)} \le C$. Passing to the limit, we get $\|\bar\rho_q\|_{L^1(\Omega\times Y)} \le C$. The positive and negative parts $(\bar\rho_q-1)^+$ and $(\bar\rho_q-1)^-$ are bounded in $L^1(\Omega\times Y)$ and they are monotonous sequences: they converge almost everywhere and for the L^1 norm. So does their difference. Point (c) is proven.

Let ψ be a test function, that is a function in $C^0(\overline{\Omega}\times\mathbb{R})$ 1-periodic with respect to the second variable. Let us moreover assume without loss of generality that $\|\psi\|_{L^\infty} \le 1$. Denote

$$M_n := \int_\Omega \psi(t,nt)\rho_n(t)\,\mathrm{d}t - \int_{\Omega\times Y} \psi(t,y)\bar\rho(t,y)\,\mathrm{d}t\mathrm{d}y$$

and

$$M_{n,q} := \int_\Omega \psi(t,nt)\rho_{n,q}(t)\,\mathrm{d}t - \int_{\Omega\times Y} \psi(t,y)\bar\rho_q(t,y)\,\mathrm{d}t\mathrm{d}y.$$

By triangle inequality, we have

$$\begin{aligned}|M_n| &\le |M_{n,q}| + \|\rho_n-\rho_{n,q}\|_{L^1(\Omega)} + \|\bar\rho-\bar\rho_q\|_{L^1(\Omega\times Y)}\\ &\le |M_{n,q}| + \sup_n \int_{|\rho_n|>q-1} |\rho_n(t)|\,\mathrm{d}t + \|\bar\rho-\bar\rho_q\|_{L^1(\Omega\times Y)}.\end{aligned}$$

Passing to the limit as n tends to ∞, using point (a), we get

$$\limsup_n |M_n| \le \sup_n \int_{|\rho_n|>q-1} |\rho_n(t)|\,\mathrm{d}t + \|\bar\rho-\bar\rho_q\|_{L^1(\Omega\times Y)}.$$

Point (d) will be proved once checked that the right hand side of this inequality tends to zero as q tends to infinity. The fact that the last addend tends to zero was proved in point(c). For proving that the first addend also tends to zero, we remark that the coerciveness assumption (3.3) implies that for any $\varepsilon>0$, for some q large enough, $|\rho|>q-1 \implies |\rho| < \varepsilon\underline{f}(\rho)$ and thus

$$\sup_n \int_{|\rho_n|>q-1} |\rho_n(t)|\,\mathrm{d}t \le \varepsilon \sup_n \int_{|\rho_n|>q-1} \underline{f}(\rho_n(t))\,\mathrm{d}t \le \varepsilon \sup_n \int_\Omega \underline{f}(\rho_n(t))\,\mathrm{d}t \le \varepsilon M.$$

□

5.2.3. Lower bound for the extensional energy

Lemma 5.9. *Let (ρ_n) be a sequence in $\mathrm{L}^1(\Omega)$ double-scale converging to $\bar\rho \in \mathrm{L}^1(\Omega\times \mathrm{Y})$. We have*

$$\liminf_n E_n^{ext}(\rho_n) \geq E^{ext}(\bar\rho).$$

Proof. We can restrict our attention to sequences (ρ_n) with bounded extensional energy and which converge in the sense of Lemma 5.8. From Lemma 5.8 we know that, for any $q\in\mathbb{N}^*$, $f(y,\rho_{n,q}) \leq f(y,\rho_n)$. Hence

$$\liminf_n E_n^{ext}(\rho_n) \geq \liminf_n \int_\Omega f(nt,\rho_{n,q}(t))\,\mathrm{d}t. \tag{5.10}$$

Let us now introduce the Moreau-Yosida approximation f_λ of f with respect to the second variable:

$$f_\lambda(y,\rho) := \inf_\xi \Big\{ f(y,\xi) + \frac{\lambda}{2}(\rho-\xi)^2 \Big\}$$

and the Fenchel conjugate of f and f_λ, still with respect to the second variable:

$$f^*(y,\psi) := \sup_\rho \big\{\psi\rho - f(y,\rho)\big\} \quad \text{and} \qquad f_\lambda^*(y,\psi) := \sup_\rho \big\{\psi\rho - f_\lambda(y,\rho)\big\}.$$

Note that, from $f(y,1)=0$ and $f(y,\rho)\geq \underline{f}(\rho)$, we can deduce $\psi \leq f^*(y,\psi) \leq \underline{f}^*(\psi)$. As $\underline{f}$ satisfies assumption (3.3), its conjugate $\underline{f}^*$ is a convex function taking finite values. So are the functions $\psi\mapsto f^*(y,\psi)$. Thus they all are continuous. We can write, for any test function ψ,

$$\begin{aligned}\liminf_n \int_\Omega f(nt,\rho_{n,q}(t))\,\mathrm{d}t &\geq \liminf_n \int_\Omega \left(\psi(t,nt)\rho_{n,q}(t) - f^*(nt,\psi(t,nt))\right)\,\mathrm{d}t\\ &\geq \int_{\Omega\times Y} \left(\psi(t,y)\bar\rho_q(t,y) - f^*(y,\psi(t,y))\right)\,\mathrm{d}t\,\mathrm{d}y.\end{aligned}$$

Here we have used the fact that $\rho_{n,q}$ double scales-converges to $\bar\rho_q$ and that the function $(t,y)\to f^*(y,\psi(t,y))$ belongs to $C^0(\overline{\Omega},\mathrm{L}^\infty_\#(\mathbb{R}))$, and thus that Remark 5.5 applies. Hence, using (5.10) we obtain

$$\liminf_n E_n^{ext}(\rho_n) \geq \sup_\psi \int_{\Omega\times Y} \left(\psi(t,y)\bar\rho_q(t,y) - f^*(y,\psi(t,y))\right)\,\mathrm{d}t\,\mathrm{d}y. \tag{5.11}$$

Passing to the supremum in ψ is easier when f^* is replaced by the regularized function f^*_λ. Clearly $f_\lambda\leq f$, hence $f^*_\lambda \geq f^*$ and, using the Fenchel identity

$$f_\lambda(y,\rho) = \rho\,\partial_2 f_\lambda(y,\rho) - f_\lambda^*(y,\partial_2 f_\lambda(y,\rho)),$$

we can write

$$\begin{aligned}\int_{\Omega\times Y}\left(\psi(t,y)\bar\rho_q(t,y) - f^*(y,\psi(t,y))\right)\,\mathrm{d}t\mathrm{d}y &\geq \int_{\Omega\times Y}\left(\psi(t,y)\bar\rho_q(t,y) - f_\lambda^*(y,\psi(t,y))\right)\,\mathrm{d}t\mathrm{d}y\\ &\geq \int_{\Omega\times Y} f_\lambda(y,\bar\rho_q(t,y))\,\mathrm{d}t\mathrm{d}y\end{aligned}$$
$$+\int_\Lambda \left((\psi(t,y)-\partial_2 f_\lambda(y,\bar\rho_q(t,y))\bar\rho_q(t,y) - f_\lambda^*(y,\psi(t,y)) + f_\lambda^*(y,\partial_2 f_\lambda(y,\bar\rho_q(t,y)))\right)\,\mathrm{d}t\mathrm{d}y$$

where $\Lambda := \{(t,y) \in \Omega \times Y :\ \partial_2 f_\lambda(y, \bar\rho_q(t,y)) \neq \psi(t,y)\}$. Indeed recall [16, 24] that, for almost every y, the regularized function $\rho \mapsto f_\lambda(y,\rho)$ is convex, belongs to $C^1(\mathbb{R})$, and satisfies $0 \le f_\lambda(y,\rho) \le \frac{\lambda}{2}(\rho-1)^2$ and $|\partial_2 f_\lambda(y,\rho)| \le 2\lambda|\rho - 1|$. As $\|\bar\rho_q - 1\|_\infty \le q$, we have $|\partial_2 f_\lambda(y, \bar\rho_q(t,y))| \le 2\lambda q$.

