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APPROXIMATE INTERNAL CONTROLLABILITY AND SYNCHRONIZATION OF A COUPLED SYSTEM OF WAVE EQUATIONS

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Abstract. Based on the uniqueness of solution to a coupled system of wave equations associated with incomplete internal observations, we establish the approximate internal synchronization by groups, the induced internal synchronization and the approximate internal synchronization in the pinning sense.

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1. Introduction

Let A be a matrix of order N and D be a full column-rank matrix of order $N \times M$. Both A and D are composed of constant entries. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^1 boundary Γ. We consider the following problem for the variable $U = (u^{(1)}, \cdots, u^{(N)})^T$:

$$\begin{cases} U'' - \Delta U + AU = D\chi_\omega H & \text{in } \mathbb{R}^+ \times \Omega, \\ U = 0 & \text{on } \mathbb{R}^+ \times \Gamma \end{cases} \tag{1.1}$$

and

$$t = 0: \quad U = \widehat{U}_0,\ U' = \widehat{U}_1 \quad \text{in } \Omega, \tag{1.2}$$

where $H = (h^{(1)}, \cdots, h^{(M)})^T$ denotes the internal controls and χ_ω is the characteristic function of the subdomain $\omega \subset \Omega$.

The approximate internal controllability was well studied in [22]. In this paper, we will further develop the theory of approximate internal synchronization and clarify the relation between the internal synchronization in the pinning sense and that in the consensus sense.

Keywords and phrases: Uniqueness of solution, incomplete observation, approximate controllability, approximate synchronization.

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The exact internal controllability of wave equations was abundantly studied in the literatures. We quote [6, 10, 30] for a single 1-d wave equation with locally distributed control in any fixed nonempty subinterval of a bounded interval. In higher dimensional case, the exact controllability was established by HUM method in [23, 32, 33] with control distributed in an ϵ-neighbourhood ω of the boundary Γ satisfying the usual multiplier control condition.

In the case of fewer internal controls, namely, when $\mathrm{rank}(D) < N$, system (1.1) is not exactly controllable (see [22], Thm. 10). In order to relax the restrictions on the domain ω and on the control matrix D, we studied in [22] the approximate controllability of system (1.1), which is equivalent to the uniqueness of solution to the adjoint system for the variable $\Phi = (\phi^{(1)}, \cdots, \phi^{(N)})^T$:

$$\begin{cases} \Phi'' - \Delta\Phi + A^T\Phi = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \Phi = 0 & \text{on } \mathbb{R}^+ \times \Gamma \end{cases} \tag{1.3}$$

associated with the initial data

$$t = 0: \quad \Phi = \Phi_0, \quad \Phi' = \Phi_1 \quad \text{in } \Omega \tag{1.4}$$

and the internal observation

$$D^T\chi_\omega\Phi \equiv 0 \quad \text{in } [0, T] \times \Omega. \tag{1.5}$$

By classic theory of semigroups (see [27]), system (1.3) forms a C^0-semigroup in the space $(L^2(\Omega) \times H^{-1}(\Omega))^N$. Moreover, it is easy to see that Kalman's rank condition:

$$\mathrm{rank}(D, AD, \cdots, A^{N-1}D) = N \tag{1.6}$$

is necessary for the uniqueness. However, when $\mathrm{rank}(D) < N$, the incomplete observation (1.5) cannot imply the nullity of all the components:

$$\chi_\omega\Phi = 0 \quad \text{in } (0, T) \times \Omega. \tag{1.7}$$

So, the uniqueness of solution to problems (1.3)–(1.4) cannot be considered as a standard continuation theorem of Holmgren's type. However, because of the commutation of the internal observation with d'Alembert operator:

$$D^T\chi_\omega\Box\Phi = \Box D^T\chi_\omega\Phi \quad \text{in } \mathcal{D}'((0, T) \times \omega), \tag{1.8}$$

it turns out that Kalman's rank condition (1.6) plays the same role as in the case of ODEs [9]. We have stablished the following result.

Theorem 1.1. *([22], Thm. 10) Assume that the adjoint problems (1.3)–(1.5) has only the trivial solution, then (A, D) satisfies Kalman's rank condition (1.6). Conversely, assume that (A, D) satisfies Kalman's rank condition (1.6), then the adjoint problems (1.3)–(1.5) has only the trivial solution, provided that $T > 2d(\Omega)$, where $d(\Omega)$ denotes the geodesic diameter of Ω.*

The above theorem is valid without any restriction on the matrix A, nor on the damping domain ω. Moreover, the observation time $T > 2d(\Omega)$ is independent of $\mathrm{rank}(D)$ and of the order N of A. This is essentially different from the case of boundary observation, where the observation time T depends on the rank of the matrix D and the number N of the equations (see [14, 31]).

Now let us comment the related literature. The situation is much complicated for the uniqueness of problems (1.3)–(1.4) associated with boundary observation:

$$D^T \partial_\nu \Phi \equiv 0 \quad \text{in } [0,T] \times \Gamma. \tag{1.9}$$

Our basic idea is to combine the uniform observability of a scalar system and Kalman's rank condition (1.6). The first attempt for realizing this idea was carried out in [13, 14] for a system of wave equations with Dirichlet boundary condition by incomplete Neumann observations. Later, this idea was used in [11, 19] for Neumann and Robin boundary conditions, and further developed in [18, 21] for an elliptic system with Neumann boundary conditions observed by incomplete Dirichlet observations. Additionally, the coupling matrix A should be nilpotent (see [2, 14, 16]), or symmetric and close to a scalar matrix etc. (see [19, 31]). Unlike the hyperbolic system, which was less studied and the obtained results are of different natures (see [1, 7, 26, 29]), the parabolic problem has been abundantly investigated in the literature. We only quote [3, 4] and the references therein for the internal control of coupled systems of heat equations with the same diffusion coefficients and constant or time-dependent coupling terms by means of Carleman estimates. We also mention [24] for the internal controllability of a system of heat equations with analytic nonlocal coupling terms and [25] for the internal observability of some parabolic equations with constant or time-dependent coupling terms using Lebeau–Robbiano strategy.

The paper is organized as follows. In Section 2, we complete the work in [22] by a charaterization of the approximate internal controllability, then we first show the equivalence of the approximate internal controllability of system (1.1) and the D-observability of adjoint system (1.3). In Section 3, we show the equivalence of the approximate internal controllability of system (1.1) and Kalman's rank condition (1.6). We then recall the relation between the approximate controllability and the exact controllability. In Section 4, we clarify the notion of indirect control by successive projections on the null space of controls. In Section 5, we study the approximate internal synchronization by p-groups of system (1.1) under the condition of C_p-compatibility. In Section 6, we establish the necessity of the condition of C_p-compatibility under the minimal rank condition. In Section 7, we further give the additional information of synchronization in the general situation. In the last section, we summarize the materials obtained for approximate internal and boundary controllabilities, and indicate some problems to be examined in the future.

2. Approximate internal controllability and D-observability

Let us first recall the following standard well-posedness result (see [5], [6] and [27]).

