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SMALL-TIME LOCAL CONTROLLABILITY OF THE BILINEAR SCHRÖDINGER EQUATION WITH A NONLINEAR COMPETITION

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Abstract. We consider the local controllability near the ground state of a 1D Schrödinger equation with bilinear control. Specifically, we investigate whether nonlinear terms can restore local controllability when the linearized system is not controllable. In such settings, it is known [K. Beauchard and M. Morancey, *Math. Control Relat. Fields* **4** (2014) 125–160, M. Bournissou, *J. Diff. Equ.* **351** (2023) 324–360] that the quadratic terms induce drifts in the dynamics which prevent small-time local controllability when the controls are small in very regular spaces. In this paper, using oscillating controls, we prove that the cubic terms can entail the small-time local controllability of the system, despite the presence of such a quadratic drift. This result, which is new for PDEs, is reminiscent of Sussmann's $\mathcal{S}(\theta)$ sufficient condition of controllability for ODEs. Our proof however relies on a different general strategy involving a new concept of tangent vector, better suited to the infinite-dimensional setting.

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1. Introduction

1.1. Description of the control system

Let $T > 0$. In this paper, we consider the 1D Schrödinger equation given by,

$$\begin{cases} i\partial_t\psi(t,x) = -\partial_x^2\psi(t,x) - u(t)\mu(x)\psi(t,x), & (t,x) \in (0,T)\times(0,1), \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T), \\ \psi(0,x) = \psi_0(x), & x \in (0,1). \end{cases} \tag{1.1}$$

This equation is used in quantum physics to describe a quantum particle stuck in an infinite potential well and subjected to a uniform electric field whose amplitude is given by $u(t)$. The function $\mu : (0,1) \to \mathbb{R}$ depicts the dipolar moment of the particle. This equation is a bilinear control system where the state is the wave function ψ such that for all time $\|\psi(t)\|_{L^2(0,1)} = 1$ and $u : (0,T) \to \mathbb{R}$ denotes a scalar control.

Keywords and phrases: Exact controllability, Schrödinger equation, bilinear control, power series expansion.

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1.2. Functional settings

Unless otherwise specified, we will work with complex-valued functions. The Lebesgue space $L^2(0,1)$ is equipped with the hermitian scalar product given by

$$\langle f, g\rangle := \int_0^1 f(x)\overline{g(x)}\mathrm{d}x, \quad \forall f, g \in L^2(0,1).$$

Let $\mathcal{S}$ be the unit-sphere of $L^2(0,1)$. The operator A is defined by

$$\mathrm{Dom}(A) := H^2(0,1) \cap H_0^1(0,1) \quad \text{and} \quad A\varphi := -\frac{d^2\varphi}{\mathrm{d}x^2}.$$

Its eigenvalues and eigenvectors are respectively given by

$$\forall j \in \mathbb{N}^*, \quad \lambda_j := (j\pi)^2 \quad \text{and} \quad \varphi_j := \sqrt{2}\sin(j\pi\cdot).$$

The family of eigenvectors $(\varphi_j)_{j\in\mathbb{N}^*}$ is an orthonormal basis of $L^2(0,1)$. We denote by,

$$\forall j \in \mathbb{N}^*, \quad \psi_j(t,x) := \varphi_j(x)\mathrm{e}^{-\mathrm{i}\lambda_j t}, \quad \forall (t,x) \in \mathbb{R}\times[0,1],$$

the solutions of the Schrödinger equation (1.1) with $u \equiv 0$ and $\psi_0 = \varphi_j$. When $j = 1$, ψ_1 is called the ground state. We also introduce the Hilbert spaces linked to the operator A, given by, for all $s \geqslant 0$,

$$H^s_{(0)}(0,1) := \mathrm{Dom}(A^{\frac{s}{2}}) \quad \text{endowed with the norm} \quad \|\varphi\|^2_{H^s_{(0)}(0,1)} := \sum_{j=1}^{+\infty} |j^s\langle\varphi,\varphi_j\rangle|^2. \tag{1.2}$$

We will also consider, for any $k \in \mathbb{N}$, $H^k((0,T),\mathbb{R})$, the usual integer-order real Sobolev space, equipped with the usual $H^k(0,T)$-norm and $H_0^k(0,T)$ the closure of $C_c^\infty(0,T)$, the set of functions with compact support inside $(0,T)$, for the norm $\|\cdot\|_{H^k(0,T)}$.

1.3. Assumptions on the dipolar moment μ

Let us state the assumptions on μ we shall consider in the paper.

($\mathrm{H_{reg}}$) The function μ is in $H^{11}((0,1),\mathbb{R})$ with the following boundary conditions

$$\mu'(0) = \mu'(1) = \mu^{(3)}(0) = \mu^{(3)}(1) = 0. \tag{1.3}$$

($\mathrm{H_{lin}}$) There exists an integer $K \in \mathbb{N}^* - \{1\}$ such that

$$\langle\mu\varphi_1, \varphi_K\rangle = 0, \tag{1.4}$$

$$\text{and there exists } c > 0 \text{ such that for all } j \in \mathbb{N}^* - \{K\}, \quad |\langle\mu\varphi_1,\varphi_j\rangle| \geqslant \frac{c}{j^7}. \tag{1.5}$$

Define, for $p = 1, 2$ and 3, the following quadratic coefficients (with respect to μ),

$$A_K^p := (-1)^{p-1}\sum_{j=1}^{+\infty}\left(\lambda_j - \frac{\lambda_1+\lambda_K}{2}\right)(\lambda_K-\lambda_j)^{p-1}(\lambda_j-\lambda_1)^{p-1}\langle\mu\varphi_1,\varphi_j\rangle\langle\mu\varphi_K,\varphi_j\rangle. \tag{1.6}$$

($\mathrm{H}_{\mathrm{quad}}$) The coefficients $(A_K^p)_{p=1,2,3}$ satisfy

$$A_K^1 = 0, \tag{1.7}$$
$$A_K^2 = 0, \tag{1.8}$$
$$A_K^3 \neq 0. \tag{1.9}$$

($\mathrm{H}_{\mathrm{cub}}$) The following cubic coefficient (with respect to μ) does not vanish

$$C_K := \sum_{j=1}^{+\infty} (\lambda_1 - \lambda_j) \langle \mu\varphi_1, \varphi_j\rangle \langle \mu'^2 \varphi_K, \varphi_j\rangle - \sum_{j=1}^{+\infty} (\lambda_j - \lambda_K) \langle \mu\varphi_j, \varphi_K\rangle \langle \mu'^2 \varphi_1, \varphi_j\rangle \neq 0. \tag{1.10}$$

Remark 1.1. The assumption ($\mathrm{H}_{\mathrm{reg}}$) is quite technical. First, under ($\mathrm{H}_{\mathrm{reg}}$), integrations by parts and Riemann–Lebesgue's lemma lead to

$$\forall q \in \mathbb{N}^*, \quad \langle \mu\varphi_q, \varphi_j\rangle = \frac{12q}{\pi^6 j^7}\left((-1)^{j+q}\mu^{(5)}(1) - \mu^{(5)}(0)\right) + \underset{j\to+\infty}{o}\left(\frac{1}{j^7}\right). \tag{1.11}$$

Thus, all the series considered in (1.6) and (1.10) converge absolutely. Actually, one can prove that the boundary conditions (1.3) are a necessary and sufficient condition for the series defined in (1.6) with $p = 3$ to converge.

Moreover, the regularity and the boundary conditions on μ are chosen to have the well-posedness of the Schrödinger equation and the controllability of the linearized equation in the same functional framework. This is complicated due to the bilinearity of the equation (see Sect. 4 or [12] for more details).

Remark 1.2. The assumptions ($\mathrm{H}_{\mathrm{lin}}$), ($\mathrm{H}_{\mathrm{quad}}$) and ($\mathrm{H}_{\mathrm{cub}}$) are strongly linked to some well-known sufficient conditions of STLC in finite dimension, written in terms of Lie brackets. For smooth vector fields X, Y in $C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, the Lie bracket $[X, Y]$ is the vector field on $\mathbb{R}^d$ defined by $[X, Y](x) := Y'(x)X(x) - X'(x)Y(x)$. We also define by induction $\mathrm{ad}_X^0(Y) := Y$ and, for all $k \in \mathbb{N}$, $\mathrm{ad}_X^{k+1}(Y) := [X, \mathrm{ad}_X^k(Y)]$. Then, formally, with $f_0(\psi) := -iA\psi$ and $f_1(\psi) := \mu\psi$, the assumptions ($\mathrm{H}_{\mathrm{quad}}$) and ($\mathrm{H}_{\mathrm{cub}}$) can be written in terms of Lie brackets (which are actually commutators here) as

$$\forall p = 1, 2, \quad \langle [\mathrm{ad}_{f_0}^{p-1}(f_1), \mathrm{ad}_{f_0}^p(f_1)]\varphi_1, \varphi_K\rangle = 0, \tag{1.12}$$
$$\langle [\mathrm{ad}_{f_0}^2(f_1), \mathrm{ad}_{f_0}^3(f_1)]\varphi_1, \varphi_K\rangle \neq 0, \tag{1.13}$$
$$\langle [\mathrm{ad}_{f_0}^1(f_1), [\mathrm{ad}_{f_0}^1(f_1), \mathrm{ad}_{f_0}^0(f_1)]]\varphi_1, \varphi_K\rangle \neq 0.$$

Notice that the Lie brackets (1.12) and (1.13) are exactly those along which the quadratic term induces a drift in the dynamics, denying $W^{3,\infty}$-STLC for finite-dimensional systems $\dot{x} = f_0(x) + uf_1(x)$, as stated in [7, Theorem 3]. All these computations can be made rigorous when μ is regular enough, but for the sake of simplicity, the details are not given.

Moreover, if we denote by, for $k \geq 1$, $\mathcal{S}_k$ the linear subspace of $L^2(0, 1)$ spanned by the iterated Lie brackets of f_0 and f_1, containing f_1 at most k times, evaluated at the equilibrium φ_1, the assumptions ($\mathrm{H}_{\mathrm{lin}}$), ($\mathrm{H}_{\mathrm{quad}}$) and ($\mathrm{H}_{\mathrm{cub}}$) imply that

$$\begin{aligned}\mathcal{S}_2 \ni [\mathrm{ad}_{f_0}^2(f_1), \mathrm{ad}_{f_0}^3(f_1)]\varphi_1 &\text{ is a linear combination of Lie brackets of } \mathcal{S}_1 \\ &\text{and of } [\mathrm{ad}_{f_0}^1(f_1), [\mathrm{ad}_{f_0}^1(f_1), \mathrm{ad}_{f_0}^0(f_1)]]\varphi_1 \in \mathcal{S}_3.\end{aligned}$$

Therefore, hypotheses ($\mathrm{H}_{\mathrm{lin}}$), ($\mathrm{H}_{\mathrm{quad}}$) and ($\mathrm{H}_{\mathrm{cub}}$) can be seen as a "*Sussmann condition* $\mathcal{S}(\theta)$" (see [19], Thm. 3.29 for instance) for the infinite-dimensional system (1.1).

The existence of a function μ satisfying ($\mathrm{H}_{\mathrm{reg}}$), ($\mathrm{H}_{\mathrm{lin}}$), ($\mathrm{H}_{\mathrm{quad}}$) and ($\mathrm{H}_{\mathrm{cub}}$) is proved in Appendix A.

1.4. Main result

First, we state the notion of small-time local controllability (STLC) used in this paper, stressing the smallness assumption imposed on the control, as it plays a key role in the validity of controllability results.

Definition 1.3. Let $(E_T, \|\cdot\|_{E_T})_{T>0}$ be a family of normed vector spaces of real functions defined on $[0,T]$ for $T>0$. The system (1.1) is said to be E-STLC around the ground state if there exists $s \in \mathbb{N}$ such that for every $T>0$ and $\eta>0$, there exists $\delta>0$ such that for every $\psi_f \in \mathcal{S} \cap H^s_{(0)}(0,1)$ with $\|\psi_f - \psi_1(T)\|_{H^s_{(0)}(0,1)} < \delta$, there exists $u \in L^2((0,T),\mathbb{R}) \cap E_T$ with $\|u\|_{E_T} < \eta$ such that the solution ψ of (1.1) associated to the initial condition φ_1 and the control u satisfies $\psi(T) = \psi_f$.

When the linearized system is controllable, using a fixed-point theorem, one can hope to prove STLC for the nonlinear system as explained in [19, Chapter 3.1] in finite dimension. When it is not the case, one needs to go further into the expansion of the solution in the spirit of [19, Chapter 8]. For the Schrödinger equation, a few STLC results are already known.

Linear behavior. Since [6], it is known that when the coefficients $(\langle \mu\varphi_1, \varphi_j\rangle)_{j\in\mathbb{N}^*}$ satisfy

$$\text{there exists a constant } c>0 \text{ such that} \quad \forall j \in \mathbb{N}^*, \quad |\langle \mu\varphi_1, \varphi_j\rangle| \geqslant \frac{c}{j^7}, \tag{1.14}$$

for every $k \in \mathbb{N}$, the Schrödinger equation is H^k_0-STLC around the ground state with targets in $H^{7+2k}_{(0)}$, using the linear test.

Quadratic behavior. In [13], the author proved that when the condition (1.14) does not hold, and more precisely, when for some $n \geq 2$ (resp. $n=1$)

$$\langle \mu\varphi_1, \varphi_K\rangle = 0, \quad A^1_K = \cdots = A^{n-1}_K = 0 \quad \text{and} \quad A^n_K \neq 0,$$

(with enough regularity on μ so that the associated series converge), the Schrödinger equation is not H^{2n-3}-STLC (resp. $W^{-1,\infty}$-STLC), due to a drift quantified by the H^{-n}-norm of the control. Let us stress that, with this result, under (H_{lin}) and (H_{quad}), the Schrödinger equation (1.1) is not H^3-STLC.

The goal of this paper is to prove the following STLC result, despite the drift, by taking advantage of the cubic term of the expansion.

Theorem 1.4. *Let μ satisfying (H_{reg}), (H_{lin}), (H_{quad}) and (H_{cub}). Then, the Schrödinger equation (1.1) is H^2_0-STLC around the ground state with targets in $H^{11}_{(0)}(0,1)$.*

The paper is organized as follows. First, in Section 2, the strategy of the proof of Theorem 1.4 is presented on a toy model in finite dimension. Then, a systematic approach to recover STLC when the linearized system misses a finite number of directions is described in Section 3. Before applying this method to the Schrödinger equation, we recall in Section 4 its well-posedness and the controllability result in projection of [12]. Then, the power series expansion of the Schrödinger equation is computed in Section 5. Finally, Section 6 is dedicated to the proof of Theorem 1.4.

1.5. State of the art

In infinite dimension, bilinear systems have long been considered non exactly controllable due to a result of Ball, Marsden, and Slemrod [1] (see [14, 36] for generalizations). But an exact controllability result [2], proved in a different functional setting, relaunched the research.

For the bilinear Schrödinger equation (1.1), *exact controllability* results were first demonstrated locally around the ground state, thanks to the *linear test* [6, 12]. These results require assumptions on the dipolar moment μ entailing that the linearized system around the ground state is exactly controllable, and they hold in arbitrarily small time, because so does the controllability of the linearized system (see [23, 34] for generalizations).

When these assumptions on μ do not hold, the linearized system misses some directions, but one may try to use higher order terms to recover controllability: this is the *power series expansion* method (see [19], Chap. 3 or [20]). This method first made it possible to recover exact controllability in *large time* (see [3, 5, 9, 15, 16, 30]), but the small-time local controllability was still open.

The contribution of the quadratic term, along a lost direction, can be a signed quadratic form, that induces a drift of the solution along this direction, preventing controllability. In the small time asymptotic, this always happens in finite dimension [7] and is also observed in infinite dimension [9, 13, 21, 22] (without being systematic [8]). For these obstructions to hold, the control needs to be small in a norm adapted to the Lie bracket along which the system drifts; thus the small-time local controllability was still open without this smallness assumption, and this is the problem under study in this article.

We consider a dipolar moment μ for which the *linearized system loses one direction*, on which appears a competition between a quadratic and a cubic term. It is known [13] that the quadratic term prevents small-time local controllability in a certain regime (in which the cubic term is negligible compared to the quadratic term). In a different regime (in which the quadratic term is negligible compared to the cubic term), we prove that *small-time local controllability* holds.

This is the first small-time local controllability result obtained on a PDE thanks to a competition between two nonlinear terms. Such mechanisms had already been understood for ODEs (see *e.g.* [35]) but the proof was not suitable for a transfer to PDEs. Thus, the first contribution of this article is a new proof strategy and the second one is to run this strategy in full on the PDE (1.1). Indeed, the infinite-dimensional nature of PDEs makes the job harder than in [35].

For approximate controllability results about the PDE (1.1), we refer to [17, 29, 31].

2. Strategy and toy model

2.1. Notation

Let $T > 0$. For $u \in L^1(0,T)$, the family $(u_n)_{n\in\mathbb{N}}$ of the iterated primitives of u is defined by induction as,

$$u_0 := u \quad \text{and} \quad \forall n \in \mathbb{N},\ u_{n+1}(t) := \int_0^t u_n(\tau)\mathrm{d}\tau, \quad t \in [0,T]. \tag{2.1}$$

Sometimes, to uniformize the notations of the primitives and derivatives of u, we will write $u^{(n)}$ when n is a negative integer to denote $u_{|n|}$, the $|n|$-th primitive of u.

2.2. Strategy

In short, when a control system satisfies assumptions of the form ($\mathrm{H}_{\mathrm{lin}}$), ($\mathrm{H}_{\mathrm{quad}}$) and ($\mathrm{H}_{\mathrm{cub}}$), the state space can be divided into two subspaces:

- the space of the *directions controllable at a linear level* (and thus at a nonlinear level by the means of the inverse mapping theorem) $\mathcal{H} = \overline{\mathrm{Span}}\,(\varphi_j;\ j \in \mathbb{N}^* - \{K\})$;

- the space of the *direction lost at the linear level* $\mathcal{M} = \mathrm{Span}\,(\varphi_K)$.

The proof of Theorem 1.4 consists first in moving along the lost direction, using a "good" *cubic term*, despite a "bad" quadratic term, by the means of oscillating controls. Then, one can use a fixed-point theorem to conclude.

2.3. A toy model

In this section, we illustrate the method used later on the Schrödinger equation (1.1) on the following polynomial toy model,

$$\begin{cases} \dot{x}_1 = u, \\ \dot{x}_2 = x_1, \\ \dot{x}_3 = x_2, \\ \dot{x}_4 = x_3^2 + x_1^2 x_2. \end{cases} \tag{2.2}$$

For this example, $\mathcal{H} = \mathrm{Span}(e_1, e_2, e_3)$ and $\mathcal{M} = \mathrm{Span}(e_4)$, where $(e_i)_{i=1,\dots,4}$ denotes the canonical basis of $\mathbb{R}^4$.

First, one can notice that the control system (2.2) *is not* H^3-STLC. Indeed, solving explicitly (2.2), the fourth component is given by,

$$x_4(T;\ u,\ 0) = \int_0^T u_3(t)^2 \mathrm{d}t + \int_0^T u_1(t)^2 u_2(t) \mathrm{d}t. \tag{2.3}$$

Moreover, using Cauchy–Schwarz and Gagliardo–Nirenberg inequalities [32], one gets the existence of $C > 0$ such that for all $T > 0$ and $u \in H^3(0,T)$,

$$\left| \int_0^T u_1(t)^2 u_2(t) \mathrm{d}t \right| \leqslant C \|u_1\|^3_{L^2(0,T)} \leqslant C\big(\|u^{(3)}\|_{L^2(0,T)} + T^{-3}\|u\|_{L^2(0,T)}\big)\|u_3\|^2_{L^2(0,T)}. \tag{2.4}$$

Thus, the quadratic term prevails on the cubic term in (2.3) when controls are small in H^3. This allows us to deny H^3-STLC for (2.2) (see for instance [7, 13] for more details on quadratic obstructions).