By Lusin's theorem, for any $\delta > 0$, there exists a test function ψ coinciding with $\partial_2 f_\lambda(y, \bar\rho_q(t,y)$ except on a set with measure smaller than δ and satisfying the same uniform bound:

$$|\Lambda| < \delta, \qquad |\psi(t,y)| \le 2\lambda q.$$

A simple computation of Fenchel conjugate gives

$$\xi + \frac{1}{2\lambda}\xi^2 \le f_\lambda^*(y,\xi) = f^*(y,\xi) + \frac{1}{2\lambda}\xi^2 \le \underline{f}^*(\xi) + \frac{1}{2\lambda}\xi^2.$$

Thus, with such a function ψ, all integrands in the last integral have a uniform bound M independent of δ and the whole integral is greater than $-4M\delta$. Taking the limit as δ tends to zero, we obtain

$$\sup_\psi \int_{\Omega\times Y} \left(\psi(t,y)\bar\rho_q(t,y) - f^*(y,\psi(t,y))\right) \mathrm{d}t\,\mathrm{d}y \ge \int_{\Omega\times Y} f_\lambda(y, \bar\rho_q(t,y))\,\mathrm{d}t\,\mathrm{d}y.$$

and thus

$$\liminf_n E_n^{ext}(\rho_n) \ge \int_{\Omega\times Y} f_\lambda(y, \bar\rho_q(t,y))\,\mathrm{d}t\,\mathrm{d}y. \tag{5.12}$$

Now we let λ tend to $+\infty$. By the monotone convergence theorem, we obtain

$$\liminf_n E_n^{ext}(\rho_n) \ge \int_{\Omega\times Y} f(y, \bar\rho_q(t,y))\,\mathrm{d}t\,\mathrm{d}y. \tag{5.13}$$

By Fatou's lemma, we can write

$$\liminf_n E_n^{ext}(\rho_n) \ge \liminf_q \int_{\Omega\times Y} f(y, \bar\rho_q(t,y))\,\mathrm{d}t\,\mathrm{d}y \ge \int_{\Omega\times Y} \liminf_q f(y, \bar\rho_q(t,y))\,\mathrm{d}t\,\mathrm{d}y.$$

Since f is lower-semi-continuous with respect to the second variable and $\bar\rho_q$ converges almost everywhere to $\bar\rho$ in $\Omega \times Y$, we finally obtain the desired result

$$\liminf_n E_n^{ext}(\rho_n) \ge \int_{\Omega\times Y} f(y, \bar\rho(t,y))\,\mathrm{d}t\,\mathrm{d}y.$$

□

5.2.4. Upper bound for the extensional energy

Lemma 5.10. *For any $\bar\rho \in L^1(\Omega\times Y)$ with finite energy $E^{ext}(\bar\rho) < +\infty$, there exists a sequence (ρ_n) in $L^1(\Omega)$ that double-scale converges to $\bar\rho$ and satisfies*

$$\limsup_n E_n^{ext}(\rho_n) \le E^{ext}(\bar\rho).$$

Proof. We extend $\bar\rho$ on the whole set $\mathbb{R}^2$ by periodicity for both variables. Then we set, for any t in Ω,

$$\rho_n(t) := \int_Y \bar\rho\left(t + \frac{y}{n}, nt\right) \mathrm{d}y. \tag{5.14}$$

We notice that, when $g \in L^1_{\ell oc}(\mathbb{R}^2)$ is a one-periodic function with respect to both variables, then, for any $n \in \mathbb{N}^*$, the following identity holds:

$$\int_{\Omega\times Y} g(t + \frac{y}{n}, nt)\,\mathrm{d}t\,\mathrm{d}y = \int_{\Omega\times Y} g(t, y)\,\mathrm{d}t\,\mathrm{d}y. \tag{5.15}$$

Applying this trick to $g(t, y) = f(y, \bar\rho(t, y))$, using the fact that ρ_n belongs to $L^1(\Omega)$ and Jensen's inequality we directly obtain $E^{ext}_n(\rho_n) \leq E^{ext}(\bar\rho)$.

In order to prove the double-scale convergence of ρ_n to $\bar\rho$ we consider a test function ψ and we extend it (actually its restriction to $\Omega \times Y$) on the whole $\mathbb{R}^2$ by periodicity. This extension is uniformly continuous on $\Omega \times \mathbb{R}$. Using (5.15) with $g = \psi\,\bar\rho$, we get the estimate

$$\begin{aligned}\left|\int_\Omega \psi(t, nt)\rho_n(t)\,\mathrm{d}t - \int_{\Omega\times Y} \psi(t, y)\bar\rho(t, y)\,\mathrm{d}t\,\mathrm{d}y\right| \\ \leq \int_{\Omega\times Y} \left|\psi(t, nt) - \psi(t + \frac{y}{n}, nt)\right| \left|\bar\rho(t + \frac{y}{n}, nt)\right| \mathrm{d}t\,\mathrm{d}y.\end{aligned}$$

For any $\varepsilon > 0$, for n large enough, the inequality $|\psi(t, nt) - \psi(t + \frac{y}{n}, nt)| \leq \varepsilon$ holds for all $(t, y) \in]0, 1 - \frac{1}{n}] \times Y$. Subdividing the last integral on this set and its complementary, we see that it is smaller than

$$\varepsilon \int_{]0,1-\frac{1}{n}]\times Y} \left|\bar\rho(t + \frac{y}{n}, nt)\right| \mathrm{d}t\,\mathrm{d}y + 2\|\psi\|_\infty \int_{]1-\frac{1}{n},1[\times Y} \left|\bar\rho(t + \frac{y}{n}, nt)\right| \mathrm{d}t\,\mathrm{d}y.$$

Using again (5.15) we know that the first addend is smaller than $\varepsilon\|\bar\rho\|_{L^1(\Omega\times Y)}$. The second addend satisfies

$$\begin{aligned}2\|\psi\|_\infty \int_{]1-\frac{1}{n},1[\times Y} \left|\bar\rho(t + \frac{y}{n}, nt)\right| \mathrm{d}t\,\mathrm{d}y &\leq 2\|\psi\|_\infty \int_{]1-\frac{1}{n},1+\frac{1}{n}[\times[n-1,n]} |\bar\rho(t, y)|\,\mathrm{d}t\,\mathrm{d}y \\ &\leq 2\|\psi\|_\infty \int_{]1-\frac{1}{n},1+\frac{1}{n}[\times Y} |\bar\rho(t, y)|\,\mathrm{d}t\,\mathrm{d}y.\end{aligned}$$

The measure of the integration domain tends to zero as n tends to infinity. Therefore

$$\limsup_n \left|\int_\Omega \psi(t, nt)\rho_n(t)\,\mathrm{d}t - \int_{\Omega\times Y} \psi(t, y)\bar\rho(t, y)\,\mathrm{d}t\,\mathrm{d}y\right| \leq \varepsilon\|\bar\rho\|_{L^1(\Omega\times Y)}.$$

The proof is concluded since ε is arbitrarily small. □

5.3. Proof of the main result

We are now in position to prove Theorem 4.1.