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ of class C^1. For any given $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$ and any given $H \in (L^1_{loc}(\mathbb{R}^+; L^2(\Omega)))^M$, problems (1.1)–(1.2) admits a unique weak solution U in the space*

$$(C^0_{loc}(\mathbb{R}^+; H^1_0(\Omega)))^N \cap (C^1_{loc}(\mathbb{R}^+; L^2(\Omega)))^N. \tag{2.1}$$

Moreover, the mapping

$$(\widehat{U}_0, \widehat{U}_1, H) \to U \tag{2.2}$$

is linear and continuous for the corresponding topologies.

Definition 2.2. System (1.1) is approximately null controllable at the time $T > 0$ in the space $(H_0^1(\Omega) \times L^2(\Omega))^N$ if for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$, there exists a sequence $\{H_n\}$ of internal controls in $(L^2(\mathbb{R}^+; L^2(\Omega)))^M$ with compact support in $[0, T]$, such that the sequence $\{U_n\}$ of corresponding solutions satisfies the following condition

$$U_n \to 0 \quad \text{in } (C^0_{loc}([T,+\infty); H^1_0(\Omega)))^N \cap (C^1_{loc}([T,+\infty); L^2(\Omega)))^N \tag{2.3}$$

as $n \to +\infty$.

Definition 2.3. The adjoint system (1.3) is D-observable on the interval $[0,T]$ in the space $(L^2(\Omega)\times H^{-1}(\Omega))^N$ if for any given initial data $(\Phi_0,\Phi_1)\in (L^2(\Omega)\times H^{-1}(\Omega))^N$, the observation

$$D^T\chi_\omega \Phi \equiv 0 \quad \text{in } [0,T]\times\omega \tag{2.4}$$

implies that $(\Phi_0,\Phi_1)\equiv 0$, that is $\Phi\equiv 0$.

In what follows, we will establish the equivalence between the approximate internal controllability of the original system (1.1) and the D-observability of the adjoint system (1.3).

Let $\mathcal{C}$ be the set of all the initial states $(V(0),V'(0))$ given by the backward problem:

$$\begin{cases} V'' - \Delta V + AV = D\chi_\omega H & \text{in } (0,T)\times\Omega, \\ V = 0 & \text{on } (0,T)\times\Gamma, \\ t = T: \quad V = V' = 0 & \text{in } \Omega \end{cases} \tag{2.5}$$

as the internal control H runs through the space $(L^2(0,T;L^2(\Omega)))^M$. Similarly to the approximate boundary controllability in [17], we can show the following result.

Proposition 2.4. *System (1.1) is approximately null controllable at the time $T>0$ in the space $(H^1_0(\Omega)\times L^2(\Omega))^N$ if and only if*

$$\overline{\mathcal{C}} = (H^1_0(\Omega)\times L^2(\Omega))^N. \tag{2.6}$$

Proof. Let

$$\mathcal{R}: \quad (\widehat{U}_0,\widehat{U}_1,H)\to (U,U')$$

denote the resolution of problem (1.1)–(1.2). By Proposition 2.1, $\mathcal{R}$ is bounded from $(H^1_0(\Omega)\times L^2(\Omega))^N \times (L^2(0,T;L^2(\Omega)))^M$ to $(C^0([0,T];H^1_0(\Omega)\times L^2(\Omega))^N$. In particular, we have

$$\|\mathcal{R}(\widehat{U}_0,\widehat{U}_1,0)(t)\| \sim \|(\widehat{U}_0,\widehat{U}_1)\|, \quad 0\leqslant t\leqslant T. \tag{2.7}$$

For any given initial data $(\widehat{U}_0,\widehat{U}_1)\in (H^1_0(\Omega)\times L^2(\Omega))^N$ and any given control function $H\in (L^2(0,T;L^2(\Omega)))^M$, we have

$$\mathcal{R}(\widehat{U}_0,\widehat{U}_1,H) = \mathcal{R}(\widehat{U}_0 - V(0),\widehat{U}_1 - V'(0),0) + \mathcal{R}(V(0),V'(0),H).$$

Noting

$$\mathcal{R}(V(0),V'(0),H)(T) = 0, \tag{2.8}$$

it follows that

$$\mathcal{R}(\widehat{U}_0,\widehat{U}_1,H)(T) = \mathcal{R}(\widehat{U}_0 - V(0),\widehat{U}_1 - V'(0),0)(T).$$

Then, by (2.7), we have

$$\|\mathcal{R}(\widehat{U}_0, \widehat{U}_1, H)(T)\| \sim \|(\widehat{U}_0 - V(0), \widehat{U}_1 - V'(0))\|. \tag{2.9}$$

Now assume that equation (2.6) holds. Then, for any given $\epsilon > 0$ and $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$, there exists $H \in (L^2(0,T; L^2(\Omega)))^M$, such that the corresponding solutions V to the backward problem (2.5) satisfies

$$\|(V(0) - \widehat{U}_0, V'(0) - \widehat{U}_1)\| \leqslant \epsilon, \tag{2.10}$$

which, together with equation (2.9), implies that

$$\|\mathcal{R}(\widehat{U}_0, \widehat{U}_1, H)(T)\| \leqslant C\epsilon. \tag{2.11}$$

Conversely, assume that for any given $\epsilon > 0$ and any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$, there exists a control function $H \in (L^2(0,T; L^2(\Omega)))^M$, such that (2.11) holds. Then, using (2.9), we get

$$\|(\widehat{U}_0 - V(0), \widehat{U}_1 - V'(0)\| \leqslant C'\epsilon,$$

which implies (2.6). The proof is thus complete. □

Theorem 2.5. *System (1.1) is approximately null controllable at the time $T > 0$ in the space $(H_0^1(\Omega) \times L^2(\Omega))^N$ if and only if the adjoint system (1.4) is D-observable on the interval $[0,T]$ in the space $(L^2(\Omega) \times H^{-1}(\Omega))^N$.*

Proof. Let Φ be a solution to the adjoint problem (1.3), respectively V be a solution to the backward problem (2.5). Then, multiplying equation (2.5) by Φ and integrating by parts, we get

$$\langle (V(0), V'(0)), (\Phi_1, -\Phi_0) \rangle = \int_0^T \int_\Omega \Phi^T D \chi_\omega H \mathrm{d}x\mathrm{d}t, \tag{2.12}$$

where $\langle \cdot, \cdot \rangle$ denote the duality between the spaces $(H_0^1(\Omega) \times L^2(\Omega))^N$ and $(H^{-1}(\Omega) \times L^2(\Omega))^N$.

Now assume that system (1.1) is not approximately null controllable. Then, by Proposition 2.4, there exists a non-trivial element $(-\Phi_1, \Phi_0) \in \mathcal{C}^\perp$. Here, the orthogonality is defined in the sense of duality, therefore, $(-\Phi_1, \Phi_0) \in (H_0^1(\Omega) \times L^2(\Omega))^N$. Let Φ be the corresponding solution to problems (1.3)–(1.4), it follows from equation (2.12) that

$$\int_0^T \int_\Omega \Phi^T D \chi_\omega H \mathrm{d}x\mathrm{d}t = 0 \tag{2.13}$$

for all $H \in (L^2(0,T; L^2(\Omega)))^M$, which implies the D-observation (2.4). We then get a contradiction.

