FORMATION OF A NONTRIVIAL FINITE-TIME STABLE ATTRACTOR IN A CLASS OF POLYHEDRAL SWEEPING PROCESSES WITH PERIODIC INPUT

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Abstract. We consider a differential inclusion known as a polyhedral sweeping process. The general sweeping process was introduced by J.-J. Moreau as a modeling framework for quasistatic deformations of elastoplastic bodies, and a polyhedral sweeping process is typically used to model stresses in a network of elastoplastic springs. Krejčí’s theorem states that a sweeping process with periodic input has a global attractor which consists of periodic solutions, and all such periodic solutions follow the same trajectory up to a parallel translation. We show that in the case of polyhedral sweeping process with periodic input the attractor has to be a convex polyhedron \( X \) of a fixed shape. We provide examples of elastoplastic spring models leading to structurally stable situations where \( X \) is a one- or two-dimensional polyhedron. In general, an attractor of a polyhedral sweeping process may be either exponentially stable or finite-time stable and the main result of the paper consists of sufficient conditions for finite-time stability of the attractor, with upper estimates for the settling time. The results have implications for the shakedown theory.

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1. Introduction

The theory of sweeping processes is a prominent approach to mathematical modeling of nonsmooth phenomena \cite{11, 19, 21, 35}. Specifically, it is well-suited for evolution problems in continuous time, where a state variable is subject to one-sided impenetrable (inequality) constraints. A sweeping process is defined as the initial value problem

\[
\begin{cases}
   -\dot{y} \in N_{C(t)}(y), \\
y(0) = y_0;
\end{cases}
\]

(1.1)

(1.2)

where \( N_{C(t)}(y) \) denotes the normal cone to a time-dependent convex set \( C(t) \) \cite{42}. Geometrically, the boundary of the set \( C(t) \) “sweeps” a point (the state variable \( y \)) not allowing the point to escape to the outside of \( C(t) \).

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In this problem, the input data are the moving set $C(t)$ as a function of time and the initial position $y_0$ of the point, while the trajectory of the swept point is the solution of the problem.

The sweeping process was proposed by French mathematician and mechanics theorist J.-J. Moreau [35]. His goal was mathematical modeling of nonsmooth phenomena in mechanics, such as elastoplasticity (e.g. one-dimensional continuous rod [36]), frictionless and frictional contact of rigid bodies [37, 38]. Moreau’s formulations are now recognized in contemporary literature on elastoplastic continuous media (e.g. [20]) and nonsmooth mechanics (e.g. [1, 3, 11]).

The sweeping process formulations allow to demonstrate an important property of the model as a dynamical system: given a periodic input, the solution is asymptotically periodic. With the mathematical level of rigor this was first shown by Krejčí [31] for a vector Stop operator. The generalization of this fact for sweeping processes with periodic moving set of the type

$$C(t) = \bigcap_{j=1}^{k} (C_j + c_j(t)), \quad (1.3)$$

was accomplished in [21] and it can be summarized as follows: there exists a closed convex set $X$, independent of time, and a periodic function $\chi(t)$ such that for every $x \in X$ function $x + \chi(t)$ is a periodic solution and each periodic solutions can be expressed as $x + \chi(t)$ for an appropriate $x \in X$. Moreover, any solution converges to $x + \chi(t)$ for some $x \in X$, and we call $X(t) = X + \chi(t)$ the attractor of the sweeping process. In other words, any sweeping process with periodic input of the type (1.3) admits an asymptotically stable periodic attractor. We refer the reader to Section 5.3 of [14] for an illustration of this fact, where authors also show the convergence to the attractor in the sense of derivatives.

The problem of efficient identification of the attractor in terms of the input datum remains open. In two mostly independent cases this problem appears to be simpler than in the general situation: a) when attractor $X$ is a singleton set and b) when the attractor is finite-time stable, i.e., when any solution will not just approach the attractor (asymptotic stability), but will merge with it at certain time.

In [21] we presented a sufficient condition for a polyhedral sweeping process to admit a singleton attractor (note, this condition, in particular, holds for the above-mentioned example in [14]). In [24] we focused on a particular case of b), and derived another type of sufficient conditions for both a) and b) to occur simultaneously: this is due to all of the solutions being swept into a vertex of $C(t)$.

It should be noted, however, that the case of a singleton attractor is far from the general or the solely important case. In the context of continuum mechanics of elastoplastic bodies Frederick & Armstrong ([17], Appendix 1) observed that, under a periodic loading, the convergence to a single stress trajectory from different initial conditions may not occur if the yield surface contains straight intervals or flat regions. This is characteristic of, for example, Tresca model (see e.g. [26], Sect. 3.3). Independently from mechanics, one can easily construct an example of a sweeping process governed by polyhedral $C(t)$ which violates the sufficient condition of [21] and has asymptotically stable one dimensional attractor, by adding one more dimension to the example of Section 5.3 in [14] and taking $C(t)$ as a triangular prism. In contrast, in [22] we provide an example of polyhedral sweeping process where the attractor appears to be one-dimensional and finite-time stable (assuming large enough amplitude of $C(t)$). Moreover, [22] proves that the attractor persists under perturbations of parameters of the underlying rheological model.

In the current paper we aim to extend the approach of [24] and generalize the sufficient conditions for finite-time stability. Specifically, in the paper [24], the effective phase of the formation of the attractor is governed by a time-independent condition on the velocity vector $c'(t)$ of the moving constraint. In other words, all faces of the constraint move in [24] with the same velocity $c'(t)$ and $c'(t)$ is supposed to obey the same sufficient conditions for all effective phases of attractor formation. In the present paper we allow different faces of the constraint to move with different velocity vectors and we allow these different velocity vectors to obey different sufficient conditions on different time intervals. This significantly extends the class of sweeping processes for which the asymptotic attractor can be detected basically covering all sweeping processes that occur in current
applications in the field. In the future this approach may lead to an explicit description of the attractor as a convex hull of a minimal number of finite-time stable solutions, which could be found numerically.

Establishing conditions for finite-time stability has its own independent value. For example, systems with controllers which enforce finite-time convergence to an equilibrium have improved precision and eliminate high-frequency control switching while still achieving robustness to parametric uncertainties in unstructured environments [47]. Multiple applications require severe response time constraints in critical situations or for better productivity and in observation problems, when finite-time convergence of the state estimate to the actual value is required [8]. Finite-time convergence implies nonuniqueness of solutions in reverse time which is not possible for Lipschitz-continuous dynamics. However, uniqueness of solutions in forward time and continuous dependence of solutions on initial conditions can be often ensured [4, 18]. The time required for solutions to reach the attractor is called the settling time. Bhat and Bernstein proposed a Lyapunov stability theorem for finite-time stability analysis of continuous autonomous systems, which also provided a basic tool for synthesizing finite-time-stable nonlinear control systems [6, 7]. In particular, several fundamental finite-time stabilization designs [16, 29, 30, 44, 49] were applied to nonlinear systems. Practical applications include finite-time tracking controller design for active suspension systems [40], robot manipulators [48] and underactuated unstable mechanical systems [41]. Problems of optimal finite-time stabilization were addressed, for example, in [25].

Important criteria for finite-time stability of systems with dry friction were established by Adly, Attouch and Cabot in works [2, 12], which are at the foundation of our work. Finite-time stability property is also frequently associated with sliding mode control of piecewise continuous systems modeled by differential inclusions because control algorithms of this type should ensure finite-time convergence to a sliding manifold [46]. In particular, the associated controllers have mechanical and electromechanical applications [13, 15].

The paper is organized as follows. In Section 2 we remind the reader a few general facts about convex sets, polyhedra and projections. We systematically present related facts on the theory of sweeping processes and its asymptotic properties, and demonstrate how incrementally added assumptions on $C(t)$ imply corresponding properties of the sweeping process and its attractor. Our final case of interest is $C(t)$ being periodically moving polyhedron with fixed normal vectors (Thm. 2.22), in which we prove the polyhedral shape of the attractor and, moreover, derive a description of $X$ in terms of the same normal vectors, which describe $C(t)$.

In Section 3 we present examples of periodically driven spring systems and the corresponding sweeping processes which have one- and two-dimensional attractors. Section 4 contains main results and their proofs. To derive the conditions on finite-time stability of $X(t)$, we separately consider convergence of solutions to the plane containing the attractor, i.e. the affine hull of $X(t)$ (Thm. 4.1) and convergence to $X(t)$ within this plane (Thm. 4.7). The results on convergence to the plane of the attractor and convergence within this plane are combined to claim finite-time stability (Cor. 4.9), covering the cases of the examples from Section 3.

2. Preliminaries

2.1. Notations

We use the following notation throughout the paper: $E$ is a general $d$-dimensional Euclidean space, $H$ is a general Hilbert space, $B_r(x)$ is the open ball of radius $r$ centered at $x$ and $B_r[x]$ is the closed ball.

For an arbitrary set $C \subset E$ we use its

- **linear hull** (see [28], A.1.3)

$$\text{lin } C := \left\{ \sum_{i=1}^{m} \lambda_i v_i : m \in \mathbb{N}, v_i \in C, \lambda_i \in \mathbb{R} \text{ for } i \in 1, m \right\};$$

- **affine hull** (see [42], p. 6)

$$\text{aff } C := \left\{ \sum_{i=1}^{m} \lambda_i v_i : m \in \mathbb{N}, v_i \in C, \lambda_i \in \mathbb{R} \text{ for } i \in 1, m \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\};$$
• **convex hull** (see [28], Proposition A.1.3.4)

\[
\text{conv} C := \left\{ \sum_{i=1}^{m} \lambda_i v_i : m \in \mathbb{N}, v_i \in C, \lambda_i \geq 0 \text{ for } i \in \overline{1,m} \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\} ;
\]

• **conical hull** (see [28], A.1.4)

\[
\text{cone} C := \left\{ \sum_{i=1}^{m} \lambda_i v_i : m \in \mathbb{N}, v_i \in C, \lambda_i \geq 0 \text{ for } i \in \overline{1,m} \right\} ;
\]

• **relative interior** (see [42], p. 6) is the interior of the set with regards to its affine hull as a topological space containing the set:

\[
\text{ri} C := \{ x \in \text{aff } C : \text{there is an } \varepsilon > 0 \text{ s.t. } B_\varepsilon(x) \cap \text{aff } C \subset C \} .
\]

### 2.2. Properties of convex sets, polytopes and projections

**Definition 2.1.** (see [42], p. 15, [32], p. 3, [5], p. 125)

Let \( H \) be a real inner product space. For a closed convex nonempty set \( C \subset H \) and a point \( x \in H \) the **outward normal cone** to \( C \) at \( x \) is

\[
N_C(x) := \{ y \in H : \langle y, c - x \rangle \leq 0 \text{ for all } c \in C \},
\]

and the **tangent cone** to \( C \) at \( x \) is

\[
T_C := \{ \lambda(c - x) : c \in C, \lambda \geq 0 \}.
\]

The normal cone to a closed convex nonempty set is intimately connected to the **metric projection** \( \text{proj}(y, C) \), which is a point from \( C \) of minimal distance to \( y \):

\[
\text{proj}(y, C) := \text{arg min}_{x \in C} \| y - x \|.
\]

The following are immediate properties of the projection.

**Proposition 2.2.** For a closed convex set \( C \subset H \) and \( y \in H \):

i) A unique projection \( \text{proj}(y, C) \) always exists.

ii) The following equivalence holds:

\[ x = \text{proj}(y, C) \iff y - x \in N_C(x). \]

iii) The function \( y \mapsto \text{proj}(y, C) \) is nonexpansive (i.e. Lipschitz-continuous with constant 1):

\[ \| \text{proj}(y_1, C) - \text{proj}(y_2, C) \| \leq \| y_1 - y_2 \|, \quad y_1, y_2 \in H. \]

**Definition 2.3.** Let \( C_1, C_2 \subset H \) be closed convex nonempty sets. The function

\[
d_H(C_1, C_2) := \max \left( \sup_{x \in C_2} \text{dist}(x, C_1), \sup_{x \in C_1} \text{dist}(x, C_2) \right) = \max \left( \sup_{x \in C_1} \inf_{y \in C_2} \| x - y \|, \sup_{x \in C_2} \inf_{y \in C_1} \| x - y \| \right)
\]

\[
= \inf \{ r > 0 : B_r(0) \subset C_2 \text{ and } B_r(0) \subset C_1 \};
\]
is called the Hausdorff distance between $C_1$ and $C_2$.

In what follows, we use the notation $F : X \rightrightarrows Y$ for a set-valued function (multifunction) which maps an element from $X$ to a subset of $Y$, see [34], Section 1.2.

**Definition 2.4.** Let $C : [0, T] \rightrightarrows H$ be a set-valued function such that its values $C(t)$ are closed nonempty subsets of $H$ for every $t$. We say that $C$ is Lipschitz-continuous with respect to the Hausdorff distance if there is an $L > 0$ such that

$$d_H(C(t_1), C(t_2)) \leq L|t_1 - t_2|, \quad t_1, t_2 \geq 0.$$ 

**Definition 2.5.** (see [42], Section 19) Let $E$ be a finite-dimensional inner product space with an inner product $\langle \cdot, \cdot \rangle$. We call a set $C \subset E$ a polyhedral convex set if

$$C = \bigcap_{i=1}^{k} \{ x \in E : \langle n_i, x \rangle \leq c_i \}$$

(2.1)

for some values $c_i \in \mathbb{R}$ and some nonzero vectors $n_i \in E, i \in \{1, k\}$.

**Definition 2.6.** (see [39], Definition 12.1) Let $C \subset E$ be a polyhedral convex set of the form (2.1) and $x \in C$. We call the index set

$$\mathcal{A}_C(x) := \{ i \in \{1, k\} : \langle n_i, x \rangle = c_i \} \subset \{1, k\}$$

the active set of $x$ in $C$ and the constraints $\langle n_i, x \rangle = c_i$ with $i \in \mathcal{A}_C(x)$ active constraints of $x$.

**Proposition 2.7.** (see [42], Corollary 23.8.1)

Let $C_1, \ldots, C_m$ be convex sets in a finite-dimensional inner product space $E$, whose relative interiors have a point in common. Then

$$N_{\bigcap_{i=1}^{m} C_i}(x) = N_{C_1}(x) + \cdots + N_{C_m}(x).$$

(2.2)

If some of the sets, say $C_1, \ldots, C_p$, are polyhedral, formula (2.2) holds if merely the sets $C_1, \ldots, C_p$, $\text{ri}C_{p+1}, \ldots, \text{ri}C_m$ have a point in common.

**Corollary 2.8.** (see [28], p. 67) Let $C \subset E$ be a polyhedral convex set of the form (2.1) and $x \in C$. Then

$$N_C(x) = \left\{ \sum_{i \in \mathcal{A}_C(x)} \lambda_i n_i : \lambda_i \geq 0, \ i \in \mathcal{A}_C(x) \right\} = \text{cone} \{ n_i : i \in \mathcal{A}_C(x) \},$$

(3.3)

$$T_C(x) = \{ y \in E : \langle n_i, y \rangle \leq 0, \ i \in \mathcal{A}_C(x) \}.$$ 

(2.4)

We will also use the following fact:

**Corollary 2.9.** Let $C \subset E$ be a polyhedral convex set of the form (2.1) and let $B \subset C$ be nonempty. Then for all $x \in \text{ri} B$ the sets $N_C(x)$ and $T_C(x)$ do not depend on $x$ and we will use the following notation throughout the
Corollary 2.9 follows directly from Corollary 2.8 and Lemma A.1 (the latter is proved in Appendix A).