Proof. Recall that

$$E_n(\chi_n) := \inf_{(\rho_n,\phi_n)} \left\{E^{ext}_n(\rho_n) + E^{bend}_n(\phi_n) : \chi_n(0) = 0;\ \chi'_n = \rho_n\, \mathrm{e}^{i(\theta^0_n+\phi_n)}\right\}.$$

Let (χ_n) be a sequence in $C(\overline{\Omega})$ with finite energy $E_n(\chi_n) < M < +\infty$. There exist sequences (ρ_n) and (ϕ_n) such that $E_n^{ext}(\rho_n) < M$, $E_n^{bend}(\phi_n) < M$ and

$$\chi_n(x) = \int_0^x \rho_n(t)\, \mathrm{e}^{i(\theta_n^0 + \phi_n(t))}\, \mathrm{d}t.$$

Lemmas 5.1 and 5.8 state that there exists $(\bar{\rho}, \phi)$ in $L^1(\Omega \times Y) \times W_l^{1,2}(\Omega)$ such that, up to a subsequence, $\|\phi_n - \phi\|_{L^\infty(\Omega)}$ tends to zero and ρ_n double-scale converges to $\bar{\rho}$. For any $0 \le x_1 < x_2 \le 1$, and for any $q \in \mathbb{N}^*$,

$$|\chi_n(x_2) - \chi_n(x_1)| \le \int_{x_1}^{x_2} |\rho_n(t)|\, \mathrm{d}t \le q(x_2 - x_1) + \sup_{k \in \mathbb{N}^*} \int_{|\rho_k| > q} |\rho_k(t)|\, \mathrm{d}t.$$

We already noticed while proving Lemma 5.8 that assumption (3.3) implies that the last addend tends to zero as q tends to infinity. Therefore the sequence (χ_n) is equicontinous.

Let $x \in \overline{\Omega}$. As $\int_\Omega \underline{f}(\rho_n(t))\, \mathrm{d}t \le \int_\Omega f(nt, \rho_n(t))\, \mathrm{d}t \le M$ and $\lim_{|\rho| \to +\infty} \frac{\underline{f}(\rho)}{\rho} = +\infty$ we can apply Lemma 5.6 which states that the sequence of functions

$$t \mapsto 1_{[0,x]}(s) \mathrm{e}^{i\phi(s)} \mathrm{e}^{i\theta^0(ns)} \rho_n(s)$$

double-scale converges to the function $(t, y) \mapsto 1_{[0,x]}(t) \mathrm{e}^{i\phi(t)} \mathrm{e}^{i\theta^0(y)} \bar{\rho}(t, y))$. Definition of double-scale convergence, applied with a constant test function, implies the convergence

$$\lim_n \left(\int_{[0,x]} \mathrm{e}^{i\phi(t)} \mathrm{e}^{i\theta^0(nt)} \rho_n(t)\, \mathrm{d}t - \int_{[0,x] \times Y} \mathrm{e}^{i\phi(t)} \mathrm{e}^{i\theta^0(y)} \bar{\rho}(t, y)\, \mathrm{d}t\, \mathrm{d}y \right) = 0.$$

We have

$$\limsup_n \left| \int_{[0,x]} \left(\mathrm{e}^{i\phi_n(t)} - \mathrm{e}^{i\phi(t)} \right) \mathrm{e}^{i\theta^0(nt)} \rho_n(t)\, \mathrm{d}t \right| \le \limsup_n \left(\|\phi_n - \phi\|_{L^\infty(\Omega)} \|\rho_n\|_{L^1(\Omega)} \right).$$

The right hand side of this inequality tends to zero, owing to the uniform convergence of ϕ_n to ϕ and to the fact that (ρ_n) has a bounded $L^1(\Omega)$ norm. Therefore, $\lim_n \chi_n(x) = \chi(x)$ where

$$\chi(x) := \int_{[0,x] \times Y} \mathrm{e}^{i\phi(t)} \mathrm{e}^{i\theta^0(y)} \bar{\rho}(t, y)\, \mathrm{d}t\, \mathrm{d}y. \tag{5.16}$$

Simple convergence of χ_n to χ is now established. By Ascoli theorem, the uniform convergence on $\overline{\Omega}$ follows. Point (a) of Theorem 4.1 is proven.

Now, let (χ_n) be a sequence converging uniformly to χ. Without loss of generality, we assume that $E_n(\chi_n)$ converges to a finite value M. Up to subsequences, there exists (ρ_n, ϕ_n) and $(\bar{\rho}, \phi)$ such that

$$E_n^{bend}(\phi_n) + E_n^{ext}(\rho_n) \le E_n(\chi_n) + \frac{1}{n} \quad \text{and} \quad \chi_n(x) = \int_0^x \rho_n(t) \mathrm{e}^{i(\theta_n^0(t) + \phi_n(t))}\, \mathrm{d}t. \tag{5.17}$$

Owing to the three compactness results established in Lemma 5.1, Lemma 5.8 and in Point (a) just stated, we have, again extracting a subsequence, the convergence for the $L^\infty(\Omega)$-norm of ϕ_n to ϕ, the double-scale convergence of ρ_n to $\bar\rho$ and the fact that χ is related to these limits by (5.16). The lower bounds established in Lemma 5.1 and Lemma 5.9 give

$$\liminf_n E_n(\chi_n) \geq \liminf_n E_n^{bend}(\phi_n) + \liminf_n E_n^{ext}(\rho_n) \geq E^{bend}(\phi) + E^{ext}(\bar\rho) \geq E(\chi).$$

As χ satisfies the constraint (5.16), point (b) is proven.

Finally, let χ with $E(\chi) < +\infty$. By definition of E, for any $m \in \mathbb{N}^*$, there exists $(\phi_m, \bar\rho_m)$ in $W_l^{1,2}(\Omega) \times L^1(\Omega \times Y)$ such that

$$E^{bend}(\phi_m) + E^{ext}(\bar\rho_m) \leq E(\chi) + \frac{1}{m} \text{ and } \chi(x) = \int_{[0,x]\times Y} \bar\rho_m(t,y) \mathrm{e}^{i(\theta^0(y)+\phi_m(t))} \,\mathrm{d}t\,\mathrm{d}y.$$

The upper bounds results established in Lemma 5.1 and Lemma 5.10 provide, for any m, sequences $(\phi_{m,n})$ and $(\bar\rho_{m,n})$ such that $(\phi_{m,n})$ converges to ϕ_m for the $L^\infty(\Omega)$-norm, $(\rho_{m,n})$ double-scale converges to $(\bar\rho_m)$ and

$$\lim_n E_n^{bend}(\phi_{m,n}) = E^{bend}(\phi_m) \quad \text{and} \quad \lim_n E_n^{ext}(\rho_{m,n}) = E^{ext}(\bar\rho_m).$$

Let us set

$$\chi_{m,n}(x) := \int_0^x \rho_{m,n}(t) \mathrm{e}^{i(\theta_n^0(t)+\phi_{m,n}(t))} \,\mathrm{d}t.$$

We have

$$\begin{aligned}\limsup_n E_n(\chi_{m,n}) &\leq \lim_n E_n^{bend}(\phi_{m,n}) + \lim_n E_n^{ext}(\rho_{m,n}) \\ &\leq E^{bend}(\phi_m) + E^{ext}(\bar\rho_m) \leq E(\chi) + \frac{1}{m}.\end{aligned}$$

We proved in Point (a) that, for any m, the sequence $\chi_{m,n}$ converges as n tends to infinity for the $L^\infty(\Omega)$-norm to the function $\int_{[0,x]\times Y} \bar\rho_m(t,y)\mathrm{e}^{i(\theta^0(y)+\phi_m(t))}\,\mathrm{d}t\,\mathrm{d}y$, that is to χ. To resume, we have both

$$\lim_m \left(\limsup_n E_n(\chi_{m,n}) \right) \leq E(\chi) \quad \text{and} \quad \lim_m \left(\lim_n \|\chi_{m,n} - \chi\|_{L^\infty(\Omega)} \right) = 0.$$

Point (c) is proven by diagonalization. □

6. Explicit computation of the limit energy

The expression of the limit energy E in terms of $\bar\rho$ is difficult to use. This is a classical situation in homogenization where the expression of the effective energy is simplified by defining a cell problem.

