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LOCAL EXPONENTIAL STABILIZATION OF ROGERS–MCCULLOCH AND FITZHUGH–NAGUMO EQUATIONS BY THE METHOD OF BACKSTEPPING

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Abstract. In this article, we study the exponential stabilization of some one-dimensional nonlinear coupled parabolic-ODE systems, namely Rogers–McCulloch and FitzHugh–Nagumo systems, in the interval $(0,1)$ by boundary feedback. Our goal is to construct an explicit linear feedback control law acting only at the right end of the Dirichlet boundary to establish the local exponential stabilizability of these two different nonlinear systems with a decay $\mathrm{e}^{-\omega t}$, where $\omega \in (0,\delta]$ for the FitzHugh–Nagumo system and $\omega \in (0,\delta)$ for the Rogers–McCulloch system and δ is the system parameter that presents in the ODE of both coupled systems. The feedback control law, derived by the backstepping method forces the exponential decay of solution of the closed-loop nonlinear system in both $\mathrm{L}^2(0,1)$ and $\mathrm{H}^1(0,1)$ norms, respectively, if the initial data is small enough. We also show that the linearized FitzHugh–Nagumo system is not stabilizable with exponential decay $\mathrm{e}^{-\omega t}$, where $\omega > \delta$.

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1. Introduction

1.1. Rogers–McCulloch and FitzHugh–Nagumo systems

Let us consider the following nonlinear coupled ODE-PDE reaction-diffusion system in the interval $(0,1)$ with nonmonotone nonlinearity of FitzHugh–Nagumo type

$$\begin{cases} u_t = u_{xx} + I(u,w) & \text{in } (0,1)\times(0,\infty), \\ w_t = \gamma u - \delta w & \text{in } (0,1)\times(0,\infty), \\ u(0,t)=0, \quad u(1,t)=q_1(t) & \text{in } (0,\infty), \\ u(x,0)=u_0(x), \ w(x,0)=w_0(x) & \text{in } (0,1). \end{cases} \tag{1.1}$$

The system (1.1) is known as monodomain equations in cardiac electrophysiology. In (1.1), $u=u(x,t)$ describes the transmembrane electrical potential of human heart, $w=w(x,t)$ is a so-called gating variable and $q_1 =$

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$q_1(t) \in L^2(0,\infty)$ is a boundary control. Nonmonotone nonlinearity of ionic current I is of the form:

$$I = I_F(u,w) = -u(u-1)(u-a) - \rho w, \tag{1.2}$$
$$I = I_R(u,w) = -u(u-1)(u-a) - \rho uw. \tag{1.3}$$

The system (1.1) with I equal to I_F (or I_R) is known as FitzHugh–Nagumo (FHN) model (or Rogers–McCulloch (RM) model). We further assume that $0 < a < 1$, $\gamma > 0$, $\delta > 0$, $\rho > 0$ are positive constants. One may see, a well-written review article [1] by Hastings in study of these mathematical models in Neurobiology of FitzHugh–Nagumo type. This type of model describes the conduction of electrical impulses in a nerve axon (see also works of J. M. Rogers and A. D. McCulloch [2]).

Cardiac fibrillation breaks the organized cardiac electrical activity that drives the heart's periodic pumping into disorganized self-sustained electrical activation patterns. A fibrillation episode results in the loss of cardiac output and, unless timely intervention is administered, death quickly ensues. The only known effective therapy for lethal disturbances in cardiac rhythm is defibrillation, the delivery of a strong electric shock to the heart. From a practical point of view, the monodomain equations are of interest since it allows one to model fibrillation processes of the human heart. The control (stimulating electrodes situated at the boundary of the domain) $q_1(t)$ here can be interpreted as an external stimulus resembling a defibrillation process, see [3–6].

We write the RM and FHN systems in infinite-dimensional ODE set up. At first, let us consider the state space $\mathbf{Z} := \mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1)$ endowed with the inner product

$$\left\langle \begin{pmatrix} u_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ w_2 \end{pmatrix} \right\rangle_{\mathbf{Z}} = \int_0^1 u_1 u_2 + \int_0^1 w_1 w_2. \tag{1.4}$$

For $q_1 = 0$, we write the system (1.1) as

$$\begin{cases} \mathbf{U}'(t) = \mathbf{A}\mathbf{U}(t) + \mathbf{F}(\mathbf{U})(t), \\ \mathbf{U}(0) = \mathbf{U}_0, \end{cases} \tag{1.5}$$

where $\mathbf{U} := \begin{pmatrix} u \\ w \end{pmatrix}$ in $\mathrm{L}^2(0,T;\mathbf{Z})$. The unbounded operator $(\mathbf{A}, D(\mathbf{A}))$ in $\mathbf{Z}$ for the RM and FHN systems respectively are as follows:

$$D(\mathbf{A}) := \{\mathbf{U} = (u,w)^T \in \mathbf{Z} : u \in \mathrm{H}_0^1(0,1) \cap \mathrm{H}^2(0,1)\},$$

$$\mathbf{A}^{RM} = \begin{pmatrix} \dfrac{d^2}{\mathrm{d}x^2} - aI_d & 0 \\ \gamma I_d & -\delta I_d \end{pmatrix}, \quad \mathbf{A}^{FHN} = \begin{pmatrix} \dfrac{d^2}{\mathrm{d}x^2} - aI_d & -\rho I_d \\ \gamma I_d & -\delta I_d \end{pmatrix},$$

where I_d is the identity map. The operators $\mathbf{F} : D(\mathbf{A}) \to \mathbf{Z}$ are defined as

$$\mathbf{F}^{RM}(u,w) = \begin{pmatrix} -u^3 + (a+1)u^2 - \rho uw \\ 0 \end{pmatrix}, \quad \mathbf{F}^{FHN}(u,w) = \begin{pmatrix} -u^3 + (a+1)u^2 \\ 0 \end{pmatrix}.$$

The spectrum of the operator $\mathbf{A}^{FHN}$ has an accumulation point (see Sect. 3.2 of this article, Prop. 2.3 in [7], Lem. 3.7 in [8] for details). In particular, the operator has two branches of eigenvalues μ_k and λ_k, where

$$\mu_k \to -\infty \quad \text{and} \quad \lambda_k \to -\delta,$$

where δ is given in (1.1). Due to this accumulating nature of the spectrum of the linearized operator, the system is not boundary null controllable (see for example Thm. 2.7 in [7], Thm. 4.2 in [9], Thm. 1 in [10], Thm. 5.6, 5.8 in [11]). Therefore in the absence of null controllability, the study of stabilization of the system (1.1) becomes more relevant and interesting. Let us first introduce the notion of exponential stabilization by feedback control.

Definition 1.1 (**Local exponential stabilization by feedback**). Let $(u_0, w_0) \in \mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1)$. The system (1.1) is called locally exponential stabilizable by feedback control around the origin in the space $\mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1)$ with a decay rate $\omega > 0$, if there exists a bounded linear map $\Pi : \mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1) \to \mathbb{R}$ such that, for some positive constant $r > 0$ the solution (u, w) of (1.1) with a control of the form $q_1(t) = \Pi \begin{pmatrix} u(\cdot,t) \\ w(\cdot,t) \end{pmatrix}$, satisfies

$$\|u(t)\|_{\mathrm{L}^2(0,1)} + \|w(t)\|_{\mathrm{L}^2(0,1)} \leq C\mathrm{e}^{-\omega t} \left(\|u_0\|_{\mathrm{L}^2(0,1)} + \|w_0\|_{\mathrm{L}^2(0,1)} \right),$$

for all $t > 0$, for some positive constant C independent of u_0 and w_0, provided that

$$\|u_0\|_{\mathrm{L}^2(0,1)} + \|w_0\|_{\mathrm{L}^2(0,1)} \leq r. \tag{1.6}$$

This paper deals with the local exponential stabilization by feedback control of the RM and FHN system using backstepping method. We will present our main results in the following sections. At first, we explain the method for linear systems.

1.2. Exponential stabilization by feedback for the Rogers–McCulloch system

We construct a feedback control law by the method of backstepping such that the solution of the closed-loop system decays exponentially towards the origin. This method has been described for various model in the book "Boundary control of PDEs, A course on backstepping designs" by Krstic and Smyshlyaev [12]. The main idea of the backstepping method relies on the construction of some suitable invertible integral transformations which convert the unstable PDEs to an exponentially stable target system. This conversion happens when the kernel function, used in the transformation satisfies some well-posed PDE. Exponential stabilizability of the original controlled system has been obtained from the stability of the target system through the invertible transformation. The stability of the target system has been established by introducing a suitable Lyapunov functional depending on that PDE. Finally, the feedback control law is deduced from the transformation by using the homogeneous boundary conditions of the target system.

Let us briefly explain the method of backsteppping for the following linearized RM system around the origin

$$\begin{cases} u_t - u_{xx} + au = 0 & \text{in } (0,1) \times (0,\infty), \\ w_t = \gamma u - \delta w & \text{in } (0,1) \times (0,\infty), \\ u(0,t) = 0, \ \ u(1,t) = q(t) & \text{in } (0,\infty), \\ u(x,0) = u_0(x) \quad w(x,0) = w_0(x) & \text{in } (0,1). \end{cases} \tag{1.7}$$

The system (1.7) with $q = 0$ is itself stable with a decay rate $\omega < \min\{a, \delta\}$. If $a < \delta$, considering the following Lyapunov functional $V(t) = \int_0^1 u^2(x,t)\mathrm{d}x + \int_0^1 w^2(x,t)\mathrm{d}x$, and by usual energy estimate we have

$$\dot{V}(t) \leq -2\omega V(t),$$

where $\omega < \min\{a, \delta\}$. This essentially gives

$$\|u(t)\|_{\mathrm{L}^2(0,1)} + \|w(t)\|_{\mathrm{L}^2(0,1)} \leq M_1 \mathrm{e}^{-\omega t} \left(\|u_0\|_{\mathrm{L}^2(0,1)} + \|w_0\|_{\mathrm{L}^2(0,1)} \right), \tag{1.8}$$

for each $t \geq 0, \omega < \min\{a, \delta\}$.

Next, utilizing backstepping based feedback law, we prove the exponential stabilizability up to decay $\mathrm{e}^{-\omega t}$, $\omega < \delta$. The Volterra integral transformation of the second kind $K : \mathrm{L}^2(0,1) \to \mathrm{L}^2(0,1)$ is defined by

$$\tilde{\sigma}(x) = (K\sigma)(x) = \sigma(x) - \int_0^x k(x,y)\sigma(y)\mathrm{d}y \quad x \in [0,1],\, \sigma \in \mathrm{L}^2(0,1), \tag{1.9}$$

where the kernel function k is the solution of the following equation in the domain $\mathcal{T} = \{(x,y) \in [0,1] \times [0,1] : 0 \leq y \leq x \leq 1\}$

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) - (\lambda - a)k(x,y) = 0 & 0 < y < x < 1, \\ 2\dfrac{\mathrm{d}}{\mathrm{d}x}k(x,x) + (\lambda - a) = 0 & 0 \leq x \leq 1, \\ k(x,0) = 0, & 0 \leq x \leq 1, \end{cases} \tag{1.10}$$

where λ is a positive constant, to be chosen sufficiently large. Using Lemma 2.2 of [13] and Lemma 4.4 of [14], we see that the equations (1.10) has unique C^2 solution. Also note that $K : \mathrm{H}^i(0,1) \to \mathrm{H}^i(0,1), i = 0,1,2$ and $K^{-1} : \mathrm{H}^i(0,1) \to \mathrm{H}^i(0,1), i = 0,1,2$ are linear bounded operators (see Lem. 2.4 in [13]). The inverse of the transformation (1.9) is given by

$$\sigma(x) = (K^{-1}\tilde{\sigma})(x) = \tilde{\sigma}(x) + \int_0^x l(x,y)\tilde{\sigma}(y)\mathrm{d}y \quad x \in [0,1], \tag{1.11}$$

where l is the solution of the following equation

$$\begin{cases} l_{xx}(x,y) - l_{yy}(x,y) + (\lambda - a)l(x,y) = 0 & 0 < y < x < 1, \\ 2\dfrac{\mathrm{d}}{\mathrm{d}x}l(x,x) + (\lambda - a) = 0 & 0 \leq x \leq 1, \\ l(x,0) = 0 & 0 \leq x \leq 1. \end{cases} \tag{1.12}$$

By using the transformation (1.9), one can show that the system (1.7) can be transformed to the following target system

$$\begin{cases} \tilde{u}_t - \tilde{u}_{xx} + \lambda \tilde{u} = 0 & \text{in } (0,1) \times (0,\infty), \\ \tilde{w}_t = \gamma \tilde{u} - \delta \tilde{w} & \text{in } (0,1) \times (0,\infty), \\ \tilde{w}(0,t) = 0, \quad \tilde{w}(1,t) = 0 & \text{in } (0,\infty), \\ \tilde{u}(x,0) = \tilde{u}_0(x), \quad \tilde{w}(x,0) = \tilde{w}_0(x) & \text{in } (0,1). \end{cases} \tag{1.13}$$

We recover the explicit feedback law by the transformation defined above, indeed, it is given by

$$q(t) = u(1,t) = \int_0^1 k(1,y)u(y,t)\mathrm{d}y. \tag{1.14}$$

Now we need to show the stability estimate for the target system (1.13). Considering the following Lyapunov functional for RM as

$$V^{RM}(t) = \int_0^1 \tilde{u}^2(x,t)\mathrm{d}x + \int_0^1 \tilde{w}^2(x,t)\mathrm{d}x \tag{1.15}$$

and choosing the damping coefficient λ large enough, one can show the following estimate (see the proof of Thm. 2.3 for details)

$$\|\tilde{u}(t)\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}(t)\|_{\mathrm{L}^2(0,1)} \le M\mathrm{e}^{-\omega t}\left(\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)}\right), \tag{1.16}$$

for each $t \ge 0$, $\omega \in [0,\delta)$ and M is a positive constant. This inequality immediately implies the exponential stabilization of (1.7) around the origin.

We will apply the aforementioned approach to prove the local exponential stabilization by feedback control of the nonlinear model (1.1).

1.2.1. Main results

Before going to state the main stabilization result of this paper, for the convenience of the reader, let us recall the RM system with Dirichlet boundary control

$$\begin{cases} u_t - u_{xx} + au = -u^3 + (a+1)u^2 - \rho uw & \text{in } (0,1)\times(0,\infty),\\ w_t = \gamma u - \delta w & \text{in } (0,1)\times(0,\infty),\\ u(0,t) = 0,\ u(1,t) = q_1(t) & \text{in } (0,\infty),\\ u(x,0) = u_0(x) \quad w(x,0) = w_0(x) & \text{in } (0,1). \end{cases} \tag{1.17}$$

Theorem 1.2 (**Local exponential stabilization by feedback for RM**). *Let us assume that $\omega \in (0,\delta)$ and k be the solution of (1.10). Then there exists $r > 0$ such that for every initial data $(u_0, w_0) \in \mathrm{L}^2(0,1)\times\mathrm{L}^2(0,1)$ with*

$$\|u_0\|_{\mathrm{L}^2(0,1)} + \|w_0\|_{\mathrm{L}^2(0,1)} \le r, \tag{1.18}$$

the system (1.17) with the feedback law

$$q_1(t) = \int_0^1 k(1,y)u(y,t)\mathrm{d}y, \tag{1.19}$$

has a unique solution $(u,w) \in \mathrm{C}^0\Big([0,\infty);\mathrm{L}^2(0,1)\Big)\cap\mathrm{L}^2\Big(0,\infty;\mathrm{H}^1(0,1)\Big)\times\mathrm{C}^0\Big([0,\infty);\mathrm{L}^2(0,1)\Big)\cap\mathrm{L}^2\Big(0,\infty;\mathrm{L}^2(0,1)\Big)$, and the solution satisfies the following

$$\|u(t)\|_{\mathrm{L}^2(0,1)} + \|w(t)\|_{\mathrm{L}^2(0,1)} \le M\mathrm{e}^{-\omega t}\left(\|u_0\|_{\mathrm{L}^2(0,1)} + \|w_0\|_{\mathrm{L}^2(0,1)}\right), \tag{1.20}$$

for all $t > 0$, where M is a positive constant independent of t, u_0 and w_0.

1.3. Exponential stabilization by feedback for the FitzHugh–Nagumo system

In this section, we state the stabilization result for FHN system. Like linearized RM system (1.7), we can show that the linearized (around origin) FHN system

$$\begin{cases} u_t - u_{xx} + au + \rho w = 0 & \text{in } (0,1)\times(0,\infty), \\ w_t = \gamma u - \delta w & \text{in } (0,1)\times(0,\infty), \\ u(0,t) = 0,\ u(1,t) = q(t) & \text{in } (0,\infty), \\ u(x,0) = u_0(x) \quad w(x,0) = w_0(x) & \text{in } (0,1), \end{cases} \tag{1.21}$$

is stable with a decay rate $\omega = \min\{a, \delta\}$ with $q = 0$. Utilizing backstepping based feedback law, one can prove the exponential stabilizability up to decay $\mathrm{e}^{-\delta t}$ (see Sect. 1.2). The choice of Lyapunov functional will be

$$V^{FHN}(t) = \frac{\gamma}{\rho}\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x + \int_0^1 \tilde{w}^2(x,t)\mathrm{d}x, \tag{1.22}$$

for this case. Next, we are going to present exponential stabilization result for the nonlinear system. At first we recall the nonlinear FHN system (1.1) and (1.2) with Dirichlet boundary control

$$\begin{cases} u_t - u_{xx} + au = -u^3 + (a+1)u^2 - \rho w & \text{in } (0,1)\times(0,\infty), \\ w_t = \gamma u - \delta w & \text{in } (0,1)\times(0,\infty), \\ u(0,t) = 0,\ u(1,t) = q_1(t) & \text{in } (0,\infty), \\ u(x,0) = u_0(x) \quad w(x,0) = w_0(x) & \text{in } (0,1). \end{cases} \tag{1.23}$$

Following the proof of the stabilization result for the RM system, we can deduce the local exponential stabilization by feedback for FHN system around the origin. Moreover, for the FHN case, one can establish the exponential stabilization by feedback control around a nonzero steady state of that system. Before stating the stabilization theorem we need to introduce the steady state of (1.23).

Definition 1.3 (**Steady state**)**.** A function $\bar{\mathbf{U}} = (\bar{u}, \bar{w}) \in \mathrm{C}^2[0,1]\times\mathrm{C}^2[0,1]$, with $\bar{u}(0) = 0$ is called a steady state of the control system (1.23) if $(\bar{u}, \bar{w})$ satisfies

$$\begin{cases} \bar{u}_{xx} + I(\bar{u}, \bar{w}) = 0 & \text{in } (0,1), \\ \gamma\bar{u} - \delta\bar{w} = 0 & \text{in } (0,1). \end{cases} \tag{1.24}$$

Let $(\bar{u}, \bar{w})$ be the steady state of the FHN system (1.23). We design a linear feedback control law, such that the solution (u, w) of (1.23) converges exponentially with a prescribed decay rate $\omega \le \delta$ to the solution $(\bar{u}, \bar{w})$ of (1.24), provided that $||(u_0, w_0) - (\bar{u}, \bar{w})||_{\mathrm{L}^2(0,1)\times\mathrm{L}^2(0,1)}$ is small enough. More precisely, we use a backstepping feedback of the form $q_1 = K(u - \bar{u})$ to stabilize the full nonlinear system. For that, let us perform the following change of variables

$$z = u - \bar{u}, \qquad v = w - \bar{w}. \tag{1.25}$$

The infinite-dimensional control system satisfied by (z, v) can be written in the following abstract form when $q_1(t) = \bar{u}(1)$

$$\begin{cases} \mathbf{V}'(t) = \mathbf{A}\mathbf{V}(t) + \mathbf{F}(\mathbf{V})(t), \\ \mathbf{V}(0) = \mathbf{V}_0, \end{cases} \tag{1.26}$$

where $\mathbf{V} := \begin{pmatrix} z \\ v \end{pmatrix}$ in $\mathrm{L}^2(0,T;\mathbf{Z})$. We write the unbounded operator $(\mathbf{A}, D(\mathbf{A}))$ in $\mathbf{Z}$ for the FHN system as follows : $D(\mathbf{A}) := \{\mathbf{V} = (z,v)^T \in \mathbf{Z} : z \in \mathrm{H}^1_0(0,1) \cap \mathrm{H}^2(0,1)\}$,

$$\bar{\mathbf{A}}^{FHN} = \begin{pmatrix} \frac{d^2}{\mathrm{d}x^2} - \bar{A}(x)I_d & -\rho I_d \\ \gamma I_d & -\delta I_d \end{pmatrix}, \quad \bar{A}(x) = a - 2\bar{u}(a+1) + 3\bar{u}^2. \tag{1.27}$$

The operator $\mathbf{F}$ is defined as

$$\mathbf{F}^{FHN}(z,v) = \begin{pmatrix} -z^3 + B(x)z^2 \\ 0 \end{pmatrix}, \quad B(x) = (a+1) - 3\bar{u}. \tag{1.28}$$

Now we are in position to state the stabilization result for nonlinear FHN system around a steady state $(\bar{u}, \bar{w})$.