Nonetheless, using the strategy described above, let us prove that (2.2) *is* H_0^2-STLC. Namely, it consists in proving that, for all $T > 0$, there exist $s > 0$ and a family of controls $(u_b)_{b\in\mathbb{R}}$ such that, for all $b \in \mathbb{R}$ small enough,

$$|x(T;\ u_b,\ 0) - be_4| \leq C|b|^{1+s} \quad \text{with} \quad \|u_b\|_{H_0^2(0,T)} \leq C|b|^s. \tag{2.5}$$

More precise statements will be given in Section 3. Let $T > 0$. The estimates (2.5) can be obtained using for example oscillating controls of the form, for all $b \in \mathbb{R}^*$,

$$u_b(t) = \mathrm{sign}(b)|b|^{\frac{7}{41}} \phi^{(3)}\left(\frac{t}{|b|^{\frac{4}{41}}}\right) \quad \text{with} \ \ \phi \in C_c^\infty(0,1) \text{ s. t. } \int_0^1 \phi''(\theta)^2 \phi'(\theta) \mathrm{d}\theta = 1. \tag{2.6}$$

These controls are fitted so that the cubic term can absorb the drift along the lost direction while remaining small in the H_0^2-norm. Indeed, substituting these controls into (2.3) and performing the change of variables $t = |b|^{\frac{4}{41}}\theta$, one gets

$$\begin{aligned} x_4(T;\ u_b,\ 0) &= |b|^{\frac{38}{41}} \int_0^{|b|^{\frac{4}{41}}} \phi^2(t/|b|^{\frac{4}{41}}) \mathrm{d}t + \mathrm{sign}(b)|b|^{\frac{37}{41}} \int_0^{|b|^{\frac{4}{41}}} \big(\phi^{(2)}(t/|b|^{\frac{4}{41}})\big)^2 \phi'(t/|b|^{\frac{4}{41}}) \mathrm{d}t \\ &= |b|^{\frac{42}{41}} \int_0^1 \phi^2(\theta) d\theta + b. \end{aligned}$$

Moreover, as ϕ is supported on $(0,1)$, one directly has,

$$(x_1(T;\ u_b,\ 0),\ x_2(T;\ u_b,\ 0),\ x_3(T;\ u_b,\ 0)) = (u_1(T), u_2(T), u_3(T)) = (0,0,0). \tag{2.7}$$

Besides, for all $b \in \mathbb{R}^*$,

$$\|u_b''\|_{L^2(0,T)} \leq \|\phi^{(5)}\|_{L^2(0,1)}|b|^{\frac{1}{41}}.$$

Notice that the map $b \mapsto u_b$ from $\mathbb{R}^*$ to $H_0^2(0,T)$ can be extended at zero with $u_0 = 0$. In the end, (2.5) holds with $s = \frac{1}{41}$ and using Brouwer's fixed-point theorem, one can conclude on the H_0^2-STLC of (2.2).

Remark 2.1. For more toy examples in finite dimension illustrating the method developed in this paper, the interested reader can refer to [11, Section 3].

Remark 2.2. At a heuristic level, assumptions ($\mathrm{H_{reg}}$), ($\mathrm{H_{lin}}$), ($\mathrm{H_{quad}}$) and ($\mathrm{H_{cub}}$) entail that, in the asymptotic of controls small in the H_0^2-norm, the leading terms of the solution ψ of the Schrödinger equation (1.1) along the lost direction are given by

$$\langle \psi(T), \varphi_K \mathrm{e}^{-\mathrm{i}\lambda_1 T}\rangle \approx -iA_K^3 \int_0^T u_3(t)^2 \mathrm{d}t + iC_K \int_0^T u_1(t)^2 u_2(t)\mathrm{d}t. \tag{2.8}$$

Thus, the behavior of the Schrödinger equation (1.1) can compare in some way to the polynomial toy model (2.2). However, let us state here the main difficulties we are going to face in the infinite-dimensional framework compared to this toy model.

- Because the solution of the Schrödinger equation (1.1) is complex-valued, under the assumption ($\mathrm{H_{lin}}$), two (real) directions are lost, φ_K and $i\varphi_K$, and not only one.
- The computations of the solution along the lost directions will be much more complicated than equation (2.3), even though still explicit (see Sect. 5).
- Using the same oscillating controls (2.6) as for the polynomial toy model (2.2), we will prove in the same way that the component of the solution of the Schrödinger equation along the first lost direction $i\varphi_K$ can be moved, that is namely,

$$|\langle \psi(T;\ u_b,\ \varphi_1), \varphi_K \mathrm{e}^{-\mathrm{i}\lambda_K T}\rangle - ib| \leq C|b|^{1+\frac{1}{41}}, \tag{2.9}$$

  see Proposition 6.1. Then, contrary to the finite-dimensional case (see (2.7)), the behavior of the solution along the remaining components $(\varphi_j)_{j\in\mathbb{N}^*-\{K\}}$ is, in general, not negligible. Thus, in a second time, we will correct exactly this infinite number of directions, using that there are controllable at the linear level, and thus at the nonlinear level. Therefore, one can find a control v_b such that,

$$\forall j \in \mathbb{N}^* - \{K\}, \quad \langle \psi(2T;\ v_b,\ \varphi_1), \varphi_j\rangle = \langle \psi_1(2T), \varphi_j\rangle.$$

  The core of the paper is to prove that such a linear correction does not destroy the work done in (2.9). This is done in Proposition 6.6, using sharp and simultaneous estimates on the linear control v_b obtained in [12, Theorem 1.2].
- The motion along the second lost direction φ_K will be built from the motion along $i\varphi_K$, see Proposition 6.8.

3. Method of control variations

Under ($\mathrm{H_{lin}}$), the linearized equation around the ground state of the Schrödinger equation (1.1) is not controllable: it misses one complex direction φ_K. This situation can be called 'controllability up to (real) codimension

two'. The goal of this section is to propose a systematic approach to deal with these situations, which is different from the ones already known for ODEs and better adapted to PDEs.

For finite-dimensional systems, the classical approach used by Kawski [28, Theorem 2.4] consists in, proving that any lost direction is a 'tangent vector' thanks to oscillating controls, and deducing STLC thanks to a time-iterative process that uses arbitrarily small time intervals. For PDEs, using small time intervals is not comfortable, due to the control-cost explosion when the time goes to zero. Therefore, in our new approach, Kawski's time-iterative process is replaced by Brouwer's fixed-point theorem. This is why our new notion of 'tangent vector' contains a continuity property.

3.1. Main result

To encompass finite and infinite-dimensional systems, STLC is discussed in terms of the surjectivity of the end-point map. Let us introduce first the functional setting. Let X be a Hilbert space over $\mathbb{R}$. Let $(E_T, \|\cdot\|_{E_T})_{T>0}$ be a family of normed vector spaces of functions defined on $[0,T]$ for $T>0$. Assume that for all $T_1, T_2 > 0$, $u \in E_{T_1}$ and $v \in E_{T_2}$, the concatenation of the two functions $u\#v$ defined by

$$u\#v := u\mathbb{1}_{(0,T_1)} + v(\cdot - T_1)\mathbb{1}_{(T_1,T_1+T_2)} \tag{3.1}$$

is in $E_{T_1+T_2}$ with moreover the following estimate:

$$\|u\#v\|_{E_{T_1+T_2}} \leqslant \|u\|_{E_{T_1}} + \|v\|_{E_{T_2}}. \tag{3.2}$$

For example, for any positive integer k, the space $H_0^k(0,T)$ satisfies this property whereas $H^k(0,T)$ does not. Finally, let $(\mathcal{F}_T)_{T>0}$ be a family of functions from $X \times E_T$ to X for $T>0$. Later, in our application, $\mathcal{F}_T$ will denote the end-point map of a control system, but the following statements hold for general operators. Now, let us introduce the new definition of tangent vector used in this paper.

Definition 3.1. A vector $\xi \in X$ is called a small-time E-continuously approximately reachable vector if there exists a continuous map $\Xi : [0,+\infty) \to X$ with $\Xi(0) = \xi$ such that for all $T>0$, there exist $C, \rho, s > 0$ and a continuous map $b \in (-\rho,\rho) \mapsto u_b \in E_T$ such that,

$$\forall b \in (-\rho,\rho), \quad \|\mathcal{F}_T(0,u_b) - b\Xi(T)\|_X \leqslant C|b|^{1+s} \quad \text{with} \quad \|u_b\|_{E_T} \leqslant C|b|^s. \tag{3.3}$$

The family $(u_b)_{b\in(-\rho,\rho)}$ (resp. the map Ξ) is called the control variations (resp. the vector variations) associated with ξ.

Remark 3.2. Let us stress that, for finite-dimensional systems, in [28], Kawski (see also the work of Frankowska [25, 26]) introduced rather the following definition: a vector ξ is said to be a tangent vector if there exist $m>0$ and a family of controls $(u_T)_{T>0}$ such that

$$\mathcal{F}_T(0,u_T) = T^m\xi + o(T^m) \quad \text{when } T \to 0. \tag{3.4}$$

Definition 3.1 is different: the final time and the amplitude of the target are unrelated. This allows constants in (3.3) badly quantified with respect to the final time T, which is not possible in (3.4).

Our systematic approach is given in the following statement.

Theorem 3.3. *Assume the following hypotheses hold.*

(A_1) *For all $T>0$, the map $\mathcal{F}_T : X \times E_T \to X$ is of class C^2 on a neighborhood of $(0,0)$ with $\mathcal{F}_T(0,0) = 0$.*
(A_2) *For all $x \in X$, the map $T \in \mathbb{R}_+ \mapsto d\mathcal{F}_T(0,0).(x,0) \in X$ can be continuously extended at zero with $d\mathcal{F}_0(0,0).(x,0) = x$.*

(A_3) *For all* $T_1, T_2 > 0$, $x \in X$, $u \in E_{T_1}$ *and* $v \in E_{T_2}$,

$$\mathcal{F}_{T_1+T_2}(x, u\#v) = \mathcal{F}_{T_2}(\mathcal{F}_{T_1}(x, u), v). \tag{3.5}$$

(A_4) *The space* $\mathcal{H} := \operatorname{Ran} \mathrm{d}\mathcal{F}_T(0,0).(0,\cdot)$ *does not depend on* T*, is closed and of finite codimension* n*.*

(A_5) *There exists* $\mathcal{M}$ *a complementary to* $\mathcal{H}$ *that admits a basis* $(\xi_i)_{i=1,\dots,n}$ *of small-time* E*-continuously approximately reachable vectors.*

Then, for all $T > 0$*,* $\mathcal{F}_T$ *is locally onto from zero: for all* $\eta > 0$*, there exists* $\delta > 0$ *such that for all* $x_f \in X$ *with* $\|x_f\|_X < \delta$*, there exists* $u \in E_T$ *with* $\|u\|_{E_T} < \eta$ *such that*

$$\mathcal{F}_T(0, u) = x_f.$$

Let us stress that similar results are already known for finite-dimensional systems (see [28], Thm. 2.4 or [35] for instance). Theorem 3.3 provides a new method for these systems but is mostly a new tool to prove the STLC of infinite-dimensional systems when a finite number of directions is lost at the linear level.

Remark 3.4. If in addition to $(A_1) - (A_5)$, we assume that

(A_6) for all $T > 0$ and $u \in E_T$, $u(T - \cdot)$ is in E_T with

$$\mathcal{F}_T(\mathcal{F}_T(0, u), u(T - \cdot)) = 0,$$

then, for all $T > 0$, $\mathcal{F}_T$ is locally onto: For all $\eta > 0$, there exists $\delta > 0$ such that for all $(x_0, x_f) \in X^2$ with $\|x_0\|_X + \|x_f\|_X < \delta$, there exists $u \in E_T$ with $\|u\|_{E_T} < \eta$ such that

$$\mathcal{F}_T(x_0, u) = x_f.$$

Indeed, let $(x_0, x_f) \in X^2$ with $\|x_0\|_X + \|x_f\|_X < \delta$. By Theorem 3.3, there exist $u, v \in E_T$ such that $\mathcal{F}_T(0, u) = x_0$ and $\mathcal{F}_T(0, v) = x_f$. Then, using successively (A_3) and (A_6), one has

$$\mathcal{F}_{2T}(x_0, u(T - \cdot)\#v) = \mathcal{F}_T(\mathcal{F}_T(x_0, u(T - \cdot)), v) = \mathcal{F}_T(0, v) = x_f.$$

Remark 3.5. The C^2-regularity of $\mathcal{F}_T$ in (A_1) is for convenience. Actually, in the proof of Theorem 3.3, one needs the following estimates: for all $T > 0$ and $R > 0$, there exists $C > 0$ such that for all $(x, u) \in X \times E_T$ with $\|x\|_X + \|u\|_{E_T} < R$,

$$\|\mathcal{F}_T(x, u) - \mathcal{F}_T(0, u) - \mathcal{F}_T(x, 0)\|_X \leqslant C\|x\|_X \|u\|_{E_T}, \tag{3.6}$$

$$\|\mathcal{F}_T(x, u) - d\mathcal{F}_T(0, 0).(x, u)\|_X \leqslant C(\|x\|_X^2 + \|u\|_{E_T}^2). \tag{3.7}$$

Both estimates follow from Taylor's formula when $\mathcal{F}_T$ is of class C^2 with $\mathcal{F}_T(0, 0) = 0$ (expanding each term around $(0, 0)$ in (3.6), one can check that there is no linear term in the expansion, the cross product on the right-hand side stems then from the bilinearity of the second-order differential).

Remark 3.6. When $\mathcal{F}_T$ denotes the end-point map of a control system,

- (A_1) is linked to the well-posedness of the system; also, we assume that $(0, 0)$ is an equilibrium of the system, the translation of our result to another equilibrium is not detailed (see Theorem 6.9 for instance);
- (A_2) is linked to the continuity of the solutions of the linearized system;
- (A_3) is related to the semigroup property of the equation;
- (A_4) means that the linearized system is controllable up to finite codimension;
- (A_5) means that the directions lost at the linear level can be recovered using oscillating controls;

- and (A_6) is linked to the time reversibility of the equation.

3.2. Proof of Theorem 3.3

The first tool is the local surjectivity of the nonlinear map $\mathcal{F}_T$ up to finite codimension.

Proposition 3.7. *Under the assumptions of Theorem 3.3, let $T > 0$, $\mathcal{N}$ be a complementary to $\mathcal{H}$ and $\mathbb{P}$ be the projection on $\mathcal{H}$ parallely to $\mathcal{N}$. Then, $\mathcal{F}_T$ is locally onto in projection on $\mathcal{H}$: there exist $\delta_0, C > 0$ and a C^1-map $\Gamma_T : B_X(0,\delta_0) \times (B_X(0,\delta_0) \cap \mathcal{H}) \to E_T$ with $\Gamma_T(0,0) = 0$ such that for all $(x_0, x_f) \in B_X(0,\delta_0) \times (B_X(0,\delta_0) \cap \mathcal{H})$,*

$$\mathbb{P}[\mathcal{F}_T(x_0, \Gamma_T(x_0, x_f))] = x_f, \tag{3.8}$$

with the size estimate

$$\|\Gamma_T(x_0, x_f)\|_{E_T} \leqslant C\left(\|x_0\|_X + \|x_f\|_X\right). \tag{3.9}$$

Proof. The proof follows by applying the inverse mapping theorem to

$$\begin{array}{ccc} X \times E_T & \to & X \times \mathcal{H} \\ (x,\ u) & \mapsto & (x, \mathbb{P}[\mathcal{F}_T(x,u)]), \end{array} \tag{3.10}$$

which is of class C^1 on a neighborhood of $(0,0)$ between Banach spaces, and whose differential at $(0,0)$ is onto by construction of the space $\mathcal{H}$. □

Then, we prove that every direction spanned by approximately reachable vectors can be recovered using higher order control variations.

Proposition 3.8. *Under the assumptions of Theorem 3.3, there exists $T^* > 0$ such that for all $T \in (0, T^*)$ and $\eta > 0$, there exist $C, s, \rho > 0$, a complementary $\mathcal{M}_T$ to $\mathcal{H}$ and a continuous map $z \in \mathcal{M}_T \cap B_X(0,\rho) \mapsto u_z \in E_T$ such that for all $z \in \mathcal{M}_T \cap B_X(0,\rho)$,*

$$\|\mathcal{F}_T(0, u_z) - z\|_X \leqslant C\|z\|_X^{1+s} \quad \textit{with} \quad \|u_z\|_{E_T} \leqslant \eta. \tag{3.11}$$

Proof. Let $T > 0$ and denote by $(T_i = \frac{iT}{n})_{i=0,\dots,n}$ a subdivision of $[0,T]$. By (A_5), there exist $C, \rho, s > 0$ and for all $i = 1, \dots, n$, two continuous maps $\Xi_i : [0, +\infty) \to X$ with $\Xi_i(0) = \xi_i$ and $b \in (-\rho, \rho) \mapsto u_b^i \in E_{T_i - T_{i-1}}$ such that for all $b \in (-\rho, \rho)$,

$$\left\|\mathcal{F}_{T_i - T_{i-1}}\left(0, u_b^i\right) - b\Xi_i(T_i - T_{i-1})\right\|_X \leqslant C|b|^{1+s} \ \text{ with } \ \|u_b^i\|_{E_{T_i - T_{i-1}}} \leqslant C|b|^s. \tag{3.12}$$

Without loss of generality, one can assume that $s \in (0,1)$. For all $c = (c_1, \dots, c_n) \in \mathbb{R}^n$ with $\|c\|_\infty < \rho$ and $k \in \{1, \dots, n\}$, we define

$$u_c^{\#^k} := u_{c_1}^1 \# u_{c_2}^2 \# \dots \# u_{c_k}^k \in E_{T_k}.$$

First, we prove by induction on $k \in \{1, \dots, n\}$ that for all $T > 0$, there exists $C > 0$ such that for all $\|c\|_\infty < \rho$,

$$\left\|\mathcal{F}_{T_k}(0, u_c^{\#^k}) - \sum_{i=1}^k c_i d\mathcal{F}_{T_k - T_i}(0,0).(\Xi_i(T_i - T_{i-1}), 0)\right\|_X \leqslant C\|c\|_\infty^{1+s}. \tag{3.13}$$

The initialization for $k=1$ follows from the definition of the family $(u_b^1)_{b\in\mathbb{R}}$ in (3.12) and (A_2). Now, assume that (3.13) holds for some $k\in\{1,\dots,n\}$. First, by (A_3), one has,

$$\mathcal{F}_{T_{k+1}}(0,u_c^{\#^{k+1}})=\mathcal{F}_{T_{k+1}-T_k}(\mathcal{F}_{T_k}(0,u_c^{\#^k}),u_{c_{k+1}}^{k+1}).$$

Thus, together with the inequality (3.6), one gets,

$$\begin{aligned}\Big\|\mathcal{F}_{T_{k+1}}(0,u_c^{\#^{k+1}})-\mathcal{F}_{T_{k+1}-T_k}(0,u_{c_{k+1}}^{k+1})-\mathcal{F}_{T_{k+1}-T_k}(\mathcal{F}_{T_k}(0,u_c^{\#^k}),0)\Big\|_X&\\ \leqslant C\|\mathcal{F}_{T_k}(0,u_c^{\#^k})\|_X\cdot\|u_{c_{k+1}}^{k+1}\|_{E_{T_{k+1}-T_k}}\leqslant C\|c\|_\infty|c_{k+1}|^s,&\quad(3.14)\end{aligned}$$

using the induction hypothesis (3.13) and the estimate (3.12). Moreover, using (3.12), one has

$$\|\mathcal{F}_{T_{k+1}-T_k}(0,u_{c_{k+1}}^{k+1})-c_{k+1}\Xi_{k+1}(T_{k+1}-T_k)\|_X\leqslant C|c_{k+1}|^{1+s},\tag{3.15}$$

and, using the Taylor expansion (3.7), one also gets,

$$\begin{aligned}\Big\|\mathcal{F}_{T_{k+1}-T_k}(\mathcal{F}_{T_k}(0,u_c^{\#^k}),0)-d\mathcal{F}_{T_{k+1}-T_k}(0,0).(\mathcal{F}_{T_k}(0,u_c^{\#^k}),0)\Big\|_X&\\ \leqslant C\|\mathcal{F}_{T_k}(0,u_c^{\#^k})\|_X^2\leqslant C\|c\|_\infty^2.&\quad(3.16)\end{aligned}$$

Besides, using the induction hypothesis (3.13), by linearity, one has

$$\begin{aligned}\Big\|d\mathcal{F}_{T_{k+1}-T_k}(0,0).(\mathcal{F}_{T_k}(0,u_c^{\#^k}),0)&\\ -\sum_{i=1}^k c_i d\mathcal{F}_{T_{k+1}-T_k}(0,0).\left(d\mathcal{F}_{T_k-T_i}(0,0).(\Xi_i(T_i-T_{i-1}),0),0\right)\Big\|_X\leqslant C\|c\|_\infty^{1+s}.&\quad(3.17)\end{aligned}$$

Differentiating (3.5), one can notice that for all $T_1,T_2>0$ and $x\in X$,

$$\mathrm{d}\mathcal{F}_{T_1}(0,0).(\mathrm{d}\mathcal{F}_{T_2}(0,0).(x,0),0)=d\mathcal{F}_{T_1+T_2}(0,0).(x,0).$$

Thus, (3.16) and (3.17) lead to

$$\left\|\mathcal{F}_{T_{k+1}-T_k}(\mathcal{F}_{T_k}(0,u_c^{\#^k}),0)-\sum_{i=1}^k c_i d\mathcal{F}_{T_{k+1}-T_i}(0,0).\left(\Xi_i(T_i-T_{i-1}),0\right)\right\|_X\leqslant C\|c\|_\infty^{1+s}.\tag{3.18}$$

Then, estimates (3.14), (3.15) and (3.18) lead to (3.13) for $k+1$. This concludes the induction and (3.13) holds for all $T>0$.