Theorem 6.1. *Let X_χ be the set of all (r,θ,ϕ) in $L^1(\Omega) \times L^\infty(\Omega) \times W_l^{1,2}(\Omega)$ satisfying the constraint*

$$\forall t \in \overline{\Omega}, \quad \int_0^t r(s)\mathrm{e}^{i\theta(s)}\,\mathrm{d}s = \chi(t).$$

For any $\chi \in C(\overline{\Omega})$, we have

$$E(\chi) = \inf_{(r,\theta,\phi)\in X_\chi} \left\{ \int_\Omega \left(\frac{1}{|\nu|}(\phi'(t))^2 + f^{hom}\big(r(t), \theta(t) - \phi(t)\big) \right) \mathrm{d}t) \right\} \tag{6.1}$$

where $f^{hom}(r,\gamma) := g_{hom}(r\mathrm{e}^{i\gamma})$ *is defined by setting, for any* $z \in \mathbb{C}$,

$$g_{hom}(z) := \inf_{\rho \in L^1(Y)} \left\{ \int_Y f(y,\rho(y))\,\mathrm{d}y : \int_Y \rho(y)\mathrm{e}^{i\theta^0(y)}\,\mathrm{d}y = z \right\}. \tag{6.2}$$

Note that, as soon as $g_{hom}(z) < +\infty$, owing to coerciveness assumption (3.3), the infimum problem (6.2) admits minimizers.

Proof. Let us denote the right hand side of (6.1) by

$$G(\chi) := \inf_{\phi \in W_l^{1,2}(\Omega)} \left\{ \int_\Omega \left(\frac{1}{|\nu|}(\phi'(t))^2 + g^{hom}\big(\chi'(t)\mathrm{e}^{-i\phi(t)}\big) \right) \mathrm{d}t) \right\}.$$

Let us first prove that $E(\chi) \geq G(\chi)$. Consider χ such that $E(\chi) < +\infty$ and $\varepsilon > 0$. There exist $\overline{\rho} \in L^1(\Omega \times Y)$ and $\phi \in W_l^{1,2}(\Omega)$ such that

$$E(\chi) + \varepsilon \geq \int_\Omega \frac{1}{|\nu|}(\phi'(t))^2\,\mathrm{d}t + \int_{\Omega\times Y} f(y,\overline{\rho}(t,y))\,\mathrm{d}t\,\mathrm{d}y \quad \text{and} \quad \int_Y \overline{\rho}(t,y)\mathrm{e}^{i\theta^0(y)}\,\mathrm{d}y = \chi'(t)\mathrm{e}^{-i\phi(t)}.$$

By definition of g^{hom}, we have $E(\chi) + \varepsilon \geq G(\chi)$. The result is obtained by letting ε tend to zero.

For proving $E(\chi) \leq G(\chi)$, we consider χ such that $G(\chi) < +\infty$. Let $\varepsilon > 0$ and choose $\phi \in W_l^{1,2}(\Omega)$ such that

$$G(\chi) + \frac{\varepsilon}{2} \geq \int_\Omega \left(\frac{1}{|\nu|}(\phi'(t))^2 + g^{hom}(z(t)) \right) \mathrm{d}t \quad \text{with} \quad z(t) := \chi'(t)\mathrm{e}^{-i\phi(t)}.$$

Let us remark that z belongs to $L^1(\Omega)$ and introduce $z_n^k := n \int_{\frac{k-1}{n}}^{\frac{k}{n}} z(t)\,\mathrm{d}t$ and the piecewise constant approximation $z_n(t) := \sum_{k=1}^n z_n^k 1_{[\frac{k-1}{n},\frac{k}{n}]}(t)$ and the corresponding placement $\chi_n \in W_l^{1,1}(\Omega)$ defined by $\chi_n' = z_n \mathrm{e}^{i\phi}$. By definition of g^{hom} there exist functions in $\rho_n^k \in L^1(Y)$ such that

$$g^{hom}(z_n^k) + \frac{\varepsilon}{2} \geq \int_Y f(y,\rho_n^k(y))\,\mathrm{d}y \quad \text{and} \quad \int_Y \rho_n^k(y)\,\mathrm{e}^{i\theta^0(y)}\mathrm{d}y = z_n^k.$$

Setting $\overline{\rho}_n(t,y) := \sum_{k=1}^n \rho_n^k(y) 1_{[\frac{k-1}{n},\frac{k}{n}]}(t)$, we have $\int_Y \overline{\rho}_n(t,y)\,\mathrm{e}^{i\theta^0(y)}\mathrm{d}y = z_n(t) = \chi_n'(t)\mathrm{e}^{-i\phi(t)}$ for almost every t in Ω. Hence

$$\begin{aligned} E(\chi_n) &\leq \int_\Omega \frac{1}{|\nu|}(\phi'(t))^2\mathrm{d}t + \int_{\Omega\times Y} f(y,\overline{\rho}_n(t,y))\,\mathrm{d}t\,\mathrm{d}y \\ &\leq \int_\Omega \frac{1}{|\nu|}(\phi'(t))^2\mathrm{d}t + \sum_{k=1}^n \frac{1}{n}\int_Y f(y,\rho_n^k(y))\,\mathrm{d}y \\ &\leq \int_\Omega \frac{1}{|\nu|}(\phi'(t))^2\mathrm{d}t + \sum_{k=1}^n \frac{1}{n} g^{hom}(z_n^k) + \frac{\varepsilon}{2}. \end{aligned}$$

By convexity, we obtain

$$E(\chi_n) \leq \int_\Omega \frac{1}{|\nu|}(\phi'(t))^2\mathrm{d}t + \int_\Omega g^{hom}(z(t)) + \frac{\varepsilon}{2} \leq G(\chi) + \varepsilon.$$

The sequence $E(\chi_n)$ is bounded. Then, up to a subsequence, χ_n converges uniformly. As χ'_n converges to χ' in the sense of distributions, χ_n converges uniformly to χ. As E is lower semi-continuous for this topology, we can pass to the limit and get $E(\chi) \leq G(\chi) + \varepsilon$. The result is proven by letting ε tends to zero. □

Remark 6.2. Expression (6.1) corresponds to the energy of a Timoshenko-type model.