Conversely, let Φ be a non-trivial solution satisfying equation (2.4). It follows from equation (2.12) that

$$\langle (V(0), V'(0)), (\Phi_1, -\Phi_0) \rangle = 0 \tag{2.14}$$

for all initial data $(V(0), V'(0)) \in \mathcal{C}$. Therefore, $(-\Phi_1, \Phi_0) \in \mathcal{C}^\perp$ with $(-\Phi_1, \Phi_0) \neq (0,0)$, namely, $\mathcal{C}$ is not dense in $(H_0^1(\Omega) \times L^2(\Omega))^N$. The proof is complete. □

3. Approximate internal controllability

As a direct consequence of Theorems 1.1 and 2.5, we have the following result.

Theorem 3.1. *([22], Thm. 13) If system (1.1) is approximately null controllable at the time T in the space $(H_0^1(\Omega)\times L^2(\Omega))^N$, then (A, D) satisfies Kalman rank condition (1.6). Conversely, under Kalman rank condition (1.6), system (1.1) is approximately null controllable at the time T in the space $(H_0^1(\Omega)\times L^2(\Omega))^N$, provided that $T > 2d(\Omega)$, where $d(\Omega)$ denotes the geodesic diameter of Ω.*

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^2 boundary Γ. For any given $x_0 \in \mathbb{R}^n$, we define

$$\Gamma(x_0) = \{x \in \Omega : \quad (x - x_0)\cdot \nu(x) > 0\}, \quad T(x_0) = 2\max_{x\in\overline{\Omega}} |x - x_0|, \tag{3.1}$$

where $\nu(x)$ is the unit outer normal vector on Γ. For a good understanding of the conception of the approximate controllability, we first recall the following result on the exact controllability.

Proposition 3.2. *([32], Thm. 2.3) Let ω be a neighbourhood of $\overline{\Gamma}(x_0)$ in Ω. Assume furthermore that D is invertible. Then system (1.1) is exactly controllable at any given time $T > T_0$ in the space $(H_0^1(\Omega)\times L^2(\Omega))^N$.*

In the case of fewer internal controls, we have the following negative result. This is the motivation for considering the approximate controllability.

Proposition 3.3. *([22], Thm. 10) Assume that rank $(D) < N$. Then no matter how large the time $T > 0$ is, system (1.1) is not exactly null controllable in the space $(H_0^1(\Omega)\times L^2(\Omega))^N$.*

Noting that the rank of D in equation (1.6) may be much smaller than N, this is the advantage to consider the approximate internal controllability. However, the following result shows that the sequence of controls $\{H_n\}$ is unbounded in general.

Proposition 3.4. *([22], Prop. 14) System (1.1) is exactly null controllable at the time T in the space $(H_0^1(\Omega)\times L^2(\Omega))^N$ by means of an internal control $H \in (L^2(0,T;L^2(\Omega)))^N$, if and only if it is approximately null controllable at the time T in the space $(H_0^1(\Omega)\times L^2(\Omega))^N$ by means of a bounded sequence $\{H_n\}$ of internal controls in $(L^2(0,T;L^2(\Omega)))^N$.*

4. Notion on indirect controls

In [30], D. Russell introduced the notion of indirect stabilization. It concerns if the dissipation induced by one of the equations can be sufficiently transmitted to the other ones in order to realize the stability of the overall system. The effectiveness of the indirect damping depends in a very complex way on all of the involved factors such as the nature of the coupling term, the order of dissipation, the hidden regularity of undamped equations, the accordance of boundary conditions and many others. We refer [8, 28] and the reference therein for the recent progress.

In this section, we try to explain the meaning of indirect controls and the mechanism of their roles for approximate controllability. The basic idea is to project system (1.1) to $\mathrm{Ker}(D^T)$ for getting a family of systems with homogeneous boundary condition. We first show the idea by a simple example:

$$\begin{cases} u_1'' - \Delta u_1 + u_2 = 0 & \text{in } \mathbb{R}^+\times\Omega,\\ u_2'' - \Delta u_2 + u_3 = 0 & \text{in } \mathbb{R}^+\times\Omega,\\ u_3'' - \Delta u_3 = 0 & \text{in } \mathbb{R}^+\times\Omega,\\ u_1 = u_2 = 0, \quad u_3 = h & \text{on } \mathbb{R}^+\times\Gamma. \end{cases} \tag{4.1}$$

First let

$$A = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}, \quad D = \begin{pmatrix} 0\\ 0\\ 1\end{pmatrix}.$$

We have

$$\mathrm{Ker}(D^T) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Then, applying the row-vectors $(1, 0, 0)$ and $(0, 1, 0)$ in $\mathrm{Ker}(D^T)$ to system (4.1), we get

$$\begin{cases} u_1'' - \Delta u_1 + u_2 = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_2'' - \Delta u_2 = -u_3 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_1 = u_2 = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{4.2}$$

The reduced system (4.2) is for the variables u_1 and u_2, so at the first step, the variable $h^{(1)} = -u_3$ can be formally regarded as an internal control appearing in system (4.2). However, the value of $h^{(1)}$ cannot be freely chosen, then we call it as an indirect internal control.

Next let

$$A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

We have

$$\mathrm{Ker}(D_1^T) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Then, applying the row-vector $(1, 0)$ in $\mathrm{Ker}(D_1^T)$ to the reduced system (4.2), at the second step we get

$$\begin{cases} u_1'' - \Delta u_1 = -u_2 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_1 = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{4.3}$$

This is a system for the variable u_1, in which the variable $h^{(2)} = -u_2$ can be regarded as an indirect internal control.

Finally, let

$$A_2 = (0), \quad D_2 = (-1). \tag{4.4}$$

Since $\mathrm{Ker}(D_2^T) = (0)$, we stop the projection.

By this way, we decompose the original system (4.1) into two sub-systems (4.2) and (4.3). Consequently, besides the direct boundary control h acting on the boundary and appearing in the original system (4.1), we find two indirect internal controls $h^{(1)}$ and $h^{(2)}$, which are hidden in the sub-systems (4.2) and (4.3), respectively.

Now we present the general procedure. Let A_0 be a matrix of order N_0, and D_0 be a matrix of order $N_0 \times M$. For $l = 1, 2, \cdots$, let

$$N_l = N_{l-1} - \mathrm{rank}(D_{l-1}). \tag{4.5}$$

Define

$$\mathrm{Ker}(D_{l-1}^T) = \mathrm{Span}\{d_1, \cdots, d_{N_l}\} \text{ and } K_{l-1} = (d_1, \cdots, d_{N_l}). \tag{4.6}$$

In particular, we have

$$D_{l-1}^T K_{l-1} = 0. \tag{4.7}$$

Noting that

$$\mathrm{Im}(K_{l-1}) \oplus \mathrm{Im}(D_{l-1}) = \mathrm{Ker}(D_{l-1}^T) \oplus \mathrm{Im}(D_{l-1}) = \mathbb{R}^{N_{l-1}}, \tag{4.8}$$

there exists a matrix A_l of order N_l and a matrix D_l of order $N_l \times M$, such that

$$K_{l-1}^T A_{l-1} = A_l K_{l-1}^T - D_l D_{l-1}^T. \tag{4.9}$$

Then, noting (4.7), we have

$$K_{l-1}^T K_{l-1} A_l^T = K_{l-1}^T A_{l-1}^T K_{l-1} \tag{4.10}$$

and

$$D_{l-1}^T D_{l-1} D_l^T = -D_{l-1}^T A_{l-1}^T K_{l-1}. \tag{4.11}$$

Since K_{l-1} is of full column-rank, then $\mathrm{Ker}(K_{l-1}) = \{0\}$. It follows from equation (4.10) that

$$A_l^T = (K_{l-1}^T K_{l-1})^{-1} K_{l-1}^T A_{l-1}^T K_{l-1}. \tag{4.12}$$

Since D_{l-1} may be not of full column-rank, $Ker(D_{l-1}) \neq \{0\}$ in general. In order to uniquely determine the matrix D_l by relation (4.11), we should require

$$\mathrm{Im}(D_l^T) \cap \mathrm{Ker}(D_{l-1}) = \{0\}. \tag{4.13}$$

We may continue the procedure of projection until $D_L = 0$ or $\mathrm{Ker}(D_L^T) = \{0\}$.