Proposition 2.10. Let $E$ be a finite-dimensional inner product space, $C \subset E$ be a nonempty convex polyhedral set, $x \in E$. Then $\text{proj}(x, C)$ is directionally differentiable at $x$, and for any $\xi \in E$

$$\frac{d}{ds} \bigg|_{s=0} \text{proj}(x + s\xi, C) = \text{proj}(\xi, C_x),$$

where

$$C_x := \{ h \in T_C(\text{proj}(x, C)) : \langle x - \text{proj}(x, C), h \rangle = 0 \}$$

is the so-called critical cone. In particular, if $x \in C$, then $C_x = T_C(x)$.

Proposition 2.10 is a special case of Theorem 3.1 from [45] for polyhedral $C$. Indeed, it follows from the statement of the Theorem and the facts that “polyhedral sets are second-order regular” and that “if the set is polyhedral, then the sigma-term vanishes” stated in [45].

Corollary 2.11. Let $y, c : [a, b] \rightarrow E$ be Lipschitz-continuous functions and $C$ be a fixed polyhedral convex set. Let

$$z(t) := \text{proj}(y(t), C + c(t)).$$

Then $z$ is Lipschitz-continuous and a.e. on $[a, b]$

$$\langle \dot{z} - \dot{z}, y - z \rangle = 0.$$  \hspace{1cm} (2.5)

Proof. Indeed, let

$$u := z - c$$

and observe that

$$u(t) = \text{proj}(y(t), C + c(t)) - c(t) = \text{proj}(y(t) - c(t), C).$$

From Proposition 2.10 we have that for any $\xi \in E$

$$\left\langle (y - c) - (z - c), \frac{d}{ds} \bigg|_{s=0} \text{proj}((y - c) + s\xi, C) \right\rangle = 0.$$
In particular, for direction $\xi = \dot{y} - \dot{c}$ we have

$$\langle y - z, \frac{d}{ds}\big|_{s=0} \proj((y - c) + s(\dot{y} - \dot{c}), \mathcal{C}) \rangle = 0. \quad (2.6)$$

Notice that $u$ is Lipschitz-continuous and from Lemma A.6 (with $f = \proj(\cdot, \mathcal{C})$ and $g = y - c$) we know that

$$\dot{u}(t) = \frac{d}{dt} \proj(y(t) - c(t), \mathcal{C}) = \frac{d}{ds}\big|_{s=0} \proj((y(t) - c(t)) + s(\dot{y}(t) - \dot{c}(t)), \mathcal{C}).$$

Thus from (2.6) we have $\langle y - z, \dot{u} \rangle = 0$, which is equivalent to (2.5).

\section{Moreau’s sweeping process}

\begin{definition} \text{(see [35], Section 5.f, [32], p. 9)} \end{definition}

Let $C : [0, T] \rightarrow H$ be a set-valued function such that its values $C(t)$ are closed convex nonempty subsets of $H$ for every $t$. Given a $y_0 \in C(0)$, we call the initial value problem (1.1)-(1.2) a (Moreau’s) sweeping process. Moreover, we say that a function $y : [0, T] \rightarrow H$ is a solution of the sweeping process (1.1)-(1.2) if it satisfies the following conditions:

\begin{enumerate}
  \item $y(0) = y_0$,
  \item $y(t) \in C(t)$ for all $t \geq 0$,
  \item $\frac{d}{dt}y(t)$ exists and $-\frac{d}{dt}y(t) \in N_{C(t)}(y(t))$ for almost all $t \geq 0$.
\end{enumerate}

The following statement provides the existence of a solution to a sweeping process (the uniqueness is stated below in Cor. 2.15).

\begin{theorem} \text{(see [32], Thm. 2)} \end{theorem}

Let $C : [0, T] \rightarrow H$ be a set-valued function such that its values $C(t)$ are closed convex nonempty subsets of $H$ for every $t$. If $C$ is Lipschitz-continuous with a constant $L$ with respect to the Hausdorff distance, then for any $y_0 \in C(0)$ there is a solution $y$ of (1.1) and $y$ is Lipschitz-continuous with the same constant $L$ (which implies that $y \in W^{1,1}(0, T, E)$).

\begin{remark} \text{If $C(t)$ is defined for $t \in [0, +\infty)$, then a solution of (1.1) can be defined on the semi-axis $t \in [0, +\infty)$ using the step-method, which involves determining a solution $y$ on an interval $[N - 1, N]$ and using $y(N)$ as the initial condition for the next interval $[N, N + 1]$.} \end{remark}

\begin{corollary} \text{(see [35], p. 285, [32], Thm. 3)} \end{corollary}

Let $y_{0,1}, y_{0,2} \in C(0)$ be two initial conditions of (1.1) and $y_1, y_2$ be the corresponding solutions. Then

$$\|y_1(t_1) - y_2(t_1)\| \geq \|y_1(t_2) - y_2(t_2)\| \quad \text{for all} \quad 0 \leq t_1 \leq t_2,$$

which implies that:

\begin{enumerate}
  \item For each initial condition the solution of (1.1)-(1.2) is unique.
  \item Each solution of (1.1) is Lyapunov stable.
  \item The solution of problem (1.1)-(1.2) continuously depends on the initial condition.
\end{enumerate}

Due to the existence and uniqueness theorems, in what follows we can say “a solution of (1.1)” meaning “a solution of (1.1)-(1.2) for some $y_0 \in C(0)$.”
2.4. Existence and general properties of the attractor of a sweeping process with a periodic input

From now on we consider sweeping processes in a finite-dimensional space, i.e. a $d$-dimensional vector space $E$ with a real-valued inner product $\langle \cdot, \cdot \rangle$.

As we discuss below, it is possible to show stronger stability properties than Lyapunov stability (Cor. 2.15) if a sweeping process satisfies the following assumptions:

i) The input multifunction $C(t)$ is $T$-periodic, i.e.

$$C(t + T) = C(t), \quad t \geq 0.$$ 

Then we call the respective sweeping process $T$-periodic.

ii) The set $C(t)$ is an intersection of translationally moving closed convex sets, i.e. it is of the form (1.3) for some fixed closed convex nonempty sets $C_j$ and Lipschitz-continuous single-valued functions $c_j$, $j \in \{1, \ldots, k\}$.

**Definition 2.16.** A set-valued function $t \mapsto Y(t)$ is called a global attractor of sweeping process (1.1) if

$$\mbox{dist}(y(t), Y(t)) \to 0 \quad \text{as} \quad t \to \infty$$

for any solution $y$ of (1.1).

The next statement (proved in [21], Section 4.1) plays an important role for the rest of the paper. The original idea of the proof of convergence to a periodic solution is due to Krejčí ([31], Theorem 3.14), who considered the case where $k = 1$ in (1.3).

**Theorem 2.17. (Periodic attractor theorem)** Consider a sweeping process (1.1) defined on $[0, +\infty)$ in a $d$-dimensional inner product space $E$, with the map $t \mapsto C(t)$ being $T$-periodic, totally bounded and Lipschitz-continuous with respect to the Hausdorff distance. Denote by $\tilde{X}$ the family of $T$-periodic solutions of (1.1) and by $X(t)$ the set of their values at $t$, i.e.:

$$X(t) := \{ x(t) : x \in \tilde{X} \} \subset C(t) \subset E,$$

$$\tilde{X} := \{ x \in W^{1,1}_{\text{loc}}([0, +\infty), E) \mbox{ is a solution of (1.1) such that } x(0) = x(T) \}.$$

Then, $X(t)$ is closed convex and nonempty for all $t \in [0, +\infty)$ and for all $x, y \in \tilde{X}$

$$\langle \dot{x}(t), x(t) - y(t) \rangle = \langle \dot{y}(t), x(t) - y(t) \rangle = 0, \quad \text{for a.a. } t \in [0, \infty).$$

Moreover, assume that $C(t)$ is an intersection of closed convex sets $C_j$ (some of them, say, the first $p$ sets, can be polyhedral) that undergo just translational motions, i.e. it is of the form (1.3) where $c_j(t)$ are single-valued $T$-periodic Lipschitz-continuous functions such that

$$\bigcap_{j=1}^p (C_j + c_j(t)) \cap \bigcap_{j=p+1}^k (\text{ri}(C_j) + c_j(t)) \neq \emptyset, \quad t \geq 0. \quad (2.7)$$

Then, for all $x, y \in \tilde{X}$

$$\dot{x}(t) = \dot{y}(t) \quad \text{for a.e. } t \in [0, \infty), \quad (2.8)$$

and for any solution of (1.1), call it $y_1$, there exists a solution $x_1 \in \tilde{X}$ such that

$$\|y_1(t) - x_1(t)\| \to 0 \quad \text{as} \quad t \to \infty, \quad (2.9)$$

and, therefore, $t \mapsto X(t)$ is a global attractor of (1.1).
We notice that (2.8) implies that each set $X(t)$, $t \geq 0$, is just a displacement of the set $X(0)$ by a vector. Specifically, the following statement holds.

**Corollary 2.18.** (see [21], Corollary 4.7) Under the conditions of Theorem 2.17 including (1.3)-(2.7), there exists a Lipschitz-continuous $T$-periodic function $\bar{x} : \mathbb{R} \to E$ such that for any $x \in \hat{X}$

$$x(t) = x(0) + \bar{x}(t), \quad t \geq 0.$$  

In particular,

$$X(t) = X(0) + \bar{x}(t), \quad t \geq 0,$$

and

$$x(t) \in \text{ri}(X(t)) \quad \text{if and only if} \quad x(0) \in \text{ri}(X(0)).$$

Equivalently, one can fix a particular periodic solution $\chi$ of (1.1) and then find a closed convex set $\mathcal{X}$ s.t. $0 \in \mathcal{X}$ and

$$X(t) = \mathcal{X} + \chi(t), \quad t \geq 0. \quad (2.10)$$

### 2.5. A polyhedral sweeping process with fixed normal vectors

We continue reviewing the properties of the attractor $X(t)$ for the sweeping of a particular type.

**Definition 2.19.** We call the sweeping process (1.1) polyhedral if $C(t)$ can be expressed as

$$C(t) = \bigcap_{i=1}^{k} \{ x \in E : \langle n_i, x \rangle \leq c_i(t) \} \quad (2.11)$$

with some fixed nonzero vectors $n_i \in E$ and Lipschitz-continuous functions $c_i : [0, +\infty) \to \mathbb{R}$. We do not consider time-dependent normal vectors in this paper.

**Remark 2.20.** Given a nonempty subset of indices $\mathcal{I} \subset \overline{1,k}$, ordered according to the natural order of $\mathbb{N}$, define a bounded linear operator $B_{\mathcal{I}} : E \to \mathbb{R}^{|\mathcal{I}|}$ and a Lipschitz-continuous function $c_{\mathcal{I}} : [0, +\infty) \to \mathbb{R}^{|\mathcal{I}|}$ componentwise by

$$(B_{\mathcal{I}} x)_{i \in \mathcal{I}} = (\langle n_i, x \rangle)_{i \in \mathcal{I}}, \quad (c_{\mathcal{I}}(t))_{i \in \mathcal{I}} = (c_i(t))_{i \in \mathcal{I}},$$

where the dimensions of $\mathbb{R}^{|\mathcal{I}|}$ are indexed by $\mathcal{I}$. Then we can represent $C(t)$ as

$$C(t) = \{ x \in E : B_{\overline{1,k}} x \leq c_{\overline{1,k}}(t) \}.$$

Furthermore, recall that a vertex of a polytope is determined by some $d$ constraints from (2.11) with linearly independent $n_i$, satisfied as equalities (see [43], Section 8.5). Therefore each vertex of $C(t)$ for a fixed $t$ can be expressed as

$$v_{\mathcal{I}}(t) := B_{\mathcal{I}}^{-1} c_{\mathcal{I}}(t)$$
for some $\mathcal{I}$ such that $|\mathcal{I}| = d$ and $\{n_i : i \in \mathcal{I}\}$ is a linearly independent family (i.e. $B_{\mathcal{I}}^{-1}$ exists).

Similarly, for any index set $\mathcal{I} \subset \overline{1,k}$ we can define the adjoint of $B_\mathcal{I}$, a bounded linear map

$$B_{\mathcal{I}}^* : \mathbb{R}^{\mathcal{I}} \rightarrow \text{lin}\{n_i : i \in \mathcal{I}\} \subset E,$$

where, again, the dimensions of $\mathbb{R}^{\mathcal{I}}$ are indexed by $\mathcal{I}$. If vectors $n_i : i \in \mathcal{I}$ are linearly independent, then the map $B_{\mathcal{I}}^*$ is invertible. Thus due to (2.3) we can represent the normal cone to (2.11) as

$$N_{C(t)}(x) = \left\{ (B_{\mathcal{A}_{C(t)}(x)}^*) \lambda : \lambda \in \mathbb{R}^{\mathcal{A}_{C(t)}(x)}, \lambda \geq 0 \right\},$$

where $\mathcal{A}_{C(t)}(x)$ is the active set of $C(t)$ at $x$, see Definition 2.6.

Using the notation of Remark 2.20 we can deduce the existence of solutions for a sweeping process with a polyhedral moving set (2.11) when all $c_i(t)$ are Lipschitz-continuous. Indeed, the existence follows from Theorem 2.13 and Lemma A.3. The following lemma provides an important characterization of trajectories for a polyhedral sweeping process.

**Lemma 2.21. (Complementarity lemma)** Consider a sweeping process (1.1) defined on $[0,T]$ in a $d$-dimensional inner product space $E$, where the map $t \mapsto C(t)$ is given by (2.11) with some fixed nonzero vectors $n_i \in E, i \in \overline{1,k}$ and Lipschitz-continuous functions $c_i : [0,T] \rightarrow \mathbb{R}$. Then for a solution $y$ there is a function $\lambda \in L^1([0,T], \mathbb{R}^k)$ such that for all $i \in \overline{1,k}$ and a.a. $t \in [0,T]$

$$\lambda_i(t) \geq 0,$$

$$c_i(t) - \langle n_i, y(t) \rangle > 0 \implies \lambda_i(t) = 0,$$

and

$$-\dot{y}(t) = B_{\mathcal{A}_{C(t)}}^* \lambda(t) = \sum_{i=1}^k \lambda_i(t) n_i.$$
on the set $\mathcal{T}_i$ (cf. (A.11) on the set $\mathcal{T}$). Lemma A.7 guarantees the existence of $l$ non-negative functions $\lambda^i \in L^1([0,T], \mathbb{R}^{|A_i|})$, each having the zero value outside of the corresponding set $\mathcal{T}_i$. Finally, by setting

$$\lambda(t) := \sum_{i=1}^{l} E_{A_i} \lambda^i(t),$$

where $E_{A_i}$ is the embedding $E_{A_i} : \mathbb{R}^{|A_i|} \to \mathbb{R}^k$ according to the index set $A_i$, we obtain a function $\lambda$ with the required properties. □

### 2.6. Attractor of a polyhedral sweeping process with a periodic input

In the case of a polyhedral sweeping process one can make an additional observation from Theorem 2.17, that whenever the periodic attractor $X(t)$ moves, it is contained in some face of $C(t)$ and it is being “pushed” by that face in a direction, orthogonal to $\text{aff } X(t)$. Moreover, we can show that the attractor of a polyhedral sweeping process is a polyhedron itself and assuming the knowledge of a single periodic solution, one can provide a simple formula for the whole attractor $X(t)$.