Theorem 1.4 (**Local exponential stabilization by feedback for FHN**). *Let $\omega \in (0,\delta]$ and k be the solution of (1.10). Then there exists $r > 0$ such that for every initial data $(u_0, w_0) \in \mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1)$ with*

$$\|u_0 - \bar{u}\|_{\mathrm{L}^2(0,1)} + \|w_0 - \bar{w}\|_{\mathrm{L}^2(0,1)} \le r, \tag{1.29}$$

the system (1.23) with the feedback law

$$q_1(t) = \int_0^1 k(1,y)\,(u - \bar{u})\,(y,t)\mathrm{d}y + \bar{u}(1), \tag{1.30}$$

has a unique solution $(u,w) \in \mathrm{C}^0\Big([0,\infty);\mathrm{L}^2(0,1)\Big) \cap \mathrm{L}^2\Big(0,\infty;\mathrm{H}^1(0,1)\Big) \times \mathrm{C}^0\Big([0,\infty);\mathrm{L}^2(0,1)\Big) \cap \mathrm{L}^2\Big(0,\infty;\mathrm{L}^2(0,1)\Big)$, and the solution satisfies the following

$$\|u(t) - \bar{u}\|_{\mathrm{L}^2(0,1)} + \|w(t) - \bar{w}\|_{\mathrm{L}^2(0,1)} \le M\mathrm{e}^{-\omega t}\left(\|u_0 - \bar{u}\|_{\mathrm{L}^2(0,1)} + \|w_0 - \bar{w}\|_{\mathrm{L}^2(0,1)}\right), \tag{1.31}$$

for all $t > 0$, where M is a positive constant independent of t, u_0 and w_0.

Remark 1.5. In the case of RM model, we show that the energy of the closed-loop system decays exponentially with a rate $\omega \in (0,\delta)$, whereas for the FHN system, critical decay rate $\omega = \delta$ can be reached. The product nonlinear term uw in the parabolic equation of the RM model prevents getting the decay up to δ by backstepping. In other hand, for the FHN case, there is a linear term w, that helps in the stability analysis, to get the desired decay. For nonlinear FHN and RM models, exponential stabilization by feedback control with decay rate $\omega > \delta$ is a challenging open problem (see Sect. 4.3 for more details). However, for the linear FHN model, we have established that exponential stabilization by feedback control with decay rate $\omega > \delta$ is not possible, see Corollary 3.13 in Section 3.2. Similar question for linear RM system, which is essentially a cascade system, is open. By backstepping method we can prove exponential stabilization by feedback control with any decay rate for the heat equation but the decay rate for ODE component beyond $\omega > \delta$ is not known.

The exponential stabilization for the FHN system leads to a weak version of local approximate controllability of that system, where the time horizon is not fixed. More precisely, let us take the target $U_1 = (u_1, w_1)$, as a steady state of (1.23). We have shown that for any $\mathcal{E} > 0$, there exists a $r > 0$ such that, whenever the initial data $\mathbf{U}_0 = (u_0, w_0)$ lies in a ball of $L^2(0,1) \times L^2(0,1)$ centered at $\mathbf{U}_1$ with radius r, there exists $T_{\mathcal{E}}(\mathcal{E}, \mathbf{U}_0, \mathbf{U}_1) > 0$ and a control $q \in L^2(0,T)$ such that

$$\|\mathbf{U}(T) - \mathbf{U_1}\|_{\mathrm{L}^2(0,1)\times\mathrm{L}^2(0,1)} \le \mathcal{E}, \tag{1.32}$$

for all $T > T_{\mathcal{E}}(\mathcal{E}, \mathbf{U}_0, \mathbf{U}_1) > 0$. This result is an improvement of the result (see Thm. 1.5) in [8], where $\mathbf{U_0}$ and $\mathbf{U_1}$ are assumed to be steady state of (1.23). In Section 3.3, we describe this notion briefly.

1.4. Motivation and literature of Backstepping method

The boundary stabilization problem considered in this paper for RM equation, is inspired by the work [15] of E. Cerpa and J.M Coron, where local stabilization of the KdV system has been studied by backstepping method. In [16], authors studied local rapid exponential stabilization for Fisher's equation using the same approach. Both the papers deal with Volterra integral transformations of second kind which is invertible indeed. To the best of the authors' knowledge, local exponential stabilization by feedback using backstepping control for the nonlinear Rogers–McCulloch system is studied for the first time in this paper within the closed-loop control framework.

In recent years backstepping method has been widely used to tackle stabilization problem for many linear PDEs, for parabolic models see [13, 17–20] and for hyperbolic models see [21–24]. However there are some works in which backstepping method has been used successfully for nonlinear systems, see [15, 16, 25–29]. In the pioneer works [26, 27, 30] more general transformations like Fredholm transformation have been used to apply the backstepping method, in which invertibility of the Fredholm transformations is not easy to show. Authors have used the null controllability of the corresponding systems to prove invertibility of the transformations, whereas in our case, we use the backstepping method by taking Volterra transformation of second kind. For the lack of null controllability of the linearized system, Fredholm operator based backstepping can not be explored here.

For PDEs, roughly speaking, backstepping is a constructive method that achieves Lyapunov stabilization by transforming the system into stable target system. It differs from optimal control methods in that it sacrifices optimality for the sake of avoiding operator Riccati equations (quadratic operator-valued equations), which are very hard to solve for infinite dimensional systems such as PDEs. In this context, we also mention Urquiza's approach [31] for time-reversible boundary control which is essentially based on general results about the algebraic Riccati equation associated with the linear quadratic regulator problem. Moreover, this method has been successfully employed in [32] for KdV equation.

Backstepping is also different from pole placement methods because, even though its objective is stabilization, which is also the objective of the pole placement methods, backstepping does not pursue precise assignment of even a finite subset of eigenvalues of the PDEs. Instead, the backstepping method achieves Lyapunov stabilization, which is often achieved by collectively shifting all the eigenvalues in a favorable direction in the complex plane, rather than by assigning individual eigenvalues. The main feature of backstepping is that it is capable of eliminating destabilizing effects that appear throughout the domain while the control is acting only from the boundary. Backstepping based feedback control given by an integral operator, can be computed explicitly and calculated numerically by successive approximation scheme and this approach requires much less computational effort, see [12, 15, 33]. Moreover, this explicit feedback law is very useful for getting the other results like null-controllability, and finite time stabilization, see [34–36].

1.5. Related works on stabilization for monodomain equations of FHN type

One of the initial works for optimal control problems related to the FitzHugh–Nagumo system is due to Barandao *et al.* [37]. It also provides an approximate controllability result for the linear model. One may see more recent significant works on optimal control problem related to FHN model [38–44] and references therein. Significant contributions on stabilization for FHN type model also came from works of T. Breiten, K. Kunisch and S. S. Rodrigues in the series of papers [3–5, 7]. Their interesting works on null controllability for FHN and RM model using moving control and on funnel control for FHN model can be found in [45, 46].

In [3], the authors considered nonlinear FHN equations in two and three dimensions around stationary solution with Neumann boundary conditions. They have studied the local exponential stabilization by feedback of the FHN model by using interior control acting in the parabolic equation using linear Riccati-based feedback law obtained from a solution of algebraic Riccati equation. To handle the nonlinearity they need to choose initial data from $\mathrm{H}^s \times \mathrm{H}^{s+1}, 1 \geq s > \frac{1}{2}$ for two dimensions and $s = 1$ for three dimensions. In [5], the authors linearized the system under a trajectory that depends on space and time. It brings a variable coefficients in the parabolic equation. They proved the exponential stabilization result using interior control for the nonlinear RM model in two and three dimensions with Neumann boundary conditions with a decay $0 < \omega < \delta$. In that paper, the authors obtained finite dimensional Riccati based feedback using differential Riccati equation for FHN and RM model around non stationary solution in two and three dimensions for initial data in $\mathrm{H}^1 \times \mathrm{L}^2$. Moreover, in [5] the authors have proved the stabilization result by taking some sharp conditions on the system parameters. In both the papers [3, 5] the authors studied the stabilization results by means of localized interior control. The strategy they have used is to stabilize the parabolic equation first and then use it to the ODE level and extract the stabilization result for the coupled system.

In [4], the authors studied boundary feedback stabilization of the FHN system locally around $(\bar{u}, \bar{w}) \in \mathrm{H}^3 \times \mathrm{L}^\infty$ with Dirichlet and Neumann boundary conditions in a two dimensional and three dimensional domain. They obtained the exponential decay rate $\omega < \delta$ when initial data lies in $\mathrm{H}^s \times \mathrm{L}^2,\ 0 < s < \frac{1}{2}$ for Dirichlet boundary conditions in two dimension case and in $\mathrm{H}^s \times \mathrm{L}^2,\ \frac{1}{2} < s \leq 1$ for Neumann boundary conditions, both in two and three dimensions. Boundary control comes from algebraic Riccati-based feedback (see Thm. 3.3 in [4]). In our case, to get the stability in L^2 norm for both components of the nonlinear system in one dimension, we need only $\mathrm{L}^2 \times \mathrm{L}^2$ regularity of initial data. Moreover, we established exponential stabilization by feedback control result for the RM system in $\mathrm{H}^1 \times \mathrm{H}^1, \mathrm{H}^1 \times \mathrm{L}^2$ norm, without taking any conditions on the system parameters.

This kind of PDE-ODE structure of coupled system poses a significant mathematical challenge in several models like heat equation with memory [47], linearized viscous Saint-Venant system [48], Stokes equation with memory [49], incompressible viscoelastic linearized Jeffrey fluids [50, 51], structurally damped wave equation [10], linearized Benjamin-Bona-Mahony equation [9], compressible linearized Navier-stokes equation [52], viscoelastic shear flows [53], transport equation with vanishing characteristic speed [54]. In contrast to usual parabolic equations, here for linearization around a stationary solution, the spectrum is no longer discrete and, as a consequence, the system is not null controllable (see [9–11] for more details). Thus our backstepping based exponential stabilization results by feedback become more important. Mathematically the monodomain equations are an evolution system consisting of a diffusion equation with a polynomial, more specifically, cubic nonlinearity coupled with an ordinary differential equation. Such systems play an important role far beyond their use to describe the electrical activity of the heart. In fact, many models of this kind arise in cellular biology, as described in [38, 55, 56] for example. In this context, we emphasize that our results are important and expect this backstepping approach will help to conclude that similar systems are exponentially stabilizable with a decay $\mathrm{e}^{-\delta t}$, where δ is the limit point of the real part of the eigenvalues of the corresponding linearized operator. This is one of the novelties of our work.

1.6. Paper organization

This paper is organized as follows. Section 2 contains the backstepping design of the linear feedback control law and exponential stabilizability for the Rogers–McCulloch system. We provide the proof of local exponential stabilization result by feedback control of RM model *i.e.* proof of the Theorem 1.2. We also mention some additional results on stabilizability of RM model in $\mathrm{H}^1 \times \mathrm{H}^1$ and $\mathrm{H}^1 \times \mathrm{L}^2$ norm. Section 3 is devoted to the proof of stabilization result (Thm. 1.4) for the FHN system. Furthermore, we show that the linearized FHN model is not exponentially stabilizable by feedback (Cor. 3.13.) with a decay rate $\mathrm{e}^{-\omega t}$, where $\omega > \delta$. A local approximate controllability result is discussed for the FHN model. In Section 4, we raise some open questions, remarks and comments in this section. Finally, for the sake of completeness, we give the details of the well-posedness of the RM equation in our setting in the Appendix.

2. Local Exponential stabilization by feedback of Rogers–McCulloch system

This section is devoted to the proof of the exponential stabilization result Theorem 1.2 for the RM system (1.17). We employ the backstepping approach, as mentioned in the Section 1.2, to establish the desired result.

2.1. Target system

Let (u,w) be the solution of the RM model (1.17). Let us define for $t \in (0,\infty)$

$$\begin{cases} \tilde{u}(\cdot,t) =: K(u(\cdot,t)), \\ \tilde{w}(\cdot,t) = K(w(\cdot,t)), \end{cases} \tag{2.1}$$

where K is given by (1.9). Then we see that, the transformations (1.9) convert the system (1.17) to the following system

$$\begin{cases} \tilde{u}_t - \tilde{u}_{xx} + \lambda \tilde{u} = -K\Big((K^{-1}\tilde{u})^3\Big) \\ \qquad\qquad + (a+1)K\Big((K^{-1}\tilde{u})^2\Big) - \rho K\Big((K^{-1}\tilde{u})(K^{-1}\tilde{w})\Big) & \text{in } (0,1)\times(0,\infty), \\ \tilde{w}_t = \gamma\tilde{u} - \delta\tilde{w} & \text{in } (0,1)\times(0,\infty), \\ \tilde{u}(0,t) = 0,\ \tilde{u}(1,t) = 0 & \text{in } (0,\infty), \\ \tilde{u}(x,0) = \tilde{u}_0(x),\ \tilde{w}(x,0) = \tilde{w}_0(x) & \text{in } (0,1), \end{cases} \tag{2.2}$$

provided the control q_1 in RM system (1.17) is given by

$$q_1(t) = u(1,t) = \int_0^1 k(1,y)u(y,t)\mathrm{d}y. \tag{2.3}$$

Proposition 2.1. *Let k be a solution of (1.10). Let (u,w) be a solution of (1.17) where the control q_1 is given by (2.3), then $(\tilde{u},\tilde{w})$, defined by (2.1) is a solution of the target system (2.2) and vice versa.*

Proof. Differentiating (2.1) with respect to t and using the fact that (u,w) satisfies equation (1.17), we obtain

$$\begin{aligned} \tilde{u}_t(x,t) &= u_t(x,t) - \int_0^x k(x,y)u_t(y,t)\mathrm{d}y \\ &= u_t(x,t) - \int_0^x k(x,y)\Big[u_{yy}(y,t) - au(y,t) - u^3(y,t) + (a+1)u^2(y,t)\Big]\mathrm{d}y \\ &\qquad + \rho\int_0^x k(x,y)u(y,t)w(y,t)\mathrm{d}y. \end{aligned}$$

Applying integration by parts for the term $\int_0^x k(x,y)u_{yy}(y,t)\mathrm{d}y$ and using the boundary values of u and k, we get

$$\begin{aligned} \tilde{u}_t(x,t) &= u_t(x,t) - \int_0^x k_{yy}(x,y)u(y,t)\mathrm{d}y + k_y(x,x)u(x,t) - k(x,x)u_x(x,t) \\ &\quad + a\int_0^x k(x,y)u(y,t)\mathrm{d}y + \int_0^x k(x,y)u^3(y,t)\mathrm{d}y - (a+1)\int_0^x k(x,y)u^2(y,t)\mathrm{d}y \\ &\qquad + \rho\int_0^x k(x,y)u(y,t)w(y,t)\mathrm{d}y. \end{aligned}$$

Differentiating the first relation of (2.1) with respect to x twice we have

$$\tilde{u}_{xx}(x,t) = u_{xx}(x,t) - k(x,x)u_x(x,t) - \frac{\mathrm{d}}{\mathrm{d}x}\Big(k(x,x)\Big)u(x,t) - k_x(x,x)u(x,t) - \int_0^x k_{xx}(x,y)u(y,t)\mathrm{d}y.$$

Since (u, w) is a solution of (1.17), combining above relations, we obtain

$$
\begin{aligned}
\tilde{u}_t - \tilde{u}_{xx} + \lambda \tilde{u} &= u_t(x,t) - u_{xx}(x,t) + a u(x,t) \\
&\quad + \int_0^x \Big[k_{xx}(x,y) - k_{yy}(x,y) - (\lambda - a) k(x,y) \Big] u(y,t) \mathrm{d}y + \left(2 \frac{\mathrm{d}}{\mathrm{d}x} k(x,x) + (\lambda - a) \right) u(x,t) \\
&\quad + \int_0^x k(x,y) u^3(y,t) \mathrm{d}y - (a+1) \int_0^x k(x,y) u^2(y,t) \mathrm{d}y + \rho \int_0^x k(x,y) u(y,t) w(y,t) \mathrm{d}y \\
&= - \left(u^3 - \int_0^x k(x,y) u^3(y,t) \mathrm{d}y \right) + (a+1) \left(u^2 - \int_0^x k(x,y) u^2(y,t) \mathrm{d}y \right) \\
&\quad - \rho u w + \rho \int_0^x k(x,y) u(y,t) w(y,t) \mathrm{d}y \\
&= -K\Big((K^{-1}\tilde{u})^3 \Big) + (a+1) K \Big((K^{-1}\tilde{u})^2 \Big) - \rho K \Big((K^{-1}\tilde{u})(K^{-1}\tilde{w}) \Big).
\end{aligned}
$$

Now computing the $\tilde{w}_t$ from (2.1) and using the fact that (u, w) is a solution of (1.17) , we obtain the $\tilde{w}$-equation of (2.2). This shows that $(\tilde{u}, \tilde{w})$ satisfies the equation (2.2). Similarly, if $(\tilde{u}, \tilde{w})$ satisfies the equation (2.2) we can show that (u, w), given by (2.1), is a solution of (1.17) where q_1 is given by (2.3). This completes the proof. □

2.2. Stability analysis

In this section, we will discuss the exponential decay of solution of the target system (2.2), which will essentially give the exponential stabilization of the closed-loop control system (1.17). First we state the well-posedness result of the target system.

Proposition 2.2. *There exist $\tilde{\delta} > 0$ and $C > 0$ such that whenever $(\tilde{u}_0, \tilde{w}_0) \in \mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1)$ satisfying*

$$
\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)} \leq \mathrm{e}^{-C\sqrt{\lambda_1}} \tilde{\delta}/2, \quad \lambda_1 = \lambda - a > 1, \tag{2.4}
$$

the target system (2.2) has a unique solution $(\tilde{u}, \tilde{w})$ in the space

$$
\tilde{X} = \mathrm{C}^0([0,\infty); \mathrm{L}^2(0,1)) \cap \mathrm{L}^2(0,\infty; \mathrm{H}^1_0(0,1)) \times \mathrm{C}^0([0,\infty); \mathrm{L}^2(0,1)) \cap \mathrm{L}^2(0,\infty; \mathrm{L}^2(0,1))
$$

and we have the following estimate

$$
\|(\tilde{u}, \tilde{w})\|_{\tilde{X}} \leq \tilde{\delta}. \tag{2.5}
$$

The well-posedness of the target system will be proved in the Section A.1 of Appendix. Assuming the well-posedness, we prove the exponential stability of the control free target system (2.2).

Theorem 2.3. *Let $\omega \in (0, \delta)$. There exists $r > 0$ such that whenever $(\tilde{u}_0, \tilde{w}_0) \in \mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1)$, with*

$$
\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)} \leq r, \tag{2.6}
$$

the target system (2.2) is exponentially stable with a decay rate ω,

$$
\|\tilde{u}(t)\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}(t)\|_{\mathrm{L}^2(0,1)} \leq N \mathrm{e}^{-\omega t} \left(\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)} \right), \tag{2.7}
$$

for each $t \geq 0$, where N is a positive constant independent of $t, \tilde{u}_0$ and $\tilde{w}_0$.