Now, using the estimate (3.13), we prove the statement of Theorem 3.8. The following map is continuous and does not vanish at zero,

$$(t_1,\dots,t_n,\hat{t}_1,\dots,\hat{t}_n)\mapsto\det\left(\mathbb{P}_{\mathcal{M}}[\mathrm{d}\mathcal{F}_{t_1}(0,0).(\Xi_1(\hat{t}_1),0)],\dots,\mathbb{P}_{\mathcal{M}}[\mathrm{d}\mathcal{F}_{t_n}(0,0).(\Xi_n(\hat{t}_n),0)]\right),$$

where $\mathbb{P}_{\mathcal{M}}$ denotes the projection on $\mathcal{M}$, defined in $(A5)$, parallely to $\mathcal{H}$. Thus, there exists $T^*>0$ such that for all $T\in[0,T^*)$, the space

$$\mathcal{M}_T:=\mathrm{Span}(d\mathcal{F}_{T-T_i}(0,0).\left(\Xi_i(T_i-T_{i-1}),0\right),\ i=1,\dots,n),$$

is a complementary to $\mathcal{H}$. Then, using (3.13), the proof is concluded with

$$u_z := u^{\#^n}_{z_1,\dots,z_n} \quad \text{for all} \quad z = \sum_{i=1}^{n} z_i d\mathcal{F}_{T-T_i}(0,0).\,(\Xi_i(T_i - T_{i-1}), 0) \in \mathcal{M}_T.$$

Notice that the continuity of the map $z \mapsto u_z$ stems from the ones of $b \mapsto u^i_b$. □

Remark 3.9. Proposition 3.8 is written this way to prepare the use of Brouwer's fixed-point theorem in the proof of Proposition 3.3. However, the proof of Proposition 3.8 gives also that every linear combination of small-time E-continuously approximately reachable vectors is a small-time E-continuously approximately reachable vector. Indeed, if $z = \sum_{i=1}^{N} z_i \xi_i$ is such a linear combination, then the estimate (3.13) entails the existence of two continuous maps

$$T \overset{\Theta}{\mapsto} \sum_{i=1}^{N} z_i d\mathcal{F}_{T-T_i}(0,0).(\Xi_i(T_i - T_{i-1}), 0) \quad \text{and} \quad b \mapsto u_b := u^{\#^N}_{(bz_1,\dots,bz_n)}$$

with $\Xi(0) = z$ (by construction of the maps Ξ_i and thanks to (A_2)) such that (3.3) holds.

Finally, one can prove Theorem 3.3: the higher order control variations constructed in Proposition 3.8 and the local surjectivity of $\mathcal{F}_T$ up to finite codimension given in Proposition 3.7 are enough to gain back the controllability lost at the linear level.

Proof of Theorem 3.3. Let $T > 0$ the final time, $T_1 \in (0, T)$ an intermediate time and $\eta > 0$ the accuracy on the control. Define $\delta := \min(\delta_0, \frac{\eta}{4C})$ where δ_0 and C are defined by Proposition 3.7. Let x_f in X with $\|x_f\|_X < \delta$. Notice that, as $\mathcal{F}_T(0,0) = 0$, it is enough to prove the theorem for T sufficiently small.

Step 1: Steering 0 *almost to* x_f. Let $\mathcal{M}_{T_1}$ be the complementary to $\mathcal{H}$ given by Proposition 3.8. The goal of this step is to construct a n-parameters family $(v_z)_{z\in\mathcal{M}_{T_1}}$ such that, for all $z \in \mathcal{M}_{T_1}$ small enough, one has

$$\mathbb{P}[\mathcal{F}_T(0, v_z)] = \mathbb{P}x_f, \tag{3.19}$$

$$\|\mathbb{P}_{\mathcal{M}_{T_1}}[\mathcal{F}_T(0, v_z)] - z\|_X \leqslant C\|z\|_X^{1+s} + C\|\mathbb{P}x_f\|_X^2, \quad \text{with } s > 0, \tag{3.20}$$

$$\|v_z\|_{E_T} \leqslant \eta, \tag{3.21}$$

where $\mathbb{P}_{\mathcal{M}_{T_1}}$ denotes the projection on $\mathcal{M}_{T_1}$ parallely to $\mathcal{H}$. By Proposition 3.8, there exist $C, \rho, s > 0$ and a continuous map $\tilde{z} \mapsto u_{\tilde{z}}$ from $\mathcal{M}_{T_1} \cap B_X(0,\rho)$ to E_{T_1} such that,

$$\forall \tilde{z} \in \mathcal{M}_{T_1} \cap B_X(0,\rho), \quad \|\mathcal{F}_{T_1}(0, u_{\tilde{z}}) - \tilde{z}\|_X \leqslant C\|\tilde{z}\|_X^{1+s} \quad \text{with} \quad \|u_{\tilde{z}}\|_{E_{T_1}} \leqslant \frac{\eta}{2}. \tag{3.22}$$

Denote by $(e_i^{T_1})_{i=1,\dots,n}$ a basis of $\mathcal{M}_{T_1}$. Then, the following map is continuous and non-vanishing at zero,

$$t \mapsto \det\left(\mathbb{P}_{\mathcal{M}_{T_1}}[d\mathcal{F}_t(0,0).(e_1^{T_1}, 0)], \dots, \mathbb{P}_{\mathcal{M}_{T_1}}[d\mathcal{F}_t(0,0).(e_n^{T_1}, 0)]\right).$$

Thus, for T small enough, $\mathbb{P}_{\mathcal{M}_{T_1}}[d\mathcal{F}_{T-T_1}(0,0).(\cdot, 0)]$ is invertible from $\mathcal{M}_{T_1}$ to $\mathcal{M}_{T_1}$ with a continuous inverse by the open mapping principle. Hence, there exists a linear continuous map h from $\mathcal{M}_{T_1}$ to $\mathcal{M}_{T_1}$ such that,

$$\forall z \in \mathcal{M}_{T_1}, \quad \mathbb{P}_{\mathcal{M}_{T_1}}[d\mathcal{F}_{T-T_1}(0,0).(h(z), 0)] = z. \tag{3.23}$$

Finally, for all $z \in \mathcal{M}_{T_1} \cap B_X(0,\rho)$, we define

$$v_z := u_{h(z)} \# \Gamma_{T-T_1}(\mathcal{F}_{T_1}(0, u_{h(z)}), \mathbb{P}x_f), \tag{3.24}$$

where $\Gamma_{T-T_1} : B_X(0,\delta_0) \times (B_X(0,\delta_0) \cap \mathcal{H})$ is constructed in Proposition 3.7 with the complementary $\mathcal{M}_{T_1}$. As $\mathcal{F}_{T_1}(0, u_{h(z)}) \to 0$ when z goes to 0, for ρ small enough, one has $\|\mathcal{F}_{T_1}(0, u_{h(z)})\|_X < \delta_0$, and thus, the family (v_z) is well-defined.

Size estimate. By (3.2), for all $z \in \mathcal{M}_{T_1} \cap B_X(0,\rho)$, v_z is in E_T with

$$\begin{aligned}\|v_z\|_{E_T} &\leqslant \|u_{h(z)}\|_{E_{T_1}} + \left\|\Gamma_{T-T_1}(\mathcal{F}_{T_1}(0, u_{h(z)}), \mathbb{P}x_f)\right\|_{E_{T-T_1}} \\ &\leqslant \frac{\eta}{2} + C\|\mathcal{F}_{T_1}(0, u_{h(z)})\|_X + C\|\mathbb{P}x_f\|_X \leqslant \frac{\eta}{2} + 2C\delta \leqslant \eta,\end{aligned}$$

using the estimates (3.22) on $u_{h(z)}$ and (3.9) on Γ_{T-T_1}. This proves (3.21).

Target almost reached. Moreover, using (A_3), one has,

$$\mathcal{F}_T(0, v_z) = \mathcal{F}_{T-T_1}\left(\mathcal{F}_{T_1}(0, u_{h(z)}), \Gamma_{T-T_1}(\mathcal{F}_{T_1}(0, u_{h(z)}), \mathbb{P}x_f)\right). \tag{3.25}$$

Therefore, by definition (3.8) of Γ_{T-T_1}, (3.19) is already satisfied. To prove (3.20), one can use (3.25) together with the inequality (3.6) to get

$$\begin{aligned}\left\|\mathbb{P}_{\mathcal{M}_{T_1}}[\mathcal{F}_T(0, v_z)] - z\right\|_X \leqslant\ & \|\mathbb{P}_{\mathcal{M}_{T_1}}[\mathcal{F}_{T-T_1}(0, \Gamma_{T-T_1}(\mathcal{F}_{T_1}(0, u_{h(z)}), \mathbb{P}x_f))]\|_X \\ &+ \|\mathbb{P}_{\mathcal{M}_{T_1}}[\mathcal{F}_{T-T_1}(\mathcal{F}_{T_1}(0, u_{h(z)}), 0) - z]\|_X \\ &+ C\|\mathcal{F}_{T_1}(0, u_{h(z)})\|_X \cdot \|\Gamma_{T-T_1}(\mathcal{F}_{T_1}(0, u_{h(z)}), \mathbb{P}x_f)\|_{E_{T-T_1}}.\end{aligned} \tag{3.26}$$

Yet, using the estimates (3.9) on Γ_{T-T_1}, (3.22) on $\mathcal{F}_{T_1}(0, u_{h(z)})$ and the continuity of h, the last term of the right-hand side of (3.26) is estimated by $C\|z\|^2 + C\|\mathbb{P}x_f\|^2$. Besides, using the Taylor expansion (3.7), the second term of the right-hand side of (3.26) is estimated by

$$\|\mathbb{P}_{\mathcal{M}_{T_1}}[d\mathcal{F}_{T-T_1}(0,0).(\mathcal{F}_{T_1}(0, u_{h(z)}), 0) - z]\|_X + \|\mathcal{F}_{T_1}(0, u_{h(z)})\|_X^2 \leqslant C\|z\|^{1+s}, \tag{3.27}$$

using the estimate (3.22), the construction (3.23) and the continuity of h. Moreover, by definition of $\mathcal{H}$ in (A_4),

$$\mathbb{P}_{\mathcal{M}_{T_1}}[d\mathcal{F}_{T-T_1}(0,0).(0, \Gamma_{T-T_1}(\mathcal{F}_{T_1}(0, u_z), \mathbb{P}x_f))] = 0.$$

Thus, using again the Taylor expansion (3.7), the first term of the right-hand side of (3.26) is estimated by

$$C\|\Gamma_{T-T_1}(\mathcal{F}_{T_1}(0, u_z), \mathbb{P}x_f)\|_X^2 \leqslant C\left(\|z\|^2 + \|\mathbb{P}x_f\|^2\right), \tag{3.28}$$

using estimate (3.9) on Γ_{T-T_1} and (3.22). Therefore, (3.26), (3.27) and (3.28) lead to (3.20).

Step 2: Steering 0 *to* x_f. Thanks to (3.19) and (3.21), to conclude the proof, it remains to prove the existence of $z \in \mathcal{M}_{T_1} \cap B_X(0,\rho)$ such that $\mathbb{P}_{\mathcal{M}_{T_1}}[\mathcal{F}_T(0, v_z)] = \mathbb{P}_{\mathcal{M}_{T_1}}[x_f]$. To that end, we apply Brouwer's fixed-point theorem to the function

$$G_{x_f} : \left| \begin{array}{rcl} \mathcal{M}_{T_1} \cap B_X(0,\rho) & \longrightarrow & \mathcal{M}_{T_1} \\ z & \longmapsto & z - \mathbb{P}_{\mathcal{M}_{T_1}}[\mathcal{F}_T(0, v_z)] + \mathbb{P}_{\mathcal{M}_{T_1}}[x_f]. \end{array}\right.$$

First, by continuity of $\mathcal{F}_T$, Γ_{T-T_1}, h and $\tilde{z} \mapsto u_{\tilde{z}}$, the map $z \mapsto v_z$ defined in (3.24) is continuous from $\mathcal{M}_{T_1}$ to E_T. Thus, G_{x_f} is continuous. It remains to prove that it stabilizes a ball. Let $\rho' \in (0,\rho)$ such that $C\rho'^s < 1/2$ and reduce $\delta > 0$ so that $C\delta^2 + \delta < \rho'/2$ where C is given in (3.20). Then, using estimate (3.20), one has, for all $z \in \mathcal{M}_{T_1} \cap B_X(0,\rho')$,

$$\|G_{x_f}(z)\|_X \leqslant C\|z\|_X^{1+s} + C\|\mathbb{P}x_f\|_X^2 + \|\mathbb{P}x_f\|_X \leqslant C\rho'^{1+s} + C\delta^2 + \delta \leqslant \rho'.$$

Thus, one can apply Brouwer's fixed-point theorem to G_{x_f} to conclude the proof. □

4. Well-posedness and STLC of the Schrödinger equation

4.1. Well-posedness of the Schrödinger equation

First, we recall the result given in [12, Theorem 2.1] about the well-posedness of the following Cauchy problem, in a functional framework suitable with Theorem 1.4,

$$\begin{cases} i\partial_t\psi(t,x) = -\partial_x^2\psi(t,x) - u(t)\mu(x)\psi(t,x) - f(t,x), & (t,x)\in(0,T)\times(0,1),\\ \psi(t,0)=\psi(t,1)=0, & t\in(0,T),\\ \psi(0,x)=\psi_0(x), & x\in(0,1). \end{cases} \tag{4.1}$$

Beforehand, recall that the spaces $H^s_{(0)}$ are defined in (1.2).

Theorem 4.1. *Let $T>0$, μ satisfying (H_{reg}), $u\in H^2_0((0,T),\mathbb{R})$, $\psi_0\in H^{11}_{(0)}(0,1)$ and $f\in H^2_0((0,T),H^7\cap H^5_{(0)}(0,1))$. There exists a unique solution of (4.1), i.e. a function $\psi\in C^2([0,T],H^7_{(0)}(0,1))$ with $\psi(T)\in H^{11}_{(0)}(0,1)$ such that the following equality holds in $H^7_{(0)}$,*

$$\forall t\in[0,T],\quad \psi(t)=\mathrm{e}^{-\mathrm{i}At}\psi_0 + i\int_0^t \mathrm{e}^{-\mathrm{i}A(t-\tau)}\left(u(\tau)\mu\psi(\tau)+f(\tau)\right)\mathrm{d}\tau.$$

Moreover, for every $R>0$, there exists $C=C>0$ such that if $\|u\|_{H^2_0(0,T)}<R$, then,

$$\|\psi(T)\|_{H^{11}_{(0)}},\ \|\psi\|_{C^2([0,T],H^7_{(0)})}\leqslant C\left(\|\psi_0\|_{H^{11}_{(0)}}+\|f\|_{H^2_0((0,T),H^7\cap H^5_{(0)})}\right).$$

Recall that, even if the Schrödinger equation (4.1) is linear in the unknown, the well-posedness is difficult to prove due to the control, entailing that the problem is not autonomous, and because the map $\varphi\mapsto\mu\varphi$ is not continuous from $H^7_{(0)}$ to $H^7_{(0)}$.

Furthermore, we will need the following continuity results to prove the well-posedness of the equations considered in the following (see Sects. 5.1 and 5.2).

Proposition 4.2. *Assume that μ satisfies (H_{reg}). Then, the operators*

$$\varphi \mapsto \mu\varphi, \tag{4.2}$$

$$\varphi \mapsto \mathrm{e}^{\mathrm{i}\alpha\mu}\varphi,\quad \alpha\in\mathbb{R}, \tag{4.3}$$

$$\varphi \mapsto 2\mu'\varphi'+\mu''\varphi, \tag{4.4}$$

map continuously $H^7\cap H^5_{(0)}$ into itself and the operator

$$\varphi \mapsto 2\mu'\varphi'+\mu''\varphi, \tag{4.5}$$

maps continuously $H^7\cap H^5_{(0)}$ into $H^6\cap H^3_{(0)}$.

Proof. Let $n=0,1,2$. By Leibniz's formula, one has,

$$(\mu\varphi)^{(2n)} = \sum_{k=0}^{n}\binom{2n}{2k}\mu^{(2k)}\varphi^{(2n-2k)} + \sum_{k=0}^{n-1}\binom{2n}{2k+1}\mu^{(2k+1)}\varphi^{(2n-2k-1)}.$$

If $\varphi \in H^7 \cap H^5_{(0)}$, then $(\mu\varphi)^{(2n)}$ vanishes at $x = 0, 1$ as for all $k \in \{0, \dots, n\}$, $\varphi^{(2n-2k)}$ do and for all $k \in \{0, \dots, n-1\}$, $\mu^{(2k+1)}$ do. This gives the continuity of (4.2) from $H^7 \cap H^5_{(0)}$ into itself. The other continuities are proved the same. □

Remark 4.3. The continuity results of Proposition 4.2 are sharp in the sense that, for instance, the operators (4.2), (4.3) and (4.5) do not map continuously $H^7_{(0)}$ into itself.

4.2. Dependency of the solution with respect to the initial condition

We write $\psi(\cdot;\ u,\ \psi_0)$ to denote the solution of (1.1) associated with control u and initial data ψ_0 to keep track of such a dependency. Then, from Theorem 4.1, one can deduce the following result about the dependency of the solution of (1.1) with respect to the initial condition.

Proposition 4.4. *Let $T > 0$, μ satisfying (H_{reg}), $\psi_0 \in H^{11}_{(0)}(0,1)$ and $\tau \in \mathbb{R}$. For all $R > 0$, there exists $C > 0$ such that for all $u \in H^2_0(0,T)$ with $\|u\|_{H^2_0(0,T)} < R$, one has*

$$\|\psi(T;\ u,\ \psi_1(\tau) + \psi_0) - \psi(T;\ u,\ \varphi_1)\mathrm{e}^{-\mathrm{i}\lambda_1\tau} - \mathrm{e}^{-\mathrm{i}AT}\psi_0\|_{H^{11}_{(0)}} \leqslant C\|u\|_{H^2_0(0,T)}\|\psi_0\|_{H^{11}_{(0)}}.$$

Proof. Define, for all $t \in [0,T]$, $\Lambda(t) := \psi(t;\ u,\ \psi_1(\tau) + \psi_0) - \psi(t;\ u,\ \varphi_1)\mathrm{e}^{-\mathrm{i}\lambda_1\tau} - \mathrm{e}^{-\mathrm{i}At}\psi_0$. Notice that Λ is the solution of,

$$i\partial_t\Lambda = -\partial_x^2\Lambda - u(t)\mu(x)\Lambda - u(t)\mu(x)\mathrm{e}^{-\mathrm{i}At}\psi_0,$$

with Dirichlet conditions and $\Lambda(0,\cdot) = 0$. Therefore, Theorem 4.1 gives the existence of $C > 0$ such that

$$\|\Lambda(T)\|_{H^{11}_{(0)}} \leqslant C\|u\|_{H^2_0(0,T)}\|\mathrm{e}^{-\mathrm{i}At}\psi_0\|_{C^2([0,T],H^7_{(0)})} = C\|u\|_{H^2_0(0,T)}\|\psi_0\|_{H^{11}_{(0)}}.$$

□

4.3. Controllability in projection with simultaneous estimates

Now, we recall a local controllability result in projection given in [12, Theorem 1.2], and obtained by the linear test. To that end, we denote by $\mathcal{H} := \overline{\mathrm{Span}_{\mathbb{C}}}\,(\varphi_j,\ j \in \mathbb{N}^* - \{K\})$ and the orthogonal projection on $\mathcal{H}$ by $\mathbb{P}\psi = \psi - \langle\psi, \varphi_K\rangle\varphi_K$ for all $\psi \in L^2(0,1)$. Besides, for any integer $k \in \mathbb{N}^*$, we introduce the following norm,

$$\|u\|_{H^{-k}(0,T)} := |u_1(T)| + \|u_k\|_{L^2(0,T)}, \quad u \in L^2(0,T), \tag{4.6}$$

where we recall that the family $(u_n)_{n\in\mathbb{N}}$ of the primitives of u is defined in (2.1). One must be careful: these norms do not coincide with the usual H^{-k}-norms (see [10], Lem. 3.3.1 for more details). However, this notation is taken to simplify the following statement by being able to write $\|\cdot\|_{H^k(0,T)}$ for positive and negative integers k.