Now let

$$N_0 = N, \quad A_0 = A, \quad D_0 = D, \quad U^{(0)} = U, \quad H^{(0)} = H.$$

Defining

$$U^{(1)} = K_0^T U^{(0)}, \quad H^{(1)} = D_0^T U^{(0)}$$

and applying K_0^T to equation (1.1) and noting equations (4.7) and (4.9) for $l = 1$, we get

$$\begin{cases} U^{(1)''} - \Delta U^{(1)} + A_1 U^{(1)} = D_1 H^{(1)} & \text{in } \mathbb{R}^+ \times \Omega, \\ U^{(1)} = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{4.14}$$

Assume that $U^{(l-1)}$ are determined for $l \geqslant 2$ by

$$\begin{cases} U^{(l-1)''} - \Delta U^{(l-1)} + A_{l-1} U^{(l-1)} = D_{l-1} H^{(l-1)} & \text{in } \mathbb{R}^+ \times \Omega, \\ U^{(l-1)} = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{4.15}$$

Let

$$U^{(l)} = K_{l-1}^T U^{(l-1)}, \quad H^{(l)} = D_{l-1}^T U^{(l-1)}. \tag{4.16}$$

For $l \geqslant 2$, applying K_{l-1}^T to equation (4.15) and noting equations (4.7) and (4.9), we get

$$\begin{cases} U^{(l)\prime\prime} - \Delta U^{(l)} + A_l U^{(l)} = D_l H^{(l)} & \text{in } \mathbb{R}^+ \times \Omega, \\ U^{(l)} = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{4.17}$$

The terms $H^{(l)}(1 \leqslant l \leqslant L)$ can be formally regarded as internal controls in the reduced systems (4.17). Thus, the original system (1.1) is directly controlled by $H^{(0)}$, and indirectly controlled by $H^{(1)}, \cdots, H^{(L)}$, which are hidden in subsystems (4.17) and intervene into the system at different steps of the reduction.

Accordingly, let $\text{rank}(D_l)$ denote the number of indirect controls $H^{(l)}$ for $1 \leqslant l \leqslant L$. Then, the following formula (see [20], Prop. 3.3)

$$\text{rank}(D, AD, \cdots, A^{N-1}D) = \sum_{l=0}^{L} \text{rank}(D_l) \tag{4.18}$$

shows that $\text{rank}(D, AD, \cdots, A^{N-1}D)$ is the total number of (direct and indirect) controls. However, the control $H^{(l)}$ is implicitly given *via* $U^{(l-1)}$ for $1 \leqslant l \leqslant L$, so, its value cannot be independently chosen, then $H^{(l)}(1 \leqslant l \leqslant L)$ will be called the indirect internal controls. Nevertheless, the following result reveals the role of the indirect internal controls $H^{(l)}(1 \leqslant l \leqslant L)$ for the approximate internal null controllability.

Proposition 4.1. *If system (1.1) is approximately null controllable, then the subsystems (4.17) are approximately null controllable for all l with $1 \leqslant l \leqslant L$.*

Proof. Noting that

$$U^{(l)} = K_{l-1}^T \cdots K_0^T U^{(0)}, \quad 1 \leqslant l \leqslant L, \tag{4.19}$$

the approximate internal controllability of system (1.1) implies well that of subsystems (4.17). Moreover, by [20], Proposition 3.4, we have

$$\text{rank}(D_l, A_l D_l, \cdots, A_l^{N_l - 1} D_l) = N_l \tag{4.20}$$

for all l with $0 \leqslant l \leqslant L$. □

Consider the following example:

$$A_0 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad D_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

In this case, K_0 can be taken as

$$K_0 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Then, applying equations (5.4)–(5.5) with $l = 1$, a straightforward computation gives

$$D_1^T = -(D_0^T D_0)^{-1} D_0^T A_0^T K_0 = (1, 0)$$

and

$$A_1^T = (K_0^T K_0)^{-1} K_0^T A_0^T K_0 = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}.$$

Similarly, applying equations (5.4)–(5.5) with $l = 2$ to

$$A_1 = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we get

$$A_2 = (-2), \quad D_2 = (2).$$

Since $\text{Ker}(D_2^T) = \{0\}$, we stop the projection. All the pairs (A_l, D_l) satisfy Kalman's rank condition (4.20) with

$$\text{rank}(D_0, A_0 D_0, A_0^2 D_0) = 3, \quad \text{rank}(D_1, A_1 D_1) = 2, \quad \text{rank}(D_2) = 1.$$

System (1.1) as well as subsystems (4.17) for $l = 1, 2$ are approximately null controllable.

5. Approximate internal synchronization by groups

When the rank of Kalman's matrix is fewer than N, the approximate internal controllability fails. As in the case of boundary controls studied in [14], we will consider the corresponding synchronization by groups.

Let $p \geqslant 1$ be an integer such that

$$0 = n_0 < n_1 < \cdots < n_p = N$$

with $n_r - n_{r-1} \geqslant 2$ for $r = 1, \cdots, p$. We re-arrange the components of the state variable U into p groups

$$(u^{(1)}, \cdots, u^{(n_1)}),\ (u^{(n_1+1)}, \cdots, u^{(n_2)}), \cdots\cdots, (u^{(n_{p-1}+1)}, \cdots, u^{(n_p)}).$$

Definition 5.1. System (1.1) is approximately synchronizable by p-groups at the time T in the space $(H_0^1(\Omega) \times L^2(\Omega))^N$ if for any given $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$, there exists a sequence $\{H_n\}$ of internal controls in $(L^2_{loc}(\mathbb{R}^+; L^2(\Omega)))^M$ with compact support in $[0, T]$, such that the corresponding sequence $\{U_n\}$ of solutions to problem (1.1)-(1.2) satisfies

$$u_n^{(k)} - u_n^{(l)} \to 0 \text{ in } C^0_{loc}([T, +\infty); H_0^1(\Omega)) \cap C^1_{loc}([T, +\infty); L^2(\Omega)) \tag{5.1}$$

as $n \to +\infty$ for all $n_{r-1} + 1 \leqslant k, l \leqslant n_r$ and $1 \leqslant r \leqslant p$.