Clearly $f^{hom}(r,\gamma)$ is 2π periodic with respect to γ. It is non negative and vanishes when $(r,\gamma) = (\ell, 0)$ (indeed choosing $\rho = 1$ solves the minimization problem in that case). At rest, the effective beam appears as a straight segment of length ℓ described by the placement map $t \in \Omega \mapsto t\ell \in \mathbb{C}$. Hence t is no more a curvilinear abscissa in the reference configuration. In order to compare the model with models found in the literature, it is pertinent to use the curvilinear abscissa $s = t/\ell$ when writing the equilibrium problem

$$\inf_{\chi \in C(\overline{\Omega})} \{E(\chi) - V(\chi)\}$$

or equivalently (their respective solutions being related by $\chi' = r\mathrm{e}^{i\theta}$)

$$\inf_{(r,\theta,\phi)\in X} \left\{ \int_\Omega \left(\frac{1}{|\nu|} (\phi'(t))^2 + f^{hom}\big(r(t), \theta(t) - \phi(t)\big) + r(t)\,\mathrm{e}^{i\theta(t)} \cdot G(t) \right) \mathrm{d}t \right\}. \tag{6.3}$$

So, we make the following change of variables: we set $\Omega_\ell := [0, \ell]$ and for any $s \in \Omega_\ell$ and $(\tilde{r}, \tilde{\Phi}) \in \mathbb{R}^2$,

$$\tilde{r}(s) := \frac{1}{\ell} r(\frac{s}{\ell}), \quad \tilde{\theta}(s) := \theta(\frac{s}{\ell}), \quad \tilde{\phi}(s) := \phi(\frac{s}{\ell}), \quad \tilde{\chi}(s) = \chi(\frac{s}{\ell}),$$

$$\tilde{f}^{hom}(\tilde{r}, \tilde{\gamma}) := \frac{1}{\ell} f^{hom}(\ell\tilde{r}, \tilde{\gamma}).$$

The elastic energy reads

$$\int_{\Omega_\ell} \left(\tilde{b}^{hom} (\tilde{\phi}'(s))^2 + \tilde{f}^{hom}(\tilde{r}(s), \tilde{\theta}(s) - \tilde{\phi}(s)) \right) \mathrm{d}t \tag{6.4}$$

where $\tilde{b}^{hom}$ is the positive constant $\tilde{b}^{hom} := \ell|\nu|^{-1}$. Now the energy vanishes when $\tilde{r} = 1$ and $\tilde{\theta} = \tilde{\phi} = 0$. We get a beam parameterized by the curvilinear abscissa at rest. When deformed, the beam is described by a placement map $\tilde{\chi}$ with $\tilde{\chi}' = \tilde{r}\mathrm{e}^{i\tilde{\theta}}$ and an extra real kinematic variable $\tilde{\phi}$. Its elastic energy contains a bending part associated to the spatial variation of $\tilde{\phi}$ with a constant stiffness $\tilde{b}^{hom}$ and a term coupling the extension $\tilde{r}$ of the beam and the discrepancy between $\tilde{\phi}$ and the orientation $\tilde{\theta}$ of the beam. This will appear clearly in the next subsections where we consider different examples for which the limit energy can be explicitly computed. These examples show that the beam can exhibit very different properties depending on its geometry and material properties at the microscopic level.

We let the reader check that, in all the considered examples, the cell problem (6.2) can be directly solved. We prefer to use instead an easy and systematic way for explicitly computing f^{hom} based on the convexity property of g_{hom}. Indeed, let us denote f^* and g^*_{hom} the Fenchel conjugates of f and g_{hom} defined by

$$f^*(y, \rho^*) := \sup_{\rho\in\mathbb{R}} \{\rho^*\,\rho - f(y,\rho)\} \tag{6.5}$$

$$g^*_{hom}(\lambda, \mu) := \sup_{(a,b)\in\mathbb{R}^2} \{\lambda a + \mu b - g_{hom}(a + ib)\}. \tag{6.6}$$

We have

$$\begin{aligned} g^*_{hom}(\lambda,\mu) &= \sup_{(a,b)\in\mathbb{R}^2}\left\{\lambda a+\mu b-\inf_{\rho\in L^1(Y)}\left\{\int_Y f(y,\rho(y))\,\mathrm{d}y : \int_Y \rho(y))\mathrm{e}^{i\theta_0(y)}\,\mathrm{d}y = a+ib\right\}\right\} \\ &= \sup_{\rho\in L^1(Y)}\int_Y \Big(\rho(y)\big(\lambda\cos(\theta_0(y))+\mu\sin(\theta_0(y))\big)-f(y,\rho(y))\Big)\,\mathrm{d}y \\ &= \int_Y \sup_{\rho\in\mathbb{R}}\Big\{\rho\big(\lambda\cos(\theta_0(y))+\mu\sin(\theta_0(y))\big)-f(y,\rho)\Big\}\,\mathrm{d}y \\ &= \int_Y f^*\big(y,\lambda\cos(\theta_0(y))+\mu\sin(\theta_0(y))\big)\,\mathrm{d}y. \end{aligned} \tag{6.7}$$

Hence, for computing f^{hom}, it is enough to compute f^* by (6.5), g^*_{hom} by (6.7), and finally to recover g_{hom} by using its lower-semi-continuity and convexity properties, which imply

$$g_{hom}(a+ib) = \sup_{(\lambda,\mu)\in\mathbb{R}^2}\{\lambda a+\mu b-g^*_{hom}(\lambda,\mu)\}. \tag{6.8}$$

6.1. Classical Euler beam

This first example is not very exciting and actually does not provide a true effective Timoshenko beam model. However, we must present it for situating our study with respect to classical beam modeling.

It is well known that, for a thin elastic object, the stiffness with respect to extension is, in general, much larger than the stiffness with respect to bending. Hence assuming that the beam is inextensible at the microscopic level is very reasonable. Incorporating this assumption in our setting is immediate: it is enough to set $f(y,\rho)=0$ if $\rho=1$, $f(y,\rho)=+\infty$ otherwise. This leads to the constraint $\rho=1$ almost everywhere. The procedure described above is easy to follow. We get successively $f^*(y,\rho^*)=\rho^*$, $g^*_{hom}(\lambda,\mu)=\lambda\ell$, and finally $g_{hom}(z)=0$ if $z=\ell$, $g_{hom}(z)=+\infty$ otherwise. That is

$$\tilde f^{hom}(\tilde r,\tilde\gamma)=0 \quad \text{if } \tilde r\mathrm{e}^{i\tilde\gamma}=1, \quad \tilde f^{hom}(\tilde r,\tilde\gamma)=+\infty \quad \text{otherwise.}$$

Up to a global multiple of π we get $\tilde\gamma=0$, that is $\tilde\phi=\tilde\theta$ and $\tilde r=1$ everywhere. The limit beam model becomes in this specific case a classical inextensible Euler beam model with the elastic energy $\int_{\Omega_\ell}\tilde b^{hom}(\tilde\theta'(\tilde s))^2\,\mathrm{d}\tilde s$. The beam is shorter and more flexible than the original one. The reader may notice that a finite bending compliance at the microscopic level does not induce a finite extensional compliance at the macroscopic level: despite its microscopic geometry which may suggest a spring, the effective model is not those of an extensional spring. This counterintuitive result is due to our assumption that the bending stiffness of the original beam is strong enough to ensure coerciveness. Considering a lower order of magnitude for this stiffness could result in a finite extensibility for the homogenized model. But that would be to the price of a vanishing effective bending stiffness. In other words one would then obtain a string model.

6.2. Linear constitutive law

Assume that the constitutive law of the beam is everywhere linear. This assumption is particularly relevant if the loading is such that the value taken by ρ remains everywhere sufficiently close to 1 so that the function f can be well approximated by: $f(y,\rho)=\frac{a(y)}{2}(\rho-1)^2$.