For the sake of explicit energy estimate of the target system (2.2), we need to use the following quantitative properties of the kernel functions (1.10) and (1.12).

Lemma 2.4 (Coron *et al.* [29, 34])**.** *Let $\lambda_1 = \lambda - a > 1$. Then there exists a constant $C_1 > 0$ such that the solutions k and l of the kernel equations* (1.10) *and* (1.12) *satisfy the following estimates*

$$\|k\|_{C^2(\mathcal{T})} \leq \mathrm{e}^{C_1\sqrt{\lambda_1}}, \quad \|l\|_{C^2(\mathcal{T})} \leq \mathrm{e}^{C_1\sqrt{\lambda_1}}. \tag{2.8}$$

Remark 2.5. See Lemma 25 in [29], Corollary 1 and Corollary 2 in [34] for more details. In [12], the representation of the kernel k has been given in terms of Bessel functions

$$k(x,y) = -\lambda_1 y \frac{I_1\left(\sqrt{\lambda_1(x^2-y^2)}\right)}{\sqrt{\lambda_1(x^2-y^2)}}, \tag{2.9}$$

where I_1 is the first order modified Bessel function. This expression helps to anticipate the above estimates (2.8).

2.2.1. Proof of Theorem 2.3

Proof. Since $\omega \in (0,\delta)$, there exists $\epsilon > 0$ such that $\omega = \delta - \epsilon$. Let $(\tilde{u},\tilde{w})$ be the unique solution of (2.2). Let us consider the Lyapunov functional as

$$V(t) = V_1(t) + V_2(t), \tag{2.10}$$

where

$$V_1(t) = \frac{1}{2}\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x, \tag{2.11}$$

and

$$V_2(t) = \frac{1}{2}\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x. \tag{2.12}$$

Differentiating (2.11) with respect to t we have

$$\frac{\mathrm{d}V_1}{\mathrm{d}t}(t) = \int_0^1 \tilde{u}(x,t)\tilde{u}_t(x,t)\mathrm{d}x. \tag{2.13}$$

Since $\tilde{u}$ satisfy the equation (2.2) we have

$$\frac{\mathrm{d}V_1}{\mathrm{d}t}(t) = \int_0^1 \tilde{u}(x,t)\tilde{u}_{xx}(x,t)\mathrm{d}x - \lambda\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x - \mathcal{I}_1 + (a+1)\mathcal{I}_2 - \rho\mathcal{I}_3, \tag{2.14}$$

where

$$\mathcal{I}_1 = \int_0^1 \tilde{u}(x,t)\,(K^{-1}\tilde{u})^3(x,t)\mathrm{d}x - \int_0^1 \tilde{u}(x,t)\left(\int_0^x k(x,y)(K^{-1}\tilde{u})^3(y,t)\mathrm{d}y\right)\mathrm{d}x, \tag{2.15}$$

$$\mathcal{I}_2 = \int_0^1 \tilde{u}(x,t)\,(K^{-1}\tilde{u})^2(x,t)\mathrm{d}x - \int_0^1 \tilde{u}(x,t)\left(\int_0^x k(x,y)(K^{-1}\tilde{u})^2(y,t)\mathrm{d}y\right)\mathrm{d}x, \tag{2.16}$$

$$\mathcal{I}_3 = \int_0^1 \tilde{u}(x,t)\,(K^{-1}\tilde{u})(x,t)\,(K^{-1}\tilde{w})(x,t)\mathrm{d}x - \int_0^1 \tilde{u}(x,t)\int_0^x k(x,y)\,(K^{-1}\tilde{u})(y,t)\,(K^{-1}\tilde{w})(y,t)\mathrm{d}y. \tag{2.17}$$

Using integration by parts we have

$$\int_0^1 \tilde{u}(x,t)\tilde{u}_{xx}(x,t)\mathrm{d}x = -\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x. \tag{2.18}$$

Now we estimate the terms $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$. To estimate this we need the bounds of the solutions l and k of (1.12) and (1.10) respectively. By using the estimates (2.8) we can have a $\tilde{C} > C_1 > 0$ (where C_1 is the same constant as in (2.8)) such that

$$\|Ku\|_{\mathrm{L}^2(0,1)} \leq \mathrm{e}^{\tilde{C}\sqrt{\lambda_1}}\|u\|_{\mathrm{L}^2(0,1)}, \quad \left\|K^{-1}\tilde{u}\right\|_{\mathrm{L}^2(0,1)} \leq \mathrm{e}^{\tilde{C}\sqrt{\lambda_1}}\|\tilde{u}\|_{\mathrm{L}^2(0,1)}, \tag{2.19}$$

$$\|Ku\|_{\mathrm{L}^\infty(0,1)} \leq \mathrm{e}^{\tilde{C}\sqrt{\lambda_1}}\|u\|_{\mathrm{L}^\infty(0,1)}, \quad \left\|K^{-1}\tilde{u}\right\|_{\mathrm{L}^\infty(0,1)} \leq \mathrm{e}^{\tilde{C}\sqrt{\lambda_1}}\|\tilde{u}\|_{\mathrm{L}^\infty(0,1)}, \tag{2.20}$$

$$\|Ku\|_{\mathrm{H}^1(0,1)} \leq \mathrm{e}^{\tilde{C}\sqrt{\lambda_1}}\|u\|_{\mathrm{H}^1(0,1)}, \quad \left\|K^{-1}\tilde{u}\right\|_{\mathrm{H}^1(0,1)} \leq \mathrm{e}^{\tilde{C}\sqrt{\lambda_1}}\|\tilde{u}\|_{\mathrm{H}^1(0,1)}. \tag{2.21}$$

Now onwards, we will use a generic positive constant C, which is independent of time t and λ (C may change line to line). Note that $C > \tilde{C}$. We also define $u := K^{-1}\tilde{u}$ and $w := K^{-1}\tilde{w}$.

Estimate of $\mathcal{I}_1$. We estimate the first term of $\mathcal{I}_1$. Using (1.9) and the bound of k, we estimate the first term of $-\mathcal{I}_1$ as follows

$$\begin{aligned}-\int_0^1 \tilde{u}(x,t)(K^{-1}\tilde{u})^3(x,t)\mathrm{d}x &= -\int_0^1 \tilde{u}(x,t)u^3(x,t)\mathrm{d}x = -\int_0^1 u^3(x,t)\left(u(x,t) - \int_0^x k(x,y)u(y,t)\mathrm{d}y\right)\mathrm{d}x \\ &\leq -\int_0^1 u^4(x,t)\mathrm{d}x + \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 |u(x,t)|^3\,dx\right)\left(\int_0^1 |u(y,t)|\,\mathrm{d}y\right).\end{aligned}$$

Using Holder's inequality for both the terms $\int_0^1 |u(x,t)|^3\,dx$ and $\int_0^1 |u(y,t)|\,\mathrm{d}y$ we have

$$-\int_0^1 \tilde{u}(x,t)(K^{-1}\tilde{u})^3(x,t)\mathrm{d}x \leq -\int_0^1 u^4(x,t)\mathrm{d}x + \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 |u(x,t)|^4\,\mathrm{d}x\right)^{\frac{3}{4}}\left(\int_0^1 |u(x,t)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

Now using Young's inequality we have

$$-\int_0^1 \tilde{u}(x,t)u^3(x,t)\mathrm{d}x \leq -\int_0^1 u^4(x,t)\mathrm{d}x + \frac{1}{3}\int_0^1 |u(x,t)|^4\,\mathrm{d}x + \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 |u(x,t)|^2\right)^2.$$

Thus continuity of the operator K^{-1} yields

$$-\int_0^1 \tilde{u}(x,t)u^3(x,t)\mathrm{d}x \leq -\frac{2}{3}\int_0^1 u^4(x,t)\mathrm{d}x + \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 |\tilde{u}(x,t)|^2\,\mathrm{d}x\right)^2. \tag{2.22}$$

Next using the maximum bound of k we estimate the second term of $-\mathcal{I}_1$ as follows

$$\begin{aligned}\int_0^1 \tilde{u}(x,t)\left(\int_0^x k(x,y)(K^{-1}\tilde{u})^3(y,t)\mathrm{d}y\right)\mathrm{d}x &= \int_0^1 \tilde{u}(x,t)\left(\int_0^x k(x,y)u^3(y,t)\mathrm{d}y\right)\mathrm{d}x \\ &\leq \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 |\tilde{u}(x,t)|\,\mathrm{d}x\right)\left(\int_0^1 |u(y,t)|^3\,\mathrm{d}y\right).\end{aligned}$$

Using Holder's inequality and Young's inequality as in the estimate of first term of $-\mathcal{I}_1$, we get

$$\int_0^1 \tilde{u}(x,t)\left(\int_0^x k(x,y)(K^{-1}\tilde{u})^3(y,t)\mathrm{d}y\right)\mathrm{d}x \le \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 |u(x,t)|^4\,\mathrm{d}x\right)^{\frac{3}{4}}\left(\int_0^1 |\tilde{u}(x,t)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\le \frac{1}{3}\int_0^1 u^4(x,t)\mathrm{d}x + \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 |\tilde{u}(x,t)|^2\,\mathrm{d}x\right)^2. \tag{2.23}$$

Combining the estimates (2.22) and (2.23) we get

$$-\mathcal{I}_1 \le -\frac{1}{3}\int_0^1 u^4(x,t)\mathrm{d}x + \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 |\tilde{u}(x,t)|^2\,\mathrm{d}x\right)^2. \tag{2.24}$$

Estimate of $\mathcal{I}_2$. Next we estimate the term $\mathcal{I}_2$. Applying Young's inequality we estimate the first term of $\mathcal{I}_2$ as follows

$$\int_0^1 \tilde{u}(x,t)(K^{-1}\tilde{u})^2(x,t)\mathrm{d}x = \int_0^1 \tilde{u}(x,t)u^2(x,t)\mathrm{d}x \le \frac{1}{3(a+1)}\int_0^1 u^4(x,t)\mathrm{d}x + \frac{3(a+1)}{4}\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x. \tag{2.25}$$

The second term $\mathcal{I}_2$ can be estimated using the bound of k as

$$-\int_0^1 \tilde{u}(x,t)\left(\int_0^x k(x,y)(K^{-1}\tilde{u})^2(y,t)\mathrm{d}y\right)\mathrm{d}x = -\int_0^1 \tilde{u}(x,t)\left(\int_0^x k(x,y)u^2(y,t)\mathrm{d}y\right)\mathrm{d}x$$
$$\le \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 u^2(y,t)\mathrm{d}y\right)\left(\int_0^1 |\tilde{u}(x,t)|\,\mathrm{d}x\right).$$

Therefore using Holder's inequality and the continuity of the operator K^{-1}, we obtain

$$-\int_0^1 \tilde{u}(x,t)\left(\int_0^x k(x,y)(K^{-1}\tilde{u})^2(y,t)\mathrm{d}y\right)\mathrm{d}x \le \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x\right)^{\frac{3}{2}}. \tag{2.26}$$

Combining (2.25) and (2.26), we obtain

$$\mathcal{I}_2 \le \frac{1}{3(a+1)}\int_0^1 u^4(x,t)\mathrm{d}x + \frac{3(a+1)}{4}\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x + \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x\right)^{\frac{3}{2}}. \tag{2.27}$$

Estimate of $\mathcal{I}_3$. We estimate the second term of $\mathcal{I}_3$ as of (2.26) to get

$$\int_0^1 \tilde{u}(x,t)\int_0^x k(x,y)(K^{-1}\tilde{u})(y,t)(K^{-1}\tilde{w})(y,t)\mathrm{d}y = \int_0^1 \tilde{u}(x,t)\int_0^x k(x,y)u(y,t)w(y,t)\mathrm{d}y\mathrm{d}x$$
$$\le \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^1 u^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^1 w^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}$$
$$\le \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x\right)\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.28}$$

Next we estimate the first term of $\mathcal{I}_3$. Using the explicit expression

$$\sigma(x) = (K^{-1}\tilde{\sigma})(x) = \tilde{\sigma}(x) + \int_0^x l(x,y)\tilde{\sigma}(y)\mathrm{d}y$$

for u and w we get

$$\begin{aligned}\int_0^1 \tilde{u}(x,t)(K^{-1}\tilde{u})(x,t)(K^{-1}\tilde{w})(x,t)\mathrm{d}x = &\underbrace{\int_0^1 \Big(\tilde{u}(x,t)\Big)^2 \tilde{w}(x,t)\mathrm{d}x}_{J_1} \\ &+ \underbrace{\int_0^1 \tilde{u}(x,t)w(x,t)\left(\int_0^x l(x,y)\tilde{u}(y,t)\mathrm{d}y\right)\mathrm{d}x}_{J_2} \\ &+ \underbrace{\int_0^1 \Big(\tilde{u}(x,t)\Big)^2 \left(\int_0^x l(x,y)\tilde{w}(y,t)\mathrm{d}y\right)\mathrm{d}x.}_{J_3}\end{aligned} \tag{2.29}$$

As before, estimating the terms J_2 and J_3 we get

$$|J_2| + |J_3| \le \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x\right)\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.30}$$

It is remain to estimate the term J_1. Since the solution of (2.2) satisfies $\tilde{u}(0,t) = 0$, we can use the Sobolev inequality

$$\|\tilde{u}(\cdot,t)\|^2_{L^\infty(0,1)} \le \int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x, \tag{2.31}$$

to bound the term J_1 as

$$|J_1| \le \left(\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x\right)\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}} \tag{2.32}$$

by Holder's inequality. Using (2.30) and (2.32) in (2.29), we bound the first term of $\mathcal{I}_3$ as

$$\begin{aligned}\int_0^1 \tilde{u}(x,t)(K^{-1}\tilde{u})(x,t)(K^{-1}\tilde{w})(x,t)\mathrm{d}x \le &\left(\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x\right)\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}} \\ &+ \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x\right)\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}.\end{aligned} \tag{2.33}$$

From (2.28) and (2.33) we get

$$|\mathcal{I}_3| \le \left(\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x\right)\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}} + \mathrm{e}^{C\sqrt{\lambda_1}}\left(\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x\right)\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.34}$$

Putting the estimates of $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ in (2.14) we obtain

$$\begin{aligned}\frac{\mathrm{d}V_1}{\mathrm{d}t}(t) \leq &- \int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x + \left((a+1)^2\frac{3}{4} - \lambda\right)\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x + \rho\left(\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x\right)\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}} \\ &+ \mathrm{e}^{C\sqrt{\lambda_1}}\left[\left(\int_0^1 |\tilde{u}(x,t)|^2\,\mathrm{d}x\right)^2 + \left(\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x\right)^{\frac{3}{2}}\right. \\ &\left. + \left(\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x\right)\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}\right].\end{aligned} \tag{2.35}$$

Differentiating (2.12) with respect to t then using the fact that $\tilde{w}$ satisfies the second equation of (2.2) we get

$$\frac{\mathrm{d}V_2}{\mathrm{d}t} = \int_0^1 \tilde{w}(x,t)\tilde{w}_t(x,t)\mathrm{d}x = -\delta\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x + \gamma\int_0^1 \tilde{u}(x,t)\tilde{w}(x,t)\mathrm{d}x.$$

We use Young's inequality for the last term in the above expression to obtain

$$\frac{\mathrm{d}V_2}{\mathrm{d}t} \leq -\delta\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x + \epsilon\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x + \frac{\gamma^2}{4\epsilon}\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x. \tag{2.36}$$

Adding (2.35) and (2.36) we obtain

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} \leq &\left(-1 + \rho\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}\right)\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x + \left(\frac{\gamma^2}{4\epsilon} + (a+1)^2\frac{3}{4} - \lambda\right)\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x \\ &- \delta\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x + \epsilon\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x + \mathcal{A}\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x\end{aligned} \tag{2.37}$$

where

$$\mathcal{A} = \mathrm{e}^{C\sqrt{\lambda_1}}\left[\left(\int_0^1 |\tilde{u}(x,t)|^2\,\mathrm{d}x\right) + \left(\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}\right].$$

We choose λ as

$$-\lambda + \frac{\gamma^2}{4\epsilon} + (a+1)^2\frac{3}{4} \leq -\delta + \frac{\epsilon}{2}. \tag{2.38}$$

Now we will use the well-posedness theory (Prop. 2.2) of solution of (2.2). By Proposition 2.2 there exists a $\tilde{\delta}$ such that for initial data satisfying

$$\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)} \leq \mathrm{e}^{-C\sqrt{\lambda_1}}\tilde{\delta}/2,$$

the solution of (2.2) satisfies

$$\|\tilde{u}(\cdot,t)\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}(\cdot,t)\|_{\mathrm{L}^2(0,1)} \leq \tilde{\delta}, \tag{2.39}$$

for all $t \geq 0$. Therefore, choosing $\tilde{\delta}$ small, we obtain

$$-1+\rho\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}} \leq -1+\rho\tilde{\delta} \leq 0. \tag{2.40}$$

Further choosing $\tilde{\delta}$ small enough, the term $\mathcal{A}$ can be estimated by (2.39) as

$$\mathcal{A} \leq \mathrm{e}^{C\sqrt{\lambda_1}}(\tilde{\delta}^2+\tilde{\delta}) \leq \frac{\epsilon}{2}. \tag{2.41}$$

Putting the estimates (2.40) and (2.41) in (2.37) and choosing λ as (2.38) we get

$$\frac{\mathrm{d}V}{\mathrm{d}t} \leq 2\left(-\delta+\epsilon\right)V(t).$$

Integrating above relation, we obtain

$$\|\tilde{u}(t)\|_{L^2(0,L)} + \|\tilde{w}(t)\|_{L^2(0,L)} \leq N\mathrm{e}^{(-\delta+\epsilon)t}\left(\|\tilde{u}_0\|_{L^2(0,1)} + \|\tilde{w}_0\|_{L^2(0,1)}\right),$$

$\forall t \geq 0$, N is a positive constant independent of $t, \tilde{u}_0$ and $\tilde{w}_0$. This completes the proof. □

2.3. Proof of Theorem 1.2

Let us consider (u_0, w_0) be the initial condition for the non-linear Rogers–McCulloch system (1.17) satisfying

$$\|u_0\|_{L^2(0,1)} + \|w_0\|_{L^2(0,1)} \leq \frac{\tilde{\delta}}{2}\mathrm{e}^{-(\tilde{C}+C)\sqrt{\lambda_1}}(=r).$$

Let us define $\tilde{u}_0 = Ku_0$ and $\tilde{w}_0 = Kw_0$. Then using the bound of the operator K (see (2.19)), we obtain

$$\|\tilde{u}_0\|_{L^2(0,1)} + \|\tilde{w}_0\|_{L^2(0,1)} \leq \mathrm{e}^{-C\sqrt{\lambda_1}}\tilde{\delta}/2.$$

Taking this $(\tilde{u}_0, \tilde{w}_0)$ as initial conditions, let $(\tilde{u}, \tilde{w})$ be the solution of (2.2). Therefore the estimate (2.7) in Theorem 2.3 holds true. Let

$$u(\cdot,t) := K^{-1}\tilde{u}(\cdot,t) \quad \text{and} \quad w(\cdot,t) := K^{-1}\tilde{w}(\cdot,t).$$

Then by Theorem 2.1, (u, w) is a solution of (1.17) with q_1 satisfying (2.3). Therefore using the bound of the operator K^{-1}, we obtain

$$\|u(t)\|_{\mathrm{L}^2(0,1)} + \|w(t)\|_{\mathrm{L}^2(0,1)} \leq M_1\left(\|\tilde{u}(t)\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}(t)\|_{\mathrm{L}^2(0,1)}\right)$$

for some positive constant M_1. Then by (2.7), for each $t \geq 0$

$$\|u(t)\|_{\mathrm{L}^2(0,1)} + \|w(t)\|_{\mathrm{L}^2(0,1)} \leq M\mathrm{e}^{-\omega t}\left(\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)}\right)$$

for some positive constant M. Then using the bound of the operator K we obtain the result. This completes the proof.