Let S_r be a full row-rank matrix of order $(n_r - n_{r-1} - 1) \times (n_r - n_{r-1})$:

$$S_r = \begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}, \quad 1 \leqslant r \leqslant p.$$

Define the $(N-p)\times N$ matrix C_p of synchronization by p-groups as

$$C_p = \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_p \end{pmatrix}. \tag{5.2}$$

Let

$$\mathrm{Ker}(C_p) = \mathrm{Span}\{e_1, \cdots, e_p\},$$

where

$$e_r = (0, \cdots, 0, \overset{(n_{r-1}+1)}{1}, \cdots, \overset{(n_r)}{1}, 0, \cdots, 0)^T, \quad 1 \leqslant r \leqslant p. \tag{5.3}$$

Then, (5.1) is equivalent to

$$C_pU_n \to 0 \text{ in } (C^0_{loc}([T, +\infty); H^1_0(\Omega)))^N \cap (C^1_{loc}([T, +\infty); L^2(\Omega)))^N \tag{5.4}$$

as $n \to +\infty$.

The convergence (5.1) or (5.4) will be called the approximate internal synchronization by p-groups in the consensus sense.

Proposition 5.2. *([17], Prop. 2.15) The matrix A satisfies the condition of C_p-compatibility:*

$$AKer(C_p) \subseteq Ker(C_p) \tag{5.5}$$

if and only if there exists a matrix A_p of order $(N-p)$, such that

$$C_pA = A_pC_p. \tag{5.6}$$

Under the condition of C_p-compatibility (5.6), applying C_p to system (1.1) and setting $W_p = C_pU$ and $D_p = C_pD$, we get the following reduced system

$$\begin{cases} W_p'' - \Delta W_p + A_pW_p = D_p\chi_\omega H & \text{in } \mathbb{R}^+ \times \Omega, \\ W_p = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{5.7}$$

Thus, the approximate internal synchronization by p-groups of system (1.1) is transformed into the approximate internal null controllability of the reduced system (5.7), which can be treated by Theorem 3.1.

Theorem 5.3. *Assume that A satisfies the condition of C_p-compatibility (5.5). Then, system (1.1) is approximately synchronizable by p-groups if and only if*

$$rank(C_p(D, AD, \cdots, A^{N-1}D)) = N - p. \tag{5.8}$$

Proof. By Theorem 3.1, the reduced system (5.7) is approximately null controllable if and only if

$$\mathrm{rank}(D_p, A_pD_p, \cdots, A_p^{N-p-1}D_p) = N - p. \tag{5.9}$$

By [17], Proposition 2.16, we have

$$\mathrm{rank}(D_p, A_pD_p, \cdots, A_p^{N-p-1}D_p) = \mathrm{rank}(C_p(D, AD, \cdots, A^{N-1}D)). \tag{5.10}$$

Then equation (5.8) is equivalent to equation (5.9). □

Remark 5.4. The matrix of synchronization by p-groups given by equation (5.2) has a specific physical meaning. However, the rank formula (5.10) is valid for any given $(N-p)\times N$ full row-rank matrix C_p, therefore, Theorem 5.3 is also valid for any given $(N-p)\times N$ full row-rank matrix C_p satisfying the condition of C_p-compatibility equation (5.5) and the rank condition (5.8).

Remark 5.5. Let us write $Im(D) = Im(D_0) + Im(D_1)$ with $Im(D_0) \in \mathrm{Ker}(C_p)$ and $Im(D_1) \in \mathrm{Im}(C_p^T)$. Since $C_pD = C_pD_1$, the matrix D_0 does not play any role for the reduced system (5.7). Without loss of generality, we may assume that $D_0 = 0$ so that $\mathrm{Im}(D) \subseteq \mathrm{Im}(C_p^T)$.

6. Condition of C_p-compatibility

The condition of C_p-compatibility equation (5.5) is a key ingredient for the synchronization of system (1.1). However, its necessity is a delicate question. In this section, we will first discuss this question under the minimum total number of controls, and further develop the study in the next section.

Recall the following fundamental result (Lem. 2.1 in [14]).

Lemma 6.1. *Let $d \geqslant 0$ be an integer. The rank condition*

$$rank(D, AD, \cdots, A^{N-1}D) = N - d \tag{6.1}$$

holds if and only if d is the largest dimension of the subspaces which are contained in $Ker(D^T)$ and invariant for A^T.

Proposition 6.2. *Assume that system (1.1) is approximately synchronizable by p-groups. Then for any given subspace V invariant for A^T and contained in $Ker(D^T)$, we have*

$$V \cap Im(C_p^T) = \{0\}. \tag{6.2}$$

Proof. Let $V = \mathrm{Span}\{E_1, \cdots, E_d\}$. We have

$$A^TE_r = \sum_{s=1}^{d} \alpha_{rs}E_s, \quad D^TE_r = 0, \quad r = 1, \cdots, p. \tag{6.3}$$

For any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$, let $\{H_n\}$ be a sequence of internal controls, which realizes the approximate internal synchronization by p-groups for system (1.1). We denote by $\{U_n\}$ the sequence of the corresponding solutions. Applying E_r to system (1.1) and noting $u_r = E_r^TU_n$ for $r = 1, \cdots, d$, we get

$$\begin{cases} u_r'' - \Delta u_r + \sum_{s=1}^d \alpha_{rs}u_s = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u_r = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \\ t = 0: \quad u_r = E_r^T\widehat{U}_0,\ u_r' = E_r^T\widehat{U}_1 & \text{in } \Omega. \end{cases} \tag{6.4}$$

Let E be a unit vector in $V \cap \mathrm{Im}(C_p^T)$. Since $E \in \mathrm{Im}(C_p^T)$, there exists $x \in \mathbb{R}^{N-p}$, such that $E = C_p^Tx$. It follows from (5.4) that

$$E^TU_n(T) = x^TC_pU_n(T) \to 0 \text{ in } H_0^1(\Omega) \text{ as } n \to +\infty. \tag{6.5}$$

On the other hand, since $E \in V$, there exist coefficients $a_1, \cdots, a_p$ not all zero such that $E = \sum_{r=1}^{d} a_r E_r$, then we have

$$E^T U_n = \sum_{r=1}^{d} a_r E_r^T U_n = \sum_{r=1}^{d} a_r u_r. \tag{6.6}$$

The homogeneous system (6.4) is independent of internal control, therefore, independent of n. It follows from equations (6.5) and (6.6) that

$$\sum_{r=1}^{d} a_r u_r(T) = 0. \tag{6.7}$$

By the time-invertibility of system (6.4), the mapping

$$(\widehat{U}_0, \widehat{U}_1) \to (u_1(T), \cdots, u_p(T)) \tag{6.8}$$

is surjective from $(H_0^1(\Omega) \times L^2(\Omega))^N$ onto $(H_0^1(\Omega))^p$. Then we deduce from equation (6.7) a contradiction $a_1 = \cdots = a_p = 0$. □

Proposition 6.3. *Assume that system (1.1) is approximately synchronizable by p-groups. Then we necessarily have*

$$rank(D, AD, \cdots, A^{N-1}D) \geqslant N - p. \tag{6.9}$$

Proof. Assume that

$$\text{rank}(D, AD, \cdots, A^{N-1}D) = N - q \quad \text{with} \quad q > p.$$

By Lemma 6.1, there exists a subspace V of dimension q, which is invariant for A^T and contained in $\text{Ker}(D^T)$. Since

$$\dim \text{Im}(C_p^T) + \dim(V) = N - p + q > N,$$

then $\text{Im}(C_p^T) \cap V \neq \{0\}$, which contradicts Proposition 6.2 □

Theorem 6.4. *Assume that system (1.1) is approximately synchronizable by p-groups under the minimal rank condition*

$$rank(D, AD, \cdots, A^{N-1}D) = N - p. \tag{6.10}$$

Then, we have

(a) $Ker(C_p)$ admits a complement which is invariant for A.