A straightforward computation gives $f^*(y,\rho^*)=\rho^*+\frac{1}{2a(y)}(\rho^*)^2$ and consequently

$$g^*_{hom}(\lambda,\mu)=\ell\lambda+\frac{1}{2}(A\lambda^2+2B\lambda\mu+C\mu^2)$$

with

$$A := \int_0^1 \frac{\cos^2(\theta^0(y))}{a(y)}\,\mathrm{d}y,\ B := \int_0^1 \frac{\cos(\theta^0(y))\sin(\theta_0(y))}{a(y)}\,\mathrm{d}y,\ C := \int_0^1 \frac{\sin^2(\theta^0(y))}{a(y)}\,\mathrm{d}y.$$

Computing the conjugate of a quadratic function is again straightforward. Noticing that the determinant $D := AC - B^2$ is positive as soon as θ^0 is not a constant function, we get

$$g_{hom}(a+ib) = \frac{1}{2}\frac{C(\ell-a)^2 + 2B(\ell-a)b + Ab^2}{D}.$$

Hence we obtain the following general effective energy density, valid when the material behavior is linear:

$$\tilde{f}^{hom}(\tilde{r},\tilde{\Phi}) = \frac{\ell}{2}\frac{C(1-\tilde{r}\cos(\tilde{\Phi}))^2 + 2B(1-\tilde{r}\cos(\tilde{\Phi}))\tilde{r}\sin(\tilde{\Phi}) + A(\tilde{r}\sin(\tilde{\Phi}))^2}{D}.$$

Recalling that $\tilde{\Phi} = \tilde{\theta} - \tilde{\phi}$, this energy corresponds to a non-linear Timoshenko model for the beam. We emphasize the fact that, even if $\tilde{f}^{hom}$ is a non-negative quadratic function of $(\tilde{r}\cos(\tilde{\Phi}), \tilde{r}\sin(\tilde{\Phi}))$, it is not a convex function of $(\tilde{r}, \tilde{\Phi})$. Instabilities can result from this lack of convexity. In order to illustrate this, we consider in Section 6.2.1 a specific test.

6.2.1. Tensile test of a beam with linear constitutive law

We focus on the geometry illustrated in Figure 1b and precisely defined by

$$\theta^0(y) := \begin{cases} \frac{\pi}{2} & \text{if } y \in [0, \frac{2}{10}] \ \cup\ [\frac{8}{10}, 1], \\ 0 & \text{if } y \in [\frac{2}{10}, \frac{3}{10}] \ \cup\ [\frac{7}{10}, \frac{8}{10}], \\ -\frac{\pi}{2} & \text{if } y \in [\frac{3}{10}, \frac{7}{10}]. \end{cases}$$

We assume that the beam is made by a single linear material characterized by a constant stiffness $a(y) = a > 0$. In that case, it is easy to compute the quantities ℓ, C, S and D. We have

$$\ell = \frac{1}{5}, \quad A = \frac{1}{5a}, \quad B = 0, \quad C = \frac{4}{5a} \quad \text{and} \quad D = \frac{4}{25a^2}$$

and thus

$$\tilde{f}^{hom}(\tilde{r},\tilde{\Phi}) = a\big((1-\tilde{r}\cos(\tilde{\Phi}))^2 + \frac{1}{4}(\tilde{r}\sin(\tilde{\Phi}))^2\big).$$

Finally, we consider a simple tensile test: the beam is submitted to a given extension while ϕ is let free at both end-points. The macroscopic equilibrium corresponds to the minimization of

$$\int_{\Omega_\ell} \Big(\tilde{f}^{hom}(\tilde{r}(s), \tilde{\theta}(s) - \tilde{\phi}(s)) + \tilde{b}^{hom}(\tilde{\phi}'(s))^2\Big)\,\mathrm{d}s$$

under the constraint $\int_{\Omega_\ell} \tilde{r}(s)\cos(\tilde{\theta}(s))\,\mathrm{d}s = k\ell$ with $k \geq 1$. Let us look for a uniform equilibrium solution. As $\tilde{f}^{hom}(\tilde{r}, \tilde{\phi})$ is a an increasing function of $\tilde{r}$ on $[1, +\infty[$, it is optimal to choose $\tilde{\theta} = 0$, $\tilde{r} = k$ and $\tilde{\phi}$ which minimizes $\tilde{f}^{hom}(k, -\tilde{\phi})$. Let us compute this optimal value and the corresponding energy $\tilde{g}^{hom}(k) := \min_{\tilde{\Phi}} \tilde{f}^{hom}(k, -\tilde{\phi})$. For $k \leq \frac{4}{3}$, the minimum is reached when $\tilde{\phi} = 0$ while, for $k > \frac{4}{3}$, $\tilde{\phi} = 0$ is a local maximum and the minimum is reached at the two opposite values $\tilde{\phi} = \pm \arccos\left(\frac{4}{3k}\right)$. These uniform equilibrium solutions are illustrated in

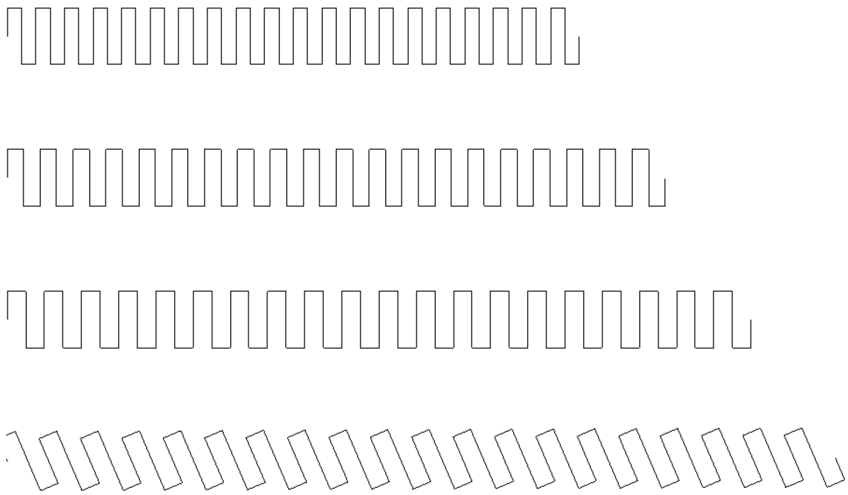

FIGURE 2. Deformation of the curve of Figure 1b when it is uniformly extended with factor 0%, 15%, 30% and 45%. The beam is assumed here to be made of a single material characterized by a linear law. One can see that, when the extension becomes larger than 33%, that is when $k > \frac{4}{3}$, the cells lean (indifferently forward or backward).

Figure 2. When the extension of the beam increases, a bifurcation occurs and the Timoshenko variable begins to be active.

The global effective elastic energy associated to these uniform extension states is now easy to compute. As long as the extension k remains in the interval $\left[1, \frac{4}{3}\right]$, it reads $\tilde{g}^{hom}(k) = \tilde{f}^{hom}(k, 0) = a(1-k)^2$ and thus is similar to the extensional energy of the beam at the microscopic level. But, when the extension becomes larger that $\frac{4}{3}$, the energy becomes $\tilde{g}^{hom}(k) = \tilde{f}^{hom}(k, \arccos\left(\frac{4}{3k}\right)) = \frac{a}{4}\left(k^2 - \frac{4}{3}\right)$ and thus the beam becomes much softer. Anyway, the function $\tilde{g}^{hom}$ remains convex. Hence the global energy is achieved by a constant function $\tilde{r}$: the equilibrium state is actually uniform in the situation considered in this subsection where ϕ has been assumed to be free on the boundary. If, instead, Dirichlet boundary conditions for ϕ have been assumed (like we did in the previous sections), then the bifurcation will happen for larger value of the extension and will be associated with non-uniform equilibrium solutions in $\tilde{\phi}$ but also in $\tilde{\theta}$ and $\tilde{r}$: the beam would not remain straight at equilibrium and a buckling phenomenon in extension would then arise.

Note that $\tilde{g}^{hom}$, being piecewise quadratic, is not a quadratic function: even if the constitutive law of the material at the microscopic level is linear, the effective behavior is no more linear. This is due to the geometrically non-linearity which comes into play at the microscopic level.