**Theorem 2.22.** (Periodic attractor theorem for a polyhedral case) Assume that a uniformly bounded set-valued function $t \mapsto C(t)$ takes nonempty values and that it is given by (2.11) with some fixed nonzero vectors $n_i \in E$ and $T$-periodic Lipschitz-continuous functions $c_i : [0, +\infty) \to \mathbb{R}$. Then the conclusions of the Periodic attractor theorem (Thm. 2.17) and Corollary 2.18 hold true. Moreover,

1) For every $t \in [0, T]$ there is an index set $P(t) \subset \mathbb{R}^k$ such that

$$P(t) \subset A_{C(t)}(x(t)) \quad \text{for any } x \in \hat{X},$$

$$P(t) = A_{C(t)}(x(t)) \quad \text{if } x(\tau) \in \text{ri } X(\tau) \text{ for some (hence all) } \tau \in [0, +\infty).$$

If $X = \{0\}$, i.e. $X(t)$ is a singleton and $\chi$ in (2.10) is the only periodic solution, then we set $P(t) = A_{C(t)}(\chi(t))$.

2) For every $t \in [0, T]$, $i \in P(t)$,

$$n_i \perp \text{aff } X(t).$$

3) Let

$$\hat{P} := \bigcup_{t \in [0, T]} P(t) \subset \mathbb{R}^k.$$ 

There is a $\lambda \in L^1([0, T], \mathbb{R}^{|\hat{P}|})$ such that for a.a. $t \in [0, T]$,

$$\lambda_i(t) \geq 0,$$

$$i \notin P(t) \implies \lambda_i(t) = 0,$$

and the function $\chi$ defined in Corollary 2.18 satisfies

$$-\dot{\chi}(t) = \sum_{i \in P(t)} \lambda_i(t)n_i.$$ (2.15)
While some of the periodic solutions can possess active constraints not from $P(t)$, as it is evident from (2.12)-(2.13), they do not contribute to the velocity. Notice that the velocity vector $\dot{\chi}(t)$, the set $P(t)$ and the multipliers $\lambda(t)$ are common for all periodic solutions $x \in \hat{X}$.

iv) There are $\mu_i \in \mathbb{R}, i \in \overline{1,k}$ given by

$$
\mu_i := \min\{c_i(t) - \langle n_i, \chi(t) \rangle : t \in [0,T]\} \geq 0, 
$$

(2.16)
such that $X$ from (2.10) can be represented as

$$
X = \bigcap_{i=1}^{k} \{x \in E : \langle n_i, x \rangle \leq \mu_i\}. 
$$

(2.17)

Proof.

i) Formula (2.13) follows from Lemma A.1 applied to $X(t)$ as $C$; then, (2.12) follows by continuity argument.

ii) follows directly from i) and the definition of the affine hull.

iii) Consider the case when $X(t)$ is not a singleton. Since this set is convex, its relative interior is nonempty. 

iii) is the statement of Complementarity Lemma 2.21 applied to a solution satisfying (2.13). The case when $X(t)$ is a singleton is simply the statement of Lemma 2.21.

iv) Let

$$
X' := \bigcap_{i=1}^{k} \{x \in E : \langle n_i, x \rangle \leq \mu_i\}. 
$$

First, let us show that $X \subset X'$. Indeed, for any $t \in [0,T]$ and any $x \in X$,

$$
x + \chi(t) \in X + \chi(t) \subset C(t),
$$

i.e. $\langle n_i, x + \chi(t) \rangle \leq c_i(t)$ for any $i \in \overline{1,k}$, hence

$$
\langle n_i, x \rangle \leq c_i(t) - \langle n_i, \chi(t) \rangle, \quad i \in \overline{1,k}.
$$

Therefore $\langle n_i, x \rangle \leq \mu_i$ and $x \in X'$.

Now let us show that $X' \subset X$. Pick an arbitrary point $x \in X'$ and let the function $y : [0, +\infty) \to E$ be given by

$$
y : t \mapsto x + \chi(t).
$$

The function $y$ is periodic and Lipschitz-continuous because $\chi$ is. To finish the proof it is enough to show that $y$ is a solution of (1.1) on $[0,T]$ as this would mean that

$$
x + \chi(t) = y(t) \in X(t) = X + \chi(t). 
$$

(2.18)

For any $i \in \overline{1,k}$ and any $t \in [0,T]$, from

$$
\langle n_i, y(t) \rangle = \langle n_i, x \rangle + \langle n_i, \chi(t) \rangle \leq \mu_i + \langle n_i, \chi(t) \rangle \leq c_i(t) - \langle n_i, \chi(t) \rangle + \langle n_i, \chi(t) \rangle = c_i(t)
$$

it follows that $y(t) \in C(t)$.

According to Definition 2.12, it remains to show that the inclusion (1.1) holds for a.a. $t \in [0,T]$. 
In the trivial case when \( \dot{\chi} \equiv 0 \) for a.a. \( t \in [0, T] \) we have \(-\dot{y} = \dot{\chi} = 0 \in N_{C(t)}(y(t))\) and \( y \) is a solution. From now, we assume that \( y \) is not a constant function, therefore the set of strongly active somewhere constraints

\[
\tilde{P} := \left\{ i \in \tilde{P} : \int_{0}^{T} \lambda_i(t)dt > 0 \right\}
\]

is nonempty. To verify (1.1) it is sufficient to show for all \( t \in [0, T] \) that

\[
\hat{P} \cap \mathcal{P}(t) \subset \mathcal{A}_{C(t)}(y(t)), \tag{2.19}
\]

as then we can use expansion (2.15):

\[
-\dot{y}(t) = -\dot{\chi}(t) = \sum_{i \in \tilde{P} \cap \mathcal{P}(t)} \lambda_i(t)n_i \in \text{cone} \{ n_i : i \in \tilde{P} \cap \mathcal{P}(t) \} \subset \text{cone} \{ n_i : i \in \mathcal{A}_{C(t)}(y(t)) \} = N_{C(t)}(y(t)), \tag{2.20}
\]

as \( \lambda_i \)'s for \( i \in \tilde{P} \setminus \tilde{P} \) vanish a.e. and can be omitted.

Assume the contrary to (2.19), that there is a \( t^* \in [0, T] \) and \( i^* \in \tilde{P} \) such that \( i^* \in \mathcal{P}(t^*) \) but \( i^* \notin \mathcal{A}_{C(t^*)}(y(t^*)) \). The latter means that

\[
\langle n_{i^*}, y(t^*) \rangle < c_{i^*}(t^*)
\]

(the opposite strict inequality cannot hold since we already established that \( y(t) \in C(t) \)). Hence, by the construction of \( y \), we have

\[
\langle n_{i^*}, x \rangle + \langle n_{i^*}, \chi(t^*) \rangle < c_{i^*}(t^*)
\]

But due to \( i^* \in \mathcal{P}(t^*) \) and (2.12),

\[
\langle n_{i^*}, \chi(t^*) \rangle = c_{i^*}(t^*)
\]

thus we actually have

\[
\langle n_{i^*}, x \rangle < 0. \tag{2.21}
\]

Since \( \chi(t) \) is a periodic solution, (2.15) implies that

\[
0 = -(\chi(T) - \chi(0)) = -\int_{0}^{T} \dot{\chi}(t)dt = \int_{0}^{T} \sum_{i \in \mathcal{P}(t)} \lambda_i(t)n_i dt = \sum_{i \in \tilde{P}} \left( \int_{0}^{T} \lambda_i(t)dt \right) n_i,
\]

and we can express \( n_{i^*} \) as

\[
\left( \int_{0}^{T} \lambda_{i^*}(t)dt \right) n_{i^*} = -\sum_{i \in \tilde{P} \setminus \{ i^* \}} \left( \int_{0}^{T} \lambda_i(t)dt \right) n_i.
\]
where all the integrals are positive numbers. Therefore it follows from (2.21) that

\[- \left\langle \sum_{i \in \tilde{\mathcal{P}} \setminus \{i^*\}} \left( \int_0^T \lambda_i(t) \, dt \right) n_i, x \right\rangle = \left( \int_0^T \lambda_i^*(t) \, dt \right) \langle n_i^*, x \rangle < 0,\]

hence

\[\sum_{i \in \tilde{\mathcal{P}} \setminus \{i^*\}} \left( \int_0^T \lambda_i(t) \, dt \right) \langle n_i, x \rangle > 0,\]

and therefore there is at least one $i \in \tilde{\mathcal{P}} \setminus \{i^*\}$ such that $\langle n_i, x \rangle > 0$. Since $x \in \mathcal{X}'$, the definition of $\mathcal{X}'$ implies

\[0 < \langle n_i, x \rangle \leq \mu_i \leq c_i(t) - \langle n_i, \chi(t) \rangle \quad \text{for all} \quad t \in [0, T],\]

thus $i \notin \mathcal{A}_{\mathcal{C}(t)}(\chi(t))$ and by (2.12) we conclude that $i \notin \mathcal{P}(t)$ for all $t \in [0, T]$. This contradicts the way we chose $i \in \tilde{\mathcal{P}} \subset \hat{\mathcal{P}}$ and the definition (2.14) of $\hat{\mathcal{P}}$. The contradiction confirms that (2.19) and, therefore, (2.20) hold true. We have shown that $y(t) = x + \chi(t)$ is a periodic solution, therefore $x \in \mathcal{X}$ by (2.18) and $\mathcal{X}' \subset \mathcal{X}$. □

**Remark 2.23.** Under the additional regularity assumption of the local permanence of the active faces, a polyhedral sweeping process (1.1), (2.11) has a better convergence property, compared to (2.9), namely

\[\lim_{q \to \infty} \|y_1(\cdot + qT) - x_1(t)\|_{H^1([0,T],E)} = 0\]

for any solution $y_1$ and its limiting periodic trajectory $x_1 \in \mathcal{X}$. Notice that this includes the $L^2$-convergence of the respective derivatives. This improvement is due to [14], see Theorem 3 and Corollary 9 specifically.

**Remark 2.24.** In order to characterize the asymptotic behavior of the sweeping process (1.1) in the context of the periodic attractor theorem for a polyhedral case (Thm. 2.22), we denote by $L$ the affine hull of the periodic attractor $\mathcal{X}(t)$:

\[L(t) = \text{aff } \mathcal{X}(t) = \mathcal{L} + \chi(t),\]

where

\[\mathcal{L} = \text{lin } \mathcal{X}\]

is a linear subspace of $E$. Then $\mathcal{L}^\perp$ is a linear subspace of $E$ of dimension $d - \dim \mathcal{X}$,

\[\dot{x} \in \mathcal{L}^\perp\]

and

\[n_i \in \mathcal{L}^\perp \quad \text{for all } i \in \hat{\mathcal{P}}.\]
The following Corollary will be useful for further considerations.

**Corollary 2.25.** In the setting of Theorem 2.22, if for some \( i \in \overline{1,k} \) and \( t_0 \in [0,T] \) the \( i \)-th constraint is active at some \( x_0 + \chi(t_0), x_0 \in \mathcal{X} \), i.e.
\[
\langle n_i, x_0 + \chi(t_0) \rangle = c_i(t_0),
\]
then the quantity (2.16) satisfies
\[
\mu_i = \langle n_i, x_0 \rangle = c_i(t_0) - \langle n_i, \chi(t_0) \rangle = \max \{ \langle n_i, x'_0 \rangle : x'_0 \in \mathcal{X} \}.
\]

**Proof.** Indeed, by (2.16) we have
\[
c_i(t_0) - \langle n_i, \chi(t_0) \rangle \geq \mu_i.
\]
On the other hand, by definition (2.17) of \( \mathcal{X} \) the inclusion \( x_0 \in \mathcal{X} \) combined with (2.23) implies
\[
c_i(t_0) - \langle n_i, \chi(t_0) \rangle = \langle n_i, x_0 \rangle \leq \mu_i,
\]
hence \( \mu_i = \langle n_i, x_0 \rangle \). Moreover, for any \( x'_0 \in \mathcal{X} \) one has \( \langle n_i, x'_0 \rangle \leq \mu_i \), from which the conclusion on the maximum follows. \( \square \)

**Corollary 2.26.** In the setting of Theorem 2.22, for every \( t \in [0,T] \)
\[
i \in \mathcal{P}(t) \implies \mu_i = 0.
\]

**Proof.** Indeed, let \( i \in \mathcal{P}(t) \), i.e. the \( i \)-th constraint is active everywhere on \( X(t) \), including \( 0 + \chi(t) \). Therefore, by Corollary 2.25 we have \( \mu_i = \langle n_i, 0 \rangle = 0 \).

The reverse implication to (2.24) is not true as we can always redenote \( \chi \) to be a periodic solution lying on any part of the boundary of \( X(t) \).

3. Motivating examples

We provide two visualizable examples, which motivate our further studies.

### 3.1. One-dimensional attractor

Let us consider a polyhedral sweeping process with a periodic input and its attractor shown in Figure 1. This sweeping process can be derived from the rheological system shown in Figure 2 using e.g. the method of [21–23]; a detailed study of a similar system can be found in [24]. More generally, such rheological systems consist of elastic-perfectly plastic springs connected according to a given graph; further, the system is subject to time-dependent displacement constraints and stress loads. In particular, the sweeping process shown on Figure 2 is produced in the three-dimensional subspace of balanced self-stresses. This subspace is embedded in \( \mathbb{R}^5 \), which represents the values of stress in each spring. Assuming for simplicity that all elastic stiffnesses (Hooke’s coefficients) of the springs are identical (we take the stiffness value equal to 1), there is a convenient orthogonal basis \( \left( \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right) \) for the parameterization of the space of self-stresses as \( \mathbb{R}^3 \), where
\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \end{pmatrix}^T, \\
v_2 &= \begin{pmatrix} -1 & 1 & 0 & -1 & 1 \end{pmatrix}^T.
\end{align*}
\]
**Figure 1.** An example of a sweeping process with a one-dimensional attractor. a) The moving set $C(t)$ (blue) in two extreme positions and the trajectory of the attractor $X(t)$ (magenta). b) The view orthogonal to the direction of motion of $C(t)$.

**Figure 2.** A rheological system in one spatial dimension. The system consists of 5 elasto-perfectly plastic springs driven by a time-dependent displacement constraint (red).

$$v_3 = \begin{pmatrix} 1 & -1 & 2 & -1 & 1 \end{pmatrix}^T.$$  

If the rheological system of Figure 2 is driven by the time-dependent length constraint (the red bar on the figure) with stress load (external forces) constantly set to zero, then the moving set of the corresponding sweeping process in the space of self-stresses has the form

$$C(t) = C + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} g(t),$$

where $g(t)$ is a scalar quantity changing proportionally to the enforced displacement constraint, and $C$ is the fixed (due to fixed values of external forces) polyhedron

$$C = \bigcap_{i=1}^{5} \{ x \in \mathbb{R}^3 : \langle n_i, x \rangle \leq c_i^+ \} \cap \bigcap_{i=6}^{10} \{ x \in \mathbb{R}^3 : \langle n_i, x \rangle \leq -c_{i-5}^- \}$$

with the normal vectors

$$n_1 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix}, \quad n_2 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix}, \quad n_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \sqrt{2} \end{pmatrix}, \quad n_4 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ -1 \sqrt{2} \end{pmatrix}, \quad n_5 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ \sqrt{2} \end{pmatrix},$$

$$n_i = -n_{i-5} \quad \text{for } i \in \overline{6, 10}.$$
The moving set \( C(t) \) (blue) of the sweeping process from Figure 1, which corresponds to the rheological model from Figure 2. The one-dimensional attractor, its negative velocity \( -\dot{\chi}(t) \) and the affine hull of the attractor \( L(t) \) (dashed line) are shown in magenta. Each pair of parallel facets with normal vectors \( n_i, n_{i+5} \) corresponds to the upper and lower elastic boundaries of the \( i \)-th spring in Figure 2.