Theorem 4.5. *Let μ satisfying (H_{reg}) and (1.5). Then, the Schrödinger equation (1.1) is STLC in projection around the ground state with controls in H^m_0 and targets in $H^{7+2m}_{(0)}$ for all $m \in \{0,1,2\}$ with the same control map.*

More precisely, for all $T > 0$, there exist C, $\delta > 0$ and a C^1-map $\Gamma_T : \Omega_0 \times \Omega_T \to H^2_0((0,T),\mathbb{R})$ where

$$\Omega_0 := \{\psi_0 \in \mathcal{S} \cap H^{11}_{(0)};\ \|\psi_0 - \varphi_1\|_{H^{11}_{(0)}} < \delta\}, \tag{4.7}$$

$$\Omega_T := \{\psi_f \in \mathcal{H} \cap H^{11}_{(0)};\ \|\psi_f - \mathbb{P}\left(\psi_1(T)\right)\|_{H^{11}_{(0)}} < \delta\}, \tag{4.8}$$

such that $\Gamma_T(\varphi_1, \psi_1(T)) = 0$ *and for every* $(\psi_0, \psi_f) \in \Omega_0 \times \Omega_T$*, the solution of* (1.1) *with control* $u := \Gamma_T(\psi_0, \psi_f)$ *and initial condition* ψ_0 *satisfies*

$$\mathbb{P}\left(\psi(T)\right) = \psi_f, \tag{4.9}$$

with the following boundary conditions on the control

$$u_2(T) = u_3(T) = 0. \tag{4.10}$$

Besides, for all m *in* $\{-3, \dots, 2\}$*, the following estimates hold*

$$\|u\|_{H^m(0,T)} \leqslant C\big(\|\psi_0 - \varphi_1\|_{H^{7+2m}_{(0)}} + \|\psi_f - \mathbb{P}(\psi_1(T))\|_{H^{7+2m}_{(0)}}\big). \tag{4.11}$$

5. Error estimates on the expansion of the solution

The goal of this section is to compute the power series expansion of the solution ψ of the Schrödinger equation (1.1), as it is the key to prove Theorem 1.4. Moreover, this expansion will be studied under the following asymptotic.

Definition 5.1. Given two scalar quantities $A(T, u)$ and $B(T, u)$, we write $A(T, u) = \mathcal{O}(B(T, u))$ if there exist $C, T^* > 0$ such that for any $T \in (0, T^*)$, there exists $\eta > 0$ such that for all $u \in H^2_0(0, T)$ with $\|u\|_{H^2_0(0,T)} < \eta$, we have $|A(T, u)| \leqslant C|B(T, u)|$.

Remark 5.2. The notation $\mathcal{O}$ introduced is not usual, but similar notations have already been used (see [8], Def. 3.1 or [7], Def. 10). Namely, it refers to the convergence $\|u\|_{H^2_0(0,T)} \to 0$. Moreover, this convergence holds uniformly with respect to the final time on a small time interval $[0, T^*]$, but the smallness assumption on the control can depend on the final time T. For instance, the following estimates hold,

$$\|u_1\|_{L^1(0,T)} = \mathcal{O}(\|u_1\|_{L^2(0,T)}), \tag{5.1}$$

$$\|u_1\|_{L^\infty(0,T)} = \mathcal{O}(T), \tag{5.2}$$

$$A = \mathcal{O}(TA + B) \quad \Rightarrow \quad A = \mathcal{O}(B). \tag{5.3}$$

5.1. The auxiliary system trick

More precisely, our goal is to prove that when μ satisfies (H_{lin}), (H_{quad}) and (H_{cub}), along the lost direction, the linear term of the expansion vanishes, and the cubic term prevails on the quadratic one (and on the terms of order higher than four). As the quadratic term namely behaves as the L^2-norm of u_3 (the third time primitive of the control), classical error estimates on the expansion involving the L^2-norm of the control u are not sharp enough. One can compute sharper estimates, involving rather the L^2-norm of the primitive u_1 of the control u by introducing the new state

$$\widetilde{\psi}(t, x) := \psi(t, x)\mathrm{e}^{-\mathrm{i}u_1(t)\mu(x)}, \quad (t, x) \in [0, T] \times [0, 1]. \tag{5.4}$$

This new state satisfies the following equation, called the auxiliary system

$$\begin{cases} i\partial_t\widetilde{\psi} = -\partial_x^2\widetilde{\psi} - iu_1(t)(2\mu'(x)\partial_x\widetilde{\psi} + \mu''(x)\widetilde{\psi}) + u_1(t)^2\mu'(x)^2\widetilde{\psi}, \\ \widetilde{\psi}(t, 0) = \widetilde{\psi}(t, 1) = 0, \\ \widetilde{\psi}(0, \cdot) = \varphi_1. \end{cases} \tag{5.5}$$

This idea was introduced in [18] and later used in [9, Section 3.2] and [13, Section 4.2] for the Schrödinger equation. This strategy can also be found in [7, Section 4] to study the quadratic behavior of scalar-input

systems or in [4, Section 7.1] to give refined error estimates for various expansions of scalar-input affine control systems. First, we state the well-posedness of the auxiliary system, stemming from Theorem 4.1.

Proposition 5.3. *Let $T > 0$, μ satisfying (H_{reg}) and u_1 in $H^3_0((0,T),\mathbb{R})$. There exists a unique solution $\widetilde{\psi}$ of (5.5) in $C^2([0,T], H^7 \cap H^5_{(0)})$, which moreover satisfies*

$$\|\widetilde{\psi}\|_{C^2([0,T],H^7\cap H^5_{(0)})} = \mathcal{O}(1). \tag{5.6}$$

Besides, the following equality holds in $H^5_{(0)}(0,1)$ for every $t \in [0,T]$,

$$\widetilde{\psi}(t) = \psi_1(t) - \int_0^t \mathrm{e}^{-\mathrm{i}A(t-\tau)} \left(u_1(\tau)\left(2\mu'\partial_x + \mu''\right)\widetilde{\psi}(\tau) + iu_1(\tau)^2\mu'^2\widetilde{\psi}(\tau)\right) \mathrm{d}\tau. \tag{5.7}$$

Due to the term $\partial_x\widetilde{\psi}$ in (5.5), in this context, the well-posedness of the auxiliary system is understood through its link (5.4) with the Schrödinger equation (rather than using the semigroup theory, see for instance [24, 33]). However, one needs to be careful: the multiplication by the exponential factor in (5.4) preserves the regularity but not the boundary conditions (see Rem. 4.3). Besides, to prove (5.7), one needs the following smoothing effect first proved in [6] and then generalized in [12].

Proposition 5.4. *Let $k \in \mathbb{N}$. There exists a non-decreasing function $C : [0,+\infty) \to (0,+\infty)$ such that for all $T \geqslant 0$ and $f \in H^k_0((0,T), H^5 \cap H^3_{(0)})$, the map $G : t \mapsto \int_0^t \mathrm{e}^{-\mathrm{i}A(t-\tau)} f(\tau)\mathrm{d}\tau$ belongs to $C^k([0,T], H^5_{(0)})$ with the following estimate,*

$$\|G\|_{C^k([0,T],H^5_{(0)})} \leqslant C\|f\|_{H^k((0,T),H^5\cap H^3_{(0)})}. \tag{5.8}$$

The complete proof of Proposition 5.3 is left to the reader.

5.2. Expansion of the auxiliary system

The first, second and third order terms of the expansion of the solution $\widetilde{\psi}$ of the auxiliary system (5.5) around the trajectory ($\psi_{\text{eq}} = \psi_1, u_{\text{eq}} = 0$) are solutions of,

$$i\partial_t\widetilde{\Psi} = -\partial_x^2\widetilde{\Psi} - iu_1(t)(2\mu'\partial_x\psi_1 + \mu''\psi_1), \tag{5.9}$$

$$i\partial_t\widetilde{\xi} = -\partial_x^2\widetilde{\xi} - iu_1(t)(2\mu'\partial_x\widetilde{\Psi} + \mu''\widetilde{\Psi}) + u_1(t)^2\mu'^2\psi_1, \tag{5.10}$$

$$i\partial_t\widetilde{\zeta} = -\partial_x^2\widetilde{\zeta} - iu_1(t)(2\mu'\partial_x\widetilde{\xi} + \mu''\widetilde{\xi}) + u_1(t)^2\mu'^2\Psi, \tag{5.11}$$

with Dirichlet boundary conditions, and the null function as initial data. Using the well-posedness result given in Theorem 4.1, Proposition 4.2 and the smoothing effect of Proposition 5.4, one can prove that

$$\widetilde{\Psi}, \quad \widetilde{\xi}, \quad \widetilde{\zeta} \quad \text{are in} \quad C^2([0,T], H^7 \cap H^5_{(0)}).$$

Moreover, one has the following estimates,

$$\|\widetilde{\Psi}\|_{C^0([0,T],H^5_{(0)}(0,1))} = \mathcal{O}\big(\|u_1\|_{L^2(0,T)}\big), \tag{5.12}$$

$$\|\widetilde{\xi}\|_{C^0([0,T],H^3_{(0)}(0,1))} = \mathcal{O}\big(\|u_1\|^2_{L^2(0,T)}\big), \tag{5.13}$$

$$\|\widetilde{\zeta}\|_{C^0([0,T],H^1_{(0)}(0,1))} = \mathcal{O}\big(\|u_1\|^3_{L^2(0,T)}\big). \tag{5.14}$$

Besides, the following equalities hold in $H^5_{(0)}$ for every $t \in [0,T]$,

$$\widetilde{\Psi}(t) = -\int_0^t \mathrm{e}^{-\mathrm{i}A(t-\tau)} u_1(\tau)[2\mu'\partial_x\psi_1(\tau) + \mu''\psi_1(\tau)]\mathrm{d}\tau, \tag{5.15}$$

$$\widetilde{\xi}(t) = -\int_0^t \mathrm{e}^{-\mathrm{i}A(t-\tau)}[u_1(\tau)(2\mu'\partial_x\widetilde{\Psi}(\tau) + \mu''\widetilde{\Psi}(\tau)) + iu_1(\tau)^2\mu'^2\psi_1(\tau)]\mathrm{d}\tau, \tag{5.16}$$

$$\widetilde{\zeta}(t) = -\int_0^t \mathrm{e}^{-\mathrm{i}A(t-\tau)}[u_1(\tau)(2\mu'\partial_x\widetilde{\xi}(\tau) + \mu''\widetilde{\xi}(\tau)) + iu_1(\tau)^2\mu'^2\widetilde{\Psi}(\tau)]\mathrm{d}\tau. \tag{5.17}$$

From this, one can deduce the behavior of the first three terms of the expansion of the auxiliary system along the lost direction. Indeed, first, the solution of (5.9) can be computed explicitly as

$$\widetilde{\Psi}(t) = \sum_{j=1}^{+\infty}\left((\lambda_j - \lambda_1)\langle\mu\varphi_1, \varphi_j\rangle \int_0^t u_1(\tau)\mathrm{e}^{\mathrm{i}(\lambda_j-\lambda_1)\tau}\mathrm{d}\tau\right)\psi_j(t), \quad t \in [0,T]. \tag{5.18}$$

Thus, one can deduce directly the following result.

Proposition 5.5. *When μ satisfies (1.4),*

$$\forall t \in [0,T], \quad \langle\widetilde{\Psi}(t), \varphi_K\rangle = 0. \tag{5.19}$$

Then, substituting the explicit form of $\widetilde{\Psi}$ given in (5.18) into (5.16), the quadratic term can also be explicitly computed as

$$\begin{aligned}\widetilde{\xi}(t) = -i\sum_{j=1}^{+\infty}&\left(\langle\mu'^2\varphi_1, \varphi_j\rangle \int_0^t u_1(\tau)^2\mathrm{e}^{\mathrm{i}(\lambda_j-\lambda_1)\tau}\mathrm{d}\tau\right)\psi_j(t)\\ &+\sum_{j=1}^{+\infty}\left(\int_0^t u_1(\tau)\int_0^\tau u_1(s)\widetilde{k}_{\mathrm{quad},j}(\tau,s)\mathrm{d}\tau ds\right)\psi_j(t),\end{aligned} \tag{5.20}$$

where, for all $j \in \mathbb{N}^*$, the quadratic kernel $\widetilde{k}_{\mathrm{quad},j}$ is given by

$$\widetilde{k}_{\mathrm{quad},j}(\tau,s) := \sum_{n=1}^{+\infty}(\lambda_1-\lambda_n)(\lambda_n-\lambda_j)\langle\mu\varphi_1,\varphi_n\rangle\langle\mu\varphi_n,\varphi_j\rangle\mathrm{e}^{\mathrm{i}((\lambda_j-\lambda_n)\tau+(\lambda_n-\lambda_1)s)}. \tag{5.21}$$

Thanks to Remark 1.1, all the quadratic kernels $\widetilde{k}_{\mathrm{quad},j}$ are bounded in $C^4(\mathbb{R}^2,\mathbb{C})$. This regularity is the key to perform integrations by parts and reveal a coercivity quantified by the H^{-3}-norm of the control, as stated in the following result.

Lemma 5.6. *If the control $u \in L^2(0,T)$ is such that $u_2(T) = u_3(T) = 0$, then,*

$$\begin{aligned}\forall j \in \mathbb{N}^*, \quad \langle\widetilde{\xi}(T), \psi_j(T)\rangle = -i\sum_{p=1}^{3}A_j^p\int_0^T u_p(t)^2\mathrm{e}^{\mathrm{i}(\lambda_j-\lambda_1)t}\mathrm{d}t\\ +\int_0^T u_3(t)\int_0^t u_3(\tau)\partial_1^2\partial_2^2\widetilde{k}_{\mathrm{quad},j}(t,\tau)\mathrm{d}\tau\mathrm{d}t.\end{aligned} \tag{5.22}$$

Therefore, when μ satisfies (H_{quad}),

$$\langle \widetilde{\xi}(T), \psi_K(T)\rangle = \mathcal{O}(\|u_3\|^2_{L^2(0,T)}). \tag{5.23}$$

Proof. Let $j \in \mathbb{N}^*$. As $A^1_j := \langle \mu'^2\varphi_1, \varphi_j\rangle$ (see (A.1) below), the computations in (5.20) give

$$\langle \widetilde{\xi}(T), \psi_j(T)\rangle = -iA^1_j \int_0^T u_1(t)^2 \mathrm{e}^{\mathrm{i}(\lambda_j-\lambda_1)t}\mathrm{d}t + \int_0^T u_1(t)\int_0^t u_1(\tau)\widetilde{k}_{\text{quad},j}(t,\tau)\mathrm{d}\tau\mathrm{d}t. \tag{5.24}$$

Besides, for all $m \in \mathbb{N}$ and H in $C^2(\mathbb{R}^2, \mathbb{C})$, if $u_{m+1}(T) = 0$, integrations by parts lead to

$$\begin{aligned}\int_0^T u_m(t)\int_0^t u_m(\tau)H(t,\tau)\mathrm{d}\tau\mathrm{d}t = &\int_0^T u_{m+1}(t)^2\left(\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(H(t,t)) - \partial_1 H(t,t)\right)\mathrm{d}t \\ &+ \int_0^T u_{m+1}(t)\int_0^t u_{m+1}(\tau)\partial_1\partial_2 H(t,\tau)\mathrm{d}\tau\mathrm{d}t.\end{aligned}$$

Therefore, (5.22) is deduced from (5.24) by applying this equality successively with $m = 1$ and $H = \widetilde{k}_{\text{quad},j}$ and with $m = 2$ and $H = \partial_1\partial_2\widetilde{k}_{\text{quad},j}$, also noticing that

$$\forall p = 2, 3, \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\partial_1^{p-2}\partial_2^{p-2}\widetilde{k}_{\text{quad},j}(t,t)\right) - \partial_1^{p-1}\partial_2^{p-2}\widetilde{k}_{\text{quad},j}(t,t) = -iA^p_j \mathrm{e}^{\mathrm{i}(\lambda_j-\lambda_1)t}.$$

□

Finally, using the explicit computations of $\widetilde{\Psi}$ and $\widetilde{\xi}$ given in (5.18) and (5.20), one gets that the third order term is given by

$$\begin{aligned}\tilde{\zeta}(T) = \sum_{j=1}^{+\infty}\Big(i\int_0^T u_1(t)^2\int_0^t u_1(\tau)\widetilde{k}^1_{\text{cub},j}(t,\tau)\mathrm{d}\tau\mathrm{d}t + i\int_0^T u_1(t)\int_0^t u_1(\tau)^2\widetilde{k}^2_{\text{cub},j}(t,\tau)\mathrm{d}\tau\mathrm{d}t \\ - \int_0^T u_1(t)\int_0^t u_1(\tau)\int_0^\tau u_1(s)\widetilde{k}^3_{\text{cub},j}(t,\tau,s)ds\mathrm{d}\tau\mathrm{d}t\Big)\psi_j(T),\end{aligned} \tag{5.25}$$

where the cubic kernels are given by

$$\widetilde{k}^1_{\text{cub},j}(t,\tau) := \sum_{n=1}^{+\infty}(\lambda_1-\lambda_n)\langle\mu\varphi_1,\varphi_n\rangle\langle\mu'^2\varphi_n,\varphi_j\rangle\mathrm{e}^{\mathrm{i}[(\lambda_j-\lambda_n)t+(\lambda_n-\lambda_1)\tau]}, \tag{5.26}$$

$$\widetilde{k}^2_{\text{cub},j}(t,\tau) := \sum_{n=1}^{+\infty}(\lambda_n-\lambda_j)\langle\mu'^2\varphi_1,\varphi_n\rangle\langle\mu\varphi_n,\varphi_j\rangle\mathrm{e}^{\mathrm{i}[(\lambda_j-\lambda_n)t+(\lambda_n-\lambda_1)\tau]}, \tag{5.27}$$

and

$$\begin{aligned}\widetilde{k}^3_{\text{cub},j}(t,\tau,s) := &\sum_{p=1}^{+\infty}\sum_{n=1}^{+\infty}(\lambda_1-\lambda_n)(\lambda_n-\lambda_p)(\lambda_p-\lambda_j) \\ &\times\langle\mu\varphi_1,\varphi_n\rangle\langle\mu\varphi_n,\varphi_p\rangle\langle\mu\varphi_p,\varphi_j\rangle\mathrm{e}^{\mathrm{i}[(\lambda_j-\lambda_p)t+(\lambda_p-\lambda_n)\tau+(\lambda_n-\lambda_1)s]}.\end{aligned} \tag{5.28}$$

5.3. Sharp error estimates for the auxiliary system

The goal of this section is to compute sharp error estimates on the expansion of the auxiliary system.

Proposition 5.7. *If μ satisfies (H_{reg}), then the following error estimates on the expansion of the auxiliary system hold,*

$$\|\widetilde{\psi} - \psi_1 - \widetilde{\Psi}\|_{L^\infty((0,T),H^2_{(0)}(0,1))} = \mathcal{O}\big(\|u_1\|^2_{L^2(0,T)}\big), \tag{5.29}$$

$$\|\widetilde{\psi} - \psi_1 - \widetilde{\Psi} - \widetilde{\xi} - \widetilde{\zeta}\|_{L^\infty((0,T),L^2(0,1))} = \mathcal{O}\big(\|u_1\|^4_{L^2(0,T)}\big). \tag{5.30}$$

Proof. Proof of (5.29). First, by Proposition 5.3, the following equality holds in $H^5_{(0)}$ for all $t \in [0,T]$,

$$\big(\widetilde{\psi} - \psi_1\big)(t) = -\int_0^t \mathrm{e}^{-\mathrm{i}A(t-\tau)}[u_1(\tau)(2\mu'\partial_x + \mu'')\widetilde{\psi}(\tau) + iu_1(\tau)^2\mu'^2\widetilde{\psi}(\tau)]\mathrm{d}\tau,$$

where every term under the integral belongs to $H^5 \cap H^3_{(0)}$. Thus, the triangle inequality and the fact that for all $s > 0$, $\mathrm{e}^{\mathrm{i}As}$ is an isometry from $H^3_{(0)}$ to $H^3_{(0)}$ give,

$$\|\widetilde{\psi} - \psi_1\|_{L^\infty H^3_{(0)}} = \mathcal{O}\big(\|u_1\|_{L^1}\|\widetilde{\psi}\|_{L^\infty H^4_{(0)}} + \|u_1\|^2_{L^2}\|\widetilde{\psi}\|_{L^\infty H^3_{(0)}}\big) = \mathcal{O}(\|u_1\|_{L^2}), \tag{5.31}$$

using estimate (5.6) on $\widetilde{\psi}$ and (5.1). Then, using (5.7) and (5.15), the following equality holds in $H^5_{(0)}$ for all $t \in [0,T]$,

$$(\widetilde{\psi} - \psi_1 - \widetilde{\Psi})(t) = -\int_0^t \mathrm{e}^{-\mathrm{i}A(t-\tau)}[u_1(\tau)\,(2\mu'\partial_x + \mu'')\,(\widetilde{\psi} - \psi_1)(\tau) + iu_1(\tau)^2\mu'^2\widetilde{\psi}(\tau)]\mathrm{d}\tau.$$

Once again, every term under the integral belongs to $H^5 \cap H^3_{(0)}$ thanks to Theorem 4.2, so using the triangle inequality, estimates (5.6) and (5.31), one has

$$\|\widetilde{\psi} - \psi_1 - \widetilde{\Psi}\|_{L^\infty H^2_{(0)}} = \mathcal{O}\big(\|u_1\|_{L^1}\|\widetilde{\psi} - \psi_1\|_{L^\infty H^3_{(0)}} + \|u_1\|^2_{L^2}\|\widetilde{\psi}\|_{L^\infty H^2_{(0)}}\big) = \mathcal{O}(\|u_1\|^2_{L^2}).$$

Proof of (5.30). Once again, the following equality holds in $H^5_{(0)}$,

$$\begin{aligned}(\widetilde{\psi} - \psi_1 - \widetilde{\Psi} - \widetilde{\xi} - \widetilde{\zeta})(t) = -\int_0^t \mathrm{e}^{-\mathrm{i}A(t-\tau)}[u_1(\tau)\,(2\mu'\partial_x + \mu'')\,(\widetilde{\psi} - \psi_1 - \widetilde{\Psi} - \widetilde{\xi})(\tau)\\ + iu_1(\tau)^2\mu'^2(\widetilde{\psi} - \psi_1 - \widetilde{\Psi})(\tau)]\mathrm{d}\tau.\end{aligned}$$

Moreover, as before, one can prove that

$$\|\widetilde{\psi} - \psi_1 - \widetilde{\Psi} - \widetilde{\xi}\|_{L^\infty H^1_{(0)}} = \mathcal{O}(\|u_1\|^3_{L^2}). \tag{5.32}$$

Therefore, (5.30) follows from (5.29) and (5.32). □

To sum up, one has the following result.

