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SHARP OBSERVABILITY INEQUALITIES FOR HYPERBOLIC SYSTEMS WITH POTENTIALS

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Abstract. This paper is devoted to a sharp internal/boundary observability inequality for a hyperbolic system with a zero order potential. For this purpose, we first establish a new Carleman estimate for hyperbolic operator in H^1-norm. Based on this Carleman estimate and a modified auxiliary optimal control problem, we obtain Carleman estimate for hyperbolic operator in L^2-norm. Then, by virtue of a modified energy estimate and a delicate treatment of the observation region, we obtain an internal observability estimate with the observability constant of the order $\exp(C||q||^{2/3}_{L^\infty(Q;\mathbb{R}^{N\times N})})$, with q the potential involved in the system. We also address the same problem for boundary observation. Compared with the related results in the literature, the main contributions of this paper are the observability constant is sharper, the waiting time T is shorter and the internal (or boundary) observation domain is smaller.

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1. Introduction

Let $n \geq 1$ and $N \geq 1$ be two integers, $T > 0$, Ω be a bounded domain of $\mathbb{R}^n$ with C^2 boundary Γ. Put $Q = (-T, T) \times \Omega$ and $\Sigma = (-T, T) \times \Gamma$.

Let $(h^{jk}(\cdot))_{1\leq j,k\leq n} \in C^2(\overline{\Omega}; \mathbb{R}^{n\times n})$ be such that

$$h^{jk}(x) = h^{kj}(x), \qquad \forall x \in \overline{\Omega}, \quad j,\ k = 1, \cdots, n, \tag{1.1}$$

and that

$$\sum_{j,k=1}^{n} h^{jk}(x)\xi^j\xi^k \geq h_0|\xi|^2, \qquad \forall (x,\xi) = (x, \xi^1, \cdots, \xi^n) \in \overline{\Omega} \times \mathbb{R}^n, \tag{1.2}$$

for some constant $h_0 > 0$.

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Consider the following hyperbolic system:

$$\begin{cases} w_{tt} - \sum_{j,k=1}^{n} (h^{jk} w_{x_j})_{x_k} = qw & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(-T) = w_0, \quad w_t(-T) = w_1 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where $w = (w_1, \cdots, w_N)^\top$ is an $\mathbb{R}^N$-valued unknown, $q \in L^\infty(-T, T; L^p(\Omega; \mathbb{R}^{N\times N}))$ for some $p \in [n, \infty]$.

The first purpose of this paper is to study the internal observability problem of (1.3), by which we mean the following: given $T > 0$ and a sub-domain $\mathcal{K}$ of Q, find (if possible) a constant $\mathcal{C}_1 = \mathcal{C}_1(q, (h^{jk})_{n\times n}, T, \mathcal{K}, \Omega) > 0$ such that the corresponding solution w of (1.3) satisfies

$$\|w_0\|^2_{L^2(\Omega)^N} + \|w_1\|^2_{H^{-1}(\Omega)^N} \le \mathcal{C}_1 \int_{\mathcal{K}} |w|^2 \mathrm{d}x\mathrm{d}t, \quad \forall (w_0, w_1) \in L^2(\Omega)^N \times H^{-1}(\Omega)^N. \tag{1.4}$$

The second purpose of this paper is to study the boundary observability problem of (1.3), by which we mean the following: given $T > 0$ and a sub-domain $\mathcal{H}$ of Σ, find (if possible) a constant $\mathcal{C}_2 = \mathcal{C}_2(q, (h^{jk})_{n\times n}, T, \mathcal{H}, \Omega) > 0$ such that the corresponding solution w of (1.3) satisfies

$$\|w_0\|^2_{H^1_0(\Omega)^N} + \|w_1\|^2_{L^2(\Omega)^N} \le \mathcal{C}_2 \int_{\mathcal{H}} \left|\frac{\partial w}{\partial \nu}\right|^2 \mathrm{d}\Gamma\mathrm{d}t, \quad \forall (w_0, w_1) \in H^1_0(\Omega)^N \times L^2(\Omega)^N. \tag{1.5}$$

Here $\nu = \nu(x) = (\nu_1, \nu_2, \ldots, \nu_n)$ denotes the unit outward normal vector of Ω at $x \in \Gamma$.

The inequality (1.4) (*resp.* (1.5)) implies that the initial energy of solutions can be estimated by the energy localized in a subdomain $\mathcal{K}$ (*resp.* $\mathcal{H}$) with an explicit observability constant. It is well known that such kind of inequalities play a key role in the unique continuation, controllability, stabilization and inverse problems of hyperbolic equations (*e.g.*, [1–10] and the rich references therein).

Remark 1.1. Clearly, whether the inequality (1.4) (*resp.* (1.5)) holds depends on the choice of the set $\mathcal{K}$ (*resp.* $\mathcal{H}$), the time T and the constant $\mathcal{C}_1$ (*resp.* $\mathcal{C}_2$), which are called the observation domain, the observation time and the observability constant, respectively. In the literature, people aims to establish observability estimate when the observation domain is as small as possible, the observation time is as short as possible, and the observability constant is as small as possible.

From [1], we know that one must put some further assumption on $(h^{jk}(\cdot))_{1\le j,k\le n}$, $\mathcal{K}$ (*resp.* $\mathcal{H}$) and T, otherwise the inequality (1.4) (*resp.* (1.5)) may not hold. We first introduce the following condition on $(h^{jk}(\cdot))_{1\le j,k\le n}$:

Condition 1.1. There exists a positive function $\varphi(\cdot) \in C^2(\overline{\Omega})$ satisfying the following:

(i) For some constant $s_0 \ge 0$, it holds

$$\sum_{j,k=1}^{n} \left\{ \sum_{j',k'=1}^{n} \left[2h^{jk'} (h^{j'k} \varphi_{x_{j'}})_{x_{k'}} - h^{jk}_{x_{k'}} h^{j'k'} \varphi_{x_{j'}} \right] \right\} \xi^j \xi^k \ge s_0 \sum_{j,k=1}^{n} h^{jk} \xi^j \xi^k, \quad \forall\, (x, \xi^1, \cdots, \xi^n) \in \overline{\Omega} \times \mathbb{R}^n; \tag{1.6}$$

(ii) There is no critical point of function $\varphi(\cdot)$ in $\overline{\Omega}$, *i.e.*,

$$\min_{x\in\overline{\Omega}} |\nabla \varphi(x)| > 0. \tag{1.7}$$

Remark 1.2. Condition 1.1 is a kind of pseudo-convex condition for the function φ with respect to $(h^{jk}(\cdot))_{1\le j,k\le n}$, which is natural for establishing a Carleman estimate for (1.3) (see [11], Sect. 28.2). Condition 1.1 is not very restrictive, *i.e.*, one can find many interesting examples with this property (*e.g.*, [3, 12]). For instance, when $(h^{jk}(\cdot))_{1\le j,k\le n} = I_n$, and for any given $x_0 \in \mathbb{R}^n\backslash\overline{\Omega}$, by choosing $\varphi(x) = |x-x_0|^2 - c_0^2$ (here c_0 is some constant satisfying $|c_0| < \min_{x\in\overline{\Omega}} |x-x_0|$), we can see that Condition 1.1 holds with $s_0 = 4$.

Before sketching what $\mathcal{K}$ and $\mathcal{H}$ are, we need some preliminaries. For the function $\varphi(\cdot)$ satisfying Condition 1.1, we put

$$\Gamma_0 \triangleq \left\{ x\in\Gamma \,\middle|\, \sum_{j,k=1}^n h^{jk}\varphi_{x_j}\nu_k > 0 \right\} \tag{1.8}$$

and $\Sigma_0 \triangleq (-T,T)\times\Gamma_0$.

Remark 1.3. It is easy to check that, if $\varphi(\cdot)\in C^2(\overline{\Omega})$ satisfies (1.6), then for any given constants $a\ge 1$ and $b\in\mathbb{R}$, the function

$$\hat{\varphi} = \hat{\varphi}(x) \triangleq a\varphi(x) + b \tag{1.9}$$

still satisfies Condition 1.1 with s_0 replaced by as_0. Clearly, the scaling and translating of $\varphi(x)$ do not change the set Γ_0. Hence, by scaling and translating, if necessary, we may assume without loss of generality that

$$\begin{cases} \text{(1.6) holds with } s_0\ge 4, \\ \dfrac{1}{4}\displaystyle\sum_{j,k=1}^n h^{jk}(x)\varphi_{x_j}\varphi_{x_k} \ge \varphi(x) > 0, \qquad \forall x\in\overline{\Omega}. \end{cases} \tag{1.10}$$

For any $\kappa>0$, we define

$$\mathcal{O}_\kappa(\Gamma_0) \triangleq \big\{ x\in\mathbb{R}^n \big|\ |x-x'|<\kappa \text{ for some } x'\in\Gamma_0 \big\}.$$

Let $\delta,\delta_0\in\mathbb{R}$ and $0<\delta_0<\delta$. Put

$$\omega \triangleq \mathcal{O}_\delta(\Gamma_0)\cap\Omega, \quad \omega_0 \triangleq \mathcal{O}_{\delta_0}(\Gamma_0)\cap\Omega, \quad Q_{\omega\backslash\omega_0} \triangleq (-T,T)\times(\omega\backslash\omega_0). \tag{1.11}$$

For any constant $\delta_1\in(0,1/2)$, let

$$\mathcal{D} \triangleq \big\{ (t,x)\in Q \big|\ \varphi(x) - t^2 > 0 \big\}. \tag{1.12}$$

Now we can give the definition of $\mathcal{K}$ and $\mathcal{H}$. Let

$$\mathcal{K} \triangleq \big[(-\delta_1 T,\delta_1 T)\times\omega_0\big] \cup (Q_{\omega\backslash\omega_0}\cap\mathcal{D}), \quad \mathcal{H} \triangleq \Sigma_0\cap\mathcal{D}. \tag{1.13}$$

In what follows, we shall denote by $C_* = C_*((h^{jk})_{n\times n}, T, \mathcal{K}, \Omega)$ and $C^* = C^*((h^{jk})_{n\times n}, T, \mathcal{H}, \Omega)$ generic positive constants, which may change from line to line.

In (1.13), we do not fix the observation time T. Due to the finite speed of propagation for solutions to (1.3), we know that T should be large enough to guarantee the inequalities (1.4) and (1.5). Now we give a characterization of T.

Set

$$R_0 \triangleq \min_{x\in\overline{\Omega}} \sqrt{\varphi(x)}, \quad R_1 \triangleq \max_{x\in\overline{\Omega}\setminus\omega} \sqrt{\varphi(x)}. \tag{1.14}$$

As said in Remark 1.3, once there is a φ satisfying Condition 1.1, one can always do some affine transformation to get a new function $\hat{\varphi}$ which fulfills both Condition 1.1 and (1.10). The Carleman estimate employing φ or $\hat{\varphi}$ as a weight function may be different. This will lead to the observation time T in the observability (1.4)/(1.5) being different. In such case, we introduce the following notation:

$$T_* \triangleq \inf\big\{R_1\big|\ \varphi(\cdot) \text{ satisfies Condition 1.1 and (1.10)}\big\}. \tag{1.15}$$

We have the following internal observability estimate for the system (1.3).

Theorem 1.4. *Let $T > T_*$ and $\mathcal{K}$ be given respectively by (1.13). Then for any $(w_0, w_1) \in L^2(\Omega)^N \times H^{-1}(\Omega)^N$, the corresponding weak solution $w \in C\big([-T,T]; L^2(\Omega)^N\big) \cap C^1\big([-T,T]; H^{-1}(\Omega)^N\big)$ of the system (1.3) satisfies the internal observability estimate (1.4) with the constant $\mathcal{C}_1$ replaced by $\mathcal{C}_*(r) \triangleq \exp\big[C_*(1 + r^{\frac{1}{3/2-n/p}})\big]$, where*

$$r \triangleq \|q\|_{L^\infty(-T,T;L^p(\Omega;\mathbb{R}^{N\times N}))}. \tag{1.16}$$

Remark 1.5. By [13], Theorem 1.2, we know that the exponent 2/3 in the constant $\mathcal{C}_*(r)$ for the special case $p = \infty$ is sharp in the sense that, when T is fixed, the observability constant has to grow, at least, at the order of $\exp\big(|q|^{\frac{2}{3}}_{L^\infty(Q;\mathbb{R}^{N\times N})}\big)$ as $|q|_{L^\infty(Q;\mathbb{R}^{N\times N})} \to \infty$.

Remark 1.6. If $\varphi(\cdot) \in C^2(\overline{\Omega})$ satisfies Condition 1.1 and (1.10), then it is easy to see that $\hat{\varphi}(\cdot) \triangleq \varphi(\cdot) - c_0^2$ (here c_0 is some constant satisfying $|c_0| < \min\limits_{x\in\overline{\Omega}} \sqrt{\varphi(x)}$) still satisfies Condition 1.1 and (1.10). For such $\hat{\varphi}$, the set $\mathcal{D}$ defined in (1.12) and T_* in (1.15) become respectively

$$\mathcal{D} = \big\{(t,x) \in Q\big|\ \varphi(x) - t^2 > c_0^2\big\}, \quad T_* = \left(\max_{x\in\overline{\Omega}\setminus\omega} \varphi(x) - \min_{x\in\overline{\Omega}} \varphi(x)\right)^{\frac{1}{2}}.$$

By Theorem 1.4, we know that (1.4) still holds for such $\mathcal{D}$ and T_*.

Remark 1.7. In the case $(h^{jk})_{n\times n} = I_n$, for any given $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$, by choosing $\varphi(x) = |x - x_0|^2 - c_0^2$ (where c_0 is some constant satisfying $|c_0| < \min_{x\in\overline{\Omega}} |x - x_0|$), it is easy to see that $\varphi(x)$ satisfies Condition 1.1 and (1.10). In this situation, we have

$$\mathcal{D} = \big\{(t,x) \in Q\big|\ |x - x_0|^2 - t^2 > c_0^2\big\}, \quad T_* = \left(\max_{x\in\overline{\Omega}\setminus\omega} |x - x_0|^2 - \min_{x\in\overline{\Omega}} |x - x_0|^2\right)^{\frac{1}{2}}. \tag{1.17}$$

In case of the boundary observability estimate for system (1.3), we set

$$\begin{cases} R^1 \triangleq \max\limits_{x\in\overline{\Omega}} \sqrt{\varphi(x)}, \\ T^* \triangleq \inf\big\{R^1\big|\ \varphi(\cdot) \text{ satisfies Condition 1.1 and (1.10) }\big\}. \end{cases} \tag{1.18}$$

We have the following result.

