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NECESSARY CONDITIONS FOR LOCAL CONTROLLABILITY OF A PARTICULAR CLASS OF SYSTEMS WITH TWO SCALAR CONTROLS

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Abstract. We consider control-affine systems with two scalar controls, such that one control vector field vanishes at an equilibrium state. We state two necessary conditions for local controllability around this equilibrium, involving the iterated Lie brackets of the system vector fields, with controls that are either bounded, small in L^∞ or small in $W^{1,\infty}$. These results are illustrated with several examples.

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1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. Let n, m in $\mathbb{N} \setminus \{0\}$ and consider a control-affine system with m controls:

$$\dot{z} = f_0(z) + \sum_{k=1}^{m} u_k(t)\, f_k(z)\,, \tag{1.1}$$

where z is the state in $\mathbb{R}^n$, $f_0, \dots, f_m$ are real analytic vector fields, and $u = (u_1, \dots, u_m)$, called the control, is assumed to be an integrable function $[0, T] \to \mathbb{R}^m$, for some $T > 0$[1]. This defines a time-varying ordinary differential equation and hence, for each choice of initial condition $z(0)$, a unique solution $t \mapsto z(t)$ on $[0, T]$, or a smaller interval $[0, T')$ in case of finite time blow-up. This system is called *controllable* if, for any two points z_0 and z_1 in the state space $\mathbb{R}^n$, or possibly a subset of it, there exist a time T and an integrable control on $[0, T]$ such that the above-mentioned solution with $z(0) = z_0$ satisfies $z(T) = z_1$. See textbooks like [6, 15, 21, 24] for

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[1] Sometimes, in the literature, one also adds some constraints on the control, *e.g.* $u : [0, T] \to \mathcal{U}$, where $\mathcal{U}$ is a convex compact subset of $\mathbb{R}^m$, see *e.g.* [24].

further precisions on controllability. We are interested in *local* controllability around an equilibrium $(z^{\mathrm{eq}}, u^{\mathrm{eq}})$:

$$f_0(z^{\mathrm{eq}}) + \sum_{k=1}^{m} u_k^{\mathrm{eq}} f_k(z^{\mathrm{eq}}) = 0\,. \tag{1.2}$$

System (1.1) is locally controllable around $(z^{\mathrm{eq}}, u^{\mathrm{eq}})$ if the above mentioned property occurs for z_0 and z_1 in arbitrarily small neighborhoods of z^{eq}, with solutions that remain in arbitrarily small neighborhoods of z^{eq}, in arbitrarily small time T, and with a control that takes values in a fixed neighborhood of u^{eq} or an arbitrarily small neighborhood of u^{eq}, or with a control that is arbitrarily close to the constant control $t \mapsto u^{\mathrm{eq}}$ in some functional space. We detail in Section 2 these various notions of local controllability.

Some (now classical for some of them) sufficient conditions [13, 14, 28] and necessary conditions [2, 3, 16, 20, 26, 28] are given in the literature in terms of finite jets of the vector fields. More precisely, these conditions allow one to decide controllability or non-controllability based on the value of a finite number of Lie brackets at z^{eq}. It is intriguing that there is a significantly wide gap between the necessary and the sufficient conditions; as pointed out in [1], it is even not clear whether or not, for a general system in this gap, local controllability depends on a finite jet of the vector fields or not.

General sufficient conditions are available [28] for m larger than 1, but the literature on necessary conditions focuses on systems with a single scalar control ($m = 1$). This paper is specifically concerned with control systems with two scalar inputs, of the form

$$\dot{z} = f_0(z) + u_1\, f_1(z) + u_2\, f_2(z)\,, \tag{1.3}$$

where the real analytic vector fields f_0, f_1, f_2 are such that f_0 and f_2 vanish at the equilibrium while f_1 does not:

$$f_0(z^{\mathrm{eq}}) = 0,\ \ f_2(z^{\mathrm{eq}}) = 0,\ \ f_1(z^{\mathrm{eq}}) \neq 0\,. \tag{1.4}$$

Such systems have two controls but the effect of one of them vanishes at the point of interest. In a sense, the contribution of this paper is to study to what extent the second control helps controllability or, conversely, to what extent obstructions to controllability of the single input system $\dot{z} = f_0(z) + u_1\, f_1(z)$ carry over when the second control u_2 is turned on.

Studying this very situation stemmed out of previous work from the authors on the controllability of magnetic micro-swimmers [10, 11, 22]. See these references for a description of these devices and their interest, for instance in micro-robotics and biomedical applications. The corresponding control systems are particular cases of (1.3)-(1.4), for which the authors have proved various controllability and non-controllability results. We believe that a more general treatment of systems of type (1.3)-(1.4), beyond the case of magnetic micro-swimmers, is of interest to the controllability problem in control theory. It is the purpose of the present paper.

The paper is structured as follows. Section 2 is devoted to precise definitions of various notions of local controllability and to recalling known controllability conditions for single-input systems. Our two main results are presented in Section 3. Section 4 illustrates the results with several examples. Section 5 is dedicated to the proofs. Finally, conclusions as well as some perspectives on further research are provided in Section 6.

2. Problem statement

2.1. Various notions of local controllability

The following definition is taken from the textbook by Coron [6], Definition 3.2, p. 125, that we modify by taking two independent small positive quantities ε and ε' while Coron imposed $\varepsilon = \varepsilon'$; as explained in Remark 2.6, the two definitions are equivalent. The notation $B(z, \eta)$ stands for the open ball with center z and radius $\eta > 0$ for some Euclidean norm on $\mathbb{R}^n$, whose choice is immaterial; L^∞ is the usual functional space, where the L^∞ norm also assumes that some Euclidean norm on $\mathbb{R}^m$ has been chosen.

Definition 2.1 (STLC)**.** The control system (1.1) is STLC at the equilibrium $(z^{\text{eq}}, u^{\text{eq}})$ if, for every $\varepsilon > 0$, $\varepsilon' > 0$, there exists $\eta > 0$ such that, for every z_0, z_1 in $B(z^{\text{eq}}, \eta)$, there exists a control $u(\cdot)$ in $\mathrm{L}^\infty([0,\varepsilon], \mathbb{R}^m)$ such that the solution $z(\cdot) : [0,\varepsilon] \to \mathbb{R}^n$ of the control system (1.1) with initial condition $z(0) = z_0$ satisfies $z(\varepsilon) = z_1$, and

$$\|u - u^{\text{eq}}\|_{\mathrm{L}^\infty([0,\varepsilon],\mathbb{R}^m)} \leqslant \varepsilon'.$$

The acronym STLC stands for *small time locally controllable.* Coron notes that it should rather be called small time locally controllable *with controls close to* u^{eq}. Historically, in the first classical papers on local controllability (*e.g.* [14, 27]), one fixes a bounded neighborhood of u^{eq} in $\mathbb{R}^m$ and a system is locally controllable if, for any time T, there is a neighborhood of z^{eq} in which any two points may be joined in time T using controls with values in the fixed bounded neighborhood of u^{eq}, *i.e.* $\|u - u^{\text{eq}}\|_{\mathrm{L}^\infty([0,\varepsilon],\mathbb{R}^m)}$ was not required to be small but only bounded, and even bounded by a number that was fixed from the definition of the system; this notion was the one initially called STLC, while STLC in the sense of Definition 2.1 was rather called *small time local controllability with small controls* [18] (small refers to the distance to u^{eq}). That was a very natural terminology, but these terms drifted, and Definition 2.1 now prevails. In the present paper, we define the notion of B-STLC (the prefix B stands for bounded control):

Definition 2.2 (B-STLC)**.** The control system (1.1) is *B-STLC at* $(z^{\text{eq}}, u^{\text{eq}})$ if there exists $\alpha > 0$ such that, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every z_0, z_1 in $B(z^{\text{eq}}, \eta)$, there exists a control $u(\cdot)$ in $\mathrm{L}^\infty([0,\varepsilon], \mathbb{R}^m)$ such that the solution $z(\cdot) : [0,\varepsilon] \to \mathbb{R}^n$ of the control system (1.1) with initial condition $z(0) = z_0$ satisfies $z(\varepsilon) = z_1$, and

$$\|u - u^{\text{eq}}\|_{\mathrm{L}^\infty([0,\varepsilon],\mathbb{R}^m)} \leqslant \alpha.$$

Remark 2.3. STLC implies B-STLC, choosing any $\alpha > 0$.

Remark 2.4. B-STLC is close to the "historical" STLC property mentioned above, the difference being that historically, a compact set was fixed *a priori* as a control constraint; forcing $\alpha = 1$ would be in the same spirit. In B-STLC, we leave α free. For a given control system that is B-STLC, let $\alpha_0 \geq 0$ be the infimum of the possible values of α in Definition 2.2. In [10, 11], the authors defined a quantitative version of B-STLC and called such a system "α_0-STLC". The system is STLC if and only if $\alpha_0 = 0$. If $\alpha_0 > 0$, one sees that a bound on the control less than α_0 makes the system non controllable, meaning that "controllability depends on the bound on the control" as illustrated *e.g.* in [10, 11], and also in [19]. We shall no longer use "α-STLC" in the sequel; it is replaced by B-STLC, that can be understood as "α-STLC for some α".

More recently, a new notion has been introduced by Beauchard and Marbach in [2], Definition 4. The idea is to ensure the smallness, not only of $u - u^{\text{eq}}$, but also of some derivatives, by requiring its norm to be bounded in Sobolev spaces. For I an interval and k a nonnegative integer, we recall that a function $f : I \to \mathbb{R}$ belongs to $\mathrm{W}^{k,\infty}(I)$ if, for all $p \in \{0, \dots, k\}$, $f^{(p)} \in \mathrm{L}^\infty(I)$, and $\mathrm{W}^{k,\infty}(I)$ is then endowed with the following norm

$$\|f\|_{\mathrm{W}^{k,\infty}} = \max_{p \in \{0,\dots k\}} \|f^{(p)}\|_{\mathrm{L}^\infty(I)},$$

that makes it a Banach space. Obviously, $\mathrm{W}^{0,\infty}(I) = \mathrm{L}^\infty(I)$.

Definition 2.5 ($\mathrm{W}^{k,\infty}$-STLC)**.** Let $k \in \mathbb{N}$. The control system (1.1) is $\mathrm{W}^{k,\infty}$*-STLC at* $(z^{\text{eq}}, u^{\text{eq}})$ if, for every $\varepsilon > 0$, $\varepsilon' > 0$, there exists $\eta > 0$ such that, for every z_0, z_1 in $B(z^{\text{eq}}, \eta)$, there exists a control $u(\cdot)$ in $\mathrm{W}^{k,\infty}([0,\varepsilon], \mathbb{R}^m)$ such that the solution $z(\cdot) : [0,\varepsilon] \to \mathbb{R}^n$ of the control system (1.1) with initial condition $z(0) = z_0$ satisfies

$z(\varepsilon) = z_1$, and

$$\|u - u^{\mathrm{eq}}\|_{\mathrm{W}^{k,\infty}([0,\varepsilon],\mathbb{R}^m)} \leqslant \varepsilon' .$$

In particular, $\mathrm{W}^{0,\infty}$-STLC is identical to STLC as introduced in Definition 2.1, while, when $k > 0$, $\mathrm{W}^{k,\infty}$-STLC is *stronger* than STLC, because it requires the control to be sufficiently smooth and the $\mathrm{W}^{k,\infty}$ norm is stronger than the L^∞ one; we shall indeed encounter systems that are L^∞-STLC, but not $\mathrm{W}^{1,\infty}$-STLC.

Remark 2.6 (on the two small positive parameters ε and ε')**.** Using these two distinct parameters to characterize smallness of time and smallness of the control in Definitions 2.1 and 2.5 (identical to definitions in *e.g.* [2]) seems quite natural. As mentioned at the very beginning of this section, the definition of STLC in Coron's textbook [6] amounts to imposing $\varepsilon = \varepsilon'$ in Definition 2.1; the resulting STLC property seems weaker at first sight but is indeed equivalent because one may adjust the controllability time by passing through the equilibrium and staying there during some arbitrary interval of time. This argument is valid because concatenating L^∞ controls on two adjacent intervals yields some L^∞ controls on the union, at no L^∞ cost.

When moving on to $\mathrm{W}^{k,\infty}$ controls, $k > 0$, no such stability by concatenation holds (continuity of the $k-1$ first derivatives fails in general), so that taking $\varepsilon = \varepsilon'$ in Definition 3.4 would yield a different (*a priori* weaker) property. We do not know whether our Theorem 3.8 would hold with that modified version of $\mathrm{W}^{1,\infty}$-STLC; the distinction is important in its proof, where we occasionally need to bound the $\mathrm{W}^{1,\infty}$ norm of the control with a parameter depending on the size of the time interval.

Stability by concatenation, and thus equivalence between Definition 2.5 and its counterpart imposing $\varepsilon = \varepsilon'$, could be recovered for $k > 0$ by considering controls in traceless spaces $\mathrm{W}_0^{k,\infty}([0,\varepsilon],\mathbb{R}^m)$, *i.e.* satisfying $u^{(p)}(0) = u^{(p)}(\varepsilon) = 0$ for $p = 0, \dots, k-1$. Such spaces (although requiring only $u^{(p)}(0) = 0$, and not $u^{(p)}(\varepsilon) = 0$) are briefly considered in [2] with regard to local controllability; they will not be further explored here.

Remark 2.7. Let us complement this discussion on the various notions of local controllability with a word on additional definitions where the "smallness" assumptions are shifted from the control to the state. In a recent paper, Boscain *et al.* [4] review and link several notions of local controllability, among which *small-time localized local controllability*, where the control is taken in L^∞, but without any boundedness assumption; the state, on the other hand, is constrained to remain within an arbitrarily small neighborhood of the equilibrium. A similar definition appears in [2], but with controls taken in L^1; the notion is then shown to be equivalent to $\mathrm{W}^{-1,\infty}$-STLC, which itself is an extension of Definition 2.5 where the $\mathrm{W}^{-1,\infty}$ norm of u is naturally taken as the L^∞ norm of the function $t \mapsto \int_0^t u$.

2.2. Notions and notations

If f and g are real analytic vector fields, their Lie bracket is denoted by $[f,g]$ (its definition is recalled at the beginning of Sect. 5.1). The "ad" operator is defined by $(\mathrm{ad} f)g = [f,g]$. The usual notation $\mathrm{ad}_f^k g$ is defined by induction with $\mathrm{ad}_f^0 g = g$ and $\mathrm{ad}_f^k g = [f, \mathrm{ad}_f^{k-1} g] = [f,[f,\cdots,[f,g],\cdots]]$; given $k \geqslant 1$ vector fields $f_1,\dots,f_k$, we use the following notation for iterated composition of $\mathrm{ad} f_i$, where one must respect the order since these operators do not commute:

$$\Big(\prod_{i=1}^{k} \mathrm{ad} f_i\Big)\, g = [f_1,[f_2,[f_3,\dots[f_k,g]\dots]]]\,. \tag{2.1}$$

The set of real analytic vector fields on $\mathbb{R}^n$ (or on an open neighborhood of the equilibrium under consideration) is a Lie algebra over $\mathbb{R}$, with multiplication given by the Lie bracket. Given a family of real analytic vector fields $\mathcal{F} = \{g_1,\dots,g_N\}$, we denote by $\mathrm{Lie}(\mathcal{F})$ the sub-algebra generated by these vector fields.

In order to correctly manipulate the Lie brackets of vector fields in $\mathcal{F}$ as algebraic objects, we then introduce the set of indeterminate symbols $\hat{\mathcal{F}} = \{\hat{g}_1,\dots,\hat{g}_N\}$ and we denote by $\mathfrak{L}(\hat{\mathcal{F}})$ the free Lie algebra of formal Lie brackets generated by the elements of $\hat{\mathcal{F}}$. The substitution of the symbols of $\hat{\mathcal{F}}$ with the associated vector fields in $\mathcal{F}$ is a natural (and usually non injective) morphism between $\mathfrak{L}(\hat{\mathcal{F}})$ and $\mathrm{Lie}(\mathcal{F})$.

This construction allows to unambiguously refer to the "order" of a Lie bracket as the number of elements in $\hat{\mathcal{F}}$ constituting it, before mapping it to $\mathrm{Lie}(\mathcal{F})$ where it could very well be equal to another vector field resulting from a formal bracket of different order. For instance, if g_1 and g_2 commute, $[\hat{g}_1, \hat{g}_2]$ is a nonzero element of order two of $\mathfrak{L}(\hat{\mathcal{F}})$, mapped to the zero vector field.

As such ambiguous statements do not play any important role in this paper, in the following, we will consistently identify these two objects and therefore "drop the hats" even when dealing with formal Lie brackets, implicitly referring to elements of $\mathfrak{L}(\hat{\mathcal{F}})$ wherever relevant – in particular, when counting the number of elements in a given Lie bracket.

Finally, we will use some condensed notations for some Lie brackets between the vector fields f_0, f_1 and f_2 defining the control systems (1.3) or (2.6) under examination: basically, f_{ij} stands for $[f_i, f_j]$, f_{ijk} for $[f_i, [f_j, f_k]]$, $f_{ij,kl}$ for $[f_{ij}, f_{kl}]$, and $f_{ij,klm}$ for $[f_{ij}, f_{klm}]$, and we will specifically use the following ones:

$$f_{01} = [f_0, f_1], \qquad f_{21} = [f_2, f_1], \tag{2.2}$$

$$f_{101} = [f_1, [f_0, f_1]], \qquad f_{121} = [f_1, [f_2, f_1]], \tag{2.3}$$

$$f_{21,01} = [[f_2, f_1], [f_0, f_1]], \tag{2.4}$$

$$f_{i1,jk1} = [[f_i, f_1], [f_j, [f_k, f_1]]]. \tag{2.5}$$

2.3. Known results for single-input systems

In this section, we set $m = 1$ in (1.1) and consider single-input control-affine systems:

$$\dot{z} = f_0(z) + u_1(t) f_1(z), \quad z \in \mathbb{R}^n \tag{2.6}$$

around the equilibrium $(z^{\mathrm{eq}}, u^{\mathrm{eq}}) = (0, 0)$, *i.e.* we assume $f_0(0) = 0$.

Definition 2.8. System (2.6) satisfies the Lie Algebra Rank Condition (LARC) at 0 if

$$\{g(0), g \in \mathrm{Lie}(f_0, f_1)\} = \mathbb{R}^n. \tag{2.7}$$

It is well-known [12, 23] (see also [27], Prop. 6.2) that, when dealing with analytic vector fields, the LARC is necessary for any form of STLC, but not sufficient. Stronger assumptions on the structure of Lie bracket spaces have to be made to obtain a sufficient condition. For $k \in \mathbb{N}$, let $\mathcal{S}_k$ be the set of all iterated Lie brackets of f_0 and f_1 in which f_1 appears at most k times, and S_k the subspace of $\mathbb{R}^n$ spanned by the value at 0 of the elements of $\mathcal{S}_k$ evaluated at 0. The following is a translation of [27], Theorem 2.1, the main result in that reference. As discussed between Definition 2.1 and Definition 2.2, STLC meant B-STLC at the time where [27] was published, so that a faithful translation of [27], Theorem 2.1 should be about B-STLC. However, the reader may check that the proof by Sussmann in [27] indeed yields small controls, and therefore proves STLC, not only in [27]'s sense, but in our sense too. We therefore write a statement[2] that is formally stronger than the one in [27], but still correct and in the spirit of [27].

Proposition 2.9 ([27], Thm. 2.1, p. 688)**.** *If System (2.6) satisfies the LARC at 0 and, for all k in $\mathbb{N}$,*

$$S_{2k+2} \subset S_{2k+1}, \tag{2.8}$$

then it is STLC.

[2] Coron does the same "abuse" when translating [28], Theorem 7.3 into [6], Theorem 3.29.