Corollary 5.8. *Let μ satisfying (H_{reg}), (H_{lin}) and (H_{quad}). Let $u \in H^2_0(0,T)$ be a control such that $u_2(T) = u_3(T) = 0$. Then, the solution $\widetilde{\psi}$ of the auxiliary system* (5.5) *satisfies*

$$\langle \widetilde{\psi}(T), \psi_K(T)\rangle - i\int_0^T u_1(t)^2 \int_0^t u_1(\tau)\widetilde{k}^1_{\mathrm{cub},K}(t,\tau)\mathrm{d}\tau\mathrm{d}t \\ - i\int_0^T u_1(t)\int_0^t u_1(\tau)^2\widetilde{k}^2_{\mathrm{cub},K}(t,\tau)\mathrm{d}\tau\mathrm{d}t = \mathcal{O}\big(\|u_3\|^2_{L^2(0,T)} + \|u_1\|^3_{L^1(0,T)}\big), \quad (5.33)$$

where we recall that $\widetilde{k}^1_{\mathrm{cub},K}$ *and* $\widetilde{k}^2_{\mathrm{cub},K}$ *are respectively defined by* (5.26) *and* (5.27).

Proof. The computations (5.19), (5.23), (5.25) and the error estimate (5.30) give that the right-hand side of (5.33) is estimated by

$$\mathcal{O}\big(\|u_3\|^2_{L^2(0,T)} + \|u_1\|^3_{L^1(0,T)} + \|u_1\|^4_{L^2(0,T)}\big).$$

Besides, for every control u such that $u_2(T) = u_3(T) = 0$, integrations by parts and Cauchy–Schwarz's inequality prove that

$$\|u_1\|^4_{L^2(0,T)} = \left(\int_0^T u'(t)u_3(t)\mathrm{d}t\right)^2 \leqslant C\|u'\|^2_{L^2(0,T)}\|u_3\|^2_{L^2(0,T)} = \mathcal{O}(\|u_3\|^2_{L^2(0,T)}),$$

recalling that we work in the asymptotic of controls small in H^2_0 (see Def. 5.1). □

5.4. Expansion of the Schrödinger equation

From the expansion of the auxiliary system, one can deduce sharp error estimates on the expansion of the Schrödinger equation. The first, second and third order terms of the expansion of the solution ψ of the Schrödinger equation (1.1) around the trajectory $(\psi_{\mathrm{eq}} = \psi_1, u_{\mathrm{eq}} = 0)$ are solutions of,

$$i\partial_t\Psi = -\partial_x^2\Psi - u(t)\mu(x)\psi_1, \quad (5.34)$$
$$i\partial_t\xi = -\partial_x^2\xi - u(t)\mu(x)\Psi, \quad (5.35)$$
$$i\partial_t\zeta = -\partial_x^2\zeta - u(t)\mu(x)\xi, \quad (5.36)$$

with Dirichlet boundary conditions, and the null function as initial data. Moreover, by identifying the same order terms in (5.4), one has the following links with the expansion of the auxiliary system,

$$\widetilde{\Psi}(t) = \Psi(t) - iu_1(t)\mu\psi_1(t), \quad (5.37)$$
$$\widetilde{\xi}(t) = \xi(t) - iu_1(t)\mu\widetilde{\Psi}(t) + \frac{u_1(t)^2}{2}\mu^2\psi_1(t), \quad (5.38)$$
$$\widetilde{\zeta}(t) = \zeta(t) - iu_1(t)\mu\widetilde{\xi}(t) + \frac{u_1(t)^2}{2}\mu^2\widetilde{\Psi}(t) + i\frac{u_1(t)^3}{6}\mu^3\psi_1(t). \quad (5.39)$$

Then, from the expansion of the auxiliary system, one can deduce sharp error estimates on the expansion of the Schrödinger equation (1.1).

Proposition 5.9. *Let* μ *satisfying* (H_{reg}). *Then,*

$$\|\psi - \psi_1 - \Psi\|_{L^\infty((0,T),L^2(0,1))} = \mathcal{O}\big(\|u_1\|^2_{L^2(0,T)} + |u_1(T)|^2\big), \quad (5.40)$$
$$\|\psi - \psi_1 - \Psi - \xi - \zeta\|_{L^\infty((0,T),L^2(0,1))} = \mathcal{O}\big(\|u_1\|^4_{L^2(0,T)} + |u_1(T)|^4\big). \quad (5.41)$$

Proof. The proofs of (5.40) and (5.41) are very similar. Thus, we only prove (5.41). Using all the links (5.4), (5.37), (5.38) and (5.39) between the expansions of the Schrödinger equation and the auxiliary system, one gets

$$\begin{aligned}(\psi-\psi_1-\Psi-\xi-\zeta)(T) &= \mathrm{e}^{\mathrm{i}u_1(T)\mu}(\widetilde{\psi}-\psi_1-\widetilde{\Psi}-\widetilde{\xi}-\widetilde{\zeta})(T)+(\mathrm{e}^{\mathrm{i}u_1(T)\mu}-1)\widetilde{\zeta}(T)\\ &\quad+(\mathrm{e}^{\mathrm{i}u_1(T)\mu}-1-iu_1(T)\mu)\widetilde{\xi}(T)+(\mathrm{e}^{\mathrm{i}u_1(T)\mu}-1-iu_1(T)\mu-\frac{u_1(T)^2}{2}\mu^2)\widetilde{\Psi}(T)\\ &\quad+(\mathrm{e}^{\mathrm{i}u_1(T)\mu}-1-iu_1(T)\mu-\frac{u_1(T)^2}{2}\mu^2-i\frac{u_1(T)^3}{6}\mu^3)\psi_1(T).\end{aligned}$$

The first term is estimated by $\|u_1\|_{L^2}^4$ thanks to the estimate (5.30) on the auxiliary system. Doing an expansion of $\mathrm{e}^{\mathrm{i}u_1(T)\mu}$, the second term (resp. the third, fourth and fifth term) is estimated by $|u_1(T)|\|\widetilde{\zeta}(T)\|_{L^2}$ (resp. $|u_1(T)|^2\|\widetilde{\xi}(T)\|_{L^2}$, $|u_1(T)|^3\|\widetilde{\Psi}(T)\|_{L^2}$ and $|u_1(T)|^2$). Then, estimates (5.12), (5.13) and (5.14) on $\widetilde{\Psi}$, $\widetilde{\xi}$ and $\widetilde{\zeta}$ together with Young inequalities lead to (5.41). □

To conclude on the error estimate on the expansion of the Schrödinger equation, one needs to estimate the boundary term $u_1(T)$. This can be done for specific motions of the solution.

Lemma 5.10. *For every u in $H_0^2(0,T)$ such that the solution of (1.1) satisfies*

$$\mathrm{Im}\left(\langle\psi(T;\ u,\ \varphi_1),\varphi_1\rangle\right)=\mathrm{Im}\left(\langle\psi_1(T),\varphi_1\rangle\right), \tag{5.42}$$

the following estimate holds

$$|u_1(T)|=\mathcal{O}\big(\|u_1\|_{L^2(0,T)}^2\big). \tag{5.43}$$

Proof. Solving explicitly (5.34), one gets

$$\langle\psi(T),\varphi_1\rangle=\langle\psi_1(T),\varphi_1\rangle+i\mathrm{e}^{-\mathrm{i}\lambda_1 T}\langle\mu\varphi_1,\varphi_1\rangle u_1(T)+\mathcal{O}\big(\|\left(\psi-\psi_1-\Psi\right)(T)\|_{L^2(0,1)}\big).$$

As $\langle\mu\varphi_1,\varphi_1\rangle\neq 0$ by (1.5), the assumption (5.42) and the estimate (5.40) of the quadratic remainder lead to

$$u_1(T)=\mathcal{O}\big(\|u_1\|_{L^2(0,T)}^2+|u_1(T)|^2\big).$$

Using (5.2) and (5.3), this estimate entails (5.43). □

Corollary 5.11. *Let μ satisfying (H_{reg}). Then, for every control $u\in H_0^2(0,T)$ such that the solution of (1.1) satisfies (5.42), the following estimate holds,*

$$\|\psi-\psi_1-\Psi-\xi-\zeta\|_{L^\infty((0,T),L^2(0,1))}=\mathcal{O}\big(\|u_1\|_{L^2(0,T)}^4\big). \tag{5.44}$$

Remark 5.12. This expansion of the solution of the Schrödinger equation (1.1) allows us to shed new light on the assumptions made on μ in Section 1.3.

- If (H_{quad}) is replaced by $A_K^1\neq 0$, by Lemma 5.6, the quadratic term is bounded by $\|u_1\|_{L^2}^2$. Thus, for controls small in $W^{-1,\infty}$, the quadratic term prevails on every cubic term, and the Schrödinger equation (1.1) is not $W^{-1,\infty}$-STLC.
- If (H_{quad}) is replaced by $A_K^1=0$ and $A_K^2\neq 0$, then Lemma 5.6 gives that the quadratic term is bounded by $\|u_2\|_{L^2}^2$. However, the cubic term cannot absorb simultaneously such a quadratic term and the quartic

term as

$$\left|\int_0^T u_1(t)^2 u_2(t)\mathrm{d}t\right|^2 \leqslant \int_0^T u_2(t)^2\mathrm{d}t \cdot \int_0^T u_1(t)^4\mathrm{d}t$$

by Cauchy–Schwarz's inequality. Thus, one cannot hope for a STLC result, unless one manages to prove a sharper error estimate than (5.44).

- Under (H_{quad}), the quadratic term is bounded by $\|u_3\|_{L^2}^2$ (see (5.23)). This time,
  - in the asymptotic of controls small in H_0^2, the cubic term can handle simultaneously such a quadratic term and the terms of order higher than four, as done in Section 6, leading to the H_0^2-STLC of the Schrödinger equation;
  - in the asymptotic of controls small in H_0^3, Gagliardo–Nirenberg inequalities prove that the quadratic term prevails on the cubic term (and on the higher-order terms), and thus one can deny H_0^3-STLC as done in [13].

  The reader can refer to the toy model in Section 2.3 for more details on this competition between the quadratic and cubic terms.

6. Motions in the lost directions

The motions in the $\pm i\varphi_K$ directions are done in two steps.

(a) First, we initiate the motion along $\pm i\varphi_K$ by noticing that when μ satisfies (H_{lin}), (H_{quad}) and (H_{cub}), along $i\varphi_K$, the solution is driven by a cubic term which enables us to move in both $+$ and $-i\varphi_K$ directions. However, at the end of this step, along the other directions $(\varphi_j)_{j\in\mathbb{N}^*-\{K\}}$, the error is possibly big (in a sense to precise).

(b) Thus, this error is corrected using the local controllability in projection result given in Theorem 4.5. However, one needs to make sure that such linear motions do not induce a too large error along $i\varphi_K$ and preserve the work done in the first step.

Then, the motions in the $\pm\varphi_K$ directions are deduced from the motions along $\pm i\varphi_K$, exploiting a rotation phenomenon.

6.1. Step (a): Motions only along the lost direction $i\varphi_K$

Proposition 6.1. *Let μ satisfying (H_{reg}), (H_{lin}), (H_{quad}) and (H_{cub}). For all $T_1 > 0$, there exist $C, \rho > 0$ and a continuous map $b \mapsto u_b$ from $\mathbb{R}$ to $H_0^2(0,T_1)$ such that,*

$$\forall b \in (-\rho,\rho), \quad |\langle \psi(T_1;\ u_b,\ \varphi_1), \psi_K(T_1)\rangle - ib| \leqslant C|b|^{1+\frac{1}{41}}. \tag{6.1}$$

Moreover, for all $p \in [1,+\infty]$ and $k \in \mathbb{Z}$, $k \geq -3$, there exists $C > 0$ such that,

$$\forall b \in (-\rho,\rho), \quad \|u_b\|_{W^{k,p}(0,T_1)} \leqslant C|b|^{\frac{1}{41}(7-4k+\frac{4}{p})}, \tag{6.2}$$

and for all $\varepsilon \in (0,\frac{3}{4})$, there exists $C > 0$ such that for all $b \in (-\rho,\rho)$,

$$\|\psi(T_1;\ u_b,\ \varphi_1) - \psi_1(T_1)\|_{H^{2m+7}_{(0)}(0,1)} \leqslant C|b|^{\frac{1}{41}(10-4m-4\varepsilon)}, \quad \forall m = -3,\dots,2. \tag{6.3}$$

Proof. Let $T_1 > 0$ and $\rho \in (0, T_1^{\frac{41}{4}})$. For all $b \in \mathbb{R}^*$, we define the control u_b by,

$$\forall t \in [0,T_1], \quad u_b(t) := \mathrm{sign}(b)|b|^{\frac{7}{41}}\phi^{(3)}\left(\frac{t}{|b|^{\frac{4}{41}}}\right)$$
$$\text{with} \quad \phi \in C_c^\infty(0,1) \quad \text{such that} \quad C_K\int_0^1 \phi''(\theta)^2\phi'(\theta)d\theta = 1, \tag{6.4}$$

where C_K is defined in (1.10). For all $b \in (-\rho,\rho)-\{0\}$, u_b is supported on $(0,|b|^{\frac{4}{41}}) \subset (0,T_1)$.

Size of the control. Let $p \in [1,+\infty]$ and $k \in \mathbb{Z}$, $k \geq -3$. Using Poincaré's inequality, substituting the explicit form of the control (6.4) and performing the change of variables $t = |b|^{\frac{4}{41}}\theta$, one has, for all $b \in (-\rho,\rho)-\{0\}$,

$$\begin{aligned}\|u_b\|^p_{W^{k,p}(0,T_1)} &\leqslant C\int_0^{|b|^{\frac{4}{41}}}\left||b|^{\frac{1}{41}(7-4k)}\phi^{(3+k)}\left(\frac{t}{|b|^{\frac{4}{41}}}\right)\right|^p \mathrm{d}t\\ &\leqslant C\|\phi^{(3+k)}\|^p_{L^p(0,1)}|b|^{\frac{1}{41}(p(7-4k)+4)}.\end{aligned}$$

Continuity of the map $b \mapsto u_b$. The continuity from $\mathbb{R}^*$ to $H^2_0(0,T_1)$ directly stems from the dominated convergence theorem. Moreover, the previous size estimate (with the H^2_0-norm) allows us to extend continuously the map at zero with $u_0 = 0$.

Expansion of the solution. As $u_1(T_1) = 0$, by (5.4), the end-point of ψ and $\widetilde{\psi}$ are the same. Thus, it suffices to prove (6.1) with $\widetilde{\psi}$ instead of ψ. Moreover, Corollary 5.8 gives the following expansion of the auxiliary system when b goes to zero,

$$\begin{aligned}\Big|\langle\widetilde{\psi}(T_1;\ u_b,\ \varphi_1),\psi_K(T_1)\rangle - i\int_0^{T_1}u_1(t)^2\int_0^t u_1(\tau)\widetilde{k}^1_{\mathrm{cub},K}(t,\tau)\mathrm{d}\tau\mathrm{d}t\\ - i\int_0^{T_1}u_1(t)\int_0^t u_1(\tau)^2\widetilde{k}^2_{\mathrm{cub},K}(t,\tau)\mathrm{d}\tau\mathrm{d}t\Big| \leqslant C|b|^{\frac{42}{41}},\end{aligned} \tag{6.5}$$

because, using (6.2), one has the following estimates,

$$\|u_3\|^2_{L^2(0,T_1)} \leqslant C|b|^{\frac{42}{41}} \qquad \text{and} \qquad \|u_1\|^3_{L^1(0,T_1)} \leqslant C|b|^{\frac{45}{41}}.$$

Then, substituting the explicit form of the control (6.4) and performing the change of variables $(t,\tau) = (|b|^{\frac{4}{41}}\theta_1, |b|^{\frac{4}{41}}\theta_2)$, the two integral terms of (6.5) are given by

$$\begin{aligned}i\,\mathrm{sign}(b)|b|\Big(\int_0^1\phi''(\theta_1)^2\int_0^{\theta_1}\phi''(\theta_2)\widetilde{k}^1_{\mathrm{cub},K}(|b|^{\frac{4}{41}}\theta_1,|b|^{\frac{4}{41}}\theta_2)\mathrm{d}\theta_2\theta_1\\ + \int_0^1\phi''(\theta_1)\int_0^{\theta_1}\phi''(\theta_2)^2\widetilde{k}^2_{\mathrm{cub},K}(|b|^{\frac{4}{41}}\theta_1,|b|^{\frac{4}{41}}\theta_2)\mathrm{d}\theta_2\theta_1\Big).\end{aligned}$$

Expanding the kernels when b goes to zero, as they are both bounded in $C^1(\mathbb{R}^2,\mathbb{C})$, one gets that (6.5) can be written as

$$\left|\langle\widetilde{\psi}(T_1;\ u_b,\ \varphi_1),\psi_K(T_1)\rangle - ib(\widetilde{k}^1_{\mathrm{cub},K}(0,0) - \widetilde{k}^2_{\mathrm{cub},K}(0,0))\int_0^1\phi''(\theta)^2\phi'(\theta)\mathrm{d}\theta\right| \leqslant C|b|^{\frac{42}{41}}.$$

Moreover, looking at the definitions of C_K, $\widetilde{k}^1_{\mathrm{cub},K}$ and $\widetilde{k}^2_{\mathrm{cub},K}$ given respectively in (1.10), (5.26) and (5.27), one can notice that

$$\widetilde{k}^1_{\mathrm{cub},K}(0,0) - \widetilde{k}^2_{\mathrm{cub},K}(0,0) = C_K.$$

Thus, the previous inequality leads to (6.1) by choice of ϕ given in (6.4).

Size of the end-point. Let $m \in \{-3,\dots,2\}$. Using the explicit form of Ψ deduced from (5.18) and (5.37), one gets,

$$\begin{aligned}\|(\psi-\psi_1)(T_1)\|_{H^{2m+7}_{(0)}} \leqslant & \left\| \left(j^{2m+7}\langle \mu\varphi_1,\varphi_j\rangle \int_0^{T_1} u_b(t)\mathrm{e}^{\mathrm{i}(\lambda_j-\lambda_1)(t-T_1)}\mathrm{d}t \right)\right\|_{\ell^2(\mathbb{N}^*)} \\ & + \|(\psi-\psi_1-\Psi)(T_1)\|_{H^{2m+7}_{(0)}}. \end{aligned} \tag{6.6}$$

Yet, for all $j \in \mathbb{N}^* - \{1\}$ and $k \in \mathbb{Z}$ with $k \geqslant -3$, by integrations by parts (integrating u when $k<0$ or differentiating u when $k \geqslant 0$), one gets,

$$\begin{aligned}\left|\int_0^{T_1} u_b(t)\mathrm{e}^{\mathrm{i}(\lambda_j-\lambda_1)t}\mathrm{d}t\right| &= \left|(\lambda_j-\lambda_1)^{-k}\int_0^{T_1} u_b^{(k)}(t)\mathrm{e}^{\mathrm{i}(\lambda_j-\lambda_1)t}\mathrm{d}t\right| \\ &\leqslant C|\lambda_j-\lambda_1|^{-k}|b|^{\frac{1}{41}(11-4k)}, \end{aligned} \tag{6.7}$$

using estimates (6.2). This also holds for $j=1$ as $u_1(T_1)=0$. By interpolation, such estimates hold for all $j \in \mathbb{N}^*$ and $k \in [-3,3]$ with a uniform constant with respect to k. Let $\varepsilon \in (0,3/4)$ and $m \in \{-3,\dots,2\}$. Taking $k = \varepsilon + m + \frac{1}{4}$ in (6.7), summing over $j \in \mathbb{N}^*$ and using Remark 1.1 to estimate the coefficients $(\langle \mu\varphi_1,\varphi_j\rangle)_{j\in\mathbb{N}^*}$, one gets

$$\left\| \left(j^{2m+7}\langle \mu\varphi_1,\varphi_j\rangle \int_0^{T_1} u_b(t)\mathrm{e}^{\mathrm{i}(\lambda_j-\lambda_1)(t-T_1)}\mathrm{d}t \right)\right\|_{\ell^2(\mathbb{N}^*)} \leqslant C\left(\sum_{j=1}^{+\infty}\frac{1}{j^{1+4\varepsilon}}\right)^{1/2}|b|^{\frac{1}{41}(10-4m-4\varepsilon)}. \tag{6.8}$$

Moreover, [12, Proposition 4.5] gives the existence of $C>0$ such that for all $m=-3,\dots,2$,

$$\|(\psi-\psi_1-\Psi)(T_1)\|_{H^{2m+7}_{(0)}} \leqslant C\|u_b\|_{H^m(0,T_1)}\|u_b\|_{H^2(0,T_1)} \leqslant C|b|^{\frac{1}{41}(10-4m)}, \tag{6.9}$$

using (6.2) to estimate the control. Then, (6.6), (6.8) and (6.9) lead to (6.3). □

6.2. Preparation of Step (b): Estimate on the evolution of the solution along the lost direction

To make sure that the linear correction of Step (b) will not destroy the motion done along the lost direction in Step (a) (see Proposition 6.1), one needs to quantify what happens along the lost direction on a time interval $[T_1,T]$, that is to quantify

$$\langle \psi(T),\psi_K(T)\rangle - \langle \psi(T_1),\psi_K(T_1)\rangle.$$

First, this estimate is computed on the quadratic term of the expansion of the Schrödinger equation.