Theorem 1.8. *Let $T > T^*$ and $\mathcal{H}$ be given respectively by (1.13). Then for any $(w_0, w_1) \in (H_0^1(\Omega))^N \times L^2(\Omega)^N$, the corresponding weak solution $w \in C([-T,T]; H_0^1(\Omega)^N) \cap C^1([-T,T]; L^2(\Omega)^N)$ of the system (1.3) satisfies the boundary observability estimate (1.5) with the constant $\mathcal{C}_2$ replaced by $\mathcal{C}^*(r) \triangleq \exp\left[C^*(1 + r^{\frac{1}{3/2-n/p}})\right]$, where r is given by (1.16).*

Remark 1.9. In the case $(h^{jk})_{n\times n} = I_n$, for any given $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$, by choosing $\varphi(x) = |x - x_0|^2 - c_0^2$ (here c_0 is some constant satisfying $|c_0| < \min_{x\in\overline{\Omega}} |x - x_0|$), it is easy to see that $\varphi(x)$ satisfies Condition 1.1 and (1.10). In this situation,

$$T^* = \left(\max_{x\in\overline{\Omega}} |x - x_0|^2 - \min_{x\in\overline{\Omega}} |x - x_0|^2\right)^{\frac{1}{2}},$$

and

$$\mathcal{D} = \left\{(t,x) \in \overline{Q} \middle| \ |x - x_0|^2 - t^2 > c_0^2\right\}, \quad \mathcal{H} = \Sigma_0 \cap \mathcal{D}.$$

Generally speaking, there are mainly four methods for establishing observability estimates for hyperbolic equations:

The first one is based on the Ingham type inequality (*e.g.*, [14]). This method works well for a variety of 1-d problems in which the Fourier representation of solutions can be used (mainly when the coefficients are time-independent) and the gap condition holds. However, it seems that it does not work for wave equations with coefficients depending both on x and t, or with a lower order potential as the system (1.3).

The second one is the classical Rellich-type multiplier approach (*e.g.*, [6]). It is used to treat wave equations with time independent lower order terms. However, it seems that it does not work for our problem since q is time dependent.

The third one is the microlocal analysis approach (*e.g.*, [1]). It is useful to solve controllability problems for many kinds of PDEs such as wave equations, Schrödinger equations and plate equations. Further, it can give a sharp sufficient condition for the observability of hyperbolic equations when q is independent of t. However, this method does not give the explicit dependence of the observability constant on the potential q. Further, it is unknown how to use this approach to establish observability estimates for (1.3) when q depends on t.

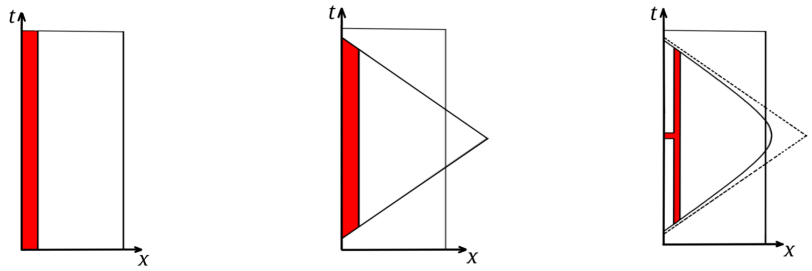
The last one is the global Carleman estimate (*e.g.*, [2, 3, 9, 10, 13, 15–18] and the rich references therein). It can be viewed as a more developed version of the classical Rellich-type multiplier approach. This approach has the advantage of being more flexible and allowing to address variable coefficients. Further, compared with the previous three methods, it is robust with respect to the lower order terms and allows to get explicit bounds on the observability constant in terms of the potentials entering in it. This is particularly important in the study of controllability problems for semilinear hyperbolic equations. Indeed, when dealing with controllability problems of semilinear equations by means of linearization and fixed point arguments, the key point is the explicit estimate of the observability constant by a suitable function of the norm of the potential (*e.g.*, [15]).

Now let us recall recent works related to the observability estimate of (1.4) obtained by Carleman estimate in the literature and explain the novelty of our works in this paper.

In [3], Chapter 4, it was proved that the internal observability estimate (1.4) holds when

$$T \gg T_*, \quad \mathcal{K} = (-T,T) \times \mathcal{O}_\delta\big(\{x \in \Gamma | (x - x_0)\cdot\nu(x) \geq 0\}\big)$$

and the observability constant $\mathcal{C}_*(r) = \exp\left[C_*(1 + r^{\frac{1}{3/2-n/p}})\right]$. Compared with Theorem 1.4, the observation time in [3], Chapter 4 is longer and the observation domain in [3], Chapter 4 is larger.



(a) (b) (c)

FIGURE 1. For the wave equation in one dimensional case, the red regions in (a) and (b) are the observation regions given in [3], Theorem 2.1 and [17], Theorem 1.2 (in the case of $q(\cdot) \in C^\infty(Q)$), respectively. The red region in (c) is our result stated in Theorem 1.4.

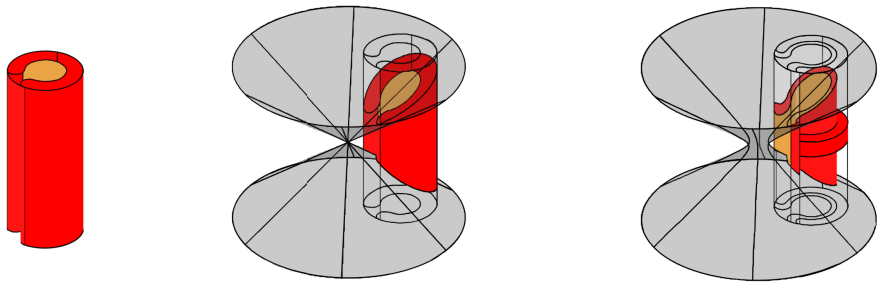


FIGURE 2. The changes of the observation regions for the wave equations in multidimensional case.

When $(h^{jk}(\cdot))_{n\times n} = I_n$ and $q(\cdot) \in C^\infty(Q)$, it was proved in [17] that the observability estimate (1.4) holds for

$$\begin{cases} T > \max\limits_{x\in\Omega} |x - x_0|, \\ \mathcal{K} = \big[(-T,T) \times \mathcal{O}_\delta\big(\{x \in \Gamma | (x - x_0) \cdot \nu(x) \geq 0\}\big)\big] \cap \big\{(t,x) \in \mathbb{R}^{1+n} \big| |x - x_0|^2 > t^2\big\}, \end{cases}$$

where $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ and the observability constant is $C\mathrm{e}^{C\|q\|^2_{L^\infty(Q)}}$. The regularity condition in [17] is relaxed in [19] as $q(\cdot) \in L^\infty(Q)$. From Theorem 1.4, we see that our observation domain is a subset of the one in [17, 19]. Further, our observability constant is sharper, which is important in the study of controllability problems for hyperbolic equations. Indeed, by Theorem 1.4, one can show the exact controllability of semilinear hyperbolic equations in which the semilinear term increases as the function $s(\ln s)^{3/2}$. On the other hand, when the observability constant is $C\mathrm{e}^{C\|q\|^2_{L^\infty(Q)}}$, one can only handle the equation with the semilinear term increases as the function $s(\ln s)^{1/2}$.

We refer to Figure 1 for the changes of observation regions in one dimensional case, and Figure 2 for the changes of observation regions in multidimensional case.

When $(h^{jk}(\cdot))_{n\times n} = I_n$, Huang–Imanuvilov–Yamamoto [20] proves the observability estimate (1.4) holds for

$$\mathcal{H} = (-T,T) \times \{x \in \Gamma | (x - x_0) \cdot \nu(x) \geq 0\}, \quad T > \left(\max_{x\in\overline{\Omega}} |x - x_0|^2 - \min_{x\in\overline{\Omega}} |x - x_0|^2\right)^{\frac{1}{2}}.$$

The observation time is the same as the one given in Theorem 1.8 and the observation domain is larger than the one given in Theorem 1.8. Further, the dependence of the observability constant on q is not provided in [20].

When $(h^{jk}(\cdot))_{n\times n} = I_n$, Shao [18] proves the observability estimate (1.4) holds for

$$\mathcal{H} = \big[(-T,T)\times\{x\in\Gamma|(x-x_0)\cdot\nu(x)\geq 0\}\big]\cap\big\{(t,x)\in\mathbb{R}^{1+n}\big|\ |x-x_0|^2>t^2\big\}$$

and give an observability constant in the same form of the one in Theorem 1.8. Nevertheless, compared with Theorem 1.8, the observation time in [18] is longer.

The rest of this paper is organized as follows. In Section 2, we prove a new Carleman estimate for hyperbolic operator in H^1-norm. In Section 3, we introduce an auxiliary optimal control problem which will play a key role in the proof of Carleman estimate for the hyperbolic operator in L^2-norm in Section 4. In Section 5, we give the proofs of our main results.

2. Carleman estimate for the hyperbolic operator in H^1-norm

This section is devoted to proving Carleman estimates for the hyperbolic operators in H^1-norm on a domain bigger than Q.

Recall (1.11) for the definitions of ω and ω_0. Let $\tilde{\delta}_1\in(\delta_0,\delta)$. Set

$$\omega_1 \triangleq \mathcal{O}_{\tilde{\delta}_1}(\Gamma_0)\bigcap\Omega. \tag{2.1}$$

Choose a $\widetilde{T}>0$ satisfying

$$\widetilde{T}>\widetilde{R}_1\triangleq\frac{1}{2}\max_{x\in\overline{\Omega}\backslash\omega_1}\left(\sum_{j,k=1}^n h^{jk}\varphi_{x_j}\varphi_{x_k}\right)^{\frac{1}{2}}. \tag{2.2}$$

Set

$$\widetilde{Q}\triangleq(-\widetilde{T},\widetilde{T})\times\Omega,\quad \widetilde{\Sigma}\triangleq(-\widetilde{T},\widetilde{T})\times\Gamma. \tag{2.3}$$

For some constant $\alpha\in(0,1)$ (which will be given later) and parameters $\lambda,\mu>0$, we choose the weight function θ as follows:

$$\theta=\mathrm{e}^{\ell},\quad \ell=\lambda\phi(\sigma),\quad \phi(\sigma)=\mathrm{e}^{\mu\sigma},\quad \sigma=\varphi(x)-\alpha t^2. \tag{2.4}$$

In the rest of this section, we will use $\widetilde{C}_*=\widetilde{C}_*((h^{jk})_{n\times n},\widetilde{T},\Omega,\omega_1)$ and $\widetilde{C}^*=\widetilde{C}^*((h^{jk})_{n\times n},\widetilde{T},\Omega,\Gamma_0)$ to denote generic positive constants which may vary from line to line. We have the following Carleman estimate.

Theorem 2.1. *Let $\widetilde{T}>\widetilde{R}_1$. Then there exist positive constants $\widetilde{C}_*>0$ and $\mu_0>0$ such that for all $\mu\geq\mu_0$, there exists a $\lambda_0=\lambda_0(\mu)>0$, so that for all $\lambda\geq\lambda_0$, and any $u\in H^1(\widetilde{Q})$ with $u_{tt}-\sum_{j,k=1}^n(h^{jk}u_{x_j})_{x_k}\in L^2(\widetilde{Q})$, $u|_{\widetilde{\Sigma}}=0$ and $\operatorname{supp}u\subset[-\widetilde{T},\widetilde{T}]\times(\overline{\Omega}\backslash\omega_1)$, it holds that*

$$\lambda\mu\int_{\widetilde{Q}}\theta^2\big(|u_t|^2+|\nabla u|^2+\lambda^2\mu^2|u|^2\big)\mathrm{d}x\mathrm{d}t\leq\widetilde{C}_*\int_{\widetilde{Q}}\theta^2\left|u_{tt}-\sum_{j,k=1}^n(h^{jk}u_{x_j})_{x_k}\right|^2\mathrm{d}x\mathrm{d}t. \tag{2.5}$$

In the case of boundary observability problems, we choose a $\widehat{T}$ satisfying

$$\widehat{T} > \widehat{R}^1 \triangleq \frac{1}{2}\max_{x\in\overline{\Omega}}\left(\sum_{j,k=1}^{n} h^{jk}\varphi_{x_j}\varphi_{x_k}\right)^{\frac{1}{2}}. \tag{2.6}$$

Put

$$\widehat{Q} \triangleq (-\widehat{T},\widehat{T})\times\Omega, \quad \widehat{\Sigma} \triangleq (-\widehat{T},\widehat{T})\times\Gamma, \quad \widehat{\Sigma}_0 \triangleq (-\widehat{T},\widehat{T})\times\Gamma_0. \tag{2.7}$$

We have the following boundary Carleman estimate.

Theorem 2.2. *Let $\widehat{T} > \widehat{R}^1$. Then there exist positive constants $\widetilde{C}^* > 0$ and $\mu_0 > 0$ such that for all $\mu \geq \mu_0$, there exists a $\lambda_0 = \lambda_0(\mu) > 0$ so that for all $\lambda \geq \lambda_0$, and any $u \in H^1(\widehat{Q})$ with $u_{tt} - \sum_{j,k=1}^{n}(h^{jk}u_{x_j})_{x_k} \in L^2(\widehat{Q})$ and $u|_{\widehat{\Sigma}} = 0$, it holds that*

$$\begin{aligned}&\lambda\mu\int_{\widehat{Q}}\theta^2\big(|u_t|^2+|\nabla u|^2+\lambda^2\mu^2u^2\big)\mathrm{d}x\mathrm{d}t\\ &\leq \widetilde{C}^*\left[\int_{\widehat{Q}}\theta^2\left|u_{tt}-\sum_{j,k=1}^{n}(h^{jk}u_{x_j})_{x_k}\right|^2\mathrm{d}x\mathrm{d}t+\lambda\mu\int_{\widehat{\Sigma}_0}\theta^2\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}\Gamma\mathrm{d}t\right].\end{aligned} \tag{2.8}$$

The proofs of Theorems 2.1 and 2.2 are based on the following weighted inequality.

Lemma 2.3. *([3], Cor. 4.1) Let $u \in C^2(\mathbb{R}^{1+n};\mathbb{R})$, $\ell \in C^3(\mathbb{R}^{1+n};\mathbb{R})$ and $\Psi \in C^1(\mathbb{R}^n;\mathbb{R})$. Set $\theta = \mathrm{e}^{\ell}$ and $v = \theta u$. Then,*

$$\begin{aligned}&\theta^2\Big|u_{tt}-\sum_{j,k=1}^{n}(h^{jk}u_{x_j})_{x_k}\Big|^2+2\mathrm{div}\,V+2M_t\\ &\geq 2\Big[\ell_{tt}+\sum_{j,k=1}^{n}(h^{jk}\ell_{x_j})_{x_k}+\Psi\Big]v_t^2-8\sum_{j,k=1}^{n}h^{jk}\ell_{tx_j}v_{x_k}v_t\\ &\quad+2\sum_{j,k=1}^{n}c^{jk}v_{x_j}v_{x_k}-2\sum_{j,k=1}^{n}h^{jk}\Psi_{x_j}vv_{x_k}+Bv^2,\end{aligned} \tag{2.9}$$

where

$$\begin{cases}A=\sum_{j,k=1}^{n}(h^{jk}\ell_{x_j}\ell_{x_k}-h^{jk}_{x_j}\ell_{x_k}-h^{jk}\ell_{x_jx_k})-\ell_t^2+\ell_{tt}-\Psi,\\ c^{jk}=\sum_{j',k'=1}^{n}\Big[2h^{jk'}(h^{j'k}\ell_{x_{j'}})_{x_{k'}}-(h^{jk}h^{j'k'}\ell_{x_{j'}})_{x_{k'}}\Big]+h^{jk}(\ell_{tt}-\Psi),\\ B=2\Big[A\Psi-(A\ell_t)_t+\sum_{j,k=1}^{n}(Ah^{jk}\ell_{x_j})_{x_k}\Big],\end{cases} \tag{2.10}$$

and

$$\begin{cases} V=[V^1,\cdots,V^k,\cdots,V^n], \\ V^k=2\displaystyle\sum_{j,j',k'=1}^n h^{jk}h^{j'k'}\ell_{x_{j'}}v_{x_j}v_{x_{k'}}+\sum_{j=1}^n h^{jk}A\ell_{x_j}v^2-\Psi v\sum_{j=1}^n h^{jk}v_{x_j} \\ \qquad -\displaystyle\sum_{j,j',k'=1}^n h^{jk}h^{j'k'}\ell_{x_j}v_{x_{j'}}v_{x_{k'}}-2\ell_t v_t\sum_{j=1}^n h^{jk}v_{x_j}+\sum_{j=1}^n h^{jk}\ell_{x_j}v_t^2, \\ M=\ell_t\Big(v_t^2+\displaystyle\sum_{j,k=1}^n h^{jk}v_{x_j}v_{x_k}\Big)-2\sum_{j,k=1}^n h^{jk}\ell_{x_j}v_{x_k}v_t+\Psi vv_t-A\ell_t v^2. \end{cases} \tag{2.11}$$

Proof of Theorem 2.1. We divide the proof into several steps.