(b) There exist $u_1, \cdots, u_p$ independent of applied internal controls, such that

$$U_n \to \sum_{r=1}^{p} u_r e_r \quad in\ (C^0_{loc}([T, +\infty); H_0^1(\Omega)))^N \cap (C^1_{loc}([T, +\infty); L^2(\Omega)))^N \tag{6.11}$$

as $n \to +\infty$.

(c) A satisfies the condition of C_p-compatibility equation (5.5).

Remark 6.5. The convergence in equation (6.11) will be called the approximate internal synchronization by p-groups in the pinning sense, and the vector-valued function $(u_1, \cdots, u_p)^T$ will be called the synchronizable state by p-groups. Theorem (6.4) means that under the minimum rank condition (6.10), the approximate internal synchronization by p-groups takes place in the pinning sense.

Proof. (a) By Lemma 6.1 with $d = p$, there exists a subspace $V = \text{Span}\{E_1, \cdots, E_p\}$ which is invariant for A^T and contained in $\text{Ker}(D^T)$. By Proposition 6.2, we have

$$\text{Im}(C_p^T) \cap V = \{0\}. \tag{6.12}$$

It follows that

$$\dim(\text{Ker}(C_p) \cap V^\perp) = \dim(\text{Ker}(C_p)^\perp \cap V) = \dim(\text{Im}(C_p^T) \cap V) = 0.$$

On the other hand, since

$$\dim\, \text{Ker}(C_p) + \dim(V^\perp) = p + N - p = N,$$

then $\text{Ker}(C_p)$ admits $V^\perp$ as a complement, which is invariant for A.

(b) Let Q_p be a matrix of order $N \times (N - p)$ such that $\text{Im}(Q_p) = V^\perp$. By equation (6.12), $\text{Im}(C_p^T) \cap (V^\perp)^\perp = \{0\}$. Then, by [17], Proposition 2.5 $\text{Im}(C_p^T)$ and $V^\perp$ are bi-orthonormal. Without loss of generality, we may assume that $C_p Q_p = I_{N-p}$. Similarly, $\text{Ker}(C_p)$ and V are bi-orthonormal, we can chose $V = \text{Span}\{E_1, \cdots, E_p\}$, such that $e_r^T E_s = \delta_{rs}$ for $1 \leqslant r, s \leqslant p$.

Now for any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$, let $\{H_n\}$ be a sequence of internal controls, which realizes the approximate internal synchronization by p-groups for system (1.1). We denote by $\{U_n\}$ the sequence of the corresponding solutions. Since $V^\perp$ is a complement of $\text{Ker}(C_p)$ and $\text{Im}(Q_p) = V^\perp$, we have $\text{Ker}(C_p) \oplus \text{Im}(Q_p) = \mathbb{R}^N$. Then, a straightforward computation gives

$$U_n = \sum_{r=1}^{p} u_r e_r + Q_p C_p U_n,$$

where for $r = 1, \cdots, p$, $u_r = E_r^T U_n$ are determined by equation (6.4) with $d = p$. Using equation (5.4), we get thus equation (6.11).

(c) Applying C_p to the equations in (1.1), we get

$$C_p U_n'' - \Delta C_p U_n + C_p A U_n = 0 \quad \text{in } [T, +\infty) \times \Omega.$$

Using equation (6.11) and passing to the limit as $n \to +\infty$, we get

$$\sum_{r=1}^{p} C_p A u_r e_r = 0 \quad \text{in } [T, +\infty) \times \Omega,$$

where $u_1, \cdots, u_p$ are given by equation (6.4). Since the mapping equation (6.8) is surjective from $(H_0^1(\Omega) \times L^2(\Omega))^N$ onto $(H_0^1(\Omega))^p$, we get thus $C_p A e_r = 0$ for $r = 1, \cdots, p$, namely, $A\text{Ker}(C_p) \subseteq \text{Ker}(C_p)$. $\square$

7. Induced internal synchronization

In this section, we will further examine the convergence (5.4). For this purpose, we introduce the following notion.

Definition 7.1. System (1.1) is induced synchronizable by the matrix $\mathcal{C}_r$, if $\mathcal{C}_r$ is a full row-rank matrix of order $(N-r)\times N(0\leqslant r\leqslant p)$, such that

(a) $\mathrm{Im}(C_p^T)\subseteq \mathrm{Im}(\mathcal{C}_r^T)$;

(b) $A\mathrm{Ker}(\mathcal{C}_r)\subseteq \mathrm{Ker}(\mathcal{C}_r)$;

(c) for any given initial data $(\widehat{U}_0,\widehat{U}_1)\in (H_0^1(\Omega)\times L^2(\Omega))^N$, there exists a sequence $\{H_n\}$ of internal controls in $(L^2_{loc}(\mathbb{R}^+;L^2(\Omega)))^M$ with compact support in $[0,T]$, such that the sequence $\{U_n\}$ of corresponding solutions to system (1.1) satisfies

$$\mathcal{C}_rU_n\to 0 \text{ in } (C^0_{loc}([T,+\infty);H_0^1(\Omega)))^{N-r}\cap (C^1_{loc}([T,+\infty);L^2(\Omega)))^{N-r}, \tag{7.1}$$

as $n\to+\infty$.

We first consider the situation that A does not satisfy the condition of C_p-compatibility equation (5.5). We extend C_p to a $(N-\widetilde{p})\times N$ full row-rank matrix $\widetilde{C}_{\tilde{p}}$ defined by

$$\mathrm{Im}(\widetilde{C}_{\tilde{p}}^T)=\mathrm{Span}\{C_p^T,A^TC_p^T,\cdots,(A^T)^{N-1}C_p^T\}. \tag{7.2}$$

Clearly, $\mathrm{Im}(C_p^T)\subseteq \mathrm{Im}(\widetilde{C}_{\tilde{p}}^T)$. By Cayley-Hamilton's theorem, we have $A^T\mathrm{Im}(C_{\tilde{p}}^T)\subseteq \mathrm{Im}(\widetilde{C}_{\tilde{p}}^T)$, or equivalently,

$$A\mathrm{Ker}(\widetilde{C}_{\tilde{p}})\subseteq \mathrm{Ker}(\widetilde{C}_{\tilde{p}}). \tag{7.3}$$

We will show now point (c) with $\mathcal{C}_r=\widetilde{C}_{\tilde{p}}$.