6.3. A mechanical diode

In [8], the very special properties of a particular elastic beam has been put forward. This study has been at the origin of the present work. From the macroscopic point of view, the peculiarity of the considered beam is that it bears extension but not contraction, behaving like a mechanical diode. From the microscopic point of view, its extensional compliance is assumed to be located only in the part of the beam orthogonal to the mean line. Let us take into account this assumption in our setting, by assuming that there exists a 1-periodic set Q such that the Lebesgue measure

$$m := |Q \cap Y| \tag{6.9}$$

is positive and such that

$$y \in Q \implies \mathrm{e}^{i\theta^0(y)} = -i. \tag{6.10}$$

This is the case for the examples presented in Figures 1b and 1c. We assume also that extension is possible only in Q

$$y \notin Q \text{ and } \rho \neq 1 \implies f(y,\rho) = +\infty. \tag{6.11}$$

and, for sake of simplicity, that the extensional behavior is uniform in Q (f does not depend on y there). A straightforward computation leads to

$$f^*(y,\rho^*) = \begin{cases} \rho^* & \text{if } \notin Q \\ f^*(\rho^*) & \text{if } y \in Q. \end{cases} \tag{6.12}$$

Hence, taking into account the fact that $\cos(\theta^0) = 0$ and $\sin(\theta^0) = -1$ on Q and reminding that $\int_Y \mathrm{e}^{i\theta^0(y)}\mathrm{d}y = \ell$,

$$\begin{aligned} g^*_{hom}(\lambda,\mu) &= \int_Y f^*(y, \lambda\cos(\theta^0(y)) + \mu\sin(\theta^0(y)))\,\mathrm{d}y \\ &= \int_Q f^*(-\mu)\,\mathrm{d}y + \int_{Y\setminus Q} \big(\lambda\cos(\theta^0(y)) + \mu\sin(\theta^0(y))\big)\,\mathrm{d}y \\ &= \int_Q \mu + f^*(-\mu)\,\mathrm{d}y + \int_Y \big(\lambda\cos(\theta^0(y)) + \mu\sin(\theta^0(y))\big)\,\mathrm{d}y \\ &= m\big(\mu + f^*(-\mu)\big) + \lambda\ell. \end{aligned}$$

Finally

$$\begin{aligned} g_{hom}(a+ib) &= \sup_\lambda \big(\lambda(a-\ell)\big) + m\sup_\mu \Big((1-\frac{b}{m})(-\mu) - f^*(-\mu)\Big) \\ &= \begin{cases} mf(1-\frac{b}{m}) & \text{if } a = \ell, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

We obtain $f^{hom}(r,\gamma) = mf\big(1 - \frac{r\sin(\gamma)}{m}\big)$ under the constraint $r\cos(\gamma) = \ell$. Using curvilinear abscissa, the effective extensional density energy reads

$$\tilde{f}^{hom}(\tilde{r},\tilde{\gamma}) := \begin{cases} \frac{m}{\ell}\, f\Big(1 - \frac{\ell\tan(\tilde{\gamma})}{m}\Big) & \text{if } \tilde{r}\cos(\tilde{\gamma}) = 1 \\ +\infty & \text{otherwise.} \end{cases} \tag{6.13}$$

Let us emphasize that any displacement of the beam with bounded energy satisfies $\tilde{r} \geq 1$: the beam is extensible but cannot contract.

Let us now resume the model, in the particular case studied in [8] where $f(\rho) = (\rho-1)^2$. The effective energy associated to a displacement $\tilde{\chi}$ parameterized by the curvilinear abscissa reads

$$\inf_{\tilde{r},\tilde{\theta},\tilde{\phi}} \left\{ \int_{\Omega_\ell} \frac{\ell}{m} \tan^2\big(\tilde{\theta}(s) - \tilde{\phi}(s)\big) + b^{hom}\big(\tilde{\phi}'(s)\big)^2\,\mathrm{d}s;\ \tilde{\chi}' = \tilde{r}\mathrm{e}^{i\tilde{\theta}};\ \tilde{r}\cos(\tilde{\theta} - \tilde{\phi}) = 1 \right\}.$$

We recover the result of [8].

6.4. A totally crushing beam

All the previous developments have been performed without using the assumption

$$\lim_{\rho\to 0^+} f(y,\rho) = +\infty, \tag{6.14}$$

forbidding ρ to vanish and become negative. One could expect that accepting this physical demand at the microscopic level would lead to a macroscopic model satisfying the same property. In this section we show that it is not always the case by focusing on the first example illustrated in Figure 1a, that is the case $\theta^0(t) = 2\sin(2\pi t)$. We assume that the function f satisfies (6.14) but that it takes finite values for any $\rho > 0$ (no part of the beam is inextensible nor have limited extension).

We subdivide Y into three parts depending on the direction θ^0: we set $A^+ = \{y \in Y : \theta^0(y) > \frac{\pi}{2}\}$, $A^- = \{y \in Y : \theta^0(y) < -\frac{\pi}{2}\}$ and $A = Y \setminus (A^+ \cup A^-)$. Owing to the symmetry $\theta^0(t) = -\theta^0(1-t)$, we have

$$\begin{aligned} a &:= -\int_{A^+} \cos(\theta^0(y))\,\mathrm{d}y = -\int_{A^-} \cos(\theta^0(y))\,\mathrm{d}y > 0, \quad \int_A \cos(\theta^0(y))\,\mathrm{d}y = \ell + a > 0,\\ b &:= \int_{A^+} \sin(\theta^0(y))\,\mathrm{d}y = -\int_{A^-} \sin(\theta^0(y))\,\mathrm{d}y > 0 \quad \text{and} \quad \int_A \sin(\theta^0(y))\,\mathrm{d}y = 0. \end{aligned}$$

Consider now the function $\rho_{\alpha,\beta,\delta}(y) := \alpha 1_{A^+}(y) + \beta 1_{A^-}(y) + \delta 1_A(y)$. We have

$$\int_Y \rho_{\alpha,\beta,\delta}(y)\mathrm{e}^{i\theta^0(y)}\,\mathrm{d}y = \alpha(-a+ib) + \beta(-a-ib) + \delta(\ell+a).$$

For any $(r,\gamma) \in \mathbb{R}^2$, we can choose positive quantities (α,β,δ) such that $r\mathrm{e}^{i\gamma} = \alpha(-a+ib)+\beta(-a-ib)+\delta(\ell+a)$ (indeed, as 0 belongs to the interior of the convex hull of $\big((-a+ib),(-a-ib),(\ell+a)\big)$, so does $r\mathrm{e}^{i\gamma}$ for $|r|$ small enough and it is then enough to multiply by a constant to reach any $r \in \mathbb{R}^+$). Hence, for any $(r,\gamma) \in \mathbb{R}^2$, we have

$$f^{hom}(r,\gamma) \le \int_Y f(y,\rho_{\alpha,\beta,\delta}(y))\,\mathrm{d}y < +\infty.$$

In particular $f^{hom}(0,\gamma) < +\infty$ and the beam can (theoretically) be completely crushed.

Let us now explain why, in the general case, the physical assumption (6.14) made at the microscopic level does not automatically lead to the same property at the macroscopic level. The point is that assumption (6.14) is not sufficient for ensuring that the microscopic model is physically well-founded. We have assumed (without using it) that the curve describing the beam at rest was simple. Nothing, in the model, ensures that the curve remains simple when it is deformed. This a well known defect of geometrically non linear problems which, generally, is lightly treated by simply saying that the model remains valid as long as no self-contact occurs.