Step 1. By (2.4), it is easy to check that

$$\ell_{x_j}=\lambda\mu\phi\varphi_{x_j},\quad \ell_{x_jx_k}=\lambda\mu^2\phi\varphi_{x_j}\varphi_{x_k}+\lambda\mu\phi\varphi_{x_jx_k},\quad \Delta\ell=\lambda\mu\phi\Delta\varphi+\lambda\mu^2\phi|\nabla\varphi|^2, \tag{2.12}$$

and

$$\ell_t=-2\lambda\mu\phi\alpha t,\quad \ell_{tt}=4\lambda\mu^2\phi\alpha^2t^2-2\lambda\mu\alpha\phi,\quad \ell_{tx_j}=-2\lambda\mu^2\phi\alpha t\varphi_{x_j}. \tag{2.13}$$

In Lemma 2.3, we choose Ψ as

$$\Psi=\ell_{tt}-\sum_{j,k=1}^n(h^{jk}\ell_{x_j})_{x_k}+2(1+\alpha)\lambda\mu\phi. \tag{2.14}$$

Then, by (2.12)–(2.14), we have

$$\begin{aligned} &2\Big[\ell_{tt}+\sum_{j,k=1}^n(h^{jk}\ell_{x_j})_{x_k}+\Psi\Big]v_t^2-8\sum_{j,k=1}^n h^{jk}\ell_{tx_j}v_{x_k}v_t \\ &=2\big[8\lambda\mu^2\phi\alpha^2t^2+(2-2\alpha)\lambda\mu\phi\big]v_t^2+16\lambda\mu^2\phi\alpha tv_t\sum_{j,k=1}^n h^{jk}\varphi_{x_j}v_{x_k} \\ &\geq 4(1-\alpha)\lambda\mu\phi v_t^2-4\lambda\mu^2\phi\Big(\sum_{j,k=1}^n h^{jk}v_{x_j}\varphi_{x_k}\Big)^2. \end{aligned} \tag{2.15}$$

Next, recalling (2.10) for the definition of c^{jk}, by (1.6), we have

$$\begin{aligned} 2\sum_{j,k=1}^n c^{jk}v_{x_j}v_{x_k} &= 2\lambda\mu\phi\sum_{j,k=1}^n\left\{\sum_{j',k'=1}^n\Big[2h^{jk'}(h^{j'k}\varphi_{x_{j'}})_{x_{k'}}-h^{jk}_{x_{k'}}h^{j'k'}\varphi_{x_{j'}}\Big]\right\}v_{x_j}v_{x_k} \\ &\quad+4\lambda\mu^2\phi\left(\sum_{j,k=1}^n h^{jk}\varphi_{x_j}v_{x_k}\right)^2-4(1+\alpha)\lambda\mu\phi\sum_{j,k=1}^n h^{jk}v_{x_j}v_{x_k} \\ &\geq 4\lambda\mu^2\phi\left(\sum_{j,k=1}^n h^{jk}\varphi_{x_j}v_{x_k}\right)^2+2(s_0-2\alpha-2)\lambda\mu\phi\sum_{j,k=1}^n h^{jk}v_{x_j}v_{x_k}. \end{aligned} \tag{2.16}$$

Combining (2.15) and (2.16), we have

$$\begin{aligned}&\theta^2\left|u_{tt}-\sum_{j,k=1}^{n}(h^{jk}u_{x_j})_{x_k}\right|^2+M_t+2\mathrm{div}\,V\\&\geq 4(1-\alpha)\lambda\mu\phi v_t^2+2(s_0-2\alpha-2)\lambda\mu\phi\sum_{j,k=1}^{n}h^{jk}v_{x_j}v_{x_k}-2\sum_{j,k=1}^{n}h^{jk}\Psi_{x_j}vv_{x_k}+Bv^2.\end{aligned}\tag{2.17}$$

Step 2. In the sequel, for $k\in\mathbb{N}$, we denote by $O(\lambda^k)$ a function of order λ^k for fixed μ and large λ. Recalling the definition of A in (2.10), we have

$$\begin{aligned}A&=\sum_{j,k=1}^{n}\left(h^{jk}\ell_{x_j}\ell_{x_k}-h^{jk}_{x_j}\ell_{x_k}-h^{jk}\ell_{x_jx_k}\right)-\ell_t^2+\ell_{tt}-\Psi\\&=\lambda^2\mu^2\phi^2\left(\sum_{j,k=1}^{n}h^{jk}\varphi_{x_j}\varphi_{x_k}-4\alpha^2t^2\right)+O(\lambda).\end{aligned}\tag{2.18}$$

From (2.10), (2.14) and (2.18), we have

$$\begin{aligned}B&=2\left[A\Psi-A_t\ell_t-A\ell_{tt}+\sum_{j,k=1}^{n}(Ah^{jk}\ell_{x_j})_{x_k}\right]\\&=4(1+\alpha)\lambda\mu\phi A+4\lambda\mu\alpha\phi tA_t+2\sum_{j,k=1}^{n}h^{jk}\lambda\mu\phi\varphi_{x_j}A_{x_k}.\end{aligned}\tag{2.19}$$

By (2.10) and (2.14), we get that

$$\begin{aligned}&4\lambda\mu\alpha\phi tA_t+2\sum_{j,k=1}^{n}h^{jk}\lambda\mu\phi\varphi_{x_j}A_{x_k}\\&=-8\lambda^3\mu^3\phi^3\alpha^2t^2\left[4\alpha+2\mu\left(\sum_{j,k=1}^{n}h^{jk}\varphi_{x_j}\varphi_{x_k}-4\alpha^2t^2\right)\right]\\&\quad+2\lambda^3\mu^3\phi^3\sum_{j,k=1}^{n}\left[h^{jk}\varphi_{x_j}\sum_{j',k'=1}^{n}\left(h^{j'k'}\varphi_{x_{j'}}\varphi_{x_{k'}}\right)_{x_k}\right]\\&\quad+4\lambda^3\mu^4\phi^3\left(\sum_{j,k=1}^{n}h^{jk}\varphi_{x_j}\varphi_{x_k}-4\alpha^2t^2\right)\sum_{j,k=1}^{n}h^{jk}\varphi_{x_j}\varphi_{x_k}+O(\lambda^2).\end{aligned}\tag{2.20}$$

Recalling that φ satisfy (1.6), and noting $h^{j'k'}=h^{k'j'}$ for $1\leq j',k'\leq n$, it is easy to check that (see [15, inequality (11.6)] for details)

$$\begin{aligned}&\sum_{j,k=1}^{n}h^{jk}\varphi_{x_j}\left(\sum_{j',k'=1}^{n}h^{j'k'}\varphi_{x_{j'}}\varphi_{x_{k'}}\right)_{x_k}\\&=\sum_{j,k,j',k'=1}^{n}\left[2h^{jk'}(h^{j'k}\varphi_{x_{j'}})_{x_{k'}}-h^{jk}_{x_{j'}}h^{j'k'}\varphi_{x_{j'}}\right]\varphi_{x_j}\varphi_{x_k}\geq s_0\sum_{j,k=1}^{n}h^{jk}\varphi_{x_j}\varphi_{x_k}.\end{aligned}\tag{2.21}$$

By (2.20) and (2.21), we have

$$
\begin{aligned}
B \geq\ & 4\lambda^3\mu^4\phi^3\left(\sum_{j,k=1}^n h^{jk}\varphi_{x_j}\varphi_{x_k} - 4\alpha^2t^2\right)^2 \\
& +2\lambda^3\mu^3\phi^3\left\{[s_0+2(1+\alpha)]\sum_{j,k=1}^n h^{jk}\varphi_{x_j}\varphi_{x_k} - \left[16\alpha^3+8(1+\alpha)\alpha^2\right]t^2\right\} + O(\lambda^2) \\
\geq\ & 2\lambda^3\mu^3\phi^3\left[s_0+2+2\alpha-4\alpha-2(1+\alpha)\right]\sum_{j,k=1}^n h^{jk}\varphi_{x_j}\varphi_{x_k} \\
& +4\lambda^3\mu^3\phi^3\left[\sqrt{\mu}\left(\sum_{j,k=1}^n h^{jk}\varphi_{x_j}\varphi_{x_k} - 4\alpha^2t^2\right) + \frac{1}{2\sqrt{\mu}}\Big(3\alpha+1\Big)\right]^2 \\
& -\lambda^3\mu^2\phi^3\Big(2\alpha+(1+\alpha)\Big)^2 + O(\lambda^2) \\
\geq\ & 2\lambda^3\mu^3\phi^3(s_0-4\alpha)\sum_{j,k=1}^n h^{jk}\varphi_{x_j}\varphi_{x_k} - \lambda^3\mu^2\phi^3\Big(3\alpha+1\Big)^2 + O(\lambda^2)
\end{aligned}
\tag{2.22}
$$

By (2.22), noting that $s_0 \geq 4$ and $\alpha < 1$, we conclude that there exists a positive constant $\mu_0 > 0$ such that for all $\mu \geq \mu_0$, one can find constants $c_0 > 0$ and $\lambda_1 > 0$, so that for any $\lambda \geq \lambda_1$,

$$
B \geq c_0\lambda^3\mu^3\phi^3\sum_{j,k=1}^n h^{jk}\varphi_{x_j}\varphi_{x_k}. \tag{2.23}
$$

Combining (2.17) and (2.23), we have

$$
\begin{aligned}
& \theta^2\left|u_{tt} - \sum_{j,k=1}^n (h^{jk}u_{x_j})_{x_k}\right|^2 + M_t + 2\mathrm{div}\, V \\
& \geq 4(1-\alpha)\lambda\mu\phi v_t^2 + 2(s_0-2\alpha-2)\lambda\mu\phi\sum_{j,k=1}^n h^{jk}v_{x_j}v_{x_k} \\
& \quad +c_0\lambda^3\mu^3\phi^3\sum_{j,k=1}^n h^{jk}\varphi_{x_j}\varphi_{x_k}v^2 - 2\sum_{j,k=1}^n h^{jk}\Psi_{x_j}vv_{x_k}.
\end{aligned}
\tag{2.24}
$$

Step 3. Noting that $\mathrm{supp}\, u \subset [-\widetilde{T},\widetilde{T}]\times(\overline{\Omega}\backslash\omega_1)$, by (2.11), we have

$$
\begin{aligned}
& \int_{\widetilde{\Sigma}} V\cdot\nu(x)\mathrm{d}x\mathrm{d}t \\
& = \lambda\mu\int_{\widetilde{\Sigma}}\left(\sum_{j,k=1}^n h^{jk}\nu^j\nu^k\right)\left(\sum_{j',k'=1}^n h^{j'k'}\phi\varphi_{x_{j'}}\nu^{k'}\right)\left|\frac{\partial v}{\partial\nu}\right|^2\mathrm{d}\Gamma\mathrm{d}t \\
& \leq \lambda\mu\int_{-\widetilde{T}}^{\widetilde{T}}\int_{\Gamma_0}\left(\sum_{j,k=1}^n h^{jk}\nu^j\nu^k\right)\left(\sum_{j',k'=1}^n h^{j'k'}\phi\varphi_{x_{j'}}\nu^{k'}\right)\left|\frac{\partial v}{\partial\nu}\right|^2\mathrm{d}\Gamma\mathrm{d}t \\
& = 0.
\end{aligned}
\tag{2.25}
$$

Recalling (2.14) for the definition of Ψ, we get

$$-2\sum_{j,k=1}^{n} h^{jk}\Psi_{x_j} v v_{x_k} \geq -\sum_{j,k=1}^{n} h^{jk} v_{x_j} v_{x_k} + O(\lambda^2)v^2. \tag{2.26}$$

Noting that $\widetilde{T} > \widetilde{R}_1$, then in (2.4), we choose $\alpha \in \left(\widetilde{R}_1/\widetilde{T}, 1\right)$. Then we obtain that

$$\begin{aligned} M(-\widetilde{T}, x) &= 2\lambda\mu\phi\alpha\widetilde{T}\left(v_t^2 + \sum_{j,k=1}^{n} h^{jk} v_{x_j} v_{x_k}\right) - 2\lambda\mu\phi\sum_{j,k=1}^{n} h^{jk}\varphi_{x_j} v_{x_k} v_t \\ &\quad + \left[4\lambda\mu^2\phi\alpha^2\widetilde{T}^2 - \sum_{j,k=1}^{n}\left(h^{jk}\lambda\mu\phi\varphi_{x_j}\right)_{x_k} + 2\lambda\mu\phi\right] v v_t \\ &\quad -2\lambda^3\mu^3\phi^3\alpha\widetilde{T}\left(\sum_{j,k=1}^{n} h^{jk}\varphi_{x_j}\varphi_{x_k} - 4\alpha^2\widetilde{T}^2\right) v^2 + O(\lambda^2)v^2 \\ &\geq \lambda\mu\phi\left[2\alpha\widetilde{T} - \left(\sum_{j,k=1}^{n} h^{jk}\varphi_{x_j}\varphi_{x_k}\right)^{\frac{1}{2}}\right]\left(v_t^2 + \sum_{j,k=1}^{n} h^{jk} v_{x_j} v_{x_k}\right) \\ &\quad +O(\lambda^2)v^2 - v_t^2 + \lambda^3\mu^3\phi^3\alpha\widetilde{T}\left(4\alpha^2\widetilde{T}^2 - \sum_{j,k=1}^{n} h^{jk}\varphi_{x_j}\varphi_{x_k}\right) v^2. \end{aligned} \tag{2.27}$$

By (2.27), one can find $\lambda_2 > 0$ such that for any $\lambda > \lambda_2$,

$$M(-\widetilde{T}, x) \geq 0. \tag{2.28}$$

Similarly, one can find $\lambda_3 > 0$ such that for any $\lambda > \lambda_3$,

$$M(\widetilde{T}, x) \leq 0. \tag{2.29}$$

Recalling that $u = \theta^{-1}v$, we get that

$$\theta^2 u_t^2 = \theta^2(\theta^{-1}v_t - \theta^{-1}\ell_t v)^2 \leq 2v_t^2 + C\lambda^2\mu^2\phi^2 v^2, \tag{2.30}$$

and

$$\theta^2|\nabla u|^2 = \theta^2\left|\theta^{-1}\nabla v - \theta^{-1}\nabla\ell v\right|^2 \leq 2|\nabla v|^2 + C\lambda^2\mu^2\phi^2 v^2. \tag{2.31}$$

Finally, integrate (2.24) on Q, by (2.26)–(2.31), one can find $\mu_0 > 0$ such that for all $\mu \geq \mu_0$, there exists $\lambda_0 \triangleq \max\{\lambda_1, \lambda_2, \lambda_3\} > 0$, so that for any $\lambda \geq \lambda_0$, the inequality (2.5) holds. □

Proof of Theorem 2.2. Noting that $\widehat{T} > \widehat{R}^1$, we can choose α in (2.4) such that $\alpha \in \left(\widehat{R}^1/\widehat{T}, 1\right)$.