Remark 2.6. The exponential stabilization result still holds if we take variable coefficients in the nonlinear term I_R of RM, that is, $I_R^s(u,w) = -u(u-\phi_2(x))(u-\phi_3(x)) - \rho uw$, with $\phi_2,\ \phi_3 \in C^1[0,1]$. The proof of exponential stabilization result is same as that of previous Theorem 1.2. We only need to choose λ as

$$-\lambda + \frac{\gamma^2}{4\epsilon} + \frac{3}{4}\,\|\phi_2+\phi_3\|^2_{\mathrm{L}^\infty(0,1)} \le -\delta + \frac{\epsilon}{2}.$$

2.4. Stabilization of Rogers–McCulloch system in $H^1 \times H^1$ and $H^1 \times L^2$

2.4.1. Pointwise-in-space boundedness and stability in $H^1 \times H^1$ Norms

We are interested in stability in $L^\infty \times L^\infty$ norm. For that we study the system in $H^1 \times H^1$. In Theorem 1.2, we have studied the stabilization of non linear Rogers–McCulloch system (1.17) in $L^2(0,1)\times L^2(0,1)$ norm for initial conditions (u_0,w_0) in $L^2(0,1)\times L^2(0,1)$. In this section, we study the stabilization of the same system in $H^1(0,1)\times H^1(0,1)$ (or $H^1(0,1)\times L^2(0,1)$) for initial condition (u_0,w_0) in $H^1_{\{0\}}(0,1)\times H^1(0,1)$ (or $H^1_{\{0\}}(0,1)\times L^2(0,1)$), where $H^1_{\{0\}}(0,1) = \{u \in H^1(0,1) : u(0)=0\}$. Before going to state the stabilization result we first write the well-posedness for the target system (2.2)

Proposition 2.7. *There exist $\tilde{\delta} > 0$ and $C > 0$ such that whenever $(\tilde{u}_0, \tilde{w}_0) \in \mathrm{H}^1_0(0,1)\times \mathrm{H}^1(0,1)$ satisfying*

$$\|\tilde{u}_0\|_{\mathrm{H}^1(0,1)} + \|\tilde{w}_0\|_{\mathrm{H}^1(0,1)} \le \mathrm{e}^{-C\sqrt{\lambda_1}}\tilde{\delta}/2, \tag{2.42}$$

the target system (2.2) has a unique solution $(\tilde{u},\tilde{w})$ in the space

$$\tilde{X}_1 = \mathrm{C}^0\left([0,\infty);\mathrm{H}^1_0(0,1)\right)\cap \mathrm{L}^2\left(0,\infty;\mathrm{H}^2(0,1)\cap\mathrm{H}^1_0(0,1)\right)\times \mathrm{C}^0\left([0,\infty);\mathrm{H}^1(0,1)\right)\cap\mathrm{L}^2\left(0,\infty;\mathrm{H}^1(0,1)\right)$$

and we have the following estimate

$$\|(\tilde{u},\tilde{w})\|_{\tilde{X}_1} \le \tilde{\delta}. \tag{2.43}$$

The proof of this result is given in the appendix. Now we will prove the following theorem.

Theorem 2.8. *Let us assume that $\omega \in (0,\delta)$. Then there exists $r > 0$ such that for every initial data (u_0,w_0),*

$$\|u_0\|_{H^1(0,1)} + \|w_0\|_{H^1(0,1)} \le r, \tag{2.44}$$

with the compatibility condition

$$u_0(1) = \int_0^1 k(1,y)u_0(y)\mathrm{d}y \tag{2.45}$$

the system (1.17) with the control of the form (2.3) has a unique solution and the solution (u,w) satisfies the following

$$\|u(t)\|_{\mathrm{H}^1(0,1)} + \|w(t)\|_{\mathrm{H}^1(0,1)} \le M\mathrm{e}^{-\omega t}\left(\|u_0\|_{\mathrm{H}^1(0,1)} + \|w_0\|_{\mathrm{H}^1(0,1)}\right), \tag{2.46}$$

for each $t \ge 0$, where M is a positive constant independent of t, u_0 and w_0.

As in the case of the Theorem 1.2, to prove above theorem it is enough to prove the following stability result for the target system (2.2).

Proposition 2.9. *Let $\omega \in (0,\delta)$. There exists $r>0$ such that whenever $(\tilde{u}_0, \tilde{w}_0) \in \mathrm{H}_0^1(0,1) \times \mathrm{H}^1(0,1)$, with*

$$\|\tilde{u}_0\|_{\mathrm{H}^1(0,1)} + \|\tilde{w}_0\|_{\mathrm{H}^1(0,1)} \leq r, \tag{2.47}$$

the target system (2.2) is exponentially stable with a decay rate ω,

$$\|\tilde{u}(t)\|_{\mathrm{H}^1(0,1)} + \|\tilde{w}(t)\|_{\mathrm{H}^1(0,1)} \leq \mathrm{e}^{-\omega t}\left(\|\tilde{u}_0\|_{\mathrm{H}^1(0,1)} + \|\tilde{w}_0\|_{\mathrm{H}^1(0,1)}\right), \tag{2.48}$$

for each $t \geq 0$.

Proof. Let $\epsilon = \delta - \omega$. Then $\epsilon > 0$. Let $(\tilde{u}, \tilde{w})$ be the solution of the target system (2.2). We consider the following Lyapunov functional as

$$V(t) = \frac{1}{2}\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x + \frac{1}{2}\int_0^1 \tilde{w}_x^2(x,t)\mathrm{d}x. \tag{2.49}$$

Differentiating both sides of (2.49) with respect to t we have

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) = \int_0^1 \tilde{u}_x(x,t)\tilde{u}_{xt}(x,t)\mathrm{d}x + \int_0^1 \tilde{w}_x(x,t)\tilde{w}_{xt}(x,t)\mathrm{d}x.$$

Now using integration by parts for the first term and $\tilde{w}$ equation of (2.2) we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) = -\int_0^1 \tilde{u}_t(x,t)\tilde{u}_{xx}(x,t)\mathrm{d}x + \gamma\int_0^1 \tilde{w}_x(x,t)\tilde{u}_x(x,t)\mathrm{d}x - \delta\int_0^1 \tilde{w}_x^2(x,t)\mathrm{d}x.$$

Using the inequality $ab \leq (\epsilon a^2) + (b^2/4\epsilon)$ for the term $\int \tilde{w}_x \tilde{u}_x$ and the $\tilde{u}$-equation of (2.2), we get

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(t) \leq &-\int_0^1 \tilde{u}_{xx}^2(x,t)\mathrm{d}x + \lambda\int_0^1 \tilde{u}(x,t)\tilde{u}_{xx}(x,t)\mathrm{d}x + \mathcal{J}_1 - (a+1)\mathcal{J}_2 + \rho\mathcal{J}_3 \\ &+\Big(-\delta+\epsilon\Big)\int_0^1 \tilde{w}_x^2(x,t)\mathrm{d}x + \frac{\gamma^2}{4\epsilon}\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x,\end{aligned} \tag{2.50}$$

where

$$\mathcal{J}_1 = \int_0^1 \tilde{u}_{xx}(x,t)K\Big((K^{-1}\tilde{u})^3\Big)(x,t)\mathrm{d}x, \tag{2.51}$$

$$\mathcal{J}_2 = \int_0^1 \tilde{u}_{xx}(x,t)K\Big((K^{-1}\tilde{u})^2\Big)(x,t)\mathrm{d}x, \tag{2.52}$$

$$\mathcal{J}_3 = \int_0^1 \tilde{u}_{xx}(x,t)K\Big((K^{-1}\tilde{u})(K^{-1}\tilde{w})\Big)(x,t)\mathrm{d}x. \tag{2.53}$$

Then using the bound of the operators K and K^{-1}, and Cauchy-Schwartz inequality, we estimate (2.51) as

$$|\mathcal{J}_1| \leq C\,\|\tilde{u}\|_{\mathrm{L}^\infty(0,1)}^3 \left(\int_0^1 \tilde{u}_{xx}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}},$$

for some constant $C>0$. Thus, use of Sobolev inequality

$$\|u\|_{\mathrm{L}^\infty(0,1)} \leq \|u_x\|_{L^2(0,1)} \tag{2.54}$$

yields

$$|\mathcal{J}_1| \leq C\left(\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x\right)^{\frac{3}{2}}\left(\int_0^1 \tilde{u}_{xx}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}.$$

Using the inequality $2ab \leq a^2+b^2$, we obtain

$$|\mathcal{J}_1| \leq \frac{1}{3}\int_0^1 \tilde{u}_{xx}^2(x,t)\mathrm{d}x + C\left(\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x\right)^3 \tag{2.55}$$

for some constant $C>0$. Like the estimate of $\mathcal{J}_1$, we estimate $\mathcal{J}_2$ as

$$(a+1)\,|\mathcal{J}_2| \leq C\left(\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x\right)\left(\int_0^1 \tilde{u}_{xx}^2(x,t)\mathrm{d}x\right)^{\frac{1}{2}}.$$

Thus using the inequality $2ab \leq a^2+b^2$, we obtain

$$(a+1)\,|\mathcal{J}_2| \leq \frac{1}{3}\int_0^1 \tilde{u}_{xx}^2(x,t)\mathrm{d}x + C\left(\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x\right)^2, \tag{2.56}$$

for some constant $C>0$. As before using the inequality $2ab \leq a^2+b^2$ and the bound of operator K, we estimate $\rho\mathcal{J}_3$ as follows

$$\rho\,|\mathcal{J}_3| \leq \frac{1}{3}\int_0^1 \tilde{u}_{xx}^2(x,t)\mathrm{d}x + C\int_0^1 (K^{-1}\tilde{u})^2(x,t)(K^{-1}\tilde{w})^2(x,t)\mathrm{d}x$$

for some positive constant C. Now using the bound of operator K^{-1} and Sobolev inequality (2.54), we get

$$\rho\,|\mathcal{J}_3| \leq \frac{1}{3}\int_0^1 \tilde{u}_{xx}^2(x,t)\mathrm{d}x + C\left(\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x\right)\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right), \tag{2.57}$$

for some positive constant C. Using integration by parts for the term $\int \tilde{u}\tilde{u}_{xx}$ in (2.50), and combining (2.55)–(2.57), we obtain

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t}(t) \leq -\lambda\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x + C_\lambda\Big(V^2+V^3\Big) + C_\lambda\left(\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x\right)\left(\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x\right)\\ + \Big(-\delta+\epsilon\Big)\int_0^1 \tilde{w}_x^2(x,t)\mathrm{d}x + \frac{\gamma^2}{4\epsilon}\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x,\end{aligned} \tag{2.58}$$

where C_λ is a positive constant depends on λ, as the positive constant C defined in (2.55)–(2.57) depends on the operator norm of K and K^{-1}. Now we choose λ large so that

$$-\lambda + \frac{\gamma^2}{4\epsilon} \leq -\delta + \frac{\epsilon}{3}. \tag{2.59}$$

For this λ and for $\epsilon > 0$, there exists a $r_1 > 0$ such that for each $t > 0$

$$\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x \leq \frac{\epsilon}{3C_\lambda}, \tag{2.60}$$

by well-posedness result Proposition 2.7 of the target system (2.2). Combining (2.58), (2.59) and (2.60), we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) \leq \Big(-\delta + \frac{2\epsilon}{3}\Big)V(t) + C\Big(V^2(t) + V^3(t)\Big) \tag{2.61}$$

for some positive constant C. Now as the proof of Proposition 2.7, we can show that the target system (2.2) is well-posed in the following setting

$$\mathrm{C}^0\Big([0,\infty);\mathrm{H}^1_0(0,1)\Big) \cap \mathrm{L}^2\Big(0,\infty;\mathrm{H}^2(0,1)\cap\mathrm{H}^1_0(0,1)\Big) \times \mathrm{C}^0\Big([0,\infty);\mathrm{H}^1(0,1)\Big) \cap \mathrm{L}^2\Big(0,\infty;\mathrm{H}^1(0,1)\Big),$$

moreover there exists $r > 0$ (choosing small enough) such that $V(t) + V^2(t) \leq \frac{\epsilon}{3C}$ provided

$$\|\tilde{u}_0\|_{\mathrm{H}^1(0,1)} + \|\tilde{w}_0\|_{\mathrm{H}^1(0,1)} \leq r. \tag{2.62}$$

Therefore we have

$$\frac{\mathrm{d}V}{\mathrm{d}t}(t) \leq (-\delta + \epsilon)V(t). \tag{2.63}$$

Hence the proof is complete. □

2.4.2. Proof of Theorem 2.8.

The proof follows directly from above Proposition and the fact that the operator K defined in (1.9) is invertible in $\mathrm{H}^1(0,1)$.

2.4.3. $H^1 \times L^2$ case.

In [8], the authors have proved approximate controllability of the FHN system in the space $\mathrm{H}^1(0,1) \times \mathrm{L}^2(0,1)$. Since they considered the initial and terminal data in a same connected component of the set of all steady state, time dependent parametrization helps to have zero initial conditions in the reduced problem. This makes the solution of the ODE regular enough. Thus by suitable energy estimate stabilization result has been obtained. In our case it can be easily checked that the RM system is exponentially stabilizable in the space $\mathrm{H}^1(0,1) \times \mathrm{L}^2(0,1)$, provided the initial data lies in the space $\mathrm{H}^1_{\{0\}}(0,1) \times \mathrm{L}^2(0,1)$. To conclude the result, one has to consider the following Lyapunov functional

$$V_1(t) = \frac{1}{2}\int_0^1 \tilde{u}^2(x,t)\mathrm{d}x + \frac{1}{2}\int_0^1 \tilde{u}_x^2(x,t)\mathrm{d}x + \frac{1}{2}\int_0^1 \tilde{w}^2(x,t)\mathrm{d}x. \tag{2.64}$$

Then by the proofs of the Theorem 2.3 and Proposition 2.9, we have exponential stabilization by feedback control in $\mathrm{H}^1 \times \mathrm{L}^2$.

3. Stabilization and related results for FitzHugh–Nagumo system

3.1. Local Exponential stabilization by feedback of the FitzHugh–Nagumo equation

This section is devoted to the proof of the stabilization theorem for FHN model *i.e.* Theorem 1.4. We first prove the stabilization of the FHN system around equilibrium state $(\bar{u}, \bar{w}) = (0, 0)$. Nonzero steady state case will follow this proof. As in the previous case, we will transform the nonlinear FitzHugh–Nagumo equation (1.23) to the natural target system

$$\begin{cases} \tilde{u}_t - \tilde{u}_{xx} + \lambda \tilde{u} + \rho \tilde{w} = \left(-u^3 + \int_0^x k(x,y) u^3(y,t) \mathrm{d}y \right) \\ \quad + (a+1) \left(u^2 - \int_0^x k(x,y) u^2(y,t) \mathrm{d}y \right) & \text{in } (0,1) \times (0,\infty), \\ \tilde{w}_t = \gamma \tilde{u} - \delta \tilde{w} & \text{in } (0,1) \times (0,\infty), \\ \tilde{u}(0,t) = 0, \quad \tilde{u}(1,t) = 0 & \text{in } (0,\infty), \\ \tilde{u}(x,0) = \tilde{u}_0(x), \quad \tilde{w}(x,0) = \tilde{w}_0(x) & \text{in } (0,1), \end{cases} \tag{3.1}$$

by the transformation (1.9), where the kernel function is the solution of the equation (1.10). By similar energy estimate as nonlinear Rogers–McCulloch case, we can easily prove the estimate

$$\|\tilde{u}(t)\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}(t)\|_{\mathrm{L}^2(0,1)} \le \mathrm{e}^{-\delta t} \left(\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)} \right). \tag{3.2}$$

As in the linearized case, (mentioned in the Introduction) here we have to take the Lyapunov functional as

$$V(t) = \frac{\gamma}{\rho} \int_0^1 \tilde{u}^2(x,t) \mathrm{d}x + \int_0^1 \tilde{w}^2(x,t) \mathrm{d}x, \tag{3.3}$$

and the corresponding damping term λ as

$$-\lambda + (a+1)^2 \frac{3}{4} \le -\delta. \tag{3.4}$$

3.1.1. Proof of Theorem 1.4

Using the invertibility of the operator defined in (1.9) and the estimate (3.2), we get the exponential stabilization result for the system (1.23) which is essentially the following estimate

$$\|u(t)\|_{\mathrm{L}^2(0,1)} + \|w(t)\|_{\mathrm{L}^2(0,1)} \le M \mathrm{e}^{-\delta t} \left(\|u_0\|_{\mathrm{L}^2(0,1)} + \|w_0\|_{\mathrm{L}^2(0,1)} \right), \tag{3.5}$$

with the feedback law

$$q_1(t) = u(1,t) = \int_0^1 k(1,y) u(y,t) \mathrm{d}y. \tag{3.6}$$

The proof for non zero steady state $(\bar{u}, \bar{w})$ case will be in similar manner. Taking the change of variable

$$z = u - \bar{u}, \qquad v = w - \bar{w}, \tag{3.7}$$

and using (1.23) and (1.24), we write the system satisfied by (z, v) as follows

$$\begin{cases} z_t - z_{xx} + A(x)z = -z^3 + B(x)z^2 - \rho v, & \text{in } (0,1)\times(0,\infty), \\ v_t = \gamma z - \delta v, & \text{in } (0,1)\times(0,\infty), \\ z(0,t) = 0,\ z(1,t) = h(t), & \text{in } (0,\infty), \\ z(x,0) = u_0(x) - \bar{u}, \quad v(x,0) = w_0(x) - \bar{w} & \text{in } (0,1), \end{cases} \tag{3.8}$$

where $A(x) = a - 2\bar{u}(a+1) + 3\bar{u}^2$ and $B(x) = (a+1) - 3\bar{u},\ h(t) = q_1(t) - \bar{u}(1)$. Performing the above proof for the (z, v) system (3.8), we can establish the stability estimate (1.31). Note that in this case the kernel function, used for the backstepping method will be:

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) - (\lambda - A(y))k(x,y) = 0 & 0 < y < x < 1, \\ 2\dfrac{\mathrm{d}}{\mathrm{d}x}k(x,x) + (\lambda - A(x)) = 0 & 0 \le x \le 1, \\ k(x,0) = 0, & 0 \le x \le 1. \end{cases} \tag{3.9}$$

Another change will happen in the choice of damping coefficient λ. As we take it in the following manner

$$-\lambda + \frac{3}{4}\|B\|^2_{\mathrm{L}^\infty(0,1)} \le -\delta, \tag{3.10}$$

where $B = (a+1) - 3\bar{u}$, defined in (1.28) and the corresponding control is as follows

$$q_1(t) = \int_0^1 k(1,y)u(y,t)\mathrm{d}y - \int_0^1 k(1,y)\bar{u}(y)\mathrm{d}y + \bar{u}(1). \tag{3.11}$$

Remark 3.1. As we mentioned in the Remark 2.6 of previous section, the exponential stabilization of FHN system (1.1)–(1.2) can be extended to the case where one can consider the nonlinear term $I_F^s(u,w) = -u(u - \phi_2(x))(u - \phi_3(x)) - \rho w$, with $\phi_2,\ \phi_3 \in C^1[0,1]$. In that case the choice of damping coefficient will be

$$-\lambda + \frac{3}{4}\|\phi_2 + \phi_3 - 3\bar{u}\|^2_{\mathrm{L}^\infty(0,1)} \le -\delta, \tag{3.12}$$

where $(\bar{u}, \bar{w})$ is a steady state of the FHN model (1.1) with nonlinearity $I = I_F^s$.