Our homogenization context makes the situation more intricate. Of course, the homogenized model is still geometrically non-linear and thus affected by the same defect. But the local problem (6.2) is itself also affected: the true physical homogenized energy should be given by problem (6.2) under the constraint that the curve $t \mapsto \int_0^t \rho(y)\mathrm{e}^{i\theta^0(y)}\,\mathrm{d}y$ remains simple. This is a much more difficult mathematical problem that we will not treat here. We simply warn the reader that the homogenized model (6.1)–(6.2) must be used only when external forces ensure (i) that the equilibrium solution $\chi = \operatorname{argmin}_\chi(\tilde{E}(\chi) + \tilde{V}(\chi))$ is a simple curve and (ii) that, at almost every point t, the associated quantities $(r(t),\Phi(t))$ are such that the solution $\rho(t,y)$ of (6.2) corresponds also to a simple curve $s \mapsto \int_0^s \rho(y)\mathrm{e}^{i\theta^0(y)}\,\mathrm{d}y$. Figure 3 illustrates the fact that this limitation is effective much before r vanishes.

We have seen that a contraction up to a vanishing length may be theoretically allowed by our homogenized model. This is true for the beam considered in this subsection but many beams do not share this ability. Let us

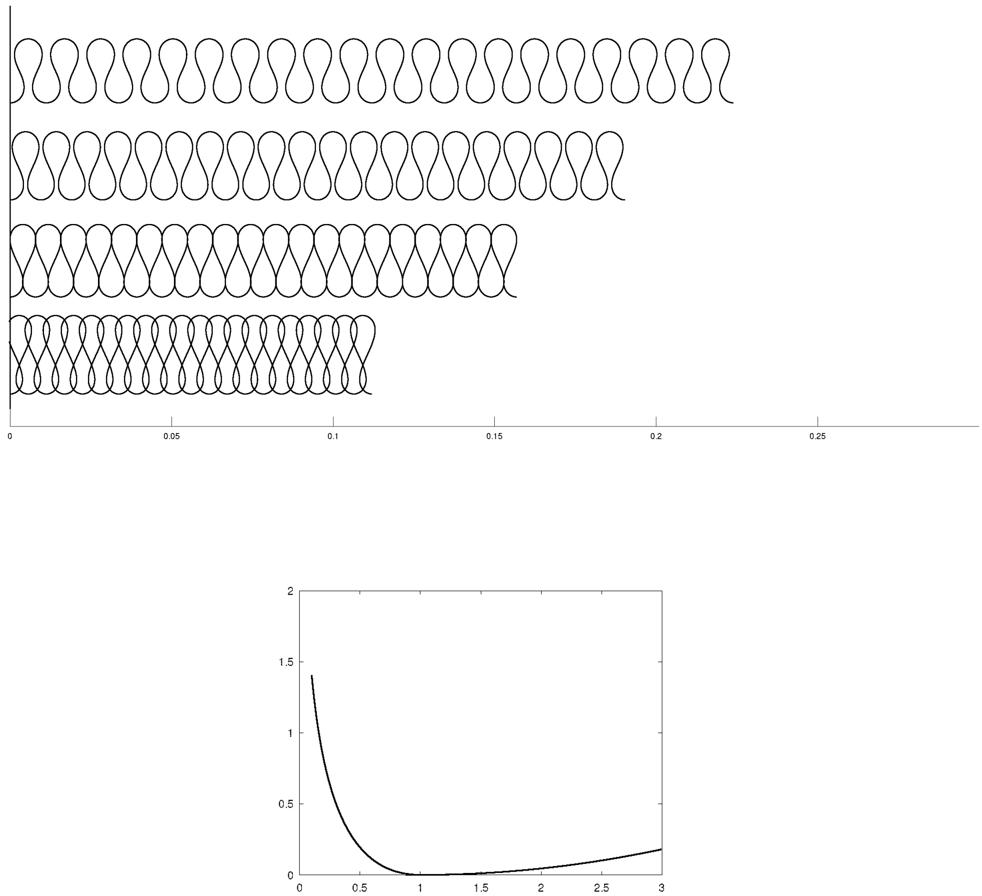

FIGURE 3. Deformation of the curve illustrated in Figure 1a when it is contracted with factor 0%, 15%, 30% and 50%. The beam is assumed here to be made of a single material characterized by the extensional energy $f(\rho)$ illustrated on the bottom. The curve on the top corresponds to the beam at rest. One can see that, when the contraction becomes larger than 30%, that is when $k < 0.7$, self-contact occurs. In that case, the energy given by formula (6.2) is no more physically acceptable. On the other hand, we emphasize the fact that the global contraction of the beam is not associated here to any local bending of the curve. Moreover, as at the microscopic level contraction is more energetic than extension, the global contraction of the beam is essentially due to the local extension of the parts of the beam going backwards.

consider two counter-examples. In the first one, the impossibility comes the microscopic geometry. If θ^0 takes values in $[-\pi/2, \pi/2]$ and function f satisfies (6.14), then $0 = |r(x)| \geq \left|\int_0^1 \rho(x,y)\cos(\theta^0(y))\,\mathrm{d}y\right|$ implies that $\rho = 0$ on a set with non-vanishing measure which is incompatible with a bounded energy. In the second counter-example the impossibility is due to the material properties at the microscopic level. Assume for instance that the function f imposes the local extension to stay inside a small interval: $f(y,\rho) < +\infty \implies \rho \in [1-a, 1+a]$ with $a < \ell$. Then we have

$$f^{hom}(r,\gamma) < +\infty \implies |r| = \left|\ell + \int_0^1 (\rho(x,y) - 1)\mathrm{e}^{i\theta^0(y)}\,\mathrm{d}y\right| \geq \ell - a > 0.$$

7. Conclusions

In this paper, we have presented a rigorous proof via Γ-convergence of the heuristic homogenization result obtained in [8]. The class of considered microstructures has been mathematically characterized by means of a rectifiable elastic curve having graded extensional and bending stiffnesses: this assumption is enough to recover the microstructure studied in [8]. Notably, the analysis developed in this paper, allows us to "distillate" those general features of duoskelion structures which are essential for obtaining the observed exotic diode behavior. Indeed, the results obtained in the present work can be applied not only to prove rigorously those obtained in [8], but also to conclude that the exotic mechanical diode behavior can be obtained when the extensional compliance is concentrated on the part of the curve that is orthogonal to the mean direction, *i.e.* the macroscopic beam's axis. At first, we prove that in the limit, when the microscopic extensional stiffness tends to infinity, then the macroscopic model reduces to the classical inextensible Euler–Bernoulli theory: extensional compliance cannot result from the bending compliance of a rapidly oscillating curve. Secondly, we show that the assumption of a linear material behavior at the microscopic level, does not result in a linear behavior after homogenization and that, therefore, a bifurcation phenomena can possibly arise even during a simple extension test. Finally, we recover as a special case the results found in [8], including those on the mechanical diode behavior. Among the various results obtained herein, it is also remarkable that, even if one prevents, through some suitable assumptions on the micro-scale stiffnesses, the length of the rectifiable elastic curve to reach zero and become negative, this demand is not preserved after homogenization.

The generalization of the results in [8] motivates and guides the investigation of different geometries than the one of the duoskelion and thus allows us to address the problem of optimizing the micro-scale geometry in the design of mechanical diodes according to technological manufacturing constraints [14]. This will pave the way towards the realization of mechanical diodes via 3D-printing [22]. It is also worth noting that the insight gained, thanks to the homogenization results presented in this paper, in the macroscopic duoskelion beam model, which has been proved to belong to the class of nonlinear models of Timoshenko type, is beneficial towards the understanding of the microscopic meaning of a more general class of continuum theories, namely micropolar ones [6].

Future challenges include the understanding of the homogenized dynamics of the beams considered in this work. Homogenization techniques could indeed shed further light on the dynamic behaviors observed in [23] and be beneficial in the search of further unusual deformation patterns or (quasi-)solitary waves in duoskelion beams [13].

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