Recalling (2.11) for the definition of V, noting that $u|_{\widehat{\Sigma}}=0$ and $v=\theta u$, by (1.8) and (1.13), we have

$$\begin{aligned}\int_{\widehat{\Sigma}} V\cdot\nu(x)\mathrm{d}x\mathrm{d}t &= \lambda\mu\int_{\widehat{\Sigma}}\phi\left(\sum_{j,k=1}^{n}h^{jk}\nu^j\nu^k\right)\left(\sum_{j',k'=1}^{n}h^{j'k'}\varphi_{x_{j'}}\nu^{k'}\right)\left|\frac{\partial v}{\partial\nu}\right|^2\mathrm{d}\Gamma\mathrm{d}t\\ &\le \lambda\mu\int_{\widehat{\Sigma}_0}\theta^2\left|\frac{\partial u}{\partial\nu}\right|^2\mathrm{d}\Gamma\mathrm{d}t.\end{aligned}\tag{2.32}$$

Then, similar to the proof of Theorem 2.1, integrating (2.24) on Q, proceeding exactly the same analysis as (2.26), (2.28)–(2.29) and (2.32), we can get the desired inequality (2.8). This completes the proof of Theorem 2.2. □

3. An auxiliary optimal control problem

To obtain Carleman estimate for the hyperbolic operator in $L^2(\widetilde{Q})$, we follow the idea in [16] to introduce an auxiliary optimal control problem. We choose $\rho^K\equiv\rho^K(x)\in C^2(\overline{\Omega})$ such that $\min_{x\in\overline{\Omega}}\rho^K(x)=1$ and that

$$\rho^K(x)=\begin{cases}1, & x\in\omega_1,\\ K, & \mathrm{dist}\,(x,\omega_1)\ge\dfrac{1}{\ln K},\end{cases}\tag{3.1}$$

where ω_1 is given in (2.1).

Let $z\in C([-\widetilde{T},\widetilde{T}],L^2(\Omega))$ satisfy the following hyperbolic equation:

$$\begin{cases} z_{tt}-\displaystyle\sum_{j,k=1}^{n}(h^{jk}z_{x_j})_{x_k}=F & \text{in } \tilde{Q},\\ z=0 & \text{on } \widetilde{\Sigma},\\ \tilde{z}(-\widetilde{T},\cdot)=\tilde{z}(\widetilde{T},\cdot)=0 & \text{in } \Omega,\\ \mathrm{supp}\,z(t,x)\subset[-\widetilde{T},\widetilde{T}]\times(\overline{\Omega}\backslash\omega_1), \end{cases}\tag{3.2}$$

where $F\in H^{-1}(\tilde{Q})$. Set $\tau=\frac{2\widetilde{T}}{m}(m\ge 3)$ and

$$z_m^j=z_m^j(x)=z(-\widetilde{T}+j\tau,x),\quad \phi_m^j=\phi_m^j(x)=\phi(-\widetilde{T}+j\tau,x),\quad j=0,1,\cdots,m.\tag{3.3}$$

Let $\{(w_m^j,r_{1m}^j,r_m^j)\}_{j=0}^m\in\left(H_0^1(\Omega)\times(L^2(\Omega))^2\right)^{m+1}$ solve the following system:

$$\begin{cases} \dfrac{w_m^{j+1}-2w_m^j+w_m^{j-1}}{\tau^2}-\displaystyle\sum_{j_1,j_2=1}^{n}\partial_{x_{j_2}}(h^{j_1j_2}\partial_{x_{j_2}}w_m^j)\\ \quad=\dfrac{r_{1m}^{j+1}-r_{1m}^j}{\tau}+\lambda z_m^j\mathrm{e}^{2\lambda\phi_m^j}+r_m^j,\quad (1\le j\le m-1) & \text{in } \Omega,\\ w_m^j=0,\quad (0\le j\le m) & \text{on } \Gamma,\\ w_m^0=w_m^m=r_m^0=r_m^m=0,\quad r_{1m}^0=r_{1m}^m & \text{in } \Omega. \end{cases}\tag{3.4}$$

Here $(r_{1m}^j,r_m^j)\in L^2(\Omega)^2$ can be regarded as controls. The set of admissible sequences for (3.4) is defined as

$$\begin{aligned}\mathcal{A}_{ad}\triangleq\Big\{\{(w_m^j,r_{1m}^j,r_m^j)\}_{j=0}^m\in\left(H_0^1(\Omega)\times L^2(\Omega)^2\right)^{m+1}\Big|\ \{(w_m^j,r_{1m}^j,r_m^j)\}_{j=0}^m \text{ fulfills (3.4)},\\ \mathrm{supp}\,\{(w_m^j,r_{1m}^j,r_m^j)\}_{j=0}^m\subset\overline{\Omega}\backslash\omega_1\Big\}.\end{aligned}\tag{3.5}$$

Clearly $\mathcal{A}_{ad} \neq \emptyset$ since $\{(0,0,-\lambda z_m^j \mathrm{e}^{2\lambda\phi_m^j})\}_{j=0}^m \in \mathcal{A}_{ad}$.

Next, we introduce a cost functional

$$\begin{aligned}\mathcal{J}\Big(\{(w_m^j, r_{1m}^j, r_m^j)\}_{j=0}^m\Big) &= \frac{\tau}{2}\int_\Omega \rho^K \frac{|r_{1m}^m|^2}{\lambda^2}\mathrm{e}^{-2\lambda\phi_m^m}\mathrm{d}x + \frac{\tau}{2}\sum_{j=1}^{m-1}\int_\Omega |w_m^j|^2 \mathrm{e}^{-2\lambda\phi_m^j}\mathrm{d}x \\ &\quad + \frac{\tau}{2}\sum_{j=1}^{m-1}\left(\int_\Omega \rho^K \frac{|r_{1m}^j|^2}{\lambda^2}\mathrm{e}^{-2\lambda\phi_m^j}\mathrm{d}x + K\int_\Omega |r_m^j|^2\mathrm{d}x\right)\end{aligned} \tag{3.6}$$

and consider the following optimal control problem:

Problem (OP): Find a $\{(\hat{w}_m^j, \hat{r}_{1m}^j, \hat{r}_m^j)\}_{j=0}^m \in \mathcal{A}_{ad}$, such that

$$\mathcal{J}\big(\{(\hat{w}_m^j, \hat{r}_{1m}^j, \hat{r}_m^j)\}_{j=0}^m\big) = \inf_{\{(w_m^j, r_{1m}^j, r_m^j)\}_{j=0}^m \in \mathcal{A}_{ad}} \mathcal{J}\big(\{(w_m^j, r_{1m}^j, r_m^j)\}_{j=0}^m\big). \tag{3.7}$$

For any $\{(w_m^j, r_{1m}^j, r_m^j)\}_{j=0}^m \in \mathcal{A}_{ad}$, by the standard regularity results for elliptic equations, one has that $w_m^j \in H^2(\Omega)\cap H_0^1(\Omega)$ for $j = 0, \cdots, m$. Moreover, we have the following technical result:

Proposition 3.1. *For any $K > 1$ and $m \geq 3$, Problem (OP) admits a unique solution $\{(\hat{w}_m^j, \hat{r}_{1m}^j, \hat{r}_m^j)\}_{j=0}^m \in \mathcal{A}_{ad}$ (which depends on K). Furthermore, for*

$$p_m^j = p_m^j(x) \triangleq K\hat{r}_m^j(x), \quad 0 \leq j \leq m,$$

one has

$$\begin{cases} \hat{w}_m^0 = \hat{w}_m^m = p_m^0 = p_m^m = 0 \quad \text{in } \Omega, \\ \hat{w}_m^j,\ p_m^j \in H^2(\Omega)\cap H_0^1(\Omega), \quad 1 \leq j \leq m-1, \end{cases} \tag{3.8}$$

and the following optimality conditions hold:

$$\frac{p_m^j - p_m^{j-1}}{\tau} + \rho^K \frac{\hat{r}_{1m}^j}{\lambda^2}\mathrm{e}^{-2\lambda\phi_m^j} = 0 \text{ in } \Omega, \qquad 1 \leq j \leq m, \tag{3.9}$$

and

$$\begin{cases} \dfrac{p_m^{j+1} - 2p_m^j + p_m^{j-1}}{\tau^2} - \displaystyle\sum_{j_1,j_2=1}^n \partial_{x_{j_2}}(h^{j_1 j_2}\partial_{x_{j_2}} p_m^j) + \hat{w}_m^i \mathrm{e}^{-2\lambda\phi_m^j} = 0 & \text{in } \Omega, \\ p_m^j = 0 & \text{on } \Gamma, \end{cases} \quad 1 \leq j \leq m-1. \tag{3.10}$$

Moreover, there is a constant $C = C(K, \lambda, z) > 0$, independent of m, such that

$$\tau\sum_{j=1}^{m-1}\int_\Omega \left(|\hat{w}_m^j|^2 + |\hat{r}_{1m}^j|^2 + |\hat{r}_m^j|^2\right)\mathrm{d}x + \tau\int_\Omega |\hat{r}_{1m}^m|^2\mathrm{d}x \leq C, \tag{3.11}$$

and that

$$\tau\sum_{j=0}^{m-1}\int_\Omega \left[\frac{(\hat{w}_m^{j+1} - \hat{w}_m^j)^2}{\tau^2} + \frac{(\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^j)^2}{\tau^2} + \frac{(\hat{r}_m^{j+1} - \hat{r}_m^j)^2}{\tau^2}\right]\mathrm{d}x \leq C, \tag{3.12}$$

and that

$$\tau \sum_{j=1}^{m-1} \int_\Omega \Big(|\nabla \hat{w}_m^j|^2 + |\nabla \hat{r}_m^j|^2 \Big) \mathrm{d}x \le C. \tag{3.13}$$

The proof of Proposition 3.1 is similar to [16], pp. 190–199 and [15], Proposition 6.1, we give it in the appendix.

4. Carleman estimate for the hyperbolic operator in $L^2(Q)$

Based on Proposition 3.1, we have the following Carleman estimate for the hyperbolic operator in L^2-norm.

Theorem 4.1. *Let $\tilde{q} \in L^\infty(-\widetilde{T}, \widetilde{T}; L^p(\Omega; \mathbb{R}^{N\times N}))$ with $p \in [n, \infty]$ and $\tilde{q}|_Q = q$. Assume that Condition 1.1 and (1.10) hold. Set $\mu = \mu_0$ as given in Theorem 2.1. Then there exists a constant $\lambda^* > 0$, such that for all $\widetilde{T} > \widetilde{R}_1$, $\lambda \ge \lambda^*$, any $\tilde{z} \overset{\triangle}{=} (\tilde{z}_1, \cdots, \tilde{z}_N) \in C([-\widetilde{T}, \widetilde{T}]; L^2(\Omega)^N)$ with $\tilde{z}_j$ satisfying (3.2) $(j = 1, \cdots, N)$, it holds*

$$\lambda \|\theta \tilde{z}\|^2_{L^2(\widetilde{Q})^N} \le C_* \left[\left\| \theta \left[\tilde{z}_{tt} - \sum_{j,k=1}^n (h^{jk} \tilde{z}_{x_j})_{x_k} - \tilde{q}\tilde{z} \right] \right\|^2_{H^{-1}(\widetilde{Q})^N} + \frac{1}{\lambda^{2(1-n/p)}} \|\theta \tilde{q} \tilde{z}\|^2_{L^2(-\widetilde{T}, \widetilde{T}; H^{-n/p}(\Omega)^N)} \right]. \tag{4.1}$$

Proof of Theorem 4.1. We divide the proof into four steps.

Step 1. First of all, recall the functions $\{(\hat{w}_m^j, \hat{r}_{1m}^j, \hat{r}_m^j)\}_{j=0}^m$ in Proposition 3.1. Put

$$\begin{cases} \tilde{w}^m(t,x) = \dfrac{1}{\tau} \displaystyle\sum_{j=0}^{m-1} \left\{ (t - j\tau) \hat{w}_m^{j+1}(x) - [t - (j+1)\tau] \hat{w}_m^j(x) \right\} \chi_{(j\tau, (j+1)\tau]}(t), \\ \tilde{r}_1^m = \hat{r}_{1m}^0(x) \chi_{\{0\}}(t) + \dfrac{1}{\tau} \displaystyle\sum_{j=0}^{m-1} \left\{ (t - j\tau) \hat{r}_{1m}^{j+1}(x) - [t - (j+1)\tau] \hat{r}_{1m}^j(x) \right\} \chi_{(j\tau, (j+1)\tau]}(t), \\ \tilde{r}^m(t,x) = \dfrac{1}{\tau} \displaystyle\sum_{j=0}^{m-1} \left\{ (t - j\tau) \hat{r}_m^{j+1}(x) - [t - (j+1)\tau] \hat{r}_m^j(x) \right\} \chi_{(j\tau, (j+1)\tau]}(t). \end{cases} \tag{4.2}$$

By (3.11), (3.12) and (3.13), one can find a subsequence of $\left\{ \left(\tilde{w}^m, \tilde{r}^m, \tilde{r}_1^m \right) \right\}_{m=1}^\infty$, which converges weakly to some $(\tilde{w}, \tilde{r}, \tilde{r}_1) \in \left[H_0^1(-T, T; L^2(\Omega)) \cap L^2(-T, T; H_0^1(\Omega)) \right]^2 \times H^1(-T, T; L^2(\Omega))$, as $m \to \infty$.