Condition (2.8) fails if for some $k \in \mathbb{N}$, some brackets in $\mathcal{S}_{2k+2}$, once evaluated at 0, do not belong to S_{2k+1}. These clearly prevent one from applying Proposition 2.9, but have no reason to be obstructions to controllability. It is however the case of some very specific such brackets. For instance, the lowest-order possible bracket in $\mathcal{S}_2$ is f_{101} (defined in (2.3)); Sussmann naturally studied this case and proved the following non-controllability result[3]:

Proposition 2.10 ([27], Prop. 6.3, p. 707)**.** *If $f_{101}(0) \notin S_1$, then System (2.6) is not B-STLC.*

This result has been extended to arbitrary values of the integer k by Stefani in [25], showing that the simplest bracket in $\mathcal{S}_{2k+2}$, namely $\mathrm{ad}_{f_1}^{2k} f_0$ (note that, for $k = 0$, $f_{101} = -\mathrm{ad}_{f_1}^{2k} f_0$), provides an obstruction:

Proposition 2.11 ([25])**.** *Assume that there exists k in $\mathbb{N}$ such that*

$$\mathrm{ad}_{f_1}^{2k+2} f_0(0) \notin S_{2k+1}.$$

Then, System (2.6) is not B-STLC.

The proof of these obstructions always consist in building a function of the state, in general one coordinate of a well chosen system of coordinates, that will be non-decreasing along any solution, irrespective of the control, at least close to equilibrium point and for small enough controls. In the cases above, the construction is possible due to a single bracket, that was subsequently called a "bad bracket". However, brackets that make Condition (2.8) fail are not, in general, obstructions to controllability and should not be called "bad brackets", even though they may have been called so in some early literature trying to generalize the two propositions above. In fact, as evidenced by an example by Kawski [17], Example 2.5.1, it is even not a good general idea to look for obstructions through single brackets having such or such properties; obstructions are more to be found through more complex objects involving the image in $\mathrm{Lie}(f_0, f_1)$ of the whole homogeneous component of the free Lie algebra $\mathfrak{L}(\hat{f}_0, \hat{f}_1)$ (see Sect. 2.1). In the present paper, the introduction of the quadratic form $D_{u_2^{\mathrm{eq}}}$ (see (3.6)) in Section 3.2 is an illustration of such an obstruction that cannot be expressed in terms of a single bracket, whereas the results in Section 3.1, that is a generalisation of Proposition 2.10 to our class of two-input systems, still relies on the single bracket f_{101} and the possibility to "cancel" it through brackets involving the second control vector field.

Let us review two more results establishing local controllability obstructions, both in the case where (2.8) fails for $k = 0$. Returning to $k = 0$, assume now that the hypothesis of Proposition 2.10 is not satisfied, *i.e.* $f_{101}(0)$ does belong to S_1. As explained for instance in [27], the next lowest-order bracket in $\mathcal{S}_2$ that can be an obstruction to controllability is $f_{01,001}$ (defined in (2.5)).

It is however noticed by H. Sussmann in [27], p. 710 (a similar counter-example is recalled as (4.5) at the beginning of Sect. 4.2) that $f_{01,001}(0) \notin S_1$ is *not* an obstruction to STLC as defined in Definition 2.1. In [16], Kawski obtained a new necessary condition by refining the space $\mathcal{S}_1$:

Proposition 2.12 ([16])**.** *Let $\mathcal{S}' = \{\mathrm{ad}_{f_0}^k(\mathrm{ad}_{f_1}^3 f_0), k \in \mathbb{N}\}$ and S' the vector subspace of $\mathbb{R}^n$ spanned by the value at 0 of the elements of $\mathcal{S}'$.*

If $f_{01,001}(0) \notin S_1 + S'$, then System (2.6) is not STLC.

More recently, the authors of [2] showed that $f_{01,001}(0) \notin S_1$ *is indeed* an obstruction, but to a stronger notion of local controllability that we just recalled in Definition 2.5.

Proposition 2.13 ([2], Thm. 3)**.** *If $f_{01,001}(0) \notin S_1$, then System (2.6) is not $\mathrm{W}^{1,\infty}$-STLC.*

Concerning systems with control in $\mathbb{R}^m, m \geqslant 2$, a general sufficient condition for local controllability, in the vein of Proposition 2.9 but more complex, can be found in [28], but no necessary condition is known, to the best of our knowledge. The main results of this paper, stated in the next section, are a step in this direction

[3]The original result assumes that $|u| \leqslant 1$, see the lines after Definition 2.2; Proposition 2.10 is hence slightly stronger than the result in [27], but follows from the same proof.

in the sense that they give an extension of the necessary conditions contained in Propositions 2.10, 2.12 and 2.13 to the case where the system has two scalar controls, and the vector field associated to the second control vanishes at the equilibrium.

3. Main results

We now consider the control-affine system (1.3) (which is also system (1.1) with $m = 2$), assuming that (1.4) is satisfied at $z^{\mathrm{eq}} = 0$, *i.e.* $f_0(0) = 0$, $f_2(0) = 0$, $f_1(0) \neq 0$, and we study local controllability for $(z, (u_1, u_2))$ close to the equilibria $(0, (0, u_2^{\mathrm{eq}}))$, with u_2^{eq} arbitrary.

Remark 3.1. The fact, due to (1.4), that *any* value of u_2^{eq} is an equilibrium value of the control for the system (1.3) is somehow unusual. Of course, once some value of u_2^{eq} is set, one can always work around the null equilibrium $(0, (0, 0))$ by performing the following affine feedback transformation on the control u_2: $\tilde{u}_2 = u_2 - u_2^{\mathrm{eq}}$. With this transformed control, system (1.3) becomes

$$\dot{z} = \tilde{f}_0(z) + u_1 \tilde{f}_1(z) + \tilde{u}_2 \tilde{f}_2(z) \tag{3.1}$$

with $\tilde{f}_0 = f_0 + u_2^{\mathrm{eq}} f_2$, $\tilde{f}_1 = f_1$ and $\tilde{f}_2 = f_2$. Assume that system (3.1) is STLC at $(0, (0, 0))$ and let ε be a positive real number, η be the associated parameter from Definition 2.1, and z_0, z_1 in $B(0, \eta)$. There exist controls u_1 and $\tilde{u}_2$ in $\mathrm{L}^\infty([0, \varepsilon])$ such that the solution of (3.1) with $z(0) = z_0$ and these controls satisfy $z(\varepsilon) = z_1$, and

$$\|u_1\|_{\mathrm{L}^\infty([0,\varepsilon],\mathbb{R})} \leqslant \varepsilon\,, \|\tilde{u}_2\|_{\mathrm{L}^\infty([0,\varepsilon],\mathbb{R})} \leqslant \varepsilon.$$

Hence, the solution of system (1.3) with $z(0) = z_0$ and controls $u_2 = u_2^{\mathrm{eq}} + \tilde{u}_2$ and u_1 satisfies $z(\varepsilon) = z_1$, and we also have $\|u_2 - u_2^{\mathrm{eq}}\|_{\mathrm{L}^\infty([0,\varepsilon],\mathbb{R})} \leqslant \varepsilon$ and $\|u_1\|_{\mathrm{L}^\infty([0,\varepsilon],\mathbb{R})} \leqslant \varepsilon$. Therefore, system (3.1) is STLC at $(0, (0, 0))$ if and only if system (1.3) is STLC at $(0, (0, u_2^{\mathrm{eq}}))$; this also holds for other types of local controllability.

Moreover, unlike STLC, the notion of B-STLC (Def. 2.2) is independent of u_2^{eq}: indeed, choosing α sufficiently large, it is easy to see that system (1.1) is B-STLC at *one* equilibrium $(0, (0, u_2^{\mathrm{eq}}))$ if and only if it is B-STLC for *all* equilibria $(0, (0, u_2^{\mathrm{eq}}))$.

Nevertheless, a key feature of our main results is that some particular values of u_2^{eq} (like the one called β in Thm. 3.2) play an important role for recovering local controllability. Therefore, we choose to state Theorem 3.2 and 3.8 without readily taking $u_2^{\mathrm{eq}} = 0$, while the proofs are conducted with $u_2^{\mathrm{eq}} = 0$ using the feedback transformation described above.

3.1. Obstruction coming from brackets of order 3

Here, we give a necessary condition that "generalizes" Proposition 2.10 for single-input systems to the class (1.1)–(1.4) of two-input systems. The bracket that carries the obstruction is again f_{101} but now, f_{121} also plays a role as it could compensate the bracket f_{101}. The family $\mathcal{S}_1$ from Proposition 2.10 has to be replaced by a larger family $\mathcal{R}_1$, made of all iterated Lie brackets of f_0, f_1 and f_2 where f_1 appears at most one time. We denote by R_1 the vector subspace of $\mathbb{R}^n$ spanned by the value at 0 of all elements of $\mathcal{R}_1$.

The necessary condition is the following.

Theorem 3.2. *Consider System (1.3) under Assumption (1.4). Assume $f_{101}(0) \notin R_1$.*

1. *If $f_{101}(0) \in R_1 + \mathrm{Span}(f_{121}(0))$, let $\beta \in \mathbb{R}$ be such that*

$$f_{101}(0) + \beta f_{121}(0) \in R_1. \tag{3.2}$$

Then, for any $u_2^{\mathrm{eq}} \in \mathbb{R}$ such that $u_2^{\mathrm{eq}} \neq \beta$, system (1.3) is not STLC at $(0, (0, u_2^{\mathrm{eq}}))$.

2. *If $f_{101}(0) \notin R_1 + \mathrm{Span}(f_{121}(0))$, then, for any $u_2^{\mathrm{eq}} \in \mathbb{R}$, system (1.3) is not B-STLC at $(0, (0, u_2^{\mathrm{eq}}))$.*

Case 2. is the one where the obstruction from Proposition 2.10 (applied to the single-input system obtained by taking $u_2 = 0$) persists, *i.e.* the second control cannot improve controllability. In Case 1., on the contrary, Relation (3.2) allows the bracket f_{121} to possibly compensate f_{101} around the particular control $u_2^{\mathrm{eq}} = \beta$; the theorem states that this value of the control is the only one around which System (1.3) may be STLC.

Remark 3.3. In Case 1., a careful inspection of the proof leads to a slightly stronger result: there is an obstruction to a weaker controllability property where the control u_1 is not required to be arbitrary small; this hybrid property could be called (B, L^∞)-STLC in the spirit of Definition 3.5 (see next section).

3.2. Obstruction coming from brackets of order 5

We now assume that the obstruction to STLC pointed out by Theorem 3.2 does not hold, and continue exploring possible obstructions to STLC. The result we give in this direction may be viewed as a "generalization" of Propositions 2.12-2.13, in the sense that it combines the point of view of Proposition 2.12 (enlarging the space S_1) and that of Proposition 2.13 (using a stronger form of STLC), and consequently identifies obstructions to various forms of local controllability, tailored to two-input systems of the type (1.3)-(1.4), that we define hereafter:

Definition 3.4. Let $k \in \mathbb{N}$. The control system (1.3) is $(\mathrm{W}^{k,\infty}, \mathrm{L}^\infty)$-STLC at $(z^{\mathrm{eq}}, (u_1^{\mathrm{eq}}, u_2^{\mathrm{eq}}))$ if, for every $\varepsilon > 0$, $\varepsilon' > 0$, there exists $\eta > 0$ such that, for every z_0, z_1 in $B(z^{\mathrm{eq}}, \eta)$, there exists a control $(u_1(\cdot), u_2(\cdot))$ in $\mathrm{W}^{k,\infty}([0,\varepsilon], \mathbb{R}) \times \mathrm{L}^\infty([0,\varepsilon], \mathbb{R})$ such that the solution $z(\cdot) : [0,\varepsilon] \to \mathbb{R}^n$ of the control system (1.3) with initial condition $z(0) = z_0$ satisfies $z(\varepsilon) = z_1$, and

$$\|u_1 - u_1^{\mathrm{eq}}\|_{\mathrm{W}^{k,\infty}([0,\varepsilon],\mathbb{R})} \leqslant \varepsilon',\ \|u_2 - u_2^{\mathrm{eq}}\|_{\mathrm{L}^\infty([0,\varepsilon],\mathbb{R})} \leqslant \varepsilon'. \tag{3.3}$$

Definition 3.5. Let $k \in \mathbb{N}$. The control system (1.3) is $(\mathrm{W}^{k,\infty}, B)$-STLC at $(z^{\mathrm{eq}}, (u_1^{\mathrm{eq}}, u_2^{\mathrm{eq}}))$ if there exists $\alpha > 0$ such that, for every $\varepsilon > 0$, $\varepsilon' > 0$, there exists $\eta > 0$ such that, for every z_0, z_1 in $B(z^{\mathrm{eq}}, \eta)$, there exists a control $(u_1(\cdot), u_2(\cdot))$ in $\mathrm{W}^{k,\infty}([0,\varepsilon], \mathbb{R}) \times \mathrm{L}^\infty([0,\varepsilon], \mathbb{R})$ such that the solution $z(\cdot) : [0,\varepsilon] \to \mathbb{R}^n$ of the control system (1.3) with initial condition $z(0) = z_0$ satisfies $z(\varepsilon) = z_1$, and

$$\|u_1 - u_1^{\mathrm{eq}}\|_{\mathrm{W}^{k,\infty}([0,\varepsilon],\mathbb{R})} \leqslant \varepsilon',\ \|u_2 - u_2^{\mathrm{eq}}\|_{\mathrm{L}^\infty([0,\varepsilon],\mathbb{R})} \leqslant \alpha. \tag{3.4}$$

Remark 3.6. The norms used for each control are different in this controllability notion, making it a form of "hybrid" small-time local controllability. It fits the nature of system (1.3), where the second control plays a particular role due to the fact that f_2 vanishes at 0.

Let us now introduce the elements of the obstruction. First of all, we assume that

$$f_{101}(0) \in R_1\,, \quad f_{121}(0) \in R_1\,, \quad f_{21,01}(0) \in R_1\,. \tag{3.5}$$

The first two points are introduced to step out of the obstruction given by Theorem 3.2; the case where a relation like (3.2) would hold is not considered. Finally, the bracket $f_{21,01}$, defined in (2.4), was not seen before; its role will be discussed below the statement of Theorem 3.8.

The obstruction *per se* is more complex than the one in Propositions 2.12-2.13: instead of coming from a single bracket $f_{01,001}$, it comes from eight different brackets of order 5; for a given value of the equilibrium control u_2^{eq}, it is convenient to define a map $D_{u_2^{\mathrm{eq}}} : \mathbb{R}^2 \to \mathbb{R}^n$ as follows:

$$\begin{aligned} D_{u_2^{\mathrm{eq}}}(\lambda_1, \lambda_2) = \quad & \lambda_1^2(f_{01,001}(0) - u_2^{\mathrm{eq}} f_{01,201}(0)) + \lambda_2^2(f_{21,021}(0) - u_2^{\mathrm{eq}} f_{21,221}(0)) \\ & - \lambda_1\lambda_2(f_{21,001}(0) + f_{01,021}(0) - u_2^{\mathrm{eq}}(f_{21,201}(0) + f_{01,221}(0))), \end{aligned} \tag{3.6}$$

and there will be an obstruction if the vector $D_{u_2^{\mathrm{eq}}}(\lambda_1, \lambda_2)$ stays, for all (λ_1, λ_2) different from $(0,0)$, strictly on the same side of some hyperplane containing a vector subspace Q of $\mathbb{R}^n$, *i.e.* if the following condition $\mathcal{C}(Q)$ holds:

$$\mathcal{C}(Q) \Leftrightarrow \begin{cases} \text{there exists a linear form } \varphi : \mathbb{R}^n \to \mathbb{R}, \text{ whose restriction to } Q \text{ is zero } (i.e.\ Q \subset \ker\varphi) \\ \text{and such that the quadratic form } (\lambda_1, \lambda_2) \mapsto \left\langle \varphi \,,\, D_{u_2^{\mathrm{eq}}}(\lambda_1, \lambda_2) \right\rangle \text{ is positive definite.} \end{cases} \tag{3.7}$$

The vector space Q will be either R_1 or larger subspaces obtained by adding either the space R', generated by four special brackets involving f_2, namely

$$R' = \mathrm{Span}(f_{01,201}(0), f_{21,221}(0), f_{21,201}(0), f_{01,221}(0)), \tag{3.8}$$

or the space R'', which is the two-input analogy of the space named S' in Proposition 2.12, and that we define as generated by the value at zero of a special family $\mathcal{W}$ of brackets containing three times f_1:

$$R'' = \mathrm{Span}\left\{g(0), g \in \mathcal{W}\right\}, \text{ with} \tag{3.9}$$

$$\mathcal{W} = \Bigg\{ \Big(\prod_{j=1}^{\ell-2\nu-\mu-1} \mathrm{ad} f_{i_{\ell-j+1}}\Big)\Big[\Big(\prod_{j=\ell-2\nu-\mu}^{\ell-2\nu-1} \mathrm{ad} f_{i_{\ell-j+1}}\Big) f_1, \Big[\Big(\prod_{j=\ell-2\nu}^{\ell-\nu-1} \mathrm{ad} f_{i_{\ell-j+1}}\Big) f_1, \Big[\Big(\prod_{j=\ell-\nu}^{\ell-1} \mathrm{ad} f_{i_{\ell-j+1}}\Big) f_1, f_{i_1}\Big]\Big]\Big],$$
$$(\ell,\mu,\nu) \in \mathbb{N}^3, 0 \leqslant \mu \leqslant \frac{1}{3}(\ell-1), \mu \leqslant \nu \leqslant \frac{1}{2}(\ell-\mu-1),\ (i_1,\dots,i_\ell) \in \{0,2\}^\ell \Bigg\}. \tag{3.10}$$

The vector fields composing the family $\mathcal{W}$ appear again in the proof of Theorem 3.8, in equation (5.65).

Remark 3.7. This setup illustrates the remark by Kawski in [17], Example 2.5.1, showing that obstructions to controllability rest on the behavior of $D_{u_2^{\mathrm{eq}}}(\lambda_1, \lambda_2)$ rather than only on the value of supposedly "bad" brackets taken individually.

We now state our main result.

Theorem 3.8. *Consider system (1.3). Assume that (1.4) and (3.5) hold and let* $u_2^{\mathrm{eq}} \in \mathbb{R}$. *Then, with condition* $\mathcal{C}(Q)$ *defined in equation (3.7), we have, at* $(0,(0,u_2^{\mathrm{eq}}))$*:*

1. *If* $\mathcal{C}(R_1)$ *holds, system (1.3) is not* $(\mathrm{W}^{1,\infty}, \mathrm{L}^\infty)$*-STLC.*
2. *If* $\mathcal{C}(R_1 + R')$ *holds, system (1.3) is not* $(\mathrm{W}^{1,\infty}, B)$*-STLC.*
3. *If* $\mathcal{C}(R_1 + R'')$ *holds, system (1.3) is not STLC (*i.e. $(\mathrm{L}^\infty, \mathrm{L}^\infty)$*-STLC).*
4. *If* $\mathcal{C}(R_1 + R' + R'')$ *holds, system (1.3) is not* (L^∞, B)*-STLC.*

The proof is given in Section 5.3. Its structure mirrors the one of the proof of Theorem 3.2, but it features additional technical ideas, most of which are adapted from the proof of Proposition 2.12 conducted in [17], pp. 40–72 to our two-input setting.

Remark 3.9. Similarly to Theorem 3.2, in Cases 1. and 3., the "critical" values of u_2^{eq} for which $\mathcal{C}(R_1)$ or $\mathcal{C}(R_1 + R'')$ fails are the only ones for which one may overcome the obstruction and recover $(\mathrm{W}^{1,\infty}, \mathrm{L}^\infty)$-STLC. However, while Theorem 3.2 shows that there exists a unique such value β, Theorem 3.8 does not guarantee existence or uniqueness of these critical values.