The values of \( c^+_i \) and \( c^-_i \) have the physical meaning of upper and lower elastic stress bounds for the \( i \)-th spring, upon reaching which the spring exhibits perfectly plastic behavior. In this example, we set
\[
c^+ = (0.5 \ 1 \ 1.6 \ 1.5 \ 1.2)^T, \quad c^- = -c^+.
\]

We should highlight, that if non-constant external force is applied at the nodes of a rheological system, then the moving set will no longer be of the form (3.1) as it will have time-dependent shape, and the general form (2.11) will be necessary to use.

Let us notice that for the vector \((1, 0, 0)^T\), which defines the direction of motion of \( C(t) \) according to (3.1), we have
\[
(1 \ 0 \ 0)^T \in \text{cone}(n_1, n_2), \quad -(1 \ 0 \ 0)^T \in \text{cone}(n_6, n_7),
\]
which enables the existence of the nontrivial (one-dimensional) attractor \( X(t) \). The trajectory shown in Figure 1 is produced using a periodic piecewise-monotone function \( g(t) \) in (3.1) with two monotone segments per period; the amplitude of \( g(t) \) is sufficiently large to eliminate stationary solutions.

Using the notation of the periodic attractor theorem for a polyhedral case (Thm. 2.22), the set \( \mathcal{P}(t) \) at different times is either empty (when \( X(t) \) is in the interior of \( C(t) \)), or \( \mathcal{P}(t) = \{1\} \) (when \( X(t) \) slides on the corresponding facet towards an edge), or \( \mathcal{P}(t) = \{1, 2\} \) (when \( X(t) \) is carried by the edge); similarly \( \mathcal{P}(t) = \{6\} \) or \( \mathcal{P}(t) = \{6, 7\} \) during the backward motion. Thus \( \hat{\mathcal{P}} = \{1, 2, 6, 7\} \). Figure 3 presents the normal vectors, the affine hull \( L(t) \) of the attractor as defined in Remark 2.24, and the relation \( \dot{\chi} \in \mathcal{L}^\perp \).

### 3.2. Two-dimensional attractor

Another polyhedral sweeping process with periodic input resulting in a two-dimensional attractor is shown in Figure 4. Its source is the rheological system in Figure 2. As now we have seven springs, the subspace of balanced self-stresses is embedded in \( \mathbb{R}^7 \), but for this particular system this subspace is also three-dimensional. The orthogonal triplet \( \left( \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \frac{v_3}{\|v_3\|} \right) \) in \( \mathbb{R}^7 \) can be used to parametrize the space of self-stresses as \( \mathbb{R}^3 \) assuming that all the stiffnesses are equal (again, we take all seven Hooke’s coefficients equal to 1), where
\[
v_1 = (1 \ 1 \ 3 \ 1 \ 2 \ 2 \ 2)^T,
\]
Figure 4. Example of a sweeping process with a two-dimensional attractor. a) The moving set $C(t)$ (blue) and the attractor $X(t)$ (magenta) in two extreme positions. b) The view orthogonal to the direction of motion of $C(t)$.

Figure 5. A rheological system with one spatial dimension composed of 7 elastic-perfectly plastic springs and driven by a time-dependent displacement constraint (the total length of the system) shown in red. All the springs are elongated along the same straight line.

\[
v_2 = (-1 -1 0 0 0 1 0)^T, \quad v_3 = (0 0 0 -1 -1 0 1)^T.
\]

Again, under the time-dependent displacement constraint one obtains the sweeping process of the form (3.1) with

\[
C = \bigcap_{i=1}^{7} \{ x \in \mathbb{R}^3 : \langle n_i, x \rangle \leq c_i^+ \} \cap \bigcap_{i=8}^{14} \{ x \in \mathbb{R}^3 : \langle n_i, x \rangle \leq -c_i^- \},
\]

\[
n_1 = n_2 = \begin{pmatrix} \frac{1}{\sqrt{21}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{pmatrix}, \quad n_3 = \begin{pmatrix} \frac{1}{\sqrt{21}} \\ 0 \\ 0 \end{pmatrix}, \quad n_4 = n_5 = \begin{pmatrix} \frac{1}{\sqrt{21}} \\ 0 \\ -\frac{1}{\sqrt{3}} \end{pmatrix}, \quad n_6 = \begin{pmatrix} \frac{2}{\sqrt{21}} \\ 0 \\ 0 \end{pmatrix}, \quad n_7 = \begin{pmatrix} \frac{2}{\sqrt{21}} \\ 0 \\ \frac{1}{\sqrt{3}} \end{pmatrix},
\]

\[
n_i = -n_{i-7} \quad \text{for } i \in \overline{8,14},
\]

and elastic boundaries

\[
c_i^+ = 1, \quad c_i^- = -1, \quad i \in \overline{1,7}.
\]

A non-trivial (two-dimensional) attractor may exist because the vector $(1, 0, 0)^T$ from (3.1) is parallel to normal vectors $n_3$ and $n_{10}$. The trajectory shown in Figure 4 is produced when $g(t)$ in (3.1) is a periodic piecewise-monotone function with two monotone segments per period and the amplitude of $g(t)$ is large enough to eliminate stationary solutions.
Figure 6. The moving set $C(t)$ (blue) of the sweeping process from Figure 4, which corresponds to the rheological model from Figure 5. The attractor and its negative velocity $-\dot{\chi}(t)$ and the plane of the attractor $L(t)$ (dashed plane) are shown in magenta. Each pair of parallel facets with normal vectors $n_i, n_{i+7}$ corresponds to the upper and lower elastic boundaries of the $i$–th spring on Figure 5.

The set $\mathcal{P}(t)$ (see Thm. 2.22) is empty when $X(t)$ is in the interior of $C(t)$, while $\mathcal{P}(t) = \{3\}$ and $\mathcal{P}(t) = \{10\}$ at different times of the backward motion. Thus $\mathcal{P} = \{3, 10\}$. Figure 6 shows the normal vectors, affine hull $L(t)$ of the attractor as defined in Remark 2.24, and relation $\dot{\chi} \in L^\perp$.

4. Main result: the finite-time stability of the attractor

4.1. Convergence to the affine hull of the attractor in finite time

The approach of Theorem 4.1 below expands on the ideas of [2]; some implications of a simpler but similar fact were studied in [24]. Recall that $L(t)$ is the affine hull of attractor $X(t)$, and $L$ is the linear hull of $X$, connected by formula (2.22).

Theorem 4.1. In the setting of the Periodic attractor theorem for a polyhedral case (Thm. 2.22) we further assume that there exist an interval $[\alpha_0, \beta_0] \subset [0, T]$, a number $\varepsilon > 0$ and index sets $\mathcal{P}_0 \subset \hat{\mathcal{P}}$, $I_0 \subset 1, k \setminus \hat{\mathcal{P}}$ such that:

A. For all $t \in [\alpha_0, \beta_0]$,

$$\mathcal{P}_0 = \mathcal{P}(t), \quad L = \{y \in E : \langle n_i, y \rangle = 0 : i \in \mathcal{P}_0\}.$$  

B. The index set $I_0 \cup \hat{\mathcal{P}}$ lists all constraints which are active somewhere at $X(t)$, and they remain active during the entire interval $(\alpha_0, \beta_0)$:

for each $i \in I_0$ and $t \in (\alpha_0, \beta_0)$ there is an $x_0 \in X$ such that

$$\langle n_i, x_0 + \chi(t) \rangle = c_i(t), \quad (4.1)$$

for all $i \in 1, k \setminus (I_0 \cup \hat{\mathcal{P}})$, $t \in [\alpha_0, \beta_0]$, $x \in X$: $$\langle n_i, x + \chi(t) \rangle < c_i(t). \quad (4.2)$$

We denote

$$Q(t) := L(t) \cap \{y \in E : \langle n_i, y \rangle \leq c_i(t) : i \in I_0\}, \quad (4.3)$$

$$K' := \text{cone}(\{n_i : i \in I_0 \cup \mathcal{P}_0\} \cup \{-n_i : i \in \mathcal{P}_0\}) = \text{cone}(\{n_i : i \in I_0\} \cup L^\perp). \quad (4.4)$$
C. For a.a. \( t \in [\alpha_0, \beta_0] \),
\[
-\dot{\chi} + B \varepsilon [0] \cap K' \subset \bigcup_{\xi \in Q(t) \cap C(t)} N_{C(t)}(\xi);
\] (4.5)

note that \( Q(t) \cap C(t) = L(t) \cap C(t) \).

Then for every solution \( y \) of (1.1) s.t.
\[
dist(y(\alpha_0), X(\alpha_0)) < \delta,
\] (4.6)

where
\[
\delta := \min \left\{ \frac{c_i(t) - \langle n_i, x \rangle}{\| n_i \|} : t \in [\alpha_0, \beta_0], x \in X(t), i \in I_0 \cup \hat{P} \right\} > 0,
\] (4.7)

the distance \( dist(y(t), Q(t)) \) is non-increasing on \([\alpha_0, \beta_0]\) and

a) if \( dist(y(\alpha_0), Q(\alpha_0)) \geq \varepsilon (\beta_0 - \alpha_0) \), then \( dist(y(\beta_0), Q(\beta_0)) \leq dist(y(\alpha_0), Q(\alpha_0)) - \varepsilon (\beta_0 - \alpha_0) \);

b) if \( dist(y(\alpha_0), Q(\alpha_0)) < \varepsilon (\beta_0 - \alpha_0) \), then there is a \( t^* \in (\alpha_0, \beta_0) \) such that \( y(t) \in Q(t) \) for all \( t \in [t^*, \beta_0] \).

Remark 4.2. To illustrate the conditions of Theorem 4.1 we consider the examples of sweeping processes from Section 3. For the first example, during the appropriate time interval we can use \( P_0 = \{1, 2\}, I_0 = \{5\} \). For the second example, \( P_0 = \{3\}, I_0 = \{6, 7\} \). The corresponding set \( Q(t) \) is shown in Figure 7.

Proof. First, notice that by (4.1) and Corollary 2.25 we have
\[
c_i(t) = \mu_i + \langle n_i, \chi(t) \rangle, \quad i \in I_0,
\]

therefore
\[
Q(t) = (L + \chi(t)) \cap \{ y \in E : \langle n_i, y \rangle \leq \mu_i + \langle n_i, \chi(t) \rangle : i \in I_0 \},
\]
i.e. we can write
\[ Q(t) = Q + \chi(t), \] (4.8)
where
\[ Q := \mathcal{L} \cap \{ y \in E : \langle n_i, y \rangle \leq \mu_i : i \in \mathcal{I}_0 \} = \left\{ y \in E : \begin{array}{l} \langle n_i, y \rangle = 0 : i \in \mathcal{P}_0 \\ \langle n_i, y \rangle \leq \mu_i : i \in \mathcal{I}_0 \end{array} \right\}. \]

Notice that \( X \subset Q \subset \mathcal{L} \).

Consider an arbitrary solution \( y \) of (1.1) satisfying (4.6). Define the following functions:
\[ z(t) := \text{proj}(y(t), Q(t)), \]
\[ V(t) := \|y(t) - z(t)\|^2 = \text{dist}^2(y(t), Q(t)). \]

We assume that \( V(t) > 0 \) for \( t \in [\alpha_0, \beta_0] \). The rest of the proof is divided into several steps.

**Step 1.** We show that \( z(t) \in C(t) \) for any \( t \in [\alpha, \beta] \). Indeed, \( y \) satisfies (4.6) and we know that the distance from \( y(t) \) to \( X(t) \) cannot increase with time because \( X(t) \) consists of solutions of (1.1), i.e.
\[ \text{dist}(y(t), X(t)) < \delta \]
for any \( t \geq \alpha_0 \). Now fix \( t \) and let \( x^* \in X(t) \) be such that \( \|y(t) - x^*\| < \delta \). Since the projection is nonexpansive and \( x^* \in X(t) \subset Q(t) \), we obtain
\[ \|x^* - z(t)\| = \|\text{proj}(x^*, Q(t)) - \text{proj}(y(t), Q(t))\| \leq \|x^* - y(t)\| < \delta, \]
therefore from (4.7) we have for \( i \in \overline{E \setminus (\mathcal{I}_0 \cup \hat{\mathcal{P}})} \)
\[ \langle n_i, z(t) \rangle = \langle n_i, z(t) - x^* \rangle + \langle n_i, x^* \rangle < \|n_i\|\delta + \langle n_i, x^* \rangle \leq c_i(t). \]
For \( i \in \mathcal{I}_0 \) the inequalities \( \langle n_i, z(t) \rangle \leq c_i(t) \) are satisfied by definition (4.3) of \( Q(t) \) and for \( i \in \hat{\mathcal{P}} \) we have
\[ \langle n_i, z(t) \rangle = \langle n_i, \chi(t) \rangle \leq c_i(t) \]
because \( z(t) - \chi(t) \in \mathcal{L} \) and \( n_i \in \mathcal{L}^\perp \). Thus (4.9) is proved.

**Step 2.** Since \( \dot{\chi}(t) \perp L(t) \) and \( z(t) \in Q(t) \subset L(t) \) we have
\[ -\dot{\chi} \in N_{Q(t)}(z). \] (4.10)
for a.a. \( t \geq 0 \). Additionally, we have by the definition of \( z \) that
\[ y - z \in N_{Q(t)}(z), \] (4.11)
\[ \varepsilon \frac{y - z}{\|y - z\|} \in N_{Q(t)}(z) \]

for \( t \in [\alpha_0, \beta_0] \). Adding this relation to (4.10), we obtain

\[ -\dot{\chi} + \varepsilon \frac{y - z}{\|y - z\|} \in N_{Q(t)}(z) \subset N_{Q(t) \cap C(t)}(z), \tag{4.12} \]

where the second inclusion follows from (4.9) and Proposition 2.7. Also, by construction of \( Q(t) \),

\[ N_{Q(t)}(z) \subset K', \]

therefore from (4.5) and (4.11) it follows that for some \( \xi \in Q(t) \cap C(t) \),

\[ -\dot{\chi} + \varepsilon \frac{y - z}{\|y - z\|} \in N_{C(t)}(\xi). \tag{4.13} \]

Applying Lemma A.5 (with \( A = Q(t) \cap C(t) \) and \( B = C(t) \)) to (4.12) and (4.13), we obtain

\[ -\dot{\chi} + \varepsilon \frac{y - z}{\|y - z\|} \in N_{C(t)}(z). \]

Therefore, due to \( y \in C(t) \),

\[ \langle -\dot{\chi} + \varepsilon \frac{y - z}{\|y - z\|}, y - z \rangle \leq 0. \tag{4.14} \]

**Step 3.** Due to (4.8), we can apply Corollary 2.11 to \( y, \chi \) and \( Q \) (as \( y, c \) and \( Q \) respectively) to deduce that a.e. on \([\alpha_0, \beta_0]\) the derivatives \( \dot{z} \) and \( \dot{V} \) exist and

\[ \langle \dot{\chi} - \dot{z}, y - z \rangle = 0. \tag{4.15} \]

On the other hand, (1.1) and (4.9) imply that

\[ \langle \dot{y}, y - z \rangle \leq 0. \tag{4.16} \]

We add (4.14), (4.15) and (4.16) to obtain

\[ \langle \dot{y} - \dot{z} + \varepsilon \frac{y - z}{\|y - z\|}, y - z \rangle \leq 0, \]

i.e.

\[ \langle \dot{y} - \dot{z}, y - z \rangle + \varepsilon \|y - z\| \leq 0. \]

Since

\[ \dot{V} = 2\langle \dot{y} - \dot{z}, y - z \rangle, \]
we have

\[ \dot{V}(t) + 2\varepsilon \sqrt{V(t)} \leq 0. \]

The statement of the theorem now follows from Lemma A.4. \(\square\)

Let us consider a few modifications of Theorem 4.1. We use the sets \(Q(t)\) defined by (4.3), \(K'\) defined by (4.4) and

\[ K := \text{cone}\{n_i : i \in I_0 \cup P_0\}. \quad (4.17) \]

**Corollary 4.3.** In the setting of the Periodic attractor theorem for a polyhedral case (Thm. 2.22), let us further assume that there exist an interval \([\alpha_0, \beta_0] \subset [0, T]\), a number \(\varepsilon > 0\) and index sets \(P_0 \subset \hat{P}, I_0 \subset \overline{I \setminus \hat{P}}\) such that condition \(A\) of Theorem 4.1 holds. Furthermore, assume that \(B^*\). The index set \(I_0 \cup \hat{P}\) lists all the constraints which are active somewhere at \(X(t)\) and they must remain active during the entire interval \((\alpha_0, \beta_0)\) at the same point, call it \(x_0\):

there is an \(x_0 \in X\) such that for each \(i \in I_0, t \in (\alpha_0, \beta_0)\):

\[ \langle n_i, x_0 + \chi(t) \rangle = c_i(t), \]

for all \(i \in \overline{I \setminus (I_0 \cup \hat{P})},\ t \in [\alpha_0, \beta_0], \ x \in X:\

\[ \langle n_i, x + \chi(t) \rangle < c_i(t). \]

\(C^*\). For a.a. \(t \in [\alpha_0, \beta_0]\)

\[ -\dot{x} + B_\varepsilon[0] \cap K' \subset K. \quad (4.18) \]

Then the conditions of Theorem 4.1 are satisfied and its conclusions hold true.