3.2. Lack of complete stabilization of linear FHN system

In this section, we will show that the linearized FHN system is not exponentially stabilizable in $\mathbf{Z}$ with decay rate $\omega > \delta$, where δ is the accumulation point of the spectrum of the corresponding linearized operator, by a boundary control belongs to $L^2(0,\infty)$. We will follow the technique of the paper [11] to prove that. Before going to describe the strategy of proof of lack of exponential stabilization result, let us first study the spectrum of the operator corresponding to the following linear FHN equation

$$\begin{cases} u_t - u_{xx} + au + \rho w = 0 & \text{in } (0,1)\times(0,\infty), \\ w_t = \gamma u - \delta w & \text{in } (0,1)\times(0,\infty), \\ u(0,t) = 0,\ u(1,t) = q(t) & \text{in } (0,\infty), \\ u(x,0) = u_0(x) \quad w(x,0) = w_0(x) & \text{in } (0,1), \end{cases} \tag{3.13}$$

Let us recall the linearized FHN operator $\mathbf{A}^{FHN}$ and we denote it by $\mathbf{A}$,

$$\mathbf{A} = \begin{pmatrix} \frac{d^2}{\mathrm{d}x^2} - aI_d & -\rho I_d \\ \gamma I_d & -\delta I_d \end{pmatrix}, \tag{3.14}$$

with

$$D(\mathbf{A}) := \{\mathbf{U} = (u, w)^T \in \mathbf{Z} : u \in \mathrm{H}_0^1(0,1) \cap \mathrm{H}^2(0,1)\}.$$

We denote the Fourier basis $\{\mathbf{\Phi}_n\}_{n\geq 1}$ in $L^2(0,1) \times L^2(0,1)$ as follows:

$$\mathbf{\Phi}_{2n}(x) = (\sin(n\pi x), 0)^T, \quad \mathbf{\Phi}_{2n-1}(x) = (0, \sin(n\pi x))^T \quad \text{for } n \geq 1.$$

Let us define the following space

$$V_n = \text{ span } \{\mathbf{\Phi}_{2n}, \mathbf{\Phi}_{2n-1}\}, \quad n \geq 1.$$

Then, for all $n \geq 1$, V_n is invariant under $\mathbf{A}$ and $\mathbf{A}_n = \mathbf{A}|_{V_n} \in L(V_n)$ has the matrix representation

$$\begin{pmatrix} -(a + n^2\pi^2) & -\rho \\ \gamma & -\delta \end{pmatrix}$$

with respect to the basis $\{\mathbf{\Phi}_{2n}, \mathbf{\Phi}_{2n-1}\}$ of V_n. Thus the characteristic polynomial for each $n \in \mathbb{N}$ is as follows:

$$x^2 + (n^2\pi^2 + a + \delta)x + \delta(n^2\pi^2 + a) + \rho\gamma = 0.$$

Hence the roots of the characteristics polynomial are given by:

$$\nu_n = \frac{1}{2}\Big[- (n^2\pi^2 + a + \delta) \pm \sqrt{(n^2\pi^2 + a - \delta)^2 - 4\rho\gamma}\Big]$$

If possible let there exists some $n_0 \in \mathbb{N}$, such that $(n_0^2\pi^2 + a - \delta)^2 = 4\rho\gamma$. Then $\forall n < n_0$, there are some complex eigenvalues but they are finite in number. For $n > n_0$, we have two branches of real eigenvalues $\{\lambda_n, \mu_n\}_{n\in\mathbb{N}}$, where

$$\mu_n \to -\infty \quad \text{ and } \quad \lambda_n \to -\delta.$$

More precisely, we have the following asymptotics of the eigenvalues:

$$\lambda_n = -\delta + O\left(\frac{1}{n^2}\right), \quad \mu_n = -n^2\pi^2 - a + O\left(\frac{1}{n^2}\right).$$

The corresponding eigenfunctions are as follows:

$$\Phi_n = \begin{pmatrix} \frac{(\delta+\lambda_n)}{\gamma} \sin(n\pi x) \\ \sin(n\pi x) \end{pmatrix} \quad \Psi_n = \begin{pmatrix} \frac{(\delta+\mu_n)}{\gamma} \sin(n\pi x) \\ \sin(n\pi x) \end{pmatrix}. \tag{3.15}$$

Remark 3.2. The spectrum of $\mathbf{A}$ and $\mathbf{A}^*$ are identical and we have the eigenfunction of $\mathbf{A}^*$:

$$\Phi_n^* = \begin{pmatrix} \frac{-(\delta+\lambda_n)}{\rho}\sin(n\pi x) \\ \sin(n\pi x) \end{pmatrix} \quad \Psi_n^* = \begin{pmatrix} \frac{-(\delta+\mu_n)}{\rho}\sin(n\pi x) \\ \sin(n\pi x) \end{pmatrix}. \tag{3.16}$$

Strategy of the proof. We assume that $\omega > \delta$ and $-\omega$ belongs to the resolvent set of the operator $\mathbf{A}$. As, we are going to show that the linearized FHN system (3.13) is not exponentially stabilizable with a decay rate $\omega > \delta$, we consider the shifted system

$$\begin{cases} u_t - u_{xx} + au + \rho w = \omega u & \text{in } (0,1)\times(0,\infty), \\ w_t = \gamma u + (\omega-\delta)w & \text{in } (0,1)\times(0,\infty), \\ u(0,t) = 0,\ u(1,t) = q(t) & \text{in } (0,\infty), \\ u(x,0) = u_0(x) \quad w(x,0) = w_0(x) & \text{in } (0,1), \end{cases} \tag{3.17}$$

Note that, system (3.13) is exponentially stabilizable with a decay rate $\mathrm{e}^{-\omega t}$ by means of boundary feedback law q if and only if the shifted system (3.17) is stabilizable by the corresponding feedback law as, (u,w,q) satisfies (3.17) if and only if $(u_1,w_1,q_1) = \mathrm{e}^{-\omega t}(u,w,q)$ satisfies (3.13). Due to this equivalence relation of these two systems, it is enough to study the shifted system (3.17).

Let us denote the shifted operator $\mathbf{A}_\omega = \mathbf{A} + \omega I_d$. The spectrum of $\mathbf{A}$ consists of an accumulating spectrum branch λ_n, that is, $\lambda_n \to -\delta$ as $n \to \infty$. Thus there exists a positive number n_ω such that $n > n_\omega$, $\lambda_n + \omega$ are positive real numbers, as $\omega > \delta$. The presence of these infinitely many positive eigenvalues of $\mathbf{A}$ will help to get the lack of exponential stabilizability of (3.17). The main idea to prove the result is to exploit the fact that if (3.17) is exponentially stabilizable in $\mathbf{Z}$ using a L^2 control, then the projected system of (3.17) onto each unstable eigenspace is also exponentially stabilizable using a control with L^2 minimal norm. The L^2 norm of this sequence of controls obtained stabilizing each n th unstable projected system is bounded. But if we assume that (3.17) is exponentially stabilizable for $\omega > \delta$, then the sequence of the L^2-norm of the minimal norm controls is divergent and hence it contradicts the fact mentioned above.

To find the projection of the system (3.17) on the finite dimensional subspaces, we utilize the classical Dirichlet lifting approach and write the aforementioned system (3.17) as an evolution equation with inhomogeneous operator equation set up. Let us first consider the stationary problem with inhomogeneous boundary condition:

$$\begin{pmatrix} \dfrac{d^2}{\mathrm{d}x^2} + (\omega-a)I_d & -\rho I_d \\ \gamma I_d & (\omega-\delta)I_d \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.18}$$
$$\sigma_1(0,t) = 0, \quad \sigma_1(1,t) = q(t).$$

Proposition 3.3. *Let $q \in \mathrm{L}^2(0,\infty)$. Then for all $t > 0$, the stationary problem (3.18) has a unique solution $\mathbf{W}(t) = (\sigma_1(t),\sigma_2(t))$ in $\mathrm{H}^1_{\{0\}}(0,1)\cap\mathrm{H}^2(0,1)\times\mathrm{H}^1_{\{0\}}(0,1)\cap\mathrm{H}^2(0,1)$. Moreover the solution satisfies the following estimate*

$$\|\sigma_1\|_{L^2(0,\infty;H^2(0,1))} + \|\sigma_2\|_{L^2(0,\infty;H^2(0,1))} \le C\,\|q\|_{L^2(0,\infty)}$$

First we denote $\mathbf{H}$ as the Hilbert space $D(\mathbf{A}^*)$ equipped with graph norm of $\mathbf{A}^*$. Let $\mathbf{H}'$ be the dual of $\mathbf{H}$ with $\mathbf{Z}$ as the pivot space. We denote $\bar{\mathbf{A}}_\omega = (\mathbf{A}^*_\omega)^*$. Essentially $\bar{\mathbf{A}}_\omega : \mathbf{Z} \mapsto D(\mathbf{A}^*)'$ is an extension of the unbounded operator to $D(\mathbf{A}^*)'$. As $-\omega$ lies in the resolvent of $\mathbf{A}$, we define the inner product on $\mathbf{H}'$ as follows:

$$\langle u_1, u_2\rangle_{\mathbf{H}'} = \left\langle (\bar{\mathbf{A}}_\omega)^{-1}u_1, (\bar{\mathbf{A}}_\omega)^{-1}u_2\right\rangle_{\mathbf{Z}}. \tag{3.19}$$

Let us define the Dirichlet operator $D_r : \mathbb{R} \mapsto \mathbf{Z}$ by

$$D_r(q(t)) = \mathbf{W}(t), \tag{3.20}$$

where $\mathbf{W}(t)$ is the solution of spatial system (3.18). Note that $D_r \in \mathcal{L}(\mathbb{R}, \mathbf{Z})$. Let us define the control operator $\mathbf{B} \in \mathcal{L}(\mathbb{R}, \mathbf{H}')$ by

$$\mathbf{B} = -\bar{\mathbf{A}}_\omega D_r. \tag{3.21}$$

Now we are ready to set our system (3.17) as an inhomogeneous evolution equation:

Proposition 3.4 (Thm. 7.4, [11])**.** *Let us assume that $q \in H^1(0,\infty)$ and $(u_0, w_0) \in \mathbf{Z}$. Then $\mathbf{U} = (u, w) \in C^0([0,\infty); \mathbf{Z})$ is the unique solution of (3.17) if and only if it is the weak solution of the evolution equation:*

$$\dot{\mathbf{U}}(t) = \bar{\mathbf{A}}_\omega \mathbf{U}(t) + \mathbf{B}q(t), \quad \mathbf{U}(0) = (u_0, w_0). \tag{3.22}$$

Moreover if $q \in L^2(0,\infty)$ and $(u_0, w_0) \in \mathbf{H}'$. Then (3.22) has a unique solution $\mathbf{U} = (u, w) \in C^0([0,\infty); \mathbf{H}')$.

Next, we will see that if the system (3.22) is stabilizable by some control $q \in L^2(0,\infty)$, then we can find a stabilizing feedback control depending continuously on the initial data of the same. First, we write the definition of open loop stabilizability with control $q \in L^2(0,\infty)$, using the wording of [57], Part I, Chapter 1, Definition 2.3.

Definition 3.5. The system (3.22) is said to be open loop stabilizable in $\mathbf{H}'$, by controls in $L^2(0,\infty)$, if for any $\mathbf{U}_0 \in \mathbf{H}'$ there exists a control $q \in L^2(0,\infty)$ such that the solution $\mathbf{U} = (u, w)$ of (3.22) satisfies

$$\int_0^\infty \|\mathbf{U}(t)\|^2_{\mathbf{H}'} < \infty.$$

We recall the result [11], Theorem 7.6 which ensures the continuous dependence of the stabilizing control on the given initial data:

Theorem 3.6 (Thm. 7.6, [11])**.** *Let us assume that $\omega > \delta$ and $-\omega$ belongs to the resolvent set of the operator $\mathbf{A}$. Also assume that the system (3.22) is open loop stabilizable in $\mathbf{H}'$ by a control $q \in L^2(0,\infty)$. Then the Riccati equation*

$$\begin{cases} X \in \mathcal{L}(\mathbf{H}', \mathbf{H}), X = X^* \geq 0, \\ \bar{\mathbf{A}}_\omega^* X + X\bar{\mathbf{A}}_\omega - XBB^*X + ((\bar{\mathbf{A}}_\omega)^*)^{-1}(\bar{\mathbf{A}}_\omega)^{-1} = 0 \end{cases} \tag{3.23}$$

admits a unique solution denoted by X^∞_{min}.

The equation

$$\dot{\mathbf{U}}(t) = \left(\bar{\mathbf{A}}_\omega - \mathbf{B}\mathbf{B}^* X^\infty_{min}\right)\mathbf{U}(t), \quad \mathbf{U}(0) = (u_0, w_0). \tag{3.24}$$

admits a unique solution in the space $\mathrm{C}^0([0,\infty); \mathbf{H}')$ and this solution satisfies

$$\|\mathbf{U}(t)\|_{\mathbf{H}'} \leq C\mathrm{e}^{-\mu t}\|\mathbf{U}_0\|_{\mathbf{H}'}, \, \mathbf{U}_0 \in \mathbf{H}' \tag{3.25}$$

for some positive constants C and μ. Moreover the stabilizing feedback control $\tilde{q}(t) = -\mathbf{B}^* X^\infty_{min}\mathbf{U}(t)$ obeys the following:

$$\|\tilde{q}\|_{L^2(0,\infty)} \leq C_1 \|\mathbf{U}_0\|_{\mathbf{H}'}. \tag{3.26}$$

Remark 3.7. Clearly exponential stabilizability by feedback implies open loop stabilizability. The above theorem ensures that the converse is also true, which means open loop stabilizability implies exponential stabilizability by feedback control. Proof can also be found in [57], Part V, Chapter 1, Section 3, Theorem 3.1.

3.2.1. Projection on the eigenspace

Recall the spectrum of the linearized operator $\mathbf{A}$. For all $n \geq n_0$, the eigenvalues are real. We project our system on the eigenspaces corresponding to the real eigenvalues. $\lambda_n + \omega, \mu_n + \omega$ are the eigenvalues of $\bar{\mathbf{A}}_\omega$ with corresponding eigenfunction Φ_n, Ψ_n respectively. The adjoint operator $\bar{\mathbf{A}}_\omega^*$ contains the same eigenvalues with the eigenvectors:

$$\Phi_n^* = \theta_n \begin{pmatrix} \frac{-(\delta+\lambda_n)}{\rho} \sin(n\pi x) \\ \sin(n\pi x) \end{pmatrix}, \quad \Psi_n^* = \tilde{\theta}_n \begin{pmatrix} \frac{-(\delta+\mu_n)}{\rho} \sin(n\pi x) \\ \sin(n\pi x) \end{pmatrix}, \tag{3.27}$$

where $\theta_n, \tilde{\theta}_n$ will be adjusted due to normalization.
Let us now project our system (3.22) on the eigenspaces corresponding to $\lambda_n, n > n_\omega$, where n_ω is defined in the strategy of proof. Let us denote

$$\mathbf{Z}_n = span\{\Phi_n\}. \tag{3.28}$$

Lemma 3.8. *The Projection map $\Pi_n : \mathbf{H}' \to \mathbf{Z}_n$, for all $n \geq n_0$ is defined as*

$$\Pi_n(u) = \langle u, \Phi_n^* \rangle_{\mathbf{H}'} \Phi_n, u \in \mathbf{H}'. \tag{3.29}$$

Proof. As $\mathbf{Z}$ is the orthogonal sum of the spaces $V_n, n \geq 1$, one can show that the eigenfunctions $\{\Phi_n\}, \{\Psi_n\}$ corresponding to the eigenvalues of $\bar{\mathbf{A}}_\omega$ form a complete orthogonal system in $\mathbf{H}'$ and similarly the eigenfunctions $\{\Phi_n^*\}, \{\Psi_n^*\}$ for $\bar{\mathbf{A}}_\omega^*$. We normalize these eigenfunctions in such a way that:

$$\langle \Phi_n^*, \Phi_m \rangle_{\mathbf{H}'} = \delta_m^n \text{ and } \langle \Psi_n^*, \Psi_m \rangle_{\mathbf{H}'} = \delta_m^n \tag{3.30}$$

so that

$$\theta_n = \frac{2\rho(\lambda_n + \omega)(-\rho\gamma + (\omega - \delta)(n^2\pi^2 + a - \omega))}{(\lambda_n + \delta)[\rho^2 - (\omega - \delta)(\lambda_n + \delta)] + \gamma(\lambda_n + \delta) + \rho(\omega - a - n^2\pi^2)}, \tag{3.31}$$

with respect to the inner product in $\mathbf{H}'$. Therefore these two sequences of eigenfunctions together forms a biorthogonal system in $\mathbf{H}'$. Therefore we can define the projection as (3.29). □

Lemma 3.9. *Let $\mathbf{B}$ be the control operator defined by (3.21). Then we have*

$$\mathbf{B}^* \Phi_n^* = (-1)^n \theta_n \frac{(\delta + \lambda_n)}{\rho} n\pi a. \tag{3.32}$$

Proof. Thanks to the definition (3.21), we have $\mathbf{B}^* = -D_r^* \bar{\mathbf{A}}_\omega^*$, where D_r^* is the adjoint operator given by the following relation:

$$\langle D_r q, \bar{\mathbf{V}} \rangle_{\mathbf{Z}} = \langle q, D_r^* \bar{\mathbf{V}} \rangle_{\mathbb{R}}. \tag{3.33}$$

We now compute $D_r^*\bar{\mathbf{V}}$, where $\bar{\mathbf{V}} \in \mathbf{Z}$ with $\bar{\mathbf{V}} = \mathbf{A}_\omega^*\mathbf{V}$, $\mathbf{V} \in \mathrm{H}_0^1(0,1) \cap \mathrm{H}^2(0,1) \times L^2(0,1)$. Thus if we set, $\bar{\mathbf{V}} = (\bar{v}_1, \bar{v}_2)$ and $\mathbf{V} = (v_1, v_2)$, then we write the above relation between $\bar{\mathbf{V}}$ and $\mathbf{V}$ as

$$\begin{pmatrix} \frac{d^2}{\mathrm{d}x^2} + (\omega - a)I_d & \gamma I_d \\ -\rho I_d & (\omega-\delta)I_d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \end{pmatrix}, \qquad (3.34)$$
$$v_1(0) = 0, \quad v_1(1) = 0.$$

Let us take inner product of (3.34) with $\mathbf{W}(t)$ and performing integration by parts we have:

$$\left\langle \mathbf{W}(t), \bar{\mathbf{V}} \right\rangle_{\mathbf{Z}} = q(t)v_1'(1) \qquad (3.35)$$

Note that $\left\langle \mathbf{W}(t), \bar{\mathbf{V}} \right\rangle_{\mathbf{Z}} = \left\langle D_r q(t), \bar{\mathbf{V}} \right\rangle_{\mathbf{Z}}$. Thus it follows that $\left\langle q(t), D_r^*\bar{\mathbf{V}} \right\rangle_{\mathbf{Z}} = q(t)v_1'(t)$. Using this expression, we compute for $\Phi_n^* = (\phi_{n,1}, \phi_{n,2})^T$

$$\langle q(t), D_r^*\mathbf{A}_\omega^*\Phi_n^* \rangle_{\mathbf{Z}} = q(t)\phi_{n,1}'(1) = -q(t)\theta_n \frac{(\delta+\mu_n)}{\rho} n\pi\cos(n\pi).$$

Thus we complete the proof. □

Proposition 3.10. *The projection of the evolution equation* (3.22) *in the space* $\mathbf{Z}_n$ *for* $n \geq n_0$ *is given by the scalar equation:*

$$\left\langle \dot{\mathbf{U}}(t), \Phi_n^* \right\rangle_{\mathbf{H}'} = (\lambda_n + \omega) \left\langle \mathbf{U}(t), \Phi_n^* \right\rangle_{\mathbf{H}'} + b_n q(t), \quad \left\langle \mathbf{U}(0), \Phi_n^* \right\rangle = \left\langle (u_0, w_0), \Phi_n^* \right\rangle_{\mathbf{H}'} = \mathbf{U_{0,n}}, \qquad (3.36)$$

where $b_n = \mathbf{B}^*\Phi_n^* = (-1)^n\theta_n \frac{(\delta+\lambda_n)}{\rho} n\pi a$.