Remark 3.10. Assumption (3.5) requires that $f_{21,01}(0) \in R_1$ for the Theorem to hold, but the same conclusion holds if we relax this requirement and include $f_{21,01}(0)$ within the subspace Q satisfying condition $\mathcal{C}$, hence the analogous statement: if $\mathcal{C}(R_1 + \mathbb{R} f_{21,01}(0))$ holds, system (1.3) is not $(\mathrm{W}^{1,\infty}, \mathrm{L}^\infty)$-STLC, and so on for the other three cases.

A similar, albeit slightly more technical, inclusion can be made in order to relax the assumption that $f_{121}(0) \in R_1$, by adding $\operatorname{Span}\{\operatorname{ad}_{f_0}^k f_{121}(0), k \in \mathbb{N}\}$ to the subspace Q in each case.

The proof is straightforwardly adaptable to these variations – see also Remark 5.7 below.

Remark 3.11. Condition $\mathcal{C}(R_1)$ (that holds in all four cases of the theorem) excludes in particular the situation where $f_{01,001}(0)$ does not belong to Q but all the other brackets $f_{i1,jk1}$ do. In this situation, since none of the brackets of order 5 containing f_2 would play any role, one could believe that the potential effect of u_2 on the controllability properties of the system is erased, and that System (2.6) would not be $(\mathrm{W}^{1,\infty}, B)$-STLC, just like in the scalar-input case.

Nevertheless, it appears in the proof of Theorem 3.8 below that, in that situation, higher-order terms in the Chen-Fliess series involving f_2 cannot be easily dominated by the term associated to $f_{01,001}(0)$. Therefore, this particular case does not seem straightforwardly reductible to the scalar-input case, and dealing with would require a different strategy than the one used in our proof.

Some illustrating and enlightening examples concerning this theorem are presented in Section 4.2.

4. Illustrating examples and applications

4.1. Examples for Theorem 3.2

In Case 2. of Theorem 3.2, the second control u_2 cannot compensate the obstruction to local controllability induced by f_{101}. The following example illustrates that case.

Example 4.1. Consider the system

$$\begin{cases} \dot{x} &= y^2 + y u_1, \\ \dot{y} &= 2y - u_1 + x u_2. \end{cases} \tag{4.1}$$

It is of the form (1.3) with

$$f_0 = \begin{pmatrix} y^2 \\ 2y \end{pmatrix}, \quad f_1 = \begin{pmatrix} y \\ -1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

Straightforward computations show that

$$R_1 = \operatorname{Span}(\mathbf{e}_2), \quad [f_1, [f_0, f_1]](0) = -6\,\mathbf{e}_1, \quad f_{121}(0) = \mathbf{e}_2,$$

so we are in Case 2. of Theorem 3.2. Therefore, for any $u_2^{\mathrm{eq}} \in \mathbb{R}$, system (4.1) is not B-STLC at $(0, (0, u_2^{\mathrm{eq}}))$.

In Case 1., Theorem 3.2 states that the system is not STLC around the equilibria $(0, (0, u_2^{\mathrm{eq}}))$, unless u_2^{eq} is equal to a particular value β, that allows the bracket f_{121} to compensate the bracket f_{101}. Around the equilibrium $(0, (0, \beta))$, the system can then be STLC, like in the next example. The method used in the following example to show STLC was introduced in [22] to show local controllability of magnetically driven micro-swimming robots – as detailed below the following example. We reproduce it here on a simpler system.

Example 4.2. Consider the system

$$\begin{cases} \dot{x} &= y^2 + y u_1 - \dfrac{2}{\alpha} y^2 u_2, \\[2ex] \dot{y} &= 2y - u_1 - \dfrac{1}{\alpha} y u_2, \end{cases} \tag{4.2}$$

for some $\alpha \neq 0$. Here we have

$$f_0 = \begin{pmatrix} y^2 \\ 2y \end{pmatrix}, \quad f_1 = \begin{pmatrix} y \\ -1 \end{pmatrix}, \quad f_2 = -\frac{1}{\alpha}\begin{pmatrix} 2y^2 \\ y \end{pmatrix}.$$

Straightforward computations show that

$$R_1 = \mathrm{Span}(\mathbf{e}_2), \quad [f_1,[f_0,f_1]](0) = -6\mathbf{e}_1, \quad f_{121}(0) = \frac{6}{\alpha}\mathbf{e}_1,$$

so we are in Case 1. of Theorem 3.2. Therefore, the system (4.2) is not STLC at $(0,(0,u_2^{\mathrm{eq}}))$ if $u_2^{\mathrm{eq}} \neq \alpha$.

Let us now study controllability at $(0,(0,\alpha))$. As explained in Remark 3.1, we make the feedback transformation $\tilde{u}_2 = u_2 = \alpha + \tilde{u}_2$, which transforms system (4.2) into

$$\begin{cases} \dot{x} & = -y^2 + yu_1 - \dfrac{2}{\alpha}y^2\tilde{u}_2, \\ \dot{y} & = y - u_1 - \dfrac{1}{\alpha}y\tilde{u}_2, \end{cases} \qquad \textit{i.e.} \text{ into (3.1) with } \tilde{f}_0 = \begin{pmatrix} -y^2 \\ y \end{pmatrix}, \tag{4.3}$$

and we show that this system is STLC at $(0,(0,0))$ (in a sense the transformation "neutralizes" the bracket f_{101}). To this end, we use the sufficient Sussmann condition for controllability [28], Theorem 7.3 with $\theta = 1$ and the notation for G_η introduced in [10], Definition III.10. Since $f_{121}(0) = \frac{6}{\alpha}\mathbf{e}_1$, the Lie brackets of order 3 generate the whole space, *i.e.* G_η is the whole tangent space if $\eta > 3$. The only Lie brackets of order at most 3 with an even number of 1 and 2 are $[f_1,[\tilde{f}_0,f_1]]$ and $[f_2,[\tilde{f}_0,f_2]]$, which are both zero and therefore belong trivially to G_3.

Hence, the Sussmann condition from [28] is satisfied and system (4.3) is STLC at $(0,(0,0))$, so system (4.2) is STLC at $(0,(0,\alpha))$.

Example 4.3 (Application to micro-swimmer robots)**.** In addition to the previous examples, let us present a practical application of Theorem 3.2. The present paper was motivated by the work on controllability of micro-swimmer robot models made in [10, 11, 22]. The two swimmers studied in these papers are made of two (respectively three) magnetized rigid segments, linked together with torsional springs, immersed in a low-Reynolds number fluid, and driven by a uniform in space, time-varying magnetic field $\mathbf{H}$. The swimmers' motion is assumed to be planar. The magnetic field $\mathbf{H}$ belongs to the swimmers' plane and can therefore be decomposed, in the moving basis associated to the first segment, in two components called $(H_\perp, H_\|)$.

Seeing the magnetic field as a control function, the dynamics of both swimmers write as control systems that are exactly of type (1.3)-(1.4):

$$\dot{\mathbf{z}} = f_0(\mathbf{z}) + H_\perp f_1(\mathbf{z}) + H_\| f_2(\mathbf{z}), \tag{4.4}$$

with the state $\mathbf{z}$ in $\mathbb{R}^4$ for the two-link swimmer (resp. $\mathbb{R}^5$ for the three-link swimmer). The detailed expressions of f_0, f_1 and f_2 with respect to the system parameters are given in [10], equations (12)–(16) (resp. [22], Appendix).

Moreover, assumptions (1.4) are satisfied. Hence, for all $H_\|$ in $\mathbb{R}$, $(0,(0,H_\|))$ is an equilibrium point (the first zero is short for $(0,0,0,0)$ in $\mathbb{R}^4$ (resp. $(0,0,0,0,0)$ in $\mathbb{R}^5$)). One also has $R_1 = \mathrm{Span}(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4)$ (resp. $R_1 = \mathrm{Span}(\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_4,\mathbf{e}_5)$) and the brackets of interest for Theorem 3.2 read:

$$[f_1,[f_0,f_1]](0) = (a_2,0,0,0) \quad (\text{resp. } [f_1,[f_0,f_1]](0) = (a_3,0,0,0,0))$$

and

$$f_{121}(0) = (b_2,0,0,0) \quad (\text{resp. } f_{121}(0) = (b_3,0,0,0,0)),$$

with a_2, a_3, b_2, b_3 constants that are nonzero under generic assumptions on the system parameters – see [10], Assumption III.2 (resp. [22], Asm. 1).

We can therefore apply Theorem 3.2, Case 1. and conclude that the two-link swimmer (resp. three-link swimmer) is not STLC at $(0,(0,H_\parallel))$ for any $H_\parallel$ such that $H_\parallel \neq a_2/b_2$ (resp. $H_\parallel \neq a_3/b_3$).

In [22], it is shown that the two-link swimmer (resp. the three-link swimmer) is indeed STLC at $(0,(0,a_2/b_2))$ (resp. $(0,(0,a_3/b_3))$), using the technique displayed in Example 4.2. However, the question of STLC at *other* equilibria of type $(0,(0,H_\parallel))$ was left open in [22], Remark 5. Theorem 3.2 allows to answer that question: $(0,(0,a_2/b_2))$ (resp. $(0,(0,a_3/b_3))$) is the *only* equilibrium of this type for which the swimmer is STLC.

Remark 4.4. Former studies on the two-link swimmer had led to the following results: in [10], it is shown that the control system (4.4) associated to the 2-link swimmer is B-STLC at $(0,(0,0))$ with controls bounded by $2a_2/b_2$; in [11], it is shown that it is moreover not STLC at $(0,(0,0))$. The proof of this last result features an explicit construction of the function Φ that is used in the proof of Theorem 3.2 below.

4.2. Examples for Theorem 3.8

We start by considering a scalar-input system, inspired by the classical example given by Sussmann in [27], Equation (6.12), p. 711:

$$\begin{cases} \dot{x} &= u_1, \\ \dot{y} &= x, \\ \dot{z} &= y, \\ \dot{w} &= x^3 + y^2 + z^2. \end{cases} \tag{4.5}$$

For this system, $f_{01,001}(0)$ is outside of S_1. Yet it is shown (see [27]) that it is B-STLC (and STLC as well, as shown for instance in [2], Example 12). Furthermore, Proposition 2.13 shows that it is *not* $\mathrm{W}^{1,\infty}$-STLC.

The following examples feature systems resembling System (4.5) with addition of a second control. Depending on the vector field f_2 associated to this second control, we will observe the role played by this second control in the two different cases of Theorem 3.8.

Example 4.5. Consider the control system

$$\begin{cases} \dot{x} &= u_1, \\ \dot{y} &= x, \\ \dot{z} &= y + xu_2, \\ \dot{w} &= x^3 + y^2 + z^2. \end{cases} \tag{4.6}$$

Straightforward computations show that $R_1 = R_1 + R' = \mathrm{Span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ (in particular, one has $f_{101}(0) = f_{121}(0) = f_{01,21}(0) = 0$).

Moreover,

$$f_{01,001}(0) = f_{21,021}(0) = -2\mathbf{e}_4 \text{ and } f_{21,001}(0) = f_{01,021}(0) = 0.$$

Therefore, by choosing $\psi = -\mathbf{e}_4^*$, we are in Case 2. of Theorem 3.8, and conclude that System (4.6) is not $(\mathrm{W}^{1,\infty}, B)$-STLC. Therefore, in this example, the second control does not help recover controllability, as in Example 4.1.

Example 4.6. Now, we consider the following system:

$$\begin{cases} \dot{x} &= u_1, \\ \dot{y} &= x, \\ \dot{z} &= y + x u_2, \\ \dot{w} &= x^3 + y^2 + z^2 + (y^2 + z^2) u_2. \end{cases} \tag{4.7}$$

All the relevant brackets are identical to those in the previous example, except that we now have

$$f_{01,201}(0) = -2\mathbf{e}_4 \text{ and } f_{21,221}(0) = -2\mathbf{e}_4.$$

Therefore, by choosing $\varphi = -\mathbf{e}_4^*$, we are in Case 1. of Theorem 3.8, and conclude that System (4.6) is not $(\mathrm{W}^{1,\infty}, \mathrm{L}^\infty)$-STLC at $(0, (0, 0))$.

However, setting $u_2 = -1$, System (4.7) becomes identical to the scalar-input system

$$\begin{cases} \dot{x} &= u_1, \\ \dot{y} &= x, \\ \dot{z} &= y - x, \\ \dot{w} &= x^3. \end{cases} \tag{4.8}$$

Let us show that this system is $\mathrm{W}^{1,\infty}$-STLC. In order to do this, we add the equation $\dot{u}_1 = v$ to the system and study controllability of this new extended system with state $(u_1, x, y, z, w) \in \mathbb{R}^5$ and control v. We check that, for this extended system, the Sussmann condition – already used above in Example 4.2 – is satisfied for $\theta = 1$ around the equilibrium $0_{\mathbb{R}^5}$. Indeed, the brackets $f_1, [f_0, f_1], [f_0, [f_0, f_1]], [f_0, [f_0, [f_0, f_1]]]$ and $[f_1, [f_0, [f_1, [f_0, [f_0, f_1]]]]]$ span the whole state space at $0_{\mathbb{R}^5}$, and all the brackets of order at most 6 with an even number of times f_1 vanish at $0_{\mathbb{R}^5}$. Therefore, the extended system is STLC around $0_{\mathbb{R}^5}$. This means that both u_1 and $\dot{u}_1 = v$ can be arbitrary small in L^∞-norm, and System (4.7) is indeed $(\mathrm{W}^{1,\infty}, \mathrm{B})$-STLC, and $(\mathrm{W}^{1,\infty}, \mathrm{L}^\infty)$-STLC at $(0, (0, -1))$.

This second example illustrates the more interesting case of our result, in which the second control, when set to be around a specific value depending of the behavior of brackets of order 5, helps recovering local controllability.

Cases 3 and 4 of Theorem 3.8 are analogous, in the sense that they only amount to change the "scalar-input" structure (*i.e.* f_0 and f_1) of the above systems (4.6) and (4.7), to create a system that fits Proposition 2.12 when $u_2 = 0$.

Finally, let us present an example partly justifying the necessity of Assumption (3.5).

Example 4.7. Theorem 3.8 requires that $f_{121}(0) \in R_1$. The following example features a case where no obstruction to local controllability can be obtained when this does not hold, even though all the other conditions to apply Theorem 3.8 are satisfied.

Consider the following system:

$$\begin{cases} \dot{x} &= u_1, \\ \dot{y} &= x, \\ \dot{z} &= y + x u_2, \\ \dot{w} &= x^3 + y^2 + z^2 + x^2 u_2. \end{cases} \tag{4.9}$$

All the relevant brackets are identical to those of Example 4.5, except this time we have

$$f_{121}(0) = -2\mathbf{e}_4,$$

so Assumption (3.5) is not satisfied.

Then, the same straightforward computations as the ones carried in the previous example show that the extended system (with the extra equation $\dot{u}_1 = v$ and (v, u_2) seen as the controls) satisfies the Sussmann condition for $\theta = 1$. Hence (4.9) is $\mathrm{W}^{1,\infty}$-STLC at 0.

This example therefore highlights the role of the bracket f_{121}, which seems to help recover some notion of local controllability in this case. Of note, in the scalar-input case, control systems similarly featuring brackets of different order interacting with each other were identified in [18].

5. Proofs of the Theorems

5.1. Notations and preliminaries

Solution associated to a control. Given a control $t \mapsto u(t) = (u_1(t), u_2(t))$ in $\mathrm{L}^\infty([0,T],\mathbb{R}^2)$, we denote by

$$t \mapsto z_u(t)$$

the solution of (1.3) with control $u(.)$ starting from the origin, *i.e.* $z_u(0) = 0$.

Vector fields and differential operators. A real analytic vector field f can equivalently be defined, in coordinates,

$$\text{either by} \quad f(x) = \begin{pmatrix} a_1(x) \\ \vdots \\ a_n(x) \end{pmatrix} \quad \text{or by} \quad f = \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k},$$

where $a_1, \dots, a_n$ are real analytic functions. The first notation views f as assigning a tangent vector to each point while the second views f as a differential operator of order 1: $f\phi = \sum_{k=1}^n a_k \frac{\partial \phi}{\partial x_k}$ for any analytic function ϕ. For two vector fields f and g, we denote by fg the differential operator of order 2 obtained by composition: $(fg)\phi = f(g\phi)$. In coordinates, and if $g = \sum_{k=1}^n b_j \frac{\partial}{\partial x_j}$, one has

$$fg = \sum_{k=1}^{n}\sum_{j=1}^{n} a_k b_j \frac{\partial^2}{\partial x_k \partial x_j} + \sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_k \frac{\partial b_j}{\partial x_k}\right)\frac{\partial}{\partial x_j}.$$

In that setting and as a differential operator, the Lie bracket (used so far in the paper, see beginning of Sect. 2.2), is a commutator: $[f,g] = fg - gf$; one recovers the usual formula from fg and gf above.

Chen-Fliess series. Let $I = (i_1, \dots, i_k) \in \{0,1,2\}^k$ be a multi-index and f_0, f_1, f_2 the three vector fields defining the system in (1.3). One defines by iterated composition the k^{th} order differential operator

$$f_I = f_{i_1} f_{i_2} \dots f_{i_k}.$$

For each $u = (u_1, u_2)$ in $\mathrm{L}^\infty([0,T],\mathbb{R}^2)$, for a multi-index $I = (i_1, \dots, i_k) \in \{0,1,2\}^k$, the iterated integral $\int_0^T u_I$ is defined as follows :

$$\int_0^T u_I = \int_0^T \int_0^{\tau_k} \int_0^{\tau_{k-1}} \cdots \int_0^{\tau_2} u_{i_k}(\tau_k) u_{i_{k-1}}(\tau_{k-1}) \dots u_{i_2}(\tau_2) u_{i_1}(\tau_1) \mathrm{d}\tau_1 \mathrm{d}\tau_2 \dots \mathrm{d}\tau_k \ , \tag{5.1}$$

where the symbol u_0 is used with the convention $u_0 \equiv 1$.

Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a real analytic function defined in a neighborhood of 0 in $\mathbb{R}^n$. The Chen-Fliess series associated to f_0, f_1, f_2, u and Φ at time T is defined as

$$\Sigma(u, f, \Phi, T) = \sum_I (f_I \Phi)(0) \int_0^T u_I, \tag{5.2}$$

where the summation is made over all the multi-indices $I = (i_1, \dots, i_k)$ in $\{0, 1, 2\}^k$ with $k \in \mathbb{N}$. The Chen-Fliess series appears in a range of works in control theory and geometry (see [5, 7, 8]). Real analyticity of the vector fields f_i and the function Φ implies (see [27], Lem. 4.2, p. 697) the following bound on the coefficients:

$$|(f_I \Phi)(0)| \leqslant C^k k!\,, \quad \text{where } k \text{ stands for the length of } I; \tag{5.3}$$

it also implies (see [27], Prop. 4.3, p. 698) that there exists $T_0(A) > 0$ such that the series converges for any $T \leqslant T_0(A)$ and any u such that $\|u\|_{\mathrm{L}^\infty([0,T])} \leqslant A$, uniformly with respect to u and T, to $\Phi(z_u(T))$, and the sum of the series provides the value of the function Φ along the solution of the system:

$$\Phi(z_u(T)) = \Sigma(u, f, \Phi, T)\,. \tag{5.4}$$

Constructing local coordinates. It is quite classical (see *e.g.* [28], proof of Prop. 6.3, in the same context as the present paper) to build local coordinates from the times in iterated flows of a family of vector fields. Since we will need precise properties of these coordinates, in different instances, we state this process as a lemma.

Lemma 5.1. *Given n real analytic vector fields $g_1, \dots, g_n$ on $\mathbb{R}^n$ such that $g_1(0), \dots, g_n(0)$ form a basis of $\mathbb{R}^n$, there exists a set of real analytic local coordinates $z \mapsto s(z) = (s_1(z), \dots, s_n(z))$ defined on a neighborhood $\mathcal{V}$ of 0 such that $s_1(0) = \dots = s_n(0) = 0$ and, for any i, j, k in $\{1, \dots, n\}$,*

$$g_1 s_i(z) = \delta_{i,1}, \quad i \in \{1, \dots, n\},\ z \in \mathcal{V}, \tag{5.5}$$

$$g_j s_i(0) = \delta_{i,j}, \quad (i, j) \in \{1, \dots, n\}^2, \tag{5.6}$$

$$g_k g_j s_i(0) = 0, \qquad (i, j, k) \in \{1, \dots, n\}^3,\ j \le k, \tag{5.7}$$

where $\delta_{i,j}$ denotes the Kronecker symbol: $\delta_{i,j}$ is equal to 1 if $i = j$ and 0 if $i \neq j$.