For a $z \in C([-\widetilde{T}, \widetilde{T}], L^2(\Omega))$ satisfying (3.2), by (3.4), Proposition 3.1, and by putting $\tilde{p} \overset{\triangle}{=} K\tilde{r}$ and standard energy estimates, we know that $\tilde{w}, \tilde{p} \in C([-\widetilde{T}, \widetilde{T}]; H_0^1(\Omega)) \cap C^1([-\widetilde{T}, \widetilde{T}]; L^2(\Omega))$, and

$$\begin{cases} \tilde{w}_{tt} - \displaystyle\sum_{j,k=1}^n (h^{jk} \tilde{w}_{x_j})_{x_k} = \tilde{r}_{1,t} + \lambda\theta^2 z + \tilde{r} & \text{in } \widetilde{Q}, \\ \tilde{p}_{tt} - \displaystyle\sum_{j,k=1}^n (h^{jk} \tilde{p}_{x_j})_{x_k} + \theta^{-2} \tilde{w} = 0 & \text{in } \widetilde{Q}, \\ \tilde{p} = \tilde{w} = 0 & \text{on } \widetilde{\Sigma}, \\ \tilde{p}(-\widetilde{T}) = \tilde{p}(\widetilde{T}) = \tilde{w}(-\widetilde{T}) = \tilde{w}(\widetilde{T}) = 0 & \text{in } \Omega, \\ \tilde{p}_t + \rho^K \theta^{-2} \dfrac{\tilde{r}_1}{\lambda^2} = 0 & \text{in } \widetilde{Q}. \end{cases} \tag{4.3}$$

By (4.3) and standard energy estimates again, one finds that $\tilde{p}_t \in C([-\widetilde{T},\widetilde{T}];H_0^1(\Omega))\cap C^1([-\widetilde{T},\widetilde{T}];L^2(\Omega))$ solves

$$\begin{cases} \tilde{p}_{ttt} - \sum_{j,k=1}^{n} (h^{jk}\tilde{p}_{tx_j})_{x_k} + (\theta^{-2}\tilde{w})_t = 0 & \text{in } \widetilde{Q}, \\ \tilde{p}_t = 0 & \text{on } \widetilde{\Sigma}, \\ \tilde{p}_{tt} + \dfrac{\rho^K}{\lambda}\theta^{-2}\Big(\dfrac{\tilde{r}_{1,t}}{\lambda} - 2\phi_t\tilde{r}_1\Big) = 0 & \text{in } \widetilde{Q}. \end{cases} \tag{4.4}$$

Apply Theorem 2.1 to $\tilde{p}$ in (4.3) and $\tilde{p}_t$ in (4.4), respectively. Noting that $\operatorname{supp}\tilde{p} \subset [-\widetilde{T},\widetilde{T}]\times(\overline{\Omega}\backslash\omega_1)$ and $\operatorname{supp}\tilde{p}_t \subset [-\widetilde{T},\widetilde{T}]\times(\overline{\Omega}\backslash\omega_1)$, we know that there is a $\lambda_0 > 0$, for any $\lambda \geq \lambda_0$, it holds

$$\lambda\int_{\widetilde{Q}} \theta^2\big(\lambda^2\tilde{p}^2 + \tilde{p}_t^2 + |\nabla\tilde{p}|^2\big)\mathrm{d}x\mathrm{d}t \leq \widetilde{C}_*\int_{\widetilde{Q}} \theta^{-2}\tilde{w}^2\mathrm{d}x\mathrm{d}t \tag{4.5}$$

and

$$\lambda\int_{\widetilde{Q}} \theta^2\big(\lambda^2\tilde{p}_t + \tilde{p}_{tt}^2 + |\nabla\tilde{p}_t|^2\big)\mathrm{d}x\mathrm{d}t \leq \widetilde{C}_*\int_{\widetilde{Q}} \theta^{-2}\big(\tilde{w}_t^2 + \lambda^2\tilde{w}^2\big)\mathrm{d}x\mathrm{d}t. \tag{4.6}$$

Step 2. By (4.3), it is easy to see that

$$-\int_{\widetilde{Q}} \tilde{r}_{1,t}\tilde{p}\mathrm{d}x\mathrm{d}t = \int_{\widetilde{Q}} \tilde{r}_1\tilde{p}_t\mathrm{d}x\mathrm{d}t = -\int_{\widetilde{Q}} \theta^{-2}\rho^K\frac{\tilde{r}_1^2}{\lambda^2}\mathrm{d}x\mathrm{d}t. \tag{4.7}$$

From (4.3) and (4.7), recalling that $\tilde{p} = K\tilde{r}$, one gets that

$$\begin{aligned} 0 &= \Big\langle \tilde{w}_{tt} - \sum_{j,k=1}^{n}\partial_{x_j}(h^{jk}\partial_{x_k}\tilde{w}) - \tilde{r}_{1,t} - \lambda\theta^2 z - \tilde{r}, \tilde{p}\Big\rangle_{L^2(\widetilde{Q})} \\ &= -\int_{\widetilde{Q}} \theta^{-2}\tilde{w}^2\mathrm{d}x\mathrm{d}t - \int_{\widetilde{Q}} \theta^{-2}\rho^K\frac{\tilde{r}_1^2}{\lambda^2}\mathrm{d}x\mathrm{d}t - \lambda\int_{\widetilde{Q}} \theta^2 z\tilde{p}\mathrm{d}x\mathrm{d}t - K\int_{\widetilde{Q}} \tilde{r}^2\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.8}$$

This, together with (4.5), implies that

$$\int_{\widetilde{Q}} \theta^{-2}\tilde{w}^2\mathrm{d}x\mathrm{d}t + \int_{\widetilde{Q}} \theta^{-2}\rho^K\frac{\tilde{r}_1^2}{\lambda^2}\mathrm{d}x\mathrm{d}t + K\int_{\widetilde{Q}} \tilde{r}^2\mathrm{d}x\mathrm{d}t \leq \frac{\widetilde{C}_*}{\lambda}\int_{\widetilde{Q}} \theta^2 z^2\mathrm{d}x\mathrm{d}t. \tag{4.9}$$

Step 3. Using (4.3) and (4.4) again, and noting that $\tilde{p}_{tt}(0,\cdot) = \tilde{p}_{tt}(T,\cdot) = 0$ in Ω, we get

$$\begin{aligned} 0 &= \Big\langle \tilde{w}_{tt} - \sum_{j,k=1}^{n}\partial_{x_j}(h^{jk}\partial_{x_k}\tilde{w}) - \tilde{r}_{1,t} - \lambda z\theta^2 - \tilde{r}, \tilde{p}_{tt}\Big\rangle_{L^2(\widetilde{Q})} \\ &= -\int_{\widetilde{Q}} \tilde{w}(\theta^{-2}\tilde{w})_{tt}\mathrm{d}x\mathrm{d}t - \int_{\widetilde{Q}} \tilde{r}_{1,t}\tilde{p}_{tt}\mathrm{d}x\mathrm{d}t - \lambda\int_{\widetilde{Q}} z\tilde{p}_{tt}\theta^2\mathrm{d}x\mathrm{d}t - \int_{\widetilde{Q}} \tilde{r}\tilde{p}_{tt}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.10}$$

Clearly, recalling (2.4) for $\theta,\ \phi$, it is easy to see that

$$(\theta^{-2})_{tt} = 2\theta^{-2}(-\lambda\phi_{tt} + 2\lambda^2\phi_t^2). \tag{4.11}$$

Then, using integration by parts, we have

$$\begin{aligned} -\int_{\widetilde{Q}} \tilde{w}(\theta^{-2}\tilde{w})_{tt}\mathrm{d}x\mathrm{d}t &= \int_{\widetilde{Q}} \left[\theta^{-2}\tilde{w}_t^2 - (\theta^{-2})_{tt}\frac{\tilde{w}^2}{2}\right]\mathrm{d}x\mathrm{d}t \\ &= \int_{\widetilde{Q}} \theta^{-2}\big(\tilde{w}_t^2 + \lambda\phi_{tt}\tilde{w}^2 - 2\lambda^2\phi_t^2\tilde{w}^2\big)\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.12}$$

Further, by (4.4), we have

$$-\int_{\widetilde{Q}} \tilde{r}_{1,t}\tilde{p}_{tt}\mathrm{d}x\mathrm{d}t = \int_{\widetilde{Q}} \theta^{-2}\tilde{r}_{1,t}\frac{\rho^K}{\lambda}\left(\frac{\tilde{r}_{1,t}}{\lambda} - 2\phi_t\tilde{r}_1\right)\mathrm{d}x\mathrm{d}t. \tag{4.13}$$

Moreover, by $\tilde{p} \triangleq K\tilde{r}$ and integration by parts, one gets

$$-\int_{\widetilde{Q}} \tilde{r}\tilde{p}_{tt}\mathrm{d}x\mathrm{d}t = K\int_{\widetilde{Q}} \tilde{r}_t^2\mathrm{d}x\mathrm{d}t. \tag{4.14}$$

Combining (4.10)–(4.14), we end up with

$$\begin{aligned} &\int_{\widetilde{Q}} \theta^{-2}\rho^K\left(\frac{\tilde{r}_{1,t}^2}{\lambda^2} - \frac{2}{\lambda}\phi_t\tilde{r}_1\tilde{r}_{1,t}\right)\mathrm{d}x\mathrm{d}t + K\int_{\widetilde{Q}} \tilde{r}_t^2\mathrm{d}x\mathrm{d}t \\ &+\int_{\widetilde{Q}} \theta^{-2}\left(\tilde{w}_t^2 + \lambda\phi_{tt}\tilde{w}^2 - 2\lambda^2\phi_t^2\tilde{w}^2\right)\mathrm{d}x\mathrm{d}t = \lambda\int_{\widetilde{Q}} \theta^2 z\tilde{p}_{tt}\mathrm{d}x\mathrm{d}t. \end{aligned} \tag{4.15}$$

Now, by (4.15) $+ \widetilde{C}_*\lambda^2 \times$ (4.9)(with a sufficiently large $\widetilde{C}_* > 0$), using the Cauchy–Schwartz inequality and noting (4.6), we obtain that

$$\int_{\widetilde{Q}} \theta^{-2}\big(\tilde{w}_t^2 + \lambda^2\tilde{w}^2\big)\mathrm{d}x\mathrm{d}t + \int_{\widetilde{Q}} \theta^{-2}\rho^K\left(\frac{\tilde{r}_{1,t}^2}{\lambda^2} + \tilde{r}_1^2\right)\mathrm{d}x\mathrm{d}t \le \widetilde{C}_*\lambda\int_{\widetilde{Q}} \theta^2 z^2\mathrm{d}x\mathrm{d}t. \tag{4.16}$$

Step 4. It follows from (4.3) that

$$\left\langle \tilde{r}_{1,t} + \lambda z\theta^2 + \tilde{r}, \theta^{-2}\tilde{w}\right\rangle_{L^2(\widetilde{Q})} = \left\langle \tilde{w}_{tt} - \sum_{j,k=1}^n \partial_{x_j}(h^{jk}\tilde{w}_{x_k}), \theta^{-2}\tilde{w}\right\rangle_{L^2(\widetilde{Q})} \tag{4.17}$$

$$= -\int_{\widetilde{Q}} \left[\tilde{w}_t(\theta^{-2}\tilde{w})_t\mathrm{d}x\mathrm{d}t - \sum_{j,k=1}^n h^{jk}(\theta^{-2}\tilde{w})_{x_j}\tilde{w}_{x_k}\right]\mathrm{d}x\mathrm{d}t \tag{4.18}$$

$$= -\int_{\widetilde{Q}} \theta^{-2}\big(\tilde{w}_t^2 + \lambda\phi_{tt}\tilde{w}^2 - 2\lambda^2\phi_t^2\tilde{w}^2\big)\mathrm{d}x\mathrm{d}t + \sum_{j,k=1}^n \int_{\widetilde{Q}} \theta^{-2}h^{jk}\tilde{w}_{x_j}\tilde{w}_{x_k}\mathrm{d}x\mathrm{d}t \tag{4.19}$$

$$-2\lambda\int_{\widetilde{Q}} \theta^{-2}\tilde{w}\sum_{j,k=1}^n h^{jk}\tilde{w}_{x_j}\phi_{x_k}\mathrm{d}x\mathrm{d}t. \tag{4.20}$$

This, together with (1.2), yields

$$\int_{\widetilde{Q}} \theta^{-2}|\nabla\tilde{w}|^2\mathrm{d}x\mathrm{d}t$$

$$\leq \widetilde{C}_* \int_{\widetilde{Q}} \left[\theta^{-2}|\tilde{r}_{1,t}+\tilde{r}||\tilde{w}|+\lambda|z\tilde{w}|+\theta^{-2}(\tilde{w}_t^2+\lambda^2\tilde{w}^2)\right]\mathrm{d}x\mathrm{d}t \tag{4.21}$$
$$\leq \widetilde{C}_* \int_{\widetilde{Q}} \left[\theta^2 z^2+\theta^{-2}\left(\frac{\tilde{r}_{1,t}^2}{\lambda^2}+\tilde{r}^2+\tilde{w}_t^2+\lambda^2\tilde{w}^2\right)\right]\mathrm{d}x\mathrm{d}t, \qquad \forall \lambda \geq \lambda_0.$$

Combining (4.9), (4.16) and (4.21), choosing the constant K in (3.1) be such that

$$K \geq \widetilde{C}_* \mathrm{e}^{2\lambda \max_{(t,x)\in[-\widetilde{T},\widetilde{T}]\times\overline{\Omega}}|\phi(t,x)|} \tag{4.22}$$

and noting that $\rho^K(x) \geq 1$ in Ω, we deduce that

$$\int_{\widetilde{Q}} \theta^{-2}(|\nabla\tilde{w}|^2+\tilde{w}_t^2+\lambda^2\tilde{w}^2)\mathrm{d}x\mathrm{d}t+\int_{\widetilde{Q}} \theta^{-2}\rho^K\left(\frac{\tilde{r}_{1,t}^2}{\lambda^2}+\tilde{r}_1^2\right)\mathrm{d}x\mathrm{d}t \leq \widetilde{C}_*\lambda\int_{\widetilde{Q}} \theta^2 z^2\mathrm{d}x\mathrm{d}t. \tag{4.23}$$

Recall that $(\tilde{w},\tilde{r}_1,\tilde{r})$ depends on K. Now we denote it by $(\tilde{w}^K,\tilde{r}_1^K,\tilde{r}^K)$ to emphasize this dependence. Fix λ and let $K\to\infty$. Since $\rho^K(x)\to\infty$ for any $x\notin\omega$, as $K\to\infty$, it follows from (4.9) and (4.23) that there exists a subsequence of $\{(\tilde{w}^K,\tilde{r}_1^K,\tilde{r}^K)\}_{K=1}^{\infty}$, which converges weakly to some $(\check{w},0,0)$ in $H_0^1(-T,T;L^2(\Omega))\cap L^2(-T,T;H_0^1(\Omega))\times H^1(-\widetilde{T},\widetilde{T};L^2(\Omega))\times L^2(\widetilde{Q})$. By (4.3), we deduce that $\check{w}$ satisfies

$$\begin{cases} \check{w}_{tt}-\sum_{j,k=1}^n (h^{jk}\check{w}_{x_j})_{x_k}=\lambda z\theta^2 & \text{in } \widetilde{Q},\\ \check{w}=0 & \text{on } \widetilde{\Sigma}\\ \check{w}(-T,\cdot)=\check{w}(T,\cdot)=0 & \text{in } \Omega. \end{cases} \tag{4.24}$$

Use (4.23) again, we get that

$$\|\theta^{-1}\check{w}\|_{H_0^1(\widetilde{Q})}^2+\lambda^2\|\theta^{-1}\check{w}\|_{L^2(\widetilde{Q})}^2 \leq \widetilde{C}_*\lambda\int_{\widetilde{Q}}\theta^2 z^2\mathrm{d}x\mathrm{d}t. \tag{4.25}$$