Proof. Acting the Projection operator on (3.22) and noting the fact that $\Pi_n(\bar{\mathbf{A}}_\omega \mathbf{U}(t)) = (\lambda_n + \omega)\left\langle \mathbf{U}(t), \Phi_n^* \right\rangle \Phi_n$ one can gen the desired projected equation. □

Lemma 3.11. *Let* $\omega > \delta$ *and* $n > n_\omega$. *If the system* (3.22) *is exponential stabilizable in* $\mathbf{H}'$ *by boundary control* $q \in L^2(0,\infty)$, *then the projected system* (3.36) *is stabilizable in* $\mathbf{Z}_n$ *with control in* $L^2(0,\infty)$. *Furthermore the control stabilizing* (3.36) *with minimal* $L^2(0,\infty)$ *norm is given by*

$$q_n^{min} = -2\frac{\lambda_n + \omega}{b_n} \mathrm{e}^{-t(\lambda_n+\omega)}\mathbf{U_{0,n}}, \quad n > n_\omega.$$

Proof. We utilize the result [58], Proposition 2, Section 5 for the scalar equation (3.36) and find the expression of the minimum norm control. Let us denote

$$W_\infty = \int_0^\infty \mathrm{e}^{-t\mathbf{A}_n}\mathbf{B}_n\mathbf{B}_n^*\mathrm{e}^{-t\mathbf{A}_n^*} \int_0^\infty \mathrm{e}^{-t(\lambda_n+\omega)} b_n^2 \mathrm{e}^{-t(\lambda_n+\omega)}\mathrm{d}t = \frac{b_n^2}{2(\lambda_n+\omega)}.$$

The expression for the minimum norm control for stabilization of (3.36) is

$$\begin{aligned} q_n^{min} &= -\mathbf{B}_n^* W_\infty^{-1} \mathrm{e}^{t(\mathbf{A}_n - \mathbf{B}_n\mathbf{B}_n^* W_\infty)}\mathbf{U_{0,n}} \\ &= -2\frac{\lambda_n+\omega}{b_n}\mathrm{e}^{-t(\lambda_n+\omega)}\mathbf{U_{0,n}}, \quad n > n_\omega. \end{aligned} \qquad (3.37)$$

□

Now we are ready to prove our main lack of stabilization result:

Theorem 3.12. *The system (3.13) is not exponential stabilizable in* $\mathbf{H}' = D(\mathbf{A}^*)'$ *with decay rate* $\omega > \delta$*, where* $-\omega$ *lies in the resolvent set of* $\mathbf{A}$*, by a boundary control in* $L^2(0,\infty)$*.*

Proof. If possible let the system (3.13) is stabilizable by a boundary control q with an exponential decay rate ω, for $\omega > \delta$ and $-\omega$ is in the resolvent set of $\mathbf{A}$. Then using Theorem 3.6 (see [11], Thm. 7.6) there exists a positive constant C such that for any initial data $\mathbf{U}_0 \in \mathbf{H}'$ we have the following estimate:

$$\|q\|_{L^2(0,\infty)} \leq C \|\mathbf{U}_0\|_{\mathbf{H}'} . \tag{3.38}$$

Now if we consider the initial data $\mathbf{Y}_n = \frac{\Phi_n}{\|\Phi_n\|_{\mathbf{Z}}}$, $n > n_\omega$, there exists a control $q_n \in L^2(0,\infty)$ satisfying (3.38) which stabilizes the system (3.22) and we have

$$\|q_n\|_{L^2(0,\infty)} \leq C, n > n_\omega, \tag{3.39}$$

as $\mathbf{Z}$ is continuously embedded in $\mathbf{H}'$. Let us now consider the scalar system (3.36) with initial data

$$\langle \mathbf{Y}_n, \Phi_n^* \rangle = \frac{1}{\|\Phi_n\|_{\mathbf{Z}}}, n > n_\omega.$$

Thus the minimal norm control is

$$\bar{q}_n^{min} = -2\frac{\lambda_n + \omega}{b_n} \mathrm{e}^{-t(\lambda_n+\omega)} \frac{1}{\|\Phi_n\|_{\mathbf{Z}}}, \quad n > n_\omega \tag{3.40}$$

and the minimal norm is

$$\left\|\bar{q}_n^{min}\right\|_{L^2(0,\infty)} = \frac{\sqrt{2(\lambda_n + \omega)}}{|b_n| \|\Phi_n\|_{\mathbf{Z}}}. \tag{3.41}$$

We know that q_n also stabilize the same equation and hence we have the following

$$\|q_n\|_{L^2(0,\infty)} \geq \left\|\bar{q}_n^{min}\right\|_{L^2(0,\infty)} . \tag{3.42}$$

Recall that $b_n = (-1)^n \theta_n \frac{(\delta+\lambda_n)}{\rho} n\pi a$. Note that θ_n is a convergent sequence and hence bounded. Also $\lambda_n + \delta = O\left(\frac{1}{n^2}\right)$. Thus $|b_n| = O\left(\frac{1}{n}\right)$. Note that

$$\|\Phi_n\|_{\mathbf{Z}} = \sqrt{\frac{1}{2}\left(\frac{\delta + \lambda_n}{\gamma} + 1\right)}. \tag{3.43}$$

Using above expressions we finally have $\{\left\|\bar{q}_n^{min}\right\|_{L^2(0,\infty)}\}$ is unbounded but the sequence $\{\|q_n\|_{L^2(0,\infty)}\}$ is bounded. This contradicts (3.42). Hence the system (3.13) is not boundary stabilizable with a exponential decay $\omega > \delta$. □

Here, using the same arguments as in the proof of Theorem 3.12, it can be proved that

Corollary 3.13. *The system (3.13) is not exponentially stabilizable by feedback* $\mathbf{Z} = L^2(0,1) \times L^2(0,1)$ *with an exponential decay rate* ω *when* $\omega > \delta$*.*

3.3. Local approximate controllability - consequence of the exponential stabilization

Let us again consider the nonlinear boundary control system for FitzHugh–Nagumo equations (1.23). In [8], it has been proved that the system (1.23) is not approximately controllable in $\mathrm{L}^2(0,1)\times\mathrm{L}^2(0,1)$ with localized interior control acting in the parabolic equation or Dirichlet boundary control. However, a partial approximate controllability result has been established for the system (1.23), when the time horizon is not fixed and $\mathbf{U_0},\mathbf{U_1}$ are steady state of (1.23).

Definition 3.14 (**(Partial) Approximate controllability**). The system (1.23) is called (partial) approximate controllable, if for any $\mathbf{U}_0=(u_0,w_0),\mathbf{U}_1=(u_1,w_1)\in\mathrm{L}^2(0,1)\times\mathrm{L}^2(0,1)$ and $\mathcal{E}>0$, there exists $T_{\mathcal{E}}(\mathcal{E},\mathbf{U}_0,\mathbf{U}_1)>0$ and a control $q_1\in\mathrm{L}^2(0,T)$, such that the solution $\mathbf{U}=(u,w)$ of the system (1.23) satisfies

$$\|\mathbf{U}(T)-\mathbf{U}_1\|_{\mathrm{L}^2(0,1)\times\mathrm{L}^2(0,1)}\leq\mathcal{E},\quad\forall\,T>T_{\mathcal{E}}.$$

Now onwards approximate controllability means partial approximate controllability. We denote by $\mathcal{S}$, the set of steady states endowed with the C^2 topology. In [8] the authors studied the following result.

Theorem 3.15 (Chowdhury *et. al.* [8]). *Let $\mathbf{U_0},\mathbf{U}_1\in\mathcal{S}$ lies in the same connected component of $\mathcal{S}$. Then there exists a localized interior control $f\in\mathrm{L}^2((0,T)\times(0,1))$ such that the system (1.23) with homogeneous Dirichlet boundary condition is approximate controllable.*

The proof of approximate controllability in large time has been established by using quasi-static deformations approach. One may look into the pioneer works [59–61], for more details about the method mentioned above. In the Section 4.7 of [8], a question regarding partial approximate controllability has been raised when one of $\mathbf{U_0},\mathbf{U_1}$ is steady state and the other is non steady state. Now by using exponential stabilization result of this article for the system (1.23), one can able to answer the local approximate controllability result by time invariant boundary feedback law. Let $\mathbf{U_1}=(\bar{u},\bar{w})$ be a steady state of the system (1.23). Let us take the initial data $\mathbf{U_0}=(u_0,w_0)\in\mathrm{L}^2(0,1)\times\mathrm{L}^2(0,1)$, which is not a steady state in general.

Corollary 3.16. *Let $\mathbf{U_1}=(\bar{u},\bar{w})\in\mathcal{S}$ and $\mathcal{E}>0$. Then there exists $r>0$ such that for all $\mathbf{U_0}=(u_0,w_0)\in\mathrm{L}^2(0,1)\times\mathrm{L}^2(0,1)$ with*

$$\|u_0-\bar{u}\|_{\mathrm{L}^2(0,1)}+\|w_0-\bar{w}\|_{\mathrm{L}^2(0,1)}<r,$$

there exists $T_{\mathcal{E}}=T(\mathcal{E},\mathbf{U_0},\mathbf{U_1})>0$ and a control $q\in\mathrm{L}^2(0,T)$, such that the solution $\mathbf{U}=(u,w)$ of the system (1.23) satisfies

$$\|\mathbf{U}(T)-\mathbf{U_1}\|_{\mathrm{L}^2(0,1)\times\mathrm{L}^2(0,1)}\leq\mathcal{E},\tag{3.44}$$

for all $T>T_{\mathcal{E}}$.

Proof. From the Theorem 1.4, whenever $\|u_0-\bar{u}\|_{\mathrm{L}^2(0,1)}+\|w_0-\bar{w}\|_{\mathrm{L}^2(0,1)}<r$, we have

$$\|u(t)-\bar{u}\|_{\mathrm{L}^2(0,1)}+\|w(t)-\bar{w}\|_{\mathrm{L}^2(0,1)}\leq\mathrm{e}^{-\omega t}\left(\|u_0-\bar{u}\|_{\mathrm{L}^2(0,1)}+\|w_0-\bar{w}\|_{\mathrm{L}^2(0,1)}\right),\forall\,t>0.\tag{3.45}$$

Therefore for any $\mathcal{E}>0$ we get the existence of $T_{\mathcal{E}}>0$ such that for all $T>T_{\mathcal{E}}$

$$\|u(T)-\bar{u}\|_{\mathrm{L}^2(0,1)}+\|w(T)-\bar{w}\|_{\mathrm{L}^2(0,1)}\leq\mathcal{E}.\tag{3.46}$$

□

4. Further remarks, comments and questions

4.1. Different boundary conditions

Throughout this paper we study all the stabilization results for the FHN and RM system with Dirichlet boundary conditions. It can be checked that all the results also hold for Dirichlet actuation with zero slope boundary and Neumann actuation with zero slope boundary cases

$$u_x(0,t) = 0, u(1,t) = q_1(t), \tag{4.1}$$
$$u_x(0,t) = 0, u_x(1,t) = q_1(t). \tag{4.2}$$

When we apply the backstepping method for these boundary conditions there will be a change in the kernel equation and the feedback law as well. For the case (4.1), the kernel function k is the solution of the following equation in the domain $\mathcal{T} = \{(x,y) \in [0,1] \times [0,1] : 0 \leq y \leq x \leq 1\}$

$$\begin{cases} k_{xx}(x,y) - k_{yy}(x,y) - (\lambda - a)k(x,y) = 0 & 0 < y < x < 1, \\ 2\dfrac{\mathrm{d}}{\mathrm{d}x}k(x,x) + (\lambda - a) = 0 & 0 \leq x \leq 1, \\ k_y(x,0) = 0 & 0 \leq x \leq 1, \\ k(0,0) = 0, \end{cases} \tag{4.3}$$

and the corresponding feedback law is

$$q_1(t) = \int_0^1 k(1,y)u(y,t)\mathrm{d}y. \tag{4.4}$$

The same k will work for the Neumann boundary case (4.2) and the corresponding control is

$$q_1(t) = k(1,1)u(1,t) + \int_0^1 k_x(1,y)u(y,t)\mathrm{d}y.$$

The well-posedness of the kernel equation (4.3) has been established in Lemma (3.2) of [13].

4.2. Exponential stabilization by feedback of Rogers–McCulloch and FitzHugh–Nagumo system with internal control

In literature stabilization problem by backstepping is restricted mainly for boundary control cases. Although there are some work in which interior control has been considered. In [62], the authors have proved exponential stabilization of the heat equation

$$\begin{cases} u_t(x,t) = au_{xx}(x,t) + \lambda(x)u + g(x)U(t), \\ u_x(0,t) + u(0,t) = 0, \quad u(1,t) = 0, \end{cases} \tag{4.5}$$

where g is a given profile and U is the scalar control. Rapid exponential stabilization by feedback for the linearized Schrodinger equation has been studied in [63] by using interior control. In the paper [64], the author explored backstepping approach for the one dimensional transport equation. However all these papers have studied the rapid exponential stabilization that is exponential stabilization by feedback with any decay rate for a single PDE. In our case, corresponding linearized operator has a accumulating spectrum branch for FHN and in particular an ODE in RM equation. There is a valid question that whether we can get a similar sort of stabilizability result with decay rate $\omega < \delta$ for FHN and RM system by backstepping method with interior control in the parabolic equation.

4.3. Nonlinear system with decay rate $\omega > \delta$

In this paper, we have shown that the FHN equation is exponentially stabilizable by feedback control with a decay $\mathrm{e}^{-\omega t}, \omega \leq \delta$. We have also proved that the linearized FHN is not exponentially stabilizable by feedback with a decay $\mathrm{e}^{-\omega t}, \omega > \delta$ (see Sect. 3.2). Now it is natural to ask whether the full nonlinear system is stabilizable with decay rate $\omega > \delta$. To the best of the authors' knowledge it is an open question.

4.4. Null controllability of the nonlinear model

Null controllability problem by means of boundary control or fixed localized interior control for FHN and RM systems is also an open problem. It is a model where the linearized system is not null controllable using boundary or fixed localized interior control. Hence for nonlinear problem, it is not known (to the best of the authors' knowledge) how to apply return method, as linearized system around any trajectory contains an ODE, which is the reason of lack of null controllability if the control is confined to a strict subset of domain (see [65] for details) or it acts on boundary. However, it is worth mentioning that in [66], the authors studied local exact controllability to the trajectories of the FHN model with a moving control.

4.5. Higher dimensional setting

Backstepping-based observer design has been already addressed for reaction-diffusion processes evolving in multidimensional spatial domains [12] (see Chap. 11), [67, 68], 2D Navier-Stokes channel flow [69]. We expect that this method is also useful for getting similar kind of results in higher dimensional FHN system (Instance: Rectangular domain in a 2D channel) with suitable boundary conditions.

4.6. Approximate controllability with backstepping based feedback control

So far backstepping method has been explored mainly for stabilization results. In [34], null controllability of the heat equation has been proved by a combination of backstepping method and Lebeau-Robbiano strategy. This approach further gives the null controllability for the KdV equation in [36]. In this context, it will be very interesting to explore this method for studying approximate controllability cases.

Appendix A.

This section is devoted to the proofs of the Propositions 2.2 and 2.7 regarding well-posedness for the target system (2.2).

A.1 Proof of Proposition 2.2

We will prove the Proposition 2.2 in three steps. At first, we recall the linear RM model with homogeneous boundary and nonzero source term. Here we establish the usual energy estimate. In next step we use the fixed point theorem to ensure the existence of the solution of the nonlinear system (2.2). The Third step is for the uniqueness of the solution of (2.2).

A.1.1 Well-posedness of the linear Rogers–McCulloch system with source term

Let us consider the space

$$\tilde{X} = \mathrm{C}^0\Big([0,\infty);\mathrm{L}^2(0,1)\Big) \cap \mathrm{L}^2\Big(0,\infty;\mathrm{H}^1_0(0,1)\Big) \times \mathrm{C}^0\Big([0,\infty);\mathrm{L}^2(0,1)\Big) \cap \mathrm{L}^2\Big(0,\infty;\mathrm{L}^2(0,1)\Big), \tag{A.1}$$

which is endowed with the norm

$$\|(\tilde{z},\tilde{v})\|_{\tilde{X}} = \|\tilde{z}\|_{C^0([0,\infty);L^2(0,1))} + \|\tilde{z}\|_{L^2(0,\infty;H^1(0,1))} + \|\tilde{v}\|_{C^0([0,\infty);L^2(0,1))} + \|\tilde{v}\|_{L^2(0,\infty;L^2(0,1))}\,. \tag{A.2}$$

First we study the following linear system with homogeneous boundary conditions,

$$\begin{cases} \tilde{z}_t - \tilde{z}_{xx} + \lambda\tilde{z} = f, & \text{in } (0,1)\times(0,\infty),\\ \tilde{v}_t = \gamma\tilde{z} - \delta\tilde{v} & \text{in } (0,1)\times(0,\infty),\\ \tilde{z}(0,t)=0,\ \ \tilde{z}(1,t)=0 & \text{in } (0,\infty),\\ \tilde{z}(x,0)=\tilde{u}_0(x) \quad \tilde{v}(x,0)=\tilde{w}_0(x) & \text{in } (0,1), \end{cases} \tag{A.3}$$

where $\tilde{u}_0, \tilde{w}_0 \in L^2(0,1)$, and $f \in L^2(0,\infty;L^2(0,1))$. We establish the following well-posedness of the above system (A.3).

Lemma A.1. *Let $(\tilde{u}_0,\tilde{w}_0) \in L^2(0,1)\times L^2(0,1)$ and $f \in L^2(0,\infty;L^2(0,1))$, then there exists a unique solution $(\tilde{z},\tilde{v}) \in \tilde{X}$ of (A.3) and the solution satisfies the following estimate*

$$\|(\tilde{z},\tilde{v})\|_{\tilde{X}} \le C\left(\|\tilde{u}_0\|_{L^2(0,1)} + \|\tilde{w}_0\|_{L^2(0,1)} + \|f\|_{L^2(0,\infty;L^2(0,1))}\right) \tag{A.4}$$

for some positive constant C.

Proof. We prove the existence of solution using semigroup theory. First we write the system (A.3) as

$$\begin{cases} \mathbf{U}'(t) = \tilde{\mathbf{A}}\mathbf{U}(t) + \mathbf{F}(t)\\ \mathbf{U}(0) = \mathbf{U}_0, \end{cases} \tag{A.5}$$

where $\mathbf{U} := \begin{pmatrix}\tilde{z}\\ \tilde{v}\end{pmatrix}$ in $L^2(0,T;\mathbf{Z})$. We write the unbounded operator $(\tilde{\mathbf{A}}, D(\tilde{\mathbf{A}}))$ in $\mathbf{Z}$ by

$$D(\tilde{\mathbf{A}}) = \left(H^2(0,1)\cap H^1_0(0,1)\right)\times L^2(0,1),$$

and

$$\tilde{\mathbf{A}} = \begin{pmatrix} \dfrac{d^2}{dx^2} - \lambda I_d & 0\\ \gamma I_d & -\delta I_d \end{pmatrix},$$

and we write $\mathbf{F} \in L^2(0,\infty;L^2(0,1)\times L^2(0,1))$ as

$$\mathbf{F}(t) = \begin{pmatrix} f(t)\\ 0\end{pmatrix}.$$

Now we will show that $\tilde{\mathbf{A}}$ is a maximal dissipative operator. Let us take $(\tilde{z},\tilde{v}) \in D(\tilde{\mathbf{A}})$. Then

$$\begin{aligned}\left\langle \tilde{\mathbf{A}}\begin{pmatrix}\tilde{z}\\ \tilde{v}\end{pmatrix}, \begin{pmatrix}\tilde{z}\\ \tilde{v}\end{pmatrix}\right\rangle &= \left\langle \begin{pmatrix}\tilde{z}_{xx} - \lambda\tilde{z}\\ \gamma\tilde{z} - \delta\tilde{v}\end{pmatrix}, \begin{pmatrix}\tilde{z}\\ \tilde{v}\end{pmatrix}\right\rangle\\ &= \int_0^1 \tilde{z}_{xx}(x)\tilde{z}(x)dx - \lambda\int_0^1 \tilde{z}^2(x)dx + \gamma\int_0^1 \tilde{v}(x)\tilde{z}(x)dx - \delta\int_0^1 \tilde{v}^2(x)dx.\end{aligned}$$

Using integration by parts for the first term $\int_0^1 \tilde{z}_{xx}\tilde{z}$ and the inequality $2ab \leq a^2 + b^2$ we get

$$\left\langle \tilde{\mathbf{A}}\begin{pmatrix}\tilde{z}\\ \tilde{v}\end{pmatrix}, \begin{pmatrix}\tilde{z}\\ \tilde{v}\end{pmatrix}\right\rangle = -\int_0^1 \tilde{z}_x^2(x)\mathrm{d}x + (-\lambda + \frac{\gamma^2}{\delta})\int_0^1 \tilde{z}^2(x)\mathrm{d}x + (-\delta + \frac{\delta}{4})\int_0^1 \tilde{v}^2(x)\mathrm{d}x \leq 0,$$

as λ is sufficiently large. Hence $\tilde{\mathbf{A}}$ is a dissipative operator.