Proof. Define a real analytic map $\mathfrak{T} : (-\varepsilon, \varepsilon)^n \to \mathbb{R}^n$, for $\varepsilon > 0$ small enough[4], by

$$\mathfrak{T}(t_1, \dots, t_n) = \mathrm{e}^{t_1 g_1} \circ \dots \circ \mathrm{e}^{t_n g_n}(0), \tag{5.8}$$

where e^{tX} denotes the flow at time t of a vector field X. Clearly, $\mathfrak{T}(0) = 0$ and the columns of the Jacobian of $\mathfrak{T}$ at 0 are the vectors $g_1(0), \dots, g_n(0)$, linearly independent, hence $\mathfrak{T}$ is a local diffeomorphism at 0, *i.e.*, by taking $\varepsilon > 0$ small enough (and smaller than above), it defines a diffeomorphism from $(-\varepsilon, \varepsilon)^n$ onto $\mathcal{V} = \mathfrak{T}((-\varepsilon, \varepsilon)^n)$. Let $s : \mathcal{V} \to (-\varepsilon, \varepsilon)^n$ be the inverse of this real analytic diffeomorphism, and let $s = (s_1, \dots, s_n)$.

With these definitions, $s_1(0) = \dots = s_n(0) = 0$ follows from $\mathfrak{T}(0) = 0$; let us now prove (5.5) to (5.7). For any $z = \mathfrak{T}(t_1, \dots, t_n) \in \mathcal{V}$ (*i.e.* $s_i(z) = t_i$, $i = 1, \dots, n$, with $|t_i| < \varepsilon$), one has $s_i(\mathrm{e}^{t g_1}(z)) = t_i + t\,\delta_{i,1}$ (if $|t + t_1| < \varepsilon$);

[4] If the vector fields f_i are complete, take $\varepsilon = \infty$ and $\mathfrak{T} : \mathbb{R}^n \to \mathbb{R}^n$. If not, define complete vector fields $\widehat{f_i} = \rho\, f_i$ with ρ a smooth cut-off function that is identically equal to 1 in a neighborhood O of the origin, and define a smooth map $\widehat{\mathfrak{T}} : \mathbb{R}^n \to \mathbb{R}^n$ from the vector fields $\widehat{f}_1, \dots, \widehat{f}_n$ as in (5.8). The vector fields are real analytic on O, although not on $\mathbb{R}^n$. Taking $\varepsilon > 0$ small enough that $\mathfrak{T}((-\varepsilon, \varepsilon)^n) \subset O$, the restriction of $\widehat{\mathfrak{T}}$ to $(-\varepsilon, \varepsilon)^n$ provides the real analytic map $\mathfrak{T}$ because it only depends on the restrictions of the vector fields $\widehat{f_i}$ to O, that are real analytic, and are also the restrictions of f_i's to O.

this implies (5.5). Taking integers j, k, and evaluating the ith coordinate of the point $\mathrm{e}^{v\,g_j} \circ \mathrm{e}^{w\,g_k}(0)$ yields, assuming $|v| < \varepsilon$, $|w| < \varepsilon$ and $|v+w| < \varepsilon$, to be sure that $\mathrm{e}^{v\,g_j} \circ \mathrm{e}^{w\,g_k}(0)$ is in $\mathcal{V}$,

$$s_i(\mathrm{e}^{v\,g_j} \circ \mathrm{e}^{w\,g_k}(0)) = v\,\delta_{i,j} + w\,\delta_{i,k} \quad \text{if } j \le k$$

(this is wrong if $j > k$ because $\mathrm{e}^{v\,g_j}$ and $\mathrm{e}^{w\,g_k}$ do not commute and $\mathfrak{T}$ is defined with that order in (5.8)). By definition of the Lie derivative, one may differentiate with respect to v to obtain the value of $g_j s_i$ at that point: $g_j s_i(\mathrm{e}^{v\,g_j} \circ \mathrm{e}^{w\,g_k}(0)) = \delta_{i,j}$. Differentiating this with respect to w to obtain the value of $g_k g_j s_i$ is correct only when $v = 0$: $g_k g_j s_i(\mathrm{e}^{w\,g_k}(0)) = 0$. These two relations imply (5.6) and (5.7) by taking $v = w = 0$. □

5.2. Proof of Theorem 3.2

Most of the section consists in proving Proposition 5.2 below, which refines the statement of Theorem 3.2. The proof relies on similar arguments to those used to prove Proposition 2.10 in [27], pp. 707–710.

Proposition 5.2. *Assume that the assumptions of Case 1. (resp. Case 2.) of Theorem 3.2 are satisfied, with $u_2^{\mathrm{eq}} = 0$, and fix an arbitrary $\alpha > 0$ (for case Case 2. only). There exists a neighborhood $\mathcal{V}$ of the origin, an analytic function*

$$\Phi : \mathcal{V} \to \mathbb{R} \quad \textit{satisfying} \quad \Phi(0) = 0,\ \mathrm{d}\Phi(0) \neq 0, \tag{5.9}$$

a constant $K > 0$, and some $T^0 > 0$ such that for any T, $0 < T < T^0$, there exists ε^0 such that, for any essentially bounded control $u : t \mapsto (u_1(t), u_2(t))$ defined on $[0,T]$ satisfying

$$\|u_1\|_{\mathrm{L}^\infty([0,T])} \le \varepsilon^0 \textit{ and } \|u_2\|_{\mathrm{L}^\infty([0,T])} \le \varepsilon^0 \ \left(\textit{resp. } \|u_1\|_{\mathrm{L}^\infty([0,T])} \le \alpha \textit{ and } \|u_2\|_{\mathrm{L}^\infty([0,T])} \le \alpha\right), \tag{5.10}$$

one has $z_u(t) \in \mathcal{V}$ for all t in $[0,T]$ and

$$\Phi(z_u(T)) \geqslant K\,\|v_1\|^2_{L^2([0,T])} \tag{5.11}$$

with

$$v_1(t) = \int_0^t u_1(s)\,\mathrm{d}s. \tag{5.12}$$

Proof of Theorem 3.2 assuming Proposition 5.2. Fix $\alpha > 0$ for Case 2., choose ε' smaller than the ε^0 given by Proposition 5.2. Then for any η, (5.9) implies that one may pick a point z_1 in $B(0,\eta)$ such that $\Phi(z_1) < 0$, and (5.11) then clearly implies that there may exist no control u satisfying (5.10) and such that $z_u(T) = z_1$. This contradicts STLC according to Definition 2.1 (resp. B-STLC according to Def. 2.2) and proves Theorem 3.2 in the case $u_2^{\mathrm{eq}} = 0$. If $u_2^{\mathrm{eq}} \neq 0$, just perform the feedback transformation suggested in Remark 3.1: $f_0 + u_1 f_1 + u_2 f_2 = \widetilde{f}_0 + u_1 f_1 + \widetilde{u}_2 f_2$ with $\widetilde{f}_0 = f_0 + u_2^{\mathrm{eq}} f_2$ and $\widetilde{u}_2 = u_2 - u_2^{\mathrm{eq}}$, apply the result to the system $\dot{z} = \widetilde{f}_0 + u_1 f_1 + \widetilde{u}_2 f_2$ at the equilibrium $z = 0$, $(u_1, \widetilde{u}_2) = (0,0)$, and deduce the result at the equilibrium $z = 0$, $(u_1, u_2) = (0, u_2^{\mathrm{eq}})$ for the original system (noting that $f_{101}(0) + \beta f_{121}(0)$ is equal to $\widetilde{f}_{101}(0) + (\beta - u_2^{\mathrm{eq}})\widetilde{f}_{121}(0)$). □

Proof of Proposition 5.2. The first step is to define the function Φ that will later be proved to have the desired properties.

Let $d_1 = \dim R_1$ and $d_2 = \dim(R_1 + \mathrm{Span}(f_{121}(0)))$ (either $d_2 = d_1$ or $d_2 = d_1 + 1$). Let $g_1, \ldots, g_n$ be n vector fields such that:

- $g_1 = f_1$,
- $(g_1(0), \ldots, g_n(0))$ is a basis of $\mathbb{R}^n$,
- in Case 1., $(g_1(0), \ldots, g_{d_1}(0))$ is a basis of R_1 and $g_{d_1+1} = f_{101}$,

- in Case 2., $(g_1(0),\dots,g_{d_2}(0))$ is a basis of $R_1+\mathrm{Span}(f_{121}(0))$ and $g_{d_2+1}=f_{101}$,

and define, from these vector fields, some local coordinates $z\mapsto(s_1(z),\dots,s_n(z))$ in a neighborhood $\mathcal{V}$ of 0 according to Lemma 5.1. Then, define Φ as follows[5] :

- in Case 1., $\Phi(\zeta)=-s_{d_1+1}(\zeta)$,
- in Case 2., $\Phi(\zeta)=-s_{d_2+1}(\zeta)$.

The function Φ is real analytic on $\mathcal{V}$ and has, by construction, the following properties:

$$\Phi(0)=0, \tag{5.13}$$

$$(\text{Case 1.})\quad \forall g\in\mathcal{R}_1, (g\Phi)(0)=0, \tag{5.14}$$

$$(\text{Case 2.})\quad \forall g\in\mathcal{R}_1\cup\{f_{121}\}, (g\Phi)(0)=0, \tag{5.15}$$

$$f_1\Phi=0 \text{ on } \mathcal{V}, \tag{5.16}$$

$$(f_{101}\Phi)(0)=-1. \tag{5.17}$$

According to (5.16) and (1.4), one also has, with $I=(i_1,\dots,i_k)$,

$$f_I\Phi(0)=0 \text{ if } i_1=0 \text{ or } i_1=2 \text{ or } i_k=1\,. \tag{5.18}$$

In order to show that Φ satisfies (5.11), we then consider the Chen-Fliess series $\Sigma(u,f,\Phi,T)$ (see (5.2)) associated to the above constructed Φ and split its terms into five different types:

$$\Sigma(u,f,\Phi,T)=P_1+P_2+P_3+P_4+P_5,$$

where each P_i contains the terms with multi-indices I defined as follows:

- $P_1 : I=(2,\dots)$, $I=(0,\dots)$, or $I=(\dots,1)$,
- $P_2 : I=(1,J)$ with J containing only 0's and 2's,
- $P_3 : I=(1,1,0)$,
- $P_4 : I=(1,1,2)$,
- P_5 : all the remaining terms.

We immediately have $P_1=0$ according to (5.18). We also have $P_2=0$. Indeed, let $I=(1,i_2,\dots,i_k)$ with $i_j=0$ or 2 for all $j\in\{2,\dots k\}$. Then, we can write that

$$f_1f_{i_2}\dots f_{i_k}=[f_1,f_{i_2}]f_{i_3}\dots f_{i_k}+f_{i_2}f_1\dots f_{i_k}, \tag{5.19}$$

and $(f_{i_2}f_1\dots f_{i_k}\Phi)(0)=0$ according to (5.18). Similarly, we have

$$[f_1,f_{i_2}]f_{i_3}\dots f_{i_k}=[[f_1,f_{i_2}],f_{i_3}]f_{i_4}\dots f_{i_k}+f_{i_3}[f_1,f_{i_2}]\dots f_{i_k}, \tag{5.20}$$

and $((f_{i_3}[f_1,f_{i_2}]\dots f_{i_k})\Phi)(0)=0$ because of assumption (1.4) and the fact that $i_3=0$ or 2. Repeating this operation $k-3$ more times, we eventually get that

$$(f_1f_{i_2}\dots f_{i_k}\Phi)(0)=([\dots[f_1,f_{i_2}],\dots,f_{i_k}]\Phi)(0). \tag{5.21}$$

But $[\dots[f_1,f_{i_2}],\dots,f_{i_k}]$ is in $\mathcal{R}_1$, so $(f_1f_{i_2}\dots f_{i_k}\Phi)(0)=0$ because of (5.14). Therefore,

$$P_1=P_2=0. \tag{5.22}$$

[5]The minus signs do not appear in [27] due to an opposite sign convention on the definition of the Lie bracket in that reference.

The terms P_3 and P_4, associated to the brackets f_{101} and f_{121}, are the key parts of the proof. Let us compute their value. For P_3, we write

$$f_{101} = -f_1 f_1 f_0 + f_1 f_0 f_1 - [f_1, f_0] f_1.$$

The last two terms on the right-hand side vanish when evaluated at 0 against Φ because of (5.16), so $(f_1 f_1 f_0 \Phi)(0) = -(f_{101}\Phi)(0) = 1$ by (5.17). Moreover, the control integral part is given by

$$\int_0^T u_{(1,1,0)} = \int_0^T \int_0^s u_1(\sigma) \int_0^\sigma u_1(\tau) \mathrm{d}\tau \mathrm{d}\sigma \mathrm{d}s = \int_0^T \int_0^s v_1'(\sigma) v_1(\sigma) \mathrm{d}\sigma \mathrm{d}s,$$

so, overall

$$P_3 = \frac{1}{2} \|v_1\|_{\mathrm{L}^2}^2, \tag{5.23}$$

with v_1 defined in equation (5.12).

Let us turn to P_4. Expanding f_{121} into a sum of third order operators and applying (5.18) yields

$$(f_1 f_1 f_2 \Phi)(0) = -(f_{121}\Phi)(0). \tag{5.24}$$

Here the two cases of the theorem differ. In Case 2., (5.15) implies $f_{121}\Phi(0) = 0$, hence $P_4 = 0$. In Case 1., this does not hold, but there is a real number β and a vector field g such that $f_{121} = -\beta f_{101} + g$, with $g \in \mathcal{R}_1$ and $\beta \neq 0$ because $f_{101}(0) \notin R_1$. Thanks to (5.24), (5.14) and (5.17), we conclude that $(f_1 f_1 f_2 \Phi)(0) = -\beta$. The control integral associated to P_4 reads

$$\int_0^T u_{(1,1,2)} = \int_0^T u_2(s) \int_0^s u_1(\sigma) \int_0^\sigma u_1(\tau) \mathrm{d}\tau \mathrm{d}\sigma \mathrm{d}s \leqslant \frac{1}{2} \|u_2\|_{\mathrm{L}^\infty} \|v_1\|_{\mathrm{L}^2}^2,$$

and we deduce the following bound on P_4:

$$|P_4| \leqslant \frac{1}{2} |\beta| \, \|u_2\|_{\mathrm{L}^\infty} \|v_1\|_{\mathrm{L}^2}^2. \tag{5.25}$$

We now state a lemma that will imply that the infinite number of terms in P_5 add up to a small remainder. It is adapted from [26], Corollary 3.1: essentially, in these iterated integrals where u_1 appears at least r times, one may always, although it is not a priori obvious, bound u_2 with its L^∞ norm in the course of proceeding with the calculations due to Stefani [26] for the single input case. The integer r is left open because we need the lemma with $r = 2$ here, and with $r = 4$ in the proof of Theorem 3.8 below.

Lemma 5.3. *Let r be an integer such that $r \geqslant 2$, let Φ be an analytic function $\mathcal{V} \to \mathbb{R}$ with $\mathcal{V}$ a neighbourhood of the origin, such that $\Phi(0) = 0$ and $f_1\Phi = 0$ on $\mathcal{V}$, and let $\sum_{\geqslant r}(u, f, \Phi, T)$ denote the sum of all the terms of the Chen-Fliess series (5.2) associated to a multi-index I containing at least r times the element 1.*

Then, for any $\alpha > 0$, there exists $T_0 > 0$ and a constant $D > 0$ such that, for all T in $[0, T_0]$, and all (u_1, u_2) satisfying $\|u_1\|_{\mathrm{L}^\infty} \leqslant \alpha$ and $\|u_2\|_{\mathrm{L}^\infty} \leqslant \alpha$, one has

$$\left| \textstyle\sum_{\geqslant r}(u, f, \Phi, T) \right| \leqslant TD \, \|v_1\|_{\mathrm{L}^r}^r . \tag{5.26}$$

Proof. Let $k \in \mathbb{N}$ such that $k \geqslant r$ and let I be a multi-index in $\{0, 1, 2\}^k$, containing 1 at least r times. Due to the assumptions made on the function Φ, the term associated to I in the Chen-Fliess series is zero if I starts with 0 or 2 or ends with 1, so we discard these cases[6] and are left with multi-indices I that, for some integer j, can be broken down into $2j$ blocks: $I=(K_1,H_1,\dots,K_j,H_j)$, where, for each m between 1 and j, H_m belongs to $\{0,2\}^{h_m}$ and K_m to $\{1\}^{k_m}$, where $h_1,\dots,h_m$ and $k_1,\dots,k_m$ are positive integers. Additionally, we call I_1 and I_2 the total number of 1's and 2's in I. In particular, $I_1=k_1+\dots+k_j$, and since we assumed that the digit 1 appear at least r times, we have $I_1\geqslant r$. The iterated integral $\int_0^T u_I$ then reads

$$\int_0^T u_I=\underbrace{\int u_*\cdots\int u_*}_{h_j\text{ times}}\underbrace{\int u_1\cdots\int u_1}_{k_j\text{ times}}\cdots\underbrace{\int u_*\cdots\int u_*}_{h_1\text{ times}}\underbrace{\int u_1\cdots\int u_1}_{k_1\text{ times}}, \tag{5.27}$$

with $*$ denoting either 0 or 2 (recall that $u_0\equiv 1$ by convention), and where we omitted the bounds on the integrals for simplicity. We will deal with this iterated integral by applying the following inequalities, valid for any integrable function $w(.)$ on $[0,T]$, any τ in $[0,T]$, and any $m\in\mathbb{N}$:

$$\Big|\underbrace{\int_0^\tau u_1\cdots\int u_1}_{m\text{ times}}\int w\Big|\leqslant\int_0^\tau\frac{(|v_1(\tau)|+|v_1(s)|)^m}{m!}\,|w(s)|\,\mathrm{d}s \tag{5.28}$$

$$\text{and}\qquad \Big|\underbrace{\int_0^\tau u_*\cdots\int u_*\int u_*}_{m+1\text{ times}}w\Big|\leqslant\|u_2\|_{\mathrm{L}^\infty}^{m_2}\,\frac{T^m}{m!}\int_0^T|w(s)|\,\mathrm{d}s, \tag{5.29}$$

where m_2 is the number of times u_2 appears in the left-hand side integral in (5.29). These are consequences of the following relations, that can easily be established by induction:

$$\underbrace{\int_0^\tau u_*\cdots\int u_*\int u_*}_{m+1\text{ times}}w\leqslant\|u_2\|_{\mathrm{L}^\infty}^{m_2}\int_0^\tau\frac{(\tau-s)^m}{m!}|w(s)|\mathrm{d}s\,,\qquad \underbrace{\int_0^\tau u_1\cdots\int u_1}_{m\text{ times}}\int w=\int_0^\tau\frac{(v_1(\tau)-v_1(s))^m}{m!}w(s)\mathrm{d}s\,.$$