Recalling $\tilde{z}=(\tilde{z}_1,\cdots,\tilde{z}_N)\in C([-\widetilde{T},\widetilde{T}];L^2(\Omega)^N)$, put $z=\tilde{z}_{j'}$, $(j'=1,\cdots,N)$ and set $\check{w}=\check{w}_{j'}$, $(j'=1,\cdots,N)$ respectively. Noting that $\tilde{q}=\tilde{q}(t,x)\in L^\infty(-\widetilde{T},\widetilde{T};L^p(\Omega;\mathbb{R}^{N\times N}))$ is a matrix, we denote it by $(\tilde{q}^{jk})_{N\times N}$, a straightforward calculation shows that

$$\begin{aligned}
&\lambda\int_{\widetilde{Q}}\theta^2\tilde{z}_{j'}^2\mathrm{d}x\mathrm{d}t=\int_{\widetilde{Q}}\tilde{z}_{j'}\left[\partial_t^2\check{w}_{j'}-\sum_{j,k=1}^n\partial_{x_k}(h^{jk}\partial_{x_j}\check{w}_{j'})\right]\mathrm{d}x\mathrm{d}t\\
&=\left\langle \partial_t^2\tilde{z}_{j'}-\sum_{j,k=1}^n\partial_{x_k}(h^{jk}\partial_{x_j}\tilde{z}_{j'})-\sum_{k'=1}^N\tilde{q}^{j'k'}\tilde{z}_{k'},\check{w}_{j'}\right\rangle_{H^{-1}(\widetilde{Q}),H_0^1(\widetilde{Q})}+\sum_{j'=1}^N\left\langle\tilde{q}^{j'k'}\tilde{z}_{j'},\check{w}_{k'}\right\rangle_{L^2(\widetilde{Q})}\\
&\leq\left\|\theta\left[\partial_t^2\tilde{z}_{j'}-\sum_{j,k=1}^n\partial_{x_k}(h^{jk}\partial_{x_j}\tilde{z}_{j'})-\sum_{k'=1}^N\tilde{q}^{j'k'}\tilde{z}_{k'}\right]\right\|_{H^{-1}(\widetilde{Q})}\left\|\theta^{-1}\check{w}_{j'}\right\|_{H_0^1(\widetilde{Q})}\\
&\quad+\sum_{j'=1}^N\left\|\tilde{q}^{j'k'}\theta z_{j'}\right\|_{L^2(-\widetilde{T},\widetilde{T};H^{-n/p}(\Omega))}\left\|\theta^{-1}\check{w}_{j'}\right\|_{L^2(-\widetilde{T},\widetilde{T};H_0^{n/p}(\Omega))}.
\end{aligned} \tag{4.26}$$

By Young's inequality, we have

$$\begin{aligned}
&\lambda^{2(1-n/p)}\big\|\theta^{-1}\check{w}_{j'}\big\|^2_{L^2(-\widetilde{T},\widetilde{T};H_0^{n/p}(\Omega))}\\
&\le \widetilde{C}_*\lambda^{2(1-n/p)}\big\|\theta^{-1}\check{w}_{j'}\big\|^{2n/p}_{L^2(-\widetilde{T},\widetilde{T};H_0^1(\Omega))}\big\|\theta^{-1}\check{w}_{j'}\big\|^{2(1-n/p)}_{L^2(\widetilde{Q})}\\
&\le \widetilde{C}_*\big(\big\|\theta^{-1}\check{w}_{j'}\big\|^2_{L^2(-\widetilde{T},\widetilde{T};H_0^1(\Omega))}+\lambda^2\big\|\theta^{-1}\check{w}_{j'}\big\|^2_{L^2(\widetilde{Q})}\big).
\end{aligned}\tag{4.27}$$

Combining with (4.25), (4.26) and (4.27), we obtain the desired estimate (4.1). This completes the proof of Theorem 4.1. □

5. Proofs of Theorem 1.4 and Theorem 1.8

In this section, we give the proofs of Theorems 1.4 and 1.8. To begin with, we define the energy of the system (1.3) with $(w_0, w_1) \in L^2(\Omega)^N \times H^{-1}(\Omega)^N$ as following:

$$E(t) \triangleq \frac{1}{2}\big(\|w(t,\cdot)\|^2_{L^2(\Omega)^N}+\|w_t(t,\cdot)\|^2_{H^{-1}(\Omega)^N}\big).\tag{5.1}$$

We recall the following known energy estimate (see [13], inequalities (2.43) and (2.44) for example).

Lemma 5.1. *For any $-T<T_0<S_0<S_0'<T_0'<T$ and $w(\cdot)\in C([-T,T];L^2(\Omega)^N)\cap C^1([-T,T];\,H^{-1}(\Omega)^N)$ satisfying (1.3), we have*

$$E(t)\le C_*\mathrm{e}^{C_* r^{\frac{1}{2-n/p}}}E(s),\quad \forall t,s\in[-T,T],\tag{5.2}$$

and

$$\int_{S_0}^{S_0'}E(t)\mathrm{d}t\le C_*(1+r)\int_{T_0}^{T_0'}\int_\Omega|w|^2\mathrm{d}x\mathrm{d}t,\tag{5.3}$$

where r is defined by (1.16).

Proof of Theorem 1.4. First, we introduce some notations:

$$\begin{cases} T_i=-\varepsilon_i T, & T_i'=\varepsilon_i T,\\ Q_i=(T_i,T_i')\times\Omega, & Q_i'=(T_i,T_i')\times\omega_1,\end{cases}\qquad i=0,1,\tag{5.4}$$

where $0<\varepsilon_0<\varepsilon_1<1$ will be given later.

Let ω_1 be given in (2.1) sufficiently close to ω such that

$$\widetilde{T}>T>\max_{x\in\overline{\Omega}\setminus\omega_1}\sqrt{\varphi(x)}.\tag{5.5}$$

Set

$$Q(b)\triangleq\{(t,x)\in(-T,T)\times(\Omega\backslash\omega_1)|\ \sigma(t,x)>b^2\}.\tag{5.6}$$

Recalling (1.14) for the definition of R_0, by (1.10), we see that $R_0 > 0$. Hence, we can choose a sufficiently small $c \in (0, R_0)$ and an $\alpha \in (0,1)$ which is close to 1 such that

$$\min_{x\in\overline{\Omega}} \varphi(x) > c^2, \tag{5.7}$$

and

$$1 - \frac{c^2}{T^2} < \alpha < 1. \tag{5.8}$$

On one hand, if $t = 0$, we have

$$\sigma(0,x) = \varphi(x) > c^2, \qquad \forall x \in \overline{\Omega}. \tag{5.9}$$

On the other hand, for any $x \in Q(c)$, we have

$$\varphi(x) - t^2 > c^2 + (\alpha - 1)T^2 > 0. \tag{5.10}$$

Combining (5.5), (5.9) and (5.10), we can take $\varepsilon_0 > 0$ sufficiently small and $\varepsilon_1 \in (0,1)$ sufficiently close to 1 such that

$$Q_0 \backslash Q_0' \subset Q(c) \subset \mathcal{D}' \subset Q_1 \backslash Q_1', \tag{5.11}$$

where

$$\mathcal{D}' \triangleq \big\{(t,x) \in Q \setminus [(-T,T)\times\omega_1] \,\big|\, \varphi(x) - t^2 > 0\big\}. \tag{5.12}$$

Now, we choose $\varepsilon > 0$ small enough such that

$$Q_0 \setminus Q_0' \subset Q(c+3\varepsilon) \subset Q(c+2\varepsilon) \subset Q(c+\varepsilon) \subset Q(c) \subset \mathcal{D}' \subset Q_1 \setminus Q_1'. \tag{5.13}$$

Let us choose a cut-off function $\chi_1 \in C_0^\infty(\widetilde{Q})$(recall (2.3) for the definition of $\widetilde{Q}$) such that

$$\chi_1(t,x) = \begin{cases} 1, & (t,x) \in Q(c+2\varepsilon) \setminus [(-T,T)\times\omega_2], \\ 0, & (t,x) \in \widetilde{Q} \setminus Q(c+\varepsilon), \end{cases} \tag{5.14}$$

where $\omega_2 \triangleq \mathcal{O}_{\tilde{\delta}_2}(\Gamma_0)$ with $\tilde{\delta}_2 \in (\tilde{\delta}_1, \delta)$. Set $y(t,s) = \chi_1(t,x)w(t,x)$ (recall that w solves (1.3)). Then we have $y \in C([-\widetilde{T},\widetilde{T}]; L^2(\Omega)^N)$ and $y(-\widetilde{T},\cdot) = y(\widetilde{T},\cdot) = 0$ in Ω. By (5.14), we know that $\operatorname{supp} y \subset [T_1, T_1'] \times (\overline{\Omega} \setminus \omega_1)$. Applying Theorem 4.1 to y, noting that $\tilde{q}|_Q = q$, we get

$$\begin{aligned}
&\int_{Q(c+2\varepsilon)\setminus[(-T,T)\times\omega_2]} \lambda\theta^2|w|^2 \mathrm{d}x\mathrm{d}t \\
&\le \int_{Q_1} \lambda\theta^2|\chi_1 w|^2 \mathrm{d}x\mathrm{d}t \\
&\le C_* \left(\left\| \theta \left\{ (\chi_1 w)_{tt} - \sum_{j,k=1}^n \partial_{x_j}\left[h^{jk}(\chi_1 w)_{x_k}\right] - q\chi_1 w \right\} \right\|^2_{H^{-1}(Q)^N} + \frac{1}{\lambda^{2(1-n/p)}} \|\theta\chi_1 q w\|^2_{L^2(-T,T;H^{-n/p}(\Omega)^N)} \right).
\end{aligned} \tag{5.15}$$

From (1.3), we obtain that

$$
\begin{aligned}
&\left\|\theta\left\{(\chi_1 w)_{tt}-\sum_{j,k=1}^{n}\partial_{x_j}\left[h^{jk}(\chi_1 w)_{x_k}\right]-q\chi_1 w\right\}\right\|_{H^{-1}(Q)^N}\\
&=\left\|\theta\left[2\chi_{1,t}w_t+w\chi_{1,tt}-\sum_{j,k=1}^{n}\partial_{x_j}(h^{jk}\chi_{1,x_k}w)-\sum_{j,k=1}^{n}h^{jk}\chi_{1,x_j}w_{x_k}\right]\right\|_{H^{-1}(Q)^N}\\
&=\sup_{|g|_{H_0^1(Q)^N}=1}\left\langle\theta\left[2\chi_{1,t}w_t+\chi_{1,tt}w-\sum_{j,k=1}^{n}\partial_{x_j}(h^{jk}\chi_{1,x_k}w)-\sum_{j,k=1}^{n}h^{jk}\chi_{1,x_j}w_{x_k}\right],g\right\rangle_{H^{-1}(Q)^N\times H_0^1(Q)^N}\\
&=\sup_{|g|_{H_0^1(Q)^N}=1}\int_Q\theta w^\top\cdot\Big[-\chi_{1,tt}g-2\chi_{1,t}g_t-2\lambda\phi_t\chi_{1,t}g\\
&\qquad+\sum_{j,k=1}^{n}\partial_{x_j}(h^{jk}\chi_{1,x_k})g+2\lambda\sum_{j,k=1}^{n}h^{jk}\phi_{x_j}\chi_{1,x_k}g+2\sum_{j,k=1}^{n}h^{jk}\chi_{1,x_j}g_{x_k}\Big]\mathrm{d}x\mathrm{d}t\\
&\le C_*\mathrm{e}^{\lambda\mathrm{e}^{\mu_0(c+2\varepsilon)^2}}\|w\|_{L^2(Q_1\setminus Q(c+2\varepsilon))^N}+C_*\|\theta w\|_{L^2([(T_1,T_1')\times\omega_2]\cap Q(c+2\varepsilon))^N}.
\end{aligned}
\tag{5.16}
$$

It follows from the Sobolev embedding theorem that

$$
\|\theta q\chi_1 w\|_{L^2(-T,T;H^{-n/p}(\Omega)^N)}\le\|\theta q\chi_1 w\|_{L^2(-T,T;L^{\frac{2p}{p+2}}(\Omega)^N)}\le C_* r\|\theta\chi_1 w\|_{L^2(Q)^N}. \tag{5.17}
$$

On the other hand, by taking $\lambda\ge(C_*r)^{\frac{1}{3/2-n/p}}$ (C_* may vary from line to line), we have

$$
\frac{C_*r}{\lambda^{1-n/p}}\|\theta\chi_1 w\|_{L^2(Q)^N}\le\lambda^{\frac12}\|\theta\chi_1 w\|_{L^2(Q)^N}. \tag{5.18}
$$

Combining (5.15)–(5.18), taking $\lambda\ge C_*r^{\frac{1}{3/2-n/p}}$, we get that

$$
\begin{aligned}
&\int_{Q(c+2\varepsilon)\setminus[(-T,T)\times\omega_2]}\lambda\theta^2|w|^2\mathrm{d}x\mathrm{d}t\\
&\le C_*\mathrm{e}^{2\lambda\mathrm{e}^{\mu_0(c+2\varepsilon)^2}}\|w\|^2_{L^2(Q_1\setminus Q(c+2\varepsilon))^N}+C_*\|\theta w\|^2_{L^2([(T_1,T_1')\times\omega_2]\cap Q(c+2\varepsilon))^N}.
\end{aligned}
\tag{5.19}
$$

Adding $\int_{(T_0,T_0')\times\omega_2}\theta^2|w|^2\mathrm{d}x\mathrm{d}t$ to each side of (5.19), recalling the definition of $\mathcal{K}$, we have

$$
\lambda\mathrm{e}^{2\lambda\mathrm{e}^{\mu_0(c+3\varepsilon)^2}}\int_{Q_0}|w|^2\mathrm{d}x\mathrm{d}t\le C_*\mathrm{e}^{2\lambda\mathrm{e}^{\mu_0(c+2\varepsilon)^2}}\int_{Q_1\setminus Q(c+2\varepsilon)}|w|^2\mathrm{d}x\mathrm{d}t+C_*\mathrm{e}^{C^*\lambda}\int_{\mathcal{K}}|w|^2\mathrm{d}x\mathrm{d}t. \tag{5.20}
$$

By Lemma 5.1, we get

$$
\int_{Q_1\setminus Q(c+2\varepsilon)}|w|^2\mathrm{d}x\mathrm{d}t\le C_*E(-T)\mathrm{e}^{C_*r^{\frac{1}{2-n/p}}}, \tag{5.21}
$$

and

$$\int_{Q_0} |w|^2 \mathrm{d}x\mathrm{d}t \geq \frac{1}{C_*(1+r)} E(-T) \mathrm{e}^{-C_* r^{\frac{1}{2-n/p}}}. \tag{5.22}$$

Taking λ large enough such that

$$\begin{cases} \frac{1}{2}\lambda\Big(\mathrm{e}^{\mu_0(c+3\varepsilon)^2} - \mathrm{e}^{\mu_0(c+2\varepsilon)^2}\Big) \geq 2C_* r^{\frac{1}{2-n/p}}, \\ \dfrac{\lambda}{C_*(1+r)} - 1 > 0. \end{cases} \tag{5.23}$$

Hence,

$$\lambda \frac{1}{C_*(1+r)} \mathrm{e}^{\lambda \mathrm{e}^{\mu_0(c+3\varepsilon)^2} - C_* r^{\frac{1}{2-n/p}}} - \mathrm{e}^{C_* r^{\frac{1}{2-n/p}} + \lambda \mathrm{e}^{\mu_0(c+2\varepsilon)^2}} > 0. \tag{5.24}$$

Finally, by (5.19) and taking $\lambda \geq C_*\Big(1 + r^{\frac{1}{3/2-n/p}}\Big)$, we have

$$E(-T) \leq C_* \mathrm{e}^{C_*\lambda} |w|^2_{L^2(\mathcal{K})^N},$$

which yields (1.4). □

Before giving the proof of Theorem 1.8, we introduce the following modified energy estimate of the system (1.3) with $(w_0, w_1) \in H_0^1(\Omega)^N \times L^2(\Omega)^N$ (see [13], inequalities (2.50) for example).