Next we prove that the operator $\tilde{\mathbf{A}}$ is a maximal. Let $(g, h) \in \mathbf{Z}$. We have to find $(\tilde{z}, \tilde{v}) \in D(\tilde{\mathbf{A}})$ such that

$$(I_D - \tilde{\mathbf{A}})\begin{pmatrix}\tilde{z}\\ \tilde{v}\end{pmatrix} = \begin{pmatrix}g\\ h\end{pmatrix}.$$

Now the above is equivalent to find a $\tilde{v} \in L^2(0,1)$ and $\tilde{z} \in H_0^1(0,1) \cap H^2(0,1)$ such that

$$-\gamma\tilde{z} - (\delta+1)\tilde{v} = h, \tag{A.6}$$

$$(\lambda+1)\tilde{z} - \tilde{z}_{xx} = g. \tag{A.7}$$

By the standard elliptic theory we have the existence of the solution $(\tilde{z}, \tilde{v}) \in D(\tilde{\mathbf{A}})$. So $\tilde{\mathbf{A}}$ is maximal dissipative and hence an infinitesimal generator of a contraction semigroup. Since $f \in \mathrm{L}^2(0, \infty, \mathrm{L}^2(0,1))$, the system (A.3) has a unique solution.

Now we show that the solution of the system (A.3) satisfy the inequality (A.4). Multiplying the first and second equations of the system (A.3) by $\tilde{z}$ and $\tilde{v}$ respectively and then integrating from 0 to 1 and adding we have

$$\begin{aligned}\int_0^1 \tilde{z}(x,t)\tilde{z}_t(x,t)\mathrm{d}x + \int_0^1 \tilde{v}(x,t)\tilde{v}_t(x,t)\mathrm{d}x &= \int_0^1 \tilde{z}(x,t)\tilde{z}_{xx}(x,t)\mathrm{d}x - \lambda\int_0^1 \tilde{z}^2(x,t)\mathrm{d}x\\ &+ \gamma\int_0^1 \tilde{v}(x,t)\tilde{z}(x,t)\mathrm{d}x - \delta\int_0^1 \tilde{v}^2(x,t)\mathrm{d}x + \int_0^1 f(x,t)\tilde{z}(x,t)\mathrm{d}x.\end{aligned}$$

Using integration by parts for the first term $\int_0^1 \tilde{z}\tilde{z}_{xx}$, the inequality $2ab \leq a^2 + b^2$ for $\int \tilde{z}\tilde{v}$ and $\int_0^1 f(x,t)\tilde{z}(x,t)\mathrm{d}x$ we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \tilde{z}^2(x,t)\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \tilde{v}^2(x,t)\mathrm{d}x + 2\int_0^1 \tilde{z}_x^2(x,t)\mathrm{d}x\\ &\leq 2(-\lambda + \frac{\gamma^2}{\epsilon})\int_0^1 \tilde{z}^2(x,t)\mathrm{d}x + 2(-\delta+\epsilon)\int_0^1 \tilde{v}^2(x,t)\mathrm{d}x + \|f(t)\|^2_{\mathrm{L}^2(0,1)} + \|\tilde{z}(t)\|^2_{\mathrm{L}^2(0,1)}\end{aligned}$$

where ϵ is small positive number. Since λ is sufficiently large, integrating with respect to t we have

$$\begin{aligned}&\int_0^1 \tilde{z}^2(x,t)\mathrm{d}x + \int_0^1 \tilde{v}^2(x,t)\mathrm{d}x + 2\int_0^t\int_0^1 \tilde{z}_x^2(x,s)\mathrm{d}x\mathrm{d}s + 2(\delta-\epsilon)\int_0^t\int_0^1 \tilde{v}^2(x,s)\mathrm{d}x\mathrm{d}s\\ &\leq \int_0^1 \tilde{u}_0^2(x)\mathrm{d}x + \int_0^1 \tilde{w}_0^2(x)\mathrm{d}x + \int_0^t \|f(s)\|^2_{\mathrm{L}^2(0,1)}\mathrm{d}s\\ &\leq \int_0^1 \tilde{u}_0^2(x)\mathrm{d}x + \int_0^1 \tilde{w}_0^2(x) + \|f\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))}, \ \forall t \geq 0.\end{aligned}$$

So,

$$\sup_{t\in[0,\infty)}\int_0^1 \tilde{z}^2(x,t)\mathrm{d}x + \sup_{t\in[0,\infty)}\int_0^1 \tilde{v}^2(x,t)\mathrm{d}x + \sup_{t\in[0,\infty)}\int_0^t\int_0^1 \tilde{z}_x^2(x,s)\mathrm{d}x\mathrm{d}s$$
$$+ 2(\gamma-\epsilon)\sup_{t\in[0,\infty)}\int_0^t\int_0^1 \tilde{v}^2(x,s)\mathrm{d}x\mathrm{d}s \le \int_0^1 \tilde{u}_0^2(x)\mathrm{d}x + \int_0^1 \tilde{w}_0^2(x)\mathrm{d}x + \|f\|^2_{\mathrm{L}^2(0,\infty);\mathrm{L}^2(0,1))}.$$

This gives the following estimate

$$\begin{aligned}&\|\tilde{z}\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} + \|\tilde{z}\|_{\mathrm{L}^2(0,\infty,\mathrm{H}^1(0,1))} + \|\tilde{v}\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} + \|\tilde{v}\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))}\\ &\le C\left(\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)} + \|f\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))}\right),\end{aligned} \tag{A.8}$$

where C is a positive constant independent of t, $\tilde{u}_0$ and $\tilde{w}_0$. This completes the proof. □

A.1.2 Existence of the solution of the target system (2.2)

Let us define a map $\mathcal{K} : \tilde{X} \to \tilde{X}$ by

$$\mathcal{K}(\tilde{u},\tilde{w}) = (\hat{u},\hat{w}), \tag{A.9}$$

where $(\hat{u},\hat{w})$ is the solution of (A.3) with $f = \mathcal{F}(\tilde{u},\tilde{w})$ where the map $\mathcal{F}$ is given by

$$\mathcal{F}(\tilde{u},\tilde{w}) = -K(K^{-1}\tilde{u})^3 + (a+1)K(K^{-1}\tilde{u})^2 - \rho K(K^{-1}\tilde{u})(K^{-1}\tilde{w}). \tag{A.10}$$

We now show that $\mathcal{K}$ has a unique fixed point to ensure the existence of solution of the target system (2.2). At first we show that $\mathcal{K}$ maps a ball in $\tilde{X}$ to itself. In order to do that we need to estimate $\|\mathcal{F}(\tilde{u},\tilde{w})\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))}$. We will estimate $\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))$-norm of each terms in the expression of $\mathcal{F}$.

Let us define

$$u(\cdot,t) =: K^{-1}(\tilde{u}(\cdot,t)) \text{ and } w(\cdot,t) =: K^{-1}(\tilde{w}(\cdot,t)).$$

As in the proof of Theorem 2.3, we will assume C is a generic positive constant.

We estimate the first term in the expression of $\mathcal{F}$ as follows

$$\left\|K(K^{-1}\tilde{u})^3\right\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))} = \left\|Ku^3\right\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))}$$

Now using the bound of the operator K and Sobolev inequality, we get

$$\begin{aligned}\left\|K(K^{-1}\tilde{u})^3\right\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))} &\le \mathrm{e}^{C\sqrt{\lambda_1}}\int_0^\infty\left(\int_0^1 u^6(x,t)\mathrm{d}x\right)\mathrm{d}t\\ &\le \mathrm{e}^{C\sqrt{\lambda_1}}\int_0^\infty \|u(t)\|^4_{\mathrm{L}^\infty(0,1)}\|u(t)\|^2_{\mathrm{L}^2(0,1)}\,\mathrm{d}t.\end{aligned} \tag{A.11}$$

We use the classical interpolation inequality (Gagliardo–Nirenberg)

$$\|\phi\|^2_{\mathrm{L}^\infty(0,1)} \le 2\|\phi\|_{\mathrm{L}^2(0,1)}\|\phi_x\|_{\mathrm{L}^2(0,1)},\ \forall\phi\in\mathrm{H}^1_{\{0\}}(0,1). \tag{A.12}$$

Using (A.12) in (A.11), we get

$$\begin{aligned}\left\| K(K^{-1}\tilde{u})^3 \right\|^2_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))} &\leq \mathrm{e}^{C\sqrt{\lambda_1}} \int_0^\infty \|u(t)\|^4_{\mathrm{L}^2(0,1)} \|u_x(t)\|^2_{\mathrm{L}^2(0,1)} \\ &\leq \mathrm{e}^{C\sqrt{\lambda_1}} \|\tilde{u}\|^4_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} \|\tilde{u}\|^2_{\mathrm{L}^2(0,\infty:\mathrm{H}^1(0,1))}. \end{aligned} \tag{A.13}$$

Therefore we have

$$\left\| K(K^{-1}\tilde{u})^3 \right\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))} \leq \mathrm{e}^{C\sqrt{\lambda_1}} \|\tilde{u}\|^2_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} \|\tilde{u}\|_{\mathrm{L}^2(0,\infty:\mathrm{H}^1(0,1))}. \tag{A.14}$$

By similar calculation we can estimate second and third terms and we have

$$\left\| (a+1)K(K^{-1}\tilde{u})^2 \right\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))} \leq \mathrm{e}^{C\sqrt{\lambda_1}} \|\tilde{u}\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} \|\tilde{u}\|_{\mathrm{L}^2(0,\infty:\mathrm{H}^1(0,1))}, \tag{A.15}$$

and

$$\left\| \rho K(K^{-1}\tilde{u})(K^{-1}\tilde{w}) \right\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))} \leq \mathrm{e}^{C\sqrt{\lambda_1}} \|\tilde{w}\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} \|\tilde{u}\|_{\mathrm{L}^2(0,\infty:\mathrm{H}^1(0,1))}. \tag{A.16}$$

Combining the estimates (A.14)–(A.15)–(A.16), we conclude that

$$\begin{aligned}\|\mathcal{F}(\tilde{u},\tilde{w})\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))} \leq \mathrm{e}^{C\sqrt{\lambda_1}} \Big[&\|\tilde{u}\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} \|\tilde{u}\|^2_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} \\ + \|\tilde{u}\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} \|\tilde{u}\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} &+ \|\tilde{u}\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} \|\tilde{w}\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} \Big], \end{aligned} \tag{A.17}$$

for some positive constant C. Therefore, using the norm on the space $\tilde{X}$, we get

$$\|\mathcal{F}(\tilde{u},\tilde{w})\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))} \leq \mathrm{e}^{C\sqrt{\lambda_1}} \left(\|(\tilde{u},\tilde{w})\|^3_{\tilde{X}} + 2\|(\tilde{u},\tilde{w})\|^2_{\tilde{X}} \right). \tag{A.18}$$

Let us consider a bounded subset $\mathcal{B}_1$ of $\tilde{X}$ as follows

$$\mathcal{B}_1 = \{(u,w) \in \tilde{X} \,|\, \|(u,w)\|_{\tilde{X}} \leq \delta_1\}. \tag{A.19}$$

We will choose δ_1 in such a way that $\mathcal{K}$ maps $\mathcal{B}_1$ to itself. Applying Lemma A.1 we obtain

$$\begin{aligned}\|\mathcal{K}(\tilde{u},\tilde{w})\|_{\tilde{X}} =& \|(\hat{u},\hat{w})\|_{\tilde{X}} \\ \leq& C\left(\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)} + \|\mathcal{F}(\tilde{u},\tilde{w})\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))} \right) \\ \leq& \mathrm{e}^{C\sqrt{\lambda_1}} \left(\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)} + \delta_1^3 + 2\delta_1^2 \right), \end{aligned}$$

for some positive constants C. Choosing

$$\left(\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)} \right) \leq \mathrm{e}^{-C\sqrt{\lambda_1}} \delta_1/2 \tag{A.20}$$

and $\delta_1^2 + 2\delta_1 \leq \frac{\mathrm{e}^{-C\sqrt{\lambda_1}}}{2}$, we get

$$\|\mathcal{K}(\tilde{u}, \tilde{w})\|_{\tilde{X}} = \|(\hat{u}, \hat{w})\|_{\tilde{X}} \leq \delta_1.$$

This proves that $\mathcal{K}$ maps $\mathcal{B}_1$ to itself. Thus to apply Banach contraction Theorem it is remain to show that $\mathcal{K}$ is contraction. Let us take $(\tilde{u}_1, \tilde{w}_1), (\tilde{u}_2, \tilde{w}_2) \in \tilde{X}$. Using the triangular inequality we have

$$\|\mathcal{F}(\tilde{u}_1, \tilde{w}_1) - \mathcal{F}(\tilde{u}_2, \tilde{w}_2)\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))} \leq \mathcal{J}_1 + (a+1)\mathcal{J}_2 - \rho\mathcal{J}_3, \tag{A.21}$$

where

$$\mathcal{J}_1 = \left\| -K\left((K^{-1}\tilde{u}_1)^3 - (K^{-1}\tilde{u}_2)^3 \right) \right\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))}, \tag{A.22}$$

$$\mathcal{J}_2 = \left\| K\left((K^{-1}\tilde{u}_1)^2 - (K^{-1}\tilde{u}_2)^2 \right) \right\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))}, \tag{A.23}$$

$$\mathcal{J}_3 = \left\| K\left((K^{-1}\tilde{u}_1)(K^{-1}\tilde{w}_1) - (K^{-1}\tilde{u}_2)(K^{-1}w_2) \right) \right\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))}. \tag{A.24}$$

Next we estimate $\mathcal{J}_1$ as follows

$$\mathcal{J}_1^2 \leq \mathrm{e}^{C\sqrt{\lambda_1}} \int_0^\infty \left\| \left(K^{-1}\tilde{u}_2 - K^{-1}\tilde{u}_1 \right) \left((K^{-1}\tilde{u}_1)^2 + K^{-1}\tilde{u}_1 K^{-1}\tilde{u}_2 + (K^{-1}\tilde{u}_2)^2 \right) \right\|_{\mathrm{L}^2(0,1))}^2 \mathrm{d}t.$$

Using the inequality

$$\left| a^3 - b^3 \right| \leq \frac{3}{2} \left| a - b \right| (a^2 + b^2),$$

we obtain

$$\left\| u^3 - v^3 \right\|_{\mathrm{L}^2(0,1))} \leq \frac{3}{2} \|u - v\|_{\mathrm{L}^2(0,1)} \left(\|u\|_{\mathrm{L}^\infty(0,1)}^2 + \|v\|_{\mathrm{L}^\infty(0,1)}^2 \right).$$

Applying above inequality we bound $\mathcal{J}_1$ as follows

$$\mathcal{J}_1^2 \leq \frac{9\mathrm{e}^{C\sqrt{\lambda_1}}}{4} \int_0^\infty \left(\left\| K^{-1}\tilde{u}_1(t) \right\|_{\mathrm{L}^\infty(0,1)}^2 + \left\| K^{-1}\tilde{u}_2(t) \right\|_{\mathrm{L}^\infty(0,1)}^2 \right)^2 \left\| K^{-1}\tilde{u}_2(t) - K^{-1}\tilde{u}_1(t) \right\|_{\mathrm{L}^2(0,1))}^2 \mathrm{d}t.$$

Now using the bound of the operator K we obtain

$$\mathcal{J}_1^2 \leq \mathrm{e}^{C\sqrt{\lambda_1}} \int_0^\infty \left(\|\tilde{u}_1(t)\|_{\mathrm{L}^\infty(0,1)}^4 + \|\tilde{u}_2(t)\|_{\mathrm{L}^\infty(0,1)}^4 \right) \|\tilde{u}_2(t) - \tilde{u}_1(t)\|_{\mathrm{L}^2(0,1))}^2 \mathrm{d}t.$$

Using the inequality (A.12) we obtain

$$\mathcal{J}_1^2 \leq \mathrm{e}^{C\sqrt{\lambda_1}} \int_0^\infty \left(\|\tilde{u}_1(t)\|_{\mathrm{L}^2(0,1)}^2 \|\tilde{u}_{1,x}(t)\|_{\mathrm{L}^2(0,1)}^2 + \|\tilde{u}_2(t)\|_{\mathrm{L}^2(0,1)}^2 \|\tilde{u}_{2,x}(t)\|_{\mathrm{L}^2(0,1)}^2 \right) \|\tilde{u}_2(t) - \tilde{u}_1(t)\|_{\mathrm{L}^2(0,1))}^2 \mathrm{d}t.$$

Thus

$$\mathcal{J}_1^2 \leq \mathrm{e}^{C\sqrt{\lambda_1}} \Big[\|\tilde{u}_1\|^2_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} \|\tilde{u}_1\|^2_{\mathrm{C}^0(0,\infty;\mathrm{L}^2(0,1))} + \|\tilde{u}_2\|^2_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} \|\tilde{u}_2\|^2_{\mathrm{C}^0(0,\infty;\mathrm{L}^2(0,1))} \Big] \|\tilde{u}_1 - \tilde{u}_2\|^2_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} . \tag{A.25}$$

Taking square root we have

$$\mathcal{J}_1 \leq \mathrm{e}^{C\sqrt{\lambda_1}} \Big[\|\tilde{u}_1\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} \|\tilde{u}_1\|_{\mathrm{C}^0(0,\infty;\mathrm{L}^2(0,1))} + \|\tilde{u}_2\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} \|\tilde{u}_2\|_{\mathrm{C}^0(0,\infty;\mathrm{L}^2(0,1))} \Big] \|\tilde{u}_1 - \tilde{u}_2\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} . \tag{A.26}$$

Now applying following inequality

$$\left\|u^2 - v^2\right\|_{\mathrm{L}^2(0,1)} \leq \|u - v\|_{\mathrm{L}^\infty(0,1)} \|u + v\|_{\mathrm{L}^2(0,1)} ,$$

we estimate the term $\mathcal{J}_2$ as

$$\mathcal{J}_2^2 \leq \mathrm{e}^{C\sqrt{\lambda_1}} \int_0^\infty \left\|K^{-1}(\tilde{u}_2(t) - \tilde{u}_1(t))\right\|^2_{\mathrm{L}^\infty(0,1)} \left\|K^{-1}(\tilde{u}_2(t) + \tilde{u}_1(t))\right\|^2_{\mathrm{L}^2(0,1)} \mathrm{d}t.$$

Using the bound of the operator K^{-1} and Sobolev inequality, we get

$$\mathcal{J}_2^2 \leq \mathrm{e}^{C\sqrt{\lambda_1}} \int_0^\infty \|\tilde{u}_2(t) - \tilde{u}_1(t)\|^2_{\mathrm{H}^1(0,1)} \|\tilde{u}_2(t) + \tilde{u}_1(t)\|^2_{\mathrm{L}^2(0,1)} \mathrm{d}t.$$

Therefore using Cauchy-Schwartz inequality we conclude that

$$\mathcal{J}_2 \leq \mathrm{e}^{C\sqrt{\lambda_1}} \|\tilde{u}_1 - \tilde{u}_2\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} \|\tilde{u}_1 + \tilde{u}_2\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} . \tag{A.27}$$

Next we estimate $\mathcal{J}_3$. We use the inequality

$$\|u_1 w_1 - u_2 w_2\|_{\mathrm{L}^2(0,1)} \leq \|u_1\|_{\mathrm{L}^\infty(0,1)} \|w_1 - w_2\|_{\mathrm{L}^2(0,1)} + \|u_1 - u_2\|_{\mathrm{L}^\infty(0,1)} \|w_2\|_{\mathrm{L}^2(0,1)}$$

to get

$$\mathcal{J}_3^2 \leq \int_0^\infty \left\|K^{-1}\tilde{u}_1(t)\right\|^2_{\mathrm{L}^\infty(0,1)} \left\|K^{-1}\Big(\tilde{w}_1(t) - \tilde{w}_2(t)\Big)\right\|^2_{\mathrm{L}^2(0,1)} \mathrm{d}t + \int_0^\infty \left\|K^{-1}(\tilde{u}_1(t) - \tilde{u}_2(t))\right\|^2_{\mathrm{L}^\infty(0,1)} \left\|K^{-1}\tilde{w}_2(t)\right\|^2_{\mathrm{L}^2(0,1)} \mathrm{d}t.$$