Proposition 6.2. *Let $0 < T_1 < T$. If μ satisfies (H_{quad}), then for all w in $H_0^2(0,T)$ such that $w_1(T_1) = w_2(T_1) = w_3(T_1) = w_2(T) = w_3(T) = 0$, the solution of (5.35) satisfies*

$$\begin{aligned}&|\langle \xi(T;\ w,\ \varphi_1), \psi_K(T)\rangle - \langle \xi(T_1;\ w,\ \varphi_1), \psi_K(T_1)\rangle| \\ &\qquad = \mathcal{O}\big(\|w_3\|^2_{L^2(0,T)} + |w_1(T)|\|w_2\|_{L^1(0,T)} + |w_1(T)|^2\big). \end{aligned} \tag{6.10}$$

Proof. Using the link with the auxiliary system (5.38) and the explicit form of $\widetilde{\Psi}$ given in (5.18), one has, for every $t \in [0,T]$,

$$\langle \xi(t), \psi_K(t)\rangle = \langle \widetilde{\xi}(t), \psi_K(t)\rangle + w_1(t)\int_0^t w_1(\tau) k_{\text{quad},t}(\tau)\mathrm{d}\tau - \frac{w_1(t)^2}{2}\langle \mu^2\varphi_1, \varphi_K\rangle \mathrm{e}^{\mathrm{i}(\lambda_K-\lambda_1)t},$$

where the quadratic kernel $k_{\text{quad},t}$ is given by,

$$k_{\text{quad},t}(\tau) = i\sum_{n=1}^{+\infty}(\lambda_n - \lambda_1)\langle \mu\varphi_1, \varphi_n\rangle\langle \mu\varphi_K, \varphi_n\rangle \mathrm{e}^{\mathrm{i}[(\lambda_n-\lambda_1)\tau + (\lambda_K-\lambda_j)t]}. \tag{6.11}$$

Thus, if the control satisfies $w_1(T_1) = 0$, then,

$$\begin{aligned}\langle \xi(T), \psi_K(T)\rangle - \langle \xi(T_1), \psi_K(T_1)\rangle &= \langle \widetilde{\xi}(T), \psi_K(T)\rangle - \langle \widetilde{\xi}(T_1), \psi_K(T_1)\rangle \\ &\quad + w_1(T)\int_0^T w_1(t) k_{\text{quad},T}(t)\mathrm{d}t - \frac{w_1(T)^2}{2}\langle \mu^2\varphi_1, \varphi_K\rangle \mathrm{e}^{\mathrm{i}(\lambda_K-\lambda_1)T}. \end{aligned} \tag{6.12}$$

Besides, using the estimate of $\widetilde{\xi}$ given in (5.23), one has

$$\langle \widetilde{\xi}(T), \psi_K(T)\rangle - \langle \widetilde{\xi}(T_1), \psi_K(T_1)\rangle = \mathcal{O}\big(\|w_3\|^2_{L^2(0,T)} + \|w_3\|^2_{L^2(0,T_1)}\big) = \mathcal{O}\big(\|w_3\|^2_{L^2(0,T)}\big).$$

As the kernel $k_{\text{quad},T}$ is bounded (independently of T), the third term of the right-hand side of (6.12) is naturally estimated by $\mathcal{O}(|w_1(T)|\|w_1\|_{L^1(0,T)})$. However, this estimate will not be sharp enough to use in the sequel of this paper (and more precisely in the proof of Proposition 6.6). Thus, one computes a sharper estimate by performing one integration by parts in the integral to get an estimate by $\mathcal{O}(|w_1(T)|\|w_2\|_{L^1(0,T)})$, noticing that $k'_{\text{quad},T}$ is still bounded thanks to Proposition 1.1. □

Then, the estimate is computed for the cubic term of the expansion of the auxiliary system.

Proposition 6.3. *Let $0 < T_1 < T$. For all w in $H_0^2(0,T)$ such that $w_2(T_1) = w_2(T) = 0$, the solution of (5.11) satisfies*

$$\begin{aligned}&\big|\langle \widetilde{\zeta}(T;\ w,\ \varphi_1), \psi_K(T)\rangle - \langle \widetilde{\zeta}(T_1;\ w,\ \varphi_1), \psi_K(T_1)\rangle\big| \\ &\qquad = \mathcal{O}\big(\|w_1\|^3_{L^1(0,T)} + \|w_1\|^2_{L^2(T_1,T)}\|w_1\|_{L^1(0,T)} + \|w_2\|_{L^\infty(T_1,T)}\|w_1\|^2_{L^2(0,T)}\big). \end{aligned} \tag{6.13}$$

Proof. Using the explicit form of $\widetilde{\zeta}$ given in (5.25), one has,

$$\begin{aligned}\langle \widetilde{\zeta}(T), \psi_K(T)\rangle - \langle \widetilde{\zeta}(T_1), \psi_K(T_1)\rangle &= i\int_{T_1}^T w_1(t)^2\int_0^t w_1(\tau)\widetilde{k}^1_{\text{cub},K}(t,\tau)\mathrm{d}\tau\mathrm{d}t \\ &\quad + i\int_{T_1}^T w_1(t)\int_0^t w_1(\tau)^2\widetilde{k}^2_{\text{cub},K}(t,\tau)\mathrm{d}\tau\mathrm{d}t + \mathcal{O}\big(\|w_1\|^3_{L^1(0,T)}\big), \end{aligned}$$

as the kernel $\widetilde{k}^3_{\mathrm{cub},K}$ defined in (5.28) is bounded in $C^0(\mathbb{R}^3,\mathbb{C})$. The first term of the right-hand side is bounded by $\|w_1\|^2_{L^2(T_1,T)}\|w_1\|_{L^1(0,T)}$ as $\widetilde{k}^1_{\mathrm{cub},K}$ is also bounded. Moreover, it would seem natural to estimate the second term by $\|w_1\|_{L^1(T_1,T)}\|w_1\|^2_{L^2(0,T)}$. However, as before, it would not provide an estimate sharp enough to use in the sequel of this work. One can compute a sharper estimate by performing one integration by parts to get that the second term of the right-hand side of the previous equality is bounded by,

$$\left|\int_{T_1}^T w_2(t)w_1(t)^2\widetilde{k}^2_{\mathrm{cub},K}(t,t)\mathrm{d}t+\int_{T_1}^T w_2(t)\int_0^t w_1(\tau)^2\partial_1\widetilde{k}^2_{\mathrm{cub},K}(t,\tau)\mathrm{d}\tau\mathrm{d}t\right| = \mathcal{O}\big(\|w_2\|_{L^\infty(T_1,T)}\|w_1\|^2_{L^2(0,T)}\big),$$

as the kernel $\widetilde{k}^2_{\mathrm{cub},K}$ defined in (5.27) is bounded in $C^1(\mathbb{R}^2,\mathbb{C})$. □

Then, one can deduce the estimate for the cubic term of the expansion of the Schrödinger equation.

Proposition 6.4. *Let $0<T_1<T$. For all w in $H^2_0(0,T)$ such that $w_1(T_1)=w_2(T_1)=w_2(T)=0$, the solution of (5.36) satisfies*

$$\begin{aligned}\big|\langle\zeta(T),\psi_K(T)\rangle-\langle\zeta(T_1),\psi_K(T_1)\rangle\big| = \mathcal{O}\big(&\|w_1\|^3_{L^1(0,T)}+\|w_1\|^2_{L^2(T_1,T)}\|w_1\|_{L^1(0,T)}\\ &+\|w_2\|_{L^\infty(T_1,T)}\|w_1\|^2_{L^2(0,T)}+|w_1(T)|\|w_1\|^2_{L^2(0,T)}+|w_1(T)|^3\big).\end{aligned}\tag{6.14}$$

Proof. Using the link with the auxiliary system (5.39), the explicit computations of $\widetilde{\Psi}$ and $\widetilde{\xi}$ given in (5.18) and (5.20), one gets, for all $t\in[0,T]$,

$$\begin{aligned}\langle\zeta(t),\psi_K(t)\rangle = \langle\widetilde{\zeta}(t),\psi_K(t)\rangle &+ w_1(t)\int_0^t w_1(\tau)^2k^1_{\mathrm{cub},t}(\tau)d\tau\\ &+ iw_1(t)\int_0^t w_1(\tau)\int_0^\tau w_1(s)k^2_{\mathrm{cub},t}(\tau,s)ds\mathrm{d}\tau-\frac{w_1(t)^2}{2}\int_0^t w_1(\tau)k^3_{\mathrm{cub},t}(\tau)\mathrm{d}\tau\\ &- i\frac{w_1(t)^3}{6}\langle\mu^3\varphi_1,\varphi_K\rangle\mathrm{e}^{\mathrm{i}(\lambda_K-\lambda_1)t},\end{aligned}$$

where the cubic kernels are given by,

$$\begin{aligned}k^1_{\mathrm{cub},t}(\tau) &:= \sum_{n=1}^{+\infty}\langle\mu'^2\varphi_1,\varphi_n\rangle\langle\mu\varphi_n,\varphi_K\rangle\mathrm{e}^{\mathrm{i}[(\lambda_n-\lambda_1)\tau+(\lambda_K-\lambda_n)t]},\\ k^2_{\mathrm{cub},t}(\tau,s) &:= \sum_{p=1}^{+\infty}\sum_{n=1}^{+\infty}(\lambda_1-\lambda_n)(\lambda_n-\lambda_p)\langle\mu\varphi_1,\varphi_n\rangle\langle\mu\varphi_n,\varphi_p\rangle\langle\mu\varphi_p,\varphi_K\rangle\\ &\qquad\times\mathrm{e}^{\mathrm{i}[(\lambda_p-\lambda_n)\tau+(\lambda_n-\lambda_1)s+(\lambda_K-\lambda_p)t]},\\ k^3_{\mathrm{cub},t}(\tau) &:= \sum_{n=1}^{+\infty}(\lambda_n-\lambda_1)\langle\mu\varphi_1,\varphi_n\rangle\langle\mu^2\varphi_n,\varphi_K\rangle\mathrm{e}^{\mathrm{i}[(\lambda_n-\lambda_1)\tau+(\lambda_K-\lambda_n)t]}.\end{aligned}$$

So, if the control satisfies $w_1(T_1)=0$, one gets

$$\langle\zeta(T),\psi_K(T)\rangle-\langle\zeta(T_1),\psi_K(T_1)\rangle=\langle\widetilde{\zeta}(T),\psi_K(T)\rangle-\langle\widetilde{\zeta}(T_1),\psi_K(T_1)\rangle$$

$$+ w_1(T)\int_0^T w_1(t)^2 k^1_{\mathrm{cub},T}(t)\mathrm{d}t + i w_1(T)\int_0^T w_1(t)\int_0^t w_1(\tau)k^2_{\mathrm{cub},T}(t,\tau)\mathrm{d}\tau\mathrm{d}t$$
$$- \frac{w_1(T)^2}{2}\int_0^T w_1(t)k^3_{\mathrm{cub},T}(t)\mathrm{d}t - i\frac{w_1(T)^3}{6}\langle \mu^3\varphi_1,\varphi_K\rangle \mathrm{e}^{\mathrm{i}(\lambda_K-\lambda_1)T}.$$

From the estimate on the auxiliary system (6.13) and the boundedness of the kernels, one deduces (6.14). □

From the behavior of the quadratic and cubic terms given in (6.10) and (6.14), the error estimate (5.44) and the estimate (5.43) on the boundary term $w_1(T)$, one can deduce the following estimate.

Theorem 6.5. *Let $0 < T_1 < T$, μ satisfying (1.4) and (H_{quad}). For all w in $H^2_0(0,T)$ such that $w_1(T_1) = w_2(T_1) = w_3(T_1) = w_2(T) = w_3(T) = 0$, if the solution of (1.1) satisfies the specific motion (5.42), then one has*

$$|\langle\psi(T),\psi_K(T)\rangle - \langle\psi(T_1),\psi_K(T_1)\rangle| = \mathcal{O}\big(\|w_3\|^2_{L^2(0,T)} + \|w_1\|^2_{L^2(0,T)}\|w_2\|_{L^1(0,T)} + \|w_1\|^3_{L^1(0,T)} + \|w_1\|^2_{L^2(T_1,T)}\|w_1\|_{L^1(0,T)} + \|w_2\|_{L^\infty(T_1,T)}\|w_1\|^2_{L^2(0,T)}\big). \quad (6.15)$$

6.3. Step (b): The vector $i\varphi_K$ is an approximately reachable vector

Proposition 6.6. *Let μ satisfying (H_{reg}), (H_{lin}), (H_{quad}) and (H_{cub}). Then, the vector $i\varphi_K$ is a small-time H^2_0-continuously approximately reachable vector associated with vector variations $\Xi(T) = i\psi_K(T)$. More precisely, for all $T > 0$, there exist $C,\rho > 0$ and a continuous map $b \mapsto w_b$ from $\mathbb{R}$ to $H^2_0(0,T)$ such that,*

$$\forall b \in (-\rho,\rho), \quad \|\psi(T;\ w_b,\ \varphi_1) - \psi_1(T) - ib\psi_K(T)\|_{H^{11}_{(0)}(0,1)} \leqslant C|b|^{1+\frac{1}{82}}, \quad (6.16)$$

with the following size estimate,

$$\forall b \in (-\rho,\rho), \quad \|w_b\|_{H^2_0(0,T)} \leqslant C|b|^{\frac{1}{41}}. \quad (6.17)$$

Proof. Construction of the control. To move along the $\pm i\varphi_K$ directions, we use non-overlapping controls. More precisely, let $0 < T_1 < T$ and define, for all $b \in \mathbb{R}$,

$$w_b := u_b\mathbb{1}_{[0,T_1]} + v_b\mathbb{1}_{[T_1,T]}, \quad (6.18)$$

where $(u_b)_{b\in\mathbb{R}}$ is the family of controls defined on $[0,T_1]$, constructed in Theorem 6.1 and

$$v_b := \Gamma_{T_1,T}\left(\psi(T_1;\ u_b,\ \varphi_1),\psi_1(T)\right),$$

where $\Gamma_{T_1,T}$ is a control operator such that the solution of (1.1) on $[T_1,T]$ with control $u = \Gamma_{T_1,T}(\psi_0,\psi_f)$ and initial condition $\psi(T_1) = \psi_0$ satisfies (4.9) with the boundary conditions on the control (4.10) and the estimates (4.11) (the construction of this operator is deduced from Theorem 4.5 by a translation of controls and a change of global phase on the state, not detailed here for the sake of simplicity).

Size of the controls. Because we use non-overlapping controls, for all $b \in \mathbb{R}$, one has

$$\|w_b\|_{H^2_0(0,T)} = \|u_b\|_{H^2_0(0,T_1)} + \|v_b\|_{H^2_0(T_1,T)} \leqslant C|b|^{\frac{1}{41}} + \|v_b\|_{H^2_0(T_1,T)}, \quad (6.19)$$

using the size estimate (6.2) on $(u_b)_{b\in\mathbb{R}}$. Besides, using the linear estimates (4.11) on $\Gamma_{T_1,T}$ and the estimates (6.3) on the end-point of the solution at time T_1, for all $\varepsilon \in (0,\frac{3}{4})$, one gets the existence of $C,\rho > 0$ such that

for all $b \in (-\rho, \rho)$ and $m \in \{-3, \dots, 2\}$,

$$\|v_b\|_{H_0^m(T_1,T)} \leqslant C \left\|\psi(T_1;\ u_b,\ \varphi_1) - \psi_1(T_1)\right\|_{H_{(0)}^{2m+7}(0,1)} \leqslant C|b|^{\frac{1}{41}(10-4m-4\varepsilon)}. \tag{6.20}$$

Taking $\varepsilon < \frac{1}{4}$, the estimates (6.19) and (6.20) imply (6.17).

Motion along $\pm i\varphi_K$. By construction of $\Gamma_{T_1,T}$ (see (4.9)), we already know that

$$\mathbb{P}(\psi(T;\ w_b,\ \varphi_1)) = \psi_1(T) = \mathbb{P}\left(\psi_1(T) + ib\psi_K(T)\right),$$

where $\mathbb{P}$ denotes the orthogonal projection on $\overline{\mathrm{Span}_{\mathbb{C}}}\left(\varphi_j,\ j \in \mathbb{N}^* - \{K\}\right)$. Thus, to prove (6.16), it only remains to prove that

$$\forall b \in (-\rho, \rho), \quad |\langle \psi(T;\ w_b,\ \varphi_1), \psi_K(T)\rangle - ib| \leqslant C|b|^{1+\frac{1}{82}}. \tag{6.21}$$

By construction of the family $(u_b)_{b\in\mathbb{R}}$, one already knows that

$$\forall b \in (-\rho, \rho), \quad |\langle \psi(T_1;\ w_b,\ \varphi_1), \psi_K(T_1)\rangle - ib| \leqslant C|b|^{1+\frac{1}{41}}.$$

Then, it remains to prove that the linear correction used on the time interval $[T_1, T]$ did not destroy this estimate, that is to prove that, for all $b \in (-\rho, \rho)$,

$$|\langle \psi(T;\ w_b,\ \varphi_1), \psi_K(T)\rangle - \langle \psi(T_1;\ w_b,\ \varphi_1), \psi_K(T_1)\rangle| \leqslant C|b|^{1+\frac{1}{82}}.$$

By Theorem 6.5, the left-hand side of the previous inequality is estimated by

$$\begin{aligned}\mathcal{O}\big(\|w_3\|^2_{L^2(0,T)} + \|w_1\|^2_{L^2(0,T)}\|w_2\|_{L^1(0,T)} + \|w_1\|^3_{L^1(0,T)} \\ + \|v_1\|^2_{L^2(T_1,T)}\|w_1\|_{L^1(0,T)} + \|v_2\|_{L^\infty(T_1,T)}\|w_1\|^2_{L^2(0,T)}\big).\end{aligned} \tag{6.22}$$

Therefore, it remains to use the estimates (6.2) on $(u_b)_{b\in\mathbb{R}}$ and (6.20) on $(v_b)_{b\in\mathbb{R}}$ to prove that the previous quantity is bounded by $|b|^{1+\frac{1}{82}}$. For example, using the estimates in H^{-3}, one has

$$\|w_3\|^2_{L^2(0,T)} = \|u_3\|^2_{L^2(0,T_1)} + \|v_3\|^2_{L^2(T_1,T)} \leqslant C\big(|b|^{\frac{42}{41}} + |b|^{\frac{44}{41}-\frac{8\varepsilon}{41}}\big) \leqslant C|b|^{\frac{42}{41}},$$

for b small enough, choosing $\varepsilon < \frac{1}{4}$. Similarly, one gets,

$$\begin{aligned}\|w_1\|^2_{L^2(0,T)}\|w_2\|_{L^1(0,T)} &\leqslant C\big(|b|^{\frac{26}{41}} + |b|^{\frac{28}{41}-\frac{8\varepsilon}{41}}\big)\big(|b|^{\frac{19}{41}} + |b|^{\frac{18}{41}-\frac{4\varepsilon}{41}}\big), \\ \|w_1\|^3_{L^1(0,T)} &\leqslant C\big(|b|^{\frac{45}{41}} + |b|^{\frac{42}{41}-\frac{12\varepsilon}{41}}\big), \\ \|v_1\|^2_{L^2(T_1,T)}\|w_1\|_{L^1(0,T)} &\leqslant C|b|^{\frac{28}{41}-\frac{8\varepsilon}{41}}\big(|b|^{\frac{15}{41}} + |b|^{\frac{14}{41}-\frac{4\varepsilon}{41}}\big).\end{aligned}$$

Choosing $\varepsilon < 1/24$, for b small enough, these three terms are bounded by $|b|^{1+\frac{1}{82}}$. Then, in (6.22), it only remains to estimate the last term. As (6.20) provides only estimates of $(v_b)_{b\in\mathbb{R}}$ in L^2-spaces, one can use a Gagliardo–Nirenberg inequality (see [32], Thm. p. 125) to estimate $\|v_2\|_{L^\infty(T_1,T)}$. More precisely, there exists $C > 0$ such that

$$\|v_2\|_{L^\infty(T_1,T)} \leqslant C\|v_1\|^{1/2}_{L^2(T_1,T)}\|v_2\|^{1/2}_{L^2(T_1,T)} + C\|v_2\|_{L^2(T_1,T)}.$$

Thus, thanks to (6.20), v_b in $W^{-2,\infty}$ is estimated as

$$\|v_2\|_{L^\infty(T_1,T)} \leqslant C|b|^{\frac{16}{41}-\frac{4\varepsilon}{41}}.$$

And, finally, one has

$$\|v_2\|_{L^\infty(T_1,T)}\|w_1\|^2_{L^2(0,T)} \leqslant C|b|^{\frac{16}{41}-\frac{4\varepsilon}{41}}\left(|b|^{\frac{26}{41}}+|b|^{\frac{28}{41}-\frac{8\varepsilon}{41}}\right)$$

which is also bounded by $|b|^{1+\frac{1}{82}}$. This concludes the proof of (6.16).