**Proof.** As condition \(B^*\) is tighter then condition \(B\) of Theorem 4.1, we only need to discuss condition \(C^*\).

Observe that \(x_0 + \chi(t) \in Q(t) \cap C(t)\) and

\[ K \subset N_{C(t)}(x_0 + \chi(t)) \subset \bigcup_{\xi \in Q(t) \cap C(t)} N_{C(t)}(\xi), \]

therefore condition \(C\) of Theorem 4.1 follows from condition \(C^*\). \(\square\)

Corollary 4.3 is convenient to use when \(Q(t)\) is a cone with the vertex at \(x_0 + \chi(t)\) as in Figure 7.

Due to polyhedral nature of \(C(t)\) it is possible to use a variant of condition (4.5) with \(K'\) replaced by \(L \subset K'\) (see (4.20) below). Under condition \(A\) of Theorem 4.1 the following set is well defined (see Cor. 2.9) and independent on \(t \in [\alpha, \beta]::

\[ K_0 := \text{cone}\{n_i : i \in P_0\} = N_{C(t)}(\text{ri} X(t)). \quad (4.19) \]

**Corollary 4.4.** In the setting of the Periodic attractor theorem for a polyhedral case (Thm. 2.22) let us further assume that there exist an interval \([\alpha_0, \beta_0] \subset [0, T]\), a number \(\varepsilon > 0\) and index sets \(P_0 \subset \hat{P}, I_0 \subset \overline{I \setminus \hat{P}}\) such that conditions \(A\) and \(B\) of Theorem 4.1 hold. Denote \(Q(t), K', K_0\) and \(K_0\) as in (4.3), (4.4), (4.17) and (4.19), respectively. Furthermore, assume that

\(A^{**}\). In addition to \(A\), the vectors \(\{n_i : i \in P_0\}\) are linearly independent.

\(C^{**}\). There exists an \(\hat{\varepsilon} > 0\) such that for a.a. \(t \in [\alpha_0, \beta_0]\)

\[ -\dot{x} + B_{\hat{\varepsilon}}[0] \cap L \subset K_0. \quad (4.20) \]
Then there is an \( \varepsilon > 0 \) such that the conditions of Theorem 4.1 are satisfied and its conclusions hold true.

Proof. We are going to show that there exists an \( \varepsilon > 0 \) for which

\[
-\hat{\chi} + B_{\varepsilon}[0] \subseteq \bigcup_{\xi \in \mathcal{Q}(t) \cap \mathcal{C}(t)} N_{\mathcal{C}(t)}(\xi),
\]

which implies (4.5).

Assumption \( A^{**} \) and (4.20) imply that for any \( t \in [\alpha, \beta] \),

\[
-\lambda_0(t) + (B^*_{P_0})^{-1}(B_{\varepsilon}[0] \cap \mathcal{L}^\perp) \subseteq \{ x \in \mathbb{R}^{[P_0]} : x_i \geq 0 \},
\]

where \( B^*_{P_0} \) is defined as in Remark 2.20 and \( \lambda_0(t) := (B^*_{P_0})^{-1}\hat{\chi}(t) \). Let \( u := (1, 1, 1, \ldots) \in \mathbb{R}^{[P_0]} \), then

\[
-\frac{\hat{\varepsilon}}{\|B^*_{P_0}u\|} B^*_{P_0}u \in B_{\varepsilon}[0] \cap \mathcal{L}^\perp,
\]

where \( B^*_{P_0}u \neq 0 \) due to \( A^{**} \), hence

\[
-\lambda_0(t) - \frac{\hat{\varepsilon}}{\|B^*_{P_0}u\|}(B^*_{P_0})^{-1}B^*_{P_0}u \in \{ x \in \mathbb{R}^{[P_0]} : x_i \geq 0 \},
\]

in other words,

\[
(-\lambda_0(t))_i \geq \frac{\hat{\varepsilon}}{\|B^*_{P_0}u\|} \quad \text{for any } i \in P_0.
\]

On the other hand, let \( r \in B_{\varepsilon}[0] \) with yet unspecified \( \varepsilon > 0 \). As \( \mathcal{Q}(t) \cap \mathcal{C}(t) \) is a convex, closed and bounded \( (\mathcal{C}(t) \) is bounded under the conditions of Thm. 2.22) subset of the Euclidean space \( E \), by the Brouwer fixed point theorem (see e.g. [10], p. 179) the map

\[
\xi \mapsto \text{proj}(\xi + r, \mathcal{Q}(t) \cap \mathcal{C}(t))
\]

from \( \mathcal{Q}(t) \cap \mathcal{C}(t) \) to itself has a fixed point, i.e. there is a point \( \xi \in \mathcal{Q}(t) \cap \mathcal{C}(t) \) such that

\[
r \in N_{\mathcal{Q}(t) \cap \mathcal{C}(t)}(\xi) = \text{cone}(\{ n_i : i \in \mathcal{A}_{\mathcal{C}(t)}(\xi) \} \cup \{ -n_i : i \in P_0 \}),
\]

where we are guaranteed to have \( P_0 \subset \mathcal{A}_{\mathcal{C}(t)}(\xi) \) by condition \( A \). There are only finitely many possible index sets \( \mathcal{A}_{\mathcal{C}(t)}(\xi) \subset \overline{1, K} \) and for each of them the cone \( N_{\mathcal{Q}(t) \cap \mathcal{C}(t)}(\xi) \) can be represented as a union of a finite number of cones, generated by linearly independent subsets of \( \{ n_i : i \in \mathcal{A}_{\mathcal{C}(t)}(\xi) \} \cup \{ -n_i : i \in P_0 \} \), see Lemma 6.5.5 in [33]. Let there be a total of \( m \) of the corresponding index sets \( \mathcal{J}_j, j \in \overline{1, m} \), accounting for all possible \( \mathcal{A}_{\mathcal{C}(t)}(\xi) \), and we do not distinguish the index sets if they differ only by having \( -n_i \) instead of \( n_i \) for the same \( i \in P_0 \). Thus we can express \( r \) in the following way for some \( j \in \overline{1, m} \):

\[
r = B^*_j \lambda_r = \sum_{i \in \mathcal{J}_j} (\lambda_r)_i n_i, \quad \text{where } \begin{cases} (\lambda_r)_i \geq 0 & \text{if } i \in \mathcal{J}_j \setminus P_0, \\ (\lambda_r)_i \in \mathbb{R} & \text{if } i \in \mathcal{J}_j \cap P_0. \end{cases}
\]
Due to the equivalence of the Euclidean norm and the sup-norm
\[ \| \lambda \|_\infty := \max_{i \in 1, n} |\lambda_i| \]
in \( \mathbb{R}^n \), there are \( c_n, C_n > 0 \) such that
\[ c_n \| \lambda \|_\infty \leq \| \lambda \| \leq C_n \| \lambda \|_\infty, \quad \lambda \in \mathbb{R}^n. \]

Since the generating normals in (4.22) are linearly independent for each \( J_j \), the map \( B^*_j \) is invertible and
\[ \| \lambda_r \|_\infty = \left\| \left( B^*_j \right)^{-1} r \right\|_\infty \leq \frac{1}{c_{[J_j]}} \left\| \left( B^*_j \right)^{-1} r \right\| \leq \frac{1}{c_{[J_j]}} \left\| \left( B^*_j \right)^{-1} \right\| \| r \|. \]
Thus, if we choose
\[ \varepsilon \leq \frac{\hat{\varepsilon}}{\| B^*_P u \|} \cdot \min_{j \in 1, m} \frac{c_{[J_j]}}{\left( B^*_j \right)^{-1}}, \]
then
\[ \| \lambda_r \|_\infty \leq \frac{\hat{\varepsilon}}{\| B^*_P u \|} \leq (-\lambda_0(t))_i \quad \text{for all } i \in \mathcal{P}_0, \]
hence the expression
\[ -\dot{x}(t) + r = -\sum_{i \in \mathcal{P}_0} (\lambda_0(t))_i n_i + \sum_{i \in J_j} (\lambda_r)_i n_i \]
\[ = \sum_{i \in \mathcal{P}_0 \cap J_j} ((-\lambda_0(t))_i + (\lambda_r)_i) n_i + \sum_{i \in \mathcal{P}_0 \setminus J_j} (-\lambda_0(t))_i n_i + \sum_{i \in J_j \setminus \mathcal{P}_0} (\lambda_r)_i n_i \]
has all the coefficients nonnegative. Both \( \mathcal{P}_0 \) and \( J_j \) belong to \( \mathcal{A}(\mathcal{C}(t), \xi) \), therefore
\[ -\dot{x}(t) + r \in \mathcal{N}(\mathcal{C}(t), \xi), \]
which concludes the proof of (4.21).

Finally, in the case when \( X(t) = Q(t) = Q(t) \cap C(t) \) (see Figure 8) it is possible to prove the following version of Theorem 4.1. Notice that it does not require \( C(t) \) to be a polytope, merely a closed convex set, but it needs \( X(t) \) to completely fill a face of \( C(t) \).

**Theorem 4.5.** In the setting of the Periodic attractor theorem (Thm. 2.17) and Corollary 2.18 let us further assume that there exist an \( \varepsilon > 0 \) and an interval \( [\alpha, \beta] \subset [0, T] \) such that
\[ -\dot{x} + B\varepsilon[0] \subset \bigcup_{\xi \in X(t)} \mathcal{N}(\mathcal{C}(t), \xi) \] (4.23)
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Figure 8. A simple example of the sweeping process illustrating Theorem 4.5. a) The moving set $C(t)$ (blue), the attractor $X(t)$ (magenta) and the normal cones $N_{C(t)}(\xi)$ at the points of the attractor $\xi \in X(t)$ (cyan). b) The left (magenta) and right (cyan) hand side of condition (4.23).

a.e. on $[\alpha, \beta]$. Then the global attractor $X(t)$ is finite-time stable. More precisely, any solution $y$ is guaranteed to reach $X(t)$ by the time

$$t^* = \left\lceil \frac{\text{dist}(y(0), X(0))}{\varepsilon(\beta - \alpha)} \right\rceil T.$$  \hspace{1cm} (4.24)

Proof. The proof follows the same steps as the proof of Theorem 4.1 with $Q(t) = X(t) = Q(t) \cap C(t)$, except for a few aspects. Namely, conditions (4.6), (4.7) and Step 1 are no longer needed because $z(t) \in Q(t)$ automatically, and (4.13) follows directly from (4.23). The global estimate (4.24) of the convergence time follows from the fact that the function $V(t) = \text{dist}^2(y(t), X(t))$ is non-increasing outside of the intervals $[\alpha + kT, \beta + kT], k \in \mathbb{N}_0$, while Lemma A.4 can be used on each of those intervals. \hfill $\Box$

An example of formation of an attractor, which fits in Theorem 4.5 can be found in [22]. Moreover, as we discuss in that papers, the dimension of the attractor and its properties are preserved under small changes of the parameters of the underlying mechanical model.

4.2. Finite-time convergence to the attractor within its affine hull

Now we provide conditions which ensure the convergence to $X(t)$ within $L(t)$. This approach is supposed to model “robust forming” of the attractor $X(t)$, where different parts of $X(t)$ are “formed” by different faces of $C(t)$ at different times as in examples presented in Section 3 (compare with “robust forming at once” in Thm. 4.5).

We will need a concept of a velocity of a face. Let us recall that every face of $C(t)$ can be identified with an index set $I \subset \Gamma, K$ of linearly independent normal vectors $\{n_i : i \in I\}$ so that the face can be written as

$$F_I(t) = C(t) \cap \{x \in E : B_Ix = c_I(t)\},$$

where $B_I$ and $c_I(t)$ are defined as in Remark 2.20.

Definition 4.6. For a face $F_I(t)$ of $C(t)$ we say that its velocity is the vector $\dot{F}_I(t) \in E$ such that

$$B_I\dot{F}_I(t) = \dot{c}_I(t),$$

$$\dot{F}_I(t) \in \text{lin}\{n_i : i \in I\}.$$
**Theorem 4.7.** In the setting of the Periodic attractor theorem for a polyhedral case (Thm. 2.22) let there be index sets $\mathcal{P}_i, \mathcal{I}_i$ and time-intervals $[\alpha_i, \beta_i] \subset [0, T]$, $i \in \overline{1, m}$ such that $\mathcal{P}_i \cap \mathcal{I}_i = \emptyset$ and
\[ 0 \leq \alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq \alpha_m < \beta_m \leq T \]
and the following conditions hold:

**A2.** For all $t \in [\alpha_i, \beta_i]$ the equality $\mathcal{P}(t) = \mathcal{P}_i$ holds and
\[ \mathcal{L} = \{y \in E : (n_j, y) = 0 : j \in \mathcal{P}_i\}. \]

**B2.** All constraints from $\mathcal{I}_i$ are active somewhere at $X(t)$ at $t = \beta_i$:
\[ \text{for every } i \in \overline{1, m} \text{ and } j \in \mathcal{I}_i \text{ there is an } x_0 \in \mathcal{X} \text{ such that } \langle n_j, x_0 + \chi(\beta_i) \rangle = c_j(\beta_i). \quad (4.25) \]

On the other hand, the constraints from $\mathcal{I}_i$ stay away from $X(t)$ prior to $\alpha_i$:
\[ \text{for every } i \in \overline{1, m} \text{ and } j \in \mathcal{I}_i, x \in \mathcal{X}, t \in [0, \alpha_i] \text{ we have } \langle n_j, x + \chi(t) \rangle < c_j(t). \quad (4.26) \]

Also (without loss of generality), we choose $\mathcal{I}_i$ to be pairwise disjoint:
\[ \text{for every } i_1, i_2 \in \overline{1, m}, i_1 \neq i_2, \text{ we have } \mathcal{I}_{i_1} \cap \mathcal{I}_{i_2} = \emptyset. \]

All of the other constraints stay away from $X(t)$ during the entire interval $[0, T]$:
\[ \text{for every } x \in \mathcal{X}, j \in \overline{1, k} \setminus \left( \hat{\mathcal{P}} \cup \bigcup_{i=1}^{m} \mathcal{I}_i \right), t \in [0, T] \text{ we have } \langle n_j, x + \chi(t) \rangle < c_j(t). \quad (4.27) \]

**C2.** Fix $i \in \overline{1, m}$. For every $t \in [\alpha_i, \beta_i]$ consider those faces and vertices of $C(t)$ for which their respective index set $\mathcal{I}$ satisfies
\[ \mathcal{P}_i \subset \mathcal{I}, \]
\[ \mathcal{I} \setminus \mathcal{P}_i \subset \mathcal{I}_i; \]
i.e. the faces and vertices of $C(t)$ which are the intersections of $L(t)$ with some of the constraints from $\mathcal{I}_i$. For every such face $F_{\mathcal{I}}(t)$ we require that for a.a. of those $t \in [\alpha_i, \beta_i]$ for which the face is nonempty,
\[ -F_{\mathcal{I}}(t) \in \text{cone}\{n_j : j \in \mathcal{I}\} = N_{C(t)}(\text{ri} F_{\mathcal{I}}(t)), \]
where the right-hand side is well-defined by Corollary 2.9. In the case where $\mathcal{I}$ defines a vertex we require for a.a. of those $t \in [\alpha_i, \beta_i]$ for which $v_{\mathcal{I}}(t)$ is an actual vertex of $C(t)$,
\[ -\dot{v}_{\mathcal{I}}(t) \in N_{C(t)}(v_{\mathcal{I}}(t)), \]
where $v_{\mathcal{I}}(t)$ is defined in Remark 2.20.