Then, as before, using the bound of the operator K^{-1} and Sobolev inequality, we obtain

$$\mathcal{J}_3^2 \leq \mathrm{e}^{C\sqrt{\lambda_1}} \int_0^\infty \|\tilde{u}_1(t)\|^2_{\mathrm{H}^1(0,1)} \|\tilde{w}_1(t) - \tilde{w}_2(t)\|^2_{\mathrm{L}^2(0,1)} \mathrm{d}t + \mathrm{e}^{C\sqrt{\lambda_1}} \int_0^\infty \|(\tilde{u}_1(t) - \tilde{u}_2(t))\|^2_{\mathrm{H}^1(0,1)} \|\tilde{w}_2(t)\|^2_{\mathrm{L}^2(0,1)} \mathrm{d}t.$$

Then using Cauchy-Schwartz inequality we conclude

$$\mathcal{J}_3 \leq \mathrm{e}^{C\sqrt{\lambda_1}} \left(\|\tilde{u}_1\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} \|\tilde{w}_1 - \tilde{w}_2\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} + \|\tilde{w}_2\|_{\mathrm{C}^0(0,\infty;\mathrm{L}^2(0,1))} \|\tilde{u}_1 - \tilde{u}_2\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} \right). \tag{A.28}$$

Therefore summing up all the estimates (A.26)–(A.28), we obtain

$$\begin{aligned} &\|\mathcal{F}(\tilde{u}_1, \tilde{w}_1) - \mathcal{F}(\tilde{u}_2, \tilde{w}_2)\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))} \\ &\quad \leq \mathrm{e}^{C\sqrt{\lambda_1}} \Big(\|(\tilde{u}_1, \tilde{w}_1)\|^2_{\tilde{X}} + \|(\tilde{u}_2, \tilde{w}_2)\|^2_{\tilde{X}} + \|(\tilde{u}_1, \tilde{w}_1)\|_{\tilde{X}} + \|(\tilde{u}_2, \tilde{w}_2)\|_{\tilde{X}} \Big) \|(\tilde{u}_1, \tilde{w}_1) - (\tilde{u}_2, \tilde{w}_2)\|_{\tilde{X}}. \end{aligned} \tag{A.29}$$

Now we consider the ball $\mathcal{B}_2$ in $\tilde{X}$ as follows

$$\mathcal{B}_2 = \{(u, w) \in \tilde{X} |\, \|(u, w)\|_{\tilde{X}} \leq \delta_2\}. \tag{A.30}$$

We will choose δ_2 in such a way that $\mathcal{K} : \mathcal{B}_2 \to \tilde{X}$ is contraction. Let $(\tilde{u}_1, \tilde{w}_1)$, $(\tilde{u}_2, \tilde{w}_2) \in \mathcal{B}_2$. Now from Lemma A.1, we have

$$\|\mathcal{K}(\tilde{u}_1, \tilde{w}_1) - \mathcal{K}(\tilde{u}_2, \tilde{w}_2)\|_{\tilde{X}} \leq \mathrm{e}^{C\sqrt{\lambda_1}} \|\mathcal{F}(\tilde{u}_1, \tilde{w}_1) - \mathcal{F}(\tilde{u}_2, \tilde{w}_2)\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))}.$$

Therefore, from (A.29) we have

$$\|\mathcal{K}(\tilde{u}_1, \tilde{w}_1) - \mathcal{K}(\tilde{u}_2, \tilde{w}_2)\|_{X} \leq 2\mathrm{e}^{C\sqrt{\lambda_1}} (\delta_2^2 + \delta_2) \|(\tilde{u}_1, \tilde{w}_1) - (\tilde{u}_2, \tilde{w}_2)\|_{X}.$$

We choose δ_2 small such that $2\mathrm{e}^{C\sqrt{\lambda_1}}(\delta_2^2 + \delta_2) < 1$. Now let us take $\tilde{\delta} < \min(\delta_1, \delta_2)$. Then we consider the bounded ball

$$\mathcal{B} = \{(u, w) \in X |\, \|(u, w)\|_{X} \leq \tilde{\delta}\}. \tag{A.31}$$

Therefore $\mathcal{K}$ is a contraction map on $\mathcal{B}$ to itself. Therefore the solution $(\tilde{u}, \tilde{w})$ of the target system (2.2) satisfy the estimate

$$\|\tilde{u}\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} + \|\tilde{u}\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} + \|\tilde{w}\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} + \|\tilde{w}\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))} < \tilde{\delta} \tag{A.32}$$

provided the initial conditions $(\tilde{u}_0, \tilde{w}_0) \in \mathrm{L}^2(0, 1) \times \mathrm{L}^2(0, 1)$ satisfies

$$\Big(\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)} \Big) \leq \mathrm{e}^{-C\sqrt{\lambda_1}} \tilde{\delta}/2 \tag{A.33}$$

where C is a positive constant

A.1.3 Uniqueness of solution of the target system (2.2)

We now prove the uniqueness of the solution. Let us take $(\tilde{u}_1, \tilde{w}_1)$ and $(\tilde{u}_2, \tilde{w}_2)$ be the solutions of the system (2.2) and let us set

$$\begin{aligned} \tilde{u} &=: \tilde{u}_1 - \tilde{u}_2, \\ \tilde{w} &=: \tilde{w}_1 - \tilde{w}_2. \end{aligned}$$

Then $(\tilde{u}, \tilde{w})$ satisfy the following equation

$$\begin{cases} \tilde{u}_t - \tilde{u}_{xx} + \lambda\tilde{u} = \mathcal{F}(\tilde{u}_1, \tilde{w}_1) - \mathcal{F}(\tilde{u}_2, \tilde{w}_2) & \text{in } (0,1)\times(0,\infty), \\ \tilde{w}_t = \gamma\tilde{u} - \delta\tilde{w} & \text{in } (0,1)\times(0,\infty), \\ \tilde{u}(0,t) = 0,\ \tilde{u}(1,t) = 0 & \text{in } (0,\infty), \\ \tilde{u}(x,0) = 0,\ \tilde{w}(x,0) = 0 & \text{in } (0,1), \end{cases} \tag{A.34}$$

where $\mathcal{F}$ is given by (A.10). Now multiplying first equation of (A.34) bu $\tilde{u}$ and the second equation of (A.34) by $\tilde{w}$ and integrating with respect to x from 0 to 1 and the adding them we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \tilde{u}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \tilde{w}^2 + 2\int_0^1 \tilde{u}_x^2 + 2\lambda\int_0^1 \tilde{u}^2 + 2\delta\int_0^1 \tilde{w}^2 \\ &= 2\gamma\int_0^1 \tilde{u}\tilde{w} + 2\int_0^1 \left(\mathcal{F}(\tilde{u}_1, \tilde{w}_1) - 2\mathcal{F}(\tilde{u}_2, \tilde{w}_2)\right)\tilde{u}, \end{aligned}$$

and we write the second term of the right hand side as

$$\int_0^1 \left(\mathcal{F}(\tilde{u}_1, \tilde{w}_1) - \mathcal{F}(\tilde{u}_2, \tilde{w}_2)\right)\tilde{u} = \sum_{i=1}^3 \mathcal{P}_i,$$

where

$$\begin{aligned} \mathcal{P}_1 &= -\int_0^1 K\left((K^{-1}\tilde{u}_1)^3 - (K^{-1}\tilde{u}_2)^3\right)\tilde{u}, \\ \mathcal{P}_2 &= (a+1)\int_0^1 K\left((K^{-1}\tilde{u}_1)^2 - (K^{-1}\tilde{u}_2)^2\right)\tilde{u}, \\ \mathcal{P}_3 &= -\rho\int_0^1 K\left((K^{-1}\tilde{u}_1)(K^{-1}\tilde{w}_1) - (K^{-1}\tilde{u}_2)(K^{-1}\tilde{w}_2)\right)\tilde{u}. \end{aligned}$$

Then similar sort of energy estimate calculation, which we already have seen several times, leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \tilde{u}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \tilde{w}^2 + c_1\int_0^1 \tilde{w}^2 \le \|\beta(t)\|_{\mathrm{L}^\infty(0,1)}\int_0^1 \tilde{u}^2,$$

where β is a function in $\mathrm{L}^2(0,\infty;\mathrm{L}^\infty(0,1))$ and c_1 is some positive constant. Since $\tilde{u}(0,x) = 0$, $\tilde{w}(0,x) = 0$, Gronwall's Lemma gives $\tilde{u} = 0, \tilde{w} = 0$. This completes the proof.

Using the invertibility of the operator (1.9) and the Proposition 2.2, we can conclude the well-posedness of the main RM system (1.1) and (1.3). For that we introduce the space

$$X = \mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1)) \cap \mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1)) \times \mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1)) \cap \mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1)),$$

which is endowed with the norm

$$\|(u,w)\|_X = \|u\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} + \|u\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} + \|w\|_{\mathrm{C}^0([0,\infty);\mathrm{L}^2(0,1))} + \|w\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))}. \tag{A.35}$$

Corollary A.2. *There exist* $\bar{\delta} > 0$, $C > 0$ *and a continuous linear feedback law* $F : \mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1) \to \mathbb{R}$, *such that for any given* $(u_0, w_0) \in \mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1)$ *with*

$$\|u_0\|_{\mathrm{L}^2(0,1)} + \|w_0\|_{\mathrm{L}^2(0,1)} \leq \frac{\bar{\delta}}{2} \mathrm{e}^{-3C\sqrt{\lambda_1}}, \tag{A.36}$$

the system (1.1) *and* (1.3) *with* $q_1(t) = F\left(u(\cdot,t), w(\cdot,t)\right)$, *has a unique solution* $(u, v) \in X$. *Moreover, the solution satisfies*

$$\|(u, w)\|_X \leq \bar{\delta}. \tag{A.37}$$

Proof. Let us take $\bar{\delta} = \mathrm{e}^{C\sqrt{\lambda_1}}\tilde{\delta}$, where $\tilde{\delta}$ is as Proposition 2.2. Using (A.36) and (2.19), we deduce

$$\|\tilde{u}_0\|_{\mathrm{L}^2(0,1)} + \|\tilde{w}_0\|_{\mathrm{L}^2(0,1)} < \frac{\bar{\delta}}{2} \mathrm{e}^{-2C\sqrt{\lambda_1}} = \frac{\tilde{\delta}}{2} \mathrm{e}^{-C\sqrt{\lambda_1}}. \tag{A.38}$$

Proposition 2.2 and estimates (2.19) conclude the corollary. □

Here we have proved the well-posedness theorem for the Rogers–McCulloch system with Dirichlet boundary conditions. We define $\mathrm{H}^1_{\{0\}}(0,1)$ as the space of functions in $H^1(0,1)$ that vanish at $x = 0$

$$\mathrm{H}^1_{\{0\}}(0,1) = \left\{ f \in \mathrm{H}^1(0,1) : f(0) = 0 \right\}.$$

Furthermore taking the initial data $(u_0, w_0) \in \mathrm{H}^1_{\{0\}}(0,1) \times \mathrm{H}^1(0,1)$, it can be proved that the system (1.1) and (1.3) is well-posed in the following space

$$X_1 = \mathrm{C}^0\left([0,\infty); \mathrm{H}^1(0,1)\right) \cap \mathrm{L}^2\left(0,\infty; \mathrm{H}^2(0,1)\right) \times \mathrm{C}^0\left([0,\infty); \mathrm{H}^1(0,1)\right) \cap \mathrm{L}^2\left(0,\infty; \mathrm{H}^1(0,1)\right).$$

We have mentioned this result in the next section. The proof for the case of FitzHugh–Nagumo model is similar to that of Corollary A.2.

A.2 Proof of Proposition 2.7

By the detailed analysis of the previous section, we can prove that if we take $(\tilde{u}_0, \tilde{w}_0) \in \mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1)$ and $f \in \mathrm{L}^2\left(0,\infty; \mathrm{L}^2(0,1)\right)$, then the system (A.3) has a unique solution in $\mathrm{C}^0([0,\infty); \mathrm{L}^2(0,1) \times \mathrm{L}^2(0,1))$. Now let us take $(\tilde{u}_0, \tilde{w}_0) \in \mathrm{H}^1_0(0,1) \times \mathrm{H}^1(0,1)$. Then by the parabolic regularity we can see that the solution $\tilde{z}$ of the first equation of (A.3) satisfies that $\tilde{z} \in \mathrm{C}^0([0,\infty); \mathrm{H}^1_0(0,1)) \cap \mathrm{L}^2(0,\infty; \mathrm{H}^2(0,1) \cap \mathrm{H}^1_0(0,1))$, see Theorem 3.2 in [70]. We infer from this and the fact $\tilde{w}_0 \in \mathrm{H}^1(0,1)$ that the solution $\tilde{v}$ of the second equation of (A.3) belongs to the space $\mathrm{C}^0([0,\infty); \mathrm{H}^1(0,1))$. Let us take $(\tilde{u}_0, \tilde{w}_0) \in \mathrm{H}^2(0,1) \cap \mathrm{H}^1_0(0,1) \times \mathrm{H}^1(0,1)$. Then we have the following estimate

$$\begin{aligned}
\int_0^1 f^2 &= \int_0^1 \left(\tilde{z}_t - \tilde{z}_{xx} + \lambda \tilde{z}\right)^2 \\
&= \int_0^1 \tilde{z}_t^2 + \int_0^1 \tilde{z}_{xx}^2 + \lambda^2 \int_0^1 \tilde{z}^2 - 2\int_0^1 \tilde{z}_t \tilde{z}_{xx} + 2\lambda \int_0^1 \tilde{z}_t \tilde{z} - 2\lambda \int_0^1 \tilde{z}_{xx} \tilde{z}.
\end{aligned}$$

Using integration by parts for the fourth and sixth term of the right hand side and using the fact $\tilde{z}(t,0) = 0 = \tilde{z}(t,1) = \tilde{z}_t(t,0) = \tilde{z}_t(t,1)$ and ignoring the positive term $\int_0^1 \tilde{z}_t^2$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda \int_0^1 \tilde{z}^2 + \int_0^1 \tilde{z}_x^2\right) + \lambda^2 \int_0^1 \tilde{z}^2 + \int_0^1 \tilde{z}_{xx}^2 + 2\lambda \int_0^1 \tilde{z}_x^2 \leq \int_0^1 f^2. \tag{A.39}$$

Now from the second equation of the system (A.3) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \tilde{v}^2 + \int_0^1 \tilde{v}_x^2\right) + 2\delta\left(\int_0^1 \tilde{v}^2 + \int_0^1 \tilde{v}_x^2\right) = 2\gamma\left(\int_0^1 \tilde{z}\tilde{v} + \int_0^1 \tilde{z}_x\tilde{v}_x\right). \tag{A.40}$$

At first we add (A.39) and (A.40) and then we apply the inequality $2\int_0^1 ab \leq \int_0^1 a^2 + \int_0^1 b^2$ for both the terms of right hand side of (A.40). We integrate (A.40) with respect to t and get a generic constant C such that

$$\begin{aligned}&\int_0^1 \tilde{z}_x^2 + \int_0^1 \tilde{z}^2 + \int_0^1 \tilde{v}^2 + \int_0^1 \tilde{v}_x^2 + \int_0^t\int_0^1 \tilde{z}^2 + \int_0^t\int_0^1 \tilde{z}_{xx}^2 + \int_0^t\int_0^1 \tilde{z}_x^2\\ &\int_0^t\int_0^1 \tilde{v}^2 + \int_0^t\int_0^1 \tilde{v}_x^2 \leq C\left(\int_0^t\int_0^1 f^2 + \int_0^1 (\tilde{u}_0)_x^2 + \int_0^1 \tilde{u}_0^2 + \int_0^1 (\tilde{w}_0)_x^2 + \int_0^1 \tilde{w}_0^2\right)\end{aligned}$$

Taking supremum over $[0,\infty)$ we get

$$\begin{aligned}&\|\tilde{z}\|_{\mathrm{C}^0([0,\infty);\mathrm{H}_0^1(0,1))} + \|\tilde{z}\|_{\mathrm{L}^2\left(0,\infty,\mathrm{H}^2(0,1)\cap\mathrm{H}_0^1(0,1)\right)} + \|\tilde{v}\|_{\mathrm{C}^0([0,\infty);\mathrm{H}^1(0,1))} + \|\tilde{v}\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))}\\ &\leq C\left(\|\tilde{u}_0\|_{\mathrm{H}^1(0,1)} + \|\tilde{w}_0\|_{\mathrm{H}^1(0,1)} + \|f\|_{\mathrm{L}^2(0,\infty;\mathrm{L}^2(0,1))}\right).\end{aligned} \tag{A.41}$$

By using density argument we can get the above estimate for $(\tilde{u}_0,\tilde{w}_0) \in \mathrm{H}_0^1(0,1)\times\mathrm{H}^1(0,1)$ and $f \in \mathrm{L}^2\left(0,\infty;\mathrm{L}^2(0,1)\right)$. Let us define a map $\mathcal{K}:\tilde{X}_1 \to \tilde{X}_1$ by

$$\mathcal{K}(\tilde{u},\tilde{w}) = (\hat{u},\hat{w}), \tag{A.42}$$

where $(\hat{u},\hat{w})$ is the solution of (A.3) with $f = \mathcal{F}(\tilde{u},\tilde{w})$, where the map $\mathcal{F}$ is given by

$$\mathcal{F}(\tilde{u},\tilde{w}) = -K(K^{-1}\tilde{u})^3 + (a+1)K(K^{-1}\tilde{u})^2 - \rho K(K^{-1}\tilde{u})(K^{-1}\tilde{w}). \tag{A.43}$$

Proceeding with similar argument as in Section A.1.2, we have that there exist a $\tilde{\delta} > 0$ and $C > 0$ such that the target system (2.2) has a unique solution $(\tilde{u},\tilde{w})$ in $\tilde{X}_1$ and we have

$$\|\tilde{u}\|_{\mathrm{C}^0([0,\infty);\mathrm{H}^1(0,1))} + \|\tilde{u}\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^2(0,1)\cap\mathrm{H}_0^1(0,1))} + \|\tilde{w}\|_{\mathrm{C}^0([0,\infty);\mathrm{H}^1(0,1))} + \|\tilde{w}\|_{\mathrm{L}^2(0,\infty;\mathrm{H}^1(0,1))} < \tilde{\delta}, \tag{A.44}$$

provided the initial conditions $(\tilde{u}_0,\tilde{w}_0) \in \mathrm{H}_0^1(0,1)\times\mathrm{H}^1(0,1)$ satisfies

$$\left(\|\tilde{u}_0\|_{\mathrm{H}^1(0,1)} + \|\tilde{w}_0\|_{\mathrm{H}^1(0,1)}\right) \leq \frac{\tilde{\delta}}{2}\mathrm{e}^{-C\sqrt{\lambda_1}}.$$

As in the previous Section the above proposition leads to the following corollary

Corollary A.3. *There exists $\bar{\delta} > 0$, there exists $C > 0$ and a continuous linear feedback law $F:\mathrm{L}^2(0,1)\times\mathrm{L}^2(0,1)\to\mathbb{R}$ such that for any given $(u_0,w_0) \in \mathrm{H}_{\{0\}}^1(0,1)\times\mathrm{H}^1(0,1)$ with*

$$\|u_0\|_{\mathrm{H}^1(0,1)} + \|w_0\|_{\mathrm{H}^1(0,1)} \leq \frac{\bar{\delta}}{2}\mathrm{e}^{-3C\sqrt{\lambda_1}}, \tag{A.45}$$

the system (1.1) and (1.3) with $q_1(t) = F\left(u(\cdot,t),w(\cdot,t)\right)$ has a unique solution $(u,v) \in X_1$. Moreover, the solution satisfies

$$\|(u,w)\|_{X_1} \leq \bar{\delta}. \tag{A.46}$$

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