Continuity of $b \mapsto w_b$. The map $b \mapsto u_b$ of Proposition 6.1 is continuous from $\mathbb{R}$ to $H^2_0(0,T_1)$. Besides, the continuity of $b \mapsto v_b$ from $\mathbb{R}$ to $H^2_0(T_1,T)$ stems from the regularity of $\Gamma_{T_1,T}$ (see Theorem 4.5) and of the solution of the Schrödinger equation with respect to the control. This gives the continuity of the map $b \mapsto w_b$ from $\mathbb{R}$ to $H^2_0(0,T)$. □

Remark 6.7. The sharp estimates (4.11) and (6.15) on the control operator $\Gamma_{T_1,T}$ and on the evolution of the solution along the lost direction are the key to move along the first lost direction $i\varphi_K$.

6.4. The vector φ_K is an approximately reachable vector

The second approximately reachable vector can be deduced from the first one using a proof inspired by the work [27, Theorem 6] of Hermes and Kawski, where the authors proved that for affine-control systems of the form $\dot{x} = f_0(x) + uf_1(x)$, if for some Lie bracket V of f_0 and f_1, $V(0)$ is a tangent vector (in the sense of (3.4)), then $[f_0, V](0)$ is also a tangent vector.

Proposition 6.8. *Let μ satisfying (H_{reg}), (H_{lin}), (H_{quad}) and (H_{cub}). Then, the vector φ_K is a small-time H^2_0-continuously approximately reachable vector associated with vector variations $\Xi(T) = \psi_K(T)$. More precisely, there exists $T^* > 0$ such that for all $T \in (0,T^*)$, there exist $C, \rho > 0$ and a continuous map $b \mapsto v_b$ from $\mathbb{R}$ to $H^2_0(0,T)$ such that,*

$$\forall b \in (-\rho,\rho), \quad \|\psi(T;\ v_b,\ \varphi_1) - \psi_1(T) - b\psi_K(T)\|_{H^{11}_{(0)}(0,1)} \leqslant C|b|^{1+\frac{1}{82}}, \tag{6.23}$$

with the following size estimate,

$$\forall b \in (-\rho,\rho), \quad \|v_b\|_{H^2_0(0,T)} \leqslant C|b|^{\frac{1}{41}}. \tag{6.24}$$

Proof. Denote by $(u_b)_{b\in\mathbb{R}}$ the control variations associated with $i\varphi_K$, constructed in Proposition 6.6. The goal is to prove the existence of $C > 0$ such that for all $(\alpha,\beta) \in \mathbb{R}^2$ small enough, one has

$$\left\|\psi(3T;\ u_\alpha\#0_{[0,T]}\#u_\beta,\ \varphi_1) - \psi_1(3T) - (i\beta e^{2i(\lambda_K-\lambda_1)T} + i\alpha)\psi_K(3T)\right\|_{H^{11}_{(0)}} \leqslant C|(\alpha,\beta)|^{1+\frac{1}{82}}. \tag{6.25}$$

Let $T \in \left(0, \frac{\pi}{2(\lambda_K-\lambda_1)}\right)$. For all $b \in \mathbb{R}$, taking $(\alpha,\beta) \in \mathbb{R}^2$ such that

$$i\beta e^{2i(\lambda_K-\lambda_1)T} + i\alpha = b,$$

one gets the existence of a family $(v_b)_{b\in\mathbb{R}}$ satisfying (6.23) and (6.24). So, it remains to prove (6.25). First, by construction of $(u_b)_{b\in\mathbb{R}}$ in Proposition 6.6, there exist $C,\rho>0$ such that for all $\alpha\in(-\rho,\rho)$,

$$\|\psi(T;\ u_\alpha,\ \varphi_1)-\psi_1(T)-i\alpha\psi_K(T)\|_{H^{11}_{(0)}}\leqslant C|\alpha|^{1+\frac{1}{82}}\quad\text{with}\quad \|u_\alpha\|_{H^2_0(0,T)}\leqslant C|\alpha|^{\frac{1}{41}}. \tag{6.26}$$

Then, on $[T,2T]$, no control is activated, so $\psi(2T)=\mathrm{e}^{-\mathrm{i}AT}\psi(T)$ and (6.26) becomes

$$\left\|\psi(2T;\ u_\alpha\#0_{[0,T]},\ \varphi_1)-\psi_1(2T)-i\alpha\psi_K(2T)\right\|_{H^{11}_{(0)}}\leqslant C|\alpha|^{1+\frac{1}{82}}. \tag{6.27}$$

Then, using the semigroup property of the Schrödinger equation, one has,

$$\psi(3T;\ u_\alpha\#0_{[0,T]}\#u_\beta,\ \varphi_1)=\psi(T;\ u_\beta,\ \psi(2T;\ u_\alpha\#0_{[0,T]},\ \varphi_1)).$$

Together with Proposition 4.4 about the dependency of the solution of the Schrödinger equation with respect to the initial condition, this gives

$$\begin{aligned}\left\|\psi(3T;\ u_\alpha\#0\#u_\beta,\ \varphi_1)-\psi(T;\ u_\beta,\ \varphi_1)\mathrm{e}^{-\mathrm{i}\lambda_1 2T}-\mathrm{e}^{-\mathrm{i}AT}(\psi(2T;\ u_\alpha\#0,\ \varphi_1)-\psi_1(2T))\right\|_{H^{11}_{(0)}}&\\ \leqslant C\|u_\beta\|_{H^2_0(0,T)}\left\|\psi(2T;\ u_\alpha\#0,\ \varphi_1)-\psi_1(2T)\right\|_{H^{11}_{(0)}}.&\end{aligned}$$

Using the estimates (6.17) on $(u_\beta)_{\beta\in\mathbb{R}}$ and (6.27), the right-hand side of the previous inequality is bounded by $|\beta|^{\frac{1}{41}}|\alpha|$. Then, using once again the estimate (6.27) and the construction of $(u_\beta)_\beta$ given in (6.16), the previous inequality entails

$$\begin{aligned}\left\|\psi(3T;\ u_\alpha\#0_{[0,T]}\#u_\beta,\ \varphi_1)-\psi_1(3T)-i\beta\psi_K(T)\mathrm{e}^{-\mathrm{i}\lambda_1 2T}-i\alpha\psi_K(3T)\right\|_{H^{11}_{(0)}}&\\ \leqslant C|\beta|^{\frac{1}{41}}|\alpha|+C|\alpha|^{1+\frac{1}{82}},&\end{aligned}$$

which gives (6.25) and this concludes the proof.

□

6.5. Proof of Theorem 1.4: The H^2_0−STLC of the Schrödinger equation

The goal of this section is to prove Theorem 1.4 using the systematic approach developed in Section 3. More precisely, we apply Theorem 3.3 with $E_T:=H^2_0(0,T)$, $X:=H^{11}_{(0)}(0,1)$ and

$$\mathcal{F}_T:(\psi_0,u)\mapsto\psi(T;\ u,\ \psi_0),$$

where ψ is the solution of the Schrödinger equation (1.1).

Remark 6.9. For the Schrödinger equation, we work around the trajectory $(\psi_{\mathrm{eq}}=\psi_1,u_{\mathrm{eq}}=0)$. The work in Section 3 can still be used by performing the change of function $\psi^*(t):=\psi(t)\mathrm{e}^{\mathrm{i}\lambda_1 t}-\varphi_1$ to work around $(0,0)$. Thus, in this setting, a vector $\xi\in X$ is called a small-time E-continuously approximately reachable vector if there exists a continuous map $\Xi:[0,+\infty)\to X$ with $\Xi(0)=\xi$ such that for all $T>0$, there exist $C,\rho,s>0$ and a continuous map $b\in(-\rho,\rho)\mapsto u_b\in E_T$ such that for all $b\in(-\rho,\rho)$,

$$\|\psi(T;\ u_b,\ \varphi_1)-\psi_1(T)-b\Xi(T)\|_X\leqslant C|b|^{1+s}\quad\text{with}\quad \|u_b\|_{E_T}\leqslant C|b|^s.$$

Let us check that the assumptions of Theorem 3.3 hold in this setting.

(A_1) By [12, Proposition 4.2], it is known that when μ satisfies (H_{reg}), the end-point map is $C^1(X \times E_T, X)$ around $(\varphi_1, 0)$. The C^2-regularity is proved similarly, and thus, the proof is not detailed.

(A_2) By [12, Proposition 4.2], the differential at $(\varphi_1, 0)$ is given by $d\mathcal{F}_T(\varphi_1, 0).(\Psi_0, v) = \Psi(T)$, where Ψ is the solution of the linearized system

$$\begin{cases} i\partial_t \Psi = -\partial_x^2 \Psi - v(t)\mu(x)\psi_1(t,x), \\ \Psi(t,0) = \Psi(t,1) = 0, \\ \Psi(0,x) = \Psi_0(x). \end{cases}$$

Thus, for all $\Psi_0 \in H^{11}_{(0)}$, the map $T \mapsto d\mathcal{F}_T(\varphi_1, 0).(\Psi_0, 0) = \mathrm{e}^{-\mathrm{i}AT}\Psi_0$ is continuous on $\mathbb{R}$ and $d\mathcal{F}_0(\varphi_1, 0).(\Psi_0, 0) = \Psi_0$.

(A_3) Using the uniqueness result stated in Theorem 4.1, one can check that, for all $T_1, T_2 > 0$, $\psi_0 \in H^{11}_{(0)}$, $u \in H^2_0(0, T_1)$ and $v \in H^2_0(0, T_2)$,

$$\psi(T_1 + T_2;\ u\#v,\ \psi_0) = \psi(T_2;\ v,\ \psi(T_1;\ u,\ \psi_0)).$$

(A_4) By [12, Proposition 4.3], when μ satisfies (H_{lin}), the reachable set of the linearized equation around the ground state is given by

$$\mathcal{H} := \overline{\mathrm{Span}_{\mathbb{C}}}\left(\psi_j(T);\ j \in \mathbb{N}^* - \{K\}\right).$$

This space does not depend on T, is closed, and is of codimension 2 in $L^2(0,1)$.

(A_5) By Propositions 6.6 and 6.8, when μ satisfies (H_{lin}), (H_{quad}) and (H_{cub}), both $i\varphi_K$ and φ_K are small-time H^2_0-continuously approximately reachable vectors.

By Theorem 3.3, when μ satisfies (H_{reg}), (H_{lin}), (H_{quad}) and (H_{cub}), the Schrödinger equation (1.1) is H^2_0-STLC around the ground state with targets in $H^{11}_{(0)}(0,1)$.

Appendix A. Existence of a function μ satisfying all the hypotheses

Remark A.1. In this appendix, the coefficients $(A^p_K)_{p=1,2,3}$ and C_K respectively defined in (1.7), (1.8), (1.9) and (1.10) are seen as quadratic or cubic forms with respect to μ. Moreover, the definition given in terms of series can be tricky to use. Thus, we use instead the expressions in terms of Lie brackets given in Remark 1.2. Computing the Lie brackets, one gets that for all μ satisfying (H_{reg}), the quadratic (resp. cubic) forms A^1_K and C_K are given by

$$A^1_K(\mu) = \langle \mu'^2 \varphi_1, \varphi_K \rangle \quad \text{and} \quad C_K(\mu) = -4\langle \mu'^2 \mu'' \varphi_1, \varphi_K \rangle. \tag{A.1}$$

The similar expression of A^2_K is quite heavy. Computing the associated Lie bracket and then 'symmetrizing' the associated quadratic form (see [13], Prop. A.3 for more details), one gets the existence of a constant $C > 0$ such that for all μ satisfying (H_{reg}), one has

$$|A^2_K(\mu) - \langle {\mu^{(3)}}^2 \varphi_1, \varphi_K \rangle| \leqslant C\|\mu\|^2_{H^2(0,1)}.$$

[13, Proposition A.3] also provides a similar approximate expression of A^3_K, but it will not be useful.

Theorem A.2. *Let $K \in \mathbb{N}^*$, $K \geqslant 2$. There exists μ satisfying (H_{reg}), (H_{lin}), (H_{quad}) and (H_{cub}).*

Remark A.3. To prove Theorem A.2, it is enough to prove the existence of a function $\mu \in H^{11}((0,1),\mathbb{R}) \cap H_0^4(0,1)$ satisfying (1.4), (1.7), (1.8), (1.9), (1.10) and

$$\operatorname{supp} \mu \subset [0,1), \tag{A.2}$$
$$\mu^{(5)}(0) \neq 0, \tag{A.3}$$
$$\forall j \in \mathbb{N}^* - \{K\}, \quad \langle \mu\varphi_1, \varphi_j \rangle \neq 0. \tag{A.4}$$

Indeed, when the boundary conditions (1.3) hold, thanks to (1.11) of Remark 1.1, assumption (1.5) is equivalent to

$$\mu^{(5)}(0) \pm \mu^{(5)}(1) \neq 0 \quad \text{and} \quad \forall j \in \mathbb{N}^* - \{K\},\ \langle \mu\varphi_1, \varphi_j \rangle \neq 0.$$

The proof of Theorem A.2 is in four steps.

- First, using Baire theorem to deal with the infinite number of non-vanishing conditions, one can find a function μ_{ref} satisfying (1.4), (1.10), (A.3) and (A.4). Notice that only the non-vanishing condition (1.9) is not treated at this stage as the strategy of the two following steps, relying on oscillating functions, would destroy this condition.
- Then, using some analyticity and the isolated zeros theorem, one constructs $\hat{\mu}_{\text{ref}}$ a perturbation of μ_{ref} satisfying (1.7) while conserving all the previous properties already satisfied by μ_{ref}.
- Similarly, one constructs then $\widetilde{\mu}_{\text{ref}}$ a perturbation of $\hat{\mu}_{\text{ref}}$ satisfying (1.8) while conserving all the previous properties satisfied by $\hat{\mu}_{\text{ref}}$.
- Finally, using the construction of a quadratic basis, from $\widetilde{\mu}_{\text{ref}}$, one constructs a new function satisfying (1.9) in addition to the previous conditions.

Proof of Theorem A.2. Let $K \in \mathbb{N}^*$, $K \geqslant 2$ and $\overline{x} \in (0,1)$ such that $\varphi_K(\overline{x}) = 0$. As $\varphi_1 > 0$ on $(0,1)$ and $\varphi_K'(\overline{x}) > 0$, one may assume the existence of $\delta > 0$ such that $\varphi_1\varphi_K > 0$ on $(\overline{x}, \overline{x}+\delta)$ and $\varphi_1\varphi_K < 0$ on $(\overline{x}-\delta, \overline{x})$. Let $\eta \in (0, \overline{x}-\delta)$ such that $\varphi_1\varphi_K \neq 0$ on $(0,\eta)$.

Step 1: Existence of μ in $H^{11} \cap H_0^4(0,1)$ supported on $[0,\eta)$ and satisfying (1.4), (1.10), (A.3) and (A.4). In this step, we work with the $H^{11}(0,1)$-topology. Denote by

$$\mathcal{E} := \left\{\mu \in H^{11}((0,1),\mathbb{R});\ \mu \equiv 0 \text{ on } \left[\frac{\eta}{2}, 1\right] \text{ and } \mu \text{ satisfies (1.4)}\right\} \cap H_0^4(0,1),$$
$$\mathcal{U} := \{\mu \in \mathcal{E};\ \mu \text{ satisfies (1.10), (A.3) and (A.4)}\}.$$

The goal of Step 1 is to prove that $\mathcal{U}$ is not empty. As $\mathcal{E}$ is not empty, it suffices to prove that $\mathcal{U}$ is dense in $\mathcal{E}$. Moreover, denoting by

$$\mathcal{C} := \{\mu \in \mathcal{E};\ C_K(\mu) \neq 0\},\ \mathcal{V} := \left\{\mu \in \mathcal{E};\ \mu^{(5)}(0) \neq 0\right\} \text{ and } \mathcal{U}_j := \{\mu \in \mathcal{E};\ \langle \mu\varphi_1, \varphi_j \rangle \neq 0\},$$

$\mathcal{U}$ is the intersection of all the open subsets $\mathcal{V}$, $\mathcal{C}$ and $\mathcal{U}_j$ for $j \in \mathbb{N}^* - \{K\}$. Thus, as $\mathcal{E}$ is a complete space (because closed in H^{11}), by Baire theorem, to prove that $\mathcal{U}$ is dense in $\mathcal{E}$, it suffices to prove that $\mathcal{V}$, $\mathcal{C}$ and $\mathcal{U}_j$ for $j \in \mathbb{N}^* - \{K\}$ are dense in $\mathcal{E}$. The density of $\mathcal{V}$ is clear. Let $j \in \mathbb{N}^* - \{K\}$.

$\mathcal{U}_j$ is dense in $\mathcal{E}$. Let μ^* in $\mathcal{E}$ such that $\langle \mu^*\varphi_1, \varphi_j \rangle = 0$ and let $\varepsilon > 0$. As the linear forms $\mu \mapsto \langle \mu\varphi_1, \varphi_K \rangle$ and $\mu \mapsto \langle \mu\varphi_1, \varphi_j \rangle$ are linearly independent (for $j \neq K$), one can find $\nu \in C_c^\infty(0, \frac{\eta}{2})$ such that $\langle \nu\varphi_1, \varphi_K \rangle = 0$ and $\langle \nu\varphi_1, \varphi_j \rangle \neq 0$. Then $\mu_\varepsilon := \mu^* + \frac{\varepsilon}{\|\nu\|}\nu$ is in $\mathcal{U}_j$ with $\|\mu_\varepsilon - \mu^*\|_{H^{11}} < \varepsilon$.

$\mathcal{C}$ is dense in $\mathcal{E}$. Let μ^* in $\mathcal{E}$ such that $C_K(\mu^*) = 0$ and let $\varepsilon > 0$. By a similar construction than the one given in [13, Theorem A.4], one can find $\nu \in C_c^\infty(0, \frac{\eta}{2})$ such that $\langle \nu\varphi_1, \varphi_K \rangle = 0$ and $C_K(\nu) \neq 0$. Then, by (A.1),

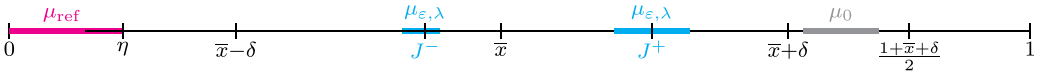

FIGURE A.1. The supports of the functions used in Step 2 are depicted.

$\varepsilon \mapsto C_K(\mu^* + \frac{\varepsilon}{\|\nu\|}\nu)$ is a polynomial of degree 3 vanishing at zero. Thus, there exists $\varepsilon^* > 0$ such that this polynomial does not vanish on $(0, \varepsilon^*)$. Hence, for all $\varepsilon \in (0,\ \varepsilon^*)$, $\mu_\varepsilon := \mu^* + \frac{\varepsilon}{\|\nu\|}\nu$ is in $\mathcal{C}$ with $\|\mu_\varepsilon - \mu^*\|_{H^{11}} < \varepsilon$.

Step 2: Existence of μ in $H^{11} \cap H_0^4$ satisfying (1.4), (1.7), (1.10), (A.2), (A.3) *and* (A.4). Let μ_{ref} in $H^{11} \cap H_0^4$ constructed at Step 1, supported on $[0, \eta)$ and satisfying (1.4), (1.10), (A.3) and (A.4). The goal of this step is to prove that if μ_{ref} does not already satisfy (1.7) (we assume then that $A_K^1(\mu_{\text{ref}}) < 0$, the case $A_K^1(\mu_{\text{ref}}) > 0$ is similar), then one can construct a perturbation of μ_{ref} satisfying (1.7) while conserving the properties already satisfied by μ_{ref}. To that end, one can consider the following 'basis' functions.