**D2.** Let $P_{\mathcal{L}}$ be the orthogonal projection map onto $\mathcal{L}$. Consider all pairs of $j_1 \in \mathcal{I}_{i_1}$ and $j_2 \in \mathcal{I}_{i_2}$ for $i_1 < i_2, i_1, i_2 \in \overline{1, m}$ for which there is a common $x_0 \in \mathcal{X}$ such that
\[ \langle n_{j_1}, x_0 \rangle = \mu_{j_1}, \quad \langle n_{j_2}, x_0 \rangle = \mu_{j_2}. \]
We require that for such \( j_1, j_2, \)
\[
(P_L n_{j_1}, P_L n_{j_2}) \geq 0.
\] (4.28)

Under the above conditions there is a \( \delta_2 > 0 \) such that every solution \( y \) which reaches \( L(t) \) by the time \( \beta_1 \) and which is from the \( \delta_2 \)-neighborhood of \( X(0) \) will reach \( X(t) \) by the time \( t = \beta_m \):
\[
y(\beta_1) \in L(\beta_1),
dist(y(0), X(0)) < \delta_2 \implies y(\beta_m) \in X(\beta_m).
\] (4.29)

With regard to condition B2 we note that we allow the constraints from \( I_i \) to be active on \( X(t) \) after the time-moment \( \alpha_i \) including the time-moments \( \beta_{i^*}, i^* > i \).

**Remark 4.8.** Again, we can illustrate the conditions of Theorem 4.7 by considering the examples of sweeping processes from Section 3. For the example shown in Figures 1–3, we can put \( \mathcal{P}_1 = \{1, 2\}, \mathcal{I}_1 = \{5\}, \mathcal{P}_2 = \{6, 7\}, \mathcal{I}_2 = \{10\} \) with \( \beta_1 \) and \( \beta_2 \) being the last time-moments of the respective intervals of monotonicity of \( g(t) \) in (3.1). Here C2 follows from (3.1) and (3.2) and D2 is satisfied trivially as there is no point in \( \mathcal{X} \) where the constraints from both \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) can be active.

For the example shown in Figures 4–6 we can put \( \mathcal{P}_1 = \{3\}, \mathcal{I}_1 = \{6, 7\}, \mathcal{P}_2 = \{10\}, \mathcal{I}_2 = \{13, 14\} \) assuming again that \( \beta_1 \) and \( \beta_2 \) are the last time-moments of the respective intervals of monotonicity of \( g(t) \) in (3.1). To check D2 we observe that
\[
P_L = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Condition (4.28) follows from
\[
(P_L n_6, P_L n_{14}) = \left\langle \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix} \right\rangle = 0,
\]
\[
(P_L n_7, P_L n_{13}) = \left\langle \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix} \right\rangle = 0,
\]
where the normal vectors from different \( I_i \) correspond to vertices of \( \mathcal{X} \) (see Fig. 4b) and Fig. 6).

**Proof of Theorem 4.7.**

**Step 1.** Using (4.26), (4.27) and compactness of \( \mathcal{X} \), assume that \( \delta_2 \) is chosen so that
\[
0 < \delta_2 < \frac{c_j(t) - \langle n_j, x + \chi(t) \rangle}{\|n_j\|} \quad \text{for all } x \in \mathcal{X}, \quad j \in \mathcal{I}_i, \quad j \in \mathcal{I}_i \cup \left( \hat{\mathcal{P}} \cup \bigcup_{i=1}^m \mathcal{I}_i \right), \quad t \in [0, \alpha_i], \quad i \in \overline{1, m},
\] (4.30)

hence
\[
\langle n_j, x \rangle < c_j(t) - \langle n_j, \chi(t) \rangle - \|n_j\|\delta_2 \quad \text{for all such } x, j, t.
\]

For further use we notice that due to (4.25), Corollary 2.25, and (2.16), we have
\[
c_j(\beta_i) - \langle n_j, \chi(\beta_i) \rangle = \mu_j \leq c_j(t) - \langle n_j, \chi(t) \rangle \quad \text{for any } i \in \overline{1, m}, \quad j \in \mathcal{I}_i, \quad t \in [0, T].
\] (4.31)
Indeed, for all $t$, we conclude that (4.34) holds for all $j$.

Step 2. Let us observe that for $t \in [\beta_1, \alpha_2]$ the set of active constraints satisfies
\[
\mathcal{A}_{C(t)}(y(t)) \subset \mathcal{I}_1 \cup \mathcal{P}(t).
\] (4.32)

Indeed, for all $i > 1$, $j \in \mathcal{I}_i$ and $j \in \overline{\mathcal{I}} \setminus \left(\hat{\mathcal{I}} \cup \bigcup_{i=1}^{m} \mathcal{I}_i\right)$
\[
\langle n_j, y(t) \rangle = \langle n_j, y(t) - x^* \rangle + \langle n_j, x^* \rangle < \|n_j\|\delta_2 + \langle n_j, x^* \rangle < c_j(t),
\]
where $x^* = \text{proj}(y(t), X(t))$.

Consider the function $y_{1,0}$ given by
\[
y_{1,0}(t) := y(\beta_1) + \chi(t) - \chi(\beta_1).
\] (4.33)

We claim that it is a solution on $[\beta_1, \alpha_2]$. To see this, notice that for every $j \in \mathcal{I}_1$,
\[
\langle n_j, y_{1,0}(t) \rangle = \langle n_j, y(\beta_1) \rangle + \langle n_j, \chi(t) \rangle - \langle n_j, \chi(\beta_1) \rangle \leq c_j(\beta_1) - \langle n_j, \chi(\beta_1) \rangle + \langle n_j, \chi(t) \rangle \leq c_j(t),
\]
where the last inequality is due to (4.31). On the other hand, (4.29) implies that $y(\beta_1) \in L(\beta_1) = \mathcal{L} + \chi(\beta_1)$, hence for $t \in [\beta_1, \alpha_2]$,
\[
y_{1,0}(t) \in \mathcal{L} + \chi(\beta_1) + \chi(t) - \chi(\beta_1) = L(t),
\]
thus (due to the properties of $\mathcal{L}$, see Rem. 2.24), for every $j \in \mathcal{P}(t)$,
\[
\langle n_j, y_{1,0}(t) \rangle = c_j(t),
\]
and for every $j \in \hat{\mathcal{I}} \setminus \mathcal{P}(t)$,
\[
\langle n_j, y_{1,0}(t) \rangle < c_j(t).
\] (4.34)

Therefore,
\[-\dot{y}_{1,0}(t) = -\chi(t) \in \text{cone}\{n_j : j \in \mathcal{P}(t)\} \subset N_{C(t)}(y_{1,0}(t)).\]

Assume now that there exists a $t' \in [\beta_1, \alpha_2]$ such that
\[
\langle n_j, y_{1,0}(t') \rangle = c_j(t')
\] (4.35)
for some $j \in \overline{\mathcal{I}} \setminus \left(\hat{\mathcal{I}} \cap \mathcal{I}_1\right)$. Without loss of generality, we can assume that $t'$ is the minimal value in $[\beta_1, \alpha_2]$ satisfying (4.35). From $y_{1,0}(\beta_1) = y(\beta_1)$ and (4.32), it follows that $t' > \beta_1$. Therefore, (4.34) holds for all $j \in \overline{\mathcal{I}} \setminus \left(\hat{\mathcal{I}} \cap \mathcal{I}_1\right)$ and $t \in [\beta_1, t')$. Moreover, $y_{1,0}$ is a solution of the sweeping process until the time-moment $t = t'$, and $y_{1,0}(t) = y(t)$ for $t \in [\beta_1, t')$ due to the uniqueness of a solution of the sweeping process. Consequently, by continuity, $y_{1,0}(t') = y(t')$ and (4.34) holds for all $j \in \overline{\mathcal{I}} \setminus \left(\hat{\mathcal{I}} \cap \mathcal{I}_1\right)$ at $t = t'$. This contradicts (4.35). Hence, we conclude that (4.34) holds for all $j \in \overline{\mathcal{I}} \setminus \left(\hat{\mathcal{I}} \cap \mathcal{I}_1\right)$ and $t \in [\beta_1, \alpha_2]$. Therefore, $y_{1,0}$ is a solution on the entire interval $t \in [\beta_1, \alpha_2]$. 
Due to the uniqueness of a solution of the sweeping process, \( y(\beta_1) = y_{1,0}(\beta_1) \) implies \( y = y_{1,0} \) on the entire interval \([\beta_1, \alpha_2]\).

**Step 3.** Similarly, for \( t \in [\alpha_2, \beta_2] \) the set of active constraints satisfies

\[
A_{C(t)}(y(t)) \subset I_1 \cup I_2 \cup P_2. \tag{4.36}
\]

Hence,

\[
A_{C(\alpha_2)}(y(\alpha_2)) = A^1 \cup A^2_{1,1} \cup P_2 \quad \text{for some} \quad A^1 \subset I_1, A^2_{1,1} \subset I_2.
\]

Consider the function \( y_{1,1} \) given by

\[
y_{1,1}(t) := y(\alpha_2) + \int_{\alpha_2}^{t} \hat{F}_{A^2_{1,1} \cup P_2}(\tau) d\tau.
\]

Notice that for every \( j \in A^2_{1,1} \cup P_2 \),

\[
\langle n_j, y_{1,1}(t) \rangle = \langle n_j, y(\alpha_2) \rangle + \left( \int_{\alpha_2}^{t} \hat{F}_{A^2_{1,1} \cup P_2}(\tau) d\tau \right) = c_j(\alpha_2) + \int_{\alpha_2}^{t} \hat{c}_j(\tau) d\tau = c_j(t). \tag{4.37}
\]

Finally we use C2 with \( I = A^2_{1,1} \cup P_2 \) to deduce that

\[-\dot{y}_{1,1}(t) = -\hat{F}_{A^2_{1,1} \cup P_2}(t) \in \text{cone}\{ n_j : j \in A^2_{1,1} \cup P_2 \} \subset N_{C(t)}(y_{1,1}(t)), \]

where the last embedding follows from (4.37).

**Step 4.** Observe that \( y_{1,1}(\alpha_2) = y(\alpha_2) \in C(t) \). Hence, as long as \( y_{1,1}(t) \in C(t) \) and \( t \in [\alpha_2, \beta_2] \), the function \( y_{1,1} \) is a solution of the sweeping process and, by uniqueness of the solution, \( y_{1,1} = y \). Unless \( y_{1,1}(t) \in C(t) \) for all \( t \in [\alpha_2, \beta_2] \) (in which case we introduce a new variable \( t'_{1,1} := \beta_2 \) and pass directly to **Step 6** of the proof), let

\[
t' := \inf \{ t \in (\alpha_2, \beta_2) : y_{1,1}(t) \notin C(t) \}. \tag{4.38}
\]

Then, for some \( j \in \overline{1, k} \setminus (A^2_{1,1} \cap P_2) \),

\[
\langle n_j, y_{1,1}(t') \rangle = c_j(t'). \tag{4.39}
\]

Moreover, for this \( j \), one can find an instance \( t'' \in (t', \beta_2) \), arbitrarily close to \( t' \), at which the \( j \)-th constraint of \( C(t) \) is violated:

\[
\langle n_j, y_{1,1}(t'') \rangle > c_j(t''), \tag{4.40}
\]

while

\[
y_{1,1}(t) = y(t) \in C(t) \quad \text{for all} \quad t \in [\alpha_0, t']. \tag{4.41}
\]
Below we consider the possibility of \( j \) being from different sets.

- \( j \in \mathcal{I}_2 \setminus \mathcal{A}^2_{1,1} \). In such a case, we denote \( t^*_1 := t' \) and consider the restriction of the function \( y_{1,1} \) to the interval \( [\alpha_2, t^*_1] \). Later on, we use a new set \( \mathcal{A}^2_{1,2} \subset \mathcal{I}_2 \) and a new function \( y_{1,2} \) as will be described in Step 5.
- For \( j \in \hat{\mathcal{P}} \setminus \mathcal{P}_2 \), we have

\[
\langle n_j, y_{1,1}(t) \rangle < c_j(t)
\]

for all \( t \in [\alpha_2, \beta_2] \) because \( y_{1,1}(t) \in L(t) \) by (4.37). Thus (4.39) cannot hold for such \( j \).
- For \( j \in \overline{I_1 \cup I_2 \cup \hat{\mathcal{P}}} \) we observe that (4.41) and (4.36) make it impossible for equality (4.39) to hold for such \( j \).
- Assume that \( j \in \mathcal{I}_1 \) and there is a \( j' \in \mathcal{A}^2_{1,1} \), for which

\[
\langle P_L n_j, P_L n_{j'} \rangle < 0. \tag{4.42}
\]

In this case, define the following \( \delta_{j_1,j_2} \) for \( j_1 \in \mathcal{I}_{i_1}, j_2 \in \mathcal{I}_{i_2}, i_1 < i_2, i_1, i_2 \in \overline{1,m} \) (in particular, for \( j_1 = j \in \mathcal{I}_1, j_2 = j' \in \mathcal{I}_2 \)):

\[
\delta_{j_1,j_2} := \min_{x_0 \in \mathcal{X}} \max \left( \frac{\text{dist}(x_0, \{ x \in E : \langle n_{j_1}, x \rangle \geq \mu_{j_1} \})}{\text{dist}(x_0, \{ x \in E : \langle n_{j_2}, x \rangle \geq \mu_{j_2} \})} \right)
\]

and observe that it is a positive number whenever \( \langle P_L n_{j_1}, P_L n_{j_2} \rangle < 0 \) due to \( D_2 \) and compactness of \( \mathcal{X} \).