Lemma 5.2. *For any $w(\cdot) \in C([-T,T]; H_0^1(\Omega)^N) \cap C^1([-T,T]; L^2(\Omega)^N)$ satisfy (1.3), we have*

$$\|(w(t), w_t(t))\|_{H_0^1(\Omega)^N \times L^2(\Omega)^N} \leq C^* \mathrm{e}^{C^* r^{\frac{1}{2-n/p}}} \|(w(s), w_t(s))\|_{H_0^1(\Omega)^N \times L^2(\Omega)^N}, \quad \forall t, s \in [-T,T], \tag{5.25}$$

where r is defined in (1.16).

Proof of Theorem 1.8. First, we introduce the following notations:

$$T_i = -\varepsilon_i T, \quad T_i' = \varepsilon_i T, \quad Q_i = (T_i, T_i') \times \Omega,$$

where $0 < \varepsilon_0 < \varepsilon_1 < 1$ (which will be given later).

Set

$$Q(b) \triangleq \{(t,x) \in (-\infty, +\infty) \times \Omega |\ \sigma(t,x) > b^2\}.$$

Let R_0 be given by (1.14). From (1.10), we know that there is a sufficiently small $c \in (0, R_0)$ and an $\alpha \in (0,1)$, close to 1, such that (5.7) and (5.8) hold. Recalling (1.18) for the definition of R^1, by (1.10), for any $T > R^1$, we know that

$$\varphi(x) < T^2, \qquad \forall x \in \Omega. \tag{5.26}$$

By (5.9) and (5.26), we can take an $\varepsilon_0 > 0$ sufficiently small and an $\varepsilon_1 \in (0,1)$ sufficiently close to 1 such that

$$Q_0 \subset Q(c) \subset \mathcal{D} \subset Q_1. \tag{5.27}$$

Now, we choose $\varepsilon > 0$ small enough such that

$$Q_0 \subset Q(c+3\varepsilon) \subset Q(c+2\varepsilon) \subset Q(c+\varepsilon) \subset Q(c) \subset \mathcal{D} \subset Q_1. \tag{5.28}$$

Recall (2.7) for the definition of $\widehat{Q}$. Let $\eta \in C_0^\infty(\widehat{Q})$ satisfying

$$\eta(t,x) = \begin{cases} 1, & (t,x) \in Q(c+2\varepsilon), \\ 0, & (t,x) \in \widehat{Q} \backslash Q(c+\varepsilon). \end{cases} \tag{5.29}$$

Set $u = \eta w$. By Theorem 2.2, we have

$$\begin{aligned} &\lambda\mu \int_Q \theta^2 \left[|(\eta w)_t|^2 + |\nabla(\eta w)|^2 + \lambda^2\mu^2\eta^2|w|^2\right] \mathrm{d}x\mathrm{d}t \\ &\leq C^* \int_Q \theta^2\eta^2 \left| w_{tt} - \sum_{j,k=1}^n (h^{jk} w_{x_j})_{x_k} \right|^2 \mathrm{d}x\mathrm{d}t + C^*\lambda\mu \int_{\Sigma_0} \theta^2 \left| \eta \frac{\partial w}{\partial \nu} \right|^2 \mathrm{d}\Gamma\mathrm{d}t \\ &+ C^* \int_Q \theta^2 \left| 2\eta_t w_t + \eta_{tt} w - 2\sum_{j,k=1}^n h^{jk}\eta_{x_j} w_{x_k} - w \sum_{j,k=1}^n (h^{jk}\eta_{x_j})_{x_k} \right|^2 \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{5.30}$$

Recalling (1.16) for the definition of r, by the Sobolev embedding theorem, we have

$$\begin{aligned} \|\theta q \eta w\|_{L^2(Q)^N} &\leq r\|\theta\eta w\|_{L^2(-T,T;L^s(\Omega))^N}, \quad 1/s + 1/p = 1/2 \\ &\leq r\|\theta\eta w\|_{L^2(-T,T;H_0^{n/p}(\Omega))^N} \\ &\leq r\|\theta\eta w\|^{n/p}_{L^2(-T,T;H_0^1(\Omega))^N} \|\theta\eta w\|^{1-n/p}_{L^2(Q)^N}. \end{aligned} \tag{5.31}$$

Hence, for any $\delta_2 > 0$, by Young's inequality,

$$\|q\theta\eta w\|^2_{L^2(Q)^N} \leq \delta_2\lambda\|\theta\eta w\|^2_{L^2(-T,T;H_0^1(\Omega))^N} + C_{\delta_2} r^{2p/(p-n)} \lambda^{-n/(p-n)} \|\theta\eta w\|^2_{L^2(Q)^N}, \tag{5.32}$$

where C_{δ_2} is a positive constant depending on ε. Consequently, for $\lambda \geq (C^* r)^{\frac{1}{3/2-n/p}}$, we have

$$(C^* r)^{2p/(p-n)} \lambda^{-n/(p-n)} \|\theta\eta w\|^2_{L^2(Q)^N} \leq \lambda^3 \|\theta\eta w\|^2_{L^2(Q)^N}. \tag{5.33}$$

Keeping in mind that $\eta \equiv 0$ outside $\mathcal{D}$ and C^* represents a generic constant, combining (5.30) and (5.32), and taking $\lambda \geq C^* r^{\frac{1}{3/2-n/p}}$, we have

$$\begin{aligned} &\lambda\mu \mathrm{e}^{2\lambda \mathrm{e}^{\mu(c+3\varepsilon)^2}} \int_{Q_0} \left(|w_t|^2 + |\nabla w|^2 + \lambda^2\mu^2|w|^2\right) \mathrm{d}x\mathrm{d}t \\ &\leq C^* \mathrm{e}^{2\lambda \mathrm{e}^{\mu(c+2\varepsilon)^2}} \|w\|^2_{L^2(Q_1\backslash Q(c+2\varepsilon))^N} + C^*\lambda\mu \int_{L^2(\Sigma_0\cap\mathcal{D})^N} \theta^2 \left| \frac{\partial w}{\partial \nu} \right|^2 \mathrm{d}\Gamma\mathrm{d}t. \end{aligned} \tag{5.34}$$

Set $\mu = \mu_0$. By (5.25), we have

$$\|w\|^2_{L^2(Q_1\backslash Q(c+2\varepsilon))^N} \leq C^* \mathrm{e}^{C^* r^{\frac{1}{2-n/p}}} \|(w_0, w_1)\|_{H_0^1(\Omega)^N \times L^2(\Omega)^N} \tag{5.35}$$

and

$$\int_{Q_0} \left(|w_t|^2 + |\nabla w|^2 + |w|^2\right) \mathrm{d}x\mathrm{d}t \geq C^* \mathrm{e}^{-C^* r^{\frac{1}{2-n/p}}} \|(w_0, w_1)\|_{(H_0^1(\Omega))^N \times L^2(\Omega)^N}. \tag{5.36}$$

Taking $\lambda > 0$ large enough such that $\frac{1}{2}\lambda\left(\mathrm{e}^{\mu_0(c+3\varepsilon)^2} - \mathrm{e}^{\mu_0(c+2\varepsilon)^2}\right) \geq 2C^* r^{\frac{1}{2-n/p}}$. Hence

$$\lambda \mathrm{e}^{\lambda \mathrm{e}^{\mu_0(c+3\varepsilon)^2} - C^* r^{\frac{1}{2-n/p}}} - \mathrm{e}^{C^* r^{\frac{1}{2-n/p}} + \lambda \mathrm{e}^{\mu_0(c+2\varepsilon)^2}} > 0. \tag{5.37}$$

Finally, by (1.13), (5.34) and taking $\lambda \geq C^*\left(1 + r^{\frac{1}{3/2-n/p}}\right)$, we have

$$\|w_0\|_{H_0^1(\Omega)^N} + \|w_1\|_{L^2(\Omega)^N} \leq C^* \mathrm{e}^{C^*\lambda} \left\| \frac{\partial w}{\partial \nu} \right\|_{L^2(\mathcal{H})^N}, \tag{5.38}$$

which yields (1.5). □

Appendix A. Appendix: Proof of Proposition 3.1

Before giving the proof of Proposition 3.1, we first recall the following known result.

Proposition A.1. *([15], Prop. 3.5) For any $\tau > 0$, $m = 2, 3, \cdots$, and $q_m^j, w_m^j \in \mathbb{C}$, $j = 0, 1, \cdots, m$, with $q_m^0 = q_m^m = 0$, one has*

$$\begin{aligned} -\sum_{j=1}^{m-1} q_m^j \frac{(w_m^{j+1} - 2w_m^j + w_m^{j-1})}{\tau^2} &= \sum_{j=0}^{m-1} \frac{(q_m^{j+1} - q_m^j)}{\tau} \frac{(w_m^{j+1} - w_m^j)}{\tau} \\ &= \sum_{j=1}^{m} \frac{(q_m^j - q_m^{j-1})}{\tau} \frac{(w_m^j - w_m^{j-1})}{\tau}. \end{aligned} \tag{A.1}$$

Proof of Proposition 3.1. Clearly, the functional $\mathcal{J}$ is weakly lower semi-continuous, strictly convex, and coercive in $(L^2(\Omega))^{3m+3}$. By the coercivity of $\mathcal{J}$, we know there exist a bounded minimizing sequence $\left\{\{w_m^{j,k}, r_{1m}^{j,k}, r_m^{j,k}\}_{j=0}^m\right\}_{k=1}^{+\infty}$ of $\mathcal{J}(\cdot)$ in $\mathcal{A}_{ad}$. Hence, there exist a sub-sequence, also denote by $\left\{\{w_m^{j,k}, r_{1m}^{j,k}, r_m^{j,k}\}_{j=0}^m\right\}_{k=1}^{+\infty}$ for simplicity of notations, satisfying

$$\{w_m^{j,k}, r_{1m}^{j,k}, r_m^{j,k}\}_{j=0}^m \text{ converges weakly to } \{\hat{w}_m^j, \hat{r}_{1m}^j, \hat{r}_m^j\}_{j=0}^m \text{ in } (L^2(\Omega))^{3m+3}.$$

On the other hand, by the regularity theory for elliptic equations, we have $\left\{\{w_m^{j,k}, r_{1m}^{j,k}, r_m^{j,k}\}_{j=0}^m\right\}_{k=1}^{+\infty}$ is also bounded in $(H_0^1(\Omega) \times L^2(\Omega)^2)^{m+1}$. Thus, it has a convergent sub-sequence. Consequently, $\{\hat{w}_m^j, \hat{r}_{1m}^j, \hat{r}_m^j\}_{j=0}^m \in (H_0^1(\Omega) \times L^2(\Omega)^2)^{m+1}$.

Meanwhile, by Mazur's Lemma, there exist a convex combination

$$\sum_{k=1}^{N_l} \lambda_{lk} \{w_m^{j,k}, r_{1m}^{j,k}, r_m^{j,k}\}_{j=0}^m \text{ converges strongly to } \{\hat{w}_m^j, \hat{r}_{1m}^j, \hat{r}_m^j\}_{j=0}^m \text{ in } (L^2(\Omega))^{3m+3} \text{ as } l \to +\infty,$$

where $\lambda_{lk}, N_l \geq 0$, $k = 1, \cdots, N_l$, and $\sum_{k=1}^{N_l} \lambda_{lk} = 1$. Then we find $\operatorname{supp}\{\hat{w}_m^j, \hat{r}_{1m}^j, \hat{r}_m^j\}_{j=0}^m \subset\subset \overline{\Omega} \backslash \omega_1$.

Further, it follows from the weak lower semi-continuity of $\mathcal{J}(\cdot)$ that $\{\hat{w}_m^j, \hat{r}_{1m}^j, \hat{r}_m^j\}_{j=0}^m$ is a solution of Problem (OP). The uniqueness follows from the strict convexity of $\mathcal{J}(\cdot)$.

The rest of the proof is divided into four steps.

Step 1. Fix $w_{0m}^j \in H^2(\Omega) \cap H_0^1(\Omega)$, $w_{1m}^j \in L^2(\Omega)$ with $w_{0m}^0 = w_{0m}^m = 0$ and $w_{1m}^0 = w_{1m}^m$ such that

$$\operatorname{supp}\{(w_{0m}^j, w_{1m}^j)\}_{j=0}^m \subset \overline{\Omega}\backslash\omega. \tag{A.2}$$

For $(\kappa_0, \kappa_1) \in \mathbb{R}^2$, we have

$$\{(\hat{w}_m^j + \kappa_0 w_{0m}^j, \hat{r}_{1m}^j + \kappa_1 w_{1m}^j, r_m^j)\}_{j=0}^m \in \mathcal{A}_{ad}, \tag{A.3}$$

where

$$\begin{cases} r_m^j = \dfrac{\hat{w}_m^{j+1} - 2\hat{w}_m^j + \hat{w}_m^{j-1}}{\tau^2} - \displaystyle\sum_{j_1,j_2=1}^n \partial_{x_{j_2}}(h^{j_1j_2}\partial_{x_{j_2}}\hat{w}_m^j) - \dfrac{\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^j}{\tau} - \lambda z_m^j \mathrm{e}^{2\lambda\phi_m^j} \\ \qquad + \kappa_0 \left[\dfrac{w_{0m}^{j+1} - 2w_{0m}^j + w_{0m}^{j-1}}{\tau^2} - \displaystyle\sum_{j_1,j_2=1}^n \partial_{x_{j_2}}(h^{j_1j_2}\partial_{x_{j_2}} w_{0m}^j)\right] \\ \qquad - \kappa_1 \dfrac{w_{1m}^{j+1} - w_{1m}^j}{\tau}, \qquad 1 \le j \le m-1, \\ r_m^0 = r_m^m = 0. \end{cases} \tag{A.4}$$

Define a function in $\mathbb{R}^2$ by

$$g(\kappa_0, \kappa_1) = \mathcal{J}(\{(\hat{w}_m^j + \kappa_0 w_{0m}^j, \hat{r}_{1m}^j + \kappa_1 w_{1m}^j, r_m^j)\}_{j=0}^m). \tag{A.5}$$

Obviously $g(\kappa_0, \kappa_1)$ has a minimum at $(0, 0)$. Hence, taking derivative of (A.5) with respect to κ_0 and κ_1, we get that

$$-K\sum_{j=1}^{m-1}\int_\Omega \hat{r}_m^j \frac{w_{1m}^{j+1} - w_{1m}^j}{\tau}\mathrm{d}x + \sum_{j=1}^m \int_\Omega \rho^K \frac{\hat{r}_{1m}^j w_{1m}^j}{\lambda^2}\mathrm{e}^{-2\lambda\phi_m^j}\mathrm{d}x = 0,$$

and

$$\sum_{j=1}^{m-1}\int_\Omega \left\{K\hat{r}_m^j \left[\frac{w_{0m}^{j+1} - 2w_{0m}^j + w_{0m}^{j-1}}{\tau^2} - \sum_{j_1,j_2=1}^n \partial_{x_{j_2}}(h^{j_1j_2}\partial_{x_{j_2}} w_{0m}^j)\right] + \hat{w}_m^j w_{0m}^j \mathrm{e}^{-2\lambda\phi_m^j}\right\}\mathrm{d}x = 0. \tag{A.6}$$

By the regularity property of solutions to elliptic equations, we have $\hat{w}_m^j, \hat{r}_m^j \in H^2(\Omega) \cap H_0^1(\Omega)$ and (3.8)–(3.10) hold.