- Let μ_0 in $C_c^\infty\left(\overline{x} + \delta, \frac{1+\overline{x}+\delta}{2}\right)$ such that $\langle \mu_0 \varphi_1, \varphi_K \rangle = 1$.
- Let J^- and J^+ two open intervals of respectively $(\overline{x} - \delta, \overline{x})$ and $(\overline{x}, \overline{x} + \delta)$. For all $\varepsilon > 0$ and $\lambda \neq 0$, we define

$$\mu_{\varepsilon,\lambda}(x) := \sqrt{\frac{\varepsilon|\lambda|}{|\varphi_1(x(\lambda))\varphi_K(x(\lambda))|}}\, g\left(\frac{x - x(\lambda)}{\varepsilon}\right), \quad x \in [0,1], \tag{A.5}$$

  where $g \in C_c^\infty(0,1)$ such that $\int_0^1 g'(y)^2 dy = 1$ and $x(\lambda) := x^+ \mathbb{1}_{\lambda>0} + x^- \mathbb{1}_{\lambda<0}$ where $x^\pm$ are in $J^\pm$ (thus, $\text{sign}(\varphi_K(x^\pm)) = \pm 1$). Notice that $\mu_{\varepsilon,\lambda}$ is supported on $(x^-, x^- + \varepsilon) \cup (x^+, x^+ + \varepsilon)$ and thus on $J^- \cup J^+$ for ε small enough. Formally, $\mu_{\varepsilon,\lambda}$ is constructed so that $A_K^1(\mu_{\varepsilon,\lambda}) \approx \lambda$.

We consider perturbations of μ_{ref} of the following form,

$$\nu_{\varepsilon,\lambda} := \mu_{\text{ref}} + \mu_{\varepsilon,\lambda} - \langle \mu_{\varepsilon,\lambda}\varphi_1, \varphi_K \rangle \mu_0, \quad \varepsilon > 0, \quad \lambda \neq 0. \tag{A.6}$$

Notice that all the functions have disjoint supports (see Fig. A.1) so that the quadratic and cubic forms can be seen as additive. Moreover, by construction, for all $\varepsilon > 0$ and $\lambda \neq 0$, $\nu_{\varepsilon,\lambda}$ already satisfies (1.4), (A.2) and (A.3).

Step 2.1: For all ε small enough, there exists $\lambda(\varepsilon) > 0$ such that $\nu_{\varepsilon,\lambda(\varepsilon)}$ satisfies (1.7). The goal is to construct a one-parameter family of functions such that the following quantity vanishes,

$$Q(\varepsilon, \lambda) := A_K^1(\nu_{\varepsilon,\lambda}) = A_K^1(\mu_{\text{ref}}) + A_K^1(\mu_{\varepsilon,\lambda}) + \langle \mu_{\varepsilon,\lambda}\varphi_1, \varphi_K \rangle^2 A_K^1(\mu_0). \tag{A.7}$$

Regularity of Q. Looking at (A.5), one could fear some lack of regularity for Q with respect to λ. However, as $A_K^1(\mu_{\text{ref}}) < 0$, one only needs to study Q on $(\mathbb{R}_+^*)^2$. Moreover, substituting the expression (A.5) and performing the change of variables $x = \varepsilon y + x^+$, one has, for all $\varepsilon > 0$ and $\lambda > 0$,

$$\begin{aligned}\langle \mu_{\varepsilon,\lambda}\varphi_1, \varphi_K \rangle &= \sqrt{\frac{\varepsilon\lambda}{\varphi_1(x^+)\varphi_K(x^+)}} \int_{x^+}^{x^+ + \varepsilon} g\left(\frac{x - x^+}{\varepsilon}\right) \varphi_1(x)\varphi_K(x) \mathrm{d}x \\ &= \varepsilon^{3/2} \sqrt{\frac{\lambda}{\varphi_1(x^+)\varphi_K(x^+)}} \int_0^1 g(y)\varphi_1(\varepsilon y + x^+)\varphi_K(\varepsilon y + x^+) \mathrm{d}y. \end{aligned} \tag{A.8}$$

Thus, the map $(\varepsilon,\lambda)\mapsto\langle\mu_{\varepsilon,\lambda}\varphi_1,\varphi_K\rangle$ is analytic on $(\mathbb{R}_+^*)^2$. Similarly, using the computation of A_K^1 given in (A.1), one gets, for all $\varepsilon>0$ and $\lambda>0$,

$$A_K^1(\mu_{\varepsilon,\lambda})=\frac{\lambda}{\varphi_1(x^+)\varphi_K(x^+)}\int_0^1 g'(y)^2\varphi_1(\varepsilon y+x^+)\varphi_K(\varepsilon y+x^+)\mathrm{d}y. \tag{A.9}$$

Hence, the map $(\varepsilon,\lambda)\mapsto A_K^1(\mu_{\varepsilon,\lambda})$ is also analytic on $(\mathbb{R}_+^*)^2$. Thus, Q is analytic on $(\mathbb{R}_+^*)^2$.

For ε small enough, $Q(\varepsilon,\cdot)$ can take both signs. Doing a Taylor expansion with respect to ε of (A.9), one gets the existence of $C>0$ such that for all $\varepsilon>0$ and $\lambda>0$,

$$\left|A_K^1(\mu_{\varepsilon,\lambda})-\lambda\right|\leqslant C\lambda\varepsilon. \tag{A.10}$$

Thus, (A.7), (A.8) and (A.10) lead to the existence of $C>0$ such that for all $\varepsilon\in(0,1)$ and $\lambda>0$,

$$|Q(\varepsilon,\lambda)-A_K^1(\mu_{\mathrm{ref}})-\lambda|\leqslant C\lambda\varepsilon. \tag{A.11}$$

Thus, there exists $\varepsilon^*>0$ such that for all $\varepsilon\in(0,\varepsilon^*)$,

$$Q\left(\varepsilon,-\frac{A_K^1(\mu_{\mathrm{ref}})}{2}\right)<0\quad\text{and}\quad Q\left(\varepsilon,-\frac{3A_K^1(\mu_{\mathrm{ref}})}{2}\right)>0.$$

For ε small enough, $Q(\varepsilon,\cdot)$ is increasing. Differentiating (A.8) and (A.9) with respect to λ and performing again an expansion with respect to ε in the spirit of (A.10), one gets the existence of $C>0$ such that for all $\varepsilon>0$ and $\lambda>0$, $|\partial_\lambda Q(\varepsilon,\lambda)-1|\leqslant C\varepsilon$. Thus, for $\varepsilon>0$ small enough, $\partial_\lambda Q(\varepsilon,\cdot)$ is positive.

Conclusion. Applying the intermediate value theorem, one gets for all $\varepsilon\in(0,\varepsilon^*)$ the existence of $\lambda=\lambda(\varepsilon)\in\left(-\frac{A_K^1(\mu_{\mathrm{ref}})}{2},-\frac{3A_K^1(\mu_{\mathrm{ref}})}{2}\right)$ such that $Q(\varepsilon,\lambda(\varepsilon))=0$, meaning that $\nu_{\varepsilon,\lambda(\varepsilon)}$ defined in (A.6) satisfies (1.7) by definition (A.7) of Q.

Step 2.2: The map $\varepsilon\mapsto\lambda(\varepsilon)$ is continuous. As for all $\varepsilon\in(0,\varepsilon^*)$, $Q(\varepsilon,\lambda(\varepsilon))=0$, recalling the definition (A.7) of Q, one has

$$\lambda(\varepsilon)=\lambda(\varepsilon)-A_K^1(\mu_{\varepsilon,\lambda(\varepsilon)})-A_K^1(\mu_{\mathrm{ref}})-\langle\mu_{\varepsilon,\lambda(\varepsilon)}\varphi_1,\varphi_K\rangle^2A_K^1(\mu_0)=:\lambda(\varepsilon)H(\varepsilon)-A_K^1(\mu_{\mathrm{ref}}).$$

Using (A.8) and (A.9), H is analytic on $(0,\varepsilon^*)$ with $|H(\varepsilon)|\leqslant C\varepsilon$ for all $\varepsilon\in(0,\varepsilon^*)$. Thus, for all ε and ε_0 in $(0,\varepsilon^*)$,

$$|\lambda(\varepsilon)-\lambda(\varepsilon_0)|\leqslant C\varepsilon|\lambda(\varepsilon)-\lambda(\varepsilon_0)|+|\lambda(\varepsilon_0)||H(\varepsilon)-H(\varepsilon_0)|.$$

Reducing ε^* if needed, the continuity of $\varepsilon\mapsto\lambda(\varepsilon)$ on $(0,\varepsilon^*)$ stems from the one of H.

Step 2.3: The map $\varepsilon\mapsto\lambda(\varepsilon)$ is analytic. Let $\varepsilon_0\in(0,\varepsilon^*)$. By construction, $Q(\varepsilon_0,\lambda(\varepsilon_0))=0$. Besides, in Step 2.1, we proved that $\partial_\lambda Q(\varepsilon_0,\lambda(\varepsilon_0))>0$ and that Q is analytic on $(0,\varepsilon^*)\times\mathbb{R}_+^*$. Hence, by the implicit function theorem, there exists an open neighborhood $\mathcal{V}$ of ε_0, an open neighborhood $\mathcal{W}$ of $\lambda(\varepsilon_0)$ and an analytic function $\Lambda:\mathcal{V}\to\mathcal{W}$ such that

$$(\varepsilon\in\mathcal{V},\ \lambda\in\mathcal{W}\text{ and }Q(\varepsilon,\lambda)=0)\quad\Leftrightarrow\quad(\varepsilon\in\mathcal{V}\text{ and }\lambda=\Lambda(\varepsilon))$$

As $\varepsilon\mapsto\lambda(\varepsilon)$ is continuous, locally $\lambda=\Lambda$ and thus, $\varepsilon\mapsto\lambda(\varepsilon)$ is analytic on $(0,\varepsilon^*)$.

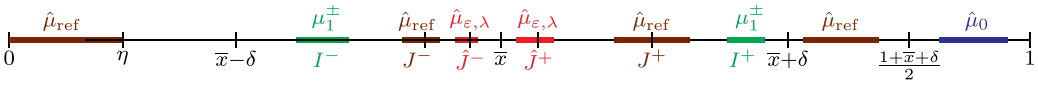

FIGURE A.2. The supports of the functions used in Step 3 are depicted.

Step 2.4: There exists ε such that $\nu_{\varepsilon,\lambda(\varepsilon)}$ satisfies (A.4) *and* (1.10). By a similar computation than the one in (A.8), one gets, for all $j \in \mathbb{N}^*$, $\varepsilon > 0$ and $\lambda > 0$,

$$\langle \nu_{\varepsilon,\lambda}\varphi_1, \varphi_j\rangle = \langle \mu_{\rm ref}\varphi_1, \varphi_j\rangle + \varepsilon^{3/2}\sqrt{\frac{\lambda}{\varphi_1(x^+)\varphi_K(x^+)}}\int_0^1 g(y)\varphi_1(\varepsilon y + x^+)\varphi_j(\varepsilon y + x^+)\mathrm{d}y \\ - \langle \mu_{\varepsilon,\lambda}\varphi_1, \varphi_K\rangle\langle \mu_0\varphi_1, \varphi_j\rangle.$$

As $(\varepsilon, \lambda) \mapsto \langle \mu_{\varepsilon,\lambda}\varphi_1, \varphi_K\rangle$ is analytic on $(\mathbb{R}_+^*)^2$ (see Step 2.1) and $\varepsilon \mapsto \lambda(\varepsilon)$ is analytic on $(0, \varepsilon^*)$ (see Step 2.3) , for all $j \in \mathbb{N}^* - \{K\}$, the map $\varepsilon \mapsto \langle \nu_{\varepsilon,\lambda(\varepsilon)}\varphi_1, \varphi_j\rangle$ is analytic on $(0, \varepsilon^*)$. It can also be extended by continuity at zero with the value $\langle \mu_{\rm ref}\varphi_1, \varphi_j\rangle \neq 0$ by construction of $\mu_{\rm ref}$. Similarly, using (A.1), $\varepsilon \mapsto C_K(\nu_{\varepsilon,\lambda(\varepsilon)})$ is analytic on $(0, \varepsilon^*)$ and can be extended continuously at zero with the value $C_K(\mu_{\rm ref}) \neq 0$. Thus, the functions $(\varepsilon \mapsto \langle \nu_{\varepsilon,\lambda(\varepsilon)}\varphi_1, \varphi_j\rangle)_{j\in\mathbb{N}^*-\{K\}}$ and $\varepsilon \mapsto C_K(\nu_{\varepsilon,\lambda(\varepsilon)})$ are analytic and non-zero on $(0, \varepsilon^*)$. Hence, by the isolated zeros theorem, there exists $\varepsilon \in (0, \varepsilon^*)$, such that for all $j \in \mathbb{N}^* - \{K\}$, $\langle \nu_{\varepsilon,\lambda(\varepsilon)}\varphi_1, \varphi_j\rangle \neq 0$ and $C_K(\nu_{\varepsilon,\lambda(\varepsilon)}) \neq 0$, meaning that $\nu_{\varepsilon,\lambda(\varepsilon)}$ satisfies (1.10) and (A.4).

Step 3: Existence of μ in $H^{11} - H_0^4$ satisfying (1.4), (1.7), (1.8), (1.10), (A.2), (A.3) *and* (A.4). The proof of Step 3 is quite similar to the one of Step 2. Let $\hat\mu_{\rm ref}$ constructed at Step 2 satisfying (1.4), (1.7), (1.10), (A.2), (A.3) and (A.4). As in Step 2, the goal is to prove that if $\hat\mu_{\rm ref}$ does not already satisfy (1.8) (we assume that $A_K^2(\hat\mu_{\rm ref}) < 0$), then one can construct a perturbation of $\hat\mu_{\rm ref}$ satisfying (1.8) while conserving the properties already satisfied by $\hat\mu_{\rm ref}$. Let $\hat J^+$ and I^+ (resp. $\hat J^-$ and I^-) open disjoint intervals of $(\overline{x} - \delta, \overline{x}) - J^-$ (resp. $(\overline{x}, \overline{x} + \delta) - J^+$). In this step, we consider the following new 'basis' functions.

- Let $\hat\mu_0$ in $C_c^\infty\left(\frac{1+\overline{x}+\delta}{2}, 1\right)$ such that $\langle \hat\mu_0\varphi_1, \varphi_K\rangle = 1$ and $A_K^1(\hat\mu_0) = 0$.
- By [13, Theorem A.4], there exists $\mu_1^\pm$ in $C_c^\infty(I^\pm)$ such that $\langle \mu_1^\pm\varphi_1, \varphi_K\rangle = 0$ and $A_K^1(\mu_1^\pm) = \pm 1$.
- For all $\varepsilon > 0$ and $\lambda \neq 0$, we define

$$\hat\mu_{\varepsilon,\lambda}(x) := \varepsilon^{5/2}\sqrt{\frac{|\lambda|}{|\varphi_1(x(\lambda))\varphi_K(x(\lambda))|}}\, g\left(\frac{x - x(\lambda)}{\varepsilon}\right),$$

  where $g \in C_c^\infty(0, 1)$ such that $\int_0^1 g^{(3)}(y)^2\mathrm{d}y = 1$ and $x(\lambda) := \hat x^+\mathbb{1}_{\lambda>0} + \hat x^-\mathbb{1}_{\lambda<0}$, where $\hat x^\pm$ are in $\hat J^\pm$. Notice that for ε small enough, the support of $\hat\mu_{\varepsilon,\lambda}$ is in $\hat J^- \cup \hat J^+$. Formally, this time, $\hat\mu_{\varepsilon,\lambda}$ is constructed so that $A_K^2(\hat\mu_{\varepsilon,\lambda}) \approx \lambda$.

In this step, we consider perturbations of $\hat\mu_{\rm ref}$ of the form,

$$\hat\nu_{\varepsilon,\lambda} := \hat\mu_{\rm ref} + \hat\mu_{\varepsilon,\lambda} - \langle \hat\mu_{\varepsilon,\lambda}\varphi_1, \varphi_K\rangle\hat\mu_0 + \sqrt{|A_K^1(\hat\mu_{\varepsilon,\lambda})|}\mu_1^{-\operatorname{sign}(A_K^1(\hat\mu_{\varepsilon,\lambda}))}, \quad \varepsilon > 0, \quad \lambda \neq 0.$$

Once again, we made sure that all the functions considered have disjoint supports (see Fig. A.2) so that the quadratic and cubic forms can be seen as additive.

Moreover, by construction, for all $(\varepsilon,\lambda)\in\mathbb{R}_+^*\times\mathbb{R}^*$, $\hat{\nu}_{\varepsilon,\lambda}$ already satisfies (1.4), (1.7), (A.2) and (A.3). Then, define

$$\hat{Q}(\varepsilon,\lambda):=A_K^2(\hat{\nu}_{\varepsilon,\lambda})=A_K^2(\hat{\mu}_{\mathrm{ref}})+A_K^2(\hat{\mu}_{\varepsilon,\lambda})+\langle\hat{\mu}_{\varepsilon,\lambda}\varphi_1,\varphi_K\rangle^2A_K^2(\hat{\mu}_0)$$
$$+|A_K^1(\hat{\mu}_{\varepsilon,\lambda})|A_K^2(\mu_1^{-\operatorname{sign}(A_K^1(\hat{\mu}_{\varepsilon,\lambda}))}). \quad \text{(A.12)}$$

The end of Step 3 is exactly the same as the one of Step 2.

- Applying the intermediate value theorem to $\hat{Q}(\varepsilon,\cdot)$, one proves the existence of a continuous map $\varepsilon\mapsto\lambda(\varepsilon)$ such that for all ε small enough, $\hat{Q}(\varepsilon,\lambda(\varepsilon))=0$ and thus, $\hat{\nu}_{\varepsilon,\lambda(\varepsilon)}$ satisfies (1.8). Notice that due to the last term in (A.12), one could fear some lack of regularity of $\hat{Q}$. However, as $\lambda\mapsto A_K^1(\hat{\mu}_{\varepsilon,\lambda})$ is continuous on $\mathbb{R}_+^*$ with a computation similar to (A.9), $\operatorname{sign}(A_K^1(\hat{\mu}_{\varepsilon,\lambda}))$ is locally constant around $\lambda=-A_K^2(\hat{\mu}_{\mathrm{ref}})$.
- Then, one uses the implicit function theorem to get that $\varepsilon\mapsto\lambda(\varepsilon)$ is analytic and thus, with the isolated zeros theorem, to get the existence of an ε such that $\hat{\nu}_{\varepsilon,\lambda(\varepsilon)}$ satisfies (1.10) and (A.4).

Step 4: Existence of μ in $H^{11}\cap H_0^4$ satisfying (1.4), (1.7), (1.8), (1.9), (1.10), (A.2), (A.3) *and* (A.4) . Let $\widetilde{\mu}_{\mathrm{ref}}$ constructed at Step 3 satisfying (1.4), (1.7), (1.8), (1.10), (A.2), (A.3) and (A.4). Assume that $A_K^3(\widetilde{\mu}_{\mathrm{ref}})=0$, otherwise $\widetilde{\mu}_{\mathrm{ref}}$ already satisfies (1.9). Let $\widetilde{J}^-$ and $\widetilde{J}^+$ two open disjoint intervals of respectively $(\overline{x}-\delta,\overline{x})-(J^-\cup\hat{J}^-\cup I^-)$ and $(\overline{x},\overline{x}+\delta)-(J^+\cup\hat{J}^+\cup I^+)$. By [13, Theorem A.4], there exists ν in $C_c^\infty(\widetilde{J}^\pm)$ such that

$$\langle\nu\varphi_1,\varphi_K\rangle=A_K^1(\nu)=A_K^2(\nu)=0 \quad \text{and} \quad A_K^3(\nu)=1.$$

Define for all $\varepsilon\in\mathbb{R}$, $\nu_\varepsilon:=\widetilde{\mu}_{\mathrm{ref}}+\varepsilon\nu$. By construction, for all $\varepsilon\in\mathbb{R}^*$, ν_ε satisfies (1.4), (1.7), (1.8), (1.9), (A.2) and (A.3) because the functions have disjoint supports. Moreover, the maps $\varepsilon\mapsto C_K(\nu_\varepsilon)$ and $\varepsilon\mapsto\langle\nu_\varepsilon\varphi_1,\varphi_j\rangle$ for all $j\in\mathbb{N}^*-\{K\}$ are polynomial, so analytic and non-vanishing at zero by construction of $\widetilde{\mu}_{\mathrm{ref}}$. So, by the isolated zeros theorem, there exists $\varepsilon\in\mathbb{R}^*$ such that ν_ε satisfies (1.10) and (A.4). □

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