Using \( \delta_{j_1,j_2} \) we require (in addition to (4.30)) that \( \delta_2 \) in the statement of the theorem is sufficiently small to ensure that

\[
\delta_2 \leq \delta_{j_1,j_2} \quad \text{for all } j_1 \in \mathcal{I}_{i_1}, j_2 \in \mathcal{I}_{i_2}, i_1 < i_2, i_1, i_2 \in \overline{1,m} \text{ such that } \langle P_L n_{j_1}, P_L n_{j_2} \rangle < 0. \tag{4.43}
\]

Recall that due to (4.41),

\[
y_{1,1}(t') = y(t') \in X(t') + B_{\delta_2}(0),
\]

i.e.

\[
y_{1,1}(t') - \chi(t') \in X + B_{\delta_2}(0).
\]

Therefore, taking

\[
x_0 := \text{proj}(y_{1,1}(t') - \chi(t'), \mathcal{X}),
\]

we obtain

\[
\| x_0 - (y_{1,1}(t') - \chi(t')) \| < \delta_2 \leq \delta_{j,j'} \leq \max \left( \frac{\text{dist}(x_0, \{ x \in E : \langle n_j, x \rangle \geq \mu_j \})}{\text{dist}(x_0, \{ x \in E : \langle n_{j'}, x \rangle \geq \mu_{j'} \})} \right). \tag{4.44}
\]

Furthermore, notice that relations (4.31) and (4.37) imply for \( j' \) that

\[
\langle n_{j'}, y_{1,1}(t') - \chi(t') \rangle = c_{j'}(t') - \langle n_{j'}, \chi(t') \rangle \geq \mu_{j'},
\]
Finally, we consider the case when
\[ y_{1,1}(t') - \chi(t') \in \{ x \in E : \langle n_j', x \rangle \geq \mu_j' \}, \]
which means that
\[ \|x_0 - (y_{1,1}(t') - \chi(t'))\| \geq \text{dist}(x_0, \{ x \in E : \langle n_j', x \rangle \geq \mu_j' \}). \]
But this does not contradict (4.44) only when
\[ \|x_0 - (y_{1,1}(t') - \chi(t'))\| < \text{dist}(x_0, \{ x \in E : \langle n_j, x \rangle \geq \mu_j \}), \]
i.e.
\[ y_{1,1}(t') - \chi(t') \notin \{ x \in E : \langle n_j, x \rangle \geq \mu_j \} \]
or, equivalently, \( \langle n_j, y_{1,1}(t') - \chi(t') \rangle < \mu_j \), which means that
\[ \langle n_j, y_{1,1}(t') \rangle < \mu_j + \langle n_j, \chi(t') \rangle \leq c_j(t'). \]
Thus, we have shown that under the conditions of the theorem equality (4.39) cannot hold with \( j \in J_1 \)
such that there is a \( j' \in A^2_{1,1} \) for which (4.42) is valid.

Finally, we consider the case when \( j \in J_1 \) and for all \( j' \in A^2_{1,1} \),
\[ \langle P_L n_j, P_{L' j'} \rangle \geq 0. \tag{4.45} \]
Recall that there is \( t'' \in (t', \beta_2) \) such that (4.40) holds. From (4.37) and A2 we see that for \( t \in [\alpha_2, t''] \),
\[ y_{1,1}(t) - \chi(t) \in L, \]
therefore on the interval \( t \in [\alpha_2, t''] \),
\[ \langle n_j, y_{1,1}(t) - \chi(t) \rangle = \langle P_L n_j, y_{1,1}(t) - \chi(t) \rangle = \langle P_L n_j, P_L(y_{1,1}(t) - \chi(t)) \rangle \]
\[ = \left\langle P_L n_j, P_L(y_{1,1}(\alpha_2) - \chi(\alpha_2)) + \int_{\alpha_2}^t P_L \hat{F}_{A^2_{1,1} \cup P_2}(\tau) - P_L \hat{\chi}(\tau) \, d\tau \right\rangle \]
\[ = \left\langle P_L n_j, P_L(y_{1,1}(\alpha_2) - \chi(\alpha_2)) + \left\langle P_L n_j, \int_{\alpha_2}^t P_L \hat{F}_{A^2_{1,1} \cup P_2}(\tau) \, d\tau \right\rangle \right\rangle \]
\[ = \left\langle P_L n_j, y_{1,1}(\alpha_2) - \chi(\alpha_2) \right\rangle = \left\langle P_L n_j, \int_{\alpha_2}^t \sum_{j' \in A^2_{1,1} \cup P_2} \lambda_{j'}(\tau) n_{j'} \, d\tau \right\rangle \]
\[ = \left\langle n_j, y_{1,1}(\alpha_2) - \chi(\alpha_2) \right\rangle - \left\langle P_L n_j, \int_{\alpha_2}^t \sum_{j' \in A^2_{1,1}} \lambda_{j'}(\tau) P_L n_{j'} \, d\tau \right\rangle \]
\[ = \left\langle n_j, y_{1,1}(\alpha_2) - \chi(\alpha_2) \right\rangle - \sum_{j' \in A^2_{1,1}} \left\langle P_L n_j, P_L n_{j'} \right\rangle \int_{\alpha_2}^t \lambda_{j'}(\tau) \, d\tau \]
Concluding the current step of the proof, we have demonstrated that $y$ on the interval $\alpha_1 \leq t \leq \beta_1$ satisfies the inequality (4.46) for $q > 1$ and $j' \in A_{1,1}^2$. Moreover, due to (4.45),

$$
\langle n_j, y_{1,1}(t) - \chi(t) \rangle \leq \mu_j,
$$

hence

$$
\langle n_j, y_{1,1}(t) \rangle \leq c_j(t). \tag{4.47}
$$

This includes $t = t''$, which contradicts (4.40). This contradiction proves that (4.47) holds on the whole interval $t \in [\alpha_2, \beta_2]$ for any $j \in I_1$ such that the inequality (4.45) is valid for all $j' \in A_{1,1}^2$.

Concluding the current step of the proof, we have demonstrated that $y_{1,1}$ is a solution of the sweeping process and

$$
y_{1,1}(t) = y(t) \in C(t)
$$
on the interval $[\alpha_2, t_{1,1}^*]$, where either $t_{1,1}^* = \beta_2$ or $t_{1,1}^* = t'$ is defined by (4.38). Moreover, in the latter case, (4.39) holds for any $j \in I_2 \setminus A_{1,1}^2$.

**Step 5.** **Step 3** and **Step 4** can be repeated with

$$
\mathcal{A}_{C(t_{1,1}^* - 1)}(y(t_{1,1}^* - 1)) = A^1 \cup A_{q,1}^1 \cup P_2 \quad \text{where} \quad A^1 \subset I_1, A_{q,1}^1 \subset I_2
$$

for $q > 1$ and

$$
y_{1,q}(t) := y(t_{1,q-1}^*) + \int_{t_{1,q-1}^*}^t \hat{F}_{A_{q,1}^1 \cup P_2}^j(\tau) d\tau.
$$

In this manner, after a finite (say, $q^*$) number of steps we will obtain $t_{1,q^*}^* = \beta_2$ because the discrete set $A_{q,1}^1 \subset I_2$ increases with $q$ due to the construction of $y_{1,q}$ and assumption **C2** (which is used in (4.46)). The same argument as in **Step 4** can be used to show that $y_{1,q}$ is a solution of the sweeping process and $y_{1,q} = y$ on the respective interval $[t_{1,q-1}^*, t_{1,q}^*]$ for $q \leq q^*$.

**Step 6.** Let us note that for each $i \in \overline{3,m}$ and $t \in [\beta_{i-1}, \alpha_i]$,

$$
\mathcal{A}_{C(t)}(y(t)) \subset I_i \cup P(t),
$$

where the last equality follows from (4.31). Also, in these formulas, $\lambda \in L^1([\alpha_2, \beta_2], \mathbb{R}^{A_{1,1}^2 \cup P_2})$ comes from **C2** and Lemma A.7 (where we take $\hat{F}_{A_2 \cup P_2}$ as $u$ and $[\alpha_2, \beta_2]$ as both $[0, T]$ and $T$), hence $\lambda_j(\tau) \geq 0$.
while for $t \in [\alpha_i, \beta_i]$,

$$\mathcal{A}_{C(t)}(y(t)) = A^1 \cup \cdots \cup A^{i-1} \cup A_{i,q} \cup P_i \quad \text{for some} \quad A^{i'} \subset I_{i'} \quad \text{with} \quad i' \in \overline{i, i-1}, \ A_{i,q} \subset I_i.$$ 

Thus, we can construct

$$y_{i,0}(t) := y(\beta_{i-1}) + \chi(t) - \chi(\beta_{i-1}),$$

$$y_{i,q}(t) := y(t_{i,q-1}^*) + \int_{t_{i,q-1}^*}^t \tilde{F}_{A_{i,q} \cup P_i}(\tau) d\tau$$

extending the argument of Step 2 to the intervals $[\beta_{i-1}, \alpha_i]$ and the argument of Steps 3-6 to the intervals $[\alpha_i, \beta_i]$. This leads to the relations

$$\langle n_j, y(\beta_m) - \chi(\beta_m) \rangle \leq \mu_j, \quad j \in I_i.$$ 

Moreover, $y(\beta_m) \in L(\beta_m)$ due to the counterparts of (4.33) and (4.37) with $j \in P_i$ at each step. These relations ensure that $y(\beta_2) - \chi(\beta_2) \in \mathcal{X}$ because, as one can see from (4.27), the constraints from $\bigcap k \setminus \left( \tilde{P} \cup \bigcup_{i=1}^m I_i \right)$ do not participate in the representation (2.16), (2.17) of $\mathcal{X}$. Thus, we have shown that if $\delta_2$ satisfies (4.30) and (4.43), we indeed have $y(\beta_m) \in X(\beta_m)$. 

**Corollary 4.9.** If the conditions of Theorem 4.1 and Theorem 4.7 hold simultaneously so that $[\alpha_0, \beta_0] \subset [\alpha_1, \beta_1], I_0 \subset I_1$ and $P_0 = P_1$, then every solution $y$ such that

$$\text{dist}(y(\alpha_0), X(\alpha_0)) < \min(\delta, \delta_2)$$

is absorbed by the attractor $X(t)$ by the time $t = \beta_m$, i.e. $y(\beta_m) \in X(\beta_m)$.

**Remark 4.10.** It can be observed that if $C(t)$ is a polyhedron, Theorem 4.5 describes a particular case of Corollary 4.9.

**Remark 4.11.** We combine Remarks 4.2, 4.8 and Corollary 4.9 to observe that attractor $X(t)$ in the examples of Figure 1 and 4 is, indeed, finite-time stable. For the systems of Figures 2 and 5, respectively, this means that, if the displacement constraint (the total length of a system in our cases) is periodic, then the stresses in the springs, are guaranteed to reach a periodic regime in finite time.

**Remark 4.12.** In addition, one can consider a situation when periodically varying stress load (external force) of bounded magnitude is applied at the nodes of the elasto-plastic systems of Figures 2 and 5. According to the construction [21–23], variations in stress load lead to the change of shape of $C(t)$ in the corresponding sweeping process, however, the normal directions to the faces do not change. Theorems 4.1 and 4.7 remain applicable in this, slightly more complicated, situation.

5. Conclusions

It is known that periodic input of a sweeping process leads to an asymptotically stable global attractor comprised of periodic solutions. For polyhedral sweeping processes we have shown that such an asymptotic attractor is a polyhedron describable in terms of normal vectors of $C(t)$ (Thm. 2.22). This allowed us to consider the formation of asymptotic attractor by independent parts of $C(t)$ and to establish conditions for finite-time convergence of an arbitrary solution to $X(t)$. Such convergence can be explained as a composition
of two phenomena: convergence to a subset \( Q(t) \) of the hyperplane \( \text{aff } X(t) \) (see Thm. 4.1, where a Lyapunov function approach is used), and motions of the faces of \( C(t) \) which sweep solutions into \( X(t) \) over a finite sequence of time-intervals (Thm. 4.7). The former theorem alone is sufficient when \( Q(t) = X(t) \), i.e. when the a face of \( C(t) \) completely coincides with \( X(t) \) during a time-interval of appropriate motion (Thm. 4.5).

By combining the conditions of Theorems 4.1 and 4.7 we cover a general case in which different faces of the attractor are formed by \( C(t) \) at different time-intervals. Specifically, we consider a finite sequence of time-intervals satisfying the conditions of Theorem 4.7 and require the conditions of Theorem 4.1 for the Lyapunov function argument only on one of the intervals. Notice that, unlike Theorem 4.1, the conditions of Theorem 4.7 require the contact of “side-forming” constraints \( I_i \) with the attractor only at the ends of respective time-intervals (condition B2). Without a condition of the type (4.5) (which implies a persistent contact of the constraints \( I_0 \) with the attractor), this generalization requires a more complicated condition to be imposed on constraints \( I_i \). Namely, each of the participating sub-faces of \( C(t) \) must move in an appropriate way (condition C2). While this actually means a combinatorial amount of conditions on velocities, such conditions look unavoidable within the proposed level of generality of the problem. Effectively, in the present form, Theorem 4.7 is limited to a simple kind of sweeping processes for which any solution in the vicinity of \( X(t) \) admits an explicit piecewise representation (that we construct in the proof of Theorem 4.7) via velocities of respected faces of \( C(t) \) over a finite number of time-intervals.

Additionally, we observed that interactions between the facets from \( I_i \) over different \( i \) must be taken into account, and we show that for finite-time stability it is enough to require the respective normal vectors to form an acute angle when projected onto the plane of the attractor (condition D2).

With such theorems we aim to further classify the asymptotic behavior in sweeping processes and predict the challenges which would arise if one aims to compute the attractor via an acute angle when projected onto the plane of the attractor (condition B2). We claim that if (A.1) holds, then, for all \( i \in \Gamma, k \),

\[
\langle n_i, x_1 \rangle = c_i \text{ if and only if } \langle n_i, x_2 \rangle = c_i.
\]

**Proof.** Assume that \( \langle n_i, x_1 \rangle = c_i \) for some \( x_1 \in \text{ri} C \) and some \( i \in \Gamma, k \), i.e. assume that the point \( x_1 \) belongs to the facet \( \{ x \in E : \langle n_i, x \rangle = c_i \} \). Let \( x_2 \) be any other point such that \( x_2 \in \text{ri} C \). We claim that if (A.1) holds, then \( \langle n_i, x_2 \rangle = c_i \).