As a first step towards a majoration of (5.27), let us first assume that $I_1=r$ in (5.27), and we consider the part of (5.27) associated to $\tilde I=(K_1,H_1,\dots,K_j)$, *i.e.* we are considering the part of the integral in (5.27) obtained by truncating the first h_j integrals. Now, we alternatively apply (5.28) and (5.29) (with $\tau=T$) to each integral block in this truncated integral. Then, we note that (5.28) is always (except the last time where it is applied with $w=v_1$) applied with a function $w(.)$ which reads $u_*\int W$, so that it can be bounded as $|w(s)|\leqslant\|u_*\|\int_0^s|W|\leqslant\|u_*\|_{\mathrm{L}^\infty}\int_0^T|W|$; then, we can pull $\int_0^T|W|$ out of the integral, and obtain:

$$\begin{aligned}\left|\int_0^T u_{\tilde I}\right|\leqslant\|u_2\|_{\mathrm{L}^\infty}^{\tilde I_2}\left(\int_0^T\frac{(|v_1(T)|+|v_1(s)|)^{k_j}}{k_j!}\mathrm{d}s\right)\left(\frac{T^{h_{j-1}-1}}{(h_{j-1}-1)!}\right)\cdots\\ \cdots\left(\frac{T^{h_1-1}}{(h_1-1)!}\right)\left(\int_0^T\frac{(|v_1(T)|+|v_1(s)|)^{k_1-1}}{(k_1-1)!}|v_1(s)|\mathrm{d}s\right).\end{aligned} \tag{5.30}$$

The right-hand side of (5.30) only depends on u_2 through its first factor; the rest of it can be dealt with exactly as in the proof of [26], Corollary 3.1 for scalar-input systems: expanding the binomial terms in (5.30) and

[6] The proof conducted in [26] does not rely on assumptions of Φ and therefore requires estimations of the iterated integrals associated to the terms that are discarded in our proof. Suitable bounds on these iterated integrals are obtained in [26], provided an estimation of the type $v_1(t)\leqslant A\int_0^t|v_1(s)|\mathrm{d}s$ (see Cor. 3.2 and Lem. 3.2 in [26]) for some constant $A\geqslant 1$. We notably do not need such an estimation here.

applying the Hölder inequality to extract the L^r norm of v_1 yields

$$\left|\int_0^T u_{\tilde I}\right| \leqslant \|u_2\|_{\mathrm{L}^\infty}^{\tilde I_2} \frac{r^r\,(2r)!\,(r+1)^{|I|+1}}{|I|!} \sum_{i=0}^{r} |v_1(T)|^{r-i}\, T^{|I|-r-i/r} \left(\int_0^T |v_1(s)|^r \mathrm{d}s\right)^{i/r}. \tag{5.31}$$

We now go back to the full integral (5.27), allowing $I_1 > r$. Then, the multi-index I can be split into two parts I' and I'' such that $I = (I', I'')$ and I'' has the same properties than $\tilde I$ above, *i.e.* it contains 1 exactly r times, and starts and finishes with 1's. Note that I' is not empty, because it contains at least H_j which is not empty. The subpart from the integral in (5.27) corresponding to I'' can then be bounded thanks to (5.31). In the remainder, *i.e.* the part corresponding to I', we bound each occurrence of u_1 and u_2 with their L^∞ norm, and make use of (5.29) and the Hölder inequality again to finally obtain

$$\left|\int_0^T u_I\right| \leqslant (2r)!\, r^{r+1}\, T\, \|u_1\|_{\mathrm{L}^\infty}^{I_1-r} \|u_2\|_{\mathrm{L}^\infty}^{I_2} \left(T^{|I|-r-2} \frac{(2r+2)^{|I|}}{(|I|-1)!}\right) \|v_1\|_{\mathrm{L}^r}^r. \tag{5.32}$$

Thanks to (5.3), we have, for some constant C independent of I,

$$|(f_I\Phi)(0)| \leqslant C^{|I|}(|I|)!. \tag{5.33}$$

Combining (5.32) and (5.33), we find an upper bound for the whole term of index I from the series:

$$\left|\left(\int_0^T u_I\right)(f_I\Phi)(0)\right| \leqslant (2r)!\, r^{r+1}\, T\, \|u_1\|_{\mathrm{L}^\infty}^{I_1-r}\, \|u_2\|_{\mathrm{L}^\infty}^{I_2}\, |I|\, C^{|I|}(2r+2)^{|I|} T^{|I|-r-2} \|v_1\|_{\mathrm{L}^r}^r. \tag{5.34}$$

Summing these terms for all possible indices, we see that

$$\left|\sum\nolimits_{\geqslant r}(u, f, \Phi, T)\right| \leqslant (2r)! r^{r+1} T \|v_1\|_{\mathrm{L}^r}^r \sum_{|I|\geqslant r+2} \|u_1\|_{\mathrm{L}^\infty}^{I_1-r} \|u_2\|_{\mathrm{L}^\infty}^{I_2} |I| C^{|I|}(2r+2)^{|I|} T^{|I|-r-2}, \tag{5.35}$$

and there exists $T_0 > 0$ such that, for all T in $[0, T_0]$, and all (u_1, u_2) satisfying $\|u_1\|_{\mathrm{L}^\infty} \leqslant \alpha$ and $\|u_2\|_{\mathrm{L}^\infty} \leqslant \alpha$, the series in (5.35) converges for all T in $[0, T_0]$. Hence, one obtains (5.26) with a constant D which depends on the sum of the series for $T = T_0$ and the other constants in (5.35). □

We now end the proof of Proposition 5.2 in both cases.

Applying Lemma 5.3 for $r = 2$, we have the existence of T_0 and D such that for $T \in [0, T_0]$,

$$|P_5| \leqslant TD\|v_1\|_{\mathrm{L}^2}^2. \tag{5.36}$$

In Case 1., using (5.22) and (5.23), we write that

$$\Sigma(u, f, \Phi, T) = \frac{1}{2}\|v_1\|_{\mathrm{L}^2}^2 + P_4 + P_5, \tag{5.37}$$

knowing, thanks to (5.25) and (5.26) that

$$|P_4 + P_5| \leqslant \frac{1}{2}|\beta|\|u_2\|_{\mathrm{L}^\infty}\|v_1\|_{\mathrm{L}^2}^2 + TD\|v_1\|_{\mathrm{L}^2}^2. \tag{5.38}$$

Let ε be a real positive number such that $\varepsilon(\frac{1}{2}|\beta| + D) \leqslant \frac{1}{4}$. Let $\varepsilon_0 = \min(T_0, \varepsilon)$. Taking a smaller T if necessary, we can assume that $z_u(t) \in \mathcal{V}$ for all t in $[0, T]$. Assume that $T \leqslant \varepsilon_0$ and $\|u_2\|_{\mathrm{L}^\infty} \leqslant \varepsilon_0$. Using (5.38) into (5.37), we obtain that $\Sigma(u, f, \Phi, T) \geqslant \frac{1}{4}\|v_1\|^2_{\mathrm{L}^2}$ for all $T \leqslant \varepsilon_0$ and $\|u_2\|_{\mathrm{L}^\infty} \leqslant \varepsilon_0$, *i.e.* we have proven (5.11) with $K = \frac{1}{4}$.

In Case 2., we obtain that

$$\Sigma(u, f, \Phi, T) = \frac{1}{2}\|v_1\|^2_{\mathrm{L}^2} + P_5. \tag{5.39}$$

Let ε be a real positive number such that $\varepsilon D \leqslant \frac{1}{4}$. Let $\varepsilon_0 = \min(T_0, \varepsilon)$. Taking a smaller T if necessary, we can assume that $z_u(t) \in \mathcal{V}$ for all t in $[0, T]$. Assume that $T \leqslant \varepsilon_0$, $\|u_1\|_{\mathrm{L}^\infty} \leqslant \alpha$ and $\|u_2\|_{\mathrm{L}^\infty} \leqslant \alpha$ (for an arbitrary $\alpha > 0$). Using (5.26) into (5.39), we obtain that $\Sigma(u, f, \Phi, T) \geqslant \frac{1}{4}\|v_1\|^2_{\mathrm{L}^2}$ for all $T \leqslant \varepsilon_0$, *i.e.* we have proven (5.11) with $K = \frac{1}{4}$.

This concludes the proof of Proposition 5.2. □

5.3. Proof of Theorem 3.8

Most of this section consists in proving Proposition 5.4 below, more precise than Theorem 3.8.

Proposition 5.4. *Assume that the assumptions of Theorem 3.8 are satisfied, and fix an arbitrary $\alpha > 0$ (needed for Cases 2. and 4. only). There exists a neighborhood $\mathcal{V}$ of the origin, an analytic function*

$$\Phi : \mathcal{V} \to \mathbb{R} \quad \textit{satisfying} \quad \Phi(0) = 0,\ \mathrm{d}\Phi(0) \neq 0, \tag{5.40}$$

a constant $K > 0$, and some $T^0 > 0$ such that for any T, $0 < T < T^0$, there exists ε^0 such that, for any essentially bounded control $u : t \mapsto (u_1(t), u_2(t))$ defined on $[0, T]$ satisfying the correct smallness assumptions depending on the case considered, and specified below, one has $z_u(t) \in \mathcal{V}$ for all t in $[0, T]$ and

$$\Phi(z_u(t)) \geqslant K \left(\|w_1\|^2_{L^2([0,t])} + \|w_2\|^2_{L^2([0,t])}\right),\ 0 \leqslant t \leqslant T \tag{5.41}$$

with

$$w_1(t) = \iint_{0 \leqslant s_2 \leqslant s_1 \leqslant t} u_1(s_1)\,\mathrm{d}s_1\,\mathrm{d}s_2\,,\ w_2(t) = \iint_{0 \leqslant s_2 \leqslant s_1 \leqslant t} u_1(s_1)\,u_2(s_2)\,\mathrm{d}s_1\,\mathrm{d}s_2. \tag{5.42}$$

The assumptions on the controls, depending on each case of the theorem, are as follows: in Case 1.,

$$\|u_1\|_{\mathrm{W}^{1,\infty}([0,T])} \leq \varepsilon^0 \quad \textit{and} \quad \|u_2\|_{\mathrm{L}^\infty([0,T])} \leq \varepsilon^0, \tag{5.43}$$

in Case 2.,

$$\|u_1\|_{\mathrm{W}^{1,\infty}([0,T])} \leq \varepsilon^0 \quad \textit{and} \quad \|u_2\|_{\mathrm{L}^\infty([0,T])} \leq \alpha, \tag{5.44}$$

in Case 3.,

$$\|u_1\|_{\mathrm{L}^\infty([0,T])} \leq \varepsilon^0 \quad \textit{and} \quad \|u_2\|_{\mathrm{L}^\infty([0,T])} \leq \varepsilon^0, \tag{5.45}$$

in Case 4.,

$$\|u_1\|_{\mathrm{L}^\infty([0,T])} \leq \varepsilon^0 \quad \textit{and} \quad \|u_2\|_{\mathrm{L}^\infty([0,T])} \leq \alpha. \tag{5.46}$$

Remark 5.5. Notice that u_1 and u_2 are not treated symmetrically in defining w_1 and w_2 in Equation 5.42 above: w_1 depends on u_1 only, while w_2 depends on u_2 *and* u_1.

Proof of Theorem 3.8 assuming Proposition 5.4. (Very similar to the proof of Thm. 3.2 assuming Prop. 5.2 in Sect. 5.2.)

Fix $\alpha > 0$ for case 2, choose ε smaller than the ε^0 given by Proposition 5.4. Then for any η, (5.40) implies that one may pick a point z_1 in $B(0, \eta)$ such that $\Phi(z_1) < 0$, and (5.41) then clearly implies that there may exist no control u satisfying one of the smallness assumptions among equations (5.43)–(5.46) (depending on which case of the Theorem in considered) and such that $z_u(T) = z_1$. This contradicts local controllability according to Definition 3.4 in Cases 1. and 3. (resp. Def. 3.5 in Cases 2. and 4.) and proves Theorem 3.2 in the case $u_2^{\mathrm{eq}} = 0$. If $u_2^{\mathrm{eq}} \neq 0$, just perform the same feedback transformation as in the proof of Proposition 5.2 and deduce the result at the equilibrium $z = 0$, $(u_1, u_2) = (0, u_2^{\mathrm{eq}})$ for the original system. □

Proof of Proposition 5.4. Let $T > 0$, $u_1 \in W^{1,\infty}([0, T])$ and $u_2 \in \mathrm{L}^\infty([0, T])$. We are going to prove (5.41) in several steps :

Step 1. We define a suitable function Φ.
Step 2. We study the associated Chen-Fliess series and, following [17], pp. 41–56, we perform a "change of basis" over the free vector space spanned by monomials f_I with index I containing two or three times "1".
Step 3. We isolate and calculate the sum of four dominating terms, called $\mathcal{P}_{\mathrm{dom}}$,
Step 4. We deal with the remaining terms in the series, by showing that they are small compared to $\mathcal{P}_{\mathrm{dom}}$,
Step 5. Finally we sum all the terms and prove (5.41).

Step 1. Construction of Φ. We start with the following lemma, stating that the assumptions made in Theorem 3.8 imply that $f_1(0)$, $f_{01}(0)$, and $f_{21}(0)$ are linearly independent.

Lemma 5.6. *If the vector fields f_0, f_1 and f_2 are such that* (1.4) *(with $z^{eq} = 0$) and* (3.5) *hold and, for some $u_2^{\mathrm{eq}} \in \mathbb{R}$ and for all $(\lambda_1, \lambda_2) \in \mathbb{R}^2 \backslash \{(0, 0)\}$, $D_{u_2^{\mathrm{eq}}}(\lambda_1, \lambda_2) \notin R_1$, then*

$$\mathrm{Rank}\{f_1(0), f_{01}(0), f_{21}(0)\} = 3\,. \tag{5.47}$$

Proof. First, notice that (5.47) is preserved by the feedback transformation $\widetilde{u}_2 = u_2 - u_2^{\mathrm{eq}}$ described in Remark 3.1, so we can assume without loss of generality that $u_2^{\mathrm{eq}} = 0$ in the statement of the Lemma.

Assume that (5.47) fails. Since $f_1(0) \neq 0$ (see (1.4)), we have three possible cases:

(i) $\mathrm{Rank}\{f_1(0), f_{21}(0)\} = 2$,
(ii) $\mathrm{Rank}\{f_1(0), f_{21}(0)\} = 1$ and $\mathrm{Rank}\{f_1(0), f_{01}(0)\} = 2$,
(iii) $\mathrm{Rank}\{f_1(0), f_{01}(0), f_{21}(0)\} = 1$.

In case *(i)*, there exists two real numbers μ, ν such that $f_{01}(0) = \mu f_{21}(0) + \nu f_1(0)$; introducing $H = f_{01} - \mu f_{21} - \nu f_1$, we have

$$f_{01} = \mu f_{21} + \nu f_1 + H\,, \quad H(0) = 0\,. \tag{5.48}$$

Taking the Lie brackets of both sides by f_0 yields

$$f_{001} = \mu f_{021} + \nu f_{01} + [f_0, H]. \tag{5.49}$$

We take the Lie bracket by f_{01}, and we use (5.48), (5.49), and the Jacobi identity in the last line, so that

$$\begin{aligned} f_{01,001} &= \mu f_{01,021} + [f_{01}, [f_0, H]] \\ &= \mu f_{01,021} + [\mu f_{21} + \nu f_1 + H, [f_0, H]] \\ &= \mu f_{01,021} + \mu[f_{21}, [f_0, H]] + \nu[f_1, [f_0, H]] + [H, [f_0, H]] \\ &= \mu f_{01,021} + \mu f_{21001} - \mu^2 f_{21021} - \mu\nu f_{2101} + \nu f_{1001} - \mu\nu f_{1021} - \nu^2 f_{101} \\ &= -\mu^2 f_{21,021} + \mu\big(f_{01,021} + f_{21,001}\big) - \mu\nu\big([f_0, f_{121}] + 2f_{21,01}\big) + \nu[f_0, f_{101}] - \nu^2 f_{101} + [H, [f_0, H]]\,. \end{aligned} \tag{5.50}$$

With D_0 the map defined in (3.6), this implies

$$D_0(1, \mu) = -\mu\nu\big([f_0, f_{121}](0) + 2f_{21,01}(0)\big) + \nu[f_0, f_{101}](0) - \nu^2 f_{101}(0) + [H, [f_0, H]](0)\,, \tag{5.51}$$

where the right-hand side is in R_1 from (3.5) and the fact that $f_0(0) = H(0) = 0$.

In case *(ii)*, we repeat the above calculations while swapping the roles of f_{01} and f_{21}, yielding an expression for $f_{21,021}$ instead of $f_{01,001}$ in equation (5.50) and obtaining that $D_0(\mu, 1) \in R_1$. Finally, case *(iii)* is actually contained in case *(i)* by taking $\mu = 0$ in equation (5.48).

Overall, we have proven in each case the existence of $(\lambda_1, \lambda_2) \neq (0, 0)$ such that $D_0(\lambda_1, \lambda_2) \in R_1$ if (5.47) fails, whence the Lemma. ☐

Remark 5.7. Let us stress here the importance of Assumption (3.5), and more precisely that $f_{101}(0) \in R_1$ and $f_{121}(0) \in R_1$, to conclude this proof and deduce that $[f_0, f_{121}](0)$ and $[f_0, f_{101}](0)$ belong to R_1 in equation (5.51). See Remark 3.10 below Theorem 3.8 for further discussion on this assumption and Section 4.2 for an example of system where $f_{121}(0) \notin R_1$, and for which the conclusion of Theorem 3.8 fails.

Remark 5.8. Interestingly, the statement of Lemma 5.6 implies that R_1 must be of dimension at least 3 for our result to hold, and therefore the obstructions of Theorem 3.8 can appear only for systems with a state of dimension at least 4 – like the examples in Section 4.2.

Now, let $d = \dim Q$ (where Q is the subspace associated to proposition $\mathcal{C}$ in Theorem 3.8 and depends on which of the four cases is considered) and construct n real analytic vector fields $g_1, \dots, g_n$ enjoying the following properties:

1. $g_1 = f_1$, $g_2 = f_{01}$, $g_3 = f_{21}$, and $(g_1(0), \dots, g_n(0))$ is an basis of $\mathbb{R}^n$,
2. $(g_1(0), \dots, g_d(0))$ is a basis of Q and g_{d+1} is such that

$$\langle \varphi, g_{d+1}(0)\rangle = 1 \quad\text{and}\quad \langle \varphi, g_j(0)\rangle = 0,\ j \neq d+1, \tag{5.52}$$

The first point is possible thanks to Lemma 5.6 and the second point is possible with the linear form φ given by property $\mathcal{C}(Q)$, see (3.7). Then define, from these $g_1, \dots, g_n$, some local coordinates $x \mapsto (s_1(x), \dots, s_n(x))$ in a neighborhood $\mathcal{V}$ of 0 according to Lemma 5.1 and define Φ as follows:

$$\Phi(\zeta) = s_{d+1}(\zeta). \tag{5.53}$$

By virtue of Lemma 5.1 and (1.4), the function Φ satisfies:

$$\Phi(0) = 0, \tag{5.54}$$

$$f_1\Phi = 0 \text{ on } \mathcal{V}, \tag{5.55}$$

$$f_I\Phi(0) = 0 \text{ if } i_1 = 0 \text{ or } i_1 = 2 \text{ or } i_k = 1\,. \tag{5.56}$$

Moreover, one has, thanks to (5.6) and (5.52) for the first point, and (5.7) for the second point,

$$\mathrm{d}\Phi(0) = \varphi, \tag{5.57}$$

$$\forall k, \ell \in \{1, \ldots, d\}, k \geqslant \ell \Rightarrow (g_k g_l \Phi)(0) = 0. \tag{5.58}$$

Step 2. Change of basis. Consider the Chen-Fliess series $\Sigma(u, f, \Phi, T)$, given by (5.2), associated to Φ constructed above, and partition the series in the sum of the terms where derivation along f_1 appears zero, one, two, three or at least four times (the subscript refers to the number of times the index 1 appears in the multi-index $I \in \{0, 1, 2\}^k$):

$$\Sigma(u, f, \Phi, T) = \mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_{\geq 4}\,. \tag{5.59}$$

As already noticed in the proof of Theorem 3.2 (Eqs. (5.19) to (5.22)), the above properties of Φ and the fact that f_0 and f_2 vanish at 0 imply $\mathcal{P}_0 = \mathcal{P}_1 = 0$; $\mathcal{P}_{\geq 4}$ will be dealt with directly in Step 4; we turn to $\mathcal{P}_2$ and $\mathcal{P}_3$.