Step 2. Replacing $\{w_{0m}^j\}_{j=0}^m$ by $\{\hat{w}_m^j\}_{j=0}^m$ in (A.6) and recalling that $p_m^j = K\hat{r}_m^j$, we have

$$\begin{aligned} 0 &= \sum_{j=1}^{m-1}\int_\Omega \left[\frac{\hat{w}_m^{j+1} - 2\hat{w}_m^j + \hat{w}_m^{j-1}}{\tau^2} - \sum_{j_1,j_2=1}^n \partial_{x_{j_2}}(h^{j_1j_2}\partial_{x_{j_2}}\hat{w}_m^j) - \frac{\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^j}{\tau} - \lambda z_m^i \mathrm{e}^{2\lambda\phi_m^j} - \hat{r}_m^j\right] p_m^j \mathrm{d}x \\ &= -\sum_{j=1}^{m-1}\int_\Omega |\hat{w}_m^j|^2 \mathrm{e}^{-2\lambda\phi_m^j}\mathrm{d}x + \sum_{j=1}^m \int_\Omega \frac{p_m^j - p_m^{j-1}}{\tau}\hat{r}_{1m}^j \mathrm{d}x - \sum_{j=1}^{m-1}\int_\Omega \big(\lambda z_m^j \mathrm{e}^{2\lambda\phi_m^j} + \hat{r}_m^j\big) p_m^j \mathrm{d}x \end{aligned} \tag{A.7}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{m-1}\left(\int_\Omega |\hat{w}_m^j|^2 \mathrm{e}^{-2\lambda\phi_m^j}\mathrm{d}x + \int_\Omega \rho^K \frac{|\hat{r}_{1m}^j|^2}{\lambda^2}\mathrm{e}^{-2\lambda\phi_m^j}\mathrm{d}x + K\int_\Omega |\hat{r}_m^j|^2\mathrm{d}x\right)\\
&\quad - \int_\Omega \rho^K \frac{|\hat{r}_{1m}^m|^2}{\lambda^2}\mathrm{e}^{-2\lambda\phi_m^m}\mathrm{d}x - \lambda\sum_{j=1}^{m-1}\int_\Omega z_m^j \mathrm{e}^{2\lambda\phi_m^j} p_m^j \mathrm{d}x.
\end{aligned}
$$

Thus, there is a constant $C = C(K,\lambda) > 0$, independent of m, such that

$$
\begin{aligned}
&\sum_{j=1}^{m-1}\left(\int_\Omega |\hat{w}_m^j|^2 \mathrm{e}^{-2\lambda\phi_m^j}\mathrm{d}x + \int_\Omega \rho^K \frac{|\hat{r}_{1m}^j|^2}{\lambda^2}\mathrm{e}^{-2\lambda\phi_m^j}\mathrm{d}x + K\int_\Omega |\hat{r}_m^j|^2\mathrm{d}x\right)\\
&\quad + \int_\Omega \rho^K \frac{|\hat{r}_{1m}^m|^2}{\lambda^2}\mathrm{e}^{-2\lambda\phi_m^m}\mathrm{d}x \le C\sum_{j=1}^{m-1}\int_\Omega |z_m^j|^2 \mathrm{e}^{4\lambda\phi_m^j}\mathrm{d}x,
\end{aligned}
\tag{A.8}
$$

which yields (3.11).

Step 3. Recalling (3.4) and replacing $\{w_{0m}^j\}_{j=0}^m$ by $\{\hat{w}_m^j\}_{j=0}^m$ in (A.6), we have

$$
\begin{aligned}
0 &= \sum_{j=1}^{m-1}\int_\Omega \left[\frac{\hat{w}_m^{j+1} - 2\hat{w}_m^j + \hat{w}_m^{j-1}}{\tau^2} - \sum_{j_1,j_2=1}^n \partial_{x_{j_2}}(h^{j_1j_2}\partial_{x_{j_2}}\hat{w}_m^j)\right.\\
&\qquad \left. - \frac{\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^j}{\tau} - \lambda z_m^j \mathrm{e}^{2\lambda\phi_m^j} - \hat{r}_m^j\right]\frac{p_m^{j+1} - 2p_m^j + p_m^{j-1}}{\tau^2}\mathrm{d}x\\
&= -\sum_{j=1}^{m-1}\int_\Omega \hat{w}_m^j \frac{\hat{w}_m^{j+1}\mathrm{e}^{-2\lambda\phi_m^{j+1}} - 2\hat{w}_m^j\mathrm{e}^{-2\lambda\phi_m^j} + \hat{w}_m^{j-1}\mathrm{e}^{-2\lambda\phi_m^{j-1}}}{\tau^2}\\
&\qquad - \left(\frac{\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^j}{\tau} + \lambda z_m^j \mathrm{e}^{2\lambda\phi_m^j} + \hat{r}_m^j\right)\frac{p_m^{j+1} - 2p_m^j + p_m^{j-1}}{\tau^2}\mathrm{d}x.
\end{aligned}
\tag{A.9}
$$

For each term in (A.9), by (A.1), we do some calculations as follows:

$$
\begin{aligned}
&-\sum_{j=1}^{m-1}\int_\Omega \left(\hat{w}_m^j \frac{\hat{w}_m^{j+1}\mathrm{e}^{-2\lambda\phi_m^{j+1}} - 2\hat{w}_m^j\mathrm{e}^{-2\lambda\phi_m^j} + \hat{w}_m^{j-1}\mathrm{e}^{-2\lambda\phi_m^{j-1}}}{\tau^2} + \hat{r}_m^j\frac{p_m^{j+1} - 2p_m^j + p_m^{j-1}}{\tau^2}\right)\mathrm{d}x\\
&= \sum_{j=0}^{m-1}\int_\Omega \left[\frac{\hat{w}_m^{j+1} - \hat{w}_m^j}{\tau}\cdot\frac{\hat{w}_m^{j+1}\mathrm{e}^{-2\lambda\phi_m^{j+1}} - \hat{w}_m^j\mathrm{e}^{-2\lambda\phi_m^j}}{\tau} + K\frac{(\hat{r}_m^{j+1} - \hat{r}_m^j)^2}{\tau^2}\right]\mathrm{d}x\\
&= \sum_{j=0}^{m-1}\int_\Omega \left[\frac{(\hat{w}_m^{j+1} - \hat{w}_m^j)^2}{\tau^2}\mathrm{e}^{-2\lambda\phi_m^j} + \frac{\hat{w}_m^{j+1} - \hat{w}_m^j}{\tau}\frac{\mathrm{e}^{-2\lambda\phi_m^{j+1}} - \mathrm{e}^{-2\lambda\phi_m^j}}{\tau}\hat{w}_m^{j+1} + K\frac{(\hat{r}_m^{j+1} - \hat{r}_m^j)^2}{\tau^2}\right]\mathrm{d}x,
\end{aligned}
\tag{A.10}
$$

and

$$
\begin{aligned}
&-\sum_{j=1}^{m-1}\int_\Omega \left(\frac{\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^j}{\tau} + \lambda z_m^j\mathrm{e}^{2\lambda\phi_m^j}\right)\frac{p_m^{j+1} - 2p_m^j + p_m^{j-1}}{\tau^2}\mathrm{d}x\\
&= -\sum_{j=1}^{m-1}\int_\Omega \left(\frac{\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^j}{\tau} + \lambda z_m^j\mathrm{e}^{2\lambda\phi_m^j}\right)\frac{1}{\tau}\left(\frac{p_m^{j+1} - p_m^j}{\tau} - \frac{p_m^j - p_m^{j-1}}{\tau}\right)\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{m-1} \int_\Omega \frac{\rho^K}{\lambda^2} \left(\frac{\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^{j}}{\tau} + \lambda z_m^j \mathrm{e}^{2\lambda\phi_m^j} \right) \frac{\hat{r}_{1m}^{j+1} \mathrm{e}^{-2\lambda\phi_m^{j+1}} - \hat{r}_{1m}^{j} \mathrm{e}^{-2\lambda\phi_m^{j}}}{\tau} \mathrm{d}x \\
&= \sum_{j=1}^{m-1} \int_\Omega \frac{\rho^K}{\lambda^2} \left[\frac{(\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^{j})^2}{\tau^2} \mathrm{e}^{-2\lambda\phi_m^j} + \frac{\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^{j}}{\tau} \cdot \frac{\mathrm{e}^{-2\lambda\phi_m^{j+1}} - \mathrm{e}^{-2\lambda\phi_m^{j}}}{\tau} \hat{r}_{1m}^{j+1} \right] \mathrm{d}x \\
&\quad + \lambda \sum_{j=1}^{m-1} \int_\Omega \frac{\rho^K}{\lambda^2} z_m^j \left(\frac{\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^{j}}{\tau} \mathrm{e}^{-2\lambda\phi_m^j} + \frac{\mathrm{e}^{-2\lambda\phi_m^{j+1}} - \mathrm{e}^{-2\lambda\phi_m^{j}}}{\tau} \hat{r}_{1m}^{j+1} \right) \mathrm{d}x.
\end{aligned} \tag{A.11}
$$

Combining (A.9)–(A.11), we have

$$
\begin{aligned}
&\sum_{j=0}^{m-1} \int_\Omega \left[\frac{(\hat{w}_m^{j+1} - \hat{w}_m^{j})^2}{\tau^2} \mathrm{e}^{-2\lambda\phi_m^j} + \frac{\rho^K}{\lambda^2} \frac{(\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^{j})^2}{\tau^2} \mathrm{e}^{-2\lambda\phi_m^j} + K \frac{(\hat{r}_m^{j+1} - \hat{r}_m^{j})^2}{\tau^2} \right] \mathrm{d}x \\
&= - \sum_{j=0}^{m-1} \int_\Omega \frac{\hat{w}_m^{j+1} - \hat{w}_m^{j}}{\tau} \cdot \frac{\mathrm{e}^{-2\lambda\phi_m^{j+1}} - \mathrm{e}^{-2\lambda\phi_m^{j}}}{\tau} \hat{w}_m^{j+1} \mathrm{d}x \\
&\quad - \sum_{j=1}^{m-1} \int_\Omega \frac{\rho^K}{\lambda^2} \cdot \frac{\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^{j}}{\tau} \cdot \frac{\mathrm{e}^{-2\lambda\phi_m^{j+1}} - \mathrm{e}^{-2\lambda\phi_m^{j}}}{\tau} \hat{r}_{1m}^{j+1} \mathrm{d}x \\
&\quad - \lambda \sum_{j=1}^{m-1} \int_\Omega \frac{\rho^K}{\lambda^2} z_m^j \left(\frac{\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^{j}}{\tau} \mathrm{e}^{-2\lambda\phi_m^j} + \frac{\mathrm{e}^{-2\lambda\phi_m^{j+1}} - \mathrm{e}^{-2\lambda\phi_m^{j}}}{\tau} \hat{r}_{1m}^{j+1} \right) \mathrm{d}x.
\end{aligned} \tag{A.12}
$$

By (A.12), we know that there is a constant $C(K, \lambda) > 0$, independent of m, such that

$$
\begin{aligned}
&\sum_{j=0}^{m-1} \int_\Omega \left[\frac{(\hat{w}_m^{i+1} - \hat{w}_m^{i})^2}{\tau^2} \mathrm{e}^{-2\lambda\phi_m^j} + \frac{\rho^K}{\lambda^2} \frac{(\hat{r}_{1m}^{j+1} - \hat{r}_{1m}^{j})^2}{\tau^2} \mathrm{e}^{-2\lambda\phi_m^j} + K \frac{(\hat{r}_m^{j+1} - \hat{r}_m^{j})^2}{\tau^2} \right] \mathrm{d}x \\
&\le C \left[\sum_{j=1}^{m-1} \int_\Omega \left(|\hat{w}_m^j|^2 + |\hat{r}_{1m}^j|^2 + K|\hat{r}_m^j|^2 + |z_m^j|^2 \right) \mathrm{d}x + \int_\Omega |\hat{r}_{1m}^m|^2 \mathrm{d}x \right].
\end{aligned} \tag{A.13}
$$

Combining (A.13) and (3.11), we get (3.12).

Step 4. Replacing $\{w_{0m}^j\}_{j=0}^m$ by $\{p_m^j\}_{j=0}^m$ in (A.6) and recalling that $p_m^j = K\hat{r}_m^j$, we have

$$
\begin{aligned}
0 &= - \sum_{j=1}^{m-1} \int_\Omega \left\{ p_m^j \left[\frac{p_m^{j+1} - 2p_m^j + p_m^{j-1}}{\tau^2} - \sum_{j_1, j_2 = 1}^{n} \partial_{x_{j_2}} (h^{j_1 j_2} \partial_{x_{j_2}} p_m^j) \right] + \hat{w}_m^j p_m^j \mathrm{e}^{-2\lambda\phi_m^j} \right\} \mathrm{d}x \\
&= \sum_{j=1}^{m} \int_\Omega \frac{(p_m^j - p_m^{j-1})^2}{\tau^2} \mathrm{d}x + \sum_{j=1}^{m-1} \int_\Omega \sum_{j_1, j_2 = 1}^{n} h^{j_1 j_2} \partial_{x_{j_1}} p_m^j \partial_{x_{j_2}} p_m^j \mathrm{d}x - \sum_{j=1}^{m-1} \int_\Omega \hat{w}_m^j p_m^j \mathrm{e}^{-2\lambda\phi_m^j} \mathrm{d}x
\end{aligned} \tag{A.14}
$$

Together with (1.2), (3.11), (3.12), recalling that $p_m^j = K\hat{r}_m^j$, we yield

$$
\tau \sum_{j=1}^{m-1} \int_\Omega |\nabla \hat{r}_m^j|^2 \mathrm{d}x \le C \tag{A.15}
$$

Then, multiply equation (3.4) by $\{\hat{w}_m^j\}_{j=0}^m$ and sum up, we get

$$\begin{aligned}
0 &= -\sum_{j=1}^{m-1}\int_\Omega \left[\frac{\hat{w}_m^{j+1}-2\hat{w}_m^j+\hat{w}_m^{j-1}}{\tau^2} - \sum_{j_1,j_2=1}^n \partial_{x_{j_2}}(h^{j_1j_2}\partial_{x_{j_2}}\hat{w}_m^j) - \frac{\hat{r}_{1m}^{j+1}-\hat{r}_{1m}^j}{\tau} - \lambda z_m^i \mathrm{e}^{2\lambda\phi_m^j} - \hat{r}_m^j\right]\hat{w}_m^j \mathrm{d}x \\
&= \sum_{j=1}^m \int_\Omega \frac{(\hat{w}_m^j - w_m^{j-1})^2}{\tau^2}\mathrm{d}x + \sum_{j=1}^{m-1}\int_\Omega \sum_{j_1,j_2=1}^n h^{j_1j_2}\partial_{x_{j_1}}\hat{w}_m^j \partial_{x_{j_2}}\hat{w}_m^j \mathrm{d}x \\
&\quad + \sum_{j=1}^{m-1}\int_\Omega \left(\frac{\hat{r}_{1m}^{j+1}-\hat{r}_{1m}^j}{\tau} + \lambda z_m^i \mathrm{e}^{2\lambda\phi_m^j} + \hat{r}_m^j\right)\hat{w}_m^j \mathrm{d}x.
\end{aligned} \tag{A.16}$$

Together with (1.2), (3.11), (3.12), (A.15), we yield (3.13).

□

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