Proposition 7.2. *Assume that system (1.1) is approximately synchronizable by p-groups. Then*

$$\mathit{rank}(\widetilde{C}_{\tilde{p}}(D,AD,\cdots,A^{N-1}D))=N-\tilde{p}. \tag{7.4}$$

Proof. It is sufficient to show that

$$\mathrm{Ker}(D,AD,\cdots,A^{N-1}D)^T\cap \mathrm{Im}(\widetilde{C}_{\tilde{p}}^T)=\{0\}.$$

Let x be a nonzero vector such that

$$x=\sum_{j=0}^{N-1}(A^T)^jC_p^Tx_j\in \mathrm{Im}(\widetilde{C}_{\tilde{p}}^T)$$

with $x_j\in\mathbb{R}^{N-p}$ for $0\leqslant j\leqslant N-1$, such that

$$D^Tx=D^TA^Tx\cdots=D^T(A^T)^{N-1}x=0.$$

For any given initial data $(\widehat{U}_0,\widehat{U}_1)\in (H_0^1(\Omega)\times L^2(\Omega))^N$, let $\{H_n\}$ be a sequence of internal controls, which realizes the approximate internal synchronization by p-groups for system (1.1). We denote by $\{U_n\}$ the sequence of corresponding solutions.

Let

$$v_i=x^TA^iU_n=\sum_{j=0}^{N-1}x_j^TC_pA^{i+j}U_n),\quad 0\leqslant i\leqslant N-1. \tag{7.5}$$

Applying C_p to the equations in system (1.1), and noting (5.4), we have

$$C_pAU_n \to 0 \quad \text{in} \quad \mathcal{D}'((T,+\infty)\times\Omega) \quad \text{as } n\to+\infty.$$

Then, applying C_pA to the equations in system (1.1), we can successively get

$$C_pA^kU_n \to 0 \quad \text{in} \quad \mathcal{D}'((T,+\infty)\times\Omega) \quad \text{as } n\to+\infty \tag{7.6}$$

for all integers $k \geqslant 0$, which implies that

$$v_i = \sum_{j=0}^{N-1} x_j^T C_p A^{i+j} U_n \to 0 \quad \text{in } \mathcal{D}'((T,+\infty)\times\Omega) \tag{7.7}$$

for $0 \leqslant i \leqslant N-1$ as $n\to+\infty$.

Now applying $(A^i)^T x$ to system (1.1) and using Cayley-Hamilton's Theorem, for $0\leqslant i\leqslant N-1$, there exist coefficients α_{ij} such that

$$\begin{cases} v_i'' - \Delta v_i + \sum_{j=0}^{N-1}\alpha_{ij}v_j = 0 & \text{in } \mathbb{R}^+\times\Omega,\\ v_i = 0 & \text{on } \mathbb{R}^+\times\Gamma,\\ t=0 \quad v_i = x^TA^i\widehat{U}_0, \quad v_i' = x^TA^i\widehat{U}_1. \end{cases} \tag{7.8}$$

Since the homogeneous system (7.8) is independent of n, it follows from equation (7.7) that

$$t\geqslant T: \quad v_i = 0, \quad 0\leqslant i\leqslant N-1. \tag{7.9}$$

By the time-invertibility of system (7.8), we get

$$x^TA^i\widehat{U}_0 = x^TA^i\widehat{U}_1 = 0 \tag{7.10}$$

for all $0\leqslant i\leqslant N-1$ and $(\widehat{U}_0,\widehat{U}_1)\in(H_0^1(\Omega)\times L^2(\Omega))^N$. In particular, we have

$$x^T\widehat{U}_0 = x^T\widehat{U}_1 = 0 \tag{7.11}$$

for all $(\widehat{U}_0,\widehat{U}_1)\in(H_0^1(\Omega)\times L^2(\Omega))^N$. We thus get a contradiction $x=0$. □

Theorem 7.3. *Assume that system (1.1) is approximately synchronizable by p-groups. Then, it is induced synchronizable by* $\widetilde{C}_{\tilde{p}}$.

Proof. Since formula (5.10) is still valid for the $(N-\tilde{p})\times N$ full row-rank matrix $\widetilde{C}_{\tilde{p}}$, Theorem 5.3 applies under the condition of $\widetilde{C}_{\tilde{p}}$-compatibility equation (7.3) and the rank condition (7.4). We get thus the convergence (7.1) with $\mathcal{C}_r = \widetilde{C}_{\tilde{p}}$. □

Corollary 7.4. *Assume that system (1.1) is approximately synchronizable by p-groups under the minimum rank condition (6.10). Then A satisfies the condition of* C_p*-compatibility equation (5.5).*

Proof. This result was already proved in Theorem 6.4. Here we give another proof, which further clarifies the necessity of the extension described by equation (7.2).

By equation (7.4), we have

$$\text{rank}(D, AD, \cdots, A^{N-1}D) \geqslant N-\tilde{p}. \tag{7.12}$$

It follows that $p \leqslant \tilde{p}$. Noting that $p \geqslant \tilde{p}$, we get $p = \tilde{p}$. Then, the extension matrix $\widetilde{C}_{\tilde{p}}$ of C_p is actually C_p itself. From equation (7.2), we deduce that $A^T \mathrm{Im}(C_p^T) \subseteq \mathrm{Im}(C_p^T)$, namely, $A\mathrm{Ker}(C_p) \subseteq \mathrm{Ker}(C_p)$. □

Now we consider the case that A satisfies the condition of C_p-compatibility equation (5.5). Let $\mathbb{D}_p$ denote the set of matrices D, which realize the approximate internal synchronization by p-groups for system (1.1). By Proposition 6.3, we have

$$\min_{D \in \mathbb{D}_p} \mathrm{rank}(D, AD, \cdots, A^{N-1}D) \geqslant N - p. \tag{7.13}$$

By Theorem 6.4, when the equality in equation (7.13) is true, $\mathrm{Ker}(C_p)$ admits a complement, which is invariant for A. In this situation, system (1.1) is approximately synchronizable by p-groups in the pinning sense and the synchronizable state by p-groups is independent of applied internal controls.

In the general case, we will extend C_p to a suitable matrix C_q^*, and bring the problem to the optimal situation considered in Theorem 6.4.

Proposition 7.5. *([17], Prop. 2.19) Assume that A satisfies the condition of C_p-compatibility equation (5.5) and that $Im(C_p^T)$ is A^T-marked (there exists a Jordan basis of A^T in $Im(C_p^T)$, which can be extended by adding new vectors to a Jordan basis of A^T in $\mathbb{C}^N$). Then there exists an $(N-q) \times N (0 \leqslant q < p)$ full row-rank matrix C_q^* such that $Ker(C_q^*)$ is the largest subspace satisfying the following two requirements.*

(a) $Ker(C_q^*)$ is contained in $Ker(C_p^T)$,

(b) $Ker(C_q^*)$ is invariant for A (then A satisfies the condition of C_q^*-compatibility) and admits a supplement which is also invariant for A.

By [17], Proposition 11.2 for any given matrix $D \in \mathbb{D}_p$, we have

$$\mathrm{rank}(D, AD, \cdots, A^{N-1}D) \geqslant N - q. \tag{7.14}$$

Moreover, in the case of equality, we have

Proposition 7.6. *([17], Prop. 11.4) Let $D \in \mathbb{D}_p$ such that*

$$rank(D, AD, \cdots, A^{N-1}D) = N - q. \tag{7.15}$$

We necessarily have the rank condition

$$rank(C_q^*(D, AD, \cdots A^{N-1}D)) = N - q. \tag{7.16}$$

So, when we make the reduction by using C_p for transforming system (1.1) to system (5.7), the rank of the matrices $(D, AD, \cdots A^{N-1}D)$ and $C_p(D, AD, \cdots A^{N-1}D)$ is reduced from $(N-q)$ to $(N-p)$. This process losses $(p-q)$ controls. However, when we make the reduction by C_q^*, the matrices $(D, AD, \cdots A^{N-1}D)$ and $C_q^*(D, AD, \cdots A^{N-1}D)$ have the same rank, therefore, there is no loss of controls in this process. In other words, the $(p-q)$ controls lost in $\mathrm{Ker}(C_p)$ is recovered in the reduction by C_q^*.