For any \( \theta \in \mathbb{R} \), define \( x_\theta \) as

\[
x_\theta = \theta x_1 + (1 - \theta) x_2.
\]
Since \( x_1, x_2 \in \text{ri} \mathcal{C} \), there exists \( \varepsilon > 0 \) such that \( x_{-\varepsilon} \in \text{ri} \mathcal{C} \) and \( x_{1+\varepsilon} \in \text{ri} \mathcal{C} \). Put \( \bar{x}_1 = x_{-\varepsilon}, \bar{x}_2 = x_{1+\varepsilon} \). Then there exist \( \theta_1, \theta_2 \in (0,1), \theta_1 \neq \theta_2 \), such that
\[
x_1 = \theta_1 \bar{x}_1 + (1 - \theta_1) \bar{x}_2, \quad x_2 = \theta_2 \bar{x}_1 + (1 - \theta_2) \bar{x}_2.
\]
(A.3)

Then, taking the inner product of the first formula of (A.3) and \( n_i \), replacing \( c_i \) by \( c_i = \theta_1 c_i + (1 - \theta_1) c_i \), and redistributing the terms, one gets
\[
\theta_1 \left( \langle n_i, \bar{x}_1 \rangle - c_i \right) = -(1 - \theta_1) \left( \langle n_i, \bar{x}_2 \rangle - c_i \right).
\]
(A.4)

Since \( \bar{x}_1, \bar{x}_2 \in \mathcal{C} \), one has \( \langle n_i, \bar{x}_1 \rangle - c_i \leq 0 \) and \( \langle n_i, \bar{x}_2 \rangle - c_i \leq 0 \). Therefore, formula (A.4) can only hold when both \( \langle n_i, \bar{x}_1 \rangle - c_i \) and \( \langle n_i, \bar{x}_2 \rangle - c_i \) vanish. Hence
\[
\langle n_i, x_2 \rangle = \theta_2 \langle n_i, \bar{x}_1 \rangle + (1 - \theta_2) \langle n_i, \bar{x}_2 \rangle = \theta_2 c_i + (1 - \theta_2) c_i = c_i.
\]

The reverse implication in (A.2) can be proved analogously. \( \square \)

**Lemma A.2.** (see also [9], Section 6) Let \( \mathcal{C}_1, \mathcal{C}_2 \subset H \) be closed convex nonempty sets. Then
\[
d_H(\mathcal{C}_1, \mathcal{C}_2) = \sup_{y \in H : \|y\| = 1} |\delta^*_C(y) - \delta^*_C(y)|,
\]
where \( \delta^*_C \) is the support function of a set \( \mathcal{C} \):
\[
\delta^*_C(y) := \sup_{x \in \mathcal{C}} \langle y, x \rangle.
\]

We provide the following short proof for the completeness of the text.

**Proof.** Indeed,
\[
\sup \left\{ |\delta^*_C(y) - \delta^*_C(y)| : \|y\| = 1 \right\} = \inf \left\{ r \geq 0 : |\delta^*_C(y) - \delta^*_C(y)| \leq r \text{ for all } y \in H : \|y\| = 1 \right\} = \inf \left\{ r \geq 0 : -r \leq \delta^*_C(y) - \delta^*_C(y) \leq r \text{ for all } y \in H : \|y\| = 1 \right\} = \inf \left\{ r \geq 0 : \delta^*_C(y) \leq \delta^*_C(y) + r \text{ and } \delta^*_C(y) \leq \delta^*_C(y) + r \text{ for all } y \in H : \|y\| = 1 \right\} = \inf \left\{ r \geq 0 : \delta^*_C(y) \leq \delta^*_C(y) + r \text{ and } \delta^*_C(y) \leq \delta^*_C(y) + r \text{ for all } y \in H : \|y\| = 1 \right\} = d_H(\mathcal{C}_1, \mathcal{C}_2),
\]
where the last equality is from Definition 2.3. \( \square \)

**Lemma A.3.** A bounded moving polyhedron (2.11) with Lipshitz-continuous functions \( c_i(t) \) is Lipschitz-continuous with respect to the Hausdorff distance.

**Proof.** By Lemma A.2 we have for \( 0 \leq t_1 < t_2 \)
\[
d_H(C(t_1), C(t_2)) = \sup_{\|y\| \leq 1} |\delta^*_{C(t_1)}(y) - \delta^*_{C(t_2)}(y)|,
\]
(A.5)
where $\delta_{C(t)}^*(y) = \sup_{x \in C(t)} \langle y, x \rangle$ is the support function of the set $C(t)$. Since $C(t)$ is a convex polyhedron, the supremum of $\delta_{C(t)}^*$ must be attained at one of its vertices (see [20], p. 330):

$$\delta_{C(t)}^*(y) = \max_{v \in V_{C(t)}} \langle y, v \rangle,$$

where $V_{C(t)}$ is the finite set of vertices of $C(t)$.

Thus (A.5) leads to

$$d_H(C(t_1), C(t_2)) = \sup_{\|y\| \leq 1} \left| \max_{v \in V_{C(t_1)}} \langle y, v \rangle - \max_{v \in V_{C(t_2)}} \langle y, v \rangle \right|. \quad (A.6)$$

Consider the case when $\max_{v \in V_{C(t_1)}} \langle y, v \rangle \geq \max_{v \in V_{C(t_2)}} \langle y, v \rangle$. Then, using the notation of Remark 2.20

$$\left| \max_{v \in V_{C(t_1)}} \langle y, v \rangle - \max_{v \in V_{C(t_2)}} \langle y, v \rangle \right| \leq \max_{\mathcal{I} \subset I} \langle y, v_\mathcal{I}(t_1) \rangle - \max_{\mathcal{I} \subset I} \langle y, v_\mathcal{I}(t_2) \rangle \leq \max_{\mathcal{I} \subset I} \langle y, v_\mathcal{I}(t_1) \rangle - \max_{\mathcal{I} \subset I} \langle y, v_\mathcal{I}(t_2) \rangle$$

because the maxima are now taken over a bigger set. Similarly, in the case of $\max_{v \in V_{C(t_2)}} \langle y, v \rangle \geq \max_{v \in V_{C(t_1)}} \langle y, v \rangle$,

$$\left| \max_{v \in V_{C(t_1)}} \langle y, v \rangle - \max_{v \in V_{C(t_2)}} \langle y, v \rangle \right| \leq \max_{\mathcal{I} \subset I} \langle y, v_\mathcal{I}(t_1) \rangle - \max_{\mathcal{I} \subset I} \langle y, v_\mathcal{I}(t_2) \rangle.$$

Hence, (A.6) implies

$$d_H(C(t_1), C(t_2)) \leq \sup_{\|y\| \leq 1} \left| \max_{\mathcal{I} \subset I} \langle y, v_\mathcal{I}(t_1) \rangle - \max_{\mathcal{I} \subset I} \langle y, v_\mathcal{I}(t_2) \rangle \right| \leq \sup_{\|y\| \leq 1} \max_{\mathcal{I} \subset I} \|y\| \|B_{\mathcal{I}^{-1}} c_\mathcal{I}(t_1) - B_{\mathcal{I}^{-1}} c_\mathcal{I}(t_2)\| \leq \max_{\mathcal{I} \subset I} \|B_{\mathcal{I}^{-1}}\| L_v |t_1 - t_2|,$$

where $L_v$ is the Lipschitz constant of $c_\mathcal{I}$.

\[\square\]

**Lemma A.4.** If an absolutely continuous function $V : [\alpha, \beta] \rightarrow \mathbb{R}_0^+$ is such that $V(\alpha) > 0$ and for some fixed $\varepsilon$ it satisfies

$$\dot{V}(t) + 2\varepsilon \sqrt{V(t)} \leq 0 \quad (A.7)$$
for a.a. \( t \in [\alpha, \beta] \) then \( V \) is a non-increasing function, for which

a) if \( \sqrt{V(\alpha)} \geq \varepsilon(\beta - \alpha) \) then \( \sqrt{V(\beta)} \leq \sqrt{V(\alpha)} - \varepsilon(\beta - \alpha) \),
b) if \( \sqrt{V(\alpha)} < \varepsilon(\beta - \alpha) \) then there is some \( t^* \in (\alpha, \beta) \) such that for all \( t \in [t^*, \beta] \) we have \( V(t) = 0 \).

**Proof.** Assume that \( V(t) > 0 \) for all \( t \in [\alpha, \beta] \). Then, integrating the expression \( \frac{\dot{V}(t)}{\sqrt{V(t)}} \) one gets

\[
\int_{\alpha}^{\beta} \frac{\dot{V}(t)}{\sqrt{V(t)}} \, dt = \int_{\alpha}^{\beta} \frac{dV(t)}{\sqrt{V(t)}} = 2 \left( \sqrt{V(\beta)} - \sqrt{V(\alpha)} \right).
\]

On the other hand, we have from (A.7)

\[
\int_{\alpha}^{\beta} \frac{\dot{V}(t)}{\sqrt{V(t)}} \, dt \leq \int_{\alpha}^{\beta} (-2\varepsilon) \, dt = -2\varepsilon(\beta - \alpha).
\]

Combine the above to get

\[
\sqrt{V(\beta)} - \sqrt{V(\alpha)} \leq -\varepsilon(\beta - \alpha)
\]

which proves a). If \( \sqrt{V(\alpha)} < \varepsilon(\beta - \alpha) \) then we have a contradiction with the assumption in the beginning of the proof and there is some \( t^* \in (\alpha, \beta) \) s.t. \( V(t^*) = 0 \). Since (A.7) implies that \( \dot{V} \leq 0 \) a.e., we have \( V(t) = 0 \) for \( t \in [t^*, \beta] \), which is b). \( \square \)

**Lemma A.5.** Let \( A \subset B \) be two closed convex nonempty sets in a Hilbert space \( H \). Assume that there are \( x, z \in A \) and \( w \subset H \) s.t.

\[
w \in N_B(x), \tag{A.8}
w \in N_A(z). \tag{A.9}
\]

Then

\[
w \in N_B(z). \tag{A.10}
\]

**Proof.** Notice that the conclusion (A.10) is equivalent to \( z = \text{proj}(w + z, B) \). Assume the contrary, there is

\[
c = \text{proj}(w + z, B), \quad c \neq z.
\]

Since (A.9) is equivalent to \( z = \text{proj}(w + z, A) \) and \( A \subset B \) we must have

\[
\|w + z - c\| < \|w + z - z\| = \|w\|,
\langle w + z - c, w + z - c \rangle < \langle w, w \rangle,
2\langle w, z - c \rangle + \langle z - c, z - c \rangle < 0,
\langle w, z - c \rangle < 0.
\]
But from the definition of the normal cone and (A.8), (A.9) we know that
\[ \langle w, c - x \rangle \leq 0, \quad \langle w, x - z \rangle \leq 0, \quad \langle w, c - z \rangle \leq 0, \quad \langle w, z - c \rangle \geq 0. \]
The contradiction completes the proof. \[\square\]

**Lemma A.6.** Consider \( f : E_1 \to E_2 \) and \( g : \mathbb{R} \to E_1 \), where \( E_1, E_2 \) are inner product spaces. If both \( g'(t_0) \) and \( (f \circ g)'(t_0) \) exist and if \( f \) is Lipschitz continuous in the neighborhood of \( g_0 = g(t_0) \), then the directional derivative \( \frac{d}{ds}|_{s=0} f(g_0 + sg'(t_0)) \) exists and
\[ \frac{d}{ds}|_{s=0} f(g_0 + sg'(t_0)) = (f \circ g)'(t_0). \]

**Proof.** We have
\[
\frac{d}{ds}|_{s=0} f(g_0 + sg'(t_0)) = \lim_{s \to 0} \frac{f(g_0 + sg'(t_0)) - f(g_0)}{s}
= \lim_{s \to 0} \left( \frac{f(g(t_0) + sg'(t_0)) - f(g(t_0 + s))}{s} + \frac{f(g(t_0 + s)) - f(g_0)}{s} \right) = (f \circ g)'(t_0),
\]
where we used the Lipschitz continuity of \( f \) to conclude that the first fraction in the limit converges to 0 as \( s \to 0 \). \[\square\]

**Lemma A.7.** In the setting of a \( d \)-dimensional Euclidean space \( E \) let us have a function \( u \in L^1([0,T],E) \) and a collection of nonzero vectors \( n_i, i \in \overline{1,k} \). Let \( T \subset [0,T] \) be a Lebesgue-measurable subset and
\[ u(t) \in \text{cone} \{ n_i : i \in \overline{1,k} \} \quad \text{for a.a. } t \in T. \quad (A.11) \]
Then there is a function \( \lambda \in L^1([0,T],\mathbb{R}^k) \) such that
\[ u(t) = \sum_{i=1}^{k} \lambda_i(t)n_i \]
and
\[ \lambda_i(t) \geq 0 \quad \text{for all } i \in \overline{1,k} \text{ a.e. on } [0,T], \]
\[ \lambda(t) = 0 \quad \text{a.e. on } [0,T] \setminus T. \]

**Proof.** From Lemma 6.5.5 in [33] we know that cone \( \{ n_i : i \in \overline{1,k} \} \) can be represented as a finite union of cones with linearly independent generating sets:
\[ \text{cone} \{ n_i : i \in \overline{1,k} \} = \bigcup_{i=1}^{l} \text{cone} \{ n_j : j \in J_i \}, \quad \text{for some } l \in \mathbb{N}, J_i \subset \overline{1,k}, i \in \overline{1,l}, \]
\{n_j : j \in J_i\} are linearly independent collections.

Moreover, since the relative boundary of each cone \(\{n_j : j \in J_i\}\) consists of the points, which are members of some other cone \(\{n_j : j \in J'\}\), \(J' \subset J_i\), we can write without loss of generality

\[
\text{cone}\{n_i : i \in \overline{1,k}\} = \{0\} \cup \bigcup_{i=1}^{l} \text{ri cone} \{n_j : j \in J_i\},
\]  

(A.12)

where the sets \(\text{ri cone} \{n_j : j \in J_i\}\) are pairwise disjoint and Borel (as a result of substraction between closed sets).

Consider

\[
T_0 := \{t \in T : u(t) = 0\} = u^{-1}(0),
\]

\[
T_i := \{t \in T : u(t) \in \text{ri cone} \{n_j : j \in J_i\}\} = u^{-1}(\text{ri cone} \{n_j : j \in J_i\}),
\]

which are measurable sets (defined up to a part of measure zero). Moreover, because of (A.11) and (A.12) we have \(T_i\) pairwise disjoint and

\[
T = \bigcup_{i=0}^{l} T_i.
\]

As in Remark 2.20, we define the bounded linear maps

\[
B^*_i : R^{|J_i|} \to \text{lin}(\{n_j : j \in J_i\}) \subset E,
\]

\[
B^*_i : \lambda \mapsto \sum_{j \in J_i} \lambda_j n_j,
\]

which are, in our case, invertible. For each \(i \in \overline{1,l}\), \(t \in T_i\) we can uniquely put

\[
\lambda(t) = E_{J_i} (B^*_{J_i})^{-1} u(t) \in \mathbb{R}^k,
\]

where \(E_{J_i}\) is simply the embedding \(E_{J_i} : \mathbb{R}^{|J_i|} \to \mathbb{R}^k\) according to the indices \(J_i\).

Since each collection \(\{n_j : j \in J_i\}\) is linearly independent, \(B^*_{J_i}\lambda(t)\) is a unique way to represent \(u(t)\) via the collection’s members at \(t \in T_i\), therefore by construction of \(T_i\) all components of \(\lambda(t)\) are non-negative.

In turn, for \(t \in [0,T] \setminus \bigcup_{i=1}^{l} T_i\),

\[
\lambda(t) = 0 \in \mathbb{R}^k.
\]

Now we demonstrate that \(\lambda : [0,T] \to \mathbb{R}^k\) is a measurable function. Let \(B \subset \mathbb{R}^k\) be a Borel set. For each \(i \in \overline{1,l}\) the set

\[
\{t \in T_i : E_i (B^*_{J_i})^{-1} u(t) \in B\}
\]
is Lebesgue-measurable, as well as $\mathcal{T}_0 \cup ([0,T] \setminus \mathcal{T})$ in the case where $0 \in B$, thus $\{ t \in [0,T] : \lambda(t) \in B \}$ is Lebesgue-measurable as well. Finally, observe, that a.e. on $[0,T]$, 

$$\|\lambda(t)\| \leq \max_{i \in \mathcal{T}\lambda} \|E_i (B_{J_i}^*)^{-1}\| \|u(t)\|,$$

hence from $u \in L^1([0,T],E)$ we conclude that $\lambda \in L^1([0,T],\mathbb{R}^k)$.

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References


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