The purpose of this step is the same as the classification in different types (P_1 to P_5) made in the proof of Theorem 3.2: classify the terms into convenient categories, in order to study them more easily. However, here, in $\mathcal{P}_2$ and $\mathcal{P}_3$, instead of simply categorizing the multi-indices I, we follow [17] and perform a re-arrangement that can be seen as the result of a change of basis on the (free) vector space generated by non-commutative monomials containing the same number of times the indeterminate f_0, the same number of times the indeterminate f_1 (two or three times, indeed) and the same number of times the indeterminate f_2; these vector spaces are homogeneous component of the free associative algebra generated by the indeterminates f_0, f_1 and f_2. In practice, this amounts to manipulating the differential operators f_I on one hand, and perform successive integrations by parts of the iterated integrals $\int u_I$ on the other hand. This suitable change of basis is performed in detail in [17], pp. 41–56 for a scalar-input system and can be adapted to the two-control system (1.3), yielding the following expressions:

$$\mathcal{P}_2 = \sum_{\ell\in\mathbb{N},\ L\in\{0,2\}^\ell} \sum_{\mu\in\{0,\ldots,\ell\}} b^{2,L}_{\mu} (W^{2,L}_{\mu}\Phi)(0), \tag{5.60}$$

$$\mathcal{P}_3 = \sum_{\ell\in\mathbb{N},\ L\in\{0,2\}^\ell} \sum_{\xi=1}^{3} \sum_{(\mu,\nu)\in\mathcal{J}(\xi,\ell)} b^{3,L}_{\xi,\mu,\nu} (W^{3,L}_{\xi,\mu,\nu}\Phi)(0), \tag{5.61}$$

where

$$\begin{aligned} &\mathcal{J}(1,\ell) = \{(\mu,\nu)\in\mathbb{N}^2,\ 0\leqslant\mu\leqslant\tfrac{1}{3}(\ell-1), \mu\leqslant\nu\leqslant\tfrac{1}{2}(\ell-\mu-1)\},\\ &\mathcal{J}(2,\ell) = \{(\mu,\nu)\in\mathbb{N}^2,\ 2\mu+\nu\leqslant\ell-1\},\\ &\mathcal{J}(3,\ell) = \{(\mu,\nu)\in\mathbb{N}^2,\ \nu\leqslant 2\mu\leqslant\ell-1\}, \end{aligned} \tag{5.62}$$

Let $\ell\in\mathbb{N}$ and $L=(i_1,\ldots,i_\ell)$ in $\{0,2\}^\ell$. The explicit expressions of the differential operators W read

$$\mathrm{W}^{2,L}_{\mu} = \Big(\prod_{j=1}^{\ell-2\mu-1}\mathrm{ad}f_{i_{\ell-j+1}}\Big)\Big[\Big(\prod_{j=\ell-2\mu}^{\ell-\mu-1}\mathrm{ad}f_{i_{\ell-j+1}}\Big)f_1, \Big[\Big(\prod_{j=\ell-\mu}^{\ell-1}\mathrm{ad}f_{i_{\ell-j+1}}\Big)f_1, f_{i_1}\Big]\Big] \ \text{ if } \ 0\leqslant 2\mu\leqslant\ell-1, \tag{5.63}$$

$$\mathrm{W}^{2,L}_{\mu} = \Big(\Big(\prod_{j=1}^{\mu}\mathrm{ad}f_{i_{\ell-j+1}}\Big)f_1\Big)\Big(\Big(\prod_{j=\mu+1}^{\ell}\mathrm{ad}f_{i_{\ell-j+1}}\Big)f_1\Big) \ \text{ if } \ \ell\leqslant 2\mu\leqslant 2\ell, \tag{5.64}$$

$$\mathrm{W}^{3,L}_{1,\mu,\nu} = -\Big(\prod_{j=1}^{\ell-2\nu-\mu-1}\mathrm{ad}f_{i_{\ell-j+1}}\Big)\Big[\Big(\prod_{j=\ell-2\nu-\mu}^{\ell-2\nu-1}\mathrm{ad}f_{i_{\ell-j+1}}\Big)f_1, \Big[\Big(\prod_{j=\ell-2\nu}^{\ell-\nu-1}\mathrm{ad}f_{i_{\ell-j+1}}\Big)f_1, \Big[\Big(\prod_{j=\ell-\nu}^{\ell-1}\mathrm{ad}f_{i_{\ell-j+1}}\Big)f_1, f_{i_1}\Big]\Big]\Big], \tag{5.65}$$

$$\mathrm{W}^{3,L}_{2,\mu,\nu} = \Big(\big(\prod_{j=1}^{\ell-2\mu-\nu-1} \mathrm{ad} f_{i_{\ell-j+1}} \big) \Big[\big(\prod_{j=\ell-2\mu-\nu}^{\ell-\mu-\nu-1} \mathrm{ad} f_{i_{\ell-j+1}} \big) f_1, \Big[\big(\prod_{j=\ell-\mu-\nu}^{\ell-\nu-1} \mathrm{ad} f_{i_{\ell-j+1}} \big) f_1, f_{i_\nu} \Big] \Big] \Big) \Big(\big(\prod_{j=\ell-\nu+1}^{\ell} \mathrm{ad} f_{i_{\ell-j+1}} \big) f_1 \Big), \tag{5.66}$$

$$\mathrm{W}^{3,L}_{3,\mu,\nu} = \Big(\big(\prod_{j=1}^{\ell-\mu-\nu} \mathrm{ad} f_{i_{\ell-j+1}} \big) f_1 \Big) \Big(\big(\prod_{j=\ell-\mu-\nu+1}^{\ell-\mu} \mathrm{ad} f_{i_{\ell-j+1}} \big) f_1 \Big) \Big(\big(\prod_{j=\ell-\mu+1}^{\ell} \mathrm{ad} f_{i_{\ell-j+1}} \big) f_1 \Big). \tag{5.67}$$

In particular, the family of operators in equation (5.65) is exactly the family $\mathcal{W}$ defined in equation (3.10) above Theorem 3.8. Their values at 0 generate the subspace R'' defined in equation (3.9), which appears in Cases 2. and 4. of the theorem.

The associated iterated integrals are respectively given by

$$b^{2,L}_{\mu} = -\tfrac{1}{2} \int u_{i_1} \int u_{i_2} \cdots \int u_{i_{\ell-2\mu}} (\int u_{i_{\ell-2\mu+1}} \cdots \int u_{i_{\ell-\mu}} \int u_1)(\int u_{i_{\ell-\mu+1}} \cdots \int u_{i_\ell} \int u_1) \ \text{ if } \ 0 \leqslant 2\mu \leqslant \ell-1, \tag{5.68}$$

$$b^{2,L}_{\mu} = (\int u_{i_1} \cdots \int u_{i_\mu} \int u_1)(\int u_{i_{\mu+1}} \cdots \int u_{i_\ell} \int u_1) \ \text{ if } \ \ell \leqslant 2\mu \leqslant 2\ell, \tag{5.69}$$

$$b^{3,L}_{1,\mu,\nu} = \delta \int u_{i_1} \cdots \int u_{i_{\ell-2\nu-\mu}} (\int u_{i_{\ell-2\nu-\mu+1}} \cdots \int u_{i_{\ell-2\nu}} \int u_1)(\int u_{i_{\ell-2\nu+1}} \cdots \int u_{i_{\ell-\nu}} \int u_1)(\int u_{i_{\ell-\nu+1}} \cdots \int u_{i_\ell} \int u_1), \tag{5.70}$$

$$b^{3,L}_{2,\mu,\nu} = \frac{1}{2} \Big(\int u_{i_1} \cdots \int u_{i_{\ell-2\mu-\nu}} (\int u_{i_{\ell-2\mu-\nu+1}} \cdots \int u_{i_{\ell-\mu-\nu}} \int u_1)(\int u_{i_{\ell-\mu-\nu+1}} \cdots \int u_{i_{\ell-\nu}} \int u_1) \Big) (\int u_{i_{\ell-\nu+1}} \cdots \int u_{i_\ell} \int u_1), \tag{5.71}$$

$$b^{3,L}_{3,\mu,\nu} = \delta (\int u_{i_1} \cdots \int u_{i_{\ell-\mu-\nu}} \int u_1)(\int u_{i_{\ell-\mu-\nu+1}} \cdots \int u_{i_{\ell-\mu}} \int u_1)(\int u_{i_{\ell-\mu+1}} \cdots \int u_{i_\ell} \int u_1). \tag{5.72}$$

The bounds on the iterated integrals have been removed to lighten notations. To avoid any notational ambiguity, let us specify that the products of two or three integrals appearing in expressions (5.68), (5.70) and in the first factor of (5.71) are meant to be part of the integrand of the integral sign on their left, whereas expressions (5.69) and (5.72) denote products of two or three independent integrals. The constant δ that appears in (5.70) and (5.72) depends on ℓ, μ and ν and take only values 1, $\frac{1}{2}$ or $\frac{1}{6}$.

Step 3. Dominating terms. The essence of the whole proof is to show that the four terms in (5.60) in $\mathcal{P}_2$, for which $\ell = 3$, $\mu = 1$ and L in $\{(0,0,0),(0,2,2),(0,0,2),(0,2,0)\}$ dominate the rest of the series. We call $\mathcal{P}_{\mathrm{dom}}$ the sum of these four terms.

The differential operators in these four terms read, with notations from (2.5),

$$W^{2,(0,0,0)}_1 = -f_{01,001},\ W^{2,(0,2,2)}_1 = -f_{21,021},\ W^{2,(0,0,2)}_1 = -f_{21,001},\ W^{2,(0,2,0)}_1 = -f_{01,021}, \tag{5.73}$$

and the associated integrals read

$$b^{2,(0,0,0)}_1 = -\frac{1}{2}\|w_1\|^2_{\mathrm{L}^2},\ b^{2,(0,2,2)}_1 = -\frac{1}{2}\|w_2\|^2_{\mathrm{L}^2},\ b^{2,(0,0,2)}_1 = b^{2,(0,2,0)}_1 = -\frac{1}{2}\int w_1 w_2, \tag{5.74}$$

where w_1 and w_2 are defined in (5.42). Now, let $\alpha = f_{01,001}\Phi(0)$, $\beta = f_{21,021}\Phi(0)$ and $\gamma = (f_{21,001}\Phi(0) + f_{01,021}\Phi(0))$. Gathering (5.73) and (5.74) into the expression of $\mathcal{P}_{\mathrm{dom}}$, we get

$$\mathcal{P}_{\mathrm{dom}} = \frac{1}{2}\left(\alpha\|w_1\|^2_{\mathrm{L}^2} + \beta\|w_2\|^2_{\mathrm{L}^2} + \gamma\Big(\int w_1 w_2\Big) \right). \tag{5.75}$$

from which we get the lower bound

$$\mathcal{P}_{\mathrm{dom}} \geqslant \frac{1}{2}\left(\alpha\|w_1\|^2_{\mathrm{L}^2} + \beta\|w_2\|^2_{\mathrm{L}^2} - |\gamma| \|w_1\|_{\mathrm{L}^2}\|w_2\|_{\mathrm{L}^2} \right), \tag{5.76}$$

which means that:

$$\mathcal{P}_{\text{dom}} \geqslant \frac{1}{2}\,\langle \varphi\,,\, D_0(\|w_1\|_{\mathrm{L}^2}, \|w_2\|_{\mathrm{L}^2})\rangle\,, \tag{5.77}$$

where D_0 is defined in (3.6), and the linear form φ appears thanks to (5.57). Finally, positive definiteness in (3.7) ensures the existence of $K>0$ such that $\langle \varphi\,,\, D_0(\lambda_1,\lambda_2)\rangle \geqslant 4K(\lambda_1^2+\lambda_2^2)$. We therefore obtain the following bound on $\mathcal{P}_{\text{dom}}$:

$$\mathcal{P}_{\text{dom}} \geqslant 2K(\|w_1\|_{\mathrm{L}^2}^2 + \|w_2\|_{\mathrm{L}^2}^2). \tag{5.78}$$

Step 4. Remaining terms in the series. The goal of this step is to show that, under the right assumptions on T and the controls, the sum of the Chen-Fliess series without the dominating terms studied in the previous step is smaller in absolute value than $K(\|w_1\|_{\mathrm{L}^2}^2 + \|w_2\|_{\mathrm{L}^2}^2)$, where K is defined in equation (5.78). Since the calculations are quite lengthy, we summarize them as follows:

- i) Terms in $\mathcal{P}_2$.
  - a. Four terms requiring special attention (namely those associated with indices L in $\{(2,0,0),(2,2,2),(2,0,2),(2,2,0)\}$).
  - b. Vanishing terms.
  - c. Case $1 \leqslant \mu \leqslant \frac{\ell-1}{2}$.
  - d. Case $\frac{\ell}{2} \leqslant \mu \leqslant \ell-2$.
  - e. Sum of all the terms from c. and d., called $\mathcal{P}_{2,2}$.
- ii) Terms in $\mathcal{P}_3$.
  - a. Vanishing terms.
  - b. Case $\xi = 1$.
  - c. Case $\xi = 2, \mu = 0$.
  - d. Case $\xi = 2, \mu \geqslant 1$.
  - e. Case $\xi = 3$.
- iii) Terms in $\mathcal{P}_{\geqslant 4}$.

Distinctions between different cases of the theorem need to be made at the following substeps:

- between Cases 1. and 3. on the one hand, and Cases 2. and 4. on the other hand, in the substep i)a,
- between Cases 1. and 2. on the one hand, and Cases 3. and 4. on the other hand, in the substep ii)b.

The rest of this step is valid for all cases.

Before starting, let us state the following lemma, that provides useful estimations on some control integrals and derived from [2], Proposition 6.

Lemma 5.9. *Let $T>0$ and $u_1 \in \mathrm{W}^{1,\infty}([0,T])$. There exists M_1 and M_2 independent of u_1 and T such that:*

$$\|v_1\|_{\mathrm{L}^4}^4 \leqslant M_1 \|u_1\|_{\mathrm{L}^\infty}^2 \|w_1\|_{\mathrm{L}^2}^2, \tag{5.79}$$

$$\|v_1\|_{\mathrm{L}^3}^3 \leqslant M_2\left(1+\frac{1}{T}\right)\|u_1\|_{\mathrm{W}^{1,\infty}} \|w_1\|_{\mathrm{L}^2}^2. \tag{5.80}$$

Proof. Both estimations are consequences of the Gagliardo-Nirenberg inequality (see for example [9], Thm. 9.3), that we recall here : let p,q,r,α,s real numbers and j,m integers satisfying $1 \leqslant q,r \leqslant +\infty$, $\frac{j}{m} \leqslant \alpha \leqslant 1$ and

$$\frac{1}{p} = j + (\frac{1}{r} - m)\alpha + \frac{1-\alpha}{q}. \tag{5.81}$$

Then, for u a function on $[0,T]$ such that $u \in \mathrm{L}^q([0,T])$ and its weak derivative $u^{(m)}$ is in $\mathrm{L}^r([0,T])$, then $u^{(j)}$ is in $L^p([0,T])$, and there exists C_1, C_2 independent of u and T such that

$$\|u^{(j)}\|_{\mathrm{L}^p} \leqslant C_1 \|u^{(m)}\|_{\mathrm{L}^r}^{\alpha} \|u\|_{\mathrm{L}^q}^{1-\alpha} + C_2 T^{\frac{1}{p}-\frac{1}{s}-j} \|u\|_{L^s}, \tag{5.82}$$

where the power in front of T is obtained by a scaling argument from the inequality applied to functions defined on $[0,1]$, as done in [2], Proposition 6. Now, for (5.79), we take $u = w_1$ (which belongs to $\mathrm{W}^{3,\infty}([0,T])$) and apply (5.82) for $p=4$, $q=2$, $r=\infty$, $\alpha=\frac{1}{2}$, $s=2$, $j=1$, and $m=2$ to get

$$\|v_1\|_{\mathrm{L}^4}^4 \leqslant C_1 \|u_1\|_{\mathrm{L}^\infty}^2 \|w_1\|_{\mathrm{L}^2}^2 + C_2 T^{-5} \|w_1\|_{\mathrm{L}^2}^4, \tag{5.83}$$

then, notice that by definition of w_1, we can write

$$T^{-5}\|w_1\|_{\mathrm{L}^2}^2 \leqslant T^{-4}\|w_1\|_{\mathrm{L}^\infty}^2 \leqslant \|u_1\|_{\mathrm{L}^\infty}^2, \tag{5.84}$$

hence (5.79) with $M_1 = C_1 + C_2$.

For (5.80), we take again $u = w_1$ and apply (5.82) for $p=3$, $q=2$, $r=\infty$, $\alpha=\frac{1}{3}$, $s=2$, $j=1$, and $m=3$, which yields

$$\|v_1\|_{\mathrm{L}^3}^3 \leqslant C_1 \|u_1\|_{\mathrm{W}^{1,\infty}} \|w_1\|_{\mathrm{L}^2}^2 + C_2 T^{-\frac{7}{2}} \|w_1\|_{\mathrm{L}^2}^3. \tag{5.85}$$

We then proceed with similar operations than in (5.84):

$$T^{-7}\|w_1\|_{\mathrm{L}^2}^2 \leqslant T^{-6}\|w_1\|_{\mathrm{L}^\infty} \leqslant T^{-2}\|u_1\|_{\mathrm{L}^\infty}^2 \leqslant T^{-2}\|u_1\|_{\mathrm{W}^{1,\infty}}^2, \tag{5.86}$$

hence (5.80) with $M_2 = \max(C_1, C_2)$. □

We are now ready to start dealing with the terms in $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_{\geqslant 4}$, following the substeps detailed above.

i) Terms in $\mathcal{P}_2$.

i) a. Four terms requiring special attention.

First of all, we will deal with the terms in $\mathcal{P}_2$ for which $\ell = 3$ and $\mu = 1$. Four of them (called the dominating terms above) have already been studied in **Step 3.**, so only four terms are remaining, namely those associated with indices L in $\{(2,0,0),(2,2,2),(2,0,2),(2,2,0)\}$. We denote their sum by $\mathcal{P}_{2,1}$. These terms require special attention because their behavior depends on which case of the theorem is considered. We have

$$W_1^{2,(2,0,0)} = -f_{01,201},\ W_1^{2,(2,2,2)} = -f_{21,221},\ W_1^{2,(2,0,2)} = -f_{21,201},\ W_1^{2,(2,2,0)} = -f_{01,221}. \tag{5.87}$$

Now, notice that in Cases 2. and 4. of Theorem 3.8, these four brackets belong to Q when evaluated at 0; therefore, by construction of Φ, one has

$$W_1^{2,(0,0,2)}\Phi(0) = W_1^{2,(2,2,2)}\Phi(0) = W_1^{2,(2,0,2)}\Phi(0) = W_1^{2,(0,2,2)}\Phi(0) = 0,$$

and $\mathcal{P}_{2,1} = 0$. On the other hand, in Cases 1. and 3. of the theorem, these terms do not vanish anymore, and we need to look at the associated integrals $b_1^{2,(2,0,0)}$, $b_1^{2,(2,2,2)}$, $b_1^{2,(2,0,2)}$, $b_1^{2,(2,2,0)}$, that read

$$b_1^{2,(2,0,0)} = -\frac{1}{2}\int u_2 w_1^2,\ b_1^{2,(2,2,2)} = -\frac{1}{2}\int u_2 w_2^2,\ b_1^{2,(2,0,2)} = b_1^{2,(2,2,0)} = -\frac{1}{2}\int u_2 w_1 w_2, \tag{5.88}$$

Let $A = \max(|W_1^{2,(0,0,2)}\Phi(0)|, |W_1^{2,(2,2,2)}\Phi(0)|, |W_1^{2,(2,0,2)}\Phi(0)|, |W_1^{2,(0,2,2)}\Phi(0)|)$; bounding u_2 by $\|u_2\|_{\mathrm{L}^\infty}$ in (5.88), we obtain that $\mathcal{P}_{2,1}$ is bounded as follows:

$$|\mathcal{P}_{2,1}| \leqslant A\|u_2\|_{\mathrm{L}^\infty}(\|w_1\|_{\mathrm{L}^2}^2 + \|w_2\|_{\mathrm{L}^2}^2). \tag{5.89}$$

Therefore, letting $T_1 = \frac{K}{6A}$, for all u_2 such that $\|u_2\|_{\mathrm{L}^\infty} \leqslant T_1$ and for all $T \in [0, T_1]$, one has

$$|\mathcal{P}_{2,1}| \leqslant \frac{K}{6}(\|w_1\|_{\mathrm{L}^2}^2 + \|w_2\|_{\mathrm{L}^2}^2). \tag{5.90}$$

i) b. Vanishing terms.