Theorem 7.7. *Assume that system (1.1) is approximately synchronizable by p-groups under the minimal rank condition (7.15). Then it is induced synchronizable by C_q^*.*

Proof. Obviously, C_q^* satisfies (a) and (b) in Definition 7.1 with $C_r = C_q^*$. In particular, A satisfies the condition of C_q^*-compatibility, by Proposition 5.2, there exists a matrix A_q^* such that $C_q^* A = A_q^* C_q^*$. Applying C_q^* to system

(1.1) and setting $W_q = C_q^* U$ and $D_q^* = C_q^* D$, we get

$$\begin{cases} W_q'' - \Delta W_q + A_q^* W_q = D_q^* \chi_\omega H & \text{in } \mathbb{R}^+ \times \Omega, \\ W_q = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{7.17}$$

By Proposition 7.6, condition (7.16) holds under the minimum rank condition (7.15). Then, applying Theorem 5.3 to the reduced system (7.17), we deduce the convergence (7.1), in which $\mathcal{C}_r$ is replaced by C_q^*. We get thus the induced synchronization by C_q^*. □

The above theorem shows that the recovered $(p-q)$ controls will provide the convergence of $(p-q)$ more components of U than the convergence (5.4). As a result, we can realize the approximate internal controllability of $(N-q)$ equations by means of $(N-q)$ controls. This is the best thing that we can do.

Corollary 7.8. *Assume that system* (1.1) *is approximately synchronizable by p-groups under the minimal rank condition* (7.15)*. Then it is approximately synchronizable by p-groups in the pinning sense.*

Proof. Let

$$\text{Span}\{e_1^*, \cdots, e_q^*\} = \text{Ker}(C_q^*). \tag{7.18}$$

For any given initial data $(\widehat{U}_0, \widehat{U}_1) \in (H_0^1(\Omega) \times L^2(\Omega))^N$, let $\{H_n\}$ be a sequence of internal controls, which realizes the approximate internal synchronization by p-groups for system (1.1). We denote by $\{U_n\}$ the sequence of the corresponding solutions. Under the minimal rank condition (7.15), by Theorems 7.7 and 6.4, there exist $u_1^*, \cdots, u_q^*$ such that

$$U_n \to \sum_{s=1}^{q} u_s^* e_s^* \quad \text{in } (C^0_{loc}([T, +\infty); H_0^1(\Omega)))^N \cap (C^1_{loc}([T, +\infty); L^2(\Omega)))^N, \tag{7.19}$$

as $n \to +\infty$. Since $\text{Ker}(C_q^*) \subset \text{Ker}(C_p)$, there exist some coefficients α_{sr} such that

$$e_s^* = \sum_{r=1}^{p} \alpha_{sr} e_r, \quad 1 \leqslant s \leqslant q. \tag{7.20}$$

Then, inserting equation (7.20) into equation (7.19) and setting

$$u_r = \sum_{s=1}^{q} \alpha_{sr} u_s^*, \quad 1 \leqslant r \leqslant p \tag{7.21}$$

we get

$$U_n \to \sum_{r=1}^{p} u_r e_r \quad \text{in } (C^0_{loc}([T, +\infty); H_0^1(\Omega)))^N \cap (C^1_{loc}([T, +\infty); L^2(\Omega)))^N, \tag{7.22}$$

as $n \to +\infty$. In other words, the primary system (1.1) is approximately synchronizable by p-groups in the pinning sense. However, since $q < p$, $u_1, \cdots, u_p$ given by equation (7.21) are linearly dependent. □

8. Synthesis and open problems

There are some commun points for the approximate boundary and internal controllabilities and synchronizations.

1. The equivalence between the controllability and the D-observability holds for these two controllabilities (Thm. 2.5, [17], Thms. 8.6 and 16.5).

2. Kalman's rank condition (1.6) is necessary for these two controllabilities (Thm. 1.1, [17], Thms. 8.9 and 16.11).

3. Under the minimal rank condition (5.8), the condition of C_p-compatibility as well as the pinning synchronization hold for these two synchronizations (Thm. 6.4, [17], Thms. 10.5 and 18.5).

The advantages of the approximate internal controllability and synchronization are as follows.

1. Kalman's rank condition (1.6) is also sufficient for the approximate internal controllability (Thm. 1.1), but insufficient in general for the approximate boundary controllability (see [17], Thms. 8.11 and 16.12).

2. The induced synchronization and the pinning synchronization hold for the approximate internal controllability (Thm. 7.7 and Cor. 7.8), but they don't hold in general for the approximate boundary controllability (see [17], Chap. 11).

3. The controllability time can be uniquely determined by the geodesic diameter of Ω and independent of both rank(D) and N for the internal controllability (Thm. 1.1). However, it depends on both rank(D) and N for the approximate boundary controllability (see [17], Thms. 8.25 and 16.15 and [31]).

Finally, we give some open problems.

1. The study can be similarly carried up for a system of wave equations with Neumann boundary condition

$$\begin{cases} U'' - \Delta U + AU = D\chi_\omega H & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_\nu U = 0 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{8.1}$$

2. System of wave equations with two coupling matrices D_1 and D_2

$$\begin{cases} U'' - \Delta U + AU = D_1\chi_\omega H_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ U = D_2 H_2 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{8.2}$$

We easily check that Kalman's rank condition (1.6) with $D = (D_1, D_2)$ is still necessary for the approximate controllability, the sufficiency will be investigated in a forthcoming work.

3. System of wave equations with coupled Robin condition

$$\begin{cases} U'' - \Delta U + AU = D_1\chi_\omega H_1 & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_\nu U + BU = D_2 H_2 & \text{on } \mathbb{R}^+ \times \Gamma. \end{cases} \tag{8.3}$$

Let $V = \text{Span}\{E_1, \cdots, E_d\}$ be a subspace which is invariant for A^T and B^T, and contained in Ker(D^T) with $D = (D_1, D_2)$. Then, there exist coefficients α_{rs} and β_{rs} such that

$$A^T E_r = \sum_{s=1}^{d} \alpha_{rs} E_s, \quad B^T E_r = \sum_{s=1}^{d} \beta_{rs} E_s, \quad r = 1, \cdots, d. \tag{8.4}$$

Applying E_r to system (8.3) and setting $u_r = (E_r, U)$, we get a homogeneous system for $r = 1, \cdots, d$:

$$\begin{cases} u_r'' - \Delta u_r + \sum_{s=1}^{d} \alpha_{rs} u_s = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \partial_\nu u_r + \sum_{s=1}^{d} \beta_{rs} u_s = 0 & \text{on } \mathbb{R}^+ \times \Gamma, \end{cases} \tag{8.5}$$

which is uncontrollable. As in [19], *"No invariant subspace of A^T and B^T is contained in Ker(D^T)"* is still a necessary condition for the approximate controllability. But the sufficiency is largely open.

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