We start by ruling out the vanishing operators, through the following lemma:

Lemma 5.10. *One has, for $\ell \geqslant 0$ and for $L \in \{0,2\}^\ell$:*

1. $(\mathrm{W}_0^{2,L}\Phi)(0) = 0$,
2. $(\mathrm{W}_\ell^{2,L}\Phi)(0) = 0$,
3. $(\mathrm{W}_{\ell-1}^{2,L}\Phi)(0) = 0$.

Proof. Let $\ell \geqslant 0$ and $L \in \{0,2\}^\ell$.

(1). $\mathrm{W}_0^{2,L}(0)$ always belongs to Q, so we use property (5.57) of Φ to conclude.

(2). The operator $\mathrm{W}_\ell^{2,L}$ takes the form hf_1 with h some vector field, so we use property (5.55) of Φ to conclude.

(3). First, we consider the case $i_1 = 0$. Then, $\mathrm{W}_{\ell-1}^{2,L} = hg_2$ with $h \in \mathcal{Q}$, where g_2 is defined in **Step 1.** and $\mathcal{Q}$ naturally denotes the set of all the brackets appearing in the definition of Q. Because h belongs to $\mathcal{Q}$ and $(g_1(0), \dots, g_d(0))$ is a basis of Q, there exists $\lambda_1, \dots, \lambda_d$ in $\mathbb{R}$ such that $h(0) = \sum_{j=1}^d \lambda_j g_j(0)$. Introducing $g_0 = h - \sum_{j=1}^d \lambda_j g_j$, we have $g_0(0) = 0$ and $h = g_0 + \sum_{j=1}^d \lambda_j g_j$. Since $g_0(0) = 0$, we immediately have $(g_0 g_2 \Phi)(0) = 0$. Moreover, for $j \in \{2, \dots, d\}$, $(g_j g_2 \Phi)(0) = 0$ by virtue of property (5.58) of Φ. Finally, noticing that $g_1 g_2 = [g_1, g_2] + g_2 g_1 = f_{101} + g_2 g_1$, and using again (5.58) as well as point (1) of the lemma (since $f_{101}(0) \in Q$), we have $(g_1 g_2 \Phi)(0) = 0$. Overall, $\mathrm{W}_{\ell-1}^{2,L}\Phi(0) = (hg_2\Phi)(0) = 0$.

In the case $i_\ell = 2$, $\mathrm{W}_{\ell-1}^{2,L} = hg_3$ with $h \in \mathcal{Q}$. Again, we write $h = g_0 + \sum_{j=1}^d \lambda_j g_j$ with $g_0(0) = 0$, and proceed as above to show that $(g_j g_3 \Phi)(0) = 0$ for all $j = 0, \dots, d$, using (5.58) and the fact that $[g_1, g_3](0) = f_{121}(0)$ and $[g_2, g_3](0) = f_{01,21}(0)$ belong to Q. ◻

Remaining terms.

According to Lemma 5.10, which rules out all the terms for which $\ell = 0$, 1 or 2, and since we also ruled out the terms for which $\ell = 3$ through $\mathcal{P}_{\mathrm{dom}}$ and $\mathcal{P}_{2,1}$, the remaining terms in $\mathcal{P}_2$ satisfy $\ell \geqslant 4$. Let $\ell \geqslant 4$ and $L \in \{0,2\}^\ell$. Let L_2 be the number of 2's in L. There are two cases to study depending on the index μ.

i) c. Case $1 \leqslant \mu \leqslant \frac{\ell-1}{2}$.

According to (5.68), we have

$$b_\mu^{2,L} = \frac{1}{2}\int u_{i_1}\int u_{i_2}\cdots\int u_{i_{\ell-2\mu}}(\int u_{i_{\ell-2\mu+1}}\cdots\int u_{i_{\ell-\mu}}\int u_1)(\int u_{i_{\ell-\mu+1}}\cdots\int u_{i_\ell}\int u_1). \tag{5.91}$$

There are three cases to study, depending on the value of $i_{\ell-\mu}$ and i_ℓ.

(1) $i_{\ell-\mu}=0$ and $i_\ell=0$. Bounding u_2 by $\|u_2\|_{L^\infty}$ gives

$$|b_\mu^{2,L}| \leqslant \frac{1}{2}\|u_2\|_{L^\infty}^{L_2} \int_0^T \frac{(T-s)^{\ell-3}}{(\ell-3)!}|w_1(s)|^2 ds \tag{5.92}$$

if $\mu=1$ and

$$|b_\mu^{2,L}| \leqslant \frac{1}{2}\|u_2\|_{L^\infty}^{L_2} \int_0^T \frac{(T-s)^{\ell-2\mu-1}}{(\ell-2\mu-1)!}\left(\int_0^s \frac{(s-\sigma)^{\mu-2}}{(\mu-2)!}|w_1(\sigma)|d\sigma\right)^2 ds \tag{5.93}$$

if $\mu>1$. We use Cauchy-Schwarz inequality and bound $(T-s)$ by T in (5.93) to get

$$|b_\mu^{2,L}| \leqslant \frac{1}{2}\|u_2\|_{L^\infty}^{L_2} \frac{T^{\ell-2\mu-1}}{(\ell-2\mu-1)!}\int_0^T \frac{(T-s)^{2\mu-3}}{(2\mu-3)(\mu-2)!^2}\int_0^s w_1^2(\sigma)d\sigma ds. \tag{5.94}$$

Bounding $(T-s)$ by T and the integral between 0 and s by the integral between 0 and T in (5.94) yields

$$|b_\mu^{2,L}| \leqslant \frac{1}{2}\|u_2\|_{L^\infty}^{L_2} \frac{T^{\ell-3}}{(2\mu-3)(l-2\mu-1)!(\mu-2)!^2}\|w_1\|_{L^2}^2. \tag{5.95}$$

Finally, we bound $\frac{1}{(\mu-2)!}$ by $\frac{\mu}{(\mu-1)!}$ in (5.95) to obtain a bound that encompasses all values of μ:

$$|b_\mu^{2,L}| \leqslant \frac{1}{2}\|u_2\|_{L^\infty}^{L_2} \frac{\mu^2 T^{\ell-3}}{(l-2\mu-1)!(\mu-1)!^2}\|w_1\|_{L^2}^2. \tag{5.96}$$

(2) $i_{\ell-\mu}=2$ and $i_\ell=2$. Bounding u_2 by $\|u_2\|_{L^\infty}$ gives

$$|b_\mu^{2,L}| \leqslant \frac{1}{2}\|u_2\|_{L^\infty}^{L_2} \int_0^T \frac{(T-s)^{\ell-3}}{(\ell-3)!}|w_2(s)|^2 ds \tag{5.97}$$

if $\mu=1$ and

$$|b_\mu^{2,L}| \leqslant \frac{1}{2}\|u_2\|_{L^\infty}^{L_2-2} \int_0^T \frac{(T-s)^{\ell-2\mu-1}}{(\ell-2\mu-1)!}\left(\int_0^s \frac{(s-\sigma)^{\mu-2}}{(\mu-2)!}|w_2(\sigma)|d\sigma\right)^2 ds \tag{5.98}$$

if $\mu>1$, and we repeat steps (5.94) and (5.95) in (5.98) to obtain the upper bound

$$|b_\mu^{2,L}| \leqslant \frac{1}{2}\|u_2\|_{L^\infty}^{L_2-2} \frac{\mu^2 T^{\ell-3}}{(l-2\mu-1)!(\mu-1)!^2}\|w_2\|_{L^2}^2. \tag{5.99}$$

(3) $i_{\ell-\mu}=2$ and $i_\ell=0$, or $i_{\ell-\mu}=0$ and $i_\ell=2$. Bounding u_2 by $\|u_2\|_{L^\infty}$ gives

$$|b_\mu^{2,L}| \leqslant \frac{1}{2}\|u_2\|_{L^\infty}^{L_2} \int_0^T \frac{(T-s)^{\ell-3}}{(\ell-3)!}|w_1(s)w_2(s)|ds \tag{5.100}$$

if $\mu = 1$ and

$$|b_\mu^{2,L}| \leqslant \frac{1}{2}\|u_2\|_{\mathrm{L}^\infty}^{L_2-1}\int_0^T \frac{(T-s)^{\ell-2\mu-1}}{(\ell-2\mu-1)!}\left(\int_0^s \frac{(s-\sigma)^{\mu-2}}{(\mu-2)!}|w_1(\sigma)|\mathrm{d}\sigma\right)\left(\int_0^s \frac{(s-\sigma)^{\mu-2}}{(\mu-2)!}|w_2(\sigma)|\mathrm{d}\sigma\right)\mathrm{d}s \tag{5.101}$$

if $\mu > 1$. Then, we repeat steps (5.94) and (5.95) in (5.101) and use the fact that $2xy \leqslant x^2 + y^2$ to obtain

$$|b_\mu^{2,L}| \leqslant \frac{1}{4}\|u_2\|_{\mathrm{L}^\infty}^{L_2-1}\frac{\mu^2 T^{\ell-3}}{(l-2\mu-1)!(\mu-1)!^2}(\|w_1\|_{\mathrm{L}^2}^2 + \|w_2\|_{\mathrm{L}^2}^2). \tag{5.102}$$

i) d. Case $\frac{\ell}{2} \leqslant \mu \leqslant \ell - 2$.

According to (5.69), we have

$$b_\mu^{2,L} = \left(\int u_{i_1}\cdots\int u_{i_\mu}\int u_1\right)\left(\int u_{i_{\mu+1}}\cdots\int u_{i_\ell}\int u_1\right).$$

Similarly to the case $2 \leqslant 2\mu \leqslant l-1$, the study of $b_\mu^{2,L}$ then splits in three cases depending on the value of $i_{\ell-\mu}$ and i_ℓ. Applying Cauchy-Schwarz inequality again leads to the following upper bounds:

(1) if $i_\mu = 0$ and $i_\ell = 0$,

$$|b_\mu^{2,L}| \leqslant \|u_2\|_{\mathrm{L}^\infty}^{L_2}\frac{T^{\ell-3}}{\sqrt{(2\mu-1)(\ell-\mu-1)}(l-\mu-1)!(\mu-2)!}\|w_1\|_{\mathrm{L}^2}^2. \tag{5.103}$$

(2) if $i_\mu = 2$ and $i_\ell = 2$,

$$|b_\mu^{2,L}| \leqslant \|u_2\|_{\mathrm{L}^\infty}^{L_2-2}\frac{T^{\ell-3}}{\sqrt{(2\mu-1)(\ell-\mu-1)}(l-\mu-1)!(\mu-2)!}\|w_2\|_{\mathrm{L}^2}^2. \tag{5.104}$$

(3) if $i_\mu = 2$ and $i_\ell = 0$, or $i_\mu = 0$ and $i_\ell = 2$,

$$|b_\mu^{2,L}| \leqslant \frac{1}{2}\|u_2\|_{\mathrm{L}^\infty}^{L_2-1}\frac{T^{\ell-3}}{\sqrt{(2\mu-1)(\ell-\mu-1)}(l-\mu-1)!(\mu-2)!}(\|w_1\|_{\mathrm{L}^2}^2 + \|w_2\|_{\mathrm{L}^2}^2). \tag{5.105}$$

i) e. Sum of all the remaining terms in $\mathcal{P}_2$.

We call $\mathcal{P}_{2,2}$ the sum of all the terms in $\mathcal{P}_2$ for which $\ell \geqslant 4$, namely

$$\mathcal{P}_{2,2} = \sum_{\ell\geqslant 4}\sum_{\mu=0}^{\ell}\sum_{L\in\{0,2\}^\ell} b_\mu^{2,L} W_\mu^{2,L}\Phi(0). \tag{5.106}$$

Using (5.3), we bound the $(W_\mu^{2,L}\Phi)(0)$ with $C^{\ell+2}(\ell+2)!$, and use the upper bounds obtained in (5.96), (5.99), (5.102), (5.103), (5.104) and (5.105):

$$|\mathcal{P}_{2,2}| \leqslant \sum_{\ell\geqslant 4} C^{\ell+2}T^{\ell-3}\sum_{\mu=0}^{\ell} B(\ell,\mu)\Bigg(\|w_1\|_{\mathrm{L}^2}^2 \sum_{\substack{L\in\{0,2\}^\ell \\ i_{\ell-\mu}=0, i_\ell=0}}\|u_2\|_{\mathrm{L}^\infty}^{L_2} + \|w_2\|_{\mathrm{L}^2}^2\sum_{\substack{L\in\{0,2\}^\ell \\ i_{\ell-\mu}=2, i_\ell=2}}\|u_2\|_{\mathrm{L}^\infty}^{L_2-2} + \frac{\|w_1\|_{\mathrm{L}^2}^2 + \|w_2\|_{\mathrm{L}^2}^2}{2}\sum_{\substack{L\in\{0,2\}^\ell \\ i_{\ell-\mu}+i_\ell=2}}\|u_2\|_{\mathrm{L}^\infty}^{L_2-1}\Bigg), \tag{5.107}$$

with $B(\ell,\mu)=\frac{\mu^2(\ell+2)!}{(\ell-2\mu-1)!(\mu-1)!^2}$ if $0\leqslant 2\mu\leqslant \ell-1$ and $\frac{(\ell+2)!}{(\ell-\mu-1)!(\mu-2)!}$ if $\ell/2\leqslant\mu\leqslant\ell$.

For given ℓ and μ, straightforward combinatorial calculation yields to

$$\sum_{\substack{L\in\{0,2\}^\ell\\ i_{\ell-\mu}=0,i_\ell=0}}\|u_2\|_{\mathrm{L}^\infty}^{L_2}=\sum_{\substack{L\in\{0,2\}^\ell\\ i_{\ell-\mu}=2,i_\ell=2}}\|u_2\|_{\mathrm{L}^\infty}^{L_2-2}=\frac{1}{2}\sum_{\substack{L\in\{0,2\}^\ell\\ i_{\ell-\mu}+i_\ell=2}}\|u_2\|_{\mathrm{L}^\infty}^{L_2-1}=(1+\|u_2\|_{\mathrm{L}^\infty})^{\ell-2}. \tag{5.108}$$

Substituting (5.108) in (5.107) yields

$$\begin{aligned}|\mathcal{P}_{2,2}|&\leqslant 2(\|w_1\|_{\mathrm{L}^2}^2+\|w_2\|_{\mathrm{L}^2}^2)\sum_{\ell\geqslant 4}C^{\ell+2}(1+\|u_2\|_{\mathrm{L}^\infty})^{\ell-2}T^{\ell-3}\sum_{\mu=0}^{\ell}B(\ell,\mu)\\&\leqslant 2C^6T(1+\|u_2\|_{\mathrm{L}^\infty})^2(\|w_1\|_{\mathrm{L}^2}^2+\|w_2\|_{\mathrm{L}^2}^2)\sum_{\ell\geqslant 0}(C(1+\|u_2\|_{\mathrm{L}^\infty})T)^\ell\sum_{\mu=0}^{\ell+4}B(\ell+4,\mu).\end{aligned}$$

For T small enough, the series above converges and therefore there exists T_2 such that for all $T\in[0,T_2]$,

$$|\mathcal{P}_{2,2}|\leqslant\frac{K}{6}(\|w_1\|_{\mathrm{L}^2}^2+\|w_2\|_{\mathrm{L}^2}^2). \tag{5.109}$$

ii) Terms in $\mathcal{P}_3$.

ii) a. Vanishing terms.

We start by ruling out the vanishing terms through the following lemma:

Lemma 5.11. *One has, for $\ell\geqslant 0$ and for $L\in\{0,2\}^\ell$:*

1. $(\mathrm{W}_{2,0,0}^{3,L}\Phi)(0)=0$,
2. $(\mathrm{W}_{2,0,1}^{3,L}\Phi)(0)=0$,
3. $(\mathrm{W}_{3,1,1}^{3,L}\Phi)(0)=0$,
4. $(\mathrm{W}_{3,\mu,0}^{3,L}\Phi)(0)=0$ *for all admissible μ.*

The proof is based on the same arguments as the proof of Lemma 5.10.

Now, let $\ell\geqslant 1$, $L\in\{0,2\}^\ell$ and L_2 the number of 2's in L.

ii) b. Case $\xi=1$.

As mentioned above, for these terms (and only for them) we will need to distinguish between Cases 1.,2. and Cases 3.,4. of the theorem. Indeed, in Cases 3. and 4., since $R''\subset Q$, and $\mathrm{W}_{1,\mu,\nu}^{3,L}(0)\in R''$ by construction, we immediately have $(\mathrm{W}_{1,\mu,\nu}^{3,L}\Phi)(0)=0$ for all admissible μ and ν. Hence, the sum of the associated terms in the Chen-Fliess series, called $\mathcal{P}_{3,1}$, is equal to zero.

On the other hand, in Cases 1.,2., these terms do not vanish, and we need to look at the iterated integrals. We will see that an assumption of smallness of $\|u_1\|_{\mathrm{W}^{1,\infty}}$ will then be required to properly bound $\mathcal{P}_{3,1}$, which explains that we can only deny $(\mathrm{W}^{1,\infty},\mathrm{L}^\infty)$-STLC or $(\mathrm{W}^{1,\infty},B)$-STLC in those two cases.

According to (5.70), we have

$$b_{1,\mu,\nu}^{3,L}=\delta\int u_{i_1}\cdots\int u_{i_{\ell-2\nu-\mu}}(\int u_{i_{\ell-2\nu-\mu+1}}\cdots\int u_{i_{\ell-2\nu}}\int u_1)(\int u_{i_{\ell-2\nu+1}}\cdots\int u_{i_{\ell-\nu}}\int u_1)(\int u_{i_{\ell-\nu+1}}\cdots\int u_{i_\ell}\int u_1). \tag{5.110}$$

We bound u_2 with $\|u_2\|_{\mathrm{L}^\infty}$ in (5.110) to obtain:

$$|b^{3,L}_{1,\mu,\nu}| \leqslant \delta \|u_2\|^{L_2}_{\mathrm{L}^\infty} \underbrace{\int \cdots \int}_{\ell-2\nu-\mu \text{ times}} \Big(\underbrace{\int \cdots \int}_{\mu \text{ times}} |v_1| \Big) \Big(\underbrace{\int \cdots \int}_{\nu \text{ times}} |v_1| \Big)^2. \tag{5.111}$$

Hence, if $\mu = \nu = 0$, (5.111) becomes

$$|b^{3,L}_{1,0,0}| \leqslant \delta \|u_2\|^{L_2}_{\mathrm{L}^\infty} \int_0^T \frac{(T-s)^{\ell-1}}{(\ell-1)!} |v_1(s)|^3 \mathrm{d}s, \tag{5.112}$$

if $\nu = 0$ and $\mu \geqslant 1$, (5.111) becomes

$$|b^{3,L}_{1,\mu,0}| \leqslant \delta \|u_2\|^{L_2}_{\mathrm{L}^\infty} \int_0^T \frac{(T-s)^{\ell-\mu-1}}{(\ell-\mu-1)!} v_1(s)^2 \int_0^s \frac{(s-\sigma)^{\mu-1}}{(\mu-1)!} |v_1(\sigma)| \mathrm{d}\sigma \mathrm{d}s, \tag{5.113}$$

and if $\nu \geqslant 1$, (5.111) becomes

$$|b^{3,L}_{1,\mu,\nu}| \leqslant \delta \|u_2\|^{L_2}_{\mathrm{L}^\infty} \int_0^T \frac{(T-s)^{\ell-2\nu-\mu-1}}{(\ell-2\nu-\mu-1)!} \Big(\int_0^s \frac{(s-\sigma)^{\nu-1}}{(\nu-1)!} |v_1(\sigma)| \mathrm{d}\sigma \Big)^2 \Big(\int_0^s \frac{(s-\sigma)^{\mu-1}}{(\mu-1)!} |v_1(\sigma)| \mathrm{d}\sigma \Big) \mathrm{d}s. \tag{5.114}$$

In (5.112), (5.113) and (5.114), we bound $(t-s)$ by t, $(s-\sigma)$ by s and s by t where necessary, which yields the following bound encompassing the three cases (5.112), (5.113) and (5.114):

$$|b^{3,L}_{1,\mu,\nu}| \leqslant A(\mu,\nu) T^{\ell-1} \|u_2\|^{L_2}_{\mathrm{L}^\infty} \int_0^T |v_1(s)|^3 \mathrm{d}s, \tag{5.115}$$

where $A(\mu,\nu)$ is equal to $\frac{\delta}{(l-1)!}$, $\frac{\delta}{(l-\mu-1)!(\mu-1)!}$ or $\frac{\delta}{(l-2\nu-\mu-1)!(\nu-1)!^2(\mu-1)!}$ depending on the cases enumerated above. Then, we use estimation (5.80) in Lemma 5.9 to obtain:

$$|b^{3,L}_{1,\mu,\nu}| \leqslant A(\mu,\nu) K \left(1 + \frac{1}{T}\right) T^{\ell-1} \|u_2\|^{L_2}_{\mathrm{L}^\infty} \|u_1\|_{\mathrm{W}^{1,\infty}} \|w_1\|^2_{\mathrm{L}^2}. \tag{5.116}$$

Recalling that $\mathcal{P}_{3,1}$ denotes the sum of all the terms of the Chen-Fliess series for which $k = 3$ and $\xi = 1$, we bound the operators $W^{3,L}_{1,\mu,\nu}$ by $C^{l+2}(l+2)!$ thanks to (5.3) and use (5.108) and the upper bound obtained in (5.116) to obtain

$$\begin{aligned} |\mathcal{P}_{3,1}| &\leqslant K \left(1 + \frac{1}{T}\right) \|u_1\|_{\mathrm{W}^{1,\infty}} \|w_1\|^2_{\mathrm{L}^2} \sum_{\ell \geqslant 2} C^{\ell+2} (\ell+2)! T^{\ell-1} \sum_{\mu,\nu} A(\mu,\nu) \sum_{L \in \{0,2\}^\ell} \|u_2\|^{L_2}_{\mathrm{L}^\infty} \\ &\leqslant K \left(1 + \frac{1}{T}\right) C^3 (1 + \|u_2\|_{\mathrm{L}^\infty}) \|u_1\|_{\mathrm{W}^{1,\infty}} \|w_1\|^2_{\mathrm{L}^2} \sum_{\ell \geqslant 0} (C(1 + \|u_2\|_{\mathrm{L}^\infty}) T)^\ell \sum_{\mu,\nu} (\ell+2)! A(\mu,\nu). \end{aligned}$$

For T small enough, the series above converges and therefore, there exists $T_{3,1}$ such that, for all $T \in [0, T_{3,1}]$, there exists $\delta(T) \leqslant T$ such that, for

$$\|u_1\|_{\mathrm{W}^{1,\infty}} \leqslant \delta(T), \tag{5.117}$$

one has

$$|\mathcal{P}_{3,1}| \leqslant \frac{K}{6}\|w_1\|_{\mathrm{L}^2}^2. \tag{5.118}$$

The series above converges for T small enough; therefore, there exists $T_{3,1}$ such that, for all $T \in [0, T_{3,1}]$, there exists $\delta(T) \leqslant T$ such that

$$\|u_1\|_{\mathrm{W}^{1,\infty}} \leqslant \delta(T) \;\Rightarrow\; |\mathcal{P}_{3,1}| \leqslant \frac{K}{6}\|w_1\|_{\mathrm{L}^2}^2. \tag{5.119}$$

Note that the bound $\delta(T)$ on $\|u_1\|_{\mathrm{W}^{1,\infty}}$ depends on the variable T, and not only on the fixed bound $T_{3,1}$, as is the case for the other estimations that we prove in this step.

ii) c. Case $\xi = 2, \mu = 0$.

Considering only the nonvanishing terms, we know that $\ell \geqslant 3$ and $\nu \geqslant 2$. According to (5.71), we have

$$b_{2,0,\nu}^{3,L} = \frac{1}{2}\Big(\int u_{i_1}\cdots\int u_{i_{\ell-\nu}} v_1^2\Big)\Big(\int u_{\ell-\nu+1}\cdots\int u_\ell \int u_1\Big). \tag{5.120}$$

There are two cases to study, depending on the value of i_ℓ.

(1) $i_\ell = 0$. We bound u_2 by $\|u_2\|_{\mathrm{L}^\infty}$ in (5.120):

$$|b_{2,0,\nu}^{3,L}| \leqslant \frac{1}{2}\|u_2\|_{\mathrm{L}^\infty}^{L_2}\Big(\int_0^T \frac{(T-s)^{\ell-\nu-1}}{(\ell-\nu-1)!}v_1(s)^2\mathrm{d}s\Big)\Big(\int_0^T \frac{(T-s)^{\nu-2}}{(\nu-2)!}w_1(s)\mathrm{d}s\Big). \tag{5.121}$$

Then, Cauchy-Schwarz inequality yields

$$|b_{2,0,\nu}^{3,L}| \leqslant \frac{1}{2}\|u_2\|_{\mathrm{L}^\infty}^{L_2}\frac{T^{\ell-\nu-\frac{1}{2}}}{\sqrt{2\ell-2\nu-1}(\ell-\nu-1)!}\|v_1\|_{\mathrm{L}^4}^2\frac{T^{\nu-\frac{3}{2}}}{\sqrt{2\nu-3}(\nu-2)!}\|w_1\|_{\mathrm{L}^2}. \tag{5.122}$$

Finally, we use estimation (5.79) in Lemma 5.9 to obtain:

$$|b_{2,0,\nu}^{3,L}| \leqslant \frac{1}{2}K\frac{T^{\ell-2}}{\sqrt{(2\ell-2\nu-1)(2\nu-3)}(\ell-\nu-1)!(\nu-2)!}\|u_2\|_{\mathrm{L}^\infty}^{L_2}\|u_1\|_{\mathrm{L}^\infty}^2\|w_1\|_{\mathrm{L}^2}^2. \tag{5.123}$$

(2) $i_\ell = 2$. We repeat steps (5.121) to (5.123) and obtain

$$|b_{2,0,\nu}^{3,L}| \leqslant \frac{1}{2}K\frac{T^{\ell-2}}{\sqrt{(2\ell-2\nu-1)(2\nu-3)}(\ell-\nu-1)!(\nu-2)!}\|u_2\|_{\mathrm{L}^\infty}^{L_2-1}\|u_1\|_{\mathrm{L}^\infty}^2\|w_1\|_{\mathrm{L}^2}\|w_2\|_{\mathrm{L}^2}, \tag{5.124}$$

hence

$$|b_{2,0,\nu}^{3,L}| \leqslant \frac{1}{4}K\frac{T^{\ell-2}}{\sqrt{(2\ell-2\nu-1)(2\nu-3)}(\ell-\nu-1)!(\nu-2)!}\|u_2\|_{\mathrm{L}^\infty}^{L_2-1}\|u_1\|_{\mathrm{L}^\infty}^2(\|w_1\|_{\mathrm{L}^2}^2+\|w_2\|_{\mathrm{L}^2}^2). \tag{5.125}$$

ii) d. Case $\xi = 2, \mu \geqslant 1$.

In this case, we notice that

$$b^{3,L}_{2,\mu,\nu} = b^{2,(i_1,\dots,i_{\ell-\nu})}_{\mu} b^{1,(i_{\ell-\nu+1},\dots,i_\ell)}_{1}. \tag{5.126}$$

Bounds for $|b^{2,(i_1,\dots,i_{\ell-\nu})}_{\mu}|$ were already found in (5.96), (5.99) and (5.102). Moreover, $|b^{1,(i_{\ell-\nu+1},\dots,i_\ell)}_{1}|$ is bounded as follows:

$$|b^{1,(i_{\ell-\nu+1},\dots,i_\ell)}_{1}| \leqslant \frac{T^{\nu+1}}{(\nu+1)!}\|u_2\|^{J_2}_{\mathrm{L}^\infty}\|u_1\|_{\mathrm{L}^\infty}, \tag{5.127}$$

where J_2 designates the number of 2's amongst $(i_{\ell-\nu+1},\dots,i_\ell)$.

Using (5.127) as well as (5.96), (5.99) and (5.102) in (5.126), we obtain the following bounds:

(1) if $i_{\ell-\mu-\nu}=0$ and $i_{\ell-\nu}=0$,

$$|b^{3,L}_{2,\mu,\nu}| \leqslant \frac{1}{2}\|u_2\|^{L_2}_{\mathrm{L}^\infty}\|u_1\|_{\mathrm{L}^\infty}\frac{\mu^2T^{\ell-2}}{(l-2\mu-\nu-1)!(\mu-1)!^2\nu!}\|w_1\|^2_{\mathrm{L}^2}. \tag{5.128}$$

(2) if $i_{\ell-\mu-\nu}=2$ and $i_{\ell-\nu}=2$,

$$|b^{3,L}_{2,\mu,\nu}| \leqslant \frac{1}{2}\|u_2\|^{L_2-2}_{\mathrm{L}^\infty}\|u_1\|_{\mathrm{L}^\infty}\frac{\mu^2T^{\ell-2}}{(l-2\mu-\nu-1)!(\mu-1)!^2\nu!}\|w_2\|^2_{\mathrm{L}^2}. \tag{5.129}$$

(3) if $i_{\ell-\mu-\nu}=0$ and $i_{\ell-\nu}=2$ or $i_{\ell-\mu-\nu}=2$ and $i_{\ell-\nu}=0$,

$$|b^{3,L}_{2,\mu,\nu}| \leqslant \frac{1}{4}\|u_2\|^{L_2-1}_{\mathrm{L}^\infty}\|u_1\|_{\mathrm{L}^\infty}\frac{\mu^2T^{\ell-2}}{(l-2\mu-\nu-1)!(\mu-1)!^2\nu!}(\|w_1\|^2_{\mathrm{L}^2}+\|w_2\|^2_{\mathrm{L}^2}). \tag{5.130}$$

Let us call $\mathcal{P}_{3,2}$ the sum of all the terms of the Chen-Fliess series for which $k=3$ and $\xi=2$. We proceed as for $\xi=1$: bounding the operators $W^{3,L}_{2,\mu,\nu}$ by $C^{l+2}(l+2)!$ thanks to (5.3) and using the upper bounds obtained in (5.123), (5.125), and (5.128) to (5.130), as well as (5.108), we obtain that there exists $T_{3,2}$ such that for all $T\in[0,T_{3,2}]$,

$$|\mathcal{P}_{3,2}| \leqslant \frac{K}{6}(\|w_1\|^2_{\mathrm{L}^2}+\|w_2\|^2_{\mathrm{L}^2}). \tag{5.131}$$

ii) e. Case $\xi=3$.

Considering only the nonvanishing terms, we know that $\mu\geqslant 2$, $\nu\geqslant 1$ and $\ell-\mu-\nu\geqslant 2$. There is again three cases to study, depending on the value of $i_{\ell-\mu-\nu}$ and i_ℓ.

(1) $i_{\ell-\mu-\nu}=0$ and $i_\ell=0$. Bounding u_2 with $\|u_2\|_{\mathrm{L}^\infty}$ yields

$$|b^{3,L}_{3,\mu,\nu}| \leqslant \delta\|u_2\|^{L_2}_{\mathrm{L}^\infty}\Big(\underbrace{\int\cdots\int}_{\ell-\mu-\nu-1\text{ times}} w_1\Big)\Big(\underbrace{\int\cdots\int}_{\mu-1\text{ times}} w_1\Big)\Big(\underbrace{\int\cdots\int}_{\nu+1\text{ times}} u_1\Big). \tag{5.132}$$

Using Cauchy-Schwarz inequality on the first two terms and equation (5.127) on the last term of (5.132), we obtain

$$|b^{3,L}_{3,\mu,\nu}| \leqslant \delta \|u_2\|^{L_2}_{\mathrm{L}^\infty} \|u_1\|_{\mathrm{L}^\infty} \frac{T^{\ell-2}}{\sqrt{(2\ell-2\mu-2\nu-3)(2\mu-3)}(\ell-2\mu-\nu-2)!(\mu-2)!\nu!} \|w_1\|^2_{\mathrm{L}^2}. \tag{5.133}$$

(2) $i_{\ell-\mu-\nu} = 2$ and $i_\ell = 2$. Repeating steps (5.132) and (5.133) yields

$$|b^{3,L}_{3,\mu,\nu}| \leqslant \delta \, \|u_2\|^{L_2-2}_{\mathrm{L}^\infty} \|u_1\|_{\mathrm{L}^\infty} \frac{T^{\ell-2}}{\sqrt{(2\ell-2\mu-2\nu-3)(2\mu-3)}(\ell-2\mu-\nu-2)!(\mu-2)!\nu!} \|w_2\|^2_{\mathrm{L}^2}. \tag{5.134}$$

(3) $i_{\ell-\mu-\nu} = 2$ and $i_\ell = 0$ or $i_{\ell-\mu-\nu} = 0$ and $i_\ell = 2$. Repeating steps (5.132) and (5.133) yields

$$|b^{3,L}_{3,\mu,\nu}| \leqslant \frac{1}{2} \delta \|u_2\|^{L_2-1}_{\mathrm{L}^\infty} \|u_1\|_{\mathrm{L}^\infty} \frac{T^{\ell-2}}{\sqrt{(2\ell-2\mu-2\nu-3)(2\mu-3)}(\ell-2\mu-\nu-2)!(\mu-2)!\nu!} (\|w_1\|^2_{\mathrm{L}^2} + \|w_2\|^2_{\mathrm{L}^2}). \tag{5.135}$$

Let us call $\mathcal{P}_{3,3}$ the sum of all the terms of the Chen-Fliess series for which $k = 3$ and $\xi = 3$. We proceed as before : bounding the operators $W^{3,L}_{3,\mu,\nu}$ by $C^{l+2}(l+2)!$ thanks to (5.3) and using the upper bounds obtained in (5.133), (5.134), and (5.135), we obtain that there exists $T_{3,3}$ such that for all $T \in [0, T_{3,3}]$,

$$|\mathcal{P}_{3,3}| \leqslant \frac{K}{6} (\|w_1\|^2_{\mathrm{L}^2} + \|w_2\|^2_{\mathrm{L}^2}). \tag{5.136}$$

iii) Terms in $\mathcal{P}_{\geqslant 4}$.

We apply Lemma 5.3 with $r = 4$, and obtain the existence of $T_4 > 0$ and some constant $D > 0$ such that for $T \in [0, T_4]$,

$$|\mathcal{P}_{\geqslant 4}| \leqslant DT \|v_1\|^4_{\mathrm{L}^4}. \tag{5.137}$$

Then, applying (5.79) from Lemma 5.9 in (5.137), we obtain that

$$|\mathcal{P}_{\geqslant 4}| \leqslant MDT \|u_1\|^2_{\mathrm{L}^\infty} \|w_1\|^2_{\mathrm{L}^2}, \tag{5.138}$$

$$\leqslant \frac{K}{6} \|w_1\|^2_{\mathrm{L}^2}, \tag{5.139}$$

for $\|u_1\|_{\mathrm{L}^\infty}$ smaller than $T_5 = \left(\frac{K}{12MDT_4}\right)^{1/2}$.

Step 5. Sum of all the terms. So far, we have obtained the following bounds for parts of the Chen-Fliess series:

- (5.90) for $\mathcal{P}_{2,1}$ and (5.109) for $\mathcal{P}_{2,2}$,
- (5.118) for $\mathcal{P}_{3,1}$, (5.131) for $\mathcal{P}_{3,2}$, and (5.136) for $\mathcal{P}_{3,3}$
- (5.139) for terms in $\mathcal{P}_{\geqslant 4}$.

These bounds are valid for T small enough and occasionally under smallness assumptions on the controls; in particular, recall that:

- (5.90) requires the smallness of $\|u_2\|_{\mathrm{L}^\infty}$ in Cases 1. and 3. – in the two other cases, we have $\mathcal{P}_{2,1} = 0$,
- (5.118) requires that $\|u_1\|_{\mathrm{W}^{1,\infty}} \leqslant \delta(T)$ in Cases 1. and 2. – in the two other cases, we have $\mathcal{P}_{3,1} = 0$,

where $\delta(T)$ is defined in (5.117) and, by the generic term "smallness", we mean upper bounds depending on the constants T_1, T_2, $T_{3,1}$, $T_{3,2}$, $T_{3,3}$, T_4, T_5. We are now ready to sum all the terms and conclude the proof.

Let $T^0 = \min(T_1, T_2, T_{3,1}, T_{3,2}, T_{3,3}, T_4, T_5)$, T in $(0, T^0)$ and $\varepsilon^0 \in (0, \delta(T))$. Let $\alpha > 0$ and u_1 and u_2 controls on $(0, T)$ satisfying one of the assumptions (5.43), (5.44), (5.45), or (5.46) in respectively Cases 1., 2., 3. or 4.. Gathering all the bounds, and using the lower bound for $\mathcal{P}_{\text{dom}}$ obtained in (5.78), we deduce that the Chen-Fliess series $\Sigma(u, f, \Phi, T)$ satisfies

$$\Sigma(u, f, \Phi, T) \geqslant \mathcal{P}_{\text{dom}} - |\mathcal{P}_{2,1}| - |\mathcal{P}_{2,2}| - |\mathcal{P}_{3,1}| - |\mathcal{P}_{3,2}| - |\mathcal{P}_{3,3}| - |\mathcal{P}_{\geqslant 4}|, \tag{5.140}$$

$$\geqslant \mathcal{P}_{\text{dom}} - 6 \cdot \frac{K}{6}(\|w_1\|^2_{\mathrm{L}^2} + \|w_2\|^2_{\mathrm{L}^2}), \tag{5.141}$$

$$\geqslant K(\|w_1\|^2_{\mathrm{L}^2} + \|w_2\|^2_{\mathrm{L}^2}), \tag{5.142}$$

which is precisely equation (5.41) in Proposition 5.4, and therefore completes the proof. □

6. Conclusion

In this paper, we explored the controllability properties of systems with two controls (1.3), satisfying the assumption (1.4). We have stated two results on these systems, Theorem 3.2 and Theorem 3.8, providing necessary conditions for local controllability around equilibria. These results extend the classical necessary conditions stated for scalar-control systems in [27]. Moreover, they are, to the best of our knowledge, the first results of this nature for non-scalar-input systems.

This work does not only present a theoretical interest for control theory. Using Theorem 3.2, we were able to completly solve an open question concerning the local controllability of magnetically controlled micro-swimming robots (see Exam. 4.3). One can use our results to easily and systematically address local controllability issues in similar applied situations.

Our necessary conditions are only based on brackets of order three and five, but there are higher-order brackets that may prevent S_2 to be contained in S_1 (in the single-input case see for instance [2]). Giving necessary conditions based on these brackets, for instance adapting the results from [2] to the situation (1.3)-(1.4), is a possible continuation of the present work. The complexity of the higher-order terms structure in the Chen-Fliess series however makes the analysis very